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by

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Crispness and Representation Theorem in Dedekind Categories

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Abstract

This paper studies notions of scalar relations and crispness of relations.

1 Introduction

Since Zadeh's invention the concept of fuzzy sets has been extensively investigated in mathematics, science and engineering. The notion of fuzzy relations is also a basic one in processing fuzzy information in relational structures, see e.g. Pedrycz [15]. Goguen [5] generalized the concepts of fuzzy sets and relations taking values from partially ordered sets. Fuzzy relational equations were initiated and applied to medical models of diagnosis by Sanchez [17].

On the other hand, the theory of relations, namely relational calculus, has a long history, see [13, 18, 19] for more details. Almost all modern formalizations of relation algebras are affected by the work of Tarski [20]. Mac Lane [12] and Puppe [16] exposed a categorical basis for the calculus of additive relations. Freyd and Scedrov [2] developed and summarized categorical relational calculus, which they called allegories. Concerning applications to the relational theory of graphs and programs, Schmidt and Ströhlein [18] gave a simple proof of a representation theorem for Boolean relation algebras satisfying the Tarski rule and the point axiom. They also wrote an excellent text book [19] on relations and graphs with many useful examples from computer science. In relational calculus one calculates with relations in an element-free style, which makes relational calculus a very useful framework for the study of mathematics [8] and theoretical computer science [1, 7, 11] and also a useful tool for applications. Some element-free formalizations of fuzzy relations and proofs of representation theorems were provided in [3, 9, 10].

In this paper we consider Dedekind categories named by Olivier and Serrato [14]. One of the aim of this paper is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness was introduced in [10] under the assumption that Dedekind categories have unit objects which are an abstraction of singleton

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(or one-point) sets. To capture the notion of crispness without such assumption, we use a notion of scalar relations. The notion of scalar relations in homogeneous relation algebras was introduced in [4]. The other aim of this paper is to prove a representation theorem for Dedekind categories. Such a theorem for Dedekind categories with a unit object satisfying strict point axiom was also proved in [10]. This paper is organized as follows:
In section 2 we first state the definition of complete Dedekind categories [14] as a categorical structure formed by L-relations [5] with sup-inf composition. Also we define a preoder among objects of Dedekind categories which compares the lattice structures on objects in a sense. Section 3 studies notions of scalars and crispness for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. In section 4 we recall the definition of L-relations, due to Goguen [5], and illustrate a few relationships between crispness and lattice structures of scalars. In section 5 we show a representation theorem for uniform Dedekind categories satisfying the strict point axiom without the assumption of existence of unit objects, and it is proved that the representation function is a bijection preserving all operations of Dedekind categories.

2 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call Dedekind categories following Olivier and Serrato [14].

Throughout this paper, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoondown Y$, and the composite of a morphism $\alpha : X \rightharpoondown Y$ followed by a morphism $\beta : Y \rightharpoondown Z$ will be written as $\alpha\beta : X \rightharpoondown Z$. We denote the identity morphism on an object X by id_X.

Definition 2.1 A *Dedekind category* $\mathcal{D}$ is a category satisfying the following:
D1. [Complete Distributive Lattice] For all pairs of objects X and Y the hom-set $\mathcal{D}(X,Y)$ consisting of all morphisms of X into Y is a complete distributive lattice with the least morphism 0_{XY} and the greatest morphism ∇_{XY}.
D2. [Involution] There is given a monotone and involutive contravariant functor $^\sharp : \mathcal{D} \rightarrow \mathcal{D}$. That is, for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$, the following involutive laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residues] For all morphisms $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoondown Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoondown Y$. □

Note that complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras.

Example 2.2 Consider a category Rel_0 whose objects are all nonempty sets and in which a hom-set $Rel_0(X,Y)$ between objects X and Y is the set of all (binary) relations on X if $X = Y$, and $\nabla_{XY} = 0_{XY}$ otherwise. That is, a hom-set $Rel_0(X, Y)$ is a singleton set when X and Y are distinct. Then it is easy to verify that the category Rel_0 is a Dedekind category. The conditions (D1) and (D2) are trivial, and (D3) and (D4) also hold as follows: If $X = Y = Z$, then (D3) and (D4) are clear. If $X = Y \neq Z$, then $\beta = 0_{YZ}$, $\gamma = 0_{XZ}$ and $\gamma \div \beta = \nabla_{XX}$. If $X \neq Y$, then $\alpha = 0_{XY}$ and $\gamma \div \beta = 0_{XY}$. □

Throughout the rest of this section, all discussions will assume a fixed Dedekind category $\mathcal{D}$. The greatest morphism ∇_{XY} is called the *universal* morphism and the least morphism 0_{XY} the *zero* morphism. A morphism is *nonzero* if it is not equal to the zero morphism. An object X is called *empty* if $\nabla_{XX} = 0_{XX}$, and *nonempty* otherwise.

Proposition 2.3 *Let $\alpha, \alpha' : X \rightharpoonup Y$ and $\beta, \beta' : Y \rightharpoonup Z$ be morphisms in $\mathcal{D}$.*

(a) $\nabla_{XX}\nabla_{XY} = \nabla_{XY}\nabla_{YY} = \nabla_{XY}$.

(b) *If $\alpha \sqcup \alpha' = \nabla_{XY}$, $\alpha \sqcap \alpha' = 0_{XY}$ and $\nabla_{XX}\alpha = \alpha$, then $\nabla_{XX}\alpha' = \alpha'$.*

(c) *If $u \sqsubseteq \mathrm{id}_X$ and $v \sqsubseteq \mathrm{id}_X$, then $u^\sharp = uu = u$ and $uv = u \sqcap v$.*

(d) *If $u \sqsubseteq \mathrm{id}_X$ and $v \sqsubseteq \mathrm{id}_Y$, then $u\alpha = \alpha \sqcap u\nabla_{XY}$ and $\alpha v = \alpha \sqcap \nabla_{XY}v$.*

Proof is omitted. □

The statement (a) in the last proposition indicates that if $\nabla_{XY} \neq 0_{XY}$, then both of X and Y are nonempty.

Proposition 2.4 *Let $\alpha : X \rightharpoonup Y$ be a morphism such that $\nabla_{XX}\alpha = \alpha$. Then the following three conditions are equivalent: (a) $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$, (b) $\nabla_{XX} = \alpha\alpha^\sharp$, (c) $\nabla_{XX} = \alpha\nabla_{YX}$.*

Proof. (a)⇒(b) If $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$, then $\nabla_{XX} = \nabla_{XX}\mathrm{id}_X \sqsubseteq \nabla_{XX}\alpha\alpha^\sharp = \alpha\alpha^\sharp$. (b)⇒(c) If $\nabla_{XX} = \alpha\alpha^\sharp$, then $\nabla_{XX} = \alpha\alpha^\sharp \sqsubseteq \alpha\nabla_{YX}$. (c)⇒(a) If $\nabla_{XX} = \alpha\nabla_{YX}$, then $\mathrm{id}_X = \mathrm{id}_X \sqcap \nabla_{XX} = \mathrm{id}_X \sqcap \alpha\nabla_{YX} \sqsubseteq \alpha(\alpha^\sharp\mathrm{id}_X \sqcap \nabla_{YX}) = \alpha\alpha^\sharp$. □

Now we define a function $\phi_W : \mathcal{D}(X, Y) \to \mathcal{D}(W, W)$ by

$$\phi_W(\xi) = \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W : W \rightharpoonup W$$

for a morphism $\xi : X \rightharpoonup Y$ and an object W of a Dedekind category $\mathcal{D}$. Then the following lemma holds:

Lemma 2.5 (a) $\phi_W(\xi)\nabla_{WZ} = \nabla_{WX}\xi\nabla_{YZ}$ *and* $\nabla_{ZW}\phi_W(\xi) = \nabla_{ZX}\xi\nabla_{YW}$ *for each object Z,*

(b) $\phi_W(\phi_X(\xi)) = \phi_W(\phi_Y(\xi)) = \phi_W(\xi)$,

(c) $\phi_W(\xi) = \phi_W(\xi^\sharp)$,

(d) *If* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $\xi \sqsubseteq \nabla_{XW}\phi_W(\xi)\nabla_{WY}$,

(e) *If* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $\phi_W(\xi) = 0_{WW}$ *is equivalent to* $\xi = 0_{XY}$.

Proof. (a) The former follows from

$$\begin{array}{rcl} \phi_W(\xi)\nabla_{WZ} & = & (\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} \\ & \sqsubseteq & \nabla_{WX}\xi\nabla_{YW}\nabla_{WZ} \\ & \sqsubseteq & \nabla_{WX}\xi\nabla_{YZ} \\ & = & \nabla_{WX}\xi\nabla_{YZ} \sqcap \nabla_{WZ} \\ & \sqsubseteq & (\nabla_{WX}\xi\nabla_{YZ}\nabla^\sharp_{WZ} \sqcap \mathrm{id}_W)\nabla_{WZ} \\ & \sqsubseteq & (\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} \\ & = & \phi_W(\xi)\nabla_{WZ}. \end{array}$$

The latter is similar.
(b) follows from

$$\begin{array}{rcll} \phi_W(\phi_X(\xi)) & = & \nabla_{WX}\phi_X(\xi)\nabla_{XW} \sqcap \mathrm{id}_W & \{\text{ Definition of } \phi_W \} \\ & = & \nabla_{WX}\nabla_{XX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & \{ \phi_X(\xi)\nabla_{XW} = \nabla_{XX}\xi\nabla_{YW} \} \\ & = & \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & \{ \nabla_{WX}\nabla_{XX} = \nabla_{WX} \} \\ & = & \phi_W(\xi) & \{\text{ Definition of } \phi_W \} \end{array}$$

and

$$\begin{array}{rcll} \phi_W(\phi_Y(\xi)) & = & \nabla_{WY}\phi_Y(\xi)\nabla_{YW} \sqcap \mathrm{id}_W & \{\text{ Definition of } \phi_W \} \\ & = & \nabla_{WX}\xi\nabla_{YY}\nabla_{YW} \sqcap \mathrm{id}_W & \{ \nabla_{WX}\phi_Y(\xi) = \nabla_{WX}\xi\nabla_{YY} \} \\ & = & \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & \{ \nabla_{YY}\nabla_{YW} = \nabla_{YW} \} \\ & = & \phi_W(\xi) & \{\text{ Definition of } \phi_W \}. \end{array}$$

$$\begin{array}{ccc} \mathcal{D}(X,Y) & \xrightarrow{\phi_X} & \mathcal{D}(X,X) \\ {\scriptstyle \phi_Y}\downarrow & & \downarrow{\scriptstyle \phi_W} \\ \mathcal{D}(Y,Y) & \xrightarrow[\phi_W]{} & \mathcal{D}(W,W) \end{array}$$

(c) follows from $\phi_W(\xi^\sharp) = (\phi_W(\xi^\sharp))^\sharp = (\nabla_{WY}\xi^\sharp\nabla_{XW} \sqcap \mathrm{id}_W)^\sharp = \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W = \phi_W(\xi)$.
(d) If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then

$$\begin{array}{rcl} \xi & = & \xi \sqcap \nabla_{XY} \\ & = & \xi \sqcap \nabla_{XW}\nabla_{WY} \\ & \sqsubseteq & \nabla_{XW}(\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WY} \\ & = & \nabla_{XW}\phi_{XYW}(\xi)\nabla_{WY}. \end{array}$$

(e) is immediate from (d). □

A binary relation $\prec$ among objects of $\mathcal{D}$ is defined as follows: For two objects X and Y, the relation $X \prec Y$ holds if and only if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$. (Note that the three conditions $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$, $\mathrm{id}_X \sqsubseteq \nabla_{XY}\nabla_{YX}$ and $\phi_{YYX}(\mathrm{id}_Y) = \mathrm{id}_X$ are mutually equivalent.) It is easy to see that $\prec$ is a preorder, that is, reflexive and transitive. For $\nabla_{XX} = \nabla_{XX}\nabla_{XX}$, and if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ and $\nabla_{YY} = \nabla_{YZ}\nabla_{ZY}$, then $\nabla_{XX} = \nabla_{XY}\nabla_{YY}\nabla_{YX} = \nabla_{XY}\nabla_{YZ}\nabla_{ZY}\nabla_{YX} \sqsubseteq \nabla_{XZ}\nabla_{ZX}$. Hence its symmetric kernel with $X \sim Y$ if and only if $X \prec Y$ and $Y \prec X$, is an equivalence relation. Remark that in the category Rel_0 of 2.2, two distinct objects are never equivalent.

Proposition 2.6 *Assume that $X \prec Y$. If $u \sqsubseteq \mathrm{id}_X$, $u \sqsubseteq \mathrm{id}_X$ and $u\nabla_{XY} \sqsubseteq v\nabla_{XY}$ for $u, v : X \rightharpoondown X$, then $u \sqsubseteq v$.*

Proof. It follows from $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ that $u = \mathrm{id}_X \sqcap u\nabla_{XX} = \mathrm{id}_X \sqcap u\nabla_{XY}\nabla_{YX}$. □

Definition 2.7 A Dedekind category $\mathcal{D}$ is *uniform* if all pairs of objects of $\mathcal{D}$ are equivalent, that is, if $X \sim Y$ for all objects X and Y of $\mathcal{D}$. □

A morphism $f : X \rightharpoondown Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) and $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function* and may be introduced as $f : X \to Y$.

Proposition 2.8 (a) *If there exists at least one total morphism $\alpha : X \rightharpoondown Y$, then $X \prec Y$.*

(b) *If there exists at least one function $f : X \to Y$, then $X \prec Y$.*

(c) *If $X \prec W$ or $Y \prec W$, then $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$.*

(d) *If $X \prec Y$ and $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then $X \prec W$.*

(e) *If $\nabla_{XY} = p^\sharp q$ for some functions $p : W \to X$ and $q : W \to Y$ and if $X \prec Y$, then $X \sim W$.*

Proof. (a) $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp \sqsubseteq \nabla_{XY}\nabla_{YX}$. (b) It is a just corollary of (a). (c) If $\nabla_{XX} = \nabla_{XW}\nabla_{WX}$, then $\nabla_{XY} = \nabla_{XX}\nabla_{XY} = \nabla_{XW}\nabla_{WX}\nabla_{XY} \sqsubseteq \nabla_{XW}\nabla_{WY}$. (d) $\nabla_{XX} = \nabla_{XY}\nabla_{YX} = \nabla_{XW}\nabla_{WY}\nabla_{YX} \sqsubseteq \nabla_{XW}\nabla_{WX}$. (e) First note that $W \prec X$ by (a). Next $\nabla_{XY} = p^\sharp q \sqsubseteq \nabla_{XW}\nabla_{WX}$ and so it follows from (d) that $X \prec W$. □

3 Scalars and Crispness

We now introduce the two notions of scalars and of s-crisp relations as a preparation for defining a concept of points with a separation property, that is, different points never meet.

Definition 3.1 A *scalar* k on X is a morphism $k : X \rightharpoondown X$ of $\mathcal{D}$ such that $k \sqsubseteq \mathrm{id}_X$ and $k\nabla_{XX} = \nabla_{XX}k$. □

A scalar k on X commutes with all endomorphisms $\alpha : X \rightharpoondown X$, that is, $k\alpha = \alpha k$, because

$$k\alpha = \alpha \sqcap k\nabla_{XX} = \alpha \sqcap \nabla_{XX}k = \alpha k.$$

It is trivial that the zero morphism $0_{XX} : X \rightharpoonup X$ and the identity morphism $\mathrm{id}_X : X \rightharpoonup X$ are scalars on X. The set of all scalars on X is denoted by $\mathcal{F}(X)$. It is clear that $\mathcal{F}(X)$ is a complete distributive lattice for all objects X. A morphism $\xi : X \rightharpoonup Y$ is called an *ideal* if $\nabla_{XX}\xi\nabla_{YY} = \xi$. The notion of ideals in relation algebras was initially introduced by Jónsson and Tarski [6]. The following lemma shows that scalars bijectively correspond to ideals.

Lemma 3.2 (a) *If $\iota : X \rightharpoonup X$ is an ideal, then $k = \iota \sqcap \mathrm{id}_X$ is a scalar on X such that $\iota = k\nabla_{XX}$.*

(b) *If k is a scalar on X, then $\iota = k\nabla_{XX}$ is an ideal such that $k = \iota \sqcap \mathrm{id}_X$.*

Proof. (a) $(\iota \sqcap \mathrm{id}_X)\nabla_{XX} \sqsubseteq \iota\nabla_{XX} = \iota = \iota \sqcap \mathrm{id}_X\nabla_{XX} \sqsubseteq (\iota\nabla^\sharp_{XX} \sqcap \mathrm{id}_X)\nabla_{XX} = (\iota \sqcap \mathrm{id}_X)\nabla_{XX}$, and so $(\iota \sqcap \mathrm{id}_X)\nabla_{XX} = \iota = \nabla_{XX}(\iota \sqcap \mathrm{id}_X)$.
(b) $\nabla_{XX}(k\nabla_{XX})\nabla_{XX} = k\nabla_{XX}\nabla_{XX}\nabla_{XX} = k\nabla_{XX}$ and $k = k\mathrm{id}_X = k = k\nabla_{XX} \sqcap \mathrm{id}_X$. □

Proposition 3.3 *Let $\xi : X \rightharpoonup Y$ be a morphism. Then*

(a) *$\phi_W(\xi)$ is a scalar on W,*

(b) *If $X \prec Y$, then $\phi_X(\phi_Y(k)) = k$ for all scalars $k \in \mathcal{F}(X)$,*

(c) *If $X \sim Y$, then $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are isomorphic as lattices,*

(d) *$\phi_X(k)\xi = \xi\phi_Y(k)$ for all scalars k on W,*

(e) *If $\xi \neq 0_{XY}$, then there is a nonzero scalar $k \in \mathcal{F}(X)$ such that $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$.*

Proof. (a) Set $W = Z$ in 2.5(a). Then $\phi_W(\xi)\nabla_{WW} = \nabla_{WX}\xi\nabla_{YW} = \nabla_{WW}\phi_W(\xi)$.
(b) First note that $\phi_Y(k)\nabla_{YX} = \nabla_{YX}k\nabla_{XX}$ by 2.5(a) and so

$$\begin{aligned}\nabla_{XY}\phi_Y(k)\nabla_{YX} &= \nabla_{XY}\nabla_{YX}k\nabla_{XX}\\ &= \nabla_{XX}k\nabla_{XX} && \{\text{by } \nabla_{XX} = \nabla_{XY}\nabla_{YX}\}\\ &= k\nabla_{XX} && \{\text{since } k \text{ is a scalar}\}.\end{aligned}$$

Hence we have

$$\begin{aligned}\phi_X(\phi_Y(k)) &= \nabla_{XY}\phi_Y(k)\nabla_{YX} \sqcap \mathrm{id}_X\\ &= k\nabla_{XX} \sqcap \mathrm{id}_X\\ &= k.\end{aligned}$$

(c) It is obvious from (b).
(d) By 2.5(a) we have $\phi_X(k)\nabla_{XY} = \nabla_{XW}k\nabla_{WY} = \nabla_{XY}\phi_Y(k)$ and consequently $\phi_X(k)\alpha = \alpha \sqcap \phi_X(k)\nabla_{XY} = \alpha \sqcap \nabla_{XY}\phi_Y(k) = \alpha\phi_Y(k)$.
(e) Set $k = \phi_X(\xi)$. Then it is clear that k is a scalar on X by (a) and $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$ by 2.5(a). And k is nonzero by 2.5(e), since ξ is nonzero. (Cf. [10, Theorem 5.4]) □

From the above Lemma 3.3(a) we have ϕ_W as a mapping $\phi_W : \mathcal{D}(X,Y) \to \mathcal{F}(W)$.

Fact 3.4

$$\begin{aligned}
\phi_W(\phi_X(\xi)) &= \nabla_{WX}\phi_X(\xi)\nabla_{XW}\sqcap \mathrm{id}_W && \{\text{Definition of } \phi_W\}\\
&= \nabla_{WX}\xi\nabla_{YW}\sqcap \mathrm{id}_W && \{\phi_X(\xi)\nabla_{XW} = \nabla_{XX}\xi\nabla_{YW}\}\\
&= \nabla_{WX}\xi\nabla_{YX}\nabla_{XW}\sqcap \mathrm{id}_W && \{\nabla_{WX}\phi_X(\xi) = \nabla_{WX}\xi\nabla_{YX}\}
\end{aligned}$$

and

$$\begin{aligned}
\phi_W(\phi_Y(\xi)) &= \nabla_{WY}\phi_Y(\xi)\nabla_{YW}\sqcap \mathrm{id}_W && \{\text{Definition of } \phi_W\}\\
&= \nabla_{WX}\xi\nabla_{YW}\sqcap \mathrm{id}_W && \{\nabla_{WY}\phi_Y(\xi) = \nabla_{WX}\xi\nabla_{YY}\}\\
&= \nabla_{WY}\nabla_{YX}\xi\nabla_{YW}\sqcap \mathrm{id}_W && \{\phi_Y(\xi)\nabla_{YX} = \nabla_{YX}\xi\nabla_{YX}\}.
\end{aligned}$$

In particular, the following holds for $\xi = \nabla_{XY}$:

$$\begin{aligned}
\nabla_{WX}\nabla_{XY}\nabla_{YW}\sqcap \mathrm{id}_W &= \nabla_{WX}\nabla_{XY}\nabla_{YX}\nabla_{XW}\sqcap \mathrm{id}_W\\
&= \nabla_{WY}\nabla_{YX}\nabla_{XY}\nabla_{YW}\sqcap \mathrm{id}_W.
\end{aligned}$$

□

Proposition 3.5 *If the Tarski rule holds in $\mathcal{D}$, that is, all nonzero morphisms $\alpha : X \rightharpoonup X$ satisfy $\nabla_{XX}\alpha\nabla_{XX} = \nabla_{XX}$, then there is no scalar on X except for the zero morphism 0_{XX} and the identity id_X.*

Proof. Let k be a nonzero scalar on X. Then, by the Tarski rule, we have

$$k\nabla_{XX} = k\nabla_{XX}\nabla_{XX} = \nabla_{XX}k\nabla_{XX} = \nabla_{XX},$$

which means that k is total, and so $\mathrm{id}_X \sqsubseteq kk^\sharp = k$ by $k \sqsubseteq \mathrm{id}_X$. □

Definition 3.6 A morphism $\alpha : X \rightharpoonup Y$ is *s-crisp* if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all nonzero scalars $k : X \rightharpoonup X$ and all morphisms $\tau : X \rightharpoonup Y$. □

It is trivial from the above definition that every universal morphism ∇_{XY} is s-crisp.

Proposition 3.7 (a) *A morphism is s-crisp if and only if its converse is s-crisp.*

(b) *The infimun of two s-crisp morphisms is s-crisp.*

(c) *If $f : X \to Y$ is a function and a morphism $\beta : Y \rightharpoonup Z$ is s-crisp, then the composite $f\beta : X \rightharpoonup Z$ is s-crisp.*

(d) *If the identity id_Y is s-crisp, then so are all functions $f : X \to Y$.*

(e) *A morphism $\alpha : X \rightharpoonup Y$ is s-crisp if and only if its relative pseudo-complemet $\alpha' \Rightarrow \alpha$ is s-crisp for every morphism $\alpha' : X \rightharpoonup Y$.*

Proof. (a) Assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha^\sharp$ for a nonzero scalar k on Y and a morphism $\tau : Y \rightharpoonup X$. Then $\phi_X(k)\tau^\sharp = \tau^\sharp k = (k\tau)^\sharp \sqsubseteq (\alpha^\sharp)^\sharp = \alpha$ and so $\tau^\sharp \sqsubseteq \alpha$, since $\phi_X(k)$ is a nonzero scalar on X by 2.5(e). Hence $\tau \sqsubseteq \alpha^\sharp$. (b) Assume that $\alpha_i : X \rightharpoonup Y$ is s-crisp for $i = 0$ or 1 and $k\tau \sqsubseteq \alpha_0 \sqcap \alpha_1$ for a nonzero scalar k on X

and a morphism $\tau : X \rightharpoonup Y$. Then we have $k\tau \sqsubseteq \alpha_0$ and $k\tau \sqsubseteq \alpha_1$, and so $\tau \sqsubseteq \alpha_0$ and $\tau \sqsubseteq \alpha_1$ by s-crispness. Hence $\tau \sqsubseteq \alpha_0 \sqcap \alpha_1$. (c) Assume that $k\tau \sqsubseteq f\beta$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Z$. First note that $\phi_Y(k)$ is a nonzero scalar by 2.5(e) and $\phi_Y(k)f^\sharp = f^\sharp k$ by 3.3(d). Then we have

$$\phi_Y(k)f^\sharp\tau = f^\sharp k\tau \sqsubseteq f^\sharp f\beta \sqsubseteq \beta$$

and so $f^\sharp\tau \sqsubseteq \beta$ by the s-crispness of β. Therefore $\tau \sqsubseteq ff^\sharp\tau \sqsubseteq f\beta$, which completes the proof. (d) is a special case of (b). (e) First assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha' \Rightarrow \alpha$ for a nonzero scalar k and morphisms $\tau, \alpha' : X \rightharpoonup Y$. Then we have

$$k(\tau \sqcap \alpha') = k\tau \sqcap \alpha' \sqsubseteq \alpha$$

and so $\tau \sqcap \alpha' \sqsubseteq \alpha$, since $\alpha : X \rightharpoonup Y$ is s-crisp. Therefore $\tau \sqsubseteq \alpha' \Rightarrow \alpha$. Conversely, if $\alpha' \Rightarrow \alpha$ is s-crisp for all morphisms $\alpha' : X \rightharpoonup Y$, then $\alpha = \nabla_{XY} \Rightarrow \alpha$ is s-crisp. This completes the proof. □

It immediately follows from the last proposition 3.7(c) that every composite of s-crisp functions is also an s-crisp function.

A morphism $\alpha : X \rightharpoonup Y$ is *complemented* if it has a complement morphism $\overline{\alpha} : X \rightharpoonup Y$ such that $\alpha \sqcup \overline{\alpha} = \nabla_{XY}$ and $\alpha \sqcap \overline{\alpha} = 0_{XY}$.

Theorem 3.8 *The following four statements are equivalent:*

(a) *If $k \neq 0_{XX}$ and $k \sqcap k' = 0_{XX}$ for scalars $k, k' \in \mathcal{F}(X)$, then $k' = 0_{XX}$,*

(b) *The zero morphism 0_{XY} is s-crisp for every object Y (that is, if $k\tau = 0_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$, then $\tau = 0_{XY}$),*

(c) *For every morphism $\alpha : X \rightharpoonup Y$, its pseudo-complement $\neg\alpha : X \rightharpoonup Y$ is s-crisp,*

(d) *Every complemented morphism $\alpha : X \rightharpoonup Y$ is s-crisp.*

Proof. (a)⇒(b) Assume that $k\tau = 0_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. Recall that $\phi_X(\tau)$ is a scalar on X. Hence we have

$$k \sqcap \phi_X(\tau) = k\phi_X(\tau) = k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \sqsubseteq k\nabla_{XX}\tau\nabla_{XY} = \nabla_{XX}k\tau\nabla_{YX} = 0_{XX}.$$

It follows from (a) that $\phi_X(\tau) = 0_{XX}$ and so $\tau = 0_{XY}$ by 2.5(e). Hence 0_{XY} is s-crisp. (b)⇒(a) is trivial. (b)⇔(c)⇔(d) is a corollary of the last lemma. □

Definition 3.9 A scalar k on X is called *linear* if and only if for every scalar k' on X an equation $k \sqcap k' = 0_{XX}$ implies $k' = 0_{XX}$. □

Let $\mathcal{W}(X)$ denote the set of all linear scalars on X. Every identity id_X is obviously linear. Note that a scalar k on X is linear if and only if its pseudo-complement $\neg k$ $(= \mathrm{id}_X \sqcap (k \Rightarrow 0_{XX}))$ in $\mathcal{F}(X)$ is equal to 0_{XX}.

Lemma 3.10 *If X is a nonempty object, then $\mathcal{W}(X)$ is a filter of $\mathcal{F}(X)$.*

Proof. 0) It is trivial that 0_{XX} is not a linear scalar, whenever X is nonempty. i) If $k_0, k_1 \in \mathcal{W}(X)$, then $k_0 \sqcap k_1 \in \mathcal{W}(X)$: Assume $(k_0 \sqcap k_1) \sqcap k' = 0_{XX}$. Then $k_0 \sqcap (k_1 \sqcap k') = 0_{XX}$ and so $k_1 \sqcap k' = 0_{XX}$, which shows $k' = 0_{XX}$. ii) If $k_0 \in \mathcal{W}(X)$ and $k_1 \in \mathcal{F}(X)$ with $k_0 \sqsubseteq k_1$, then $k_1 \in \mathcal{W}(X)$: Assume $k_1 \sqcap k' = 0_{XX}$. Then $k_0 \sqcap k' = 0_{XX}$ and so $k' = 0_{XX}$. □

So the set of linear scalars on X is a sublattice of the lattice $\mathcal{F}(X)$ of all scalars on X, and as such it is distributive.

Definition 3.11 A morphism $\alpha : X \rightharpoonup Y$ is *l-crisp* if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all linear scalars $k : X \rightharpoonup X$ and all morphisms $\tau : X \rightharpoonup Y$. □

Proposition 3.12 *Every zero morphism* 0_{XY} *is l-crisp.*

Proof. Assume that $k\tau = 0_{XY}$ for a linear scalar on X and a morphism $\tau : X \rightharpoonup Y$. Then $k \sqcap \phi_X(\tau) = k\phi_X(\tau) = k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \sqsubseteq k\nabla_{XX}\tau\nabla_{YX} \sqsubseteq \nabla_{XX}k\tau\nabla_{YX} = 0_{XY}$ and so $\phi_X(\tau) = 0_{XX}$. Hence $\tau = 0_{XY}$ by 2.5(e). □

4 L-Relations

Let L be a complete distributive lattice (or, a complete Heyting algebra) with least element 0 and greatest element 1. The supremum (least upper bound) and the infimum (greatest lower bound) of a family $\{k_\lambda\}$ of elements in L will be denoted by $\vee_\lambda k_\lambda$ and $\wedge_\lambda k_\lambda$, respectively. For two elements $a, b \in L$ the relative pseudo-complement of a relative to b will be written as $a \Rightarrow b$. Now recall some fundamentals on L-relations [5].

Let X and Y be sets. An L-relation R from X into Y, written $R : X \rightharpoonup Y$, is a function $R : X \times Y \rightarrow L$. The set of all L-relations from X into Y will be denoted by L-$Rel(X, Y)$. An L-relation R is contained in an L-relation S, written $R \subseteq S$, if $R(x, y) \leq S(x, y)$ for all $(x, y) \in X \times Y$. The zero relation O_{XY} and the universal relation ∇_{XY} are L-relations with $O_{XY}(x, y) = 0$ and $\nabla_{XY}(x, y) = 1$ for all $(x, y) \in X \times Y$, respectively. It is trivial that $\subseteq$ is a partial order, and $O_{XY} \subseteq R \subseteq \nabla_{XY}$ for all L-relations R. For a family $\{R_\lambda\}_\lambda$ of L-relations we define L-relations $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ as follows:

$$(\cup_\lambda R_\lambda)(x, y) = \vee_\lambda R_\lambda(x, y)$$

and

$$(\cap_\lambda R_\lambda)(x, y) = \wedge_\lambda R_\lambda(x, y)$$

for all $(x, y) \in X \times Y$. It is obvious that $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ are the least upper bound and the greatest lower bound of a family $\{R_\lambda\}_\lambda$, respectively, with respect to the order $\subseteq$. The composite $RS (= R; S) : X \rightharpoonup Z$ of an L-relation $R : X \rightharpoonup Y$ followed by an L-relation $S : Y \rightharpoonup Z$ is defined by

$$(RS)(x, z) = \vee_{y \in Y}[R(x, y) \wedge S(y, z)]$$

for all $(x, z) \in X \times Z$. This composition of L-relations is called sup-inf composition. The composition is associative, i.e. the equation $(RS)T = R(ST)$ holds for all L-relations R, S and T. The identity relation id_X of a set X is an L-relation such that $\mathrm{id}_X(x, x') = 1$ if $x = x'$ and $\mathrm{id}_X(x, x') = 0$ otherwise. The unit laws $\mathrm{id}_X R = R$ and $R\mathrm{id}_Y = R$ hold for all $R : X \rightharpoondown Y$. The converse (or transpose) $R^\sharp : Y \rightharpoondown X$ of an L-relation $R : X \rightharpoondown Y$ is defined by

$$R^\sharp(y, x) = R(x, y)$$

for all $(y, x) \in Y \times X$. For L-relations $S : Y \rightharpoondown Z$ and $T : X \rightharpoondown Z$, the residue $T \div S : X \rightharpoondown Y$ is defined by

$$(T \div S)(x, y) = \wedge_{z \in Z}[S(y, z) \Rightarrow T(x, z)]$$

for all $(x, y) \in X \times Y$. The readers can easily see that L-relations and their operations defined above satisfy all axioms of Dedekind categories; only D3 (Dedekind formula) and D4 (Residues) are not so obvious, and will be proved in the following:

Proposition 4.1 *Let $R : X \rightharpoondown Y, S : Y \rightharpoondown Z$ and $T : X \rightharpoondown Z$ be L-relations. Then*

(a) *$RS \cap T \subseteq R(S \cap R^\sharp T)$ (Dedekind formula),*

(b) *$RS \subseteq T$ if and only if $R \subseteq T \div S$.*

Proof. (a) Since $R^\sharp(y, x) \wedge T(x, z) \leq (R^\sharp T)(y, z)$, we obtain for all $(x, z) \in X \times Z$ that

$$\begin{aligned}
(RS \cap T)(x, z) &= \vee_{y \in Y}[R(x, y) \wedge S(y, z)] \wedge T(x, z) \\
&= \vee_{y \in Y}[R(x, y) \wedge S(y, z) \wedge T(x, z)] \\
&= \vee_{y \in Y}[R(x, y) \wedge S(y, z) \wedge R^\sharp(y, x) \wedge T(x, z)] \\
&\leq \vee_{y \in Y}[R(x, y) \wedge S(y, z) \wedge (R^\sharp T)(y, z)] \\
&= \vee_{y \in Y}[R(x, y) \wedge (S \cap R^\sharp T)(y, z)] \\
&= [R(S \cap R^\sharp T)](x, z).
\end{aligned}$$

(b) follows from the following equivalence:

$$\begin{aligned}
RS \subseteq T &\iff \forall x \forall z : (RS)(x, z) \leq T(x, z) \\
&\iff \forall x \forall z : \vee_{y \in Y}[R(x, y) \wedge S(y, z)] \leq T(x, z) \\
&\iff \forall x \forall z \forall y : R(x, y) \wedge S(y, z) \leq T(x, z) \\
&\iff \forall x \forall z \forall y : R(x, y) \leq S(y, z) \Rightarrow T(x, z) \\
&\iff \forall x \forall y : R(x, y) \leq \wedge_{z \in Z}[S(y, z) \Rightarrow T(x, z)] \\
&\iff \forall x \forall y : R(x, y) \leq (T \div S)(x, y) \\
&\iff R \subseteq T \div S.
\end{aligned}$$

□

Obviously an L-relation $k : X \rightharpoondown X$ is a scalar on X if and only if

$$\forall x, x' \in X : k(x, x) = k(x', x') \text{ and } x \neq x' \Rightarrow k(x, x') = 0.$$

An L-relation $R : X \rightharpoondown Y$ is called *0-1* crisp [5] if $R(x, y) = 0$ or $R(x, y) = 1$ for all $(x, y) \in X \times Y$. Of course O_{XY}, ∇_{XY} and id_X are 0-1 crisp. For a 0-1 crisp L-relation $R : X \rightharpoondown Y$ define an L-relation $\overline{R} : X \rightharpoondown Y$ by $\overline{R}(x, y) = 0$ if $R(x, y) = 1$ and $\overline{R}(x, y) = 1$ otherwise. Then $R \cup \overline{R} = \nabla_{XY}$ and $R \cap \overline{R} = O_{XY}$. This fact means that all 0-1 crisp L-relations are complemented.

Proposition 4.2 *All s-crisp L-relations are 0-1 crisp.*

Proof. Let an L-relation $R : X \rightharpoonup Y$ be s-crisp. Assume that $a = R(x_0, y_0)$ is not equal to $0 \in L$ for some point $(x_0, y_0) \in X \times Y$. Consider a scalar k on X such that $k(x, x') = a$ if $x = x'$ and $k(x, x') = 0$ otherwise, and an L-relation $T : X \rightharpoonup Y$ such that $T(x, y) = a \Rightarrow R(x, y)$ for all $(x, y) \in X \times Y$. Then we have $kT \sqsubseteq R$, since

$$(kT)(x, y) = a \wedge (a \Rightarrow R(x, y)) \leq R(x, y)$$

for all $(x, y) \in X \times Y$. Hence $T \sqsubseteq R$ follows from the fact that $R : X \rightharpoonup Y$ is s-crisp. Finally we have $1 = (a \Rightarrow a) = T(x_0, y_0) \leq R(x_0, y_0)$, which shows R is 0-1 crisp. □

The converse of the last proposition does not hold in general. Its necessary and sufficient condition is given by the following:

Proposition 4.3 *For L-relations the following statements are equivalent:*

C0. $\forall a, b \in L : a \wedge b = 0 \Rightarrow a = 0$ *or* $b = 0$.

K0. All 0-1 crisp L-relations are s-crisp.

Proof. First assume that C0 and $kT \sqsubseteq R$ for a scalar k on X, an L-relation $T : X \rightharpoonup Y$, and a 0-1 crisp L-relation $R : X \rightharpoonup Y$. To prove that R is s-crisp we have to show that $T(x, y) \leq R(x, y)$ for all $(x, y) \in X \times Y$. Since $R(x, y) = 0$ or 1 by the 0-1 crispness of R it is enough to show that if $R(x, y) = 0$ then $T(x, y) = 0$. But $(kT)(x, y) = k(x, x) \wedge T(x, y) \leq R(x, y)$. Hence when $R(x, y) = 0$, we have $T(x, y) = 0$ from C0 and $k(x, x) \neq 0$. Conversely assume that K0 and $a \wedge b = 0$ for $a, b \in L$. Define a scalar k on a singleton set $I = \{*\}$ and an L-relation $R : I \rightarrow I$ by $k(*, *) = a$ and $T(*, *) = b$, respectively. Then $kT = 0_{II}$ and so $k = 0_{II}$ or $T = 0_{II}$ since 0_{II} is s-crisp by the assumption K0. □

Proposition 4.4 *For L-relations the following statements are equivalent:*

C1. $\forall a, b \in L : a \wedge b = 0$ *and* $a \vee b = 1 \Rightarrow a = 0$ *or* $b = 0$.

K1. All complemented L-relations are 0-1 crisp.

K2. All L-relations which are functions are 0-1 crisp.

Proof. Trivial. □

Definition 4.5 An element x of a lattice L is called *linear* if $x \wedge y = 0$ implies $y = 0$ for $y \in L$. □

Let $k : X \rightharpoonup X$ be an L-relation on a nonemptyset X. If k is a linear scalar, then $k(x, x)$ is linear in L for all $x \in X$.

Assume that $k(x, x) \wedge a = 0$ for $a \in L$. Now consider a scalar $k' : X \rightharpoonup X$ such that $k'(x, x') = a$ if $x = y$, and $k'(x, x') = 0$ otherwise. Then $k \sqcap k' = O_{XX}$ and so $k' = O_{XX}$ by the linearity of k. Hence $a = 0$, which proves that $k(x, x)$ is linear.

Proposition 4.6 *All 0-1 crisp L-relations are l-crisp.*

Proof. Let an L-relation $R : X \rightharpoonup Y$ be 0-1 crisp and assume that $kT \sqsubseteq R$ for a linear scalar k on X and an L-relation $T : X \rightharpoonup Y$. We have to show that $T(x,y) \leq R(x,y)$ for all $(x,y) \in X \times Y$. Now $k(x,x) \wedge T(x,y) \leq R(x,y) = (kT)(x,y) \subseteq R(x,y)$, and since $k(x,x)$ is linear, it follows that $R(x,y) = 0$ implies $T(x,y) = 0$, which is sufficient since $R(x,y)$ can only be 0 or 1 by 0-1 crispness. □

The converse of the above proposition does not hold: Consider a Boolean lattice L having a nontrivial element s such that $s \neq 0$ and $s \neq 1$, and define an L-relation $R_s : X \rightharpoonup X$ by $R(x,x') = s$ if $x = x'$ and $R(x,x') = 0$ otherwise. Then it is clear that R_s is l-crisp, but not 0-1 crisp. Generally for a Boolean lattice L every L-relation is l-crisp since the identity id_X is a unique linear csalar on X.

5 Representation Theorem

In this section we first introduce the concept of points in Dedekind categories. Then some useful properties on points, due to Schmidt and Ströhlein [18], and a point axiom will be stated to show a representation theorem in uniform Dedekind categories. In particular, the point axiom induces a function assigning a concrete L-relation between the sets of point relations to an abstract relation in Dedekind categories. In view of [4, 9, 18] the concept of points in Dedekind categories is defined as follows:

Definition 5.1 Let $\mathcal{D}$ be a Dedekind category. A point x of X is an s-crisp function $x : X \to X$ such that $\nabla_{XX} x = x$. □

We will denote the set of all points of X by $\chi(X)$.

Lemma 5.2 *Let x and x' be points of X. Then*

(a) *If $\nabla_{XX}\rho = \rho$ and $\rho \sqsubseteq x$ for a morphism $\rho : X \rightharpoonup X$, then $\rho = kx$ for a unique scalar k on X.*

(b) *If $x \neq x'$, then $x \sqcap x' = 0_{XX}$ and $xx'^\sharp = 0_{XX}$.*

Proof. (a) First set $k = \phi_X(\rho x^\sharp)$. Then by 3.3(a) k is a scalar on X, and $k = \rho x^\sharp \sqcap \mathrm{id}_X$ from $\nabla_{XX} x = x$ and $\nabla_{XX}\rho = \rho$. Moreover we have

$$\rho = \rho \sqcap x \sqsubseteq (\rho x^\sharp \sqcap \mathrm{id}_X)x \sqsubseteq \rho x^\sharp x \sqsubseteq \rho.$$

Finally the uniqueness of k follows from $k = k\nabla_{XX} \sqcap \mathrm{id}_X = kx\nabla_{XX} \sqcap \mathrm{id}_X = \rho\nabla_{XX} \sqcap \mathrm{id}_X$.

(b) It is enough to show that if $x \sqcap x' \neq 0_{XX}$ then $x = x'$. As $x \sqcap x' \sqsubseteq x$ and $\nabla_{XX}(x \sqcap x') = x \sqcap x'$, by (a) there is a unique scalar $k : X \rightharpoonup X$ such that $x \sqcap x' = kx$. If $x \sqcap x' \neq 0_{XX}$, then $k \neq 0_{XX}$ and so $x \sqsubseteq x'$, because $kx \sqsubseteq x'$ and x' is s-crisp. If $x \sqcap x' = 0_{XX}$, then $xx'^\sharp = xx'^\sharp \sqcap \nabla_{XX} \sqsubseteq (x \sqcap \nabla_{XX}x')x'^\sharp = (x \sqcap x')x'^\sharp = 0_{XX}$. This completes the proof. □

Set $L = \mathcal{F}(W)$ for a fixed object W. Then L is a complete disributive lattice. A function $\chi(\alpha) : \chi(X) \times \chi(Y) \to L$ assigning $\chi(\alpha)(x, y) = \phi_W(x\alpha y^\sharp) \in L$ to a pair (x, y) of points x of X and y of Y, gives an L-relation of $\chi(X)$ into $\chi(Y)$. Thus we have a function $\chi : \mathcal{D}(X, Y) \to L\text{-}Rel(\chi(X), \chi(Y))$.

Proposition 5.3 *If $\mathcal{D}$ is a uniform Dedekind category, then the function $\chi : \mathcal{D}(X, Y) \to L\text{-}Rel(\chi(X), \chi(Y))$ satisfies the following properties:*

(a) $\chi(0_{XY}) = O_{\chi(X)\chi(Y)}$, $\chi(\nabla_{XY}) = \nabla_{\chi(X)\chi(Y)}$ *and* $\chi(\mathrm{id}_X) = \mathrm{id}_{\chi(X)}$,

(b) $\chi(\alpha \sqcup \alpha') = \chi(\alpha) \cup \chi(\alpha')$ *and* $\chi(\alpha \sqcap \alpha') = \chi(\alpha) \cap \chi(\alpha')$,

(c) $\chi(\alpha^\sharp) = \chi(\alpha)^\sharp$,

(d) $\chi(\alpha)\chi(\beta) = \chi(\alpha[\sqcup_{y\in\chi(Y)} y^\sharp y]\beta)$.

(e) *The function $\chi : \mathcal{D}(X, Y) \to L\text{-}Rel(\chi(X), \chi(Y))$ is surjective.*

Proof. Recall that $\chi(\alpha)(x, y) = \phi_W\phi_X(x\alpha y^\sharp) = \phi_W\phi_Y(x\alpha y^\sharp)$ by 2.5(b).
(a) It is immediate that $\chi(0_{XY})(x, y) = 0_{WW}$. Note that $x\nabla_{XY}y^\sharp = \nabla_{XY}$ from $x\nabla_{XX} = \nabla_{XX}$ and $y\nabla_{YY} = \nabla_{YY}$. The second equality follows from

$$\begin{array}{rcll} \phi_W(x\nabla_{XY}y^\sharp) & = & \phi_W(\nabla_{XY}) & \{\text{by } x\nabla_{XX} = \nabla_{XX} \text{ and } y\nabla_{YY} = \nabla_{YY}\} \\ & = & \phi_W\phi_X(\nabla_{XY}) & \{\text{by } 2.5(\text{c})\} \\ & = & \phi_W(\mathrm{id}_X) & \{\text{by } X \sim Y\} \\ & = & \mathrm{id}_W & \{\text{by } X \sim W\} \end{array}$$

and the third holds from $\phi_X(x\mathrm{id}_X x'^\sharp) = \nabla_{XX}xx'^\sharp\nabla_{XX} \sqcap \mathrm{id}_X = xx'^\sharp \sqcap \mathrm{id}_X$ and 5.2(b).
(b) The former equality is trivial from $\phi_W(x(\alpha \sqcup \alpha')y^\sharp) = \phi_W(x\alpha y^\sharp) \sqcup \phi_W(x\alpha' y^\sharp)$, and the latter follows from

$$\begin{array}{rcl} \phi_W(x(\alpha \sqcap \alpha')y^\sharp) & = & \nabla_{WX}(x\alpha y^\sharp \sqcap x\alpha' y^\sharp)\nabla_{YW} \sqcap \mathrm{id}_W \\ & \sqsubseteq & \nabla_{WX}x\alpha y^\sharp\nabla_{YW} \sqcap \nabla_{WX}x\alpha' y^\sharp\nabla_{YW} \sqcap \mathrm{id}_W \\ & (= & \phi_W(x\alpha y^\sharp) \sqcap \phi_W(x\alpha' y^\sharp)\ ) \\ & \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap \nabla_{XW}\nabla_{WX}x\alpha' y^\sharp\nabla_{YW}\nabla_{WY})\nabla_{YW} \sqcap \mathrm{id}_W \\ & \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap \nabla_{XX}x\alpha' y^\sharp\nabla_{YY})\nabla_{YW} \sqcap \mathrm{id}_W \\ & \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap x\alpha' y^\sharp)\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \phi_W(x(\alpha \sqcap \alpha')y^\sharp). \end{array}$$

(c) It directly follows from 2.5(c).
(d) First note that $\chi(\alpha)(x, y) \sqcap \chi(\beta)(y, z) = \chi(\alpha y^\sharp y\beta)(x, z)$ for $(x, y, z) \in \chi(X) \times$

$\chi(Y) \times \chi(Z)$, since

$$\begin{array}{rcl}
\phi_W(x\alpha y^\sharp) \sqcap \phi_W(y\beta z^\sharp) & = & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp (\nabla_{YW} \sqcap y\alpha^\sharp x^\sharp \nabla_{XW} \nabla_{WY} y\beta z^\sharp \nabla_{ZW}) \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& = & \nabla_{WX} x\alpha y^\sharp y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& (= & \phi_W(x\alpha y^\sharp y\beta z^\sharp)\ ) \\
& = & (\nabla_{WX} x\alpha y^\sharp \sqcap \nabla_{WZ} z\beta^\sharp y^\sharp)(y\alpha^\sharp x^\sharp \nabla_{XW} \sqcap y\beta z^\sharp \nabla_{ZW}) \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& = & \phi_W(x\alpha y^\sharp) \sqcap \phi_W(y\beta z^\sharp).
\end{array}$$

Therefore we have

$$\begin{array}{rcl}
\chi(\alpha)\chi(\beta)(x,z) & = & \sqcup_{y\in\chi(Y)}[\chi(\alpha)(x,y) \sqcap \chi(\beta)(y,z)] \\
& = & \sqcup_{y\in\chi(Y)}\chi(\alpha y^\sharp y\beta)(x,z) \\
& = & \chi(\alpha[\sqcup_{y\in\chi(Y)} y^\sharp y]\beta)(x,z).
\end{array}$$

(e) Let $R : \chi(X) \rightharpoonup \chi(Y)$ be an L-relation. Noticing $L = \mathcal{F}(W)$ we define a morphism $\alpha_R : X \rightharpoonup Y$ by

$$\alpha_R = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(R(x,y)) x^\sharp \nabla_{XY} y.$$

Then we have $\phi_X(x_0\alpha_R y_0^\sharp) = \phi_X(R(x_0,y_0))$ from $\phi_X(x_0\alpha_R y_0^\sharp)\nabla_{XY} = x_0\alpha_R y_0{}^\sharp = \phi_X(R(x_0,y_0))\nabla_{XY}$. Hence

$$\chi(\alpha_R)(x_0,y_0) = \phi_W(x_0\alpha_R y_0^\sharp) = \phi_W\phi_X(x_0\alpha_R y_0^\sharp) = \phi_W\phi_X(R(x_0,y_0)) = R(x_0,y_0),$$

which completes the proof. □

Definition 5.4 A Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if

$$\sqcup_{x\in\chi(X)} x = \nabla_{XX}$$

for all objects X. □

Assume that $\sqcup_{x\in\chi(X)} x = \nabla_{XX}$. Then it follows from $\mathrm{id}_X \sqcap x \sqsubseteq (\mathrm{id}_X x^\sharp \sqcap \mathrm{id}_X)x \sqsubseteq x^\sharp x$ that $\mathrm{id}_X = \mathrm{id}_X \sqcap \nabla_{XX} = \mathrm{id}_X \sqcap (\sqcup_{x\in\chi(X)} x) = \sqcup_{x\in\chi(X)}(\mathrm{id}_X \sqcap x) \sqsubseteq \sqcup_{x\in\chi(X)} x^\sharp x$. Hence $\sqcup_{x\in\chi(X)} x^\sharp x = \mathrm{id}_X$. Conversely assume that $\sqcup_{x\in\chi(X)} x^\sharp x = \mathrm{id}_X$. Then $\nabla_{XX} = \nabla_{XX}\mathrm{id}_X = \nabla_{XX}(\sqcup_{x\in\chi(X)} x^\sharp x) = \sqcup_{x\in\chi(X)} \nabla_{XX} x^\sharp x = \sqcup_{x\in\chi(X)} \nabla_{XX} x = \sqcup_{x\in\chi(X)} x$. Therefore the condition $\sqcup_{x\in\chi(X)} x = \nabla_{XX}$ is equivalent to $\sqcup_{x\in\chi(X)} x^\sharp x = \mathrm{id}_X$.

Proposition 5.5 *If a Dedekind category $\mathcal{D}$ satisfies the strict point axiom, then for all objects X the identity morphism id_X is complemented. Moreover, if the statement (a) of Theorem 3.8 is valid in $\mathcal{D}$, then id_X is s-crisp.*

Proof. Assume that $\nabla_{XX} = \sqcup_{x\in\chi(X)} x$. Then it is obvious that

$$\nabla_{XX} = \nabla_{XX}\nabla_{XX} = (\sqcup_{x\in\chi(X)} x^\sharp)(\sqcup_{y\in\chi(X)} y) = \mathrm{id}_X \sqcup (\sqcup_{x\neq y\in\chi(X)} x^\sharp y).$$

Here note that for $x \neq y \in \chi(X)$ we have $\mathrm{id}_X \sqcap x^\sharp y \sqsubseteq x^\sharp(x\mathrm{id}_X \sqcap y) = 0_{XX}$. Hence this shows that $\sqcup_{x\neq y\in\chi(X)} x^\sharp y$ is the complement of id_X. □

Theorem 5.6 (Representation Theorem) *Assume that $\mathcal{D}$ is a uniform Dedekind and satisfies the strict point axiom. Then every morphism $\alpha : X \rightharpoonup Y$ has a unique representation*

$$\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y,$$

where $k_{x,y}$ is a scalar on X for all $(x,y) \in \chi(X) \times \chi(Y)$.

Proof. Note that $x\alpha y^\sharp = \phi(x\alpha y^\sharp)\nabla_{XY}$ for $x \in \chi(X)$ and $y \in \chi(Y)$, because $x\alpha y^\sharp = \nabla_{XX} x\alpha y^\sharp \nabla_{YY} = \phi_X(x\alpha y^\sharp)\nabla_{XY}$ by 2.5(a). We now show the uniqueness of the representation. Assume $\alpha = \sqcup_{x\in\chi(X)}\sqcup_{y\in\chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y$. Then for all $(x,y) \in \chi(X)\times\chi(Y)$ we have $k_{x,y}\nabla_{XY} = x\alpha y^\sharp = \phi_X(x\alpha y^\sharp)\nabla_{XY}$ and so $k_{x,y} = \phi_X(x\alpha y^\sharp)$ by 2.6. Hence it suffices to see that $\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(x\alpha y^\sharp) x^\sharp \nabla_{XY} y$. Since $\mathrm{id}_X = \sqcup_{x\in\chi(X)} x^\sharp x$ and $\mathrm{id}_Y = \sqcup_{y\in\chi(Y)} y^\sharp y$ by the strict point axiom, we have

$$\begin{aligned}
\alpha &= \mathrm{id}_X \alpha \mathrm{id}_Y \\
&= (\sqcup_{x\in\chi(X)} x^\sharp x)\alpha(\sqcup_{y\in\chi(Y)} y^\sharp y) \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp x\alpha y^\sharp y \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp \phi_X(x\alpha y^\sharp)\nabla_{XY} y \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(x\alpha y^\sharp) x^\sharp \nabla_{XY} y.
\end{aligned}$$

This completes the proof. □

Corollary 5.7 *A uniform Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if the function $\chi : \mathcal{D}(X,X) \to L\text{-}Rel(\chi(X),\chi(X))$ is injective for all objects X.*

Proof. First assume that the function χ is injective. Then it follows from Prop. 5.3(a) and (d) that $\mathrm{id}_X = \sqcup_{x\in\chi(X)} x^\sharp x$, which is equivalent to $\nabla_X = \sqcup_{x\in\chi(X)} x$. Secondly assume that the point axiom and consequently the representation theorem 5.7 hold. Let $\chi(\alpha) = \chi(\alpha')$ for $\alpha, \alpha' : X \rightharpoonup Y$. Then $\phi_W(x\alpha y^\sharp) = \phi_W(x\alpha' y^\sharp)$ for all $(x,y) \in \chi(X) \times \chi(Y)$. Since $\mathcal{D}$ is uniform, $\phi_X(x\alpha y^\sharp) = \phi_Y(x\alpha' y^\sharp)$ for all $(x,y) \in \chi(X) \times \chi(Y)$and so $\alpha = \alpha'$ by the virtue of the representation theorem. □

From the proof of 2.3(a) it is easy to see that $\nabla_{XY} \neq O_{XY}$ for all nonempty objects X and Y (Cf. [10, Condition P1 in Definition 4.4]) if $\mathcal{D}$ has a unit object I and satisfies the strict point axiom.

As a result we have proved that a Dedekind category which has a unit object satisfying the strict point axiom is equivalent to a subcategory of a category of L-relations.

Let I and X be objects in $\mathcal{D}$. An I-point of X is an s-crisp function $p : I \to X$ such that $p = \nabla_{II} p$. Thus, when I is a unit object in $\mathcal{D}$, an I-point of X is just an s-crisp function from I to X. The set of all I-points of X will be denoted by $Q(X)$.

Proposition 5.8 *Let I and X be objects in $\mathcal{D}$. Then*

(a) *If $X \prec I$, then a morphism $x = \nabla_{XI} p : X \rightharpoonup X$ is a point of X for an I-point $p : I \rightharpoonup X$ of X,*

(b) *If* $I \prec X$*, then a morphism* $p = \nabla_{IX} x : I \rightharpoonup X$ *is an* I*-point of* X *for a point* $x : X \rightharpoonup X$ *of* X*,*

(c) *If* $X \sim I$*, then* $\nabla_{IX} = \sqcup_{p \in Q(X)} p$ *is equivalent to* $\nabla_{XX} = \sqcup_{x \in \chi(X)} x$*.*

Proof. (a) First note that $\nabla_{XX} x = \nabla_{XX} \nabla_{XI} p = \nabla_{XI} p = x$, $x^\sharp x = (\nabla_{XI} p)^\sharp (\nabla_{XI} p) = p^\sharp \nabla_{IX} \nabla_{XI} p \sqsubseteq p^\sharp \nabla_{II} p = p^\sharp p \sqsubseteq \mathrm{id}_X$, and $x x^\sharp = (\nabla_{XI} p)(\nabla_{XI} p)^\sharp = \nabla_{XI} p p^\sharp \nabla_{IX} \sqsupseteq \nabla_{XI} \nabla_{IX} = \nabla_{XX}$ by $X \prec I$. Next assume that $k\tau \sqsubseteq \nabla_{XI} p (= x)$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup X$. Then $\phi_I(k) \nabla_{IX} \tau = \nabla_{IX} k \tau \sqsubseteq \nabla_{IX} \nabla_{XI} p \sqsubseteq \nabla_{II} p = p$ and so $\nabla_{IX} \tau \sqsubseteq p$, since $\phi_I(k) \neq 0_{II}$ by 2.5(e) and p is s-crisp. Hence $\tau \sqsubseteq \nabla_{XX} \tau = \nabla_{XI} \nabla_{IX} \tau \sqsubseteq \nabla_{XI} p = x$ by $X \prec I$.
(b) First note that $\nabla_{II} p = \nabla_{II} \nabla_{IX} x = \nabla_{IX} x = p$, $p^\sharp p = (\nabla_{IX} x)^\sharp (\nabla_{IX} x) = x^\sharp \nabla_{XI} \nabla_{IX} x \sqsubseteq x^\sharp \nabla_{XX} x = x^\sharp x \sqsubseteq \mathrm{id}_X$, and $p p^\sharp = (\nabla_{IX} x)(\nabla_{IX} x)^\sharp = \nabla_{IX} x x^\sharp \nabla_{XI} = \nabla_{IX} \nabla_{XX} \nabla_{XI} = \nabla_{II} \sqsupseteq \mathrm{id}_I$ by $I \prec X$. Next assume that $k\tau \sqsubseteq \nabla_{IX} x (= p)$ for a nonzero scalar k on I and a morphism $\tau : I \rightharpoonup X$. Then $\phi_X(k) \nabla_{XI} \tau = \nabla_{XI} k \tau \sqsubseteq \nabla_{XI} \nabla_{IX} x \sqsubseteq \nabla_{XX} x = x$ and so $\nabla_{XI} \tau \sqsubseteq x$, since $\phi_X(k) \neq 0_{XX}$ by 2.5(e) and x is s-crisp. Hence $\tau \sqsubseteq \nabla_{II} \tau = \nabla_{IX} \nabla_{XI} \tau \sqsubseteq \nabla_{IX} x = p$ by $I \prec X$.
(c) First assume that $\nabla_{IX} = \sqcup_{p \in Q(X)} p$. Then

$$\sqcup_{x \in \chi(X)} x = \sqcup_{p \in Q(X)} \nabla_{XI} p = \nabla_{XI} \sqcup_{p \in Q(X)} p = \nabla_{XI} \nabla_{IX} = \nabla_{XX}$$

by $X \prec I$. Conversely assume that $\nabla_{XX} = \sqcup_{x \in \chi(X)} x$. Then

$$\sqcup_{p \in Q(X)} p = \sqcup_{x \in \chi(X)} \nabla_{IX} x = \nabla_{IX} \sqcup_{x \in \chi(X)} x = \nabla_{IX} \nabla_{XX} = \nabla_{IX}.$$

□

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