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Rewriting Fuzzy Graphs

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Abstract. This paper studies a fuzzy graph rewriting with single pushout approach from a viewpoint of fuzzy relational calculus. Two possible kinds of matchings for fuzzy graph rewritings are given, namely, a rigorous matching which just generalizes matchings for crisp (or ordinary) graph rewritings, and an ε-matching which represents rather ambiguous or fuzzy one. Finally the pullback structure of fuzzy graphs are analyzed for pullback rewritings.

1. Introduction

Since the late sixties the algebraic theory of graph grammar, motivated from the study of graph grammars, has been studied by many researchers, for example, Rosenfeld, Montanari, Courcelle, Schneider, Ehrig and Kreowski. Since then the idea to transform graphs with so-called double pushout derivations has been applied to various fields [3, 6, 7, 8] of computer science. In 1984 Raoult [16] proposed another idea for graph transformations, so-called single pushout rewritings [13, 2], making use of a notion of partial morphisms of term graphs, and discussed about the Church-Rosser property and critical pairs of production rules by a categorical setting. So far the single pushout rewritings has been extensively developed from various view points, for example by [2, 4, 5, 11, 12, 16].

The aim of this paper is to generalize relational graph rewritings [14] of (crisp or ordinary) graphs and to formalize a fuzzy graph rewriting with single pushout approach from a viewpoint of fuzzy relational calculus [9, 10]. A fuzzy graph here means a pair of a set of nodes and a fuzzy (connection) relation on the nodes. Fuzzy Graphs are used to represent fuzzy relations between objects such as fuzzy dynamic programming and fuzzy citation diagram of documents [15]. To formalize a fuzzy graph rewriting we first discuss about the algebraic and logical structure of fuzzy relations. A partial morphism between fuzzy graphs is a partial function between sets of nodes preserving fuzzy graph structures. As an application of relational calculus we show that the category of fuzzy graphs and their partial morphisms has pushouts. Based on these motivations we study a fuzzy graph rewriting with single pushout approach. In general the definition (or choice) of matchings to production rules changes the aspect of graph rewritings. Thus we propose two possible kinds of matchings for fuzzy graph rewriting. The former is a rigorous matching, which leads a fact that if the production rule is a partial morphism, then the rewriting square is a pushout. The latter is rather ambiguous or fuzzy one, called an ε-matching, where $0 < \varepsilon < 1$. The fuzzy graph rewriting with using ε-matching generates an ε-similar fuzzy graph regarded as a fuzzy approximation to pushout rewritings. Finally the authors will analyze the pullback structures of fuzzy graphs for pullback rewritings, initiated by Bauderon[1].

In section 2 we briefly review some fundamentals of fuzzy relation algebras as preliminaries. In section 3 two notions of relative crispness and ε-crispness of fuzzy relations are introduced. The former is equivalent to the existence of relative complements, and the latter is a fuzzy approximation of relative crispness. The basic properties of relative crispness and ε-crispness are studied. In section 4 a formalization of fuzzy graph rewritings are proposed as a generalization of relational graph rewritings due to [14]. Analogously we obtain a fundamental fact that if the production rule is a partial morphism then the rewriting square is a pushout. The section 5 first introduces two possible candidates of matchings for fuzzy graphs: rigorous matchings and ε-matchings. Then the main results in this paper are proved. In section 6 we finally mention a fact that the category of fuzzy 8 and total morphisms between them has pullbacks. This fact suggests a possibility of the pullback approach [1] to fuzzy graph rewritings.

2. Fuzzy Relations

The fuzzy relation algebra is investigated by [9, 10] in order to prove the representation theorem. In this section we will review fuzzy relation algebra.

Let A and B be sets. A **fuzzy relation** α from A to B, denoted by $\alpha : A \rightharpoonup B$, is a function from the Cartesian product $A \times B$ to the unit interval $[0, 1]$. We denote the set of all fuzzy relations from A to B by $\mathbf{FRel}(A, B)$. The **zero** relation $O_{A,B} : A \rightharpoonup B$ and the **universal** relation $\nabla_{A,B} : A \rightharpoonup B$ are fuzzy relations with $O_{A,B}(a, b) = 0$ and $\nabla_{A,B}(a, b) = 1$ for any $(a, b) \in A \times B$, respectively. We abbreviate $O_{A,B}$ and $\nabla_{A,B}$ to O and ∇ if their domains are understood from the contexts. The **identity** relation id_A on A is a fuzzy relation from A to A defined by

$$\mathrm{id}_A(a, a') = \begin{cases} 1 & \text{if } a = a', \\ 0 & \text{otherwise} \end{cases}$$

for any $a, a' \in A$.

A fuzzy relation $\alpha : A \rightharpoonup B$ is contained by $\beta : A \rightharpoonup B$, denoted by $\alpha \sqsubseteq \beta$, if and only if $\alpha(a,b) \leq \beta(a,b)$ for any $(a,b) \in A \times B$. Obviously $O \sqsubseteq \alpha \sqsubseteq \nabla$ for all $\alpha : A \rightharpoonup B$. The relation $\sqsubseteq$ is clearly a partial order on $\mathbf{FRel}(A,B)$.

The infimum $\sqcap_\lambda \alpha_\lambda$ and the supremum $\sqcup_\lambda \alpha_\lambda$ of a family $\{\alpha_\lambda : A \rightharpoonup B\}_\lambda$ of fuzzy relations are given by

$$(\sqcap_\lambda \alpha_\lambda)(a,b) = \bigwedge_\lambda [\alpha_\lambda(a,b)] \quad \text{and} \quad (\sqcup_\lambda \alpha_\lambda)(a,b) = \bigvee_\lambda [\alpha_\lambda(a,b)],$$

where $\wedge$ and $\vee$ denote the infimum and the supremum of real numbers, respectively. For shorthand we write $\alpha \sqcap \beta$ and $\alpha \sqcup \beta$ for the infimum and the supremum of $\{\alpha, \beta\}$.

It is well–known that $\langle \mathbf{FRel}(A,B), \sqsubseteq, \sqcap, \sqcup \rangle$ is a complete distributive lattice with zero element, that is, a Heyting algebra. For α and β in $\mathbf{FRel}(A,B)$, the pseudo complement of α relative to β is given by

$$(\alpha \Rightarrow \beta)(a,b) = \begin{cases} 1 & \text{if } \alpha(a,b) \leq \beta(a,b) \\ \beta(a,b) & \text{otherwise.} \end{cases}$$

We denote $\alpha \Rightarrow O$ by $\neg a$.

For a fuzzy relation $\alpha : A \rightharpoonup B$ the **inverse (transposed) relation** $\alpha^\sharp : B \rightharpoonup A$ is defined by

$$\alpha^\sharp(y,x) \stackrel{\text{def}}{=} \alpha(x,y).$$

The following properties hold: $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, $(\alpha^\sharp)^\sharp = \alpha$ and $\alpha \sqsubseteq \alpha'$ implies $\alpha^\sharp \sqsubseteq \alpha'^\sharp$. The **composite** of fuzzy relations $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ is defined by

$$(\alpha\beta)(a,b) \stackrel{\text{def}}{=} \bigvee_{b \in B} [\alpha(a,b) \wedge \beta(b,c)]$$

Note that the composition is distributive over the 0 operator, that is, $\alpha(\sqcup_\lambda \beta_\lambda) = \sqcup_\lambda \alpha\beta_\lambda$, and $O\alpha = \alpha O = O$. Also the **Dedekind formula**, equivalent to the Schröder rule [17] for Boolean relation algebras, holds.

$$\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp \gamma)$$

for $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : A \rightharpoonup C$.

Now in order to introduce the notion of crispness for fuzzy relations we define a scalar multiplication, that is, for a fuzzy relation $\alpha : A \rightharpoonup B$ and a scalar $k \in [0,1]$

$$(k\alpha)(a,b) \stackrel{\text{def}}{=} k\alpha(a,b) \text{ for } (a,b) \in A \times B\ .$$

We call a fuzzy relation $\alpha : A \rightharpoonup B$ **crisp** if $\alpha(a,b) = 1$ or 0 for any $(a,b) \in A \times B$. It is easy to see that a fuzzy relation α is crisp iff $\alpha \sqcap k\nabla = k\alpha$ iff there exists a relation β such that $\alpha \sqcup \beta = \nabla$ and $\alpha \sqcap \beta = O$. Clearly $\beta = \neg\alpha$. Note that if β is crisp then $\alpha \Rightarrow \beta$ is crisp, and in particular $\neg\alpha$ is always crisp. In this paper we set $\beta \dot{-} \alpha = \beta \sqcap \neg\alpha$, called by pseudo subtraction. The pseudo subtraction can be written as

$$(\beta \dot{-} \alpha)(a,b) = \begin{cases} \beta(a,b) & \text{if } \alpha(a,b) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If a crisp relation $\alpha : A \rightharpoonup B$ satisfies univalency $\alpha^\sharp\alpha \sqsubseteq id_B$ it is called a **partial function**. If a partial function $\alpha : A \rightharpoonup B$ is called **function** if it satisfies totality $\mathrm{id}_A \sqsubseteq \alpha\alpha^\sharp$. We use Roman letters $f, g, h, \cdots$, for partial functions and functions. We denote a partial function f from A to B by $f : A \to B$.

The **domain** of a fuzzy relation $\alpha : A \rightharpoonup B$ is a fuzzy relation $d(\alpha) = \alpha\alpha^\sharp \sqcap \mathrm{id}_A$.

Proposition 2.1. [14] Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be fuzzy relations and $f : A \to B$ a partial function.

1. $d(\alpha\beta)d(\alpha) = d(\alpha\beta)$.
2. $d(f\beta)f = fd(\beta)$.

3. Relative Crispness of Fuzzy Relations

In this section we introduce relative crispness of fuzzy relations. A fuzzy relation $\alpha : A \rightharpoonup B$ is called **crisp relative to** a fuzzy relation $\gamma : A \rightharpoonup B$ if $\alpha \sqcup k\gamma = k\alpha$ for all $k \in [0,1]$, which is equivalent to a condition that $\alpha(a,b) \neq 0$ implies $\alpha(a,b) = \gamma(a,b)$ for $(a,b) \in A \times B$. Also it is known that a fuzzy relation $\alpha : A \rightharpoonup B$ is crisp relative to $\gamma : A \rightharpoonup B$ if and only if there exists a unique fuzzy relation $\delta : A \rightharpoonup B$ such that $\alpha \sqcup \delta = \gamma$ and $\alpha \sqcap \delta = O$. Such a fuzzy relation δ, the complement of α relative to γ, can be given by

$$\delta(a,b) = \begin{cases} \gamma(a,b) & \text{if } \alpha(a,b) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have the following fundamental fact.

Proposition 3.1. If α is crisp relative to γ, then $(\gamma \dot{-} \alpha) \sqcup \alpha = \gamma$ and $(\gamma \dot{-} \alpha) \sqcap \alpha = O$.

It is easily proved if fuzzy relations $\alpha, \alpha' : A \rightharpoonup B$ are crisp relative to $\gamma : A \rightharpoonup B$ then $\alpha \sqcap \alpha'$ and $\alpha \sqcup \alpha'$ are crisp relative to γ. In order to formalize ambiguous matchings which will be defined in section 5 for fuzzy graphs we define ε−**crispness** of fuzzy relations as an approximation of relative crispness introduced above.

Definition 3.1. Given $0 < \varepsilon < 1$. A fuzzy relation $\alpha : A \rightharpoonup B$ is ε−crisp relative to a fuzzy relation $\gamma : A \rightharpoonup B$ if $\alpha(a,b) \neq 0$ implies $\gamma(a,b) \neq 0$ and $|\alpha(a,b) - \gamma(a,b)| \leq \varepsilon$ for $(a,b) \in A \times B$.

It is obvious that if a fuzzy relation $\alpha : A \rightharpoonup B$ is crisp relative to $\gamma : A \rightharpoonup B$ then α is ε−crisp relative to γ. We say that two fuzzy relations are *ε−similar* if each of them is ε−crisp relative to the other.

Proposition 3.2. Let α, β and γ be fuzzy relations from A to B. If α and β are ε−crisp relative to γ, then both of $\alpha \sqcup \beta$ and $\alpha \sqcap \beta$ are ε−crisp relative to γ.

Proof : Suppose that $(\alpha \sqcup \beta)(a,b) \neq 0$. Then $\alpha(a,b) \neq 0$ or $\beta(a,b) \neq 0$. If $\alpha(a,b) \neq 0$ and $\beta(a,b) \neq 0$, then

$$\beta(a,b) \neq 0 \quad \text{and} \quad |\alpha(a,b) - \gamma(a,b)| \leq \varepsilon \quad \text{and} \quad |\beta(a,b) - \gamma(a,b)| \leq \varepsilon.$$

So that $|(\alpha \sqcup \beta)(a,b) - \gamma(a,b)| \leq \varepsilon$. Else if $\alpha(a,b) \neq 0$ or $\beta(a,b) = 0$ then

$$\gamma(a,b) \neq 0 \quad \text{and} \quad |\alpha(a,b) - \gamma(a,b)| \leq \varepsilon.$$

Hence $|(\alpha \sqcup \beta)(a,b) - \gamma(a,b)| \leq \varepsilon$. The case of $\alpha \sqcap \beta$ can be proved similarly. □

Lemma 3.1. If $\alpha : A \rightharpoonup B$ is ε−crisp relative to $\gamma : A \rightharpoonup B$, then $(\gamma \dot{-} \alpha) \sqcup \alpha$ is ε−crisp relative to γ and γ is ε−crisp relative to $(\gamma \dot{-} \alpha) \sqcup \alpha$ (that is, $(\gamma \dot{-} \alpha) \sqcup \alpha$ and γ are ε−similar).

Proof : If $\alpha(a,b) \neq 0$ then $\gamma(a,b) \neq 0$ and $|\alpha(a,b) - \gamma(a,b)| \leq \varepsilon$ by the assumption. Note that

$$[(\gamma \dot{-} \alpha) \sqcup \alpha](a,b) = \begin{cases} \gamma(a,b) & \text{if } \alpha(a,b) = 0, \\ \alpha(a,b) & \text{otherwise.} \end{cases}$$

Obviously $(\gamma \dot{-} \alpha) \sqcup \alpha$ is ε−crisp relative to γ. Conversely if $\gamma(a,b) \neq 0$ and $\alpha(a,b) = 0$, then $[(\gamma \dot{-} \alpha) \sqcup \alpha](a,b) = \gamma(a,b) \neq 0$. Else if $\gamma(a,b) \neq 0$ and $\alpha(a,b) \neq 0$, then $[(\gamma \dot{-} \alpha) \sqcup \alpha](a,b) = \alpha(a,b) \neq 0$, and

$$|\gamma(a,b) - [\alpha \sqcup (\gamma \dot{-} \alpha)](a,b)| = |\gamma(a,b) - \alpha(a,b)| \leq \varepsilon.$$

Hence γ is ε−crisp relative to $(\gamma \dot{-} \alpha) \sqcup \alpha$. □

Lemma 3.2. Let $\alpha : A \rightharpoonup A$ and $\gamma : A \rightharpoonup A$ be fuzzy relations and $f : A \to B$ a partial function. If α and γ are ε–similar, then $f^\sharp \alpha f$ and $f^\sharp \gamma f$ are ε–similar.

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, shift left, dotted, "f^\sharp \alpha f"] \arrow[r, shift right, dotted, "f^\sharp \gamma f"'] & B \end{tikzcd}$$

Proof : For $b, b' \in B$ we have

$$\begin{aligned} (f^\sharp \alpha f)(b, b') &= \bigvee_{a, a' \in A} [f^\sharp(b, a) \wedge \alpha(a, a') \wedge f(a', b')] \\ &= \bigvee \{\alpha(a, a') \mid b = f(a) \text{ and } b' = f(a') \text{ for } a, a' \in A\ \} \end{aligned}$$

and

$$\begin{aligned} (f^\sharp \gamma f)(b, b') &= \bigvee_{a, a' \in A} [f^\sharp(b, a) \wedge \gamma(a, a') \wedge f(a', b')] \\ &= \bigvee \{\gamma(a, a') \mid b = f(a) \text{ and } b' = f(a') \text{ for } a, a' \in A\ \}. \end{aligned}$$

Suppose $(f^\sharp \alpha f)(b, b') \neq 0$. Then $b = f(a)$, $b' = f(a')$ and $\alpha(a, a') \neq 0$ for some $a, a' \in A$. Hence by assumption $(f^\sharp \gamma f)(b, b') \neq 0$. For $a_0, a_0' \in A$ such that $b = f(a_0)$, $b' = f(a_0')$ and $\alpha(a_0, a_0') \neq 0$ we have

$$\alpha(a_0, a_0') \leq \gamma(a_0, a_0') + \varepsilon \leq (f^\sharp \gamma f)(b, b') + \varepsilon.$$

Therefore

$$(f^\sharp \alpha f)(b, b') \leq (f^\sharp \gamma f)(b, b') + \varepsilon,$$

and conversely

$$(f^\sharp \gamma f)(b, b') \leq (f^\sharp \alpha f)(b, b') + \varepsilon,$$

which shows that $f^\sharp \alpha f$ is ε–crisp relative to $f^\sharp \gamma f$. Again analogously $f^\sharp \gamma f$ is ε–crisp relative to $f^\sharp \alpha f$. $\Box$

4. Rewriting for Fuzzy Graphs

Mizoguchi and Kawahara[14] discussed graph rewriting with ordinary (namely, crisp) relations. This section presents a formalization of fuzzy graph rewritings in the same manner as [14]. In this paper we will deal with "ordinary" sets of nodes and "fuzzy" relations of edges, and partial morphisms are (ordinary) partial functions preserving fuzzy graph structures.

A **fuzzy graph** $\langle A, \alpha \rangle$ is a pair of a set A and a fuzzy relation $\alpha : A \rightharpoonup A$. A **partial morphism** of a fuzzy graph $\langle A, \alpha \rangle$ into a fuzzy graph $\langle B, \beta \rangle$ is a partial function $f : A \to B$ satisfying $d(f)\alpha f \sqsubseteq f\beta$ (see [11]).

We denote the category of fuzzy graphs and their partial morphisms by **Pfn($\mathcal{F}$-Graph)**. Using a fact [16] that the category **Pfn** of sets and partial functions has pushouts, we can prove the following theorem.

Theorem 4.1. The category **Pfn($\mathcal{F}$-Graph)** has pushouts.

Let us consider the following diagrams, which are in **Pfn($\mathcal{F}$-Graph)** and **Pfn**. In the left hand side f and g are partial morphisms of fuzzy graphs. Construct a pushout in **Pfn**, which is in the middle.

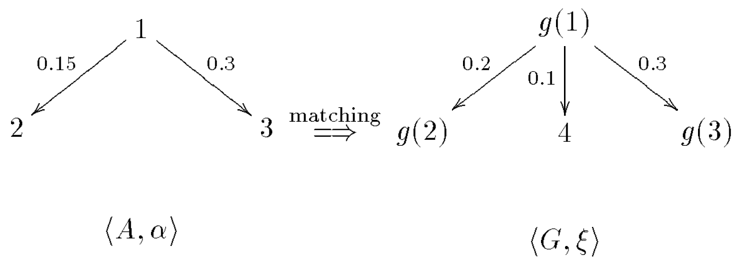
$$
\begin{tikzcd}
\langle A, \alpha \rangle \arrow[r, "f"] \arrow[d, "g"'] & \langle B, \beta \rangle \\
\langle C, \gamma \rangle &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dotted, "h"] \\
C \arrow[r, dotted, "k"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle A, \alpha \rangle \arrow[r, "f"] \arrow[d, "g"'] & \langle B, \beta \rangle \arrow[d, "h"] \\
\langle C, \gamma \rangle \arrow[r, "k"'] & \langle D, \delta \rangle
\end{tikzcd}
$$

Finally define $\delta = k^\sharp \gamma k \sqcup h^\sharp \beta h$ as a fuzzy graph structure on D. Then we obtain the pushout square in the right hand side.

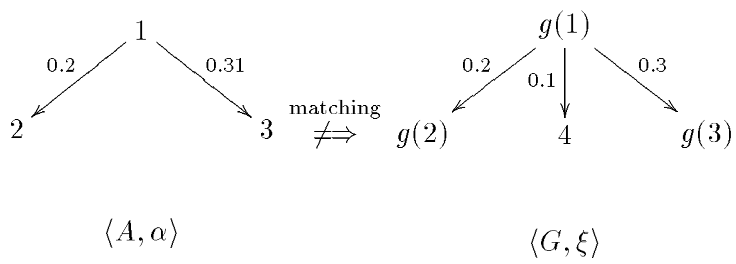
Rewriting consists of two notions. One is a "rewriting rule" which is a correspondence between nodes and can be formalized as partial functions on sets of nodes. The other is a "matching" into which rewriting rules are applied. Matchings have to indicate appropriate subgraphs in objective graphs.

Definition 4.1. Let $\langle A, \alpha \rangle$ and $\langle B \beta \rangle$ be fuzzy graphs. A **rewriting rule** is a triple $p = (\langle A, \alpha \rangle, \langle B, \beta \rangle, f : A \rightharpoonup B)$, where f is a partial function (which is not necessarily a partial morphism).

Let $\langle A, \alpha \rangle$ and $\langle G, \xi \rangle$ be fuzzy graphs. A **matching** from $\langle A, \alpha \rangle$ into $\langle G, \xi \rangle$ is an injective morphism $g : \langle A, \alpha \rangle \to \langle G, \xi \rangle$ (that is, $g^\sharp g \sqsubseteq \mathrm{id}_G$, $gg^\sharp = \mathrm{id}_A$ and $\alpha g \sqsubseteq g\beta$). From the discussion in [14] one may define matchings as partial and injective morphisms.



In the above example (A, α) is matched into (G, ξ) by g, but is not matched in the next example.



Given a rewriting rule $p = (\langle A, \alpha\rangle, \langle B, \beta\rangle, f : A \to B)$ and a matching $g : \langle A, \alpha\rangle \to \langle G, \xi\rangle$, we construct a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dotted, "h"] \\ G \arrow[r, dotted, "k"'] & H \end{tikzcd}$$

in **Pfn**. The graph $\langle G, \xi\rangle$ is said to be **rewritten into** $\langle H, \eta\rangle$ **by applying a rewriting rule** p **along a matching** g if a relation η is defined as $\eta = k^\sharp(\xi \mathbin{\dot{-}} g^\sharp \alpha g)k \sqcup h^\sharp \beta h$. Applying rewriting rule is viewed as a **rewriting square**:

$$\begin{tikzcd} \langle A, \alpha\rangle \arrow[r, "f"] \arrow[d, "g"'] & \langle B, \beta\rangle \arrow[d, "h"] \\ \langle G, \xi\rangle \arrow[r, "k"'] & \langle H, \eta\rangle \end{tikzcd}$$

Remark that a rewriting square is not necessary a pushout in **Pfn($\mathcal{F}$-Graph)**. Mizoguchi and Kawahara [14] showed that rewriting squares are pushouts in **Pfn(Graph)** if rewriting rules are partial morphisms of graphs. To extend this fact for fuzzy graph rewritings we need to defined the following.

Definition 4.2. Let $\langle A, \alpha\rangle$ and $\langle G, \xi\rangle$ be fuzzy graphs. An injective function $g : A \to G$ is a **rigorous** matching from $\langle A, \alpha\rangle$ into $\langle G, \xi\rangle$ if $g^\sharp \alpha g$ is crisp relative to ξ.

Note that for a rigorous matching $g : \langle A, \alpha\rangle \to \langle G, \xi\rangle$ the following holds:

$$(\xi \mathbin{\dot{-}} g^\sharp \alpha g) = \xi \sqcup g^\sharp \alpha g \text{ and } (\xi \mathbin{\dot{-}} g^\sharp \alpha g) \sqcap g^\sharp \alpha g = O.$$

Theorem 4.2. Let $p = (\langle A, \alpha\rangle, \langle B, \beta\rangle, f : A \to B)$ be a rewriting rule and $g : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ a rigorous matching. If $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ is a partial morphism, then a rewriting square is a pushout in **Pfn($\mathcal{F}$-Graph)**.

Proof : As f is a partial morphism we have $f^\sharp \alpha f = f^\sharp d(f)\alpha f \sqsubseteq f^\sharp f \beta \sqsubseteq \beta$ by $f = d(f)f$ and $k^\sharp g^\sharp \alpha g k = h^\sharp f^\sharp \alpha f h \sqsubseteq h^\sharp \beta h \sqsubseteq \eta$. Hence it follows that

$$\begin{aligned} \eta &= k^\sharp(\xi \dot{-} g^\sharp \alpha g)k \sqcup h^\sharp \beta h \\ &= k^\sharp(\xi \dot{-} g^\sharp \alpha g)k \sqcup k^\sharp g^\sharp \alpha g k \sqcup h^\sharp \beta h \\ &= k^\sharp[(\xi \dot{-} g^\sharp \alpha g) \sqcup g^\sharp \alpha g]k \sqcup h^\sharp \beta h. \end{aligned}$$

We need to show that $\eta = k^\sharp \xi k \sqcup h^\sharp \beta h$. But $(\xi \dot{-} g^\sharp \alpha g) \sqcup g^\sharp \alpha g = \xi$ because $g^\sharp \alpha g$ is crisp relative to ξ. The proof completes. □

5. Main Theorem

In this section we state the main theorem in this paper. First we define ambiguous matchings called ε–matching. Resultant graph applied by an ambiguous matching is ε–similar to a pushout of fuzzy graphs.

Definition 5.1. A matching $g : \langle A, \alpha\rangle \to \langle G, \xi\rangle$ is an ε**–matching** from $\langle A, \alpha\rangle$ to $\langle G, \xi\rangle$ if and only if a relation $g^\sharp \alpha g$ is ε–crisp relative to ξ.

By Lemma 3.2 we can prove the following.

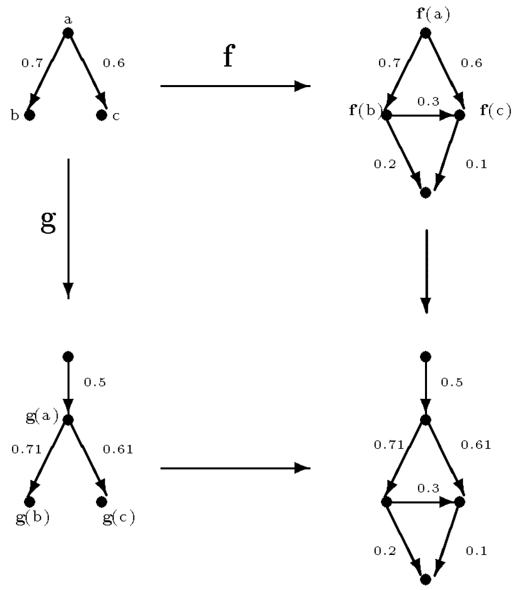
Theorem 5.1. Let $p = (\langle A, \alpha\rangle, \langle B, \beta\rangle, f : A \to B)$ be a rewriting rule. If f in p is a partial morphism of fuzzy graphs and $g : \langle A, \alpha\rangle \to \langle G, \xi\rangle$ is an ε–matching, then $\eta = k^\sharp \xi k \sqcup h^\sharp \beta h$ and $\hat{\eta} = k^\sharp(\xi \dot{-} g^\sharp \alpha g)k \sqcup h^\sharp \beta h$ are ε–similar. (That is, a fuzzy graph $\langle H, \hat{\eta}\rangle$ gives an approximation of a fuzzy graph $\langle H, \eta\rangle$.)

Proof : From the computation in the proof of Lemma 3.2 we have

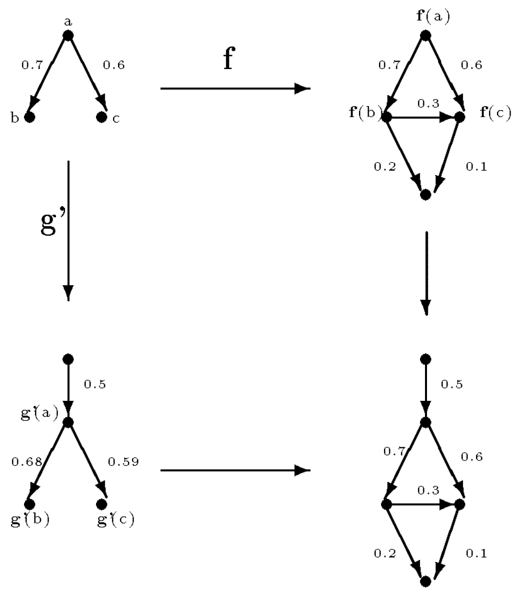
$$\hat{\eta} = k^\sharp[(\xi \dot{-} g^\sharp \alpha g) \sqcup g^\sharp \alpha g]k \sqcup h^\sharp \beta h.$$

Since $g^\sharp \alpha g$ is ε–crisp relative to ξ, two fuzzy relations $(\xi \dot{-} g^\sharp \alpha g) \sqcup g^\sharp \alpha g$ and ξ are ε–similar from Lemma 3.1. Next it follows from Lemma 4.2 that $k^\sharp[(\xi \dot{-} g^\sharp \alpha g) \sqcup g^\sharp \alpha g]k$ and $k^\sharp \xi k$ are ε–similar, and hence η and $\hat{\eta}$ are ε–similar. This completes the proof. □

Here we give examples of rewriting squares of fuzzy graphs, which shows ε–similarity of resultant graphs and pushouts. Let $\varepsilon = 0.2$. The first example exhibits a pushout square since the rewriting rule f is a partial morphism of fuzzy graphs.



The second example exhibits a rewriting square with an ambiguous matching. Observe that the resultant graph of the second one is ε–similar to the resultant one in the first one.



6. Pullbacks of Fuzzy Graphs

In this section we finally mention just a fact that the category **Pfn($\mathcal{F}$-Graph)** of fuzzy graphs and partial morphisms between them has pullbacks. This fact suggests a possibility of the pullback approach [1] to fuzzy graph rewritings. Recall an elementary fact that the category **Pfn** of sets and partial functions has pullbacks.

Theorem 6.1. The category **Pfn($\mathcal{F}$-Graph)** of fuzzy graphs and partial morphisms between them has pullbacks.

Proof : Assume that $h : \langle B, \beta\rangle \to \langle H, \eta\rangle$ and $k : \langle G, \xi\rangle \to \langle H, \eta\rangle$ be two partial morphisms of fuzzy graphs. Then we construct a pullback of $h : B \to H$ and $k : G \to H$ in the category **Pfn**, illustrated by the following:

$$\begin{tikzcd} A \arrow[r, dotted, "f"] \arrow[d, dotted, "g"'] & B \arrow[d, "h"] \\ G \arrow[r, "k"'] & H. \end{tikzcd}$$

in the category of fuzzy graphs and partial morphisms. Define a fuzzy relation $\alpha = f\beta f^\sharp \sqcap g\xi g^\sharp$ as a fuzzy graph structure on A. We will prove the following square

$$\begin{tikzcd} \langle A, \alpha\rangle \arrow[r, "f"] \arrow[d, "g"'] & \langle B, \beta\rangle \arrow[d, "h"] \\ \langle G, \xi\rangle \arrow[r, "k"'] & \langle H, \eta\rangle. \end{tikzcd}$$

is a pushout in **Pfn($\mathcal{F}$-Graph)**. To check this fact we first show that $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ and $g : \langle A, \alpha\rangle \to \langle G, \xi\rangle$ are morphisms of fuzzy graphs. But it is trivial from $d(f)\alpha f \sqsubseteq f\beta f^\sharp f \sqsubseteq f\beta$ by $f^\sharp f \sqsubseteq id_B$. Let $f' : \langle A', \alpha'\rangle \to \langle B, \beta\rangle$ and $g' : \langle A', \alpha'\rangle \to \langle G, \xi\rangle$ be two partial morphisms of fuzzy graphs such that $f'h = g'k$. Since A is a pullback in the category **Pfn**, there exists a unique partial function $t : A' \to A$ such that $tf = f'$ and $tg = g'$. Note that $t = f'f^\sharp \sqcap g'g^\sharp$ by the pullback property and so $d(t) \sqsubseteq d(f'f^\sharp) \sqsubseteq d(f')$. Then we have

$$\begin{aligned} d(t)\alpha' t &= d(t)\alpha'(f'f^\sharp \sqcap g'g^\sharp) \\ &\sqsubseteq d(t)\alpha' f'f^\sharp \sqcap d(t)\alpha' g'g^\sharp \\ &\sqsubseteq d(f')\alpha' f'f^\sharp \sqcap d(g')\alpha' g'g^\sharp \\ &\sqsubseteq f'\beta f^\sharp \sqcap g'\xi g^\sharp \\ &= tf\beta f^\sharp \sqcap tg\xi g^\sharp \end{aligned}$$

$$= t(f\beta f^\sharp \sqcap g\xi g^\sharp)$$
$$= t\alpha,$$

and consequently t is a partial morphism of fuzzy graphs, which completes the proof. □

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