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Abstract

The resolvent problem of the linearized compressible Navier-Stokes equation around a given constant state is considered in an infinite layer $\mathbf{R}^{n-1} \times (0,a)$, $n \geq 2$, under the no slip boundary condition for the momentum. It is proved that the linearized operator is sectorial in $W^{1,p} \times L^p$ for $1 < p < \infty$. The L^p estimates for the resolvent are established for all $1 \leq p \leq \infty$. The estimates for the *high frequency part* of the resolvent are also derived, which lead to the exponential decay of the corresponding part of the semigroup.

1. Introduction

This paper studies the following resolvent problem

$$(\lambda + L)u = f \tag{1.1}$$

in an infinite layer $\Omega = \mathbf{R}^{n-1} \times (0,a)$, $n \geq 2$, where $\lambda \in \mathbf{C}$ is a parameter, $f = f(x)$ is a given function with values in $\mathbf{R}^{n+1}$, $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ is the unknown function with $\phi = \phi(x) \in \mathbf{R}$ and $m = {}^T(m^1(x), \cdots, m^n(x)) \in \mathbf{R}^n$, and L is an operator defined by

$$L = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla & -\nu \Delta I_n - \tilde{\nu} \nabla \mathrm{div} \end{pmatrix}$$

with positive constants ν and γ and a nonnegative constant $\tilde{\nu}$. Here $x = \begin{pmatrix} x' \\ x_n \end{pmatrix} \in \Omega$ with $x' \in \mathbf{R}^{n-1}$, $x_n \in (0,a)$; the superscript ${}^T\cdot$ stands for the transposition; I_n is the $n \times n$ identity matrix; and div, ∇ and Δ are the usual divergence, gradient and Laplacian with respect to x. We consider (1.1) under the boundary condition

$$m|_{\partial\Omega} = 0. \tag{1.2}$$

Problem (1.1)–(1.2) is the resolvent problem associated with the linearization of the compressible Navier-Stokes equation around a motionless state with a positive constant density, where ϕ and m are the Laplace transform of the perturbation of the density and the momentum, respectively.

In this paper we establish the L^p estimates for the solution of (1.1)–(1.2) for all $1 \le p \le \infty$. The estimates show that $-L$ generates an analytic semigroup in $W^{1,p} \times L^p$ for $1 < p < \infty$. We also establish the estimates for the *high frequency part* of the resolvent, which lead to the exponential decay of the corresponding part of the semigroup. The *low frequency part* is investigated in [5]. The precise statement of the main results of this paper will be given in section 2.

The resolvent problem in an infinite layer was studied in [1, 2, 3] in the case of the incompressible Stokes equation. They established the L^p estimates of the resolvent for $1 < p < \infty$, which yields the exponential decay of the Stokes semigroup in L^p norms. To analyze the resolvent, they considered the Fourier transform in $x' \in \mathbf{R}^{n-1}$ and derived solution formulae for the resolvent problem. The L^p estimates were then obtained by applying the Fourier multiplier theorem.

In this paper we will also consider the Fourier transform $(\lambda + \widehat{L}_{\xi'})^{-1}$ of the resolvent in $x' \in \mathbf{R}^{n-1}$, where $\xi' \in \mathbf{R}^{n-1}$ denotes the phase space variable. In section 3 we derive an integral representation of $(\lambda + \widehat{L}_{\xi'})^{-1}$. In the derivation we make use of the invariance of (1.1) under the orthogonal group and Green's formula for (1.1), which naturally leads to a decomposition of the solution into the two parts; one is the solution under the slip boundary condition; and the other is the term arising from the viscous friction stress due to the no slip boundary condition (1.2). (Cf. the solution formula for the half-space problem given in [6, 7].) In section 4 we investigate the resolvent problem based on the integral representation. The L^p estimates for $1 < p < \infty$ are obtained by applying the Fourier multiplier theorem as in [1, 2, 3]. We obtain the L^p estimates for $p = 1, \infty$ based on the Riemann-Lebesgue lemma as in the analysis of the Cauchy problem for (1.1) given in [8]. We also establish the estimates for the high frequency part $|\xi'| >> 1$. In contrast to the incompressible Stokes problem, $\widehat{L}_{\xi'}$ has different characters between the cases $|\xi'| >> 1$ and $|\xi'| << 1$. The spectral analysis near the origin is given in [5]. It is shown in [5] that the continuous spectrum reaches the origin $\lambda = 0$.

2. Main Results

We first introduce some notation which will be used throughout the paper. For a domain D and $1 \le p \le \infty$ we denote by $L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let ℓ be a nonnegative integer. The symbol $W^{\ell,p}(D)$ denotes the ℓ th order L^p Sobolev space on D with norm $\|\cdot\|_{W^{\ell,p}(D)}$. When $p = 2$, the space $W^{\ell,2}(D)$ is denoted by $H^\ell(D)$ and its norm is denoted by $\|\cdot\|_{H^\ell(D)}$. $C_0^\ell(D)$ stands for the set of all C^ℓ functions which have compact support in D. We denote by $W_0^{1,p}(D)$ the completion of $C_0^1(D)$ in $W^{1,p}(D)$. In particular, $W_0^{1,2}(D)$ is denoted by $H_0^1(D)$.

We simply denote by $L^p(D)$ (resp., $W^{\ell,p}(D)$, $H^\ell(D)$) the set of all vector fields $m = {}^T(m^1, \cdots, m^n)$ on D with $m^j \in L^p(D)$ (resp., $W^{\ell,p}(D)$, $H^\ell(D)$), $j = 1, \cdots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{W^{\ell,p}(D)}$, $\|\cdot\|_{H^\ell(D)}$). For $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ with $\phi \in W^{k,p}(D)$ and $m = {}^T(m^1, \cdots, m^n) \in W^{\ell,q}(D)$, we define $\|u\|_{W^{k,p}(D)\times W^{\ell,q}(D)}$ by $\|u\|_{W^{k,p}(D)\times W^{\ell,q}(D)} = \|\phi\|_{W^{k,p}(D)} + \|m\|_{W^{\ell,q}(D)}$. When $k = \ell$ and $p = q$, we simply write $\|u\|_{W^{k,p}(D)\times W^{k,p}(D)} = \|u\|_{W^{k,p}(D)}$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $W^{\ell,p}(\Omega)$, $H^\ell(\Omega)$) as L^p (resp., $W^{\ell,p}$, H^ℓ). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case $D = (0, a)$ we denote the norm of $L^p(0, a)$ by $|\cdot|_p$. The inner product of $L^2(0, a)$ is denoted by

$$(f, g) = \int_0^a f(x_n)\overline{g(x_n)}\, dx_n, \quad f, g \in L^2(0, a).$$

Here $\overline{g}$ denotes the complex conjugate of g. We will also denote the bilinear pairing f and g by

$$((f, g)) = \int_0^a f(x_n) g(x_n)\, dx_n.$$

The norms of $W^{\ell,p}(0, a)$ and $H^\ell(0, a)$ are denoted by $|\cdot|_{W^{\ell,p}}$ and $|\cdot|_{H^\ell}$, respectively.

We often write $x \in \Omega$ as $x = \begin{pmatrix} x' \\ x_n \end{pmatrix}$, $x' = {}^T(x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$. Partial derivatives of a function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u;\, |\alpha| = k)$.

We denote the $k \times k$ identity matrix by I_k. In particular, when $k = n+1$, we simply write I for I_{n+1}. We also define $(n+1) \times (n+1)$ diagonal matrices

Q_0, $\widetilde{Q}$, Q' and Q_n by

$$Q_0 = \mathrm{diag}\,(1, 0, \cdots, 0, 0), \quad \widetilde{Q} = \mathrm{diag}\,(0, 1 \cdots, 1, 1),$$

$$Q' = \mathrm{diag}\,(0, 1, \cdots, 1, 0), \quad Q_n = \mathrm{diag}\,(0, 0, \cdots, 0, 1).$$

For $u = \begin{pmatrix} \phi \\ m \end{pmatrix} \in \mathbf{R}^{n+1}$ with $m = \begin{pmatrix} m' \\ m^n \end{pmatrix} \in \mathbf{R}^n$, we have

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ m \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ m' \\ 0 \end{pmatrix}, \quad Q_n u = \begin{pmatrix} 0 \\ 0 \\ m^n \end{pmatrix}.$$

We next introduce some notation about integral operators. For a function $f = f(z)$ $(z \in \mathbf{R}^k)$, we denote its Fourier transform by $\mathscr{F}_z f$:

$$(\mathscr{F}_z f)(\zeta) = \int_{\mathbf{R}^k} f(z) e^{-i\zeta \cdot z}\, dz$$

In particular, when $k = n - 1$ $(x' \in \mathbf{R}^{n-1})$, we denote $\mathscr{F}_{x'} f$ by $\widehat{f}$,

$$\widehat{f} = (\mathscr{F}_{x'} f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-i\xi' \cdot x'}\, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}_\zeta^{-1}$:

$$(\mathscr{F}_\zeta^{-1} f)(z) = (2\pi)^{-k} \int_{\mathbf{R}^k} f(\zeta) e^{i\zeta \cdot z}\, d\zeta.$$

For a function $K(x_n, y_n)$ on $(0, a) \times (0, a)$ we will denote by Kf the integral operator $\int_0^a K(x_n, y_n) f(y_n)\, dy_n$. Similarly, for a function $K(x', x_n, y_n)$ on $\mathbf{R}^{n-1} \times (0, a) \times (0, a)$ we will denote by Kf the integral operator $\int_{\mathbf{R}^{n-1}} \int_0^a K(x' - y', x_n, y_n) f(y_n)\, dy' dy_n$.

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ we will denote the set $\{\lambda \in \mathbf{C};\ |\arg(\lambda - \Lambda)| \le \theta\}$ by $\Sigma(\Lambda, \theta)$:

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C};\ |\arg(\lambda - \Lambda)| \le \theta\}.$$

We now state the main results of this paper.

Theorem 2.1. *Let* $1 < p < \infty$. *There exists a number* $\theta \in (\frac{\pi}{2}, \pi)$ *such that for any* $\eta > 0$ *problem* (1.1)–(1.2) *has a unique solution* $u \in W^{1,p} \times$

$\left(W^{2,p} \cap W_0^{1,p}\right)$ *for any* $f \in W^{1,p} \times L^p$*, provided that* $\lambda \in \Sigma(\eta, \theta)$*. Furthermore,* $u = (\lambda + L)^{-1} f$ *satisfies the following estimates uniformly in* $\lambda \in \Sigma(\eta, \theta)$*:*

$$\|\partial_x^k (\lambda + L)^{-1} f\|_p \leq C \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_p}{(|\lambda| + 1)^{1 - \frac{k}{2}}} \right\}, \quad k = 0, 1,$$

and

$$\|\partial_x^2 \widetilde{Q} (\lambda + L)^{-1} f\|_2 \leq C \|f\|_{W^{1,p} \times L^p}.$$

Furthermore, if $\widetilde{Q} f\Big|_{x_n = 0, a} = 0$*, then there holds*

$$\|\partial_x \widetilde{Q} (\lambda + L)^{-1} f\|_p \leq \frac{C}{|\lambda| + 1} \|f\|_{W^{1,p}}.$$

In the application to the nonlinear problem we will also use the following $H^s \to L^\infty$ estimates of the resolvent.

Theorem 2.2. *Let* θ *be the number given in Theorem* 2.1 *and let* η *be a positive number. Then the following estimates hold uniformly in* $\lambda \in \Sigma(\eta, \theta)$*:*

$$\|\partial_x^k Q_0 (\lambda + L)^{-1} f\|_\infty \leq C \left\{ \frac{\|Q_0 f\|_{H^{[\frac{n}{2}] + 1 + k}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_{H^{[\frac{n}{2}] + k}}}{(|\lambda| + 1)^{\frac{3}{4}}} \right\}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \widetilde{Q} (\lambda + L)^{-1} f\|_\infty \leq C \left\{ \frac{\|Q_0 f\|_{H^{[\frac{n}{2}] + k}}}{(|\lambda| + 1)^{\frac{3}{4}}} + \frac{\|\widetilde{Q} f\|_{H^{[\frac{n}{2}] - 1 + k}}}{(|\lambda| + 1)^{\frac{\varepsilon}{4}}} \right\}, \quad k = 0, 1.$$

Here ε *is some number satisfying* $0 < \varepsilon < \frac{1}{3}$*.*

As for the L^p estimates for $p = 1, \infty$, we have the following result.

Theorem 2.3. *Let* θ *be the number given in Theorem* 2.1 *and let* η *be a positive number. Then the following estimates hold uniformly in* $\lambda \in \Sigma(\eta, \theta)$*: for* $p = 1, \infty$*,*

$$\|\partial_x^k Q_0 (\lambda + L)^{-1} f\|_p \leq \frac{C}{|\lambda| + 1} \|f\|_{W^{k+1,p} \times W^{k,p}}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \widetilde{Q} (\lambda + L)^{-1} f\|_p \leq C \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_p}{(|\lambda| + 1)^{1 - \frac{k}{2}}} \right\}, \quad k = 0, 1.$$

Furthermore, if $\widetilde{Q}f\Big|_{x_n=0,a} = 0$, *then, there hold, for* $p = 1, \infty$,

$$\|\partial_x \widetilde{Q}(\lambda + L)^{-1} f\|_p \le \frac{C}{|\lambda| + 1} \|f\|_{W^{1,p}}.$$

We see from Theorem 2.1 that $-L$ generates the analytic semigroup $\mathscr{U}(t)$ in $W^{1,p} \times L^p$ for $1 < p < \infty$. Based on Theorems 2.1–2.3 we have the following estimates of $\mathscr{U}(t)$ for $0 < t \le 1$.

Corollary 2.4. *Let* $\ell = 0, 1$. *Then there hold the estimates*

$$\|\partial_x^\ell \mathscr{U}(t) u_0\|_p \le C t^{-\frac{\ell}{2}} \|u_0\|_{W^{\ell,p} \times L^p}, \quad 1 < p < \infty,$$

$$\|\partial_x^\ell \mathscr{U}(t) u_0\|_\infty \le C t^{-(1-\varepsilon)} \|u_0\|_{H^{[\frac{n}{2}]+1+\ell} \times H^{[\frac{n}{2}]+\ell}}$$

and

$$\|\partial_x^\ell \mathscr{U}(t) u_0\|_p \le C t^{-\frac{\ell}{2}} \|u_0\|_{W^{\ell+1,p} \times W^{\ell,p}}, \quad p = 1, \infty,$$

for $0 < t \le 1$ *with some constant* $0 < \varepsilon < 1$, *provided that* u_0 *belongs to the Sobolev spaces indicated on the right-hand side of each inequality above. Furthermore, if* $\widetilde{Q}u_0|_{x_n=0,a} = 0$, *then*

$$\|\partial_x \mathscr{U}(t) u_0\|_1 \le C \|u_0\|_{W^{2,1} \times W^{1,1}}$$

holds for $0 \le t \le 1$.

To investigate problem (1.1)–(1.2) we consider the Fourier transform in $x' \in \mathbf{R}^{n-1}$. Applying the Fourier transform, we have the following boundary value problem for functions $\phi(x_n)$ and $m(x_n)$ on the interval $(0, a)$:

$$\lambda u + \widehat{L}_{\xi'} u = f, \tag{2.1}$$

where $u = \begin{pmatrix} \phi(x_n) \\ m'(x_n) \\ m^n(x_n) \end{pmatrix}$, $f = \begin{pmatrix} f^0(x_n) \\ f'(x_n) \\ f^n(x_n) \end{pmatrix}$ and $\widehat{L}_{\xi'}$ is a closed operator on $H^1(0,a) \times L^2(0,a)$ defined by $\widehat{L}_{\xi'} = \widehat{A}_{\xi'} + \widehat{B}_{\xi'}$ with domain of definition $D(\widehat{L}_{\xi'}) = H^1(0,a) \times (H^2(0,a) \cap H_0^1(0,a))$. Here

$$\widehat{A}_{\xi'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \widetilde{\nu}\xi'^T \xi' & -i\widetilde{\nu}\xi' \partial_{x_n} \\ 0 & -i\widetilde{\nu}^T \xi' \partial_{x_n} & \nu(|\xi'|^2 - \partial_{x_n}^2) - \widetilde{\nu}\partial_{x_n}^2 \end{pmatrix}$$

and

$$\widehat{B}_{\xi'} = \begin{pmatrix} 0 & i\gamma^T\xi' & \gamma\partial_{x_n} \\ i\gamma\xi' & 0 & 0 \\ \gamma\partial_{x_n} & 0 & 0 \end{pmatrix}.$$

In the analysis of the semigroup generated by $-L$, it is convenient to decompose the phase space $\{\xi' \in \mathbf{R}^{n-1}\}$ into the parts $|\xi'| << 1$ and $|\xi'| >> 1$, since $\widehat{L}_{\xi'}$ has different characters between the cases $|\xi'| << 1$ and $|\xi'| >> 1$. Motivated by this we introduce the following decomposition. Let r be a positive number. We take a function $\chi(\xi') \in C^\infty(\mathbf{R}^{n-1})$ satisfying $0 \le \chi \le 1$ on $\mathbf{R}^{n-1}$, $\chi(\xi') = 0$ for $|\xi'| \le \frac{r}{2}$ and $\chi(\xi') = 1$ for $|\xi'| \ge r$. We define the operator $R^{(1)}(\lambda)$ by

$$R^{(1)}(\lambda)f = \mathscr{F}_{\xi'}^{-1}\left[\chi(\xi')(\lambda + \widehat{L}_{\xi'})^{-1}\widehat{f}\right].$$

The estimates in Theorems 2.1–2.3 hold for a negative η with $(\lambda + L)^{-1}$ replaced by $R^{(1)}(\lambda)$.

Theorem 2.5. *Let r be a positive number.*

(i) *There exist positive numbers $\widetilde{\eta}$ and $\widetilde{\theta}$ with $\widetilde{\theta} \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \ge r$.*

(ii) *Let $1 < p < \infty$ and define $R^{(1)}(\lambda)$ as above. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$:*

$$\|\partial_x^k R^{(1)}(\lambda)f\|_p \le \left\{\frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda|+1} + \frac{\|\widetilde{Q}f\|_p}{(|\lambda|+1)^{1-\frac{k}{2}}}\right\}, \quad k = 0, 1,$$

and

$$\|\partial_x^2 \widetilde{Q} R^{(1)}(\lambda)f\|_2 \le C\|f\|_{W^{1,p}\times L^p}.$$

Theorem 2.6 *Let $\widetilde{\eta}$ and $\widetilde{\theta}$ be the numbers as in Theorem 2.5. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$:*

$$\|\partial_x^k Q_0 R^{(1)}(\lambda)f\|_\infty \le C\left\{\frac{\|Q_0 f\|_{H^{[\frac{n}{2}]+1+k}}}{|\lambda|+1} + \frac{\|\widetilde{Q}f\|_{H^{[\frac{n}{2}]+k}}}{(|\lambda|+1)^{\frac{3}{4}}}\right\}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \widetilde{Q} R^{(1)}(\lambda)f\|_\infty \le C\left\{\frac{\|Q_0 f\|_{H^{[\frac{n}{2}]+k}}}{(|\lambda|+1)^{\frac{3}{4}}} + \frac{\|\widetilde{Q}f\|_{H^{[\frac{n}{2}]-1+k}}}{(|\lambda|+1)^{\frac{\varepsilon}{4}}}\right\}, \quad k = 0, 1.$$

Here ε is some number satisfying $0 < \varepsilon < \frac{1}{3}$.

Theorem 2.7. *Let $p = 1, \infty$. Let $\widetilde{\eta}$ and $\widetilde{\theta}$ be the numbers as in Theorem 2.5. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$:*

$$\|\partial_x^k Q_0 R^{(1)}(\lambda) f\|_p \leq \frac{C}{|\lambda| + 1} \|f\|_{W^{k+1,p} \times W^{k,p}}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \widetilde{Q} R^{(1)}(\lambda) f\|_p \leq C \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_p}{(|\lambda| + 1)^{1-\frac{k}{2}}} \right\}, \quad k = 0, 1.$$

3. An integral representation of the resolvent

In this section we derive an integral representation of the resolvent. For this purpose we take the Fourier transform of (1.1)–(1.2) in $x' \in \mathbf{R}^{n-1}$ and consider the boundary value problem (2.1).

We begin with the following observation on the resolvent of $-\widehat{L}_{\xi'}$. If $\lambda \neq 0$, then, by the first row of equation (2.1), ϕ is written as

$$\phi = \frac{1}{\lambda} \left\{ f^0 - i\gamma \xi' \cdot m' - \gamma \partial_{x_n} m^n \right\}. \tag{3.1}$$

Substituting (3.1) into the second and third rows of (2.1), we obtain

$$\mathcal{L}(\lambda, \xi') m = F, \tag{3.2}$$

where $\mathcal{L}(\lambda, \xi') = \mathcal{A}(\lambda, \xi') + \mathcal{B}(\lambda, \xi')$ with domain of definition $D(\widetilde{\mathcal{L}}(\lambda, \xi')) = H^2(0, a) \cap H_0^1(0, a)$ and $F = \lambda \begin{pmatrix} f' \\ f^n \end{pmatrix} - \gamma \begin{pmatrix} i\xi' \\ \partial_{x_n} \end{pmatrix} f^0$. Here

$$\mathcal{A}(\lambda, \xi') = \begin{pmatrix} \{\lambda^2 + \nu\lambda(|\xi'|^2 - \partial_{x_n}^2)\} I_{n-1} & 0 \\ 0 & \lambda^2 + \nu\lambda(|\xi'|^2 - \partial_{x_n}^2) \end{pmatrix},$$

$$\mathcal{B}(\lambda, \xi') = (\tilde{\nu}\lambda + \gamma^2) \begin{pmatrix} \xi'^T \xi' & -i\xi' \partial_{x_n} \\ -i^T \xi' \partial_{x_n} & -\partial_{x_n}^2 \end{pmatrix}.$$

We thus deduce that (2.1) is equivalent to (3.1) and (3.2) if $\lambda \neq 0$. We also write (3.2) as

$$\widetilde{\mathcal{L}}(\lambda, \xi') m = \widetilde{F}, \tag{3.3}$$

where

$$\widetilde{\mathcal{L}}(\lambda,\xi') = -\partial_{x_n}^2 I_n + \widetilde{\mathcal{A}}(\lambda,\xi') + \widetilde{\mathcal{B}}(\lambda,\xi')\partial_{x_n}$$

with domain of definition $D(\widetilde{\mathcal{L}}(\lambda,\xi')) = H^2(0,a)\cap H_0^1(0,a)$ and $\widetilde{F} = \begin{pmatrix} \frac{1}{\nu\lambda}F' \\ \frac{1}{\nu_1\lambda+\gamma^2}F^n \end{pmatrix}$.

Here

$$\widetilde{\mathcal{A}}(\lambda,\xi') = \begin{pmatrix} \frac{1}{\nu}(\lambda+\nu|\xi'|^2)I_{n-1} + \frac{1}{\nu\lambda}(\widetilde{\nu}\lambda+\gamma^2)\xi'^T\xi' & 0 \\ 0 & \frac{\lambda(\lambda+\nu|\xi'|^2)}{\nu_1\lambda+\gamma^2} \end{pmatrix}$$

and

$$\widetilde{\mathcal{B}}(\lambda,\xi') = \begin{pmatrix} 0 & -i\frac{\widetilde{\nu}\lambda+\gamma^2}{\nu\lambda}\xi' \\ -i\frac{\widetilde{\nu}\lambda+\gamma^2}{\nu_1\lambda+\gamma^2}{}^T\xi' & 0 \end{pmatrix}.$$

Here and in what follows we write ν_1 for $\nu+\widetilde{\nu}$:

$$\nu_1 = \nu + \widetilde{\nu}.$$

Lemma 3.1. *Assume that $\lambda \neq 0$ and $\nu_1\lambda+\gamma^2 \neq 0$. Then $\lambda \in \rho(-\widehat{L}_{\xi'})$ if and only if* $\operatorname{Ker}\mathcal{L}(\lambda,\xi') = \{0\}$.

Proof. Suppose that $\lambda \in \rho(-\widehat{L}_{\xi'})$. Let $m \in H^2(0,a)\cap H_0^1(0,a)$ satisfy $\mathcal{L}(\lambda,\xi')m = 0$, namely, $\widetilde{\mathcal{L}}(\lambda,\xi')m = 0$. We define ϕ as in (3.1) with $f^0 = 0$ and set $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$. It then follows that $(\lambda+\widehat{L}_{\xi'})u = 0$. Since $\lambda \in \rho(-\widehat{L}_{\xi'})$, we see that $u = 0$, in particular, $m = 0$. This shows that $\operatorname{Ker}\mathcal{L}(\lambda,\xi') = \{0\}$. Conversely, suppose that $\operatorname{Ker}\mathcal{L}(\lambda,\xi') = \{0\}$, namely, $\operatorname{Ker}\widetilde{\mathcal{L}}(\lambda,\xi') = \{0\}$. We define $\widetilde{F}$ as in (3.3). Since (2.1) is equivalent to (3.1) and (3.3), it suffices to show the unique existence of the solution $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ of (3.1) and (3.3) satisfying $|u|_{H^1\times H^2} \leq C|f|_{H^1\times L^2}$ with some $C>0$. Since $\operatorname{Ker}\widetilde{\mathcal{L}}(\lambda,\xi') = \{0\}$, the standard theory of elliptic equations shows that there exists a unique solution $m \in H^2(0,a)\cap H_0^1(0,a)$ of (3.3) and m satisfies

$$|m|_{H^2} \leq C(\lambda,\xi')\left|\widetilde{F}\right|_2 \leq C(\lambda,\xi')|f|_{H^1\times L^2}.$$

It then follows from (3.1) that $\phi \in H^1(0,a)$ and $|\phi|_{H^1} \leq C(\lambda,\xi')|f|_{H^1\times L^2}$. We thus conclude that $\lambda \in \rho(-\widehat{L}_{\xi'})$. This completes the proof.

We next give a fundamental set of solutions of the ordinary differential equation (3.2) with $F=0$. For this purpose we introduce some notation. We set

$$\lambda_{1,0} = -\nu|\xi'|^2$$

and

$$\lambda_{\pm,0} = -\frac{\nu_1}{2}|\xi'|^2 \pm \frac{1}{2}\sqrt{\nu_1^2|\xi'|^4 - 4\gamma^2|\xi'|^2},$$

and define

$$\mu_1 = \mu_1(\lambda, |\xi'|^2) = \sqrt{\frac{\lambda + \nu|\xi'|^2}{\nu}}$$

and

$$\mu_2 = \mu_2(\lambda, |\xi'|^2) = \sqrt{\frac{\lambda^2 + \nu_1|\xi'|^2\lambda + \gamma^2|\xi'|^2}{\nu_1\lambda + \gamma^2}}.$$

Remark 3.2. We observe that $\mu_1 = \sqrt{\frac{\lambda - \lambda_{1,0}}{\nu}}$ and $\mu_2 = \sqrt{\frac{(\lambda-\lambda_{+,0})(\lambda-\lambda_{-,0})}{(\nu_1\lambda+\gamma^2)}}$. Furthermore, $\lambda_{-,0} = \overline{\lambda_{+,0}}$ with $\operatorname{Im}\lambda_{+,0} = \gamma|\xi'|\sqrt{1 - \frac{\nu_1^2}{4\gamma^2}|\xi'|^2}$ when $|\xi'| < 2\gamma/\nu_1$ and $\lambda_{\pm,0} \in \mathbf{R}$ when $|\xi'| > 2\gamma/\nu_1$, and it holds that

$$\lambda_{\pm,0} = -\frac{\nu_1}{2}|\xi'|^2 \pm i|\xi'| + O(|\xi'|^3)$$

as $|\xi'| \to 0$, and

$$\lambda_{+,0} = -\frac{\gamma^2}{\nu_1} + O(|\xi'|^{-2}), \quad \lambda_{-,0} = -\nu_1|\xi'|^2 + O(1)$$

as $|\xi'| \to \infty$.

Proposition 3.3. *Assume that $\lambda \neq 0$, $\tilde{\nu}\lambda + \gamma^2 \neq 0$, $\nu_1\lambda + \gamma^2 \neq 0$, $\lambda \neq \lambda_{1,0}$ and $\lambda \neq \lambda_{\pm,0}$. Then the following functions $v_1, \cdots, v_{2n}$ form a fundamental set of solutions of the ordinary differential equation in (3.2) with $F = 0$:*

$$v_j(x_n) = \begin{pmatrix} \boldsymbol{e}'_j \cosh \mu_1 x_n \\ -i\frac{\xi_j}{\mu_1}\sinh \mu_1 x_n \end{pmatrix}, \quad v_n(x_n) = \begin{pmatrix} i\frac{\xi'}{\mu_2}\sinh \mu_2 x_n \\ \cosh \mu_2 x_n \end{pmatrix},$$

$$v_{n+j}(x_n) = \begin{pmatrix} \boldsymbol{e}'_j \mu_1 \sinh \mu_1 x_n \\ -i\xi_j \cosh \mu_1 x_n \end{pmatrix}, \quad v_{2n}(x_n) = \begin{pmatrix} i\xi' \cosh \mu_2 x_n \\ \mu_2 \sinh \mu_2 x_n \end{pmatrix},$$

where $j = 1, \cdots, n-1$ and $\boldsymbol{e}'_j = {}^T(0, \cdots, \overset{j}{1}, \cdots, 0) \in \mathbf{R}^{n-1}$.

Remark. We note that $v_1, \cdots, v_{2n}$ are analytic in λ and $|\xi'|^2$.

Proof. Setting $w = {}^T(w^1, \cdots, w^{2n})$ with $w^j = m^j$, $w^{n+j} = \partial_{x_n} m^j$, $j = 1, \cdots, n$, we see that the ordinary differential equation in (3.2) with $F = 0$ is equivalent to

$$\frac{dw}{dx_n} = \mathcal{M}(\lambda, \xi')w, \tag{3.4}$$

where

$$\mathcal{M}(\lambda,\xi') = \begin{pmatrix} 0 & I_n \\ \widetilde{\mathcal{A}}(\lambda,\xi') & \widetilde{\mathcal{B}}(\lambda,\xi') \end{pmatrix}$$

with $\widetilde{\mathcal{A}}(\lambda,\xi')$ and $\widetilde{\mathcal{B}}(\lambda,\xi')$ defined in (3.3).

To obtain a fundamental set of solutions of (3.4) we first consider the characteristic equation of $\mathcal{M}(\lambda,\xi')$. Let T' be an $(n-1)\times(n-1)$ orthogonal matrix and set

$$\mathcal{T} = \begin{pmatrix} T' & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & T' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It then follows that $\mathcal{M}(\lambda,\xi') = \mathcal{T}^{-1}\mathcal{M}(\lambda,T'\xi')\mathcal{T}$.

We take T' in such a way that $T'\xi' = |\xi'|\boldsymbol{e}'_{n-1}$. With this T' we find that

$$\begin{aligned} \det\left(\mu I_{2n} - \mathcal{M}(\lambda,\xi')\right) &= \det\left(\mu I_{2n} - \mathcal{M}(\lambda,T'\xi')\right) \\ &= \det\left(\mu^2 I_n - \mu\widetilde{\mathcal{B}}(\lambda,T'\xi') - \widetilde{\mathcal{A}}(\lambda,T'\xi')\right) \\ &= (\mu^2-\mu_1^2)^{n-1}(\mu^2-\mu_2^2). \end{aligned}$$

Therefore, the eigenvalues of $\mathcal{M}(\lambda,\xi')$ are given by $\pm\mu_1$ and $\pm\mu_2$. Note that $\mu_1 \neq \pm\mu_2$ since $\lambda \neq 0$ and $\widetilde{\nu}\lambda+\gamma^2 \neq 0$. Furthermore, $\mu_j \neq 0$, j=1,2, since $\lambda \neq \lambda_{1,0}$ and $\lambda \neq \lambda_{\pm,0}$.

We next look for eigenvectors associated with $\pm\mu_j$. To obtain eigenvectors for μ_1 we consider the problem

$$\mathcal{M}(\lambda,\xi')\begin{pmatrix} X \\ Y \end{pmatrix} = \mu_1\begin{pmatrix} X \\ Y \end{pmatrix},$$

where $X, Y \in \mathbf{C}^n$. This is equivalent to

$$(\widetilde{\mathcal{A}}(\lambda,\xi') + \mu_1\widetilde{\mathcal{B}}(\lambda,\xi') - \mu_1^2 I_n)X = 0.$$

We write $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}$. Since $\widetilde{\nu}\lambda+\gamma^2 \neq 0$, we have $\xi'\cdot X' - i\mu_1 X_n = 0$. This implies that $X_{j,+} = \begin{pmatrix} \mu_1\boldsymbol{e}'_j \\ -i\xi_j \end{pmatrix}$, $j=1,\cdots,n-1$, are eigenvectors for μ_1. Similarly, one can see that $X_{j,-} = \begin{pmatrix} -\mu_1\boldsymbol{e}'_j \\ -i\xi_j \end{pmatrix}$, $j=1,\cdots,n-1$, are eigenvectors for $-\mu_1$.

We next look for eigenvector for μ_2. As in the case of μ_1, it suffices to find a nontrivial solution $X \in \mathbf{C}^n$ of

$$(\widetilde{\mathcal{A}}(\lambda,\xi') + \mu_2\widetilde{\mathcal{B}}(\lambda,\xi') - \mu_2^2 I_n)X = 0.$$

Since $\widetilde{\nu}\lambda + \gamma^2 \neq 0$, it follows that

$$\begin{cases} \frac{\lambda}{\nu_1\lambda+\gamma^2}X_j + \frac{1}{\lambda}(\xi'\cdot X')\xi_j - i\mu_2\frac{\xi_j}{\lambda}X_n = 0, \quad j = 1,\cdots,n-1, \\ -i\mu_2(\xi'\cdot X') - |\xi'|^2 X_n = 0. \end{cases}$$

This implies that $X_{n,+} = \begin{pmatrix} \xi' \\ -i\mu_2 \end{pmatrix}$ is an eigenvector for μ_2. Similarly, we can find that $X_{n,-} = \begin{pmatrix} \xi' \\ i\mu_2 \end{pmatrix}$ is an eigenvector for $-\mu_2$.

Since $\mu_1 \neq \pm\mu_2$ and $\mu_j \neq 0$, $j = 1, 2$, we have a fundamental set of solutions of the ordinary differential equation in (3.2) with $F = 0$: $u_{j,\pm} = X_{j,\pm}e^{\pm\mu_1 x_n}$ $(j = 1,\cdots,n-1)$, $u_{n,\pm} = X_{n,\pm}e^{\pm\mu_2 x_n}$. One can now obtain the basis $v_1,\cdots,v_{2n}$ by setting $v_j = \frac{1}{2\mu_1}(u_{j,+} - u_{j,-})$, $v_{n+j} = \frac{1}{2}(u_{j,+} + u_{j,-})$ $(j = 1,\cdots,n-1)$, $v_n = \frac{i}{2\mu_2}(u_{n,+} - u_{n,-})$ and $v_{2n} = \frac{i}{2}(u_{n,+} + u_{n,-})$. This completes the proof.

We next give a characterization of the resolvent set $\rho(-\widehat{L}_{\xi'})$. We define complex valued functions b_j $(j = 1, 2, 3)$ by

$$b_1(\lambda,\xi',x_n) = b_1(\lambda,|\xi'|^2,x_n) = \cosh\mu_1 x_n - \cosh\mu_2 x_n,$$

$$b_2(\lambda,\xi',x_n) = b_2(\lambda,|\xi'|^2,x_n) = \mu_1\sinh\mu_1 x_n - \frac{|\xi'|^2}{\mu_2}\sinh\mu_2 x_n,$$

$$b_3(\lambda,\xi',x_n) = b_3(\lambda,|\xi'|^2,x_n) = \mu_2\sinh\mu_2 x_n - \frac{|\xi'|^2}{\mu_1}\sinh\mu_1 x_n,$$

with $\mu_j = \mu_j(\lambda,\xi')$, $j = 1, 2$. We set

$$D(\lambda,\xi') = D(\lambda,|\xi|^2) = b_3(\lambda,\xi',a)b_2(\lambda,\xi',a) + |\xi'|^2 b_1(\lambda,\xi',a)^2.$$

In the following we will frequently abbreviate $b_j(\lambda,\xi',x_n)$ to $b_j(x_n)$. Note that b_j $(j = 1, 2, 3)$ are analytic in λ and $|\xi'|^2$, and hence, so is D. We also set

$$\lambda_{1,k} = -\nu|\xi^{(k)}|^2$$

for $\xi' \in \mathbf{R}^{n-1}$ and $k = 1, 2, \cdots$, where

$$|\xi^{(k)}|^2 = |\xi'|^2 + a_k^2, \quad a_k = \frac{k\pi}{a}.$$

Lemma 3.4. *Assume that $\lambda \neq 0$, $\widetilde{\nu}\lambda + \gamma^2 \neq 0$, $\nu_1\lambda + \gamma^2 \neq 0$, $\lambda \neq \lambda_{\pm,0}$ and $\lambda \neq \lambda_{1,k}$ for any $k = 0, 1, 2, \cdots$. Then $\lambda \in \rho(-\widehat{L}_{\xi'})$ if and only if $D(\lambda,\xi') \neq 0$.*

Proof. By Proposition 3.3 the solution of the equation in (3.2) with $F = 0$ is written as

$$m = c_1 v_1 + \cdots + c_{2n} v_{2n}$$

for some $c_j \in \mathbf{C}$, $j = 1, \cdots, 2n$. This m satisfies the boundary condition $m|_{x_n=0,a} = 0$ if and only if

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_{2n} \end{pmatrix} = 0 \quad \text{with} \quad A = \begin{pmatrix} A_1(0) & A_2(0) \\ A_1(a) & A_2(a) \end{pmatrix}, \tag{3.5}$$

where $A_1(x_n) = (v_1(x_n), \cdots, v_n(x_n))$ and $A_2(x_n) = (v_{n+1}(x_n), \cdots, v_{2n}(x_n))$. Note that $A_1(0) = I_n$. In view of Lemma 3.1, $\lambda \in \rho(-\widehat{L}_{\xi'})$ if and only if (3.5) has only the trivial solution $c_j = 0$, $j = 1, \cdots, 2n$, namely, $\det A \neq 0$.

Let us compute $\det A$. Since $A_1(0) = I_n$, by a well known formula, we have

$$\det A = \det (A_2(a) - A_1(a)A_2(0)).$$

A direct calculation gives

$$\begin{aligned} & A_2(a) - A_1(a)A_2(0) \\ & = \begin{pmatrix} \mu_1 \sinh \mu_1 a I_{n-1} - \frac{\xi'^T \xi'}{\mu_2} \sinh \mu_2 a & -i\xi' b_1(a) \\ -i^T \xi' b_1(a) & b_3(a) \end{pmatrix}. \end{aligned}$$

Let T' be the $(n-1) \times (n-1)$ orthogonal matrix given in the proof of Proposition 3.3 such as $T'\xi' = |\xi'| \boldsymbol{e}'_{n-1}$ and set

$$T = \begin{pmatrix} T' & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} & T(A_2(a) - A_1(a)A_2(0))T^{-1} \\ & = \begin{pmatrix} \mu_1 \sinh \mu_1 a I_{n-2} & 0 & 0 \\ 0 & b_2(a) & -i|\xi'| b_1(a) \\ 0 & -i|\xi'| b_1(a) & b_3(a) \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \det A &= \det (T(A_2(a) - A_1(a)A_2(0))T^{-1}) \\ &= \{\mu_1 \sinh \mu_1 a\}^{n-2} D(\lambda, \xi'). \end{aligned} \tag{3.6}$$

Therefore, we see that $\lambda \in \rho(-\widehat{L}'_\xi)$ if and only if $D(\lambda, \xi') \neq 0$. This completes the proof.

To obtain a solution formula for (2.1) we next consider the Fourier series expansion of the solution $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ of (2.1). We expand ϕ and m into the Fourier series as

$$\phi = \frac{1}{2}\phi_0 + \sum_{k=1}^{\infty} \phi_k \cos a_k x_n, \quad m' = \frac{1}{2}m'_0 + \sum_{k=1}^{\infty} m'_k \cos a_k x_n$$

and

$$m^n = \frac{1}{2}m^n_0 + \sum_{k=1}^{\infty} m^n_k \sin a_k x_n,$$

where $a_k = k\pi/a$; and $m^n_0 = 0$,

$$\phi_k = \frac{2}{a}\int_0^a \phi(x_n) \cos a_k x_n \, dx_n, \quad m'_k = \frac{2}{a}\int_0^a m'(x_n) \cos a_k x_n \, dx_n$$

and

$$m^n_k = \frac{2}{a}\int_0^a m^n(x_n) \sin a_k x_n \, dx_n.$$

Similarly, we expand $f = \begin{pmatrix} f^0 \\ f' \\ f^n \end{pmatrix}$ as

$$f^0 = \frac{1}{2}f^0_0 + \sum_{k=1}^{\infty} f^0_k \cos a_k x_n, \quad f' = \frac{1}{2}f'_0 + \sum_{k=1}^{\infty} f'_k \cos a_k x_n$$

and

$$f^n = \frac{1}{2}f^n_0 + \sum_{k=1}^{\infty} f^n_k \sin a_k x_n,$$

where the Fourier coefficients are defined as above.

Similarly to $\lambda_{1,k}$, we also introduce $\lambda_{\pm,k}$ by

$$\lambda_{\pm,k} = -\frac{\nu_1}{2}|\xi^{(k)}|^2 \pm \frac{1}{2}\sqrt{\nu_1^2|\xi^{(k)}|^4 - 4\gamma^2|\xi^{(k)}|^2},$$

for $\xi' \in \mathbf{R}^{n-1}$ and $k = 1, 2, \cdots$. Here, as above, $|\xi^{(k)}|^2 = |\xi'|^2 + a_k^2$.

Remark 3.5. We note that $\lambda_{\pm,k}$ have properties similar to those of $\lambda_{\pm,0}$, namely, $\lambda_{\pm,k}$ are the two roots of $\lambda^2 + \nu_1|\xi^{(k)}|^2\lambda + \gamma^2|\xi^{(k)}|^2 = 0$; $\lambda_{-,k} = \overline{\lambda_{+,k}}$

with $\mathrm{Im}\,\lambda_{+,k} = \gamma|\xi^{(k)}|\sqrt{1-\frac{\nu_1^2}{4\gamma^2}|\xi^{(k)}|^2}$ when $|\xi^{(k)}| < 2\gamma/\nu_1$ and $\lambda_{\pm,k} \in \mathbf{R}$ when $|\xi^{(k)}| > 2\gamma/\nu_1$; and

$$\lambda_{+,k} = -\frac{\gamma^2}{\nu_1} + O(|\xi^{(k)}|^{-2}), \quad \lambda_{-,k} = -\nu_1|\xi^{(k)}|^2 + O(1)$$

as $|\xi^{(k)}| \to \infty$. In contrast to the case $k = 0$, we find that there exists a positive number η_0 such that

$$\lambda_{1,k},\ \lambda_{\pm,k} \notin \{\lambda;\, \mathrm{Re}\,\lambda \geq -\eta_0\}$$

for all $k \geq 1$ and $\xi' \in \mathbf{R}^{n-1}$.

The Fourier expansions described above are based on the reflection symmetry of equation (1.1), but they do not fit in the boundary condition (1.2). This leads to a decomposition of the Fourier coefficients ϕ_k, m_k' and m_k^n into two parts, one of which involves the boundary values of the x_n derivative of m' at $x_n = 0, a$.

Proposition 3.6. *Let $\xi' \neq 0$. Assume that $\lambda \neq 0$, $\tilde{\nu}\lambda + \gamma^2 \neq 0$, $\nu_1\lambda + \gamma^2 \neq 0$, $\lambda \neq \lambda_{1,k}$ and $\lambda \neq \lambda_{\pm,k}$ for any $k = 0, 1, 2, \cdots$. Then the Fourier coefficients $u_k = \begin{pmatrix} \phi_k \\ m_k \end{pmatrix}$, $k = 0, 1, 2, \cdots$, are given by*

$$u_k = \widehat{L}_k(\lambda, \xi')^{-1} f_k + \widehat{L}_k(\lambda, \xi')^{-1} Y_k.$$

Here

$$\widehat{L}_k(\lambda,\xi')^{-1} = \frac{1}{\lambda-\lambda_{1,k}} \begin{pmatrix} 0 & 0 \\ 0 & P_{0,k} \end{pmatrix} + \frac{1}{(\lambda-\lambda_{+,k})(\lambda-\lambda_{-,k})} \begin{pmatrix} \lambda + \nu_1|\xi^{(k)}|^2 & -i\gamma^T\xi' & -\gamma a_k \\ -i\gamma\xi' & \lambda\frac{\xi'^T\xi'}{|\xi^{(k)}|^2} & -i\lambda\frac{a_k\xi'}{|\xi^{(k)}|^2} \\ \gamma a_k & i\lambda\frac{a_k{}^T\xi'}{|\xi^{(k)}|^2} & \lambda\frac{a_k^2}{|\xi^{(k)}|^2} \end{pmatrix}$$

where $P_{0,k}$ is an $n \times n$ matrix defined by

$$P_{0,k} = I_n - \begin{pmatrix} \frac{\xi'^T\xi'}{|\xi^{(k)}|^2} & -i\frac{a_k\xi'}{|\xi^{(k)}|^2} \\ i\frac{a_k{}^T\xi'}{|\xi^{(k)}|^2} & \frac{a_k^2}{|\xi^{(k)}|^2} \end{pmatrix}$$

and Y_k is given by

$$Y_k = \begin{pmatrix} 0 \\ \frac{2\nu}{a}\{(-1)^k \partial_{x_n} m'(a) - \partial_{x_n} m'(0)\} \\ 0 \end{pmatrix}.$$

Proof. Since

$$\begin{aligned} \frac{2}{a}\int_0^a \partial_{x_n} w(x_n) \cos a_k x_n \, dx_n &= \frac{2}{a}\{(-1)^k w(a) - w(0)\} \\ &\quad + a_k \frac{2}{a}\int_0^a w(x_n) \sin a_k x_n \, dx_n \end{aligned}$$

and

$$\frac{2}{a}\int_0^a \partial_{x_n} w(x_n) \sin a_k x_n \, dx_n = -a_k \frac{2}{a}\int_0^a w(x_n) \cos a_k x_n \, dx_n,$$

it is not so difficult to obtain the desired expression of u_k from (2.1). We omit the details.

We next compute the term Y_k which involves the boundary values of the x_n derivative of m'. To do so, we make use of Green's formula for (3.2). We define $(n-1) \times (n-1)$ matrices

$$P'_{1,0} = \frac{\xi'^T \xi'}{|\xi'|^2}, \quad P'_{0,0} = I_{n-1} - P'_{1,0}.$$

Proposition 3.7. *Let the assumption of Proposition* 3.6 *be satisfied. Then Y_k has the form*

$$Y_k = \frac{2}{a}\int_0^a \{(-1)^k B(\lambda, \xi', y_n) + \check{B}(\lambda, \xi', a - y_n)\} f(y_n) \, dy_n,$$

where $B(\lambda, \xi', y_n)$ and $\check{B}(\lambda, \xi', y_n)$ are $(n+1) \times (n+1)$ matrices of the form

$$B(\lambda, \xi', y_n) = \begin{pmatrix} 0 & 0 & 0 \\ \boldsymbol{b}_0(\lambda, \xi', y_n) & B'(\lambda, \xi', y_n) & \boldsymbol{b}_n(\lambda, \xi', y_n) \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\check{B}(\lambda, \xi', y_n) = B(\lambda, \xi', y_n)\mathrm{diag}(1, \cdots, 1, -1).$$

Here $\boldsymbol{b}_0(\lambda, \xi', y_n)$ *and* $\boldsymbol{b}_n(\lambda, \xi', y_n)$ *are* $(n-1)$*-vectors of the form*

$$\boldsymbol{b}_0(\lambda, \xi', y_n) = i\xi'\beta_0(\lambda, \xi', y_n)$$

with

$$\beta_0(\lambda, \xi', y_n) = \frac{\gamma\lambda}{\nu_1\lambda + \gamma^2}\frac{1}{D(\lambda, \xi')}\{b_3(a)\frac{1}{\mu_2}\sinh\mu_2 y_n + b_1(a)\cosh\mu_2 y_n\}$$

and

$$\boldsymbol{b}_n(\lambda, \xi', y_n) = -\frac{i\xi'}{D(\lambda, \xi')}\{b_3(a)b_1(y_n) - b_1(a)b_3(y_n)\}\,;$$

and $B'(\lambda, \xi', y_n)$ *is an* $(n-1)\times(n-1)$ *matrix defined by*

$$B'(\lambda, \xi', y_n) = -\frac{\sinh\mu_1 y_n}{\sinh\mu_1 a}P'_{0,0} - \beta_1(\lambda, \xi', y_n)P'_{1,0}$$

with

$$\beta_1(\lambda, \xi', y_n) = \frac{1}{D(\lambda, \xi')}\{b_3(a)b_2(y_n) + |\xi'|^2 b_1(a)b_1(y_n)\}.$$

Proof. In the proof we denote $Y_k = \begin{pmatrix} 0 \\ \widetilde{Y}_k \end{pmatrix}$. Since $\lambda \neq 0$, (2.1) is equivalent to (3.1) and (3.2). Consider the following equation

$$\widetilde{M}(\lambda, \xi')m = 0, \tag{3.7}$$

where

$$\widetilde{M}(\lambda, \xi') = \begin{pmatrix} \{\lambda^2 + \nu\lambda(|\xi'|^2 - \partial_{x_n}^2)\}I_{n-1} + (\tilde{\nu}\lambda + \gamma^2)\xi'^T\xi' & i(\tilde{\nu}\lambda + \gamma^2)\xi'\partial_{x_n} \\ i(\tilde{\nu}\lambda + \gamma^2)^T\xi'\partial_{x_n} & \lambda^2 + \nu\lambda|\xi'|^2 - \{\nu_1\lambda + \gamma^2\}\partial_{x_n}^2 \end{pmatrix}.$$

As in the proof of Proposition 3.3, one can see that the following functions $\tilde{v}_1, \cdots, \tilde{v}_{2n}$ form a fundamental set of solutions of (3.7):

$$\tilde{v}_j(x_n) = \begin{pmatrix} \boldsymbol{e}'_j\cosh\mu_1 x_n \\ i\frac{\xi_j}{\mu_1}\sinh\mu_1 x_n \end{pmatrix}, \quad \tilde{v}_n(x_n) = \begin{pmatrix} -i\frac{\xi'}{\mu_2}\sinh\mu_2 x_n \\ \cosh\mu_2 x_n \end{pmatrix},$$

$$\tilde{v}_{n+j}(x_n) = \begin{pmatrix} \boldsymbol{e}'_j\mu_1\sinh\mu_1 x_n \\ i\xi_j\cosh\mu_1 x_n \end{pmatrix}, \quad \tilde{v}_{2n}(x_n) = \begin{pmatrix} -i\xi'\cosh\mu_2 x_n \\ \mu_2\sinh\mu_2 x_n \end{pmatrix},$$

where $j = 1, \cdots, n-1$. We note that $\widetilde{v}_1, \cdots, \widetilde{v}_{2n}$ are analytic in λ and $|\xi'|^2$.

By Green's formula for (3.2), we see that

$$\begin{aligned} ((F, v_j)) &= ((m, \widetilde{M}(\lambda, \xi')\widetilde{v}_j)) - \lambda\nu \left[\partial_{x_n} m' \cdot \widetilde{v}_j'\right]_{x_n=0}^{x_n=a} \\ &\quad -\{\nu_1\lambda + \gamma^2\} \left[\partial_{x_n} m^n \widetilde{v}_j^n\right]_{x_n=0}^{x_n=a} \\ &= -\partial_{x_n} V(a) \cdot \widetilde{v}_j(a) + \partial_{x_n} V(0) \cdot \widetilde{v}_j(0), \end{aligned}$$

where

$$V(x_n) = \begin{pmatrix} \nu\lambda m'(x_n) \\ \{\nu_1\lambda + \gamma^2\} m^n(x_n) \end{pmatrix}.$$

We thus obtain

$$\tilde{A} \begin{pmatrix} \partial_{x_n} V(0) \\ \partial_{x_n} V(a) \end{pmatrix} = \begin{pmatrix} \widetilde{F}^{(1)} \\ \widetilde{F}^{(2)} \end{pmatrix}$$

with

$$\widetilde{A} = \begin{pmatrix} A^{(1)}(0) & -A^{(1)}(a) \\ A^{(2)}(0) & -A^{(2)}(a) \end{pmatrix}, \quad F^{(j)} = ((A^{(j)}, F)) \quad (j = 1, 2),$$

$$A^{(1)}(x_n) = \begin{pmatrix} {}^T\widetilde{v}_1(x_n) \\ \vdots \\ {}^T\widetilde{v}_n(x_n) \end{pmatrix}, \quad A^{(2)}(x_n) = \begin{pmatrix} {}^T\widetilde{v}_{n+1}(x_n) \\ \vdots \\ {}^T\widetilde{v}_{2n}(x_n) \end{pmatrix}.$$

For a moment we assume that $\widetilde{A}$ is invertible. We write the inverse of $\widetilde{A}$ as

$$\widetilde{A}^{-1} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \partial_{x_n} V(0) \\ \partial_{x_n} V(a) \end{pmatrix} = \int_0^a \begin{pmatrix} S_1 A^{(1)}(y_n) + S_2 A^{(2)}(y_n) \\ S_3 A^{(1)}(y_n) + S_4 A^{(2)}(y_n) \end{pmatrix} F(y_n)\, dy_n,$$

and, hence,

$$\begin{aligned} \widetilde{Y}_k &= \frac{2}{a\lambda} \widetilde{Q}' \{(-1)^k \partial_{x_n} V(a) - \partial_{x_n} V(0)\} \\ &= \frac{2}{a\lambda} \int_0^a \widetilde{Q}' \{(-1)^k (S_3 A^{(1)}(y_n) + S_4 A^{(2)}(y_n)) \\ &\qquad -(S_1 A^{(1)}(y_n) + S_2 A^{(2)}(y_n))\} F(y_n)\, dy_n. \end{aligned}$$

Here and in what follows we denote the $n \times n$ matrix $\mathrm{diag}(1, \cdots, 1, 0)$ by $\widetilde{Q}'$:

$$\widetilde{Q}' = \mathrm{diag}(1, \cdots, 1, 0).$$

By a direct calculation we have

$$A^{(1)}(x_n) = \begin{pmatrix} \cosh \mu_1 x_n I_{n-1} & i\frac{\xi'}{\mu_1} \sinh \mu_1 x_n \\ -i\frac{{}^T\xi'}{\mu_2} \sinh \mu_2 x_n & \cosh \mu_2 x_n \end{pmatrix}$$

and

$$A^{(2)}(x_n) = \begin{pmatrix} \mu_1 \sinh \mu_1 x_n I_{n-1} & i\xi' \cosh \mu_1 x_n \\ -i{}^T\xi' \cosh \mu_2 x_n & \mu_2 \sinh \mu_2 x_n \end{pmatrix}.$$

We thus obtain

$$\widetilde{A} = \begin{pmatrix} I_n & -A^{(1)}(a) \\ A^{(2)}(0) & -A^{(2)}(a) \end{pmatrix}.$$

We define $A(y_n)$ by

$$\begin{aligned} A(y_n) &= A^{(2)}(0)A^{(1)}(y_n) - A^{(2)}(y_n) \\ &= \begin{pmatrix} -\mu_1 \sinh \mu_1 y_n I_{n-1} + \frac{|\xi'|^2}{\mu_2} \sinh \mu_2 y_n P'_{1,0} & -ib_1(y_n)\xi' \\ -ib_1(y_n){}^T\xi' & -b_3(y_n) \end{pmatrix}. \end{aligned}$$

One can see that if $A(a)$ is invertible, then so is $\tilde{A}$, and $\tilde{A}^{-1}$ is given by

$$\tilde{A}^{-1} = \begin{pmatrix} I_n - A^{(1)}(a)A(a)^{-1}A^{(2)}(0) & A^{(1)}(a)A(a)^{-1} \\ -A(a)^{-1}A^{(2)}(0) & A(a)^{-1} \end{pmatrix}.$$

We now verify that $A(a)$ is invertible under the assumption of Proposition 3.7. We denote the $n \times n$ diagonal matrix $\mathrm{diag}\,(1, \cdots, 1, -1)$ by $\check{I}_n$. We observe that $A^{(j)}(x_n) = {}^T\left[\check{I}_n A_j(x_n)\check{I}_n\right]$, $j = 1, 2$, where $A_j(x_n)$ are the matrices given in the proof of Lemma 3.4. Therefore, we have

$$\widetilde{A} = {}^T\left[\mathrm{diag}\,(\check{I}_n, -\check{I}_n)\, A\, \mathrm{diag}\,(\check{I}_n, \check{I}_n)\right].$$

It then follows from (3.6) that

$$\det \widetilde{A} = (-1)^n \det A = (-1)^n \{\mu_1 \sinh \mu_1 a\}^{n-2} D(\lambda, \xi'),$$

and hence, by Lemma 3.4, $A(a)$ is invertible under the assumption of Proposition 3.7.

We thus obtain

$$\widetilde{Y}_k = \frac{2}{a\lambda}\int_0^a \{(-1)^k \widetilde{B}^{(1)}(\lambda,\xi',y_n) + \widetilde{B}^{(2)}(\lambda,\xi',y_n)\}F(y_n)\,dy_n, \tag{3.8}$$

where $\widetilde{B}^{(1)}(\lambda,\xi,y_n) = -\widetilde{Q}'A(a)^{-1}A(y_n)$ and $\widetilde{B}^{(2)}(\lambda,\xi',y_n) = \widetilde{Q}'(A^{(1)}(y_n) - A^{(1)}(a)A(a)^{-1}A(y_n))$.

We next show that

$$-A(a)^{-1}A(y_n) = K^{(1)}(\lambda,\xi',y_n) + K^{(2)}(\lambda,\xi',y_n), \tag{3.9}$$

where

$$K^{(1)}(\lambda,\xi',y_n) = -\frac{\sinh\mu_1 y_n}{\sinh\mu_1 a}\begin{pmatrix} P'_{0,0} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} & K^{(2)}(\lambda,\xi',y_n) \\ = \; & -\tfrac{1}{D(\lambda,\xi')}\begin{pmatrix} \{b_3(a)b_2(y_n) + |\xi'|^2 b_1(a)b_1(y_n)P'_{1,0} & i\{b_3(a)b_1(y_n) - b_1(a)b_3(y_n)\}\xi' \\ i\{b_2(a)b_1(y_n) - b_1(a)b_2(y_n)\}^T\xi' & b_2(a)b_3(y_n) + |\xi'|^2 b_1(a)b_1(y_n) \end{pmatrix}. \end{aligned}$$

Let us prove (3.9). Noting that $P'_{1,0}$ is an orthogonal projection onto the subspace spanned by ξ', namely, $P_{1,0}\xi' = \xi'$, ${}^T\xi' P'_{1,0} = {}^T\xi'$ and ${P'_{1,0}}^2 = P'_{1,0}$, we see that

$$A(a)^{-1} = -\frac{1}{\mu_1\sinh\mu_1 a}\begin{pmatrix} P'_{0,0} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{D(\lambda,\xi')}\begin{pmatrix} -b_3(a)P'_{1,0} & ib_1(a)\xi' \\ ib_1(a)^T\xi' & -b_2(a) \end{pmatrix},$$

since $\mu_1\sinh\mu_1 a \neq 0$ and $D \neq 0$. We now obtain (3.9) by a direct computation.

We next prove $\widetilde{B}^{(2)}(\lambda,\xi',a-y_n) = \widetilde{B}^{(1)}(\lambda,\xi',y_n)\check{I}_n$. To do so, we write $\widetilde{B}^{(j)}(\lambda,\xi',y_n)$ as

$$\widetilde{B}^{(j)}(\lambda,\xi',y_n) = \begin{pmatrix} B'^{(j)}(\lambda,\xi',y_n) & \boldsymbol{b}_n^{(j)}(\lambda,\xi',y_n) \\ 0 & 0 \end{pmatrix} \tag{3.10}$$

for $j = 1, 2$. Then we need to show

$$B'^{(1)}(\lambda,\xi',y_n) = B'^{(2)}(\lambda,\xi',a-y_n) \tag{3.11}$$

and

$$\boldsymbol{b}_n^{(1)}(\lambda,\xi',y_n) = -\boldsymbol{b}_n^{(2)}(\lambda,\xi',a-y_n). \tag{3.12}$$

To prove (3.11) and (3.12) we make use of the following reflection symmetry of problem (3.2). For $m(x_n) = \begin{pmatrix} m'(x_n) \\ m^n(x_n) \end{pmatrix}$ we define $\check{m}(x_n)$ by

$$\check{m}(x_n) = \begin{pmatrix} m'(a-x_n) \\ -m^n(a-x_n) \end{pmatrix}.$$

Let $m(x_n)$ be a solution of problem (3.2) with $F(x_n) = \lambda \begin{pmatrix} f'(x_n) \\ f^n(x_n) \end{pmatrix} - \gamma \begin{pmatrix} i\xi' \\ \partial_{x_n} \end{pmatrix} f^0(x_n)$. Then $\check{m}(x_n)$ is a solution of problem (3.2) with $F(x_n)$ replaced by $\check{F}(x_n) = \begin{pmatrix} F'(a-x_n) \\ -F^n(a-x_n) \end{pmatrix}$. Since $\partial_{x_n}(\check{m})'(a) = -\partial_{x_n} m'(0)$ and $\partial_{x_n}(\check{m})'(0) = -\partial_{x_n} m'(a)$, we have

$$(-1)^k \partial_{x_n}(\check{m})'(a) - \partial_{x_n}(\check{m})'(0) = (-1)^k\{(-1)^k \partial_{x_n} m'(a) - \partial_{x_n} m'(0)\}. \tag{3.13}$$

We next define $B'_k(\lambda, \xi', y_n)$ and $\boldsymbol{b}_{n,k}(\lambda, \xi', y_n)$ by

$$B'_k(\lambda, \xi', y_n) = (-1)^k B'^{(1)}(\lambda, \xi', y_n) + B'^{(2)}(\lambda, \xi', y_n) \tag{3.14}$$

and

$$\boldsymbol{b}_{n,k}(\lambda, \xi', y_n) = (-1)^k \boldsymbol{b}_n^{(1)}(\lambda, \xi', y_n) + \boldsymbol{b}_n^{(2)}(\lambda, \xi', y_n). \tag{3.15}$$

Since $\boldsymbol{b}_{n,k}(\lambda, \xi', 0) = \boldsymbol{b}_{n,k}(\lambda, \xi', a) = 0$, integrating by parts, we see from (3.8) that

$$\begin{aligned} &\frac{2}{a}\lambda\nu\{(-1)^k \partial_{x_n}(\check{m})'(a) - \partial_{x_n}(\check{m})'(0)\} \\ &= \frac{2}{a}\int_0^a [B'_k(\lambda, \xi', y_n)\{\lambda f'(a-y_n) - i\gamma\xi' f^0(a-y_n)\} \\ &\qquad -\lambda \boldsymbol{b}_{n,k}(\lambda, \xi', y_n) f^n(a-y_n) \\ &\qquad +\gamma\partial_{y_n} \boldsymbol{b}_{n,k}(\lambda, \xi', y_n) f^0(a-y_n)]\, dy_n \\ &= \frac{2}{a}\int_0^a [B'_k(\lambda, \xi', a-y_n)\{\lambda f'(y_n) - i\gamma\xi' f^0(y_n)\} \\ &\qquad -\lambda \boldsymbol{b}_{n,k}(\lambda, \xi', a-y_n) f^n(y_n) \\ &\qquad +\gamma\partial_{y_n} \boldsymbol{b}_{n,k}(\lambda, \xi', a-y_n) f^0(y_n)]\, dy_n \end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
&\frac{2}{a}\lambda\nu\{(-1)^k\partial_{x_n}m'(a)-\partial_{x_n}m'(0)\}\\
(3.17)\qquad &= \frac{2}{a}\int_0^a[B'_k(\lambda,\xi',y_n)\{\lambda f'(y_n)-i\gamma\xi' f^0(y_n)\}\\
&\qquad +\lambda \boldsymbol{b}_{n,k}(\lambda,\xi',y_n)f^n(y_n)+\gamma\partial_{y_n}\boldsymbol{b}_{n,k}(\lambda,\xi',y_n)f^0(y_n)]\,dy_n.
\end{aligned}
$$

Combining (3.13), (3.16) and (3.17) we obtain $B'_k(\lambda,\xi',a-y_n)=(-1)^kB'_k(\lambda,\xi',y_n)$ and $\boldsymbol{b}_{n,k}(\lambda,\xi',a-y_n)=(-1)^{k+1}\boldsymbol{b}_{n,k}(\lambda,\xi',y_n)$. This, together with (3.14) and (3.15), implies that

$$
\begin{aligned}
&(-1)^kB'^{(1)}(\lambda,\xi',a-y_n)+B'^{(2)}(\lambda,\xi',a-y_n)\\
(3.18)\qquad &= B'^{(1)}(\lambda,\xi',y_n)+(-1)^kB'^{(2)}(\lambda,\xi',y_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&(-1)^k\boldsymbol{b}_n^{(1)}(\lambda,\xi',a-y_n)+\boldsymbol{b}_n^{(2)}(\lambda,\xi',a-y_n)\\
(3.19)\qquad &= -\boldsymbol{b}_n^{(1)}(\lambda,\xi',y_n)-(-1)^k\boldsymbol{b}_n^{(2)}(\lambda,\xi',y_n).
\end{aligned}
$$

It follows from (3.18), with k being odd and even respectively, that

$$
\begin{cases}
-B'^{(1)}(\lambda,\xi',a-y_n)+B'^{(2)}(\lambda,\xi',a-y_n)=B'^{(1)}(\lambda,\xi',y_n)-B'^{(2)}(\lambda,\xi',y_n),\\
B'^{(1)}(\lambda,\xi',a-y_n)+B'^{(2)}(\lambda,\xi',a-y_n)=B'^{(1)}(\lambda,\xi',y_n)+B'^{(2)}(\lambda,\xi',y_n),
\end{cases}
$$

which implies that $B'^{(2)}(\lambda,\xi',a-y_n)=B'^{(1)}(\lambda,\xi',y_n)$. This shows (3.11). Similarly, one can prove (3.12) by using (3.19).

We finally set $B'(\lambda,\xi',y_n)=B'^{(1)}(\lambda,\xi',y_n)$, $\boldsymbol{b}_n(\lambda,\xi',y_n)=\boldsymbol{b}_n^{(1)}(\lambda,\xi',y_n)$ and $\boldsymbol{b}_0(\lambda,\xi',y_n)=-\frac{\gamma}{\lambda}\{iB'(\lambda,\xi',y_n)\xi'-\partial_{y_n}\boldsymbol{b}_n(\lambda,\xi',y_n)\}$. Then, by a direct calculation, we see that

$$
\boldsymbol{b}_0=\frac{i\gamma\lambda\xi'}{\nu_1\lambda+\gamma^2}\frac{1}{D(\lambda,\xi')}\left\{b_3(a)\frac{1}{\mu_2}\sinh\mu_2y_n+b_1(a)\cosh\mu_2y_n\right\},
$$

and the desired expression of Y_k is obtained. This completes the proof.

We now give an integral representation of $(\lambda+\widehat{L}_{\xi'})^{-1}f$. To do so, we introduce some functions. We define $g^D_{\mu_j}(x_n,y_n)$ $(j=1,2)$ by

$$
g^D_{\mu_j}(x_n,y_n)=\frac{1}{\mu_j\sinh\mu_ja}\sinh\mu_j(a-x_n)\sinh\mu_jy_n,\quad y_n\le x_n,
$$

and x_n, y_n exchanged for $x_n \leq y_n$. Similarly, we define $g^N_{\mu_j}(x_n, y_n)$ by

$$g^N_{\mu_j}(x_n, y_n) = \frac{1}{\mu_j \sinh \mu_j a} \cosh \mu_j(a - x_n) \cosh \mu_j y_n, \quad y_n \leq x_n,$$

and x_n, y_n exchanged for $x_n \leq y_n$. We set

$$g^M_{\mu_1,\mu_2}(x_n, y_n) = g^M_{\mu_1}(x_n, y_n) - g^M_{\mu_2}(x_n, y_n), \quad M = D, N.$$

Note that $g^D_{\mu_j}$ and $g^N_{\mu_j}$ are the Green functions of the equation $\mu_j^2 v - \partial^2_{x_n} v = 0$ under the Dirichlet and Neumann boundary conditions at $\{x_n = 0, a\}$, respectively. We also define $h_{\mu_j}(x_n)$ and $h_{\mu_1,\mu_2}(x_n)$ by

$$h_{\mu_j}(x_n) = \frac{1}{\mu_j \sinh \mu_j a} \cosh \mu_j x_n$$

and

$$h_{\mu_1,\mu_2}(x_n) = h_{\mu_1}(x_n) - h_{\mu_2}(x_n).$$

We will denote the Dirac measure with point mass at x_n by δ_{x_n}, namely,

$$\delta_{x_n} f = \int_0^a \delta(x_n - y_n) f(y_n)\, dy_n = f(x_n).$$

Theorem 3.8. *Let λ satisfy $\lambda \neq 0$, $\tilde{\nu}\lambda + \gamma^2 \neq 0$, $\nu_1\lambda + \gamma^2 \neq 0$. Assume that $\lambda \neq \lambda_{1,k}$ and $\lambda \neq \lambda_{\pm,k}$ for any $k = 0, 1, 2, \cdots$. Assume also that $\lambda \in \rho(-\widehat{L}'_\xi)$. Then the solution $(\lambda + \widehat{L}_{\xi'})^{-1} f$ of (2.1) is written as*

$$(\lambda + \widehat{L}_{\xi'})^{-1} f = \widehat{G}(\lambda, \xi') f + \widehat{K}(\lambda, \xi') f$$

with integral operators $\widehat{G}(\lambda, \xi')$ and $\widehat{K}(\lambda, \xi')$ defined by

$$(\widehat{G}(\lambda, \xi') f)(x_n) = \int_0^a \widehat{G}(\lambda, \xi', x_n, y_n) f(y_n)\, dy_n$$

and

$$(\widehat{K}(\lambda, \xi') f)(x_n) = \int_0^a \widehat{K}(\lambda, \xi', x_n, y_n) f(y_n)\, dy_n.$$

Here $\widehat{G}(\lambda,\xi',x_n,y_n)$ *is an* $(n+1)\times(n+1)$ *matrix of the form*

$$\begin{aligned}
&\widehat{G}(\lambda,\xi',x_n,y_n)\\
&= \tfrac{\nu_1}{d(\lambda)}\delta(x_n-y_n)Q_0\\
&\quad +\tfrac{\gamma}{d(\lambda)}\begin{pmatrix} \frac{\gamma\lambda}{d(\lambda)}g^N_{\mu_2}(x_n,y_n) & -i^T\xi' g^N_{\mu_2}(x_n,y_n) & -\partial_{x_n}g^D_{\mu_2}(x_n,y_n)\\ -i\xi' g^N_{\mu_2}(x_n,y_n) & 0 & 0\\ -\partial_{x_n}g^N_{\mu_2}(x_n,y_n) & 0 & 0 \end{pmatrix}\\
&\quad +\begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{1}{\nu}g^N_{\mu_1}(x_n,y_n)I_{n-1} & 0\\ 0 & 0 & \frac{1}{\nu}g^D_{\mu_1}(x_n,y_n) \end{pmatrix}\\
&\quad +\begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{\xi'^T\xi'}{\lambda}g^N_{\mu_1,\mu_2}(x_n,y_n) & -\frac{i\xi'}{\lambda}\partial_{x_n}g^D_{\mu_1,\mu_2}(x_n,y_n)\\ 0 & -\frac{i\xi'}{\lambda}\partial_{x_n}g^N_{\mu_1,\mu_2}(x_n,y_n) & -\frac{1}{\lambda}\partial^2_{x_n}g^D_{\mu_1,\mu_2}(x_n,y_n) \end{pmatrix},
\end{aligned}$$

where $d(\lambda)=\nu_1\lambda+\gamma^2$ *and* $\mu_j=\mu_j(\lambda,\xi')$, $j=1,2$; *and*

$$\widehat{K}(\lambda,\xi',x_n,y_n)=\widehat{H}(\lambda,\xi',x_n,y_n)+\check{H}(\lambda,\xi',a-x_n,a-y_n),$$

where

$$\check{H}(\lambda,\xi',a-x_n,a-y_n)=\widehat{H}(\lambda,\xi',a-x_n,a-y_n)\mathrm{diag}\,(I_n,-1)$$

and

$$
\begin{aligned}
&H(\lambda, \xi', x_n, y_n) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \frac{i\xi'}{\nu} h_{\mu_1}(x_n)\beta_0(y_n) & \frac{1}{\nu} h_{\mu_1}(x_n) B'(y_n) & \frac{1}{\nu} h_{\mu_1}(x_n) \boldsymbol{b}_n(y_n) \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{\gamma|\xi'|^2}{d(\lambda)} h_{\mu_2}(x_n)\beta_0(y_n) & \frac{i\gamma^T\xi'}{d(\lambda)} h_{\mu_2}(x_n)\beta_1(y_n) & -\frac{i\gamma^T\xi'}{d(\lambda)} h_{\mu_2}(x_n)\boldsymbol{b}_n(y_n) \\ \frac{i|\xi'|^2\xi'}{\lambda} h_{\mu_1,\mu_2}(x_n)\beta_0(y_n) & 0 & 0 \\ -\frac{|\xi'|^2}{\lambda}\partial_{x_n} h_{\mu_1,\mu_2}(x_n)\beta_0(y_n) & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\xi'^T\xi'}{\lambda} h_{\mu_1,\mu_2}(x_n)\beta_1(y_n) & \frac{|\xi'|^2}{\lambda} h_{\mu_1,\mu_2}(x_n)\boldsymbol{b}_n(y_n) \\ 0 & -\frac{i'^T\xi'}{\lambda}\partial_{x_n} h_{\mu_1,\mu_2}(x_n)\beta_1(y_n) & \frac{i^T\xi'}{\lambda}\partial_{x_n} h_{\mu_1,\mu_2}(x_n)\boldsymbol{b}_n(y_n) \end{pmatrix}
\end{aligned}
$$

with $d(\lambda) = \nu_1\lambda + \gamma^2$, $\mu_j = \mu_j(\lambda, \xi')$, $j = 1, 2$, $\beta_j(y_n) = \beta_j(\lambda, \xi', y_n)$, $j = 0, 1$, $B'(y_n) = B'(\lambda, \xi', y_n)$ *and* $\boldsymbol{b}_n(y_n) = \boldsymbol{b}_n(\lambda, \xi', y_n)$.

Proof. By assumption we see that the Fourier series of $(\lambda + \widehat{L}_{\xi'})^{-1} f$ takes the form as in Proposition 3.6 with Y_k given in Proposition 3.7.

The Fourier sine and cosine series of $w^M(x_n) = \int_0^a g_\mu^M(x_n, y_n) v(y_n)\, dy_n$ with $M = D, N$ are given by

$$
w^D(x_n) = \sum_{k=1}^{\infty} \frac{1}{\mu^2 + a_k^2} v_{s,k} \sin a_k x_n
$$

and

$$
w^N(x_n) = \frac{1}{2} v_{c,0} + \sum_{k=1}^{\infty} \frac{1}{\mu^2 + a_k^2} v_{c,k} \cos a_k x_n,
$$

respectively. Here $v_{s,k}$ and $v_{c,k}$ are the Fourier sine and cosine coefficients of v respectively. Since

$$
\frac{1}{\lambda - \lambda_{1,k}} = \frac{1}{\nu} \frac{1}{\mu_1^2 + a_k^2},
$$

$$
\frac{1}{(\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k})} = \frac{1}{\nu_1\lambda + \gamma^2} \frac{1}{\mu_2^2 + a_k^2}
$$

and

$$
\frac{1}{\lambda - \lambda_{1,k}} \frac{1}{|\xi^{(k)}|^2} - \frac{\lambda}{(\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k})} \frac{1}{|\xi^{(k)}|^2} = -\frac{1}{\lambda}\left(\frac{1}{\mu_1^2 + a_k^2} - \frac{1}{\mu_2^2 + a_k^2}\right),
$$

we see that the Fourier coefficients of $\widehat{G}(\lambda,\xi')f$ are equal to $\widehat{L}_k(\lambda,\xi')^{-1}f_k$.

Also, since the Fourier cosine coefficients of $e^{\mu x_n}$ are given by

$$\frac{2}{a}\frac{\mu\{(-1)^k e^{\mu a}-1\}}{\mu^2+a_k^2}, \quad k=0,1,2,\cdots,$$

we see from Propositions 3.6 and 3.7 that the Fourier coefficients of $\widehat{K}(\lambda,\xi')f$ are equal to $\widehat{L}_k(\lambda,\xi')^{-1}Y_k$. This completes the proof.

4. Preliminary estimates

In this section we prepare some estimates for the analysis of the integral kernel given in Theorem 3.8.

We first give an estimate of the resolvent set by the energy method.

Proposition 4.1. (i) *There exists a positive constant c_1 such that the set*

$$\Sigma_0 \equiv \{\lambda;\, \mathrm{Re}\,\lambda + c_1\,|\mathrm{Im}\,\lambda|^2 > 0\}$$

is in the resolvent set $\rho(-\widehat{L}_{\xi'})$ for all ξ'.

(ii) *There exists a positive number η_1 such that the set*

$$\{\lambda;\, \mathrm{Re}\,\lambda \geq -\eta_1\} \cap \{\lambda;\, \mathrm{Re}\,\lambda \leq 0,\ \mathrm{Im}\,\lambda = 0\}^c$$

is in the resolvent set $\rho(-\widehat{L}_{\xi'})$ for all ξ'. Here and in what follows, for a set E, the symbol E^c denotes the complementary set of E.

Proof. Let us consider problem (2.1). We note that $\widehat{A}_{\xi'}$ is self adjoint in $L^2(0,a)$ and $\widehat{B}_{\xi'}$ is skew-symmetric and the following relations hold:

$$\begin{aligned}(\widehat{A}_{\xi'}u,u) &= \nu|\xi'|^2|m|_2^2+\nu|\partial_{x_n}m|_2^2+\tilde{\nu}|i\xi'\cdot m'+\partial_{x_n}m^n|_2^2,\\ (\widehat{B}_{\xi'}u,u) &= 2i\gamma\mathrm{Im}\,(i\xi'\cdot m'+\partial_{x_n}m^n,\phi).\end{aligned} \tag{4.1}$$

Let $u=\begin{pmatrix}\phi\\ m\end{pmatrix}$ be a solution of (2.1). Taking the L^2-inner product of (2.1) with u we see from (4.1) that

$$\begin{aligned}&\lambda|u|_2^2+\nu|\xi'|^2|m|_2^2+\nu|\partial_{x_n}m|_2^2+\tilde{\nu}|i\xi'\cdot m'+\partial_{x_n}m^n|_2^2\\ &\quad+2i\gamma\mathrm{Im}\,(i\xi'\cdot m'+\partial_{x_n}m^n,\phi)=(f,u).\end{aligned} \tag{4.2}$$

The real part of (4.2) gives

$$\begin{aligned}&\mathrm{Re}\,\lambda|u|_2^2+\nu|\xi'|^2|m|_2^2+\nu|\partial_{x_n}m|_2^2+\tilde{\nu}|i\xi'\cdot m'+\partial_{x_n}m^n|_2^2\\ &\quad=\mathrm{Re}\,(f,u)\leq\varepsilon|u|_2^2+\tfrac{1}{\varepsilon}|f|_2^2\end{aligned} \tag{4.3}$$

for any $\varepsilon > 0$. On the other hand, the imaginary part of (4.2) gives

$$\mathrm{Im}\,\lambda |u|_2^2 = \mathrm{Im}\,(f, u) - 2\gamma \mathrm{Im}\,(i\xi' \cdot m' + \partial_{x_n} m^n, \phi),$$

from which we obtain

$$|\mathrm{Im}\,\lambda|^2\, |u|_2^2 \le 2\left\{|f|_2^2 + \gamma^2 |i\xi' \cdot m' + \partial_{x_n} m^n|_2^2\right\}. \tag{4.4}$$

It then follows from (4.3) and (4.4) that

$$\left(\mathrm{Re}\,\lambda + c_1\, |\mathrm{Im}\,\lambda|^2 - \varepsilon\right) |u|_2^2 + \frac{\nu}{2}\left(|\xi'|^2 |m|_2^2 + |\partial_{x_n} m|_2^2\right) \le C_\varepsilon |f|_2^2 \tag{4.5}$$

for any $\varepsilon > 0$ with some constant $c_1 = c_1(\nu, \gamma) > 0$.

We next estimate $|\partial_{x_n}\phi|_2$. Differentiating the first row of (2.1) with respect to x_n we have

$$\lambda \partial_{x_n}\phi + \gamma \partial_{x_n}^2 m^n = \partial_{x_n} f^0 - i\gamma\xi' \cdot \partial_{x_n} m' \tag{4.6}$$

We also see from the third row of (2.1) that

$$-\nu_1 \partial_{x_n}^2 m^n + \gamma \partial_{x_n}\phi = g, \tag{4.7}$$

where $g = f^n - \{\lambda m^n + \nu|\xi'|^2 m^n - i\tilde{\nu}\xi' \cdot \partial_{x_n} m'\}$. By adding (4.7) $\times\ \frac{\gamma}{\nu_1}$ to (4.6) we obtain

$$\left(\lambda + \frac{\gamma^2}{\nu_1}\right)\partial_{x_n}\phi = \partial_{x_n} f^0 - i\gamma\xi' \cdot \partial_{x_n} m' + \frac{\gamma}{\nu_1} g. \tag{4.8}$$

This implies that if $\lambda \ne -\frac{\gamma^2}{\nu_1}$, then

$$|\partial_{x_n}\phi|_2 \le \frac{C}{\left|\lambda + \frac{\gamma^2}{\nu_1}\right|}\left\{|f|_{H^1\times L^2} + |\lambda||m|_2 + |\xi'|^2|m|_2 + |\xi'||\partial_{x_n} m|_2\right\}. \tag{4.9}$$

We thus deduce from (4.5) and (4.9) that

$$|u|_{H^1\times L^2} \le C|f|_{H^1\times L^2} \tag{4.10}$$

for some $C > 0$, provided that $\lambda \in \Sigma_0$. This, together with (2.1), yields

$$|m|_{H^2} \le \widetilde{C}|f|_{H^1\times L^2} \tag{4.11}$$

with some constant $\widetilde{C} = \widetilde{C}(\eta_0, \theta_0, \xi') > 0$. Since $\lambda \ne 0$ and $\nu_1\lambda + \gamma^2 \ne 0$ when $\lambda \in \Sigma_0$, it follows from (4.10) and (4.11) that $\mathrm{Ker}\,\mathcal{L} = \{0\}$ for $\lambda \in \Sigma_0$. Lemma 3.1 then implies that $\Sigma_0 \subset \rho(-\widehat{L}_{\xi'})$. This completes the proof of (i).

We next prove (ii). In view of Lemma 3.1 it suffices to prove that (3.2) is uniquely solvable for any $F \in L^2(0,a)$ if $\lambda \in \{\lambda;\, \mathrm{Re}\,\lambda \geq -\eta_1\} \cap \{\lambda;\, \mathrm{Re}\,\lambda \leq 0, \mathrm{Im}\,\lambda = 0\}^c$ with some $\eta_1 > 0$.

We take the L^2-inner product of (3.2) with m. Then, integrating by parts, we have

$$(4.12)\quad (\lambda^2+\nu\lambda|\xi'|^2)|m|_2^2+\nu\lambda|\partial_{x_n}m|_2^2+(\tilde{\nu}\lambda+\gamma^2)|i\xi'\cdot m'+\partial_{x_n}m^n|_2^2 = (F,m).$$

The imaginary part of (4.12) gives

$$\mathrm{Im}\,\lambda\Big\{(2\mathrm{Re}\,\lambda+\nu|\xi'|^2)|m|_2^2+\nu|\partial_{x_n}m|_2^2+\tilde{\nu}|i\xi'\cdot m'+\partial_{x_n}m^n|_2^2\Big\} = \mathrm{Im}\,(F,m).$$

It follows that if $\mathrm{Im}\,\lambda \neq 0$, then

$$(4.13)\quad \Big(2\mathrm{Re}\,\lambda+\nu|\xi'|^2\Big)|m|_2^2+\nu|\partial_{x_n}m|_2^2+\tilde{\nu}|i\xi'\cdot m'+\partial_{x_n}m^n|_2^2 = \frac{\mathrm{Im}\,(F,m)}{\mathrm{Im}\,\lambda}.$$

By the Poincaré inequality, there exists a positive constant η_1 such that the left-hand side of (4.13) is bounded from below by

$$\Big(2\mathrm{Re}\,\lambda+\nu|\xi'|^2+4\eta_1\Big)|m|_2^2+\frac{\nu}{2}|\partial_{x_n}m|_2^2,$$

while the right-hand side of (4.13) is bounded from above by

$$\eta_1|m|_2^2+\frac{C}{|\mathrm{Im}\,\lambda|^2}|F|_2^2.$$

We thus conclude that

$$\Big(2\mathrm{Re}\,\lambda+\nu|\xi'|^2+3\eta_1\Big)|m|_2^2+\frac{\nu}{2}|\partial_{x_n}m|_2^2 \leq \frac{C}{|\mathrm{Im}\,\lambda|^2}|F|_2^2,$$

in particular, if $\lambda \in \mathrm{Re}\,\lambda \geq -\eta_1-\frac{\nu|\xi'|^2}{2}$, $\mathrm{Im}\,\lambda \neq 0$ and $F=0$, then $m=0$. This, together with (i), implies that $\{\lambda;\, \mathrm{Re}\,\lambda \geq -\eta_1-\frac{\nu|\xi'|^2}{2}\} \cap \{\lambda;\, \mathrm{Re}\,\lambda \leq 0, \mathrm{Im}\,\lambda = 0\}^c \subset \rho(-\widehat{L}_{\xi'})$. This completes the proof.

We next investigate $D(\lambda,\xi')$. We first estimate $|D(\lambda,\xi')|$ from below for small λ.

Proposition 4.2. (i) *Let r_1 and R_1 be any positive numbers with $r_1 < R_1$. Then there exists a positive number $\Lambda_1 = \Lambda_1(r_1,R_1)$ such that the inequality*

$$|D(\lambda,\xi')| \geq \frac{a^4}{24\nu^2}r_1^2|\lambda|^2$$

holds for $|\lambda| \le \Lambda_1$ *and* $r_1 \le |\xi'| \le R_1$.

(ii) *There are positive numbers* Λ_2 *and* $R_2 = R_2(\Lambda_2)$ *such that the inequality*

$$|D(\lambda,\xi')| \ge \frac{1}{16\nu^2}\left(\frac{\sinh|\xi'|a}{|\xi'|}\right)^2 |\lambda|^2$$

holds for $|\lambda| \le \Lambda_2$ *and* $|\xi'| \ge R_2$.

Proof. Assume that $|\nu_1\lambda+\gamma^2| \ge \frac{\gamma^2}{2\nu_1} \equiv c_2$. Let Λ and r be positive numbers satisfying

$$\frac{\Lambda}{\nu r^2} < 1 \quad \text{and} \quad \frac{\Lambda^2}{c_2 r^2} < 1. \tag{4.14}$$

Then for $|\lambda| \le \Lambda$ and $|\xi'| \ge r$, we have

$$\left|\frac{\lambda}{\nu|\xi'|^2}\right| \le \frac{\Lambda}{\nu r^2} < 1 \quad \text{and} \quad \left|\frac{\lambda^2}{\nu_1\lambda+\gamma^2}\frac{1}{|\xi'|^2}\right| \le \frac{\Lambda^2}{c_2 r^2} < 1,$$

and hence,

$$\mu_1 = |\xi'| + \frac{\lambda}{2\nu}\frac{1}{|\xi'|} - \frac{\lambda^2}{8\nu^2}\frac{1}{|\xi'|^3} + O\left(|\lambda|^3|\xi'|^{-5}\right)$$

and

$$\mu_2 = |\xi'| + \mu_2^{(1)}\frac{1}{|\xi'|} + \mu_2^{(2)}\frac{1}{|\xi'|^3} + O\left(|\lambda|^6|\xi'|^{-5}\right).$$

Here

$$\mu_2^{(1)} = \frac{\lambda^2}{2(\nu_1\lambda+\gamma^2)} \quad \text{and} \quad \mu_2^{(2)} = -\frac{\lambda^4}{8(\nu_1\lambda+\gamma^2)^2}.$$

It then follows that

$$\begin{aligned}\sinh\mu_j a \;&=\; \sinh|\xi'|a + \frac{e_j^{(1)}}{|\xi'|}\cosh|\xi'|a + \frac{e_j^{(2)}}{|\xi'|^2}\sinh|\xi'|a + \frac{e_j^{(3)}}{|\xi'|^3}\cosh|\xi'|a \\ &\quad + O\left(|\lambda|^3|\xi'|^{-4}e^{|\xi'|a}\right)\end{aligned}$$

and

$$\begin{aligned}\cosh\mu_j a \;&=\; \cosh|\xi'|a + \frac{e_j^{(1)}}{|\xi'|}\sinh|\xi'|a + \frac{e_j^{(2)}}{|\xi'|^2}\cosh|\xi'|a + \frac{e_j^{(3)}}{|\xi'|^3}\sinh|\xi'|a \\ &\quad + O\left(|\lambda|^3|\xi'|^{-4}e^{|\xi'|a}\right),\end{aligned}$$

where $j = 1, 2$,

$$e_1^{(1)} = \tfrac{\lambda a}{2\nu}, \quad e_2^{(1)} = \tfrac{\lambda^2 a}{2(\nu_1\lambda+\gamma^2)}, \quad e_1^{(2)} = \tfrac{\lambda^2 a^2}{8\nu^2}, \quad e_2^{(2)} = \tfrac{\lambda^4 a^2}{8(\nu_1\lambda+\gamma^2)^2}$$

and

$$e_1^{(3)} = O\left(|\lambda|^2\right), \quad e_2^{(3)} = O\left(|\lambda|^4\right).$$

We thus obtain

$$\text{(4.15)} \quad \frac{1}{\lambda^2} D(\lambda, \xi') = -\frac{a^2}{4\nu^2} + d_0(\lambda) + \left(\frac{1}{4\nu^2} + d_1(\lambda)\right)\left(\frac{\sinh |\xi'|a}{|\xi'|}\right)^2 + d_2(\lambda, \xi'),$$

where $d_0(\lambda)$, $d_1(\lambda)$ and $d_2(\lambda, \xi')$ are some functions satisfying

$$\text{(4.16)} \qquad |d_0(\lambda)| \le C|\lambda|, \quad |d_1(\lambda)| \le C|\lambda|, \quad |d_2(\lambda, \xi')| \le C\frac{|\lambda|}{|\xi'|^3} e^{2|\xi'|a}$$

uniformly in λ and ξ' with $|\lambda| \le \Lambda$ and $|\xi'| \ge r$ for some $C = C(r_1) > 0$.

Let $r_1 > 0$ and $R_1 > 0$ be fixed. Since

$$\left(\frac{\sinh |\xi'|a}{|\xi'|}\right)^2 - a^2 = \left(\frac{\sinh |\xi'|a}{|\xi'|} + a\right)\left(\frac{\sinh |\xi'|a}{|\xi'|} - a\right) \ge \frac{a^4}{3}|\xi'|^2$$

for $|\xi'| > 0$, we see from (4.15) and (4.16) that if $r_1 \le |\xi'| \le R_1$, then

$$\left|\frac{1}{\lambda^2} D(\lambda, \xi')\right| \ge \frac{a^4}{12\nu^2}|\xi'|^2 - C|\lambda|\left\{1 + \frac{e^{2R_1 a}}{|\xi'|^3}\right\},$$

provided that $|\lambda| \le \Lambda_1$ for some Λ_1 satisfying (4.14) with $\Lambda = \Lambda_1$ and $r = r_1$. Here $C = C(r_1) > 0$. Therefore, there exists a constant $\Lambda_1 = \Lambda(r_1, R_1) > 0$ such that

$$|D(\lambda, \xi')| \ge \frac{a^4}{24\nu^2} r_1^2 |\lambda|^2$$

for λ and ξ' with $|\lambda| \le \Lambda_1$ and $r_1 \le |\xi'| \le R_1$. This proves (i).

We next prove (ii). From (4.16) we see that there exists a positive number Λ_2 such that

$$\left|\frac{1}{4\nu^2} + d_1(\lambda)\right| \ge \frac{1}{8\nu^2}$$

for $|\lambda| \le \Lambda_2$. Furthermore, there exists a positive number $R_2 = R_2(\Lambda_2)$ such that (4.14) is satisfied with $\Lambda = \Lambda_2$ and $r = R_2$, and the inequalities

$$\left|-\frac{a^2}{4\nu^2} + d_0(\lambda)\right| \le \frac{1}{32\nu^2}\left(\frac{\sinh |\xi'|a}{|\xi'|}\right)^2, \quad |d_2(\lambda, \xi')| \le \frac{1}{32\nu^2}\left(\frac{\sinh |\xi'|a}{|\xi'|}\right)^2$$

hold for all $|\lambda| \le \Lambda_2$ and $|\xi'| \ge R_2$. It then follows from (4.15) that

$$\left|\frac{1}{\lambda^2} D(\lambda, \xi')\right| \ge \frac{1}{16\nu^2}\left(\frac{\sinh |\xi'|a}{|\xi'|}\right)^2$$

for all $|\lambda| \le \Lambda_2$ and $|\xi'| \ge R_2$. This completes the proof.

By Propositions 4.1 and 4.2 we have the following consequence on $\rho(-\widehat{L}_{\xi'})$.

Lemma 4.3. (i) *There exists a number $\theta_1 \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(\eta, \theta_1) \subset \rho(-\widehat{L}_{\xi'})$ for any $\eta > 0$ and $\xi' \in \mathbf{R}^{n-1}$. Furthermore, the integral representation of $(\lambda + \widehat{L}_{\xi'})^{-1}$ in Theorem 3.8 holds for $\lambda \in \Sigma(\eta, \theta_1)$ for any $\eta > 0$ and $\xi' \in \mathbf{R}^{n-1}$.*

(ii) *For any $r > 0$ there exist positive numbers η_2 and θ_2 with $\theta_2 \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\eta_2, \theta_2) \subset \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \ge r$. Furthermore, the integral representation of $(\lambda + \widehat{L}_{\xi'})^{-1}$ in Theorem 3.8 holds for $\lambda \in \Sigma(-\eta_2, \theta_2)$ and $|\xi'| \ge r$.*

Proof. The first assertion of (i) is an easy consequence of Proposition 4.1. By Remarks 3.2 and 3.5, changing θ_1 suitably if necessary, we see that $\lambda_{1,k}, \lambda_{\pm,k} \notin \Sigma(\eta, \theta_1)$ for any $k = 0, 1, 2, \cdots$. Therefore, the integral representation in Theorem 3.8 holds for $\lambda \in \Sigma(\eta, \theta_1)$.

Similarly, we see from Lemma 3.4 and Propositions 4.1 and 4.2 that for any $r > 0$ there exist positive numbers η_2 and θ_2 with $\theta_2 \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\eta_2, \theta_2) - \{0\} \subset \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \ge r$. Furthermore, by Remarks 3.2 and 3.5, changing η_2 and θ_2 suitably if necessary, we deduce that $\lambda_{1,k}, \lambda_{\pm,k} \notin \Sigma(-\eta_2, \theta_2)$ for any $k = 0, 1, 2, \cdots$, and the integral representation in Theorem 3.8 holds for $\lambda \in \Sigma(-\eta_2, \theta_2) - \{0\}$ and $|\xi'| \ge r$.

Let us prove $0 \in \rho(-\widehat{L}_{\xi'})$. Assume that ξ' satisfies $|\xi'| \ge r$. We note that $(\lambda + \widehat{L}_{\xi'})^{-1}$ is analytic in $\Sigma(-\eta_2, \theta_2) - \{0\}$. Furthermore, $\widehat{G}(\lambda, \xi')$ and $\widehat{K}(\lambda, \xi')$ are also analytic in $\Sigma(-\eta_2, \theta_2) - \{0\}$. Therefore, it suffices to prove that $\widehat{G}(\lambda, \xi')$ and $\widehat{K}(\lambda, \xi')$ are bounded in $\Sigma(-\eta_2, \theta_2) - \{0\}$.

Since

$$\mu_1 - \mu_2 = \frac{\lambda}{\mu_1 + \mu_2} \left\{ \frac{1}{\nu} + \frac{\lambda}{\nu_1 \lambda + \gamma^2} \right\}, \tag{4.17}$$

we see that $\widetilde{Q}\widehat{G}(\lambda, \xi')\widetilde{Q}$ is bounded in $\Sigma(-\eta_2, \theta_2) - \{0\}$. It is easy to see that the other components of $\widehat{G}(\lambda, \xi')$ are bounded in $\Sigma(-\eta_2, \theta_2) - \{0\}$, and hence, $\widehat{G}(\lambda, \xi')$ is bounded in $\Sigma(-\eta_2, \theta_2) - \{0\}$.

As for $\widehat{K}(\lambda, \xi')$, we easily see that $b_j(\lambda, \xi', y_n) = O(\lambda)$ as $\lambda \to 0$ for $j = 1, 2, 3$. This, together with Proposition 4.2, implies that $\beta_j(\lambda, \xi', y_n) = O(1)$ $(j = 0, 1)$, $B'(\lambda, \xi', y_n) = O(1)$ and $\boldsymbol{b}_n(\lambda, \xi', y_n) = O(1)$ as $\lambda \to 0$. Furthermore, using (4.17), we also see that $h_{\mu_1,\mu_2}(x_n) = O(\lambda)$ as $\lambda \to 0$. It then follows that $\widehat{K}(\lambda, \xi')$ is bounded in $\Sigma(-\eta_2, \theta_2) - \{0\}$ and assertion (ii) is proved. This completes the proof.

We next derive estimates for $|D(\lambda,\xi')|$ from below when $|\lambda|+|\xi'|^2$ is large. In the following we specify branches of $\mu_j(\lambda,\xi')$, $j=1,2$, as a function of λ. As for $\mu_1(\lambda,\xi')$ we take the principal branch of the square root of $\frac{\lambda+\nu|\xi'|^2}{\nu}$, i.e., for $\lambda\notin(-\infty,-\nu|\xi'|^2]$ we denote by $\mu_1(\lambda,\xi')$ the square root of $\frac{\lambda+\nu|\xi'|^2}{\nu}$ with $\operatorname{Re}\mu_1(\lambda,\xi')>0$. As for $\mu_2(\lambda,\xi')$ we take the branch in the following way. When $|\xi'|<\frac{2\gamma}{\nu_1}$, we use the branch specified by the requirement

$$\arg(\lambda-\lambda_{\pm,0})=\mp\tfrac{\pi}{2} \text{ at } \lambda=\operatorname{Re}\lambda_{+,0} \text{ and } \arg(\lambda+\tfrac{\gamma^2}{\nu_1})=0 \text{ at } \lambda=0$$

and take the branch cut

$$\{\lambda;\ \operatorname{Re}\lambda\le-\tfrac{\gamma^2}{\nu_1},\ \operatorname{Im}\lambda=0\}\cup\{\lambda\in\Gamma_{\gamma,\nu_1};\ \operatorname{Re}\lambda\le\operatorname{Re}\lambda_{\pm,0}\}.$$

Here Γ_{γ,ν_1} is the circle defined by $\Gamma_{\gamma,\nu_1}=\{\lambda;\ \left|\lambda+\frac{\gamma^2}{\nu_1}\right|=\frac{\gamma^2}{\nu_1}\}$. When $|\xi'|\ge\frac{2\gamma}{\nu_1}$, we use the branch specified by $\arg(\lambda-\lambda_{\pm,0})=\arg(\lambda+\frac{\gamma^2}{\nu_1})=0$ at $\lambda=0$ and take the branch cut

$$\{\lambda;\ \operatorname{Re}\lambda\le-\tfrac{\gamma^2}{\nu_1},\ \operatorname{Im}\lambda=0\}.$$

As shown in [6], it holds $\operatorname{Re}\mu_2(\lambda,\xi')>0$ for λ outside the branch cut.

We introduce a function $D_1(\lambda,\xi')$ defined by

$$D_1(\lambda,\xi')=e^{-(\mu_1+\mu_2)a}D(\lambda,\xi').$$

To investigate the resolvent $(\lambda+L)^{-1}=\mathscr{F}^{-1}\left[(\lambda+L'_\xi)^{-1}\right]$ for large $\lambda+|\xi'|^2$, it is convenient to consider $D_1(\lambda,\xi')$ rather than $D(\lambda,\xi')$.

Lemma 4.4. (i) *There are positive numbers η_3 and θ_3 with $\theta_3\in(\frac{\pi}{2},\pi)$ such that the inequality*

$$|D_1(\lambda,\xi')|\ge C\frac{|\lambda|^2}{|\lambda|+1+|\xi'|^2}$$

holds uniformly in $\lambda\in\Sigma(-\eta_3,\theta_3)\cap\{\lambda;\ |\lambda|\ge\delta\}$ and $|\xi'|\ge R_3$ for any $\delta>0$ with some constant $R_3=R_3(\delta)>0$.

(ii) *There are positive numbers η_4 and θ_4 with $\theta_4\in(\frac{\pi}{2},\pi)$ such that the inequality*

$$|D_1(\lambda,\xi')|\ge C\frac{|\lambda|^2}{|\lambda|+1+|\xi'|^2}$$

holds uniformly in $\lambda\in\Sigma(\eta_4,\theta_4)$ and $\xi'\in\mathbf{R}^{n-1}$.

Proof. We first observe that there are positive numbers η_3, $\theta_3\in(\frac{\pi}{2},\pi)$ and $\widetilde{R}_3$ such that

$$\operatorname{Re}\mu_j\ge C(|\lambda|+1+|\xi'|^2)^{\frac{1}{2}}\qquad(j=1,2) \tag{4.18}$$

for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \geq \widetilde{R}_3$. Furthermore, there exist positive numbers $\widetilde{\eta}_4$ and $\theta_4 \in (\frac{\pi}{2}, \pi)$ such that (4.18) also holds for $\lambda \in \Sigma(\widetilde{\eta}_4, \theta_4)$ and $\xi' \in \mathbf{R}^{n-1}$. In fact, it is easy to show the inequalities (4.18) for $j = 1$ with appropriate η_3, $\widetilde{\eta}_4$, θ_3, θ_4 and $\widetilde{R}_3$. As for $j = 2$ one can find (4.18) by using the observation in Remark 3.2. It is also possible to see that

$$|\mu_j| \leq C(|\lambda| + 1 + |\xi'|^2)^{\frac{1}{2}} \tag{4.19}$$

for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and ξ' with $|\xi'| \geq \widetilde{R}_3$, and for $\lambda \in \Sigma(\widetilde{\eta}_4, \theta_4)$ and $\xi' \in \mathbf{R}^{n-1}$.

Consider next the quadratic equation $\omega^2 + (2\nu + \widetilde{\nu})|\xi'|^2\omega + \gamma^2|\xi'|^2 = 0$ for ω. This equation has two roots

$$\omega_\pm = -\frac{1}{2}(2\nu + \widetilde{\nu})|\xi'|^2 \pm \frac{1}{2}\sqrt{(2\nu + \widetilde{\nu})^2|\xi'|^4 - 4\gamma^2|\xi'|^2}.$$

Therefore, as in Remark 3.2, we see that

$$\omega_\pm = -\frac{1}{2}(2\nu + \widetilde{\nu})|\xi'|^2 \pm i\gamma|\xi'| + O(|\xi'|^3) \quad \text{as} \quad |\xi'| \to 0,$$

and $\omega_\pm \in \mathbf{R}$ for $|\xi'| \geq \frac{2\gamma}{2\nu + \widetilde{\nu}}$, and

$$\omega_+ = -\frac{\gamma^2}{2\nu + \widetilde{\nu}} + O(|\xi'|^{-2}), \quad \omega_- = -(2\nu + \widetilde{\nu})|\xi'|^2 + O(1) \quad \text{as} \quad |\xi'| \to \infty.$$

We thus deduce, by suitably changing $\theta_3, \theta_4, \widetilde{R}_3, \eta_3$ and $\widetilde{\eta}_4$ if necessary, that

$$\begin{aligned} |\lambda^2 + (2\nu + \widetilde{\nu})|\xi'|^2\lambda + \gamma^2|\xi'|^2| &= |(\lambda - \omega_+)(\lambda - \omega_-)| \\ &\geq C(|\lambda| + 1 + |\xi'|^2)(|\lambda| + 1) \end{aligned} \tag{4.20}$$

for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and ξ' with $|\xi'| \geq \widetilde{R}_3$, and for $\lambda \in \Sigma(\widetilde{\eta}_4, \theta_4)$ and $\xi' \in \mathbf{R}^{n-1}$.

We now estimate $D_1(\lambda, \xi')$. We write $D_1(\lambda, \xi')$ as

$$D_1(\lambda, \xi') = \frac{1}{4}\left(\frac{|\xi'|^4}{\mu_1\mu_2} - 2|\xi'|^2 + \mu_1\mu_2\right) + \widetilde{D}_1(\lambda, \xi'), \tag{4.21}$$

where

(4.22)

$$\begin{aligned} \widetilde{D}_1(\lambda, \xi') \;=\; \frac{1}{4}\Bigg\{ & 8|\xi'|^2 e^{-(\mu_1+\mu_2)a} + \left(\frac{|\xi'|^4}{\mu_1\mu_2} - 2|\xi'|^2 + \mu_1\mu_2\right) e^{-2(\mu_1+\mu_2)a} \\ & - \left(\frac{|\xi'|^4}{\mu_1\mu_2} + 2|\xi'|^2 + \mu_1\mu_2\right)\left(e^{-2\mu_1 a} + e^{-2\mu_2 a}\right)\Bigg\}. \end{aligned}$$

Since

$$\left|\frac{|\xi'|^4}{\mu_1\mu_2} - 2|\xi'|^2 + \mu_1\mu_2\right| = \left|\frac{1}{\mu_1\mu_2}\right|\left|\frac{(\mu_1\mu_2)^2 - |\xi'|^4}{\mu_1\mu_2 + |\xi'|^2}\right|^2$$

and

$$(\mu_1\mu_2)^2 - |\xi'|^4 = \frac{\lambda\{\lambda^2 + (2\nu + \widetilde{\nu})|\xi'|^2\lambda + \gamma^2|\xi'|^2\}}{\nu_1\lambda + \gamma^2},$$

we see from (4.19) and (4.20) that

$$\frac{1}{4}\left|\frac{|\xi'|^4}{\mu_1\mu_2} - 2|\xi'|^2 + \mu_1\mu_2\right| \geq C_0\frac{|\lambda|^2}{|\lambda| + 1 + |\xi'|^2} \tag{4.23}$$

uniformly for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and ξ' with $|\xi'| \geq \widetilde{R}_3$, and for $\lambda \in \Sigma(\widetilde{\eta}_4, \theta_4)$ and $\xi' \in \mathbf{R}^{n-1}$.

We also see from (4.18) and (4.19) that

$$\left|e^{-\mu_j a}\right| \leq C_k\left(\mathrm{Re}\,\mu_j a\right)^{-k} \leq C_k(|\lambda| + 1 + |\xi'|^2)^{-\frac{k}{2}}$$

for any k. This, together with (4.22), implies that for any $\delta > 0$ there exists a positive number $R_3 = R_3(\delta)$ with $R_3 \geq \widetilde{R}_3$ such that if $\lambda \in \Sigma(-\eta_3, \theta_3) \cap \{\lambda;\, |\lambda| \geq \delta\}$ and $|\xi'| \geq R_3$, then

$$\left|\widetilde{D}_1(\lambda, \xi')\right| \leq \frac{C|\lambda|^2}{\delta^2(|\lambda| + 1 + |\xi'|^2)^2} \leq \frac{C_0}{2}\frac{|\lambda|^2}{|\lambda| + 1 + |\xi'|^2}.$$

Combining this with (4.23), we have

$$|D_1(\lambda, \xi')| \geq \frac{C_0}{2}\frac{|\lambda|^2}{|\lambda| + 1 + |\xi'|^2}$$

for $\lambda \in \Sigma(-\eta_3, \theta_3) \cap \{\lambda;\, |\lambda| \geq \delta\}$ and ξ' with $|\xi'| \geq R_3$. This proves (i).

We can also find a positive number η_4 with $\eta_4 \geq \widetilde{\eta}_4$ such that

$$\left|\widetilde{D}_1(\lambda, \xi')\right| \leq \frac{C|\lambda|^2}{\eta_4^2(|\lambda| + 1 + |\xi'|^2)^2} \leq \frac{C_0}{2}\frac{|\lambda|^2}{|\lambda| + 1 + |\xi'|^2}$$

for all $\lambda \in \Sigma(\eta_4, \theta_4)$ and $\xi' \in \mathbf{R}^{n-1}$, and hence,

$$|D_1(\lambda, \xi')| \geq \frac{C_0}{2}\frac{|\lambda|^2}{|\lambda| + 1 + |\xi'|^2}$$

for all such λ and ξ'. This completes the proof.

To estimate each component of the integral kernel in Theorem 3.8, we will use the following lemma.

Lemma 4.5. *Let $\alpha' \in \mathbf{Z}^{n-1}$ be any multi-index with $|\alpha'| \le n$ and let θ_4 be the number given in Lemma* 4.4*. Then there exists a positive number η_4 such that the following estimates hold uniformly in $\lambda \in \Sigma(\eta_4, \theta_4)$ and $\xi' \in \mathbf{R}^{n-1}$:*

$$\text{(i)} \qquad \left|\partial_{\xi'}^{\alpha'} \mu_j\right| \le C(|\lambda| + 1 + |\xi'|^2)^{\frac{1}{2} - \frac{|\alpha'|}{2}} \quad (j = 1, 2),$$

$$\text{(ii)} \qquad \left|\partial_{\xi'}^{\alpha'} (\mu_1 - \mu_2)\right| \le C|\lambda|(|\lambda| + 1 + |\xi'|^2)^{-\frac{1}{2} - \frac{|\alpha'|}{2}},$$

$$\text{(iii)} \qquad \left|\partial_{\xi'}^{\alpha'} \left(\mu_1\mu_2 - |\xi'|^2\right)\right| \le C|\lambda|(|\lambda| + 1 + |\xi'|^2)^{-\frac{|\alpha'|}{2}},$$

$$\text{(iv)} \qquad \left|\partial_{\xi'}^{\alpha'} e^{-\mu_j x_n}\right| \le C(|\lambda| + 1 + |\xi'|^2)^{-\frac{|\alpha'|}{2}} e^{-\frac{1}{2}\mathrm{Re}\,\mu_j x_n} \quad (j = 1, 2)$$

for all $0 \le x_n \le a$,

$$\text{(v)} \qquad \left|\partial_{\xi'}^{\alpha'} \left(\frac{1}{D_1(\lambda, \xi')}\right)\right| \le C|\lambda|^{-2}(|\lambda| + 1 + |\xi'|^2)^{1 - \frac{|\alpha'|}{2}},$$

$$\text{(vi)} \qquad \begin{aligned} &\left|\partial_{\xi'}^{\alpha'} \left(e^{-\mu_1 x_n} - e^{-\mu_2 x_n}\right)\right| \\ \le\ & C|\lambda|(|\lambda| + 1 + |\xi'|^2)^{-1 - \frac{|\alpha'|}{2}} \left\{e^{-\frac{1}{8}\mathrm{Re}\,\mu_1 x_n} + e^{-\frac{1}{8}\mathrm{Re}\,\mu_2 x_n}\right\} \end{aligned}$$

for all $0 \le x_n \le a$.

The inequalities (i)–(vi) *also hold uniformly in $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \ge R_3$ for some $R_3 > 0$, where η_3 and θ_3 are the numbers given in Lemma* 4.4*.*

Proof. Let η_4 and θ_4 are the numbers given in Lemma 4.4 (ii). Then a direct calculation gives the inequality in (i). The inequality in (ii) follows from (i), (4.17) and (4.18).

Let us prove (iii). Suppose first that $|\lambda| \le C_0|\xi'|^2$, where C_0 is a positive constant to be determined later. We write $\mu_1\mu_2 - |\xi'|^2$ as

$$\mu_1\mu_2 - |\xi'|^2 = \frac{(\mu_1\mu_2)^2 - |\xi'|^4}{\mu_1\mu_2 + |\xi'|^2}. \tag{4.24}$$

Since

$$(\mu_1\mu_2)^2 - |\xi'|^4 = \frac{\lambda^3}{\nu(\nu_1\lambda + \gamma^2)} + \lambda\left(\frac{1}{\nu} + \frac{\lambda}{\nu_1\lambda + \gamma^2}\right)|\xi'|^2,$$

we see that

$$\left|\partial_{\xi'}^{\alpha'}\left((\mu_1\mu_2)^2-|\xi'|^4\right)\right|\le C|\lambda|(|\lambda|+|\xi'|^2)^{1-\frac{|\alpha'|}{2}}. \tag{4.25}$$

On the other hand, since $\left|\frac{\lambda^2}{\nu_1\lambda+\gamma^2}\right|\le C|\lambda|$ for $\lambda\in\Sigma(\eta_4,\theta_4)$, taking C_0 sufficiently small, we find that

$$\left|\frac{\lambda}{\nu}\right|\le\frac{|\xi'|^2}{2},\quad \left|\frac{\lambda^2}{\nu_1\lambda+\gamma^2}\right|\le\frac{|\xi'|^2}{2}$$

for λ and ξ' with $|\lambda|\le C_0|\xi'|^2$. Applying the mean value theorem to $\mu_1=|\xi'|\sqrt{1+\frac{\lambda}{\nu|\xi'|^2}}$ and $\mu_2=|\xi'|\sqrt{1+\frac{\lambda^2}{(\nu_1\lambda+\gamma^2)|\xi'|^2}}$, we obtain

$$\mu_1\mu_2+|\xi'|^2=2|\xi'|^2+\tfrac{\lambda}{\nu}q_1(\lambda,\xi')+\tfrac{\lambda^2}{\nu_1\lambda+\gamma^2}q_2(\lambda,\xi')+\tfrac{\lambda^3}{\nu(\nu_1\lambda+\gamma^2)|\xi'|^2}q_3(\lambda,\xi'),$$

where $q_j(\lambda,\xi')$, $j=1,2,3$, are some functions satisfying

$$|q_j(\lambda,\xi')|\le C,\quad j=1,2,3,$$

for λ and ξ' with $|\lambda|\le C_0|\xi'|^2$. Therefore, taking C_0 smaller if necessary, we see that

$$\left|\mu_1\mu_2+|\xi'|^2\right|\ge 2|\xi'|^2-C|\lambda|\ge|\xi'|^2\ge C(|\lambda|+1+|\xi'|^2) \tag{4.26}$$

uniformly for $\lambda\in\Sigma(\eta_4,\theta_4)$ and ξ' with $|\lambda|\le C_0|\xi'|^2$. It then follows from (i) and (4.24)–(4.26) that

$$\left|\partial_{\xi'}^{\alpha'}\left(\mu_1\mu_2-|\xi'|^2\right)\right|\le C|\lambda|(|\lambda|+1+|\xi'|^2)^{-\frac{|\alpha'|}{2}}$$

uniformly for $\lambda\in\Sigma(\eta_4,\theta_4)$ and ξ' with $|\lambda|\le C_0|\xi'|^2$. As for the case $|\lambda|\ge C_0|\xi'|^2$, we easily see that

$$\left|\partial_{\xi'}^{\alpha'}\left(\mu_1\mu_2-|\xi'|^2\right)\right|\le C(|\lambda|+|\xi'|^2)^{1-\frac{|\alpha'|}{2}}\le C|\lambda|(|\lambda|+1+|\xi'|^2)^{-\frac{|\alpha'|}{2}}.$$

This proves (iii).

We next consider (iv). The case $\alpha'=0$ is trivial. Let $|\alpha'|\ge1$. Then, by (i) and (4.18), we have

$$\begin{aligned}\left|\partial_{\xi'}^{\alpha'}e^{-\mu_j x_n}\right| &\le C\sum_{\ell=1}^{|\alpha'|}e^{-\mathrm{Re}\,\mu_j x_n}\prod_{\alpha_1+\cdots+\alpha_\ell=\alpha'}\left|\partial_{\xi'}^{\alpha_1}\mu_j x_n\right|\cdots\left|\partial_{\xi'}^{\alpha_\ell}\mu_j x_n\right|\\ &\le C\sum_{\ell=1}^{|\alpha'|}x_n^\ell e^{-\mathrm{Re}\,\mu_j x_n}\left(|\lambda|+1+|\xi'|^2\right)^{\frac{\ell}{2}-\frac{|\alpha'|}{2}}\\ &\le C\sum_{\ell=1}^{|\alpha'|}(\mathrm{Re}\,\mu_j)^{-\ell}e^{-\frac12\mathrm{Re}\,\mu_j x_n}\left(|\lambda|+1+|\xi'|^2\right)^{\frac{\ell}{2}-\frac{|\alpha'|}{2}}\\ &\le C\left(|\lambda|+1+|\xi'|^2\right)^{-\frac{|\alpha'|}{2}}e^{-\frac12\mathrm{Re}\,\mu_j x_n}.\end{aligned}$$

This shows (iv).

As for (v), since $\frac{|\xi'|^4}{\mu_1\mu_2} - 2|\xi'|^2 + \mu_1\mu_2 = \frac{1}{\mu_1\mu_2}\left(\mu_1\mu_2 - |\xi'|^2\right)^2$, we see from (i) and (iii) that

$$\left|\partial_{\xi'}^{\alpha'}\left(\frac{|\xi'|^4}{\mu_1\mu_2} - 2|\xi'|^2 + \mu_1\mu_2\right)\right| \leq C|\lambda|^2(|\lambda| + 1 + |\xi'|^2)^{-1-\frac{|\alpha'|}{2}}.$$

This, together with (4.21) and (4.22), implies that

$$\left|\partial_{\xi'}^{\alpha'} D_1(\lambda, \xi')\right| \leq C|\lambda|^2(|\lambda| + 1 + |\xi'|^2)^{-1-\frac{|\alpha'|}{2}}. \tag{4.27}$$

Combining Lemma 4.4 (ii) and (4.27), we obtain the desired inequality in (v).

We finally prove (vi). Since

$$e^{-\mu_1 x_n} - e^{-\mu_2 x_n} = (\mu_2 - \mu_1)x_n \int_0^1 e^{-\{\theta\mu_1 + (1-\theta)\mu_2\}x_n}\, d\theta,$$

we see from (ii) and (iv) that

$$\begin{aligned}
&\left|\partial_{\xi'}^{\alpha'}\left(e^{-\mu_1 x_n} - e^{-\mu_2 x_n}\right)\right| \\
&\leq \; Cx_n|\lambda|(|\lambda| + 1 + |\xi'|^2)^{-\frac{1}{2}-\frac{|\alpha'|}{2}} \int_0^1 e^{-\frac{1}{2}\mathrm{Re}\,\{\theta\mu_1 + (1-\theta)\mu_2\}x_n}\, d\theta \\
&\leq \; Cx_n|\lambda|(|\lambda| + 1 + |\xi'|^2)^{-\frac{1}{2}-\frac{|\alpha'|}{2}} \left(e^{-\frac{1}{4}\mathrm{Re}\,\mu_1 x_n} + e^{-\frac{1}{4}\mathrm{Re}\,\mu_2 x_n}\right) \\
&\leq \; C|\lambda|(|\lambda| + 1 + |\xi'|^2)^{-1-\frac{|\alpha'|}{2}} \left(e^{-\frac{1}{8}\mathrm{Re}\,\mu_1 x_n} + e^{-\frac{1}{8}\mathrm{Re}\,\mu_2 x_n}\right).
\end{aligned}$$

The desired inequalities are thus proved for $\lambda \in \Sigma(\eta_4, \theta_4)$.

Let us consider the case $\lambda \in \Sigma(-\eta_3, \theta_3)$. In the same way as above, one can show the inequalities (i)–(iv) and (vi) for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \geq R_3$, where R_3 is the positive number given in Lemma 4.4 (i). As for (v), we take $\delta = \Lambda_2$ in Lemma 4.4 (i) with Λ_2 being the number given in Proposition 4.2 (ii). Similarly to the case $\lambda \in \Sigma(\eta_4, \theta_4)$, by using Lemma 4.4 (i), we can also prove the inequality (v) for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \geq R_3$, provided that $|\lambda| \geq \delta = \Lambda_2$. In case $|\lambda| \leq \Lambda_2$, since $e^{-(\mu_1+\mu_2)a}e^{2|\xi'|a} = e^{O(|\xi'|^{-1})}$, we see from Proposition 4.2 (ii) that

$$|D_1(\lambda, \xi')| \geq C\frac{|\lambda|^2}{|\xi'|^2}$$

for large $|\xi'|$. Furthermore, similarly to the case $\lambda \in \Sigma(\eta_4, \theta_4)$, one can obtain (4.27) with a more detailed computation for $\widetilde{D_1}(\lambda, \xi')$ given in (4.22) for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and large $|\xi'|$. The desired inequality (v) then follows by changing R_3 suitably large if necessary. This completes the proof.

We next derive the estimates for the integral kernel of $\widehat{G}(\lambda, \xi')$. We set

$$
(4.28)\quad \begin{cases}
g^{(1)}_{\mu_j}(x_n, y_n) = \dfrac{1}{2\mu_j} e^{-\mu_j|x_n - y_n|}, \\
g^{(2)}_{\mu_j}(x_n, y_n) = \dfrac{1}{2\mu_j}\left(e^{-\mu_j(x_n+y_n)} + e^{-\mu_j\{(a-x_n)+(a-y_n)\}}\right), \\
g^{(3)}_{\mu_j}(x_n, y_n) = \dfrac{1}{2\mu_j}\dfrac{e^{-2\mu_j a}}{1-e^{-2\mu_j}}\left(e^{-\mu_j|x_n-y_n|} + e^{\mu_j|x_n-y_n|}\right), \\
g^{(4)}_{\mu_j}(x_n, y_n) = \dfrac{1}{2\mu_j}\dfrac{e^{-2\mu_j a}}{1-e^{-2\mu_j}}\left(e^{-\mu_j(x_n+y_n)} + e^{-\mu_j\{(a-x_n)+(a-y_n)\}}\right),
\end{cases}
$$

where $\mu_j = \mu_j(\lambda, \xi')$, $j = 1, 2$. In what follows we will denote $|\lambda| + 1 + |\xi'|^2$ by $\sigma(\lambda, \xi')$:

$$\sigma(\lambda, \xi') = |\lambda| + 1 + |\xi'|^2.$$

Lemma 4.6. *Let $g^{(k)}_{\mu_j}(x_n, y_n)$, $j = 1, 2$, $k = 1, \cdots, 4$, be defined in (4.28). Then*

$$g^D_{\mu_j}(x_n, y_n) = \sum_{k=1}^{4}(-1)^{k+1} g^{(k)}_{\mu_j}(x_n, y_n), \quad g^N_{\mu_j}(x_n, y_n) = \sum_{k=1}^{4} g^{(k)}_{\mu_j}(x_n, y_n),$$

and the following estimates hold for any multi-index $\alpha' \in \mathbf{Z}^{n-1}$ with $|\alpha'| \le [\frac{n-1}{2}]+1$ and any nonnegative integer ℓ uniformly in $\lambda \in \Sigma(\eta_4, \theta_4)$, $\xi' \in \mathbf{R}^{n-1}$ and $x_n, y_n \in [0, a]$:

$$(\mathrm{i})\qquad \left|\partial^{\alpha'}_{\xi'}\left[(\partial^\ell_{x_n} g^{(k)}_{\mu_j})(x_n, y_n)\right]\right| \le C_\ell \sigma(\lambda, \xi')^{\frac{\ell}{2}-\frac{1}{2}-\frac{|\alpha'|}{2}} E^{(k)}(x_n, y_n)$$

for $k = 1, 2$ and $j = 1, 2$, and

$$
(\mathrm{ii})\qquad \begin{aligned}
&\left|\partial^{\alpha'}_{\xi'}\left[(\partial^\ell_{x_n} g^{(k)}_{\mu_1})(x_n, y_n) - (\partial^\ell_{x_n} g^{(k)}_{\mu_2})(x_n, y_n)\right]\right| \\
&\le \; C|\lambda|\sigma(\lambda, \xi')^{\frac{\ell}{2}-\frac{3}{2}-\frac{|\alpha'|}{2}} E^{(k)}(x_n, y_n),
\end{aligned}
$$

for $k = 1, 2$. Here $E^{(k)}(x_n, y_n)$, $k = 1, 2$, are the functions defined by

$$E^{(1)}(x_n, y_n) = e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}|x_n - y_n|},$$

$$E^{(2)}(x_n, y_n) = e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(x_n+y_n)} + e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}\{(a-x_n)+(a-y_n)\}}$$

with some constant $c > 0$ independent of $\lambda \in \Sigma(\eta_4, \theta_4)$, $\xi' \in \mathbf{R}^{n-1}$ and $x_n, y_n \in [0, a]$. Furthermore, for any nonnegative integer q, there hold the estimates

$$\text{(iii)} \qquad \left|\partial_{\xi'}^{\alpha'}\left[(\partial_{x_n}^{\ell} g_{\mu_j}^{(k)})(x_n, y_n)\right]\right| \le C_{\ell,q}\sigma(\lambda, \xi')^{-\frac{q+|\alpha'|}{2}}, \quad k = 3, 4, \ j = 1, 2.$$

The inequalities (i)–(iii) *also hold uniformly in $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \ge R_3$ for some $R_3 > 0$, where η_3 and θ_3 are the numbers given in Lemma* 4.4.

Proof. A direct computation shows that $g_{\mu_j}^M(x_n, y_n)$, $M = D, N$, are written as above. One can prove inequalities (i) and (iii) by a direct application of Lemma 4.5.

As for (ii), we write $g_{\mu_1}^{(1)}(x_n, y_n) - g_{\mu_2}^{(1)}(x_n, y_n)$ as

$$\begin{aligned} g_{\mu_1}^{(1)}(x_n, y_n) - g_{\mu_2}^{(1)}(x_n, y_n) &= \frac{1}{2}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) e^{-\mu_1|x_n - y_n|} \\ &\quad + \frac{1}{2\mu_2}\left(e^{-\mu_1|x_n-y_n|} - e^{-\mu_2|x_n-y_n|}\right). \end{aligned}$$

Since $\partial_{x_n}^{\ell} e^{-\mu_j|x_n-y_n|} = (-\mu_j \operatorname{sgn}(x_n - y_n))^{\ell} e^{-\mu_j|x_n-y_n|}$, we can obtain inequality (ii) for $k = 1$. The case $k = 2$ can be proved similarly. This completes the proof.

In the same manner we can estimate $h_{\mu_j}(x_n)$.

Lemma 4.7. *Let $\alpha' \in \mathbf{Z}^{n-1}$ be a multi-index with $|\alpha'| \le [\frac{n-1}{2}] + 1$ and let ℓ be a nonnegative integer. Then the following estimates hold uniformly in $\lambda \in \Sigma(\eta_4, \theta_4)$, ξ' and $x_n \in [0, a]$:*

$$\text{(i)} \qquad \left|\partial_{\xi'}^{\alpha'}\partial_{x_n}^{\ell} h_{\mu_j}(x_n)\right| \le C_{\ell}\sigma(\lambda, \xi')^{\frac{\ell}{2} - \frac{1}{2} - \frac{|\alpha'|}{2}} e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(a - x_n)}$$

for $j = 1, 2$, and

$$\text{(ii)} \qquad \left|\partial_{\xi'}^{\alpha'}\partial_{x_n}^{\ell} h_{\mu_1,\mu_2}(x_n)\right| \le C_{\ell}\sigma(\lambda, \xi')^{\frac{\ell}{2} - \frac{3}{2} - \frac{|\alpha'|}{2}} e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(a - x_n)}.$$

The inequalities (i) *and* (ii) *also hold uniformly in $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \ge R_3$ for some $R_3 > 0$, where η_3 and θ_3 are the numbers given in Lemma* 4.4.

Proof. The proof is similar to that of Lemma 4.6. We omit it.

We next estimate $\beta_j(\lambda, \xi', y_n)$, $B'(\lambda, \xi', y_n)$ and $\boldsymbol{b}_n(\lambda, \xi', y_n)$.

Lemma 4.8. *Let $\alpha' \in \mathbf{Z}^{n-1}$ be a multi-index with $|\alpha'| \le [\frac{n-1}{2}]+1$. Then the following estimates hold uniformly in $\lambda \in \Sigma(\eta_4, \theta_4)$, $\xi' \in \mathbf{R}^{n-1}$ and $y_n \in [0,a]$:*

$$\text{(i)} \qquad \left|\partial_{\xi'}^{\alpha'} \beta_0(\lambda, \xi', y_n))\right| \le C_\ell \,(|\lambda|+1)^{-1}\, \sigma(\lambda,\xi')^{-\frac{|\alpha'|}{2}} e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(a-y_n)},$$

$$\text{(ii)} \qquad \left|\partial_{\xi'}^{\alpha'} \beta_1(\lambda, \xi', y_n)\right| \le C\sigma(\lambda,\xi')^{-\frac{|\alpha'|}{2}} e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(a-y_n)},$$

$$\text{(iii)} \qquad \left|\partial_{\xi'}^{\alpha'} B'(\lambda, \xi', y_n)\right| \le C\sigma(\lambda,\xi')^{-\frac{|\alpha'|}{2}} e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(a-y_n)},$$

$$\text{(iv)} \qquad \left|\partial_{\xi'}^{\alpha'} \boldsymbol{b}_n(\lambda, \xi', y_n)\right| \le C\sigma(\lambda,\xi')^{-\frac{|\alpha'|}{2}} e^{-c\sigma(\lambda,\xi')^{\frac{1}{2}}(a-y_n)}.$$

The inequalities (i)–(iv) *also hold uniformly in $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \ge R_3$ for some $R_3 > 0$, where η_3 and θ_3 are the numbers given in Lemma* 4.4.

Proof. A direct calculation shows that

$$\begin{aligned} \beta_0(\lambda,\xi',y_n) &= \tfrac{\gamma\lambda}{\nu_1\lambda+\gamma^2}\tfrac{1}{4D_1(\lambda,\xi')}\tfrac{\mu_1\mu_2-|\xi'|^2}{\mu_1\mu_2}e^{-\mu_2(a-y_n)} \\ &\quad +\tfrac{1}{4D_1(\lambda,\xi')}\widetilde{\beta}_0(\lambda,\xi',y_n), \end{aligned} \tag{4.29}$$

where $\widetilde{\beta}_0(\lambda,\xi',y_n)$ is written as the sum of terms of the form

$$e^{-\mu_k a-\mu_1 z_n-\mu_2 w_n} \times \left\{\text{polynomial in } \mu_1, \mu_2, \frac{1}{\mu_1}, \frac{1}{\mu_2} \text{ and } \xi'\right\}, \; k=1,2,$$

with z_n and w_n being linear functions of y_n that satisfy $z_n \ge 0$, $w_n \ge 0$ and $\mathrm{Re}\,(\mu_k a+\mu_1 z_n+\mu_2 w_n) \ge \mathrm{Re}\,\mu_2(a-y_n)$ for $y_n \in [0,a]$. The desired inequality in (i) is obtained by applying Lemma 4.5 to (4.29). Similarly, by a direct computation, we see that $\beta_1(\lambda,\xi',y_n)$ and $\boldsymbol{b}_n(\lambda,\xi',y_n)$ can be written as

$$\begin{aligned} \beta_1(\lambda,\xi',y_n) &= \tfrac{\mu_1\mu_2-|\xi'|^2}{4D_1(\lambda,\xi')}\left\{\tfrac{\mu_1\mu_2-|\xi'|^2}{\mu_1\mu_2}e^{-\mu_2(a-y_n)} - \left(e^{-\mu_1(a-y_n)} - e^{-\mu_2(a-y_n)}\right)\right\} \\ &\quad +\widetilde{\beta}_1(\lambda,\xi',y_n) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{b}_n(\lambda,\xi',y_n) &= -\tfrac{i\xi'}{4D_1(\lambda,\xi')}\tfrac{\mu_1\mu_2-|\xi'|^2}{\mu_1}\left(e^{-\mu_1(a-y_n)} - e^{-\mu_2(a-y_n)}\right) \\ &\quad +i\xi'\widetilde{\beta}_n(\lambda,\xi',y_n), \end{aligned}$$

where $\widetilde{\beta}_j(\lambda,\xi',y_n)$, $j=1,n$, are written as the sum of terms of the same form as that for $\widetilde{\beta}_0(\lambda,\xi',y_n)$ with z_n and w_n satisfying $z_n\ge 0$, $w_n\ge 0$ and $\mathrm{Re}\,(\mu_k a+\mu_1 z_n+\mu_2 w_n)\ge \mathrm{Re}\,\mu_\ell(a-y_n)$ for $y_n\in[0,a]$, $k=1,2$ and $\ell=1$ or 2. The desired inequalities in (ii) and (iv) now follow from Lemma 4.5.

We next consider inequality (iii). We write $B'(\lambda,\xi',y_n)$ as

$$B'(\lambda,\xi',y_n)=-\frac{\sinh\mu_1 y_n}{\sinh\mu_1 a}I_{n-1}-\beta(\lambda,\xi',y_n)P'_{1,0},$$

where

$$\beta(\lambda,\xi',y_n)=\frac{\sinh\mu_1 y_n}{\sinh\mu_1 a}-\beta_1(\lambda,\xi',y_n).$$

A direct computation shows that $\beta(\lambda,\xi',y_n)$ is written as

$$\begin{aligned}\beta(\lambda,\xi',y_n) &= -\tfrac{|\xi'|^2}{4D_1(\lambda,\xi')}\tfrac{\mu_1\mu_2-|\xi'|^2}{\mu_1\mu_2}\left(e^{-\mu_1(a-y_n)}-e^{-\mu_2(a-y_n)}\right)\\ &\quad+\tfrac{|\xi'|^2}{4D_1(\lambda,\xi')(1-e^{-2\mu_1 a})}\widetilde{\beta}(\lambda,\xi',y_n),\end{aligned}$$

where $\widetilde{\beta}(\lambda,\xi',y_n)$ is written as the sum of the same form as that for $\widetilde{\beta}_1(\lambda,\xi',y_n)$. Consequently we have

$$\begin{aligned}&B'(\lambda,\xi',y_n)\\ &= \tfrac{1}{\mu_1}e^{-\mu_1(a-y_n)}I_{n-1}+\tfrac{\xi'^T\xi'}{4D_1(\lambda,\xi')}\tfrac{\mu_1\mu_2-|\xi'|^2}{\mu_1\mu_2}\left(e^{-\mu_1(a-y_n)}-e^{-\mu_2(a-y_n)}\right)\\ &\quad+\tfrac{1}{\mu_1}\tfrac{e^{-\mu_1 a}}{1-e^{-2\mu_1 a}}\left(e^{-\mu_1(2a-y_n)}-e^{-\mu_1 y_n}\right)I_{n-1}+\tfrac{\xi'^T\xi'}{4D_1(\lambda,\xi')(1-e^{-2\mu_1 a})}\widetilde{\beta}(\lambda,\xi',y_n).\end{aligned}$$

The desired inequality in (iii) now follows from Lemma 4.5.

Similarly to above, one can prove the desired inequalities for $\lambda\in\Sigma(-\eta_3,\theta_3)$ and $|\xi'|\ge R_3$ with $|\lambda|\ge\delta$, where δ is any positive number.

Let us consider the case $|\lambda|\le\delta$. In this case a direct computation shows that

$$\left|\partial_{\xi'}^{\alpha'}\widetilde{\beta}_j(\lambda,\xi',y_n)\right|\le C|\lambda|^2 e^{-c|\xi'|a},\quad j=0,1,n,$$

for small $|\lambda|$ and large $|\xi'|$. Combining this with Lemma 4.5, we can obtain the desired inequalities for $\lambda\in\Sigma(-\eta_3,\theta_3)$ and $|\xi'|\ge R_3$ with some large $R_3>0$. This completes the proof.

5. Proof of the main results

In this section we prove the main results stated in section 2.

To prove Theorem 2.1 we will apply the following Fourier multiplier theorem.

Lemma 5.1. (Fourier multiplier theorem) *Let $1 < p < \infty$ and let s be an integer satisfying $s \geq [\frac{k}{2}] + 1$. Suppose that $\Psi(\zeta) \in C^s(\mathbf{R}^k - \{0\}) \cap L^\infty(\mathbf{R}^k)$ and that there exists a positive constant C_0 such that*

$$|\zeta|^{|\alpha|} \left|\partial_\zeta^\alpha \Psi(\zeta)\right| \leq C_0$$

for all $\zeta \in \mathbf{R}^k - \{0\}$ and $|\alpha| \leq s$. Then the operator $\mathscr{F}_\zeta^{-1}[\Psi(\zeta)(\mathscr{F}f)(\zeta)]$ is extended to a bounded linear operator on $L^p(\mathbf{R}^k)$ and there holds the estimate

$$\|\mathscr{F}_\zeta^{-1}[\Psi(\zeta)(\mathscr{F}f)(\zeta)]\|_{L^p(\mathbf{R}^k)} \leq CC_0\|f\|_{L^p(\mathbf{R}^k)}.$$

See, e.g., [4] for the proof.

We will also use the following lemma concerning integral operator.

Lemma 5.2. *Let $\Phi(x_n, y_n)$ be a measurable function on $(0, a) \times (0, a)$.*

(i) *Suppose that there exists a positive constant M such that $\int_0^a |\Phi(x_n, y_n)|\, dy_n \leq M$ for a.e. x_n and $\int_0^a |\Phi(x_n, y_n)|\, dx_n \leq M$ for a.e. y_n. Let $1 \leq p \leq \infty$. Then it holds that*

$$\left|\int_0^a \Phi(\cdot, y_n) f(y_n)\, dy_n\right|_p \leq M|f|_p.$$

(ii) *Suppose that there exists a positive constant M such that $(\int_0^a |\Phi(x_n, y_n)|^2\, dy_n)^{\frac{1}{2}} \leq M$ for a.e. x_n. Then it holds that*

$$\left|\int_0^a \Phi(\cdot, y_n) f(y_n)\, dy_n\right|_\infty \leq M|f|_2.$$

The proof of the above lemma is well known. We omit it.

To obtain the estimates for derivatives of the resolvent we still need some consideration. We proceed as in [3].

Lemma 5.3. *There holds*

$$\mathscr{F}_{\xi'}^{-1}\left[\int_0^a \frac{1}{2\mu_j} e^{-\mu_j|x_n - y_n|} \widehat{f}(\xi', y_n)\, dy_n\right](x', x_n) = \mathscr{F}_\xi^{-1}\left[\frac{1}{\mu_j^2 + \xi_n^2} \mathscr{F}Ef(\xi)\right](x', x_n)$$

for $j = 1, 2$, where $\xi = (\xi', \xi_n)$ and $Ef(x', x_n)$ is the zero-extension of $f(x', x_n)$ to $\mathbf{R}^n$.

Proof. Since $\mathscr{F}_{x_n}\left[\frac{1}{2\mu_j}e^{-\mu_j|x_n|}\right] = \frac{1}{\mu_j^2+\xi_n^2}$, we obtain the desired relation. This completes the proof.

We will also use the following lemma ([3, Lemma 2.6]).

Lemma 5.4. ([3, Lemma 2.6]) *Let $1 < p < \infty$ and define the operator T by*

$$Tf(x_n) = \int_0^a \frac{f(y_n)}{x_n + y_n}\, dy_n$$

for $x_n \in (0,a)$ and $f \in L^p(0,a)$. Then there exists a positive constant $C = C(p)$ such that

$$|Tf|_p \leq C|f|_p.$$

This lemma follows from the fact that Tf can be written in a form of the Hilbert transformation on $\mathbf{R}$. See [3, Lemma 2.6].

We are now in a position to Prove Theorem 2.1.

Proof of Theorem 2.1. Let η be any positive number. We will prove Theorem 2.1 with θ satisfying $\frac{\pi}{2} < \theta < \min\{\theta_1, \theta_3, \theta_4\}$, where θ_j $(j = 1,3,4)$ are the numbers given in Lemmas 4.3 and 4.4. We note that $\Sigma(\eta,\theta) \subset \Sigma(\eta,\theta_1) \cap \Sigma(-\eta_3,\theta_3)$ and that $\overline{\Sigma(\eta,\theta) - \Sigma(\eta_4,\theta_4)}$ is a compact set. Here η_3 and η_4 are the numbers given in Lemma 4.4.

We see from Theorem 3.8 that there exists a solution u of problem (1.1)–(1.2) which takes the form $u = R(\lambda)f$, where $\widehat{R}(\lambda,\xi') = (\lambda + \widehat{L}_{\xi'})^{-1} = \widehat{G}(\lambda,\xi') + \widehat{K}(\lambda,\xi')$ with $\widehat{G}(\lambda,\xi')$ and $\widehat{K}(\lambda,\xi')$ being the integral operators given in Theorem 3.8. We first prove that the estimates in Theorem 2.1 hold for $u = R(\lambda)f$. We then prove the uniqueness of solutions of problem (1.1)–(1.2).

Let us estimate $u = R(\lambda)f$. We first consider the case $\lambda \in \Sigma(\eta_4,\theta_4)$. We note that one can see from the form of $\widehat{R}(\lambda,\xi')$ that k-th order derivatives of $Q_0R(\lambda)f$ are estimated as $(k+1)$-th order ones of $\widetilde{Q}R(\lambda)f$. So we here give the proof of the estimates for $\widetilde{Q}R(\lambda)f$ only.

Hereafter we set $G(\lambda) = \mathscr{F}_{\xi'}^{-1}[\widehat{G}(\lambda,\xi')]$ and $K(\lambda) = \mathscr{F}_{\xi'}^{-1}[\widehat{K}(\lambda,\xi')]$. We also write $G(\lambda) = G^{(1)}(\lambda) + \cdots + G^{(4)}(\lambda)$, where $G^{(\ell)}(\lambda)$ is the matrix with $g_{\mu_j}^D$ and $g_{\mu_j}^N$ in $G(\lambda)$ replaced by $(-1)^{\ell+1}g_{\mu_j}^{(\ell)}$ and $g_{\mu_j}^{(\ell)}$, respectively. Here $g_{\mu_j}^{(\ell)}$ are the functions defined in (4.28).

Let us consider $\widetilde{Q}R(\lambda)\widetilde{Q}f$. We first estimate $\partial_x^k\widetilde{Q}G(\lambda)\widetilde{Q}f$ for $k \leq 1$. Since $\partial_{x_n}^2\left(g_{\mu_1,\mu_2}^M\widetilde{Q}\widehat{f}\right) = \left(\partial_{x_n}^2 g_{\mu_1}^M - \partial_{x_n}^2 g_{\mu_2}^M\right)\left[\widetilde{Q}\widehat{f}\right]$, we see from Lemma 4.6 that, for

$|\beta'| + \ell \le 1$ and $j = 1, 2$,

$$\begin{aligned}
&\left|\partial_{\xi'}^{\alpha'}\left[(i\xi')^{\beta'}\partial_{x_n}^{\ell}\widetilde{Q}\widehat{G}^{(j)}(\lambda, \xi' x_n, y_n)\widetilde{Q}\right]\right| \\
&\le\ C\sigma(\lambda, \xi')^{\frac{|\beta'|+\ell}{2}-\frac{1}{2}-\frac{|\alpha'|}{2}}\left\{E^{(1)}(x_n, y_n) + E^{(2)}(x_n, y_n)\right\} \\
&\le\ C(|\lambda| + 1)^{\frac{|\beta'|+\ell}{2}-\frac{1}{2}}\mathcal{E}(\lambda, x_n, y_n)|\xi'|^{-|\alpha'|},
\end{aligned}$$

where

$$\mathcal{E}(\lambda, x_n, y_n) = e^{-c(|\lambda|+1)^{\frac{1}{2}}|x_n-y_n|} + e^{-c(|\lambda|+1)^{\frac{1}{2}}(x_n+y_n)} + e^{-c(|\lambda|+1)^{\frac{1}{2}}\{(a-x_n)+(a-y_n)\}}.$$

Since

$$\sup_{0\le x_n\le a}\int_0^a \mathcal{E}(\lambda, x_n, y_n)\,dy_n + \sup_{0\le y_n\le a}\int_0^a \mathcal{E}(\lambda, x_n, y_n)\,dx_n \le C(|\lambda| + 1)^{-\frac{1}{2}},$$

we see from Lemmas 5.1 and 5.2 that

$$\|\partial_x^k\widetilde{Q}G^{(j)}(\lambda)\widetilde{Q}f\|_p \le \frac{C}{(|\lambda| + 1)^{1-\frac{k}{2}}}\|\widetilde{Q}f\|_p, \quad j = 1, 2,$$

for $k \le 1$. Similarly, one can estimate $\partial_x^k\widetilde{Q}G^{(j)}(\lambda)\widetilde{Q}f$ $(j = 3, 4)$ and $\partial_x^k\widetilde{Q}K(\lambda)\widetilde{Q}f$ for $k \le 1$.

We next consider $\partial_x^2\widetilde{Q}R(\lambda)\widetilde{Q}f$. We first estimate $\partial_x^2\widetilde{Q}G^{(1)}(\lambda)\widetilde{Q}f$. By Lemma 5.3, we have

$$\mathscr{F}_{\xi'}^{-1}\left[g_{\mu_j}^{(1)}\widetilde{Q}\widehat{f}\right](x', x_n) = \mathscr{F}_{\xi}^{-1}\left[\frac{1}{\mu_j^2 + \xi_n^2}\mathscr{F}_x[\widetilde{Q}Ef]\right](x', x_n), \quad j = 1, 2,$$

and, in particular,

$$\mathscr{F}_{\xi'}^{-1}\left[g_{\mu_1,\mu_2}^{(1)}\widetilde{Q}\widehat{f}\right](x', x_n) = \mathscr{F}_{\xi}^{-1}\left[M(\lambda, \xi)\mathscr{F}_x[\widetilde{Q}Ef]\right](x', x_n),$$

where Ef denotes the zero extension of f to $\mathbf{R}^n$ and

$$M(\lambda, \xi) = \frac{1}{\mu_1^2 + \xi_n^2} - \frac{1}{\mu_2^2 + \xi_n^2}$$

with $\mu_j = \mu_j(\lambda, \xi')$, $j = 1, 2$. Since $\mu_1^2 + \xi_n^2 = \frac{\lambda+\nu|\xi|^2}{\nu}$, $\mu_2^2 + \xi_n^2 = \frac{\lambda^2+\nu_1|\xi|^2\lambda+\gamma^2|\xi|^2}{\nu_1\lambda+\gamma^2}$ with $|\xi|^2 = |\xi'|^2 + \xi_n^2$, and $\nu_1 = \nu + \widetilde{\nu}$, we have

$$M(\lambda, \xi) = -\frac{\lambda(\widetilde{\nu}\lambda + \gamma^2)}{(\lambda + \nu|\xi|^2)(\lambda^2 + \nu_1|\xi|^2\lambda + \gamma^2|\xi|^2)}.$$

Therefore, in view of the observation in Remarks 3.2 and 3.5, we obtain $\left|\partial_\xi^\alpha\left(\frac{\xi^\beta}{\mu_j^2+\xi_n^2}\right)\right| \leq C|\xi|^{-|\alpha|}$ for $|\beta|=2$ and $\left|\partial_\xi^\alpha\left(\frac{\xi^\beta}{\lambda}M(\lambda,\xi)\right)\right| \leq C|\xi|^{-|\alpha|}$ for $|\beta| = 4$ uniformly in $\lambda \in \Sigma(\eta_4,\theta_4)$. It then follows from Lemma 5.1 that

$$\|\partial_x^2\widetilde{Q}G^{(1)}(\lambda)\widetilde{Q}f\|_p \leq C\|\widetilde{Q}f\|_p,$$

We next consider $\partial_x^2\widetilde{Q}G^{(2)}(\lambda)\widetilde{Q}f$. For $|\beta'|+\ell=2$, we see from Lemma 4.6 that

$$\begin{aligned}&\left|\partial_{\xi'}^{\alpha'}\left[(i\xi')^{\beta'}\partial_{x_n}^\ell\widetilde{Q}\widehat{G}^{(2)}(\lambda,\xi',x_n,y_n)\widetilde{Q}\right]\right|\\ &\leq \; C\sigma(\lambda,\xi')^{\frac{1}{2}-\frac{|\alpha'|}{2}}E^{(2)}(x_n,y_n)\\ &\leq \; C\left\{\tfrac{1}{x_n+y_n}+\tfrac{1}{(a-x_n)+(a-y_n)}\right\}|\xi'|^{-|\alpha'|}.\end{aligned}$$

It then follows from Lemma 5.1 that

$$\begin{aligned}&\left\|\partial_x^2\widetilde{Q}G^{(2)}(\lambda)\widetilde{Q}f(\cdot,y_n)\right\|_{L^p(\mathbf{R}^{n-1})}\\ &\leq \; C\int_0^a\left\{\frac{1}{x_n+y_n}+\frac{1}{(a-x_n)+(a-y_n)}\right\}\left\|\widetilde{Q}f(\cdot,y_n)\right\|_{L^p(\mathbf{R}^{n-1})}\,dy_n,\end{aligned}$$

and hence, by Lemma 5.4, we obtain

$$\left\|\partial_x^2\widetilde{Q}G^{(2)}(\lambda)\widetilde{Q}f\right\|_p \leq C\left\|\widetilde{Q}f\right\|_p.$$

In the same way one can also obtain the desired estimates for $\partial_x^2\widetilde{Q}G^{(j)}(\lambda)\widetilde{Q}f$ $(j=3,4)$ and $\partial_x^2\widetilde{Q}K(\lambda)\widetilde{Q}f$.

We next consider $\widetilde{Q}R(\lambda)Q_0f$. Similarly to above, one can prove $\left\|\partial_x^k\widetilde{Q}R(\lambda)Q_0f\right\|_p \leq \frac{C}{|\lambda|+1}\|Q_0f\|_p$ for $k\leq 1$. As for $\partial_x^2\widetilde{Q}R(\lambda)Q_0f$, we first note that $\partial_{x_n}g_{\mu_2}^N Q_0\widehat{f} = g_{\mu_2}^D[\partial_{x_n}Q_0\widehat{f}]$, which is obtained by integration by parts. It then follows that

$$\widetilde{Q}\widehat{G}(\lambda,\xi')Q_0\widehat{f} = -\frac{\gamma}{\nu_1\lambda+\gamma^2}\begin{pmatrix}0\\ g_{\mu_2}^N[\widehat{\nabla' Q_0 f}]\\ g_{\mu_2}^D[\partial_{x_n}\widehat{Q_0 f}]\end{pmatrix},$$

where $\nabla' = {}^T(\partial_{x_1},\cdots,\partial_{x_{n-1}})$. Therefore, similarly to above, we obtain

$$\|\partial_x^2\widetilde{Q}G(\lambda)Q_0f\|_p \leq \frac{C}{|\lambda|+1}\|Q_0f\|_{W^{1,p}}.$$

As for $\partial_x^2\widetilde{Q}K(\lambda)Q_0f$, we write

$$\widetilde{Q}\widehat{H}(\lambda,\xi')Q_0\widehat{f} = \begin{pmatrix}0\\ \left(\frac{1}{\nu}h_{\mu_1}\beta_0+\frac{|\xi'|^2}{\lambda}h_{\mu_1,\mu_2}\right)[\widehat{\nabla' Q_0 f}]\\ \frac{i\xi'}{\lambda}\partial_{x_n}h_{\mu_1,\mu_2}[\widehat{\nabla' Q_0 f}]\end{pmatrix}.$$

The desired estimate for $\partial_x^2 \widetilde{Q} K(\lambda) Q_0 f$ now follows similarly to above.

As mentioned before, one can prove the estimates for $\partial_x^k Q_0 R(\lambda) f$ with $k \le 1$ similarly to the estimates for $\partial_x^{k+1} \widetilde{Q} R(\lambda) f$. We thus omit the details.

Let us prove the last inequality in Theorem 2.1. We assume that $\widetilde{Q} f\Big|_{x_n=0,a} = 0$. It suffices to prove

$$\|\partial_{x_n} \widetilde{Q} R(\lambda) \widetilde{Q} f\|_p \le \frac{C}{|\lambda|+1} \|\widetilde{Q} f\|_{W^{1,p}}.$$

Since $\widetilde{Q} f\Big|_{x_n=0,a} = 0$, by integration by parts, we also have

$$\partial_{x_n} g^D_{\mu_j} \widetilde{Q} \widehat{f} = g^N_{\mu_j} \left[\partial_{x_n} \widetilde{Q} \widehat{f} \right], \quad j = 1, 2.$$

Therefore, we see from Lemmas 4.6, 5.1 and 5.2 that

$$\|\partial_{x_n} \widetilde{Q} G(\lambda) \widetilde{Q} f\|_p \le \frac{C}{|\lambda|+1} \|\partial_{x_n} \widetilde{Q} f\|_p.$$

As for $\widetilde{Q} K(\lambda) \widetilde{Q} f$, we note that an application ∂_{x_n} to $h_{\mu_j}(x_n)$ yields one of the factors μ_j, $j = 1, 2$. But, since $B'(\lambda, \xi', y_n)$ and $\boldsymbol{b}_n(\lambda, \xi', y_n)$ are written as linear combinations of $e^{\pm \mu_j y_n}$ ($j = 1, 2$), we see that if $\widetilde{Q} f\Big|_{x_n=0,a} = 0$, then

$$\int_0^a e^{\pm \mu_j y_n} \widetilde{Q} \widehat{f}(y_n)\, dy_n = \mp \frac{1}{\mu_j} \int_0^a e^{\pm \mu_j y_n} \partial_{y_n} \widetilde{Q} \widehat{f}(y_n)\, dy_n, \quad j = 1, 2.$$

Therefore, we can gain one of the factors μ_j^{-1}, $j = 1, 2$. It then follows that

$$\|\partial_{x_n} \widetilde{Q} K(\lambda) \widetilde{Q} f\|_p \le \frac{C}{|\lambda|+1} \|\partial_{x_n} \widetilde{Q} f\|_p.$$

We have thus obtained the desired estimates when $\lambda \in \Sigma(\eta_4, \theta_4)$.

We next consider the case $\lambda \in \Sigma(\eta, \theta) - \Sigma(\eta_4, \theta_4)$. We fix a positive number R_3 in such a way that Lemmas 4.5–4.8 hold for $\lambda \in \Sigma(-\eta_3, \theta_3)$ and $|\xi'| \ge R_3$.

We decompose $R(\lambda) f$ in the following way. Let $\widetilde{\chi}(\xi')$ be a C^∞ function on $\mathbf{R}^{n-1}$ satisfying $0 \le \widetilde{\chi} \le 1$ on $\mathbf{R}^{n-1}$, $\widetilde{\chi}(\xi') = 0$ for $|\xi'| \le R_3$ and $\widetilde{\chi}(\xi') = 1$ for $|\xi'| \ge 2R_3$. We write $R(\lambda) f$ as

$$R(\lambda) f = \widetilde{R}^{(0)}(\lambda) f + \widetilde{R}^{(\infty)}(\lambda) f,$$

where

$$\widetilde{R}^{(0)}(\lambda) f = \mathscr{F}_{\xi'}^{-1} \left[(1 - \widetilde{\chi}(\xi')) \widehat{R}(\lambda, \xi') \widehat{f} \right]$$

and

$$\widetilde{R}^{(\infty)}(\lambda)f = \mathscr{F}_{\xi'}^{-1}\left[\widetilde{\chi}(\xi')\widehat{R}(\lambda,\xi')\widehat{f}\right].$$

Similarly to the case of $\lambda \in \Sigma(\eta_4,\theta_4)$, we can obtain the desired estimates for $\widetilde{R}^{(\infty)}(\lambda)f$ since $\Sigma(\eta,\theta) \subset \Sigma(-\eta_3,\theta_3)$. The desired estimates also hold for $\widetilde{R}^{(0)}(\lambda)f$. In fact, by Lemmas 3.4 and 4.3, we see that $D(\lambda,\xi') \neq 0$ on the compact set $\overline{\Sigma(\eta,\theta) - \Sigma(\eta_4,\theta_4)} \times \{\xi';\ |\xi'| \leq 2R_3\}$. Therefore, $\widehat{R}(\lambda,\xi',x_n,y_n)$ is analytic, and so we can obtain the desired estimates for $\widetilde{R}^{(0)}(\lambda)f$. We thus conclude that $u = R(\lambda)f$ satisfies the estimates in Theorem 2.1.

It remains to prove the uniqueness of solutions of problem (1.1)–(1.2). To prove the uniqueness, we consider the adjoint problem

$$(\overline{\lambda} + L^*)w = g, \quad \widetilde{Q}w\Big|_{x_n=0,a} = 0, \tag{5.1}$$

where

$$L = \begin{pmatrix} 0 & -\gamma\mathrm{div} \\ -\gamma\nabla & -\nu\Delta I_n - \widetilde{\nu}\nabla\mathrm{div} \end{pmatrix}.$$

Similarly to the case of problem (1.1)–(1.2), we consider the Fourier transform of (5.1) in $x' \in \mathbf{R}^{n-1}$:

$$(\overline{\lambda} + \widehat{L}^*_{\xi'})\widehat{w} = \widehat{g}, \quad \widetilde{Q}\widehat{w}\Big|_{x_n=0,a} = 0.$$

It is easily verified that $(\overline{\lambda} + \widehat{L}^*_{\xi'})^{-1}$ has an integral representation $\widehat{R}^*(\overline{\lambda},\xi')$ corresponding to Theorem 3.8, which is written in terms of the components of $\widehat{R}(\overline{\lambda},\xi')$. Therefore, $R^*(\overline{\lambda})g = \mathscr{F}_{\xi'}^{-1}\left[\widehat{R}^*(\overline{\lambda},\xi')\widehat{g}\right]$, which is a solution of (5.1), has the same estimates as those for $R(\lambda)f$. We thus conclude that for any $\lambda \in \Sigma(\eta,\theta)$ and $g \in W^{1,p} \times L^p$ problem (5.1) has a solution $w \in W^{1,p} \times \left(W^{2,p} \cap W_0^{1,p}\right)$.

Assume now that $u \in W^{1,p} \times \left(W^{2,p} \cap W_0^{1,p}\right)$ and $(\lambda + L)u = 0$. Let $g \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\Omega)$ and let w be a solution of (5.1) satisfying $w \in W^{1,p'} \times \left(W^{2,p'} \cap W_0^{1,p'}\right)$ with $1/p' = 1 - 1/p$. It then follows that

$$0 = ((\lambda + L)u, w) = (u, (\overline{\lambda} + L^*)w) = (u, g).$$

This implies that $u = 0$, and the uniqueness of problem (1.1)–(1.2) holds for $\lambda \in \Sigma(\eta,\theta)$. This completes the proof.

We next give a proof of Theorem 2.2.

Proof of Theorem 2.2. We here prove the second inequality in Theorem 2.2 for $\partial_{x_n}Q_nG(\lambda)f$ only, since the other cases can be proved similarly by applying Lemmas 4.6–4.8 and 5.2.

We first consider the case $\lambda \in \Sigma(\eta_4, \theta_4)$. Let $E^{(j)}(x_n, y_n)$, $j = 1, 2$, be the functions defined in Lemma 4.6 and set $E(x_n, y_n) = E^{(1)}(x_n, y_n) + E^{(2)}(x_n, y_n)$. Then we have

$$\sup_{0 \le x_n \le a} \left(\int_0^a |E(x_n, y_n)|^2 \, dy_n \right)^{\frac{1}{2}} \le C\sigma(\lambda, \xi')^{-\frac{1}{4}}.$$

Since $\partial_{x_n}^2 \left(g_{\mu_2}^N Q_0 \widehat{f} \right) = \left(\partial_{x_n}^2 g_{\mu_2}^N \right) \left[Q_0 \widehat{f} \right] - Q_0 \widehat{f}$, we see from Lemmas 4.6 and 5.2 that

$$\begin{aligned}
\|\partial_{x_n} Q_n G(\lambda) Q_0 f\|_\infty &\le \tfrac{C}{|\lambda|+1} \left\{ \|Q_0 f\|_\infty + \left\| \left(\partial_{x_n}^2 g_{\mu_2}^N \right) \left[Q_0 \widehat{f} \right] \right\|_{L^1_{\xi'} L^\infty_{x_n}} \right\} \\
&\le \tfrac{C}{|\lambda|+1} \left\{ \|Q_0 f\|_\infty + \left\| (|\lambda| + 1 + |\xi'|^2)^{\frac{1}{4}} \left| Q_0 \widehat{f}(\xi') \right|_2 \right\|_{L^1_{\xi'}} \right\} \\
&\le C \left\{ \tfrac{1}{|\lambda|+1} \|Q_0 f\|_\infty + \tfrac{1}{(|\lambda|+1)^{\frac{3}{4}}} \|Q_0 f\|_{H^{[\frac{n}{2}]+1}} \right\} \\
&\le \tfrac{C}{(|\lambda|+1)^{\frac{3}{4}}} \|Q_0 f\|_{H^{[\frac{n}{2}]+1}}.
\end{aligned}$$

We next consider $\partial_{x_n} Q_n G(\lambda) Q' f$. Similarly to above, we see from Lemmas 4.6 and 5.2 that, for $0 < \varepsilon < 1$,

$$\begin{aligned}
\|\partial_{x_n} Q_n G(\lambda) Q' f\|_\infty &\le C \left\| \partial_{x_n} Q_n \widehat{G}(\lambda, \xi') Q' \widehat{f} \right\|_{L^1_{\xi'} L^\infty_{x_n}} \\
&\le C \left\| (|\lambda| + 1 + |\xi'|^2)^{-\frac{3}{4}} |\xi'| \left| Q' \widehat{f}(\xi') \right|_2 \right\|_{L^1_{\xi'}} \\
&\le C \left\| (|\lambda| + 1)^{-\frac{\varepsilon}{4}} (|\lambda| + 1 + |\xi'|^2)^{-\frac{1}{4}(1-\varepsilon)} \left| Q' \widehat{f}(\xi') \right|_2 \right\|_{L^1_{\xi'}} \\
&\le \tfrac{C}{(|\lambda|+1)^{\frac{\varepsilon}{4}}} \left\| Q' \widehat{f} \right\|_{H^s}
\end{aligned}$$

with $s > \frac{n}{2} - \varepsilon$. Therefore, we can take $s = \left[\frac{n}{2} \right]$. Similarly, we can obtain

$$\|\partial_{x_n} Q_n G(\lambda) Q_n f\|_\infty \le \frac{C}{(|\lambda| + 1)^{\frac{\varepsilon}{4}}} \|f'(\xi')\|_{H^{[\frac{n}{2}]}}.$$

We next consider the case $\lambda \in \Sigma(\eta, \theta) - \Sigma(\eta_4, \theta_4)$. We decompose $(\lambda + L)^{-1} f$ into $(\lambda + L)^{-1} f = \widetilde{R}^{(0)}(\lambda) f + \widetilde{R}^{(\infty)}(\lambda) f$ as in the proof of Theorem 2.1. It then follows that $\widetilde{R}^{(\infty)}(\lambda) f$ can be estimated as above. One can also see that $\widetilde{R}^{(0)}(\lambda) f$ has the desired estimates since $D(\lambda, \xi') \neq 0$ on the compact set $\overline{\Sigma(\eta, \theta) - \Sigma(\eta_4, \theta_4)} \times \{\xi'; |\xi'| \le 2R_3\}$. This completes the proof.

We next prove the L^p estimates of the resolvent for $p = 1, \infty$. For this purpose we prepare the following lemma.

Lemma 5.5. *Let $\ell = 0, 1$ and let $\widehat{\Phi}(\xi', x_n)$ be a function satisfying*

$$|\partial_{\xi'}^{\alpha'} \widehat{\Phi}(\xi', \cdot)|_1 \le C\sigma(\lambda, \xi')^{\frac{\ell}{2}-1-\frac{|\alpha'|}{2}}$$

for all α' with $|\alpha'| \le n$. Then $\Phi(x', x_n) = (\mathscr{F}_{\xi'}^{-1}\widehat{\Phi})(x', x_n)$ satisfies

$$\|\Phi\|_1 \le C(|\lambda| + 1)^{\frac{\ell}{2}-1}.$$

Proof. The proof is based on the Riemann-Lebesgue lemma as in the estimates of solutions to the Cauchy problem given in [8].

Since $2 - \ell + n > n - 1$, by integration by parts, we see that

$$\begin{aligned} |\Phi(x', \cdot)|_1 &= \left| (2\pi)^{n-1} \int_{\mathbf{R}^{n-1}} \widehat{\Phi}(\xi', x_n) \left(\sum_{j=1}^{n-1} \frac{x_j}{i|x'|^2} \partial_{\xi_j} \right)^n e^{ix'\cdot\xi'}\, d\xi' \right|_{L^1_{x_n}} \\ &\le C|x'|^{-n} \int_{\mathbf{R}^{n-1}} \sum_{|\alpha'|=n} \left| \partial_{\xi_j}^{\alpha'} \widehat{\Phi}(\xi', x_n) \right|_{L^1_{x_n}} d\xi' \\ &\le C|x'|^{-n} \int_{\mathbf{R}^{n-1}} \left(|\lambda| + 1 + |\xi'|^2 \right)^{\frac{\ell}{2}-1-\frac{n}{2}} d\xi' \\ &\le C|x'|^{-n}(|\lambda| + 1)^{-\frac{3}{2}+\frac{\ell}{2}}. \end{aligned}$$

This implies that

$$\begin{aligned} \| |\Phi(x', \cdot)|_1 \|_{L^1(|x'| \ge (|\lambda|+1)^{-\frac{1}{2}})} &\le C(|\lambda| + 1)^{-\frac{3}{2}+\frac{\ell}{2}} \| |x'|^{-n} \|_{L^1(|x'| \ge (|\lambda|+1)^{-\frac{1}{2}})} \\ &\le C(|\lambda| + 1)^{-1+\frac{\ell}{2}}. \end{aligned}$$

We next show that

$$\| |\Phi(x', \cdot)|_1 \|_{L^1(|x'| \le (|\lambda|+1)^{-\frac{1}{2}})} \le C(|\lambda| + 1)^{-1+\frac{\ell}{2}}.$$

In case $\ell = 0$, similarly to above, we have

$$\begin{aligned} |\Phi(x', \cdot)|_1 &= \left| (2\pi)^{n-1} \int_{\mathbf{R}^{n-1}} \widehat{\Phi}(\xi', x_n) \left(\sum_{j=1}^{n-1} \frac{x_j}{i|x'|^2} \partial_{\xi_j} \right)^{n-2} e^{ix'\cdot\xi'}\, d\xi' \right|_{L^1_{x_n}} \\ &\le C|x'|^{-(n-2)} \int_{\mathbf{R}^{n-1}} \sum_{|\alpha'|=n-2} \left| \partial_{\xi_j}^{\alpha'} \widehat{\Phi}(\xi', x_n) \right|_{L^1_{x_n}} d\xi' \\ &\le C|x'|^{-(n-2)} \int_{\mathbf{R}^{n-1}} \left(|\lambda| + 1 + |\xi'|^2 \right)^{\frac{\ell}{2}-1-\frac{n-2}{2}} d\xi' \\ &\le C|x'|^{-(n-2)}(|\lambda| + 1)^{-\frac{1}{2}}. \end{aligned}$$

This implies

$$\begin{aligned} \| |\Phi(x',\cdot)|_1 \|_{L^1(|x'|\le(|\lambda|+1)^{-\frac12})} &\le C(|\lambda|+1)^{-\frac12} \left\| |x'|^{-(n-2)} \right\|_{L^1(|x'|\le(|\lambda|+1)^{-\frac12})} \\ &\le C(|\lambda|+1)^{-1}. \end{aligned}$$

The proof of the lemma for $\ell = 0$ is thus complete.

Let us consider the case $\ell = 1$. We first observe that

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} \partial_{\xi'}^{\alpha'} \widehat{\Phi}(\xi', x_n) e^{ix'\cdot\xi'}\, d\xi' &= -\int_{\mathbf{R}^{n-1}} \partial_{\xi'}^{\alpha'} \widehat{\Phi}(\xi', x_n) e^{ix'\cdot(\xi' + \frac{x'}{|x'|^2}\pi)}\, d\xi' \\ &= -\int_{\mathbf{R}^{n-1}} \partial_{\xi'}^{\alpha'} \widehat{\Phi}(\xi' - z', x_n) e^{ix'\cdot\xi'}\, d\xi'. \end{aligned} \tag{5.2}$$

Here and in what follows we write $z' = \frac{x'}{|x'|^2}\pi$. From (5.2) we see that

$$\begin{aligned} &|\Phi(x')|_1 \\ &\le C|x'|^{-(n-2)} \int_{\mathbf{R}^{n-1}} \sum_{|\alpha'|=n-2} \left| \partial_{\xi'}^{\alpha'} \left(\widehat{\Phi}(\xi' - z', x_n) - \widehat{\Phi}(\xi', x_n) \right) \right|_{L^1_{x_n}} d\xi'. \end{aligned} \tag{5.3}$$

By assumption, we have

$$\left| \partial_{\xi'}^{\alpha'} \left(\widehat{\Phi}(\xi' - z', x_n) - \widehat{\Phi}(\xi', x_n) \right) \right|_{L^1_{x_n}} \le C\Phi^{(1)}(\xi', x'), \tag{5.4}$$

where

$$\Phi^{(1)}(\xi', x') = \left(|\lambda| + 1 + |\xi'|^2 \right)^{-\frac{n-1}{2}} + \left(|\lambda| + 1 + |\xi' - z'|^2 \right)^{-\frac{n-1}{2}}.$$

We also have

$$\begin{aligned} &\left| \partial_{\xi'}^{\alpha'} \left(\widehat{\Phi}(\xi' - z', x_n) - \widehat{\Phi}(\xi', x_n) \right) \right|_{L^1_{x_n}} \\ &\le C|z'| \sum_{|\beta'|=n-1} \int_0^1 \left| \partial_{\xi'}^{\beta'} \widehat{\Phi}(\xi' - \theta z', x_n) \right|_{L^1_{x_n}} d\theta \\ &\le C|x'|^{-1} \Phi^{(2)}(\xi', x'), \end{aligned} \tag{5.5}$$

where

$$\Phi^{(2)}(\xi', x') = \int_0^1 \left(|\lambda| + 1 + |\xi' - \theta z'|^2 \right)^{-\frac{n}{2}} d\theta.$$

Let τ be a number satisfying $1 - \frac{1}{n} < \tau < 1$. It then follows from (5.4) and (5.5) that, for $|\alpha'| = n - 2$,

$$\begin{aligned} &\int_{\mathbf{R}^{n-1}} \left| \partial_{\xi'}^{\alpha'} \left(\widehat{\Phi}(\xi' - z', x_n) - \widehat{\Phi}(\xi', x_n) \right) \right|_{L^1_{x_n}} d\xi' \\ &\le C|x'|^{-(1-\tau)} \int_{\mathbf{R}^{n-1}} \widehat{\Phi}^{(1)}(\xi', x')^{\tau} \widehat{\Phi}^{(2)}(\xi', x')^{1-\tau}\, d\xi' \\ &= C|x'|^{-(1-\tau)} \int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \widehat{\Phi}^{(1)}(\xi', x')^{\tau} \widehat{\Phi}^{(2)}(\xi', x')^{1-\tau}\, d\xi', \end{aligned} \tag{5.6}$$

where

$$\Omega_1 = \{\xi';\, |\xi'| \geq 2|z'|\}\,,$$
$$\Omega_2 = \{\xi';\, |\xi'| \leq 2|z'|\} \cap \{\xi';\, |\xi' - z'| \geq 2|z'|\}\,,$$
$$\Omega_3 = \{\xi';\, |\xi'| \leq 2|z'|\} \cap \{\xi';\, |\xi' - z'| \leq 2|z'|\}\,.$$

On Ω_1 we have $|\xi' - z'| \geq \frac{1}{2}|\xi'|$ and $|\xi' - \theta z'| \geq \frac{1}{2}|\xi'|$ for any $\theta \in [0,1]$. It follows that

(5.7)
$$\begin{aligned}\int_{\Omega_1} \widehat{\Phi}^{(1)}(\xi',x')^{\tau}\widehat{\Phi}^{(2)}(\xi',x')^{1-\tau}\,d\xi' &\leq C\int_{\mathbf{R}^{n-1}} \left(|\lambda| + 1 + |\xi'|^2\right)^{-\frac{n}{2}+\frac{\tau}{2}} d\xi' \\ &\leq C(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}.\end{aligned}$$

On Ω_2 we have $|\xi' - z'| \geq |\xi'|$ and $|\xi' - \theta z'| \geq |\xi' - z'| - (1-\theta)|z'| \geq \frac{1}{2}|\xi'|$ for any $\theta \in [0,1]$. Therefore, as above, we obtain

$$\int_{\Omega_2} \widehat{\Phi}^{(1)}(\xi',x')^{\tau}\widehat{\Phi}^{(2)}(\xi',x')^{1-\tau}\,d\xi' \leq C(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}. \tag{5.8}$$

As for the integral on Ω_3, we have

$$\begin{aligned}&\int_{\Omega_3} \widehat{\Phi}^{(1)}(\xi',x')^{\tau}\widehat{\Phi}^{(2)}(\xi',x')^{1-\tau}\,d\xi' \\ &\leq C\int_{\{|\xi'|\leq 2|z'|\}} |\xi'|^{-(n-1)\tau}(|\lambda|+1)^{-\frac{n}{2}(1-\tau)}d\xi' \\ &\quad + C\int_{\{|\xi'-z'|\leq 2|z'|\}} |\xi'-z'|^{-(n-1)\tau}(|\lambda|+1)^{-\frac{n}{2}(1-\tau)}d\xi' \\ &\leq C(|\lambda|+1)^{-\frac{n}{2}(1-\tau)}|x'|^{-(1-\tau)(n-1)}.\end{aligned} \tag{5.9}$$

Consequently we see from (5.6)–(5.9) that

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}} \left|\partial_{\xi'}^{\alpha'}\left(\widehat{\Phi}(\xi'-z',x_n) - \widehat{\Phi}(\xi',x_n)\right)\right|_{L^1_{x_n}} d\xi' \\ &\leq C\left\{(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}|x'|^{-(1-\tau)} + (|\lambda|+1)^{-\frac{n}{2}(1-\tau)}|x'|^{-(1-\tau)n}\right\}.\end{aligned}$$

This, together with (5.3), implies that

$$\begin{aligned}|\Phi(x')|_1 \leq\ & C\Big\{(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}|x'|^{-(n-1)+\tau} \\ &+ (|\lambda|+1)^{-\frac{n}{2}(1-\tau)}|x'|^{-(n-2)-(1-\tau)n}\Big\}.\end{aligned}$$

Since $1 - \frac{1}{n} < \tau$, integrating this over $\{x';\, |x'| \leq (|\lambda|+1)^{-\frac{1}{2}}\}$, we obtain

$$\big\||\Phi(x',\cdot)|_1\big\|_{L^1(|x'|\leq(|\lambda|+1)^{-\frac{1}{2}})} \leq C(|\lambda|+1)^{-\frac{1}{2}}.$$

This completes the proof.

We now prove the L^p estimates of the resolvent for $p = 1, \infty$.

Proof of Theorem 2.3. We here prove the estimates only for $\lambda \in \Sigma(\eta_4, \theta_4)$. The other case can be treated by using the decomposition $(\lambda + L)^{-1} f = \widetilde{R}^{(0)}(\lambda) f + \widetilde{R}^{(\infty)}(\lambda) f$ as in the proof of Theorem 2.1.

Let us denote the integral kernels of $(\lambda + L)^{-1}$, $G(\lambda) f$ and $K(\lambda) f$ by $R(\lambda, x' - y', x_n, y_n)$, $G(\lambda, x' - y', x_n, y_n)$ and $K(\lambda, x' - y', x_n, y_n)$, respectively.

Let $E(x_n, y_n) = E^{(1)}(x_n, y_n) + E^{(2)}(x_n, y_n)$ be defined as in the proof of Theorem 2.2. It holds that

$$\sup_{0 \le y_n \le a} \int_0^a E(x_n, y_n)\, dx_n + \sup_{0 \le x_n \le a} \int_0^a E(x_n, y_n)\, dy_n \le C \sigma(\lambda, \xi')^{-\frac{1}{2}}.$$

Therefore, by Lemmas 4.6–4.8 and 5.5, we have

$$\begin{aligned} \|\partial_x^k \widetilde{Q} (\lambda + L)^{-1} \widetilde{Q} f\|_1 &\le \sup\nolimits_{0 \le y_n \le a} \|\partial_x^k \widetilde{Q} R(\lambda, \cdot, \cdot, y_n)\|_1 \|\widetilde{Q} f\|_1 \\ &\le \tfrac{C}{(|\lambda|+1)^{1-\frac{k}{2}}} \|\widetilde{Q} f\|_1, \quad k = 0, 1, \end{aligned}$$

and

$$\begin{aligned} \|\partial_x^k \widetilde{Q} (\lambda + L)^{-1} \widetilde{Q} f\|_\infty &\le \sup\nolimits_{0 \le x_n \le a} \|\partial_x^k \widetilde{Q} R(\lambda, \cdot, x_n, \cdot)\|_1 \|\widetilde{Q} f\|_\infty \\ &\le \tfrac{C}{(|\lambda|+1)^{1-\frac{k}{2}}} \|\widetilde{Q} f\|_\infty, \quad k = 0, 1. \end{aligned}$$

Similarly one can estimate $Q_0(\lambda + L)^{-1} f$ to obtain the desired estimate.

We next consider $\partial_x Q_0 (\lambda + L)^{-1} \widetilde{Q} f$. We here estimate only $\partial_x Q_0 G(\lambda) Q_n f$. Since $\partial_{x'}^{\alpha'} G(\lambda) f = G(\lambda) \partial_{x'}^{\alpha'} f$, we see from Lemmas 4.6 and 5.5 that

$$\|\partial_{x'}^{\alpha'} Q_0 G(\lambda) Q_n f\|_p \le \frac{C}{|\lambda| + 1} \|\partial_{x'}^{\alpha'} Q_n f\|_p, \quad p = 1, \infty, \ |\alpha'| = 1.$$

Since $\partial_{x_n}^2 (g_{\mu_2}^{(D)} \widehat{f}) = \mu_2^2 g_{\mu_2}^{(D)} \widehat{f} - \widehat{f}$ and $\mu_2^2 = \frac{\lambda^2}{\nu_1 \lambda + \gamma^2} + |\xi'|^2$, we have

$$\begin{aligned} &\partial_{x_n} Q_0 G(\lambda) Q_n f \\ &\quad = -\tfrac{\gamma}{\nu_1 \lambda + \gamma^2} \left\{ \tfrac{\lambda^2}{\nu_1 \lambda + \gamma^2} Q_0 G(\lambda) Q_n f + \nabla' Q_0 G(\lambda) [\nabla' Q_n f] - Q_n f \right\}. \end{aligned}$$

It then follows from Lemmas 4.6 and 5.5 that

$$\|\partial_{x_n} Q_0 G(\lambda) Q_n f\|_p \le \frac{C}{|\lambda| + 1} \left\{ \|Q_n f\|_p + \|\nabla' Q_n f\|_p \right\}, \quad p = 1, \infty.$$

The remaining part of $\partial_x Q_0(\lambda+L)^{-1}\widetilde{Q}f$ can be estimated similarly. One can also estimate $\partial_x \widetilde{Q}(\lambda+L)^{-1}Q_0 f$ in a similar manner.

We next estimate $\partial_x Q_0(\lambda+L)^{-1}Q_0 f$. Consider first $Q_0 K(\lambda)Q_0 f$. We set

$$\widetilde{k}(\lambda, x', x_n, y_n) = \mathscr{F}^{-1}\left[h_{\mu_2}(\lambda, \xi', x_n)\beta_0(\lambda, \xi', y_n)\right](x'),$$

$$\widetilde{K}(\lambda)f = \textstyle\int_{\mathbf{R}^{n-1}} \widetilde{k}(x'-y', x_n, y_n)Q_0 f(y', y_n)\, dy' dy_n.$$

Then

$$Q_0 K(\lambda)Q_0 f = -\frac{\gamma}{\nu_1\lambda+\gamma^2}\nabla'\widetilde{K}(\lambda)\left[\nabla' Q_0 f\right].$$

It then follows from Lemmas 4.7, 4.8 and 5.5 that

$$\|\partial_{x'} Q_0 K(\lambda)Q_0 f\|_p \le \frac{C}{|\lambda|+1}\|\partial_{x'}\nabla' Q_0 f\|_p, \quad p = 1, \infty.$$

Also, since

$$Q_0 K(\lambda)Q_0 f = -\frac{\gamma}{\nu_1\lambda+\gamma^2}\widetilde{K}(\lambda)\left[\Delta' Q_0 f\right],$$

we similarly obtain, by Lemmas 4.7, 4.8 and 5.5,

$$\|\partial_{x_n} Q_0 K(\lambda)Q_0 f\|_p \le \frac{C}{|\lambda|+1}\|\Delta' Q_0 f\|_p.$$

As for $\partial_x Q_0 G(\lambda)Q_0 f$, one can treat it as in the case of $\partial_x \widetilde{Q}(\lambda+L)^{-1}\widetilde{Q}f$.

Let us finally consider $\partial_x \widetilde{Q}(\lambda+L)^{-1}f$, assuming that $\widetilde{Q}f\Big|_{x_n=0,a} = 0$. As for $\partial_x \widetilde{Q}(\lambda+L)^{-1}\widetilde{Q}f$, we rewrite it as in the proof of the last inequality of Theorem 2.1. One can then estimate it similarly to the case of $\partial_x^k \widetilde{Q}(\lambda+L)^{-1}\widetilde{Q}f$ with $k \le 1$ to obtain the desired inequality. The estimate for $\partial_x \widetilde{Q}(\lambda+L)^{-1}Q_0 f$ has been already obtained. This completes the proof.

We finally prove Theorems 2.5–2.7.

Proof of Theorems 2.5–2.7. We prove Theorems 2.5–2.7 for $\widetilde{\eta} = \min\{\eta_2, \eta_3\}$ and $\frac{\pi}{2} < \widetilde{\theta} < \min\{\theta_2, \theta_3, \theta_4\}$, where η_j $(j = 2, 3)$ and θ_j $(j = 2, 3, 4)$ are the numbers given in Lemmas 4.3 and 4.4. For this $\{\widetilde{\eta}, \widetilde{\theta}\}$ we see from Lemma 4.3 that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \ge r$. Furthermore, we deduce that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \Sigma(-\eta_2, \theta_2) \cap \Sigma(-\eta_3, \theta_3)$ and that $\overline{\Sigma(-\widetilde{\eta}, \widetilde{\theta}) - \Sigma(\eta_4, \theta_4)}$ is a compact set. Here η_4 and θ_4 are the numbers given in Lemma 4.4.

In view of the proof of Theorems 2.1–2.3, the desired estimates for $R^{(1)}(\lambda)f$ hold for $\lambda \in \Sigma(\eta_4, \theta_4)$.

In case $\lambda \in \Sigma(-\tilde{\eta}, \tilde{\theta}) - \Sigma(\eta_4, \theta_4)$, we decompose $R^{(1)}(\lambda)f$ in the following way. Let $\chi(\xi')$ and $\tilde{\chi}(\xi')$ be the cut-off functions given in the definition of $R^{(1)}(\lambda)f$ and in the proof of Theorem 2.1, respectively. We write $R^{(1)}(\lambda)f$ as

$$R^{(1)}(\lambda)f = \tilde{R}^{(2)}(\lambda)f + \tilde{R}^{(\infty)}(\lambda)f,$$

where

$$\tilde{R}^{(2)}(\lambda)f = \mathscr{F}_{\xi'}^{-1}\left[\chi(\xi')(1-\tilde{\chi}(\xi'))\hat{R}(\lambda,\xi')\hat{f}\right]$$

and

$$\tilde{R}^{(\infty)}(\lambda)f = \mathscr{F}_{\xi'}^{-1}\left[\chi(\xi')\tilde{\chi}(\xi')\hat{R}(\lambda,\xi')\hat{f}\right].$$

By Lemma 3.4 and Lemma 4.3 (ii) we see that $D(\lambda, \xi') \neq 0$ on the compact set $\overline{\Sigma(-\tilde{\eta}, \tilde{\theta}) - \Sigma(\eta_4, \theta_4)}$. Therefore, as above, one can see that $R^{(1)}(\lambda)f$ has the desired estimates for $\lambda \in \Sigma(-\tilde{\eta}, \tilde{\theta}) - \Sigma(\eta_4, \theta_4)$ as in the proof of Theorems 2.1–2.3. This completes the proof.

References

[1] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, J. Math. Soc. Japan **55** (2003), pp. 469–497.

[2] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer part 2, $\lambda = 0$ case, J. Math. Fluid Mech. **5** (2003), pp. 245–274.

[3] H. Abels and M. Wiegner, Resolvent estimates for the Stokes operator on an infinite layer, Differential and Integral Equations **18** (2005), pp. 1081–1110.

[4] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin, Heidelberg, New York (1976).

[5] Y. Kagei, Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer, preprint.

[6] Y. Kagei and T. Kobayashi, On large time behavior of solutions to the Compressible Navier-Stokes Equations in the half space in $\mathbf{R}^3$, Arch. Rational Mech. Anal. **165** (2002), pp. 89–159.

[7] Y. Kagei, and T. Kobayashi, Asymptotic behavior of solutions to the compressible Navier-Stokes equations on the half space, Arch. Rational Mech. Anal. **177** (2005), pp. 231–330.

[8] T. Kobayashi and Y. Shibata, Remarks on the rate of decay of solutions to linearized compressible Navier-Stokes equations, Pacific J. Math. **207** (2002), pp. 199–234.