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1 Introduction

The term earthquake is commonly used to describe sudden slip on a fault within the Earth that produces seismic waves. The inverse problem of earthquakes consists of analyzing records of seismic waves to obtain detailed information on the earthquake rupturing process. The solution of this problem is far from trivial for various reasons [1].

This paper deals with a simple model of earthquakes — called Random Domino Automaton (RDA) [2] — built on simple mechanism present in tectonic earthquakes, i.e those earthquakes that occur in response to plate motions (or other predominantly shearing sources). As stress builds up across a fault surface, friction on the fault prevents the surface from sliding until the strength (some maximum shear stress) of the fault is exceeded. Then the two sides of the fault snap back (elastic rebound), releasing some or all of the elastic strain built up. This slip occurs at several kilometers per second, releasing high-frequency seismic waves. These are by far the most common type of earthquakes and the most destructive [3].

The RDA model aims to reproduce some universal statistical temporal properties of earthquakes. It is a slowly driven system with avalanches in the form of stochastic cellular automaton, i.e. totally discrete (in space-time and in values) dynamical system. Depending on the parameters, it is able to produce a range of avalanche distributions [2]. It may be regarded as an extension of Drosel-Schwabl forest-fires model [4]. In its simplest version RDA was proposed to test the reconstruction procedure for the Ito equation [5, 6] and then was studied as a earthquakes model [7]. Inverse problem for finite RDA [8] was considered in the paper [9] using Markov Chains terminology. It's algebraic structure is related to Motzkin numbers [10], and recently a similarly constructed stochastic cellular automaton related to Catalan numbers was proposed [11]. Moreover, for a special choice of parameters, RDA exhibits two Self Organized Criticality - like states [12], and this property may help explain mega-earthquakes [13].

2 1D Random Domino Automaton model

Particles occupy the sites (cells) of a finite subset of 1-dimensional lattice $\Omega \subset \mathbb{Z}$. Sites may be vacant or occupied by a single particle (ball). The particles are indistinguishable. The number of sites is denoted by $N = |\Omega|$, and periodic boundary conditions are assumed.

Clusters are formed by sequentially occupied cells, and they are are separated by empty clusters. Size of a cluster (or an empty cluster) is equal to the number of occupied (or empty) cells contained in it. The number of clusters of size i in the system is denoted by n_i, and the number of empty clusters of size i by n_i^0. The density of the system ρ and the total number of clusters n read

$$\rho = \frac{1}{N} \sum_{i \geq 1} i n_i, \quad \text{and} \quad n = \sum_{i \geq 1} n_i. \tag{1}$$

The last one is equal to the total number of empty clusters n^0, assuming periodic boundary conditions and also, that there are at least one cluster and one empty cluster present in the system.

There are three kinds of empty cells, distinguished according to the number of occupied nearest neighbors. An empty cell may be the nearest neighbor for 0,1 or 2 occupied cells. Total numbers of empty cells of these three kinds are denoted by x_0, x_1 and x_2 respectively, and thus

$$\sum_{i \geq 1} i n_i^0 = x_0 + x_1 + x_2 = (1 - \rho) N. \tag{2}$$

A given empty cell may change its state to occupied — depending on the number of its occupied nearest neighbors, it creates a new cluster (of size 1), or enlarges an existing adjacent cluster, or merges two adjacent clusters. Thus we call respective empty cells creating, enlarging and merging.

From the above definitions of x_i, $i = 0, 1, 2$, it follows

$$x_0 = \sum_{i \geq 3} (i - 2) n_i^0 = (1 - \rho) N - 2n + n_1^0, \tag{3}$$

$$x_1 = 2 \sum_{i \geq 2} n_i^0 = 2(n - n_1^0), \tag{4}$$

$$x_2 = n_1^0. \tag{5}$$

The first equation 3 is a consequence of counting empty cells from "interiors" of empty clusters and the second one (4) follows from counting of remaining edge cells. We point out the identity

$$n = \frac{1}{2} \left(x_1 + 2x_2 \right), \tag{6}$$

which reflects the simple fact, that each cluster has two empty cells as nearest neighbors. These cells may be enlarging or merging cells only, and each merging cell is a neighbor for two clusters. This constraint (6) follows from equations (4) and (5).

Discrete time dynamics is defined as follows. In each time step a particle is added to the system, and it hits one cell. Assume, that each cell — occupied or not, and anywhere located — has the same probability to be hit.

- If the site receiving a new particle is empty, then there three various actions are possible: creation of a new cluster, or enlarging of adjacent cluster, or merging two adjacent clusters, depending of the type of the empty cell. We set constant values of probabilities for this actions: c_0, c_1 and c_2 accordingly. The incoming ball may be also scattered away with probability $(1 - c_0)$, or $(1 - c_1)$, or $(1 - c_2)$, respectively.

- If the cell receiving the ball is occupied, and belongs to the cluster of size i, the whole cluster is removed with probability $\mu(i) = \mu_i$ depending of the size i of the cluster. Otherwise the ball is scattered with probability $(1 - \mu_i)$.

The dependence of the avalanche probability μ_i of the size of the hit cluster distinguish essentially RDA from Drossel-Schwabl forest-fires model [4], and consequences of this generalization are far reaching.

The RDA is a Markov chain, and its space of states is irreducible, aperiodic and recurrent. Thus statistically **stationary state** is well defined, and it is possible to derived respective balance equations by using "flow in = flow out" principle and counting respective probabilities. Below we present the balance equations for ρ, n, x_i and n_i for statistically stationary state of the automaton in mean field approximation for the special choice $c_0 = c_1 = c$.

The balance equation for density ρ is

$$cx_0 + cx_1 + c_2x_2 = \sum_{i\geq 1} \mu_i n_i i^2. \tag{7}$$

The balance equation for n is

$$cx_0 - c_2x_2 = \sum_{i\geq 1} \mu_i n_i i. \tag{8}$$

These two equations are exact. The following equations: for creating cells x_0, enlarging cells x_1 and merging cells x_2 use the mean field approximation and are

$$\begin{aligned} 3cx_0 &= \sum_{i\geq 1} \mu_i n_i i^2 + \frac{x_1}{n} \sum_{i\geq 1} \mu_i n_i i, & (9)\\ 2cx_0 - 2cx_1 &= \frac{x_1 - 2x_2}{n} \sum_{i\geq 1} \mu_i n_i i, & (10)\\ cx_1 - c_2 x_2 &= \frac{2x_2}{n} \sum_{i\geq 1} \mu_i n_i i. & (11) \end{aligned}$$

Remark 1. Not all of the equations derived above are independent. Because of the relation (2), a combination of equations (9), (10), (11) gives the equation (7) for the density ρ. The relation (6), imply that a combination of equations (10) and (11) must be consistent with the equation (8) for the total number of clusters n.

The balance equations for n_i's are

$$n_1 = \frac{1}{\mu_1 + Y} \left(c_0 x_0\right), \qquad (12)$$

$$n_2 = \frac{1}{2\mu_2 + Y} \left(c_1 x_1 \frac{n_1}{n}\right), \qquad (13)$$

$$n_i = \frac{1}{i\mu_i + Y} \left(c_1 x_1 \frac{n_{i-1}}{n} + c_2 x_2 \sum_{k=1}^{i-2} \frac{n_k}{n} \frac{n_{i-k-1}}{n}\right) \quad \text{for } i \geq 3, \qquad (14)$$

where

$$Y = \frac{1}{n} \left(c_1 x_1 + 2c_2 x_2\right).$$

Remark 2. Equations (12)–(14) sum up to balance equation (7) for ρ.

3 The inverse problem for RDA

The inverse problem for RDA is to find the rebound parameters (probabilities μ_i and c_i) that result in the given stationary distribution w_i of avalanches. Here w_i is relative frequency of appearance of an avalanche of a given size i depending of its size.

The probability of avalanche of size i is proportional to the number of cells contained in clusters of size i (i.e. $i \cdot n_i$) and respective rebound parameter μ_i, thus

$$w_i \sim \mu_i n_i i, \qquad \text{or} \qquad w_i := \frac{\hat{\mu}_i \hat{n}_i i}{\sum_i \hat{\mu}_i \hat{n}_i i}, \qquad (15)$$

where we assume a normalization $\sum_i w_i = 1$.

A key observation for solving this problem is, that it is possible to express parameters μ_i as functions of n_i, c_i and x_i, as can be seen from the form of equations (12)–(14). Also removing fractions from these equations leads to relation between w_i and n_i.

The details of solving inverse problem will be presented in another paper. Here we present just an example of the application of the developed procedure for an exponential distribution of avalanches in the form

$$w_i := \frac{1}{2} \cdot \left(\frac{2}{3}\right)^i . \tag{16}$$

The coefficient 1/2 is normalization constant, $\sum w_i = 1$, and average size of avalanche $\eta =< i_w >= 3$. The value of I is set to 1. The density is $\rho \approx 0.41289$, and $\hat{x}_0 = 1.4$, $\hat{x}_1 = 0.8$, $\hat{x}_2 = 1.2$. The Figure 1 presents calculated distributions of $\hat{n}_i$ and $\hat{\mu}_i$. The value of $\hat{c}$ is equal to 1. All the calculations performed for this example were exact, and reconstructed distribution of w_i from calculated values of $\hat{\mu}_i$ and $\hat{c}$ coincide with (16) exactly. The procedure is working exactly.

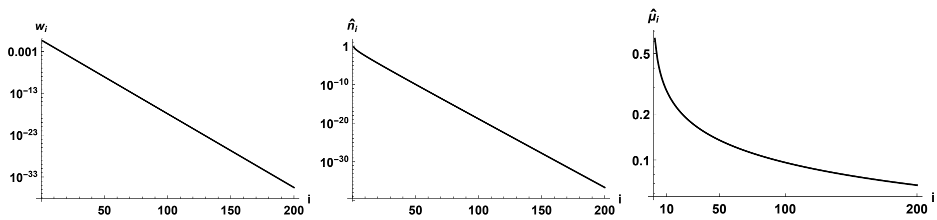

Figure 1: Inverse problem for an exponential distribution of avalanches. For the given distribution of avalanches (left), it was reconstructed the distribution of clusters (middle) and respective rebound parameters (right), that produce exactly the staring distribition of avalanches — these two curves overlap.

4 Conclusion

The developed procedure of reconstruction of rebound parameters based on a distribution of avalanches proved to work exactly in the simple case of exponential distribution. Nevertheless, the procedure is general and works also for other distributions, like resembling Gutenberg-Richter law, which will be presented in detail in the future.

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