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Abstract. We consider the Labyrinth model, which is a separable two-dimensional quasicrystal model. The spectrum of the Labyrinth model is given by products of two Cantor sets. We will show that the spectrum of the Labyrinth model is an interval for coupling constants sufficiently close to 1.

1. Products of two Cantor sets

For any gap U of a Cantor set, we denote the right (resp. left) endpoint of U by U^R (resp. U^L). If gaps U_1, U_2 satisfy $U_1^R < U_2^L$, we denote $U_1 < U_2$.

Definition 1.1. Let K be a Cantor set. We define the *thickness* of K by

$$\inf_{U_1<U_2} \max\left\{\frac{U_2^L - U_1^R}{|U_1|}, \frac{U_2^L - U_1^R}{|U_2|}\right\},$$

where the infimum is taken for all pairs of gaps of K, with at least one of them being a finite gap. We denote this value by $\tau(K)$.

In [**1**], the author considered products of two Cantor sets and obtained the optimal estimate in terms of thickness that $K \cdot L$ is an interval. We call K a 0-*Cantor set* if $K_+, K_- \neq \phi$, $\inf K_+ = 0$ and $\inf K_- = 0$.

Theorem 1.1 (Theorem 1.4 in [**1**]). *Let K, L be 0-Cantor sets. Then, if*

$$2\left(\tau(K)+1\right)\left(\tau(L)+1\right) \leqslant \left(\tau(K)\tau(L)-1\right)^2,$$

$K \cdot L$ is an interval. In particular, if

$$\tau(K) = \tau(L) \geqslant 1+\sqrt{2},$$

then $K \cdot L$ is an interval. Furthermore, for any $M, N > 0$ with

$$2(M+1)(N+1) > (MN-1)^2,$$

there exist 0-Cantor sets K, L such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of two intervals.

2. Spectrum of the Labyrinth model

Let $\lambda > 1$ and define

$$\omega_\lambda(n) = \begin{cases} \lambda & \text{if } n\alpha \mod 1 \in [1-\alpha, 1) \\ 1 & \text{if } n\alpha \mod 1 \in [0, 1-\alpha), \end{cases}$$

where $\alpha = \frac{-1+\sqrt{5}}{2}$ is the inverse of the golden mean. The *Labyrinth model* is given by the following self-adjoint operator that acts on $\ell^2(\mathbb{Z}^2)$:

$$\begin{aligned}(H_{\lambda_1,\lambda_2}\psi)(m,n) = \omega_{\lambda_1}(m+1)\omega_{\lambda_2}(n+1)\psi(m+1,n+1) \\ + \omega_{\lambda_1}(m+1)\omega_{\lambda_2}(n)\psi(m+1,n-1) \\ + \omega_{\lambda_1}(m)\omega_{\lambda_2}(n+1)\psi(m-1,n+1) \\ + \omega_{\lambda_1}(m)\omega_{\lambda_2}(n)\psi(m-1,n-1).\end{aligned}$$

In [**2**], the author proved the following:

Theorem 2.1 (Theorem 1.1 in [**2**]). *The spectrum $\sigma(H_{\lambda_1,\lambda_2})$ is given by products of two Cantor sets, and is an interval for sufficiently large λ_1, $\lambda_2 > 1$.*

References

[1] Y. Takahashi, Products of two Cantor sets, *Nonlinearity* **30** (2017), 2114–2137.

[2] Y. Takahashi, Quantum and spectral properties of the Labyrinth model, *J. Math. Phys.* **57**, 2016.