A Simple Linear Regression Approach to Structural Change: A Note on the Initial Value of Kalman Filter Algorithm

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A Simple Linear Regression Approach to Structural Change

—A Note on the Initial Value of Kalman Filter Algorithm—

Kosuke Oya

After briefly reviewing some aspects of economic structural change analysis with linear regression model, a simple linear regression model with time varying parameters that is supplement of structural change analysis is discussed. Specially, the problems relating to the initial value of Kalman Filter algorithm used for the recursive estimation of linear regression model recursively is discussed.

1. Introduction

There are many works on economic structural change analysis with linear regression model. These works have made the relation of variables in the model fixed during periods in which data are available. Namely, structural parameters in the model are constant. The reason why we have taken such treatment is that we have to specify the model, that is a replication which reflects a real structure or an approximation of it, to extract the economic mechanism from data and realize the relation of economic variables. Even if the model including the mechanism of structural change could be specified, the model itself is still fixed. Therefore the analysis with linear regression model is valid only when structure of the model coincides with a real economic structure or is an approximation of it. Further, prediction with that model is valid, only when the same prerequisite is satisfied during prediction term.

The problem is which kind of economic structural changes have occurred. The economic structural change has two contrastive types. One is a qualitative economic structural change. Generally the model used for the analysis about quailtative one is thought that variables in the model should be changed, if the model doesn't reflect a real economic structure well. The other is a quantitative one. The model used for the analysis about quantitative one is thought that variables in the model should not be changed, even if it doesn't reflect well. The former model has fixed parameters, the latter model has variable parameters (time varying parameter regression model).

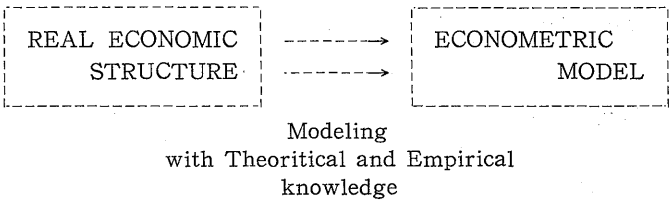

Figure 1

See following Figure 1. If some structural change occures after the modeling of the real economic structure has been completed, the econometric model that has been constructed with the knowledge of the real economic structure before some economic structural change occures is no longer valid. In view of the qualitative economic structural change, the variables in the model should be changed. In view of the quantitative one, they should not be changed, the parameters (coefficients) of variables in the model should be changed. In this paper, an approach to the quantitative economic structural change will be given based upon a linear regression model with time varying parameters. In section 2, the model will be presented and discussed. Section 3 will discusse a method for avoiding the initial value problem of Kalman Filter algorithm. Section 4 will give a simple economic application. The final section gives summary.

2. General formulation of the model

An approach to analyze the quantitative economic structural change regarding econometric model is based upon recursive estimation or Bayesian analysis.[1] In this paper, recursive estimation is discussed. Recursive estimation for the quantitative economic strucutrual change analysis can be derived as the state space model by Kalman Filter algorithm. The Kalman Filter algorithm was derived in Kalman (1960) and has been an important technique in modern control theory. Linear econometric regression models with time varying parameters can be viewed as a special case of state space model as will be shown below. The state space model consists of two sets of equations: 'transition' equations and 'measurement' equations. The linear regression equations are viewed as 'measurement' equations. The equations of time varying parameters are viewed as 'transition' equations. To facilitate analysis, a simple regression equation is considered as follows:

$$y_t = X_t \beta_t + e_t, \qquad t=1,2,\ldots,T \quad (2.1)$$

where y_t is a scalar endogenous variable, X_t is a k-vector of exogenous variables, e_t is a scalar disturbance term and β_t is a k-vector of unknown parameters. β_t is supposed to follows as AR(1) process scheme

$$\beta_t = \Phi \beta_{t-1} + u_t \quad (2.2)$$

where u_t is a k-vector of disturbance term, Φ is a transition$(k \times k)$matrix that its i-th diagonal element is ϕ_{ii}, $|\phi_{ii}| \leq 1$[2] and all off-diagonal elements are zero. Assumptions on the disturbances are as follows:

$e_t \sim N(0,R)$

$u_t \sim N(\mathbf{o},Q)$

$E(e_t e_s)=0, t \neq s$

$E(e_t u_t)=\mathbf{o}$

$E(u_t u_s')=O, t \neq s$

O is the k×k zero matrix, **o** is the k×1 zero vector.

The 'measurement' equation(2.1) and the 'transiton' equation(2.2) contain unknown parameters as follows:

β_t;k-vector of coefficients of exogenous variables

R; variance of e_t

Φ; transiton matrix

Q; variance matrix of u_t

To estimate the unknown parameters, Generalized Least Squares (GLS) method[3)] is basically used, because Q—the variance matrix of the error term—has no longer homoscedasticity. Then recurcive estimation is derived as Kalman Filter algorithm which produces recursive variance of the estimation error of the state variables and recursive estimators of the coefficients of exogenous variables β_t (the estimators of the state variables). This equivalance between GLS and Kalman Filter algorithm was discussed in Sant (1977). A method for avoiding the estimation of the initial value of Kalman Filter algorithm is presented in the next section.

3. Kalman Filter algorithm and its initial value problem

Let us consider β_t—the k-vector of unknown coefficients—as a state variable. The Kalman Filter algorithm can be specified as follows.

(1) $b_{t/t}=b_{t/t-1}+K_t\mu_t$

(2) $b_{t/t-1}=\Phi b_{t-1/t-1}$

(3) $\mu_t=y_t-X_t b_{t/t-1}$

(4) $S_t=X_t P_{t/t-1}X_t'+R$

(5) $K_t=P_{t/t-1}X_t'S_t^{-1}$

(6) $P_{t/t}=[I-K_t X_t]P_{t/t-1}$

(7) $P_{t/t-1}=\Phi P_{t-1/t-1}\Phi+Q, \quad t=1,2,\ldots,T$

$b_{t/s}$; the minimum mean square estimator of β_t at period s, given by the conditional expected value of β_t, $E[\beta_t \mid y_1,y_2,\ldots,y_s]$

K_t ; the kalman gain at period t

μ_t ; the innovation at period t

S_t ; the variance of the innovation at period t

$P_{t/s}$; the variance matrix of the estimation error of β_t at period s

I ; the $k\times k$ identity matrix

Problem of adapting Kalman Filter algorithm to recursive estimation of unknown parameter vector β_t is that the hyper-parameters (initial values of this algorithm, i. e. the values of the variance of the disturbance term of 'measurment' equation (2.1) and 'transition' equation (2.2), the transition matrix of (2.2)) are unknown.

For using the Kalman Filter algorithm to estimate the unknown parameter vector β_t recursively, the hyper-parameters should be given. To identify this algorithm, we should estimate the hyper-parameters using data information without arbitrariness. The likelihood function is used to estimate the hyper-parameters. Adapting the Kalman Filter algorithm to the recursive estimation of unknown parameter vector β_t, we can get the log likelihood function as follows.[4)]

$$L(\theta)=\text{constant}-\frac{1}{2}\sum_{t=1}^{T}\left[\log S_t+\frac{\mu_t^2}{S_t}\right] \quad (3.1)$$

where θ is unknown hyper-parameters, Φ,R,Q,$b_{0/0}$ and $P_{0/0}$.

By maximuizing (3.1) with respect to the unknown hyper-parameters, we can get the maximum likelihood estimators of them to identify the Kalman Filter algorithm. And the estimator of the state variable $b_{t/t}$ can be derived from that algorithm. Among the hyper-parameters, $b_{0/0}$ and $P_{0/0}$ —the initial values of the algorithm—are necessary only to derive the estimator of the state variable $b_{t/t}$ And the estimator of the state variable $b_{t/t}$ is very sensitive to the initial values. Hence in this paper, regarding the initial values of the Kalman Filter algorithm as the 'nuisance' parameters, the approach without the estimators of the initial values is specified.[5)]

To carry out this approach, the log likelihood function (3.1) is concentrated with respect to $b_{0/0}$. Ignoring the constant term and rewriting (3.1) as (3.2)

$$L(\theta)=-\frac{1}{2}\sum_{t=1}^{T}\left[\log(X_tP_{t/t-1}X_t'+R)\quad\frac{(y_t-X_tb_{t/t-1})^2}{(X_tP_{t/t-1}X_t'+R)}\right]\qquad(3.2)$$

From the Kalman Filter algorithm (1), (2), (3)

$$b_{t/t-1}=A_t+B_t\Phi b_{0/0},^{6)}\qquad t=1,2,\ldots,T\qquad(3.3)$$

where $A_t=\Phi K_{t-1}y_{t-1}+\sum_{L=2}^{t-1}[\prod_{M=L}^{t-1}\Phi(I-K_MX_M)\Phi K_{L-1}y_{L-1}]$

$$B_t=\prod_{M=1}^{t-1}\Phi(I-K_MX_M)$$

$A_1=\mathbf{o}$

$B_1=I$

$\sum_{L=2}^{t-1}(\cdot)=\mathbf{o}$, for $t-1<2$

$\prod_{M=1}^{T}(\cdot)$; a premultiplying operator in ascending subscript order.

e. g. $\prod_{M=1}^{T}z_M=z_Tz_{T-1}\cdots z_1$

Inserting (3.3) into (3.2) and differentiating (3.2) with respect to $b_{0/0}$ yields

$$S(b_{0/0})=\frac{\partial L}{\partial b_{0/0}}=-\sum_{t-1}^{T}\left[\frac{\Phi B_t' X_t' X_t A_t}{(X_t P_{t/t-1} X_t'+R)}+\frac{\Phi B_t' X_t' X_t B_t \Phi b_{0/0}}{(X_t P_{t/t-1} X_t'+R)}-\frac{\Phi B_t' X_t' y_t}{(X_t P_{t/t-1} X_t'+R)}\right]$$

The ML estimator $b_{0/0}$ can be obtained as a solution of $s(b_{0/0})=0$. Hence, we have (3.4) as

$$\hat{b}_{0/0}=\Phi^{-1}\left[\sum_{t=1}^{T}\frac{B_t' X_t' X_t B_t}{(X_t P_{t/t-1} X_t'+R)}\right]^{-1}\left[\sum_{t=1}^{T}\frac{\Phi B_t' X_t'(y_t-X_t A_t)}{(X_t P_{t/t-1} X_t'+R)}\right] \quad (3.4)$$

Inserting (3.4) into (3.3), we have

$$\hat{b}_{t/t-1}=A_t+B_t\Phi\hat{b}_{0/0} \quad (3.5)$$

Inserting (3.5) into (3.2) yields the concentrated log likelihood function (3.6) as follows:

$$L(\theta^*)=-\frac{1}{2}\sum_{t=1}^{T}\left[\log(X_t P_{t/t-1} X_t'+R)+\frac{(y_t-X_t\hat{b}_{t/t-1})^2}{(X_t P_{t/t-1} X_t'+R)}\right] \quad (3.6)$$

where θ^* is Φ, R, Q and $P_{0/0}$.

Here regarding $b_{0/0}$ as an unknown constant, $P_{0/0}$—the variance matrix of estimation error of β_0—can be equal to zero matrix.[7] The derived concentrated log likelihood function (3.6) is function of Φ, R and Q. Therefore we can get the estimators of unknown hyper-parameters except for the initial values of the Kalman Filter by maximizing the concentrated log likelihood function (3.6) with respect to the initial values and derive the estimator of the state variable β_t from the Kalman Filter algorithm.

4. Economic application

In this section, the approach discussed above will be applied to the recursive estimation of the Phillips curve in Japan. This application is only to illustrate the behavior of recursive estimates of the cofficients of exogenous variables not to test the model, namely not to test the natural rate of unemployment hypothesis.

Let us consider the Phillips curve model in Japan as

$$\dot{w}_t = \alpha_t + \beta_t \left(\frac{1}{U_t}\right) + \gamma_t \dot{CPI}_t \qquad (4.1)$$

where $\dot{w}_t$, u_t and $\dot{CPI}_t$ are rate of change of wage index (all industry), unemployment rate and rate of change of consumer price index. Type of data is annual. See following Table 1.

Table 1 Data of $\dot{w}_t$, $\frac{1}{U_t}$ and $\dot{CPI}_t$

YEAR	$\dot{W}_t$	$1/U_t$	$\dot{CPI}_t$	YEAR	$\dot{W}_t$	$1/U_t$	$\dot{CPI}_t$
1953	15.33	0.54	6.69	1970	16.86	0.87	7.71
1954	7.63	0.44	6.42	1971	14.74	0.82	6.09
1955	4.65	0.40	−1.07	1972	15.89	0.72	4.52
1956	7.59	0.44	0.30	1973	21.72	0.79	11.77
1957	4.23	0.52	3.23	1974	26.62	0.73	24.44
1958	3.12	0.49	−0.47	1975	13.95	0.53	11.81
1959	6.06	0.45	1.05	1976	12.45	0.50	9.33
1960	7.33	0.62	3.64	1977	8.59	0.49	7.99
1961	11.09	0.70	5.34	1978	6.23	0.45	3.86
1962	9.98	0.78	6.74	1979	6.33	0.48	3.56
1963	10.46	0.78	7.66	1980	6.65	0.50	8.05
1964	10.19	0.85	3.82	1981	5.59	0.45	4.92
1965	9.43	0.82	6.69	1982	4.45	0.43	2.63
1966	10.69	0.76	5.06	1983	3.85	0.38	1.83
1967	11.87	0.79	3.96	1984	4.53	0.37	2.26
1968	13.74	0.85	5.37	1985	3.59	0.38	2.02
1969	15.72	0.89	5.18				

Note: w_t = wage index, $\dot{w}_t = (w_t - w_{t-1})/w_{t-1}$, CPI_t = consumer price index, $\dot{CPI}_t = (CPI_t - CPI_{t-1})/CPI_{t-1}$. w_t, u_t and CPI_t are taken from NEEDS (brought out by the NIHON KEIZAI).

To compare the analysis by the time varying parameter regression model with the analysis by a linear regression with constant parameters, we estimate the equation (4.1) by ordinary least squares method as follows:

$$\dot{w}_t = \underset{(-1.57)}{-2.503} + \underset{(5.07)}{13.813}\left(\frac{1}{U_t}\right) + \underset{(7.19)}{0.754}\ \dot{CPI}_t \qquad (4.2)$$

$R^2 = 0.802$, $SE = 2.524$,

where R^2=adjusted coefficient of determination, SE=standard error, t-value in parenthesis.

We regard the equation (4.1) as the 'measurement' equation and postulate that the coefficients of the equation (4.1) follow an AR(1) process as the 'transition' equation. To facilitate this analysis, the transition matrix is supposed to be equal to identity matrix. Hence although the stochastic process of the coefficients is non-stationary, it can be considered as stationary, if we think that the stochastic process starts nearly before the observations are available.

In the concentrated log likelihood function (3.6), the unknown hyper-parameters are R and Q. Note that the equation (4.1) has an intercept term. The disturbance term of the equaton (4.1) can be combined with the disturbance term of the intercept term to form a disturbance term. Therefore it is sufficient for us to estimate its variance. So we can get R, that is variance of the disturbance term of the equation (4.1), equal to zero.

Firstly, we estimate the unknown hyper-parameter Q using the concentrated log likelihood function (3.6). Secondly the Kalman Filter algorithm is identified and the state variables, namely the coefficients of exogenous variables are derived from that algorithm.

The estimated hyper-parameter Q is as follows:

$$Q=\begin{bmatrix} 2.73 & 0 & 0 \\ 0 & 1.71 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}$$

Table 2 shows the estimates of coefficients of exogenous variables.

Using the estimates of the coefficients and hyper-parameters, we apply the well known method of smoothing[8]. The method applied in this section is called the fixed-interval smoothing algorithm. This algo-

Table 2 The estimates of coefficients

YEAR	α_t	β_t	γ_t	YEAR	α_t	β_t	γ_t
1953	4.56	13.11	0.55	1970	4.14	12.35	0.26
1954	−0.16	12.07	0.38	1971	3.17	12.16	0.27
1955	0.19	12.14	0.36	1972	6.37	11.93	0.21
1956	1.92	12.73	0.39	1973	6.04	12.08	0.52
1957	−1.71	10.77	0.11	1974	6.04	12.31	0.48
1958	−2.04	10.68	0.13	1975	1.29	13.52	0.47
1959	0.86	11.03	0.24	1976	1.35	13.53	0.47
1960	0.36	10.28	0.17	1977	−1.09	12.85	0.42
1961	1.61	11.33	0.29	1978	−1.12	12.85	0.42
1962	0.65	10.34	0.19	1979	−1.26	12.73	0.42
1963	0.80	10.41	0.20	1980	−1.70	12.53	0.26
1964	0.65	10.25	0.22	1981	−1.33	12.53	0.25
1965	0.16	10.14	0.14	1982	−1.56	12.51	0.26
1966	2.04	10.52	0.12	1983	−1.31	12.44	0.26
1967	2.70	11.01	0.11	1984	−0.72	12.50	0.28
1968	3.11	11.52	0.17	1985	−1.59	12.08	0.28
1969	3.84	12.30	0.18				

Table 3 The estimates of the coefficients by the smoother

YEAR	α_t	β_t	γ_t	YEAR	α_t	β_t	γ_t
1953	2.89	12.82	0.82	1970	0.50	11.29	0.85
1954	−1.56	11.32	0.65	1971	0.44	11.28	0.84
1955	0.71	11.82	0.73	1972	3.76	11.94	0.79
1956	2.11	12.12	0.65	1973	2.75	12.01	0.81
1957	−2.98	10.40	0.56	1974	2.96	11.92	0.61
1958	−1.94	10.96	0.64	1975	2.96	13.26	0.34
1959	0.33	11.59	0.50	1976	2.67	13.34	0.34
1960	−1.31	11.39	0.44	1977	−0.12	13.07	0.28
1961	−0.55	12.54	0.54	1978	−0.59	12.94	0.27
1962	−1.80	11.50	0.42	1979	−0.89	12.94	0.28
1963	−1.76	11.53	0.42	1980	−1.68	12.30	0.27
1964	−1.45	11.66	0.45	1981	−1.56	12.35	0.32
1965	−2.81	10.25	0.57	1982	−1.64	12.35	0.32
1966	−1.40	10.56	0.79	1983	−1.37	12.37	0.31
1967	0.15	10.87	0.78	1984	−0.66	12.17	0.30
1968	0.30	10.98	0.77	1985	−1.59	12.08	0.28
1969	0.88	11.70	0.86				

rithm gives better estimators than the estimators by Kalman Filter algorithm, if the model which these algorithms are applied to is valid. Table 3 shows the estimates of the parameters by the smoother. The rate of change of wage index and its estimate by the Kalman Filter

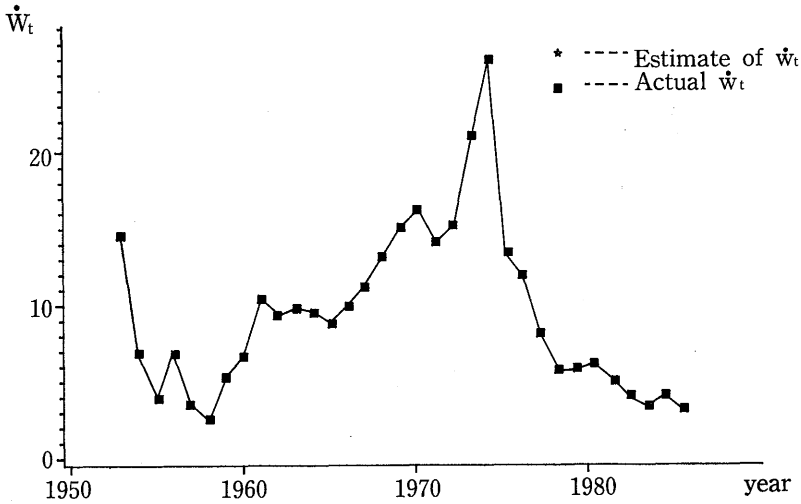

Figure 2-1

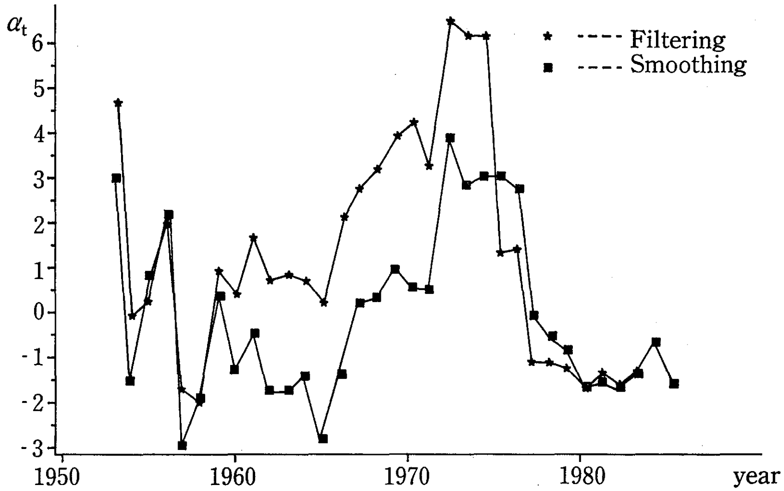

Figure 2-2

algorithm are presented in Figure 2-1. Figure 2-2, Figure 2-3 and Figure 2-4 show the two kinds of estimates of the intercept of the equation (4.1), the coefficient of the reciprocal of unemployment rate and the coefficient of the rate of change of consumer price index by the

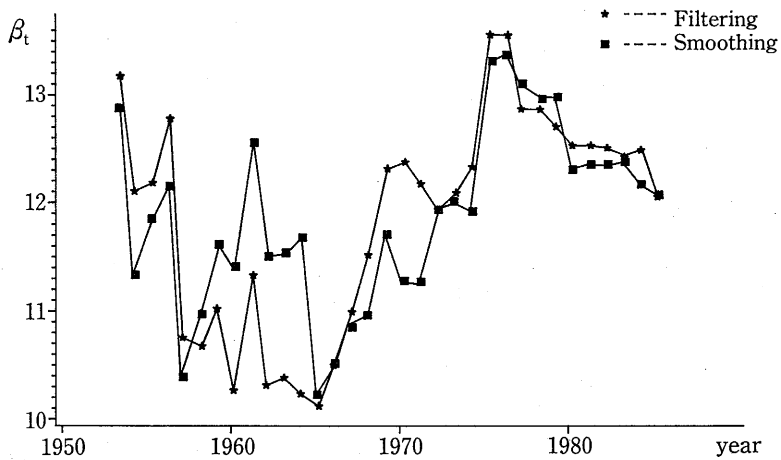

Figure 2-3

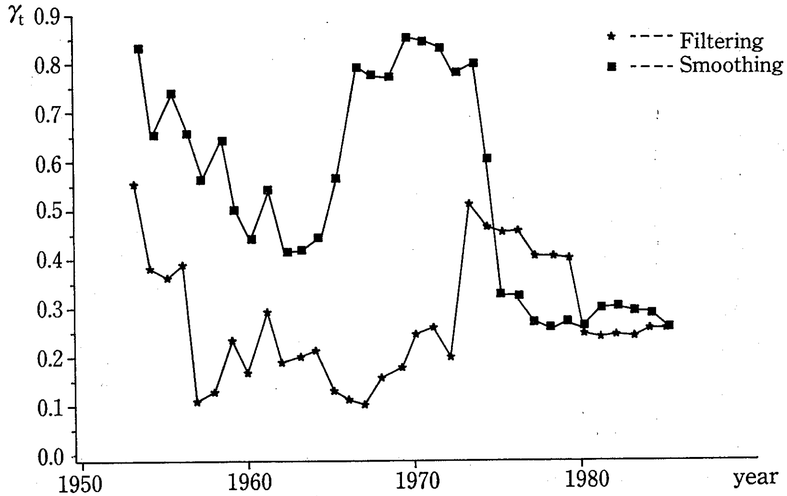

Figure 2-4

Kalman Filter algorithm and smoother respectively. Both estimates by the Kalman Filter algorithm and smoother are appropriate for illustrating the behavior of coefficients of the exogenous variables in the equation (4.1).

In this paper, concerning to find the maximum likelihood estimators of the hyper-parameters, we use the optimization of nonlinear programming using penalty functions and complex method[9].

5. Summary and Conclusions

We have divided the economic structural change into two types, one is the qualitative economic structural change and the other is the quantitative one. Although both types of economic structural change have been analyzed with usual regression model that has constant coefficients, the quantitative economic structural change should not be analyzed with usual regression model because the usual regression model can not capture the change of parameters in the model. For analyzing the quantitative economic structural change, we have derived the recursive estimation method by Kalman Filter algorithm without estimation of its initial values.

We obtain the following conclusions.

(1) There are two types of economic structural change. Specially for the analysis of quantitative economic structural change the recursive estimation method or the Bayesian method is suitable.

(2) The initial values of the Kalman Filter algorithm are unneccessary for the quantitative economic structural change analysis, because we can specify the concentrated log likelihood function

with respect to those values.

(3) We should derive a criterion to distinguish between the qualitative economic structural change and the quantitative one.

Notes

1) See reference [3], [7], [9].

2) If $|\phi_{ii}|=1$, then the stochastic process of β_{it}, i.e. the i-th element of β_t, is non-stationary.

3) The unknown parameter vetor β_t of the equation (2.1), (2.2) is taken to be stochastic, not constant. Hence GLS estimator is the best within the class of estimators which are linear and unconditionally unbiased.
An estimator is unconditionally unbiased (u-unbiased) if its estimation error has zero expectation.

4) The likelihood function of the unknown parameters as shown in Harvey (1981) can be specified as constitution of the decomposed joint distribution of T independent prediction errors (innovations).

5) The hyper-parameters estimation method using panel data was suggested in Rosenberg (1973), Liu and Tiao (1980).

6) See appendix of reference [6].

7) The concentrated log likelihood function (3.6) is exact unless $P_{0/0}$ is equal to zero matrix.

8) See reference [1].

9) See reference [5].

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