Effective Demand and Prices: Variations on a Theme by Sraffa

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Effective Demand and Prices

——Variations on a Theme by Sraffa——

Seiji Nagata

Contents

Introduction

This paper deals with a problem concerning the determination of both output and prices, paying attention to the conditions of reproduction. Given a semi-positive surplus product vector, a positive output vector is uniquely determined through L-System. On the other hand, for the determination of prices, we have only to give a nonnegative labour share in S- or R-System; then relative prices, including the rate of profit, are uniquely determined.

Thus, L-System seems to have very little connection with S- or R-System directly. But once we synthesize both systems, making use of M-System as a coupler, in consideration of expectations and realization, these systems develop into K-System. In K-System, given an expected rate of profit r^e, an output vector is uniquely determined in the consequence of anticipative production intended by capitalist, and so employment and the working day are determined so as to be compatible with the output determined in that way; next, the working day determines the rate of exploitation and the labour share, therefore ultimately determines relative prices and the realized rate of profit r. This process is illustrated with the Figure 1 below.

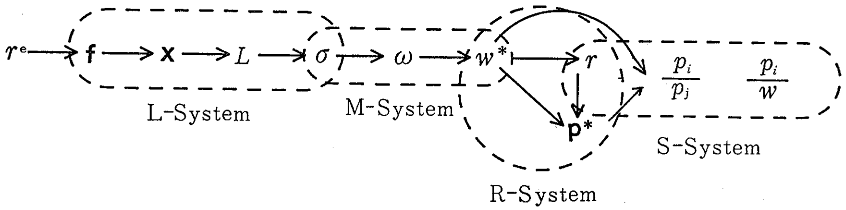

Figure 1. Determination of Variables in K-System

The symbols used here are defined as follows

r^e : the expected rate of profit,

$\mathbf{f}$: the surplus product vector,

$\mathbf{x}$: the output vector,

L : the quantity of employed labour,

σ : the working day,

ω : the rate of exploitation,

w^* : the labour share in R-System,

r : the realized rate of profit,

$\mathbf{p}^*$: the price vector in R-System,

p_i : the price of the ith commodity,

w : the wage rate.

§1. Assumptions and Some Basic Propositions

Let **x**, **A**, $\mathbf{a}_0$, and **c** be as follows;

x : the output vector,

$\mathbf{x}=[x_i]$,

x_i = the output of the ith commodity,

A : the input-output matrix,

$\mathbf{A}=[a_{ij}]$,

a_{ij} = the quantity of the ith commodity required by the jth industry as input for the production of one unit of the jth commodity,

$\mathbf{a}_0$: the labour input coefficient vector,

$\mathbf{a}_0=[a_{0j}]$,

a_{0j} = the quantity of labour required by the jth industry as input for the production of one unit of the jth commodity,

c : the necessary consumption vector,

$\mathbf{c}=[c_i]$,

c_i = the quantity of the ith commodity required for the reproduction of one unit of labour power.

The following four assumptions are laid down.

Assumption 1. $\mathbf{A}\geqq\mathbf{0}$ and **A** is indecomposable[1). (1. 1)

Assumption 2. $\mathbf{a}_0\geq\mathbf{0}$. (1. 2)

Assumption 3. $\mathbf{c}\geq\mathbf{0}$. (1. 3)

Let us define the augmented input-output matrix as follows;

$$\tilde{\mathbf{A}}=\mathbf{A}+\mathbf{c}\mathbf{a}_0{}'=[a_{ij}+c_i a_{0j}]\geq\mathbf{0}\ . \qquad (1.\ 4)$$

That is, the typical element $\tilde{a}_{ij}$ of $\tilde{\mathbf{A}}$ represents the quantity of the ith commodity required by the jth industry as input, including the necessary consumption, for the production of one unit of the jth commodity.

Assumption 4. $\tilde{\mathbf{A}}$ is productive. Namely,

$$\exists \mathbf{x} \geqq \mathbf{0}\,;\ \mathbf{x} > \tilde{\mathbf{A}}\mathbf{x}, \tag{1. 5}$$

Let us define the surplus product vector **f** as

$$\mathbf{f} = (\mathbf{I} - \tilde{\mathbf{A}})\mathbf{x},$$

then

$$\mathbf{f} > \mathbf{0}. \tag{1. 6}$$

Assumption 4 means that there is a nonnegative level of production which can yield a positive surplus product. Incidentally,

$$\mathbf{x} > \mathbf{0}^{2)}. \tag{1. 7}$$

Under these assumptions, we obtain two basic propositions.

Proposition 1. $\tilde{\mathbf{A}}$ is indecomposable. (1. 8)

Proof. Suppose that $\tilde{\mathbf{A}}$ were decomposable. Then there exists a nonempty proper subset J of $\{1, 2, \cdots, n\}$ such that

$$\forall (i, j) \in J^c \times J;\ \tilde{a}_{ij} = 0.$$

Therefore

$$0 = \tilde{a}_{ij} \geqq a_{ij} \geqq 0.$$

Accordingly, there exists a nonempty proper subset J of $\{1, 2, \cdots, n\}$ such that

$$\forall (i, j) \in J^c \times J;\ a_{ij} = 0.$$

This contradicts the indecomposability of **A**. Q.E.D.

Proposition 2. There exists the positive inverse matrix of $\mathbf{I} - \tilde{\mathbf{A}}$, that is

$$\exists (\mathbf{I} - \tilde{\mathbf{A}})^{-1} > \mathbf{0}. \tag{1. 9}$$

Proof. (1. 4) and (1. 8) show that $\tilde{\mathbf{A}}$ is an indecomposable and nonnegative matrix, so that, according to Frobenius-Perron's Theorem, the following two conditions (i), (ii) are equivalent[3].

(i) $(\exists \mathbf{f} \geq 0)(\exists \mathbf{x} \geqq \mathbf{0})((\mathbf{I} - \tilde{\mathbf{A}})\mathbf{x} = \mathbf{f})$.

(ii) $\exists (\mathbf{I} - \tilde{\mathbf{A}})^{-1} > \mathbf{0}$.

Since equations (1. 5) and (1. 6) satisfy condition (i), condition (ii) also holds. Q.E.D.

Notes

1) A nonnegative $n \times n$ matrix $\mathbf{A}=[a_{ij}]$ is called decomposable if there exists a nonempty proper subset J of $\{1,2, \cdots, n\}$ such that

$$\forall (i, j) \in J^c \times J;\ a_{ij}=0.$$

On the other hand, $\mathbf{A} \geqq \mathbf{0}$ is called indecomposable, if it is not decomposable and is not the zero matrix of order 1. See Nikaido [13] p. 105. This definition of indecomposability is identical with the following one:

if there exists a chain of sectors $\{k_0, k_1, k_2, \cdots, k_\nu\}$ joining $k_0=i$ to $k_\nu=j$ such that any two consecutive sectors k_s and k_{s+1} are directly connected in the sense that $a_{k_s k_{s+1}}>0$.

The proof is shown in Nikaido [13] p. 109. Therefore, the economic implication of indecomposability is that all of the commodities consisting the matrix $\mathbf{A}$ are *basic* products in the sense that each of them enters, no matter whether directly or indirectly, into the production of all commodities. See Sraffa [16] pp. 7-8 and Shiozawa [17] pp. 90-94.

2) Suppose that

$$\exists i;\ x_i \neq 0,$$

then

$$x_i=0 \leqq \sum_j a_{ij} x_j.$$

This contradicts (1. 5).

3) Frobenius-Perron's Theorem concerning an indecomposable matrix $\mathbf{A} \geqq \mathbf{0}$ implies that the following conditions (I)-(VII) are mutually equivalent.

(I) $(\exists \mathbf{c} \geq \mathbf{0})(\exists \mathbf{x} \geqq \mathbf{0})((\rho \mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{c})$.

(II) $(\exists \mathbf{c} > \mathbf{0})(\exists \mathbf{x} \geqq \mathbf{0})((\rho \mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{c})$.

(III) $(\forall \mathbf{c} \geq \mathbf{0})(\exists \mathbf{x} \geq \mathbf{0})((\rho \mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{c})$.

(IV) Any principal minor of $\rho \mathbf{I}-\mathbf{A}$ is positive.

(V) $\rho>\lambda(\mathbf{A})$, where $\lambda(\mathbf{A})$ denotes the maximum nonnegative eigenvalue of $\mathbf{A}$.

(VI) $\exists (\rho \mathbf{I}-\mathbf{A})^{-1}>\mathbf{0}$.

(VII) $(\rho \mathbf{I}-\mathbf{A})^{-1}=\frac{1}{\rho} \sum_{\nu=0}^{\infty} \frac{\mathbf{A}^\nu}{\rho^\nu}$.

The proof is shown in Nikaido [11] pp. 119-140. Moreover, for any indecomposable matrix $\mathbf{A} \geqq \mathbf{0}$, there holds the following.

(VIII) $\lambda(\mathbf{A})>0$, and there exists a positive eigenvector $\mathbf{x}>\mathbf{0}$ associated with $\lambda(\mathbf{A})$, besides, any eigenvector associatied with $\lambda(\mathbf{A})$ is unique only up to a scale factor.

(IX) The nonnegative eigenvalue problem

$$\mathbf{Ay}=\mu\mathbf{y},\ \mu\geqq 0,\ \mathbf{y}\geq\mathbf{0}$$

has no solution except $\mu=\lambda(\mathbf{A})$.

(X) If $\mathbf{A}_1\geq\mathbf{A}_2\geqq\mathbf{0}$ and at least one of these matrices is indecomposable, then $\lambda(\mathbf{A}_1)>\lambda(\mathbf{A}_2)$.

(XI) $\lambda(\mathbf{A})$ is a simple root of the characteristic equation.

(XII) $\lambda(\mathbf{A})=\lambda(\mathbf{A}')$.

The proof is shown in Nikaido [11] pp. 120–138.

Incidentally, according to Frobenius-Perron's Theorem, for any indecomposable matrix $\mathbf{A}\geqq\mathbf{0}$ there only needs Assumption 4′ below instead of Assumption 4 so as to fulfill Proposition 2.

Assumption 4′ $\exists\mathbf{x}\geqq\mathbf{0}\,;\ \mathbf{x}\geq\tilde{\mathbf{A}}\mathbf{x}$.

The economic implication of Assumption 4′ is that there exists a nonnegative level of production yielding a positive surplus product of at least one sector.

§ 2. Quantity System or L-System[1)]

Denoting the necessary consumption function by $\mathbf{C}=\mathbf{C}(\mathbf{x})$, we get from the definition in the preceding section

$$\mathbf{C}(\mathbf{x})=\mathbf{c}\mathbf{a}_0'\mathbf{x}. \qquad (2.\ 1)$$

There is a relation amongst commodities such that

$$\mathbf{Ax}+\mathbf{C}+\mathbf{f}=\mathbf{x}, \qquad (2.\ 2)$$

or

$$\tilde{\mathbf{A}}\mathbf{x}+\mathbf{f}=\mathbf{x}. \qquad (2.\ 3)$$

According to Propostion 2, there exists $(\mathbf{I}-\tilde{\mathbf{A}})^{-1}>\mathbf{0}$ uniquely, so that with a semi-positive surplus product vector $\mathbf{f}\geq\mathbf{0}$,

$$\mathbf{x}=(\mathbf{I}-\tilde{\mathbf{A}})^{-1}\mathbf{f}>\mathbf{0}. \qquad (2.\ 4)$$

Thus, all we have to do is to give a semi-positive surplus product vector, and then a positive output vector $\mathbf{x}$ is uniquely determined. We can expand $(\mathbf{I}-\tilde{\mathbf{A}})^{-1}$ in the form of the matrix power series[2)], so

$$\mathbf{x}=\mathbf{f}+\tilde{\mathbf{A}}\mathbf{f}+\tilde{\mathbf{A}}^2\mathbf{f}+\cdots. \qquad (2.\ 5)$$

This implies that $\mathbf{x}$ is equal to the product required directly or indirectly for the production of the surplus product $\mathbf{f}$. Hence, we can interpret the matrix $(\mathbf{I}-\tilde{\mathbf{A}})^{-1}>\mathbf{0}$ as the matrix multiplier corresponding to the surplus product vector.

On the other hand, for the production of $\mathbf{x}$ there needs labour L such that

$$\mathbf{a}_0'\mathbf{x}=L. \tag{2. 6}$$

Substituting (2. 4) into (2. 6), we obtain

$$\mathbf{a}_0'(\mathbf{I}-\tilde{\mathbf{A}})^{-1}\mathbf{f}=L>0. \tag{2. 7}$$

Therefore, fixing a surplus product $\mathbf{f}\geq\mathbf{0}$, required labour $L>0$ is uniquely determined. In a similar way to (2. 5),

$$L=\mathbf{a}_0'\mathbf{f}+\mathbf{a}_0'\tilde{\mathbf{A}}\mathbf{f}+\mathbf{a}_0'\tilde{\mathbf{A}}^2\mathbf{f}+\cdots, \tag{2. 8}$$

and so L equals labour required directly or indirectly for the production of the surplus product $\mathbf{f}$.

Let us define N and σ as follows;

N: the number of the employed workers,

σ : the working day.

And we assume that there holds a relation between L and N such that

$$L=\sigma N, \tag{2. 9}$$

$$\sigma=\begin{cases}\sigma^* & \text{if} \quad L<\sigma^*N^*,\\ L/N^* & \text{if} \quad L\geqq\sigma^*N^*,\end{cases} \tag{2. 10}$$

$$N=\begin{cases}L/\sigma^* & \text{if} \quad L<\sigma^*N^*,\\ N^* & \text{if} \quad L\geqq\sigma^*N^*,\end{cases} \tag{2. 11}$$

where

σ^* : the standard working day,

N^* : the number of the workers.

In other words,

i) $L<\sigma^*N^*\Longrightarrow$at the standard working day only L/σ^* workers will be employed, but the remained N^*-L/σ^* workers will be unemployed;

ii) $L \geqq \sigma^* N^* \Longrightarrow$ all of the workers will be employed and so full employment holds, but the working day is prolonged up to $\sigma = L/N^*$.

Under the situation i) there holds

$$\mathbf{f}_1 \leqq \mathbf{f}_2 \Rightarrow \mathbf{x}_1 < \mathbf{x}_2 \Rightarrow L_1 < L_2 \Rightarrow N_1 < N_2, \qquad (2.\ 12)$$

so we ascertain that if we were able to increase **f**, the number of the employed workers would be increased. Now we see that *there really can be an unemployment suffered from deficiency of effectice demand.*

Notes

1) The initial L is an abbreviation for Leontief.

2) See Frobenius-Perron's Theorem in note 3) of §1, in particular the condition (VII).

§ 3. Value System or M-System[1]

Let us define the value of the *i*th product as the sum of direct labour input and indirect labour input used in the means of production required for the production of one unit of the *i*th product. That is, denoting the value vector by **v**, there should hold

$$\mathbf{a}_0 + \mathbf{A}'\mathbf{v} = \mathbf{v}. \qquad (3.\ 1)$$

Since $\mathbf{A} \geqq \mathbf{0}$ is an indecomposable and productive matrix,

$$\exists \mathbf{v} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{a}_0 > \mathbf{0}^{2)}, \qquad (3.\ 2)$$

and

$$\mathbf{v} = \mathbf{a}_0 + \mathbf{a}_0\mathbf{A}' + \mathbf{a}_0(\mathbf{A}')^2 + \cdots. \qquad (3.\ 3)$$

Therefore, there exists uniquely the positive value vector which represents labour required directly or indirectly for the production of one unit of each commodity.

Substituting (2. 1), (2. 2), and(3. 2) into (2. 6), we obtain

$$L = \mathbf{a}_0'\mathbf{x}$$
$$= \mathbf{a}_0'((\mathbf{I}-\mathbf{A})^{-1}\mathbf{c}\mathbf{a}_0'\mathbf{x} + (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f})$$
$$= \mathbf{v}'\mathbf{c}\mathbf{a}_0'\mathbf{x} + \mathbf{v}'\mathbf{f}. \qquad (3.\ 4)$$

This implies that required labour L is equal to the value of the necessary consumption vector $\mathbf{v}'\mathbf{c}\mathbf{a}_0'\mathbf{x}$ plus the value of the surplus product vector $\mathbf{v}'\mathbf{f}$. And

$$\mathbf{v}'\mathbf{x} = \mathbf{v}'\tilde{\mathbf{A}}\mathbf{x} + \mathbf{v}'(\mathbf{I}-\tilde{\mathbf{A}})\mathbf{x}$$
$$= \mathbf{v}'\tilde{\mathbf{A}}\mathbf{x} + \mathbf{v}'\mathbf{f}$$
$$= \mathbf{v}'\mathbf{A}\mathbf{x} + \mathbf{v}'\mathbf{c}\mathbf{a}_0'\mathbf{x} + \mathbf{v}'\mathbf{f}$$
$$= \mathbf{v}'\mathbf{A}\mathbf{x} + L. \qquad (3.\ 5)$$

Thus, the value of the gross product $\mathbf{v}'\mathbf{x}$ is equal to the value of input as the means of production $\mathbf{v}'\mathbf{A}\mathbf{x}$ plus the value of reqiured labour L. Now we find that '*Production of Commodities by Means of Commodities*' *really implies* '*Production of Commodities by Means of Labour Power Commodity*'.

Notes

1) The initial M is presented to Marx.

2) According to Assumption 1, $\mathbf{A}$ is indecomposable. And, since

$$\exists \mathbf{x} \geqq \mathbf{0};\ (\mathbf{I}-\mathbf{A})\mathbf{x} \geq (\mathbf{I}-\tilde{\mathbf{A}})\mathbf{x} > \mathbf{0},$$

$\mathbf{A}$ is productive. Therefore, an application of Frobenius-Perron's Theorem to this case gives

$$\exists (\mathbf{I}-\mathbf{A})^{-1} > \mathbf{0}.$$

Since $(\mathbf{I}-\mathbf{A}')^{-1} = ((\mathbf{I}-\mathbf{A})^{-1})'$,

$$\exists (\mathbf{I}-\mathbf{A}')^{-1} > \mathbf{0}.$$

§ 4. Price System or S-System[1]

Now, let us discuss if there exists such a combination of price vector $\mathbf{p}$, uniform rate of profit r, and surplus wage rate w as is compatible with the production of the product $\mathbf{x}$ given by L-System. In this case, such a combination must satisfy

$$p_i x_i = (1+r)\left(\sum_{j=1}^{n}(x_{ji}+L_i c_j)\,p_j + wL_i\right), \tag{4. 1}$$

where

x_{ij} : the quantity of the ith commodity required by the jth industry as input for the production of x_j units of the jth commodity,

L_i : the quantity of labour required by the ith industry as input for the production of x_i units of the ith commodity.

Dividing both sides of (4. 1) by x_i and expressing it by vector notation, this problem is convertible into the problem if there exists a combination of $\mathbf{p}\geq\mathbf{0}$, $r\geqq 0$, and $w\geqq 0$ which satisfies (4. 2) below.

$$\mathbf{p}=(1+r)(\tilde{\mathbf{A}}'\mathbf{p}+\mathbf{a}_0 w). \tag{4. 2}$$

Going up three steps, we discuss this problem.

i) the case with $r=0$

$$\mathbf{p}=\tilde{\mathbf{A}}'\mathbf{p}+\mathbf{a}_0 w$$
$$=(\mathbf{I}-\tilde{\mathbf{A}}')^{-1}\mathbf{a}_0 w>\mathbf{0}^{2)}. \tag{4. 3}$$

Post-multiplying both sides of the transpose of (4. 3) by $\mathbf{f}$ and considering (2. 7), we obtain

$$\mathbf{p}'\mathbf{f}=\mathbf{a}_0'(\mathbf{I}-\tilde{\mathbf{A}})^{-1}\mathbf{f}w$$
$$=Lw.$$

Thus, the surplus wages are on such level that workers could buy all of the surplus product.

ii) the case with $w=0$

In this case, (4. 2) reduces to

$$\mathbf{p}=(1+r)\tilde{\mathbf{A}}'\mathbf{p}. \tag{4. 4}$$

To solve this problem, let us consider the following nonnegative eigenvalue problem;

$$\tilde{\mathbf{A}}'\mathbf{p}=\lambda\mathbf{p},\ \lambda\geqq 0,\ \mathbf{p}\geq\mathbf{0}. \tag{4. 5}$$

According to Frobenius-Perron's Theorem, there exists the unique solution $\lambda=\lambda(\tilde{\mathbf{A}}')>0$ for the problem (4. 5), where $\lambda(\tilde{\mathbf{A}}')$ denotes the maximum nonnegative eigenvalue of $\tilde{\mathbf{A}}'$, and a positive eigenvector $\mathbf{p}>\mathbf{0}$ associated with $\lambda(\tilde{\mathbf{A}}')$ exists uniquely only up to a scale factor. Since the solution $\lambda=\lambda(\tilde{\mathbf{A}}')$ for the nonnegative eigenvalue problem (4. 5) is unique, r which satisfies (4. 4) must satisfy the following relation;

$$\frac{1}{1+r}=\lambda(\tilde{\mathbf{A}}').$$

Denoting this solution by R,

$$R=\frac{1}{\lambda(\tilde{\mathbf{A}}')}-1>0^{3)}. \qquad (4.\ 6)$$

Since the price vector $\mathbf{p}>\mathbf{0}$ is unique only up to a scale factor, the relative prices are uniquely determined.

iii) the case with $0<r<R$

$$\mathbf{p}=(1+r)(\tilde{\mathbf{A}}'\mathbf{p}+\mathbf{a}_0 w), \qquad (4.\ 7)$$

so

$$\left(\frac{1}{1+r}\mathbf{I}-\tilde{\mathbf{A}}'\right)\mathbf{p}=\mathbf{a}_0 w. \qquad (4.\ 8)$$

Therefore, iff[4] there exists

$$\left(\frac{1}{1+r}\mathbf{I}-\tilde{\mathbf{A}}'\right)^{-1}>\mathbf{0}, \qquad (4.\ 9)$$

we can solve (4. 8) as follows;

$$\mathbf{p}=\left(\frac{1}{1+r}\mathbf{I}-\tilde{\mathbf{A}}'\right)^{-1}\mathbf{a}_0 w>\mathbf{0}. \qquad (4.\ 10)$$

Here, according to Frobenius-Perron's Theorem, we know that if r satisfies such condition as $0\leqq r<R$, then (4. 9) holds; for

$$\exists\left(\frac{1}{1+r}\mathbf{I}-\tilde{\mathbf{A}}'\right)^{-1}>\mathbf{0}\Longleftrightarrow\frac{1}{1+r}>\lambda(\tilde{\mathbf{A}}')=\frac{1}{1+R}$$
$$\Longleftrightarrow R>r\geqq 0.$$

Thus we realize that R is the greatest of all r which satisfies

(4. 2). So we call R the maximum rate of profit. R is the rate of profit which is uniquely determined from the character of $\tilde{\mathbf{A}}$.

In the consequence of i)-iii), we see that *the set of all nonnegative rate of profit which satisfies* (4. 2) *is identical with the interval* $[0, R]$.

Notes

1) The initial S implies an abbreviation for Sraffa.

2) Suppose that w were zero, then

$$\begin{aligned}\mathbf{p}&=\tilde{\mathbf{A}}'\mathbf{p},\\ &=(\mathbf{I}-\tilde{\mathbf{A}}')^{-1}\cdot\mathbf{0},\\ &=\mathbf{0}.\end{aligned}$$

This contradicts our requirement $\mathbf{p}\geq\mathbf{0}$.

3) From Assumption 4, $\tilde{\mathbf{A}}$ is productive, and then, according to Frobenius-Perron's Theorem, there holds

$$1>\lambda(\tilde{\mathbf{A}})=\lambda(\tilde{\mathbf{A}}').$$

4) The word *iff* implies *if and only if.*

§5. Standard System or R-System[1]

Let us investigate if there exists such a product vector $\mathbf{q}$ as could bring a uniform physical rate of surplus $\Pi>0$ throughout the economy[2]. Such a product must satisfy

$$\tilde{\mathbf{A}}\mathbf{q}(1+\Pi)=\mathbf{q}. \tag{5. 1}$$

In a simillar way to (4. 4), we can solve this problem with the help of a nonnegative eigenvalue problem; and

$$\begin{aligned}\exists\Pi&=\frac{1}{\lambda(\tilde{\mathbf{A}})}-1\\ &=\frac{1}{\lambda(\tilde{\mathbf{A}}')}-1\\ &=R>0.\end{aligned} \tag{5. 2}$$

In fact, the maximum rate of profit R is the unique solution for (5. 1), and an eigenvector $\mathbf{q}>\mathbf{0}$ associated with R is unique only up to a scale factor. In the standard system,

$$(\mathbf{I}-\tilde{\mathbf{A}})\mathbf{q}=R\tilde{\mathbf{A}}\mathbf{q},$$

$$\mathbf{f}=R(\mathbf{q}-\mathbf{f}),$$

so

$$R=\frac{f_1}{q_1-f_1}=\cdots=\frac{f_n}{q_n-f_n}. \qquad (5.\ 3)$$

Certainly, we confirm that R is the uniform physical rate of surplus throughout the economy. This ratio has no connection with any scale factor of $\mathbf{q}$. Furthermore,

$$\tilde{\mathbf{A}}\mathbf{q}(1+R)=\mathbf{q},$$

$$\mathbf{q}'\tilde{\mathbf{A}}'\mathbf{v}(1+R)=\mathbf{q}'\mathbf{v},$$

$$R=\frac{\mathbf{q}'\mathbf{v}-\mathbf{q}'\tilde{\mathbf{A}}'\mathbf{v}}{\mathbf{q}'\tilde{\mathbf{A}}'\mathbf{v}}$$

$$=\frac{\mathbf{f}'\mathbf{v}}{\mathbf{q}'\tilde{\mathbf{A}}'\mathbf{v}}. \qquad (5.\ 4)$$

Therefore, R also indicates the ratio of the value of the surplus product to the sum of the value of the means of production and the value of the necessary consumption in the standard system.

Next, we deal with a combination of the standard system normalized by (5. 6) and the price system normalized by (5. 8).

$$\tilde{\mathbf{A}}\mathbf{q}^*(1+R)=\mathbf{q}^*, \qquad (5.\ 5)$$

$$\mathbf{a}_0'\mathbf{q}^*=\mathbf{v}'\mathbf{f}, \qquad (5.\ 6)$$

$$(\tilde{\mathbf{A}}'\mathbf{p}^*+\mathbf{a}_0w^*)(1+r)=\mathbf{p}^*, \qquad (5.\ 7)$$

$$\mathbf{p}^{*\prime}(\mathbf{I}-\tilde{\mathbf{A}})\mathbf{q}^*=\mathbf{v}'\mathbf{f}; \qquad (5.\ 8)$$

where

$$\mathbf{q}^*=\frac{\mathbf{v}'\mathbf{f}}{\mathbf{a}_0'\mathbf{q}}\mathbf{q}, \qquad (5.\ 9)$$

$$\mathbf{p}^*=\frac{\mathbf{v}'\mathbf{f}}{\mathbf{p}'(\mathbf{I}-\tilde{\mathbf{A}})\mathbf{q}^*}\mathbf{p}, \qquad (5.\ 10)$$

$$w^*=\frac{\mathbf{v}'\mathbf{f}}{\mathbf{p}'(\mathbf{I}-\tilde{\mathbf{A}})\mathbf{q}^*}w. \qquad (5.\ 11)$$

Pre-multiplying both sides of (5. 5) by $\mathbf{p}^*$, and post-multiplying both sides of the transpose of (5. 7) by $\mathbf{q}^*$, thereafter, subtracting the former from the latter, we get

$$\mathbf{p}^{*\prime}\tilde{\mathbf{A}}\mathbf{q}^{*}(r-R)+(1+r)w^{*}\mathbf{a}_0{}'\mathbf{q}^{*}=\mathbf{0}. \tag{5. 12}$$

On the other hand, from (5. 5) and (5. 8)

$$\mathbf{p}^{*\prime}\tilde{\mathbf{A}}\mathbf{q}^{*}R=\mathbf{v}'\mathbf{f}. \tag{5. 13}$$

Substituting (5. 6) and (5. 13) into (5. 12) and rearranging the terms, we finally obtain the relation between r and w^* such that

$$r=\frac{R(1-w^*)}{1+w^*R}, \tag{5. 14}$$

or

$$r=\frac{1+R}{1+w^*R}-1. \tag{5. 15}$$

Thus, r is a monotonically decreasing function of w^*.

Let us consider the implication of w^* in detail, paying attention to the value side of the system. Multiplying both the numerator and the denominator of (5. 11) by the scalar $\mathbf{v}'\mathbf{f}/(\mathbf{p}'(\mathbf{I}-\tilde{\mathbf{A}})\mathbf{q}^*)$, we obtain

$$w^*=\frac{w^*\mathbf{v}'\mathbf{f}}{\mathbf{v}'\mathbf{f}}; \tag{5. 16}$$

that is, w^* implies the share of wages in the surplus product in terms of value. It is easy to connect w^* with the rate of exploitation defined as the ratio of profit to wages in terms of value. First, we define the profit share of the surplus product in terms of value ω^* as the proportion of the surplus product in terms of value which goes to profit.

$$\omega^*=\frac{(1-w^*)\mathbf{v}'\mathbf{f}}{\mathbf{v}'\mathbf{f}}$$

$$=1-w^*. \tag{5. 17}$$

By definition, the rate of exploitation ω is expressed as follows;

$$\omega=\frac{(1-w^*)\mathbf{v}'\mathbf{f}}{\mathbf{v}'\mathbf{c}\mathbf{a}_0{}'\mathbf{x}+w^*\mathbf{v}'\mathbf{f}}$$

$$=\frac{(1-w^*)\mathbf{v}'\mathbf{f}}{L-(1-w^*)\mathbf{v}'\mathbf{f}}$$

$$=\frac{\omega^*\theta}{1-\omega^*\theta}, \tag{5. 18}$$

where θ is defined as the value of the surplus product per unit of labour, or the ratio of the surplus product to the net product in terms of value.

$$\theta = \frac{\mathbf{v'f}}{L}. \tag{5. 19}$$

This ratio is exclusively determined by L-System and obviously satisfies

$$0 < \theta < 1. \tag{5. 20}$$

Therefore, if ω^* is positive,

$$\omega < \frac{\omega^*}{1-\omega^*}. \tag{5. 21}$$

The time is ripe to expose the hidden connection between the rate of profit r and the rate of exploitation ω. We already have the result in the preceding section that any nonnegative rate of profit belongs to the interval $[0, R]$, and, conversely, that any element contained in this interval must be a rate of profit that satisfies (4. 2). We also found out three relations; the relation between r and w^* expressed as (5.14), the relation between w^* and ω^* expressed as (5. 17), and the relation between ω^* and ω expressed as (5. 18). Making use of these results, we obtain the following.

i) $r=R \Longleftrightarrow w^*=0 \Longleftrightarrow \omega^*=1 \Longleftrightarrow \omega=\theta/(1-\theta).$ (5. 22)

ii) $0<r_1<r_2<R \Longleftrightarrow 1>w_1^*>w_2^*>0$

$\Longleftrightarrow 0<\omega_1^*<\omega_2^*<1$

$\Longleftrightarrow 0<\omega_1<\omega_2<\theta/(1-\theta).$ (5. 23)

iii) $r=0 \Longleftrightarrow w^*=1 \Longleftrightarrow \omega^*=0 \Longleftrightarrow \omega=0.$ (5. 24)

Consequently,

$$r>0 \Longleftrightarrow \omega>0, \tag{5. 25}$$

that is, the rate of profit is positive iff the rate of exploitation is positive. In other words, *positive profit exists iff there exists exploitation in the production process.* We also understand that $\theta/(1-\theta)$ is the greatest of all rate of exploitation, so we call $\theta/(1-\theta)$ the maximum rate of exploitation. This rate corresponds to the maximum rate of profit, and is restricted by the value of the necessary consumption. Furthermore, according to (5. 23), the greater the rate of profit becomes, the greater the rate of exploitation is; and vice versa.

After a roundabout but worthwhile trip to the standard system, we at last can determine the prices in the acutual price system. In the standard system, we have only to give $w^*>0$, therefore, to give a positive rate of exploitation $\omega<\theta/(1-\theta)$, then the rate of profit r is uniquely determined from (5. 14), and the price vector $\mathbf{p}^*$ defined as (5. 10) is uniquely determined as follows;

$$\mathbf{p}^*=\left(\frac{1}{1+r}\mathbf{I}-\tilde{\mathbf{A}}'\right)^{-1}\mathbf{a}_0 w^*>\mathbf{0}. \qquad (5.\ 26)$$

Utilizing the price vector in the standard system, we can uniquely determine the relative prices in the actual price system;

$$p_i/p_j=p_i^*/p_j^*, \qquad (5.\ 27)$$

$$p_i/w=p_i^*/w^*. \qquad (5.\ 28)$$

If $w^*=0$, so $w=0$, then the relative prices are determined directly as the discussion in §4 ii) shows.

Notes

1) The initial R implies the maximum rate of profit defined in the preceding section. As we shall see later, the physical rate of surplus in every sector in the standard system is equal to R. Alternatively, since the surplus product in the standard system fills the role of '*an invariable standard of value*' for which Ricardo had been searching through life, we may interpret the initial R as an abbreviation for Ricardo. See Sraffa 〔16〕 pp. 18-31.

2) Let us call the system represented as (5. 1) the standard system, $\Pi>0$ satisfying (5. 1) the standard ratio, and $\mathbf{q}>\mathbf{0}$ the standard commodity. See Sraffa 〔16〕 pp. 18-31.

§ 6. Expectations and Realization ——K-System[1] ——

Pre-multiplying both sides of (2. 2) by $\mathbf{p}'$, we get

$$\mathbf{p}'\tilde{\mathbf{A}}\mathbf{x}+\mathbf{p}'\mathbf{f}=\mathbf{p}'\mathbf{x}. \tag{6. 1}$$

On the other hand, post-multiplying both sides of the transpose of(4. 2) by $\mathbf{x}$, we obtain

$$\mathbf{p}'\mathbf{x}=(1+r)\mathbf{p}'\tilde{\mathbf{A}}\mathbf{x}+w\mathbf{a}_0'\mathbf{x}. \tag{6. 2}$$

According to (6. 1) and (6. 2), there holds

$$\mathbf{p}'\mathbf{c}\mathbf{a}_0'\mathbf{x}+\mathbf{p}'\mathbf{f}=\mathbf{p}'\mathbf{c}\mathbf{a}_0'\mathbf{x}+w\mathbf{a}_0'\mathbf{x}+r(\mathbf{p}'\tilde{\mathbf{A}}+w\mathbf{a}_0')\mathbf{x}, \tag{6. 3}$$

that is,

National Income = Wages + Profit. (6. 4)

And

$$\mathbf{p}'\mathbf{f}=w\mathbf{a}_0'\mathbf{x}+r(\mathbf{p}'\tilde{\mathbf{A}}+w\mathbf{a}_0')\mathbf{x}, \tag{6. 5}$$

or

The Sum of the Surplus Product in Terms of Prices
= Surplus Wages + Profit. (6. 6)

Incidentally, (6. 3) and (6. 5) are identities, although an apostle of the general equilibrium theory may be anxious to regard them as equilibrium equations which determine r or w. For, while $\mathbf{x}$ is uniquely determined by $\mathbf{f}$ as (2. 4) shows, (4. 1) implies that $\mathbf{p}$ is determined so as to be compatible with the good working of distribution of product $\mathbf{x}$ determined in that way between wages and profit; thus, by definition, (4. 1) must always satisfy (2. 4), so w^* have to be given exogenously.

Now, let us discuss the problem of the determination of $\mathbf{f}$ and w which are exogenous variables in L-System and R-System respectively. First, we assume that $\mathbf{f}$ is a monotonically increasing function of the expected rate of profit r^e;

$$\mathbf{f}=\mathbf{f}(r^e), \tag{6. 7}$$

$$r^e \geqq 0, \tag{6. 8}$$

$$r_1{}^e < r_2{}^e \Longrightarrow \mathbf{f}(r_1{}^e) \leq \mathbf{f}(r_2{}^e). \tag{6. 9}$$

That is, the greater the expected rate of profit is, the more encouraged the production plan held in capitalist's mind becomes. Thus we consider the vector **f** as a representation of *anticipative production*[2)] intended by capitalist.

Next, we consider w^* as a monotonically increasing function of the working day σ;

$$w^* = w^*(\sigma), \tag{6. 10}$$

$$\sigma = \sigma^* \Longrightarrow w^*(\sigma) = 0, \tag{6. 11}$$

$$\sigma^* < \sigma_1 < \sigma_2 \Longrightarrow 0 < w^*(\sigma_1) < w^*(\sigma_2) < 1. \tag{6. 12}$$

In other words, when the working day is the standard one, or untill full employment, the relative share of surplus wages is equal to zero, but once full employment holds, bonus must be paid to workers according to the prolonged working day. This assumption may be well-grounded when we translate it into value terms as follows;

$$\omega = \omega(\sigma), \tag{6. 13}$$

$$\sigma = \sigma^* \Longrightarrow \omega(\sigma) = \theta/(1-\theta), \tag{6. 14}$$

$$\sigma^* < \sigma_1 < \sigma_2 \Rightarrow \theta/(1-\theta) > \omega(\sigma_1) > \omega(\sigma_2) > 0. \tag{6. 15}$$

Therefore, when unemployment prevails, employed workers are exploited at maximal rate, on the other hand, once full employment holds, under the pressure of workers' demand for higher wages the rate of exploitation must be lowered, in spite of capitalist's reluctance, according to the prolonged working day.

Now that we have obtained (6. 7) and (6. 10), the whole system including L-, M-, S-, and R-System is complete except for only one exogenous variable r^e. Given the expected rate of profit r^e, the surplus product vector **f** representing anticipative production intended by capitalist is determined from (6. 7), then, according to (2. 4), (2. 7), (2.

10), and (2. 11), the output vector **x**, the required labour L, the working day σ, and the number of the employed workers N are determininined in L-System. Thus the Quantity System is complete. Next, utilizing σ determined in this way, the rate of exploitation ω is determined from (6. 13) in the Value System which, irrespective of prices, always accompanies L-System or, in terms of the Standard System, the relative share of surplus wages w^* is determined from (6. 10). Substituting this w^* into (5. 14), (5. 26), (5. 27), and (5. 28) in R-System, we obtain the rate of profit r, relative prices p_i/p_j, and prices in terms of the wage unit p_i/w; with this also the Price System is complete. As we have seen so far, while the output **x**, therefore the number of the employed workers N too, is determined by capitalist on his expectation for the rate of profit, the realized or actual rate of profit r is determined in the consequence of the struggle for higher share of the surplus product between capitalist and workers, or, in terms of value, in the consequence of workers' struggle for lower rate of exploitation against capitalist. It was Keynes who considered the discrepancy between expectation and realization a primary factor which causes the instability peculiar to the capitalistic production.

Notes

1) The initial K is presented to Keynes.
2) This word is a borrowing from Fujino 〔2〕.

§7. Concluding Remarks
——Towards a Dynamic Economics[1]——

In K-System mentioned above, there exists no nexus which unites the expected rate of profit r^e with the realized rate of profit r. An explication of the linkage between them may command an extensive view towards a dynamic economics. Some directions are shown as follows.

i) Development of Productivity in Capitalism and the Knife-Edge Problem

When wages are equal to the value of labour power, in other words, when wages are equal to the cost of reproduction of labour power, or, when the surplus wages are zero, the rate of profit, accordingly the rate of exploitation too, is maximal and the existing system is equal to the standard one. In this situation, production is performed at the maximum and uniform rate of physical surplus. If capitalist utilizes the surplus product for the development of more productive technology in search of higher rate of profit, a technique which can yield higher maximum rate of profit will be adopted. But this process won't continue forever; the day will come when the economy dashes against the full employment ceiling and the surplus wages are no longer zero, so that the rate of exploitation is no longer maximal, under the pressure of workers' demand for higher wages. In the consequence, the realized rate of profit will fall. If the realized rate of profit were far less than the expected one, the level of anticipative production intended by capitalist would also fall and there could happen mass unemployment. In this case, the surplus wages would return to zero again; but had the expected rate of profit still been depressed, there would happen a secular stagnation untill business recovery comes.

ii) The Problem of Fiscal Expenditure

In case r were less than r^e, capitalist would cause the state to collect taxes from the surplus wages and to utilize this taxes for the development of more productive technology. In the consequence, the maximum rate of profit would increase, and r might possibly move nearer to r^e.

Note

1) This expression is a borrowing from Harrod [4].

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