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Ei, Shin-Ichiro
Faculty of Mathematics, Kyushu University

Ikeda, Kota
Graduate School of Advanced Mathematical Science, Meiji University

Miyamoto, Yasuhito
Department of Mathematics, Tokyo Institute of Technology

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DYNAMICS OF A BOUNDARY SPIKE FOR THE SHADOW GIERER-MEINHARDT SYSTEM

Shin-Ichiro Ei

Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan

Kota Ikeda

Graduate School of Advanced Mathematical Science
Meiji University, Kawasaki, 214-8571, Japan

Yasuhito Miyamoto

Department of Mathematics, Tokyo Institute of Technology, Tokyo, 152-8551, Japan

Abstract. The Gierer-Meinhardt system is a mathematical model describing the process of hydra regeneration. The authors of [3] showed that if an initial value is close to a spiky pattern and its peak is far away from the boundary, the solution of the shadow Gierer-Meinhardt system, called a *interior spike solution*, moves towards a point on boundary which is the closest to the peak. However it has not been studied how a solution close to a spiky pattern with the peak on the boundary, called a *boundary spike solution* moves along the boundary. In this paper, we consider the shadow Gierer-Meinhardt system and dynamics of a boundary spike solution. Our results state that a boundary spike moves towards a critical point of the curvature of the boundary and approaches a stable stationary solution.

1. **Introduction.** We consider the following *Gierer-Meinhardt system*:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \dfrac{A^p}{H^q}, & x \in \Omega,\ t > 0, \\ \tau H_t = d\Delta H - H + \dfrac{A^r}{H^s}, & x \in \Omega,\ t > 0, \\ \partial_\nu A = \partial_\nu H = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1}$$

where A and H represent the scaled activator concentration and inhibitor one, respectively, Ω is a bounded domain in $\mathbb{R}^2$ with the smooth boundary $\partial\Omega$, ϵ, d, and τ are positive parameters, and the exponents p, q, r and s satisfy

$$p > 1, q > 0, r > 0, s \geq 0, \text{ and } \gamma := \frac{qr}{p-1} - s - 1 > 0,$$

ν is the inner normal unit vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$ is the directional derivative in the direction of the vector ν. Under these assumptions, the Gierer-Meinhardt system has the possibility to exhibit Turing's instability, which means that a homogeneous state becomes unstable by the presence of diffusion (see [24]). Hence we expect that a spatially inhomogeneous state (namely, a spatial pattern) will appear in the Gierer-Meinhardt system. In fact, some mathematicians proved the existence

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of a stationary solution with some spiky pattern, which is the sharply localized concentration of the activator. This system seems to generate spiky patterns in a wide range of parameters, as suggested in [16].

Here we take $d \to \infty$ in the second equation of the Gierer-Meinhardt system (1) and formally have

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \dfrac{A^p}{\Xi^q}, & x \in \Omega,\ t > 0, \\ \tau \Xi_t = -\Xi + \dfrac{1}{|\Omega| \Xi^s} \displaystyle\int_\Omega A^r dx, & t > 0, \\ \partial_\nu A = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2}$$

This system is called the *shadow* Gierer-Meinhardt system, which was first introduced in Nishiura [22] and has been studied by various authors as follows: In Wei [25], it was shown that there exists a stationary solution of (2) with a boundary spike layer such that the peak is close to a non-degenerate local maximum point of the curvature of $\partial\Omega$, where the curvature of $\partial\Omega$ is measured in the direction of ν. One of the authors showed in [18] that if $r = p+1$ and τ is sufficiently small, a stationary solution with a boundary spike layer near a non-degenerate local maximum point of the curvature of $\partial\Omega$ is stable. The problems of existence and stability of spikes have large literature. If the readers are interested in these problems, see [5], [12], [14], [15], [19], [21], [27], and [28], and references cited therein.

The authors of [3] considered the dynamics of a solution of (2) with a spike located at an interior point of Ω and showed that the spike moves exponentially slowly towards the point on the boundary that is the closest to the spike as long as the distance between the spike and the boundary of Ω is larger than $2\epsilon|\log \epsilon|$. From the stability result of [17], it is expected that after the spike reaches the boundary, it moves towards a local maximum point of the curvature of $\partial\Omega$. Indeed, it was shown in [13] by formal analysis that the motion of boundary spike solutions is determined by a reduced ordinary differential equation like (12) and occurs on a slow time scale of $O(\epsilon^3)$. However, this was not rigorously shown so far.

Similar dynamics of boundary spike solutions for various equations. In [1], some free boundary problem in a 2-dimensional bounded domain, called Mullins-Sekerka evolution problems, was considered. The authors of [2] studied the global dynamics of spike state in the Allen-Cahn equation by the construction of an approximately invariant manifold. Many results for the dynamics of boundary spike solutions imply that the spike moves along the boundary on a slow time scale, and the motion is generically governed by the curvature of the boundary.

Recently, (1) was investigated in [8] under special conditions. The technique developed there for the proof is not applicable to (2) because of the non-local terms in (2). Moreover, the dynamics of boundary spikes is quite different from each other in (1) and (2). In fact, a boundary spike solution with multi-peaks can exist stably in (1) while one with multi-peaks must be unstable as shown in Theorem 3.3.

In the present paper, we consider the dynamics of a boundary spike solution with one peak on the boundary, called *a single-spike boundary solution* while we call *a single-spike interior solution* as a spike solution with one peak interior of Ω. Since a single-spike boundary solution moves along the boundary, we investigate the motion of the peak on the boundary whose location on the boundary is denoted by $h(t) \in \partial\Omega$. As described in (12) or Theorem 3.1, $h(t)$ moves towards a local maximum point of the curvature of the boundary according to the gradient of the curvature. Moreover, we also know from (12) or Theorem 3.1, that the speed of the motion of $h(t)$ is ϵ^3-order. Thus, we can say about the total dynamics of a single-spike solution that any single-spike interior solution of (2) first approaches the closest point of the boundary and after the spike reaches the boundary, it moves to a local maximum point of the curvature of the boundary. Our result also implies that any single-spike solution of (2) located near a local minimum point of the curvature of the boundary is *unstable*.

This paper is organized as follows: In Section 2, we will give the formal derivation of the motion of a single-spike boundary solution. Main results are mentioned in Section 3, in which it is shown that the movement of a single-spike boundary solution is essentially described by $h_t = \epsilon^3 M_0 \kappa_\sigma(h)$ for a constant $M_0 > 0$, where $\kappa(\sigma)$ is the curvature of the boundary with the arclength parameter σ of the boundary and $h(t)$ corresponds to the location of the peak on the boundary of a single-spike boundary solution.

The spectrum of a linearized operator with respect to a single-spike boundary solution is also given in the section because it is important in order to investigate the motion of the peak according to Theorem 3.1.

If there exists a multi-spikes boundary solution with two peaks on the boundary, it is strongly unstable because the linearized operator with respect to the solution has positive eigenvalues, which is also mentioned in Theorem 3.3. This result emphasizes that only single-spike boundary solution can be stable for (2) and that multi-spikes boundary solutions quickly collapse.

Proofs are given in Sections 4, 5 and 6.

2. **Setting and the derivation of the motion of a boundary spike.** In this section, we rescale (2). Let $A(t,x) = \Xi^{\frac{q}{p-1}}(t)u(t,x)$. Then (2) becomes

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} &=& \epsilon^2 \Delta u - u + u^p - \dfrac{q}{\tau(p-1)}\left(-1 + \dfrac{\Xi^\gamma}{|\Omega|}\displaystyle\int_\Omega u^r dx\right)u, \\ \tau\dfrac{\partial \Xi}{\partial t} &=& \left(-1 + \dfrac{\Xi^\gamma}{|\Omega|}\displaystyle\int_\Omega u^r dx\right)\Xi, \\ \partial_\nu u &=& 0, \qquad x \in \partial\Omega. \end{array}\right. \tag{3}$$

In (3), we again change the variable by $\Xi(t) = \epsilon^{-2/\gamma}|\Omega|^{1/\gamma}\xi(t)$, we have

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} &=& \epsilon^2 \Delta u - u + u^p - \dfrac{q}{\tau(p-1)}\left(-1 + \dfrac{\xi^\gamma}{\epsilon^2}\displaystyle\int_\Omega u^r dx\right)u, \\ \tau\dfrac{\partial \xi}{\partial t} &=& \left(-1 + \dfrac{\xi^\gamma}{\epsilon^2}\displaystyle\int_\Omega u^r dx\right)\xi, \\ \partial_\nu u &=& 0, \qquad x \in \partial\Omega. \end{array}\right. \tag{4}$$

In the remaining of this section, we will give the formal derivation of the motion of a boundary spike. Mathematically rigorous results on it will be stated in next sections.

We assume that the boundary $\partial\Omega$ of Ω is a sufficiently smooth closed curve given by $\{\Gamma(\sigma) \in \mathbb{R}^2;\ 0 \le \sigma \le \sigma_0\}$ with $\Gamma(0) = \Gamma(\sigma_0)$, where σ is the arc length parameter of $\partial\Omega$. Then we can have a tubular neighborhood of $\partial\Omega$ as $x = \Gamma(\sigma) + z\nu(\sigma)$, where $\nu = \nu(\sigma)$ is the inward normal unit vector of $\partial\Omega$ at $\Gamma(\sigma)$. Here and hereafter, we deal with the parameter σ as $\sigma \pmod{\sigma_0}$, that is, any $\sigma \in \mathbb{R}$ is identified with $\sigma' \in [0, \sigma_0]$. Let $\kappa = \kappa(\sigma)$ be the curvature of $\partial\Omega$ at $\Gamma(\sigma)$ measured in the direction of ν. Now, we assume the mass of u concentrates at some point on the boundary

$\partial\Omega$, say $\sigma = h(t)$, and take the stretched coordinate $z = \epsilon\mu$ and $\sigma = h(t) + \epsilon l$. Then (4) is

$$\begin{cases} u_t - \dfrac{h_t}{\epsilon} u_l = u_{\mu\mu} - \dfrac{\epsilon\kappa}{1-\epsilon\mu\kappa} u_\mu + \dfrac{1}{1-\epsilon\mu\kappa}\left(\dfrac{1}{1-\epsilon\mu\kappa} u_l\right)_l \\ \qquad\qquad - u + u^p - \dfrac{q}{\tau(p-1)}\left(-1 + \xi^\gamma \langle u^r, 1-\epsilon\mu\kappa\rangle\right) u, \\ \tau\xi_t = \left(-1 + \xi^\gamma \langle u^r, 1-\epsilon\mu\kappa\rangle\right)\xi, \\ u_\mu = 0, \ \mu = 0 \end{cases} \tag{5}$$

for $t > 0$ and $(l,\mu) \in \mathbb{R}^2_+$, where $\mathbb{R}^2_+ := \{(l,\mu) \in \mathbb{R}^2;\ \mu > 0\}$, $\langle u, v\rangle := \int_{\mathbb{R}^2_+} uv d\mu dl$, and $\kappa = \kappa(h(t) + \epsilon l)$, and we note that we take approximately $\int_\Omega f dx \sim \epsilon^2 \int_{\mathbb{R}^2_+} f \cdot (1-\epsilon\mu\kappa) d\mu dl$.

Let $U := (u,\xi)$ and write (5) by $U_t - \frac{h_t}{\epsilon} U_l = F(U) + \epsilon\mathbb{B}(\epsilon)U + \epsilon G(U;\epsilon)$, where

$$F(U) := \begin{pmatrix} \Delta_{\mu,l} u - u + u^p - \frac{q}{\tau(p-1)}\left(-1 + \xi^\gamma \langle u^r, 1\rangle\right) u \\ \dfrac{1}{\tau}\left(-1 + \xi^\gamma \langle u^r, 1\rangle\right)\xi \end{pmatrix},$$

$$\mathbb{B}(\epsilon)U := \begin{pmatrix} B(\epsilon)u \\ 0 \end{pmatrix},$$

$$G(U;\epsilon) := \begin{pmatrix} \frac{q}{\tau(p-1)} \xi^\gamma \langle u^r, \mu\kappa\rangle u \\ -\frac{1}{\tau}\xi^\gamma \langle u^r, \mu\kappa\rangle \xi \end{pmatrix} = \xi^\gamma \langle u^r, \mu\kappa\rangle \begin{pmatrix} \frac{q}{\tau(p-1)} u \\ \frac{1}{\tau}\xi \end{pmatrix}$$

and $\epsilon B(\epsilon)u := -\dfrac{\epsilon\kappa}{1-\epsilon\mu\kappa} u_\mu + \dfrac{1}{1-\epsilon\mu\kappa}\left(\dfrac{1}{1-\epsilon\mu\kappa} u_l\right)_l - u_{ll}$.

Theorem 2.1. *([25]) $F(U) = 0$ has a solution, say $u = S(\rho)$ and $\xi = \zeta$, where $\rho = \sqrt{\mu^2 + l^2}$. $S(\rho)$ is positive and exponentially decaying with respect to ρ, that is, $S(\rho) \to \frac{1}{\sqrt{\rho}} e^{-\rho}$ as $\rho \to +\infty$.*

Note that $\zeta^\gamma \langle S^r, 1\rangle = 1$ holds. Let $\mathbb{S} := (S, \zeta)$ and $\frac{h_t}{\epsilon} = H(h)$. Since $h_t = O(\epsilon)$ holds, we may assume $U_t = \epsilon U_T$ for $T := \epsilon t$. Expanding $\mathbb{B}(\epsilon) = \mathbb{B}_1 + \epsilon\mathbb{B}_2 + \cdots$, $G(U;\epsilon) = G_1(U) + \epsilon G_2(U) + \cdots$ and substituting $U = \mathbb{S} + \epsilon U_1 + \epsilon^2 U_2 + \cdots$, $H(h) = H_0(h) + \epsilon H_1(h) + \cdots$ into (5), we have $H_0 = 0$ from the coefficients of ϵ^0. Hence we may assume $U_t = \epsilon^2 U_T$ for $T := \epsilon^2 t$ by the redefinition of time scale. Next considering terms of order ϵ^1, we have

$$-H_1 \mathbb{S}_l = \mathcal{L}U_1 + \mathbb{B}_1\mathbb{S} + G_1(\mathbb{S}), \tag{6}$$

where $\mathcal{L} := F'(\mathbb{S})$, the linearized operator with respect to $\mathbb{S}$. In fact, it is explicitly expressed by

$$\mathcal{L} := \begin{pmatrix} L - \frac{qr}{\tau(p-1)} \left\langle S^{r-1}, \cdot \right\rangle \zeta^\gamma S & -\frac{q\gamma}{\tau(p-1)\zeta} S \\ \frac{r}{\tau}\left\langle S^{r-1}, \cdot \right\rangle \zeta^{\gamma+1} & \frac{\gamma}{\tau} \end{pmatrix}, \tag{7}$$

where $L := \Delta - 1 + pS^{p-1}$ with the Neumann boundary condition. We note that $\mathcal{L}\mathbb{S}_l = 0$ because $F(U)$ is free from translation with respect to l. Moreover, it is easily checked that the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ satisfies $\mathcal{L}^*\mathbb{S}_l = 0$. Hence, (6) implies

$$-H_1 \langle \mathbb{S}_l, \mathbb{S}_l\rangle = \langle \mathbb{B}_1\mathbb{S} + G_1(\mathbb{S}), \mathbb{S}_l\rangle. \tag{8}$$

Lemma 2.2. *The right hand side of (8) is zero.*

Proof. $\kappa = \kappa(h+\epsilon l)$ is expanded as $\kappa = \kappa(h) + \epsilon l \kappa_\sigma(h) + \cdots$. Hence $B(\epsilon)u = B_1 u + \epsilon B_2 u + \cdots$ is given by $B_1 u = \kappa(h)\{-u_\mu + 2\mu u_{ll}\}$ and $B_2 u = \kappa_\sigma(h)\{\mu u_l - l u_\mu + 2\mu l u_{ll}\} + \kappa^2(h)\{-\mu u_\mu + 3\mu^2 u_{ll}\}$ and so on. Similarly, $G_1(U) = \kappa(h)\xi^\gamma \langle u^r, \mu\rangle \cdot \begin{pmatrix} \frac{q}{\tau(p-1)}u \\ \frac{1}{\tau}\xi \end{pmatrix}$ and $G_2(U) = \kappa_\sigma(h)\xi^\gamma \langle u^r, \mu l\rangle \begin{pmatrix} \frac{q}{\tau(p-1)}u, \\ \frac{1}{\tau}\xi \end{pmatrix}$ hold.

Since $\mathbb{S}_l = (\cos\theta S_\rho, 0)$, $\mathbb{S}_\mu = (\sin\theta S_\rho, 0)$ and

$$S_{ll} = \frac{\sin^2\theta}{\rho} S_\rho + \cos^2\theta S_{\rho\rho} \tag{9}$$

hold, the direct calculation of the right hand side of (8) gives this proof. □

Thus, we get $H_1 = 0$.

Next we shall consider H_2. The coefficients of ϵ^2 in (5) leads to

$$-H_2\mathbb{S}_l = \mathcal{L}U_2 + \mathbb{B}_1 U_1 + \mathbb{B}_2\mathbb{S} + G_1'(\mathbb{S})U_1 + G_2(\mathbb{S}) + \frac{1}{2}F''(\mathbb{S})(U_1)^2$$

and

$$-\langle \mathbb{S}_l, \mathbb{S}_l\rangle H_2 = \langle \mathbb{B}_2\mathbb{S} + G_2(\mathbb{S}), \mathbb{S}_l\rangle + \left\langle \mathbb{B}_1 U_1 + G_1'(\mathbb{S})U_1 + \frac{1}{2}F''(\mathbb{S})(U_1)^2, \mathbb{S}_l \right\rangle. \tag{10}$$

In order to obtain H_2, we have to solve U_1. Let $E := Ker\mathcal{L} = span\{\mathbb{S}_l\}$ and $E^\perp := \{U;\ \langle U, \mathbb{S}_l\rangle = 0\}$. Now we may assume $U_j \in E^\perp$. Since U_1 satisfies

$$0 = \mathcal{L}U_1 + \mathbb{B}_1\mathbb{S} + G_1(\mathbb{S}), \tag{11}$$

and Lemma 2.2 implies $\mathbb{B}_1\mathbb{S} + G_1(\mathbb{S}) \in E^\perp$, U_1 is given by $U_1 = \kappa(h)\Phi_1$, where $\Phi_1 \in E^\perp$ is the unique solution of

$$0 = \mathcal{L}\Phi_1 + \begin{pmatrix} -S_\mu + 2\mu S_{ll} \\ 0 \end{pmatrix} + \zeta^\gamma \langle S^r, \mu\rangle \begin{pmatrix} \frac{q}{\tau(p-1)}S, \\ \frac{1}{\tau}\zeta \end{pmatrix},$$

which shows $\Phi_1 = \Phi_1(l,\mu)$ is even with respect to l. Hence U_1 is also even for l and we have $\left\langle \mathbb{B}_1 U_1 + G_1'(\mathbb{S})U_1 + \frac{1}{2}F''(\mathbb{S})(U_1)^2, \mathbb{S}_l \right\rangle = 0$ in (10). The direct calculations give $\langle \mathbb{S}_l, \mathbb{S}_l\rangle = \frac{\pi}{2}\int_0^\infty \rho(S_\rho)^2 d\rho$ and

$$\langle \mathbb{B}_2\mathbb{S} + G_2(\mathbb{S}), \mathbb{S}_l\rangle = -\frac{2}{3}\kappa_\sigma(h)\int_0^\infty \rho^2(S_\rho)^2 d\rho.$$

Thus, (10) shows $H_2 = \frac{2}{3\langle \mathbb{S}_l, \mathbb{S}_l\rangle}\int_0^\infty \rho^2(S_\rho)^2 d\rho\,\kappa_\sigma(h)$, that is,

$$h_t = \epsilon^3 M_0 \kappa_\sigma(h), \tag{12}$$

where $M_0 := \frac{4\int_0^\infty \rho^2(S_\rho)^2 d\rho}{3\pi\int_0^\infty \rho(S_\rho)^2 d\rho}$.

3. **Main results.** In this section, we use same notations and symbols as in Section 2.

Define $\Omega(\delta) := \{x = \Gamma(\sigma) + z\nu(\sigma),\ 0 \le \sigma \le \sigma_0,\ 0 \le z < \delta\}$. We fix sufficiently small $\delta > 0$ and represent $\Omega = \Omega_0 \cup \Omega_1$, where $\Omega_1 := \Omega(2\delta)$ and $\Omega_0 := \Omega\backslash\Omega(\delta)$. Hereafter in this section, c and c_j denote general constants independent of ϵ and δ. Let $\chi_0(x)$ and $\chi_1(x)$ be cut-off functions such that $0 \le \chi_j(x) \le 1$, $\chi_0(x) + \chi_1(x) = 1$, $\chi_0(x) = 1$ and $\chi_1(x) = 0$ for $x \in \Omega\backslash\Omega_1$, $\chi_0(x) = 0$ and $\chi_1(x) = 1$ for $x \in \Omega(\delta)$.

In Ω_1, we can define the tubular neighborhood $\sigma = \Sigma(x)$ and $z = Z(x)$ by $x = \Gamma(\sigma) + z\nu(\sigma)$. Define $S(x;h) := \chi_1(x)S(\frac{\rho(x;h)}{\epsilon})$ and $\mathbb{S}(x;h) := (S(x;h), \zeta)$, where $\rho(x;h) := \sqrt{(\sigma-h)^2 + z^2}$ for $\sigma = \Sigma(x)$ and $z = Z(x)$. Here, we extend $\rho(x;h)$ to the whole domain Ω so as to satisfy $c_1 Dist\{x, \Gamma(h)\} \leq \rho(x;h) \leq c_2 Dist\{x, \Gamma(h)\}$ for positive constants c_1, c_2.

Let $\Omega_2 := \Omega_0 \cap \Omega_1$. Since $S(\rho)$ satisfies $S(\rho) \to e^{-\rho}/\sqrt{\rho}$ as $\rho \to +\infty$, $S(x;h) \leq O(e^{-\delta/\epsilon})$ holds in Ω_2. Define a positive function $\chi_2(\rho)$ satisfying $\chi_2(\rho) = O(e^{-\rho/2})$ as $\rho \to +\infty$ and $\mathbb{X}_2(\rho) := diag(\chi_2(\rho), 1)$. Let $X(h) := \{u;\ u(x) = \chi_2(\rho(x;h)/\epsilon)v(x),\ v \in L^\infty(\Omega)\}$ and $\mathbb{X}(h) := \{(u,\xi);\ u \in X(h),\ \xi \in \mathbb{R}\}$ with the norm $\|(u,\xi)\|_{\mathbb{X}(h)} := \|\chi_2^{-1}(\rho(x;h)/\epsilon)u(x)\|_\infty + |\xi|$.

Theorem 3.1. *Let*

$$\omega_a := \{z \in \mathbb{C};\ \mathrm{Re} z > -a\} \backslash \{0\},$$
$$\omega_{a',a_1} := \{be^{i\theta} \in \mathbb{C};\ |\theta| < \pi/2 + a', b > a_1\}.$$

Suppose that there are $a_0 > 0$, $a_0' > 0$, $a_1 > 0$ such that the following hold: $\mathcal{L}$ has a simple zero eigenvalue, the set $\omega_{a_0} \cup \omega_{a_0',a_1}$ is in the resolvent set of $\mathcal{L}$, and there is $C > 0$ such that $\left\|(\lambda - \mathcal{L})^{-1}\right\| \leq C/|\lambda|$ for $\lambda \in \omega_{a_0',a_1}$. Then the solution U of (4) satisfies

$$U(t,x) = \mathbb{S}(x;h(t)) + \mathbb{X}_2(\rho(x;h)/\epsilon)V(t,x)$$

uniformly for $t > 0$ and $x \in \Omega$ with $\|V(t)\|_\infty = O(\epsilon^3)$ if an initial data $U(0) \in \mathbb{X}(h(0))$ and $\|U(0) - \mathbb{S}(\cdot;h(0))\|_{\mathbb{X}(h(0))} < \delta$ for sufficiently small $\delta > 0$. Moreover,

$$h_t = \epsilon^3 M_0 \kappa_\sigma(h) + O(\epsilon^4) \tag{13}$$

holds.

Theorem 3.2. *Assume that $p = r - 1$. If τ is small, then the assumptions in Theorem 3.1 hold, i.e.,*
(i) *0 is a simple eigenvalue of $\mathcal{L}$ and there are $a_0 > 0$, $a_0' > 0$, $a_1 > 0$ such that the set $\omega_{a_0} \cup \omega_{a_0',a_1}$ is in the resolvent set of $\mathcal{L}$, and*
(ii) *there is $C > 0$ such that*

$$\left\|(\lambda - \mathcal{L})^{-1}\right\| < C/|\lambda| \quad \text{for} \quad \lambda \in \omega_{a_0',a_1}.$$

Hence, the conclusions in Theorem 3.1 hold.

Above results are of solutions with single peak on the boundary. If solution of (2) has two peaks on the boundary, it is strongly unstable. We shall prove this expectation.

In (2), we set $A = \epsilon^{-qn/(p-1)\gamma}u$ and $\Xi = \epsilon^{-n/\gamma}\xi$ and consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \epsilon^2 \Delta u - u + \dfrac{u^p}{\xi^q}, \\ \tau \dfrac{d\xi}{dt} = -\xi + \dfrac{1}{\epsilon^n \xi^s |\Omega|} \displaystyle\int_\Omega u^r dx, \\ \partial_\nu u = 0. \end{cases} \tag{14}$$

In Theorems 3.1 and 3.2, Ω is just a two dimensional domain. On the other hand, when we study the instability of two peaks, it is an n-dimensional domain for $n \geq 1$. Here we suppose that there is a stationary solution such as

$$u(x) \sim \zeta^{\frac{q}{p-1}} \left\{ S\left(\frac{x-h_1}{\epsilon}\right) + S\left(\frac{x-h_2}{\epsilon}\right) \right\}, \qquad \xi \sim \zeta,$$

where $h_1, h_2 \in \overline{\Omega}$, S is a unique positive radially symmetric solution of

$$\Delta S - S + S^p = 0 \quad \text{in } \mathbb{R}^n$$

as given in Theorem 2.1, and

$$\zeta = \left(\frac{|\Omega|}{2\int_{\mathbb{R}^n} S^r dy} \right)^{1/\gamma}.$$

The function S is called *ground state solution* and it has an exponentially decaying property.

In order to study the stability, we naturally introduce the following linearized eigenvalue problem :

$$\begin{cases} \lambda\phi = \epsilon^2 \Delta\phi - \phi + p\dfrac{u^{p-1}}{\xi^q}\phi - q\dfrac{u^p}{\xi^{q+1}}\eta, \\ \tau\lambda\eta = -\eta + \dfrac{r}{\epsilon^n \xi^s |\Omega|}\displaystyle\int_\Omega u^{r-1}\phi dx - \dfrac{s\eta}{\epsilon^n \xi^{s+1}|\Omega|}\displaystyle\int_\Omega u^r dx, \\ \partial_\nu \phi = 0. \end{cases} \tag{15}$$

We shall look for a positive eigenvalue of this problem and prove the instability of two peaks.

Theorem 3.3. *Let $\partial\Omega \in C^2$, $r > 1$. In addition, suppose that there is a stationary solution (u, ξ) of (14) such that u has 2-spikes, i.e.,*

$$\left\| u(x) - \zeta^{\frac{q}{p-1}} \left\{ S(\frac{x-h_1}{\epsilon}) + S(\frac{x-h_2}{\epsilon}) \right\} \right\|_{L^\infty(\Omega)} \to 0, \quad |\xi - \zeta| \to 0$$

for $h_i \in \overline{\Omega}$ which satisfies $(h_1 - h_2)/\epsilon \to \infty$ as $\epsilon \to 0$. Then (15) has at least one eigenvalue around $\lambda = \lambda_1$, where λ_1 is a unique positive eigenvalue of

$$\lambda_1 \psi = \Delta\psi - \psi + pS^{p-1}\psi \quad \text{in } \mathbb{R}^n. \tag{16}$$

Thanks to $\lambda_1 > 0$, this theorem says that any solution with two peaks in the shadow Gierer-Meinhardt system is always strongly unstable. In fact, it seems that any solution with K-peaks for $K \geq 2$ is strongly unstable and the associated linearized system has $(K-1)$ eigenvalues around λ_1. Hence it is sufficient to study the dynamics of a solution with single peak in the shadow Gierer-Meinhardt system.

4. **Proof of Theorem 3.1.** We write (4) as $U_t = \mathbb{F}(U)$, where $U = {}^t(u, \xi)$. Let $K_1[\xi, u] := -1 + \frac{\xi^\gamma}{\epsilon^2}\int_\Omega u^r dx$. Now we consider functions $u(x) = \chi_2(\rho(x;h)/\epsilon)v(x)$ for $v \in L^\infty(\Omega)$. Then

$$\begin{aligned} K_1[\xi, u] &= -1 + \frac{\xi^\gamma}{\epsilon^2}\int_{\Omega_1} u^r dx + O(e^{-c\delta/\epsilon}\|v\|_\infty^r) \\ &= -1 + \frac{\xi^\gamma}{\epsilon^2}\int_0^{\sigma_0}\int_0^{2\delta} u^r(\sigma, z)(1 - \kappa(\sigma)z) d\sigma dz + O(e^{-c\delta/\epsilon}\|v\|_\infty^r) \\ &= -1 + \xi^\gamma \int_{-\sigma_0/2\epsilon}^{\sigma_0/2\epsilon}\int_0^{2\delta/\epsilon} u^r(\sigma, z)(1 - \epsilon\kappa(h + \epsilon l)\mu) dl d\mu + O(e^{-c\delta/\epsilon}\|v\|_\infty^r) \\ &= K_2^+[\xi, u] - \epsilon K_3^+[\xi, u] + O(e^{-c\delta/\epsilon}\|v\|_\infty^r) \end{aligned}$$

holds for small $\delta > 0$, where $\sigma = h + \epsilon l$, $z = \epsilon\mu$ and $K_2^+[\xi, u] := -1 + \xi^\gamma \int_{\mathbb{R}^2_+} u^r dl d\mu$, $K_3^+[\xi, u] := \xi^\gamma \int_{\mathbb{R}^2_+} u^r(\sigma, z)\kappa(h + \epsilon l)\mu dl d\mu$. Here we assume $u(x) = \chi_2(\rho(x;h)/\epsilon)v(x)$ $= \chi_2(\rho(x;h)/\epsilon)v(\sigma, z)$ in Ω_1 is extended to $\mathbb{R}^2_+$ appropriately with respect to the variables σ and z. Specifically, we extend $v(\sigma, z)$ in Ω_1 smoothly to $\widetilde{v}(\sigma, z)$ in $\mathbb{R}^2_+$ such that $\widetilde{v}(\sigma, z) = v(\sigma, z)$ in Ω_1, $\|\widetilde{v}\|_{L^\infty(\mathbb{R}^2_+)} \leq \|v\|_{L^\infty(\Omega_1)}$ and $\rho(x;h)$ is extended by $\rho = \sqrt{(\sigma-h)^2+z^2}$ for $(\sigma, z) \in \mathbb{R}^2_+$ as it is. Throughout this paper, functions are extended to those in $\mathbb{R}^2_+$ like this manner without notes.

Let $U_1 = U_1(l,\mu) = \kappa(h)\Phi_1(l,\mu)$ and $U_2 = U_2(l,\mu)$ be the functions constructed in Section 2, where $\sigma = h + \epsilon l$ and $z = \epsilon\mu$. Define $\Psi(x;h) := \mathbb{X}_1(x)\{\mathbb{S}(l,\mu) + \epsilon U_1(l,\mu) + \epsilon^2 U_2(l,\mu)\} = \mathbb{S}(x;h) + \epsilon\Psi_1(x;h,\epsilon)$, where $\mathbb{X}_1(x) := diag(\chi_1(x), 1)$. Substituting $U = \Psi(x;h) + \mathbb{X}_2(\rho(x;h)/\epsilon)V(t,x)$ into $U_t = \mathbb{F}(U)$, we have

$$h_t\partial_h(\Psi + Y) + Y_t = \mathbb{F}(\Psi) + \mathbb{F}'(\Psi)Y + N(Y) \tag{17}$$

with $|N_1(Y)(x)| \le c\chi_2(\rho(x;h)/\epsilon)\|V\|^2_\infty$ for $Y = \mathbb{X}_2(\rho(x;h)/\epsilon)V$, where $\|V\|_\infty = \|(v(x),\xi)\|_\infty := \|v\|_\infty + |\xi|$ and $N(Y) = {}^t(N_1(Y), N_2(Y))$. Here we note that $|\partial^2_{uu}(K_1[\xi,S])u^2(x)| \le c\chi_2(\rho(x;h)/\epsilon)\|v\|^2_\infty$ holds for $u = \chi_2(\rho(x;h)/\epsilon)v$ because of the exponentially decaying properties of S. Since the first components of $\mathbb{S}$, U_1 and U_2 are $O(e^{-c\delta/\epsilon})$ in Ω_2, Ψ satisfies

$$\mathbb{F}(\Psi) - \epsilon^3 H_2\partial_h\Psi = \epsilon^3\Psi_3 \tag{18}$$

for some $\Psi_3 = \Psi_3(x;h) = (\psi_3(x;h), \xi_3) \in \mathbb{X}(h)$. Let $\mathbb{L}(h) := \mathbb{F}'(\mathbb{S}(x;h))$. $\mathbb{L}(h)$ is given by

$$\mathbb{L}(h)\begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} \mathcal{A}(h)u - \frac{q}{\tau(p-1)}\left\{K_1[\zeta, S(;h)]u + \frac{r\zeta^\gamma S(;h)}{\epsilon^2}\int_\Omega S(x;h)^{r-1}udx \right. \\ \left. + \frac{\gamma\zeta^{\gamma-1}S(;h)}{\epsilon^2}\int_\Omega S(x;h)^r dx\xi\right\} \\ \frac{1}{\tau}\left\{K_1[\zeta, S(;h)]\xi + \frac{r\zeta^{\gamma+1}}{\epsilon^2}\int_\Omega S(x;h)^{r-1}udx + \frac{\gamma\zeta^\gamma}{\epsilon^2}\int_\Omega S(x;h)^r dx\xi\right\} \end{pmatrix} \tag{19}$$

for $u \in X(h)$, where $S(;h) = S(x;h)$ and $\mathcal{A}(h)u := \epsilon^2\Delta u - u + pS(;h)^{p-1}u$. Since $|S(x;h)| \le O(e^{-\rho(x;h)/\epsilon})$ and $K_2^+[\zeta, S] = 0$ hold, we have for small $\delta > 0$

$$\begin{aligned}
K_1[\zeta, S(;h)] &= K_2^+[\zeta,S] - \epsilon K_3^+[\zeta,S] + O(e^{-c\delta/\epsilon}) \\
&= -\epsilon K_3^+[\zeta,S] + O(e^{-c\delta/\epsilon}) \\
&= -\epsilon\zeta^\gamma\langle S^r, \mu\kappa\rangle + O(e^{-c\delta/\epsilon}), \\
\int_\Omega S(x;h)^{r-1}udx &= \int_{\Omega(\delta)} S(x;h)^{r-1}udx + O(e^{-c\delta/\epsilon}) \\
&= \epsilon^2\int_{\mathbb{R}^2_+} S^{r-1}(\rho)u(l,\mu)(1-\epsilon\mu\kappa)dld\mu + O(e^{-c\delta/\epsilon}) \\
&= \epsilon^2\int_{\mathbb{R}^2_+} S^{r-1}(\rho)u(l,\mu)dld\mu \\
&\quad -\epsilon^3\int_{\mathbb{R}^2_+} S^{r-1}(\rho)u(l,\mu)\mu\kappa dld\mu + O(e^{-c\delta/\epsilon}) \\
&= \epsilon^2\left\langle S^{r-1}, u\right\rangle - \epsilon^3\left\langle S^{r-1}\mu\kappa, u\right\rangle + O(e^{-c\delta/\epsilon}), \\
\int_\Omega S(x;h)^r dx &= \epsilon^2\int_{\mathbb{R}^2_+} S^r(\rho)(1-\epsilon\mu\kappa)dld\mu + O(e^{-c\delta/\epsilon}) \\
&= \frac{\epsilon^2}{\zeta^\gamma} - \epsilon^3\int_{\mathbb{R}^2_+} S^r(\rho)\mu\kappa dld\mu + O(e^{-c\delta/\epsilon}) \\
&= \frac{\epsilon^2}{\zeta^\gamma} - \epsilon^3\langle S^r, \mu\kappa\rangle + O(e^{-c\delta/\epsilon}),
\end{aligned}$$

where $\rho = \sqrt{l^2+\mu^2}$ and $\kappa = \kappa(h+\epsilon l)$. Thus $\mathbb{L}(h)$ is represented as

$$\begin{aligned}\mathbb{L}(h)U &= \begin{pmatrix} \mathcal{A}(h)u - \frac{q}{\tau(p-1)}\left\{-\epsilon\zeta^\gamma \langle S^r,\kappa\rangle u + r\zeta^\gamma S(\left\langle S^{r-1},u\right\rangle \right. \\ \left. -\epsilon\left\langle S^{r-1}\mu\kappa,u\right\rangle) + \frac{\gamma S}{\zeta}(1-\epsilon\zeta^\gamma\langle S^r,\mu\kappa\rangle)\xi\right\} \\ \frac{1}{\tau}\left\{-\epsilon\zeta^\gamma\langle S^r,\mu\kappa\rangle\xi + r\zeta^{\gamma+1}(\left\langle S^{r-1},u\right\rangle\right. \\ \left. -\epsilon\left\langle S^{r-1}\mu\kappa,u\right\rangle) + \gamma(1-\epsilon\zeta^\gamma\langle S^r,\mu\kappa\rangle)\xi\right\}\end{pmatrix} + O(e^{-c\delta/\epsilon})U \\ &= \mathbb{A}(h)U + K_4^+U + \epsilon K_5^+U + O(e^{-c\delta/\epsilon})U,\end{aligned}$$

where $U = {}^t(u,\xi)$ and

$$\begin{aligned}\mathbb{A}(h)U &:= \begin{pmatrix}\mathcal{A}(h)u \\ 0\end{pmatrix}, \\ K_4^+U &:= \begin{pmatrix} -\frac{q}{\tau(p-1)}(r\zeta^\gamma S\left\langle S^{r-1},u\right\rangle + \frac{\gamma S}{\zeta}\xi) \\ \frac{1}{\tau}(r\zeta^{\gamma+1}\left\langle S^{r-1},u\right\rangle + \gamma\xi)\end{pmatrix} \\ &= \frac{r\zeta^{\gamma+1}\left\langle S^{r-1},u\right\rangle + \gamma\xi}{\tau\zeta}\begin{pmatrix} -\frac{q}{p-1}S \\ \zeta\end{pmatrix},\end{aligned}$$

$$\begin{aligned}K_5^+U &:= \begin{pmatrix} \frac{q}{\tau(p-1)}(\zeta^\gamma\langle S^r,\mu\kappa\rangle u + r\zeta^\gamma S\left\langle S^{r-1}\mu\kappa,u\right\rangle + \frac{\gamma S}{\zeta}\zeta^\gamma\langle S^r,\mu\kappa\rangle\xi) \\ -\frac{1}{\tau}(\zeta^\gamma\langle S^r,\mu\kappa\rangle\xi + r\zeta^{\gamma+1}\left\langle S^{r-1}\mu\kappa,u\right\rangle + \gamma\zeta^\gamma\langle S^r,\mu\kappa\rangle\xi)\end{pmatrix} \\ &= \zeta^\gamma\langle S^r,\mu\kappa\rangle\begin{pmatrix}\frac{q}{\tau(p-1)}u \\ -\frac{1}{\tau}\xi\end{pmatrix} - \frac{r\zeta^\gamma\left\langle S^{r-1}\mu\kappa,u\right\rangle + \gamma\zeta^{\gamma-1}\langle S^r,\mu\kappa\rangle\xi}{\tau}\begin{pmatrix}-\frac{q}{p-1}S \\ \zeta\end{pmatrix}.\end{aligned}$$

Lemma 4.1. *The spectral set, say $I(h)$ of $\mathbb{L}(h)$ consists of $I_1(h)$ and $I_2(h)$ such that $I_1(h) \subset \{|\lambda| \le c\sqrt{\delta}\}$ and $I_2(h) \subset \{Re\lambda < -\alpha_1\}$ for a positive constant α_1.*

Proof. Let $\Omega_1' := \Omega(\delta)$ and first consider $\mathbb{L}(h)$ in Ω_1'. Since $\mathcal{A}(h)$ is expressed as $\mathcal{A}(h) = L + \epsilon B(\epsilon)$ in Ω_1' by using the tubular coordinate (l,μ), $\mathbb{L}(h)$ is represented as $\mathcal{L} + \epsilon\mathbb{B}(\epsilon) + \epsilon K_5^+ + K_6$ for $0 \le l \le \sigma_0/\epsilon$ and $0 \le \mu \le \delta/\epsilon$, where $B(\epsilon)$ and $\mathbb{B}(\epsilon)$ are in Section 2 and K_6 is an operator with $O(e^{-c\delta/\epsilon})$ operator norm. Note that $\mathcal{L} = \begin{pmatrix} L & 0 \\ 0 & 0\end{pmatrix} + K_4^+$. We may assume all of the above operators are appropriately extended to those in $\mathbb{R}^2_+$. In fact, such extensions are trivially done for K_j^+ and K_6 while the extension of $B(\epsilon)$ is not trivial. $B(\epsilon)$ is precisely expressed as $B(l,z;h,\epsilon)$. Since the domain $(l,\mu) \in [0,\sigma_0/\epsilon]\times[0,\delta/\epsilon]$ is connected at $l=0$ and σ_0/ϵ, we can extend it to the operator with periodic coefficients with respect to l coordinate, that is, the operator for $(l,\mu) \in (-\infty,\infty)\times[0,\delta/\epsilon]$ satisfying $B(l+\sigma_0/\epsilon,z;h,\epsilon) = B(l,z;h,\epsilon)$. For z coordinate, by multiplying the cut-off function $\chi_1(x)$, we have the operator defined in $\mathbb{R}^2_+$. Moreover, $|\epsilon B(\epsilon)u| \le O(\delta)\|u\|_{C^2}$ for $(l,\mu) \in [0,\sigma_0/\epsilon]\times[0,\delta/\epsilon]$ holds and hence we may assume $\epsilon B(\epsilon)$ is extended to $\mathbb{R}^2_+$ satisfying this same estimate, that is, $|\epsilon B(\epsilon)u| \le O(\delta)\|u\|_{C^2(\mathbb{R}^2_+)}$ holds.

By the assumption of this theorem, the set $\omega_{a_0} \cup \omega_{a_0',a_1}$ is in the resolvent set of $\mathcal{L}$. Then it is easily checked that

$$\|\epsilon(\lambda-\mathcal{L})^{-1}\mathbb{B}(\epsilon)\| \le c\delta(1+\frac{1}{|\lambda|}) \tag{20}$$

holds for $\lambda \in \omega_{a_0}\cup\omega_{a_0',a_1}$ and a constant $c > 0$.

Define $\mathcal{L}_1 := \mathcal{L} + \epsilon\mathbb{B}(\epsilon) + \epsilon K_5^+ + K_6$ and consider it as an operator in $L^\infty(\mathbb{R}^2_+)$ in the same manner as $B(\epsilon)$ above. That is, $\mathcal{L}_1$ is an operator with periodic coefficients with respect to l.

Suppose $\lambda \in \omega_{a_0} \cup \omega_{a_0',a_1}$ and consider $(\lambda - \mathcal{L}_1)U = g$. Then $(\lambda - \mathcal{L}_1)U = g$ becomes

$$\{Id - (\lambda - \mathcal{L})^{-1}\epsilon\mathbb{B}(\epsilon) - \epsilon(\lambda - \mathcal{L})^{-1}K_5^+ - (\lambda - \mathcal{L})^{-1}K_6\}U = (\lambda - \mathcal{L})^{-1}g. \quad (21)$$

(20) implies

$$\|\epsilon(\lambda - \mathcal{L})^{-1}\mathbb{B}(\epsilon)\| \le c\delta(1 + \frac{1}{|\lambda|}) \le c'(\delta + \sqrt{\delta})$$

for constants $c,\ c' > 0$ if $|\lambda| \ge c''\sqrt{\delta}$ for $c'' > 0$. Since K_5^+ is bounded and $\|K_6\| \le O(e^{-c\delta/\epsilon})$, the operator in the left hand side of (21) is invertible, which shows

$$\|(\lambda - \mathcal{L}_1)^{-1}\| \le \frac{c}{|\lambda|} \quad (22)$$

for $\lambda \in \omega_{a_0} \cup \omega_{a_0',a_1}$ with $|\lambda| \ge c'\sqrt{\delta}$, where c and c' are positive constants. (22) holds in $L^\infty(\mathbb{R}^2_+)$ and if $g \in L^\infty(\mathbb{R}^2_+)$ is periodic with respect to l argument, the function U of the equation $(\lambda - \mathcal{L}_1)U = g$ is also periodic with respect to l argument by the periodicity of the operator $\mathcal{L}_1$ with respect to the same argument. Thus we may also assume (22) in $L^\infty(\mathbb{R}'_+)$, where $\mathbb{R}'_+ := \{0 \le l \le \sigma_0/\epsilon,\ 0 \le \mu < \infty\}$. We denote by $\mathbb{L}_1(h)$ the operator corresponding to $\mathcal{L}_1$ expressed with the original coordinate $x \in \Omega_1$. Multiplying S byχ_1, we may assume $\mathbb{L}_1(h)$ is defined in $\Omega_3 := \Omega(3\delta)$ and $\mathbb{L}_1(h) = \mathbb{L}(h)$ in Ω_3.

Let $\mathcal{A}_0 u := \epsilon^2\Delta u - u$, $\mathbb{A}_0 := \begin{pmatrix} \mathcal{A}_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{L}_0(h) := \mathbb{A}_0 + K_4^0 + \epsilon K_5^0$ in $L^\infty(\Omega)$ with the Neumann boundary condition, where

$$\begin{aligned} K_4^0 U &:= \frac{r\zeta^{\gamma+1}\left\langle S^{r-1}, u\right\rangle + \gamma\xi}{\tau}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ K_5^0 U &:= \zeta^\gamma \left\langle S^r, \kappa\right\rangle \begin{pmatrix} \frac{q}{\tau(p-1)}u \\ -\frac{1}{\tau}\xi \end{pmatrix} - \frac{r\zeta^\gamma\left\langle S^{r-1}\mu\kappa, u\right\rangle + \gamma\zeta^{\gamma-1}\left\langle S^r, \mu\kappa\right\rangle\xi}{\tau}\begin{pmatrix} 0 \\ \zeta \end{pmatrix} \end{aligned}$$

for $U = {}^t(u, \xi)$. $\mathbb{L}_0(h)$ is clearly invertible in $L^\infty(\Omega)$ for sufficiently small $\epsilon > 0$ and the spectral set is in the left hand side uniformly apart from the imaginary axis. Hence we assume $I^0(h) \subset \{Re\lambda < -\alpha_2\}$ for a positive constant α_2, where $I^0(h)$ is the spectral set of $\mathbb{L}_0(h)$.

Define $\Omega_0' := \Omega\backslash\Omega_1$ and $D(\lambda) := \chi_1(x)(\lambda - \mathbb{L}_1(h))^{-1}\widetilde{\chi}_1(x) + \chi_0(x)(\lambda - \mathbb{L}_0(h))^{-1}$ for $\lambda \in \omega_{a_0} \cup \omega_{a_0',a_1}$ with $|\lambda| \ge c'\sqrt{\delta}$, where $\widetilde{\chi}_1(x)$ is a cut-off function satisfying $0 \le \widetilde{\chi}_1(x) \le 1$, $\widetilde{\chi}_1(x) = 1$ for $x \in \Omega_1$ and $\widetilde{\chi}_1(x) = 0$ for $x \in \Omega\backslash\Omega_3$. Here we can assume $a_0 < \alpha_2$. Then we have

$$(\lambda - \mathbb{L}(h))D(\lambda) = Id + O(e^{-c\delta/\epsilon}) \quad (23)$$

in $\Omega_1' \cup \Omega_0'$ and therefore it suffices to consider $D(\lambda)$ in $\Omega_2 := \Omega_1 \cap \Omega_0$ because of $\Omega = \Omega_1' \cup \Omega_0' \cup \Omega_2$.

Let $U_1 = {}^t(u_1, \xi_1) := (\lambda - \mathbb{L}_1(h))^{-1}\widetilde{\chi}_1 g$ and $U_0 = {}^t(u_0, \xi_0) := (\lambda - \mathbb{L}_0(h))^{-1}g$ for $g \in L^\infty(\Omega)$. In Ω_2, $|\mathbb{L}_1(h)U - \mathbb{L}_0(h)U| \le O(e^{-c\delta/\epsilon})$ holds. Let $\Omega_{1/2} := \Omega(\frac{1}{2}\delta)$ and $\Omega_4 := \Omega\backslash\Omega_{1/2}$.

Proposition 1. *For λ with $Re\lambda > -a_2 (> -\alpha_2)$,*

$$\|U_1 - U_0\|_{C^2(\Omega_2)} \le c\epsilon^2(\|U_1\|_{C^0(\Omega_3)} + \|U_0\|_{C^0(\Omega_4)}),$$

where $\|U_j\|_{C^k(\Omega_j)} := \|u_j\|_{C^k(\Omega_j)} + |\xi_j|$.

Proof. Let $\Omega'' := \Omega_3 \cap \Omega_4$ and $P(r_0, x_0) := \{x \in \Omega_1;\ |x - x_0| < r_0\}$. Now we fix any $x_0 \in \Omega_2$ and consider two balls $P(r_0\epsilon, x_0) \subset P(r_1\delta, x_0) \subset \Omega''$ for positive constants r_0, r_1.

In Ω_2, $\mathbb{L}_1(h) = \mathbb{L}_0(h) + C_\dagger(h)$ with $\|C_\dagger(h)\| \leq O(e^{-c\delta/\epsilon})$ for $c > 0$. Hence $(\lambda - \mathbb{L}_0(h))(U_1 - U_0) = C_\dagger(h)U_1$ in Ω_2 holds, specially, in $P(r_1\delta, x_0)$. Let $\widetilde{\mathcal{A}}_0 u := \Delta u - u$ and $\widetilde{\mathbb{A}}_0 := \begin{pmatrix} \widetilde{\mathcal{A}}_0 & 0 \\ 0 & 0 \end{pmatrix}$. Taking the stretched coordinate $y := (x - x_0)/\epsilon$, we see the equation $(\lambda - \mathbb{L}_0(h))(U_1 - U_0) = C_\dagger(h)U_1$ in $P(r_1\delta, x_0)$ is

$$(\lambda - \widetilde{\mathbb{L}}_0)(\widetilde{U}_1 - \widetilde{U}_0) = \widetilde{C}_\dagger(h)\widetilde{U}_1,\ y \in \widetilde{P}_1 \tag{24}$$

with $\|\widetilde{C}_\dagger(h)\| \leq O(e^{-c\delta/\epsilon})$, where $\widetilde{\mathbb{L}}_0 := \widetilde{\mathbb{A}}_0 + K_4^0 + \epsilon K_5^0$, $\widetilde{U}_j(y) := U_j(x)$ with $y := (x - x_0)/\epsilon$ and $\widetilde{P}_1 := P(r_1\delta/\epsilon, 0)$. Since ϵ is sufficiently small, $\widetilde{P}_1$ is nearly the whole $\mathbb{R}^2$ and the invertibility of $\widetilde{\mathbb{L}}_0$ in $X_0 := L^\infty(\mathbb{R}^2)$ is clear. We may assume λ is in the resolvent set of $\widetilde{\mathbb{L}}_0$ and $\widetilde{U}_0$, $\widetilde{U}_1$, $\widetilde{C}_1(h)$ are extended to $\mathbb{R}^2$ with the estimates $\|\widetilde{U}_j\|_{X_0} \leq \|\widetilde{U}_j\|_{C^0(\widetilde{P}_1)} \leq \|U_j\|_{C^0(\Omega_j)}$ and $\|\widetilde{C}_\dagger(h)\| \leq O(e^{-c\delta/\epsilon})$. K_4^0 and K_5^0 are considered in this case as functionals with respect to $u \in X_0$. In fact, they are $O(\epsilon^2)$ functionals because $\langle S, u\rangle = O(\epsilon^2)$ for $u \in L^\infty(\mathbb{R}^2_+)$. Let $\widetilde{W} := (\lambda - \widetilde{\mathbb{L}}_0)^{-1}\widetilde{C}_\dagger(h)\widetilde{U}_1 \in X_0$. Then we have

$$\|\widetilde{W}\|_{C^2(\mathbb{R}^2)} \leq O(e^{-c\delta/\epsilon})\|\widetilde{U}_1\|_{X_0} \leq O(e^{-c\delta/\epsilon})\|U_1\|_{C^0(\Omega_3)}$$

and

$$(\lambda - \widetilde{\mathbb{L}}_0)(\widetilde{U}_1 - \widetilde{U}_0 - \widetilde{W}) = 0,\ y \in \widetilde{P}_1. \tag{25}$$

Let $\widetilde{V} = {}^t(\widetilde{v}, \xi) := \widetilde{U}_1 - \widetilde{U}_0 - \widetilde{W}$ and we show

$$\begin{cases} \|\widetilde{v}\|_{C^2(\widetilde{P}_0)} = \epsilon^2\|v\|_{C^2(P_0)} \leq O(e^{-c\delta/\epsilon})\|v\|_{C^0(P_1)}, \\ |\xi| \leq O(\epsilon^2)\|V\|_{C^0(P_1)}, \end{cases} \tag{26}$$

where $\widetilde{P}_0 := P(r_0, 0)$, $P_0 := P(r_0\epsilon, x_0)$, $P_1 := P(r_1\delta, x_0)$ and $V = {}^t(v(x), \xi) := {}^t(\widetilde{v}(y), \xi)$.

Since $(\lambda - \widetilde{\mathbb{L}}_0)\widetilde{V} = 0$, $(\lambda - \widetilde{\mathcal{A}}_0)\widetilde{v} = 0$ for $y \in \widetilde{P}_1$. Let $\widetilde{v}(y) = \Sigma_{n=-\infty}^{\infty} b_n(\rho)e^{in\theta}$ and $\widetilde{v}|_{\partial\widetilde{P}_1} = \Sigma_{n=-\infty}^{\infty} b_n^* e^{in\theta}$ for $y = \rho e^{i\theta}$. Then each $b_n(\rho)$ satisfies

$$\begin{cases} b_n'' + \dfrac{1}{\rho}b_n' - \dfrac{n^2}{\rho^2}b_n - (\lambda + 1)b_n = 0, \\ b_n'(0) = 0,\ b_n(r_1\delta/\epsilon) = b_n^*. \end{cases} \tag{27}$$

Solutions of (27) are given by the Bessel functions $Z_n(\beta\rho)$, where Z_n is a solution of

$$Z_n'' + \frac{1}{\rho}Z_n' - (1 + \frac{n^2}{\rho^2})Z_n = 0$$

and $\beta := \sqrt{\lambda + 1}$.

Now we shall show $Re\beta > c$ for a constant $c > 0$. We may assume $Re(\lambda + 1) > -a_2 + 1 > c' > 0$ for a constant $c' > 0$. Then $Re\beta = \sqrt{|\lambda + 1|}\cos\frac{1}{2}\theta > \sqrt{c'}\frac{1}{\sqrt{2}} > c$ for $c > 0$, where $-\frac{\pi}{2} < \theta := arg(\lambda + 1) < \frac{\pi}{2}$. Since $Z_n(\rho) = O(\frac{1}{\sqrt{\rho}}e^{\pm\rho})$ as $\rho \to \infty$, $|b_n(\rho)| \leq O(e^{-c\rho})|b_n^*| \leq O(e^{-c\rho})\|v\|_{C^0(P_1)}$. Then we have

$$\|\widetilde{v}\|_{C^2(\widetilde{P}_0)} \leq O(e^{-c\delta/\epsilon})\|v\|_{C^0(P_1)}.$$

Since $\widetilde{v} \in X_0$, $|\xi| \leq O(\epsilon^2)\|V\|_{C^0(P_1)}$ is obvious by substituting $\widetilde{v}$ in K_4^0 and K_5^0. This means (26).

Thus, we see

$$\begin{aligned}\|U_1 - U_0\|_{C^2(P_0)} &= \|V - W\|_{C^2(P_1)}\\ &\leq O(\frac{1}{\epsilon^2}e^{-c\delta/\epsilon} + \epsilon^2)\|V - W\|_{C^0(P_1)}\\ &\leq O(\epsilon^2)(\|U_1\|_{C^0(\Omega_3)} + \|U_0\|_{C^0(\Omega_4)}),\end{aligned}$$

which gives the proof. □

We can express $U_0 = U_1 + C_0(h)U_0 + C_1(h)U_1$ in Ω_2 with $\|C_j(h)U\|_{C^2(\Omega_2)} \leq O(\epsilon^2)\|U\|_{C^0(\Omega)}$. Hence it follows in Ω_2

$$(\lambda - \mathbb{L}(h))(\chi_1 U_1 + \chi_0 U_0) = g + C_2(h)g,$$

where

$$C_2(h)g := ((\lambda - \mathbb{L}(h))\chi_0\{C_0(h)(\lambda - \mathbb{L}_0(h))^{-1}g + C_1(h)(\lambda - \mathbb{L}_1(h))^{-1}\widetilde{\chi}_1 g\}$$

and $\|C_2(h)g\|_{C^0(\Omega_2)} \leq O(\epsilon^2)\|g\|_{C^0(\Omega)}$ is satisfied. Since $(\lambda-\mathbb{L}(h))(\chi_1 U_1+\chi_0 U_0) = g$ in $\Omega_1' \cup \Omega_0'$ as in (23), we can assume $C_2(h)$ is defined in Ω with the same estimate.

Thus $(\lambda - \mathbb{L}(h))D(\lambda) = Id + C_2(h)$ holds. Since $\|C_2(h)\| \leq O(\epsilon^2)$, $Id + C_2(h)$ is invertible and we have $(\lambda - \mathbb{L}(h))D(\lambda)(Id + C_2(h))^{-1} = Id$. This means

$$(\lambda - \mathbb{L}(h))^{-1} = D(\lambda)(Id + C_2(h))^{-1} = D(\lambda)(Id + C_3(h)) \tag{28}$$

with $\|C_3(h)\| \leq O(\epsilon^2)$ and λ is in the resolvent set of $\mathbb{L}(h)$. □

Noting $\mathbb{L}(h)$ is a sectorial operator, we define the projections

$$Q(h) := \frac{1}{2\pi i}\int_{\Gamma_1}(\lambda - \mathbb{L}(h))^{-1}d\lambda,$$

$R(h) := Id - Q(h)$ and the eigenspaces $E(h) := Q(h)L^2(\Omega)$, $E^{\perp}(h) := R(h)L^2(\Omega)$, where Γ_1 is a closed circle surrounding $I_1(h)$ in the region $\{Re\lambda > -\alpha_1\}$. Let $\Phi_0(h)(x) := \partial_h\mathbb{S}(x;h) = (\partial_h S(x;h), 0)$, $\Phi_0^*(h)(x) := \frac{1}{\int_\Omega(\partial_h S(x;h))^2dx}\partial_h\mathbb{S}(x;h) = \frac{1}{<\mathbb{S}_l,\mathbb{S}_l>}\partial_h\mathbb{S}(x;h) + O(e^{-c\delta/\epsilon})$ and $Q_0(h)U := \int_\Omega u(x)\phi_0^*(x;h)dx\partial_h\mathbb{S}(x;h)$ for $U = {}^t(u,\xi)$, where $\Phi_0^*(h)(x) = (\phi_0^*(x;h), 0)$.

Lemma 4.2. *$\|Q(h) - Q_0(h)\| \leq c\epsilon^2$ holds.*

Proof. This is shown in quite a similar way to Lemma 5.2 in [7] by using (28). □

Lemma 4.2 implies that $I_1(h) = \{\lambda_0\}$ and $E(h) = span\{\Phi(h)\}$ for $\lambda_0 \in \mathbb{R}$ and a function $\Phi(h)$. The following lemma is also shown in quite a similar way to Lemma 5.2 in [7].

Lemma 4.3. *$\lambda_0 = \lambda_0(h,\epsilon) = O(\epsilon)$ and $\Phi(h)(x) = \partial_h\mathbb{S}(x;h) + O(\epsilon) \in \mathbb{X}(h)$ hold.*

Let $\Phi^*(h)$ be the eigenfunction of $\mathbb{L}^*(h)$, the adjoint operator of $\mathbb{L}(h)$ such that $\mathbb{L}^*(h)\Phi^*(h) = \lambda_0\Phi^*(h)$ and $< \Phi(h), \Phi^*(h) >_\Omega = 1$, where $< U, U' >_\Omega := \int_\Omega u(x)u'(x)dx + \xi\xi' = 1$ for $U = {}^t(u(x), \xi)$ and $U' = {}^t(u'(x), \xi')$. Note that $\Phi^*(h)(x) = \Phi_0^*(h)(x) + O(\epsilon) \in \mathbb{X}(h)$ and $E^{\perp}(h) = \{U;\ < U, \Phi^*(h) >_\Omega = 0\}$ hold.

Let $\epsilon\mathbb{B}^*(h)Y := \mathbb{F}'(\Psi(x;h))Y - \mathbb{L}(h)Y$. Note that $|(B_1^*(h)Y)(x)| \leq c\chi_2(\rho(x;h)/\epsilon) \cdot\|V\|_\infty$ holds for $Y = \mathbb{X}_2(\rho(x;h)/\epsilon)V$ and $\mathbb{B}^*(h) = {}^t(B_1^*(h), B_2^*(h))$. Then (17) is now written as

$$h_t\partial_h(\Psi + Y) + Y_t = \mathbb{F}(\Psi) + \mathbb{L}(h)Y + \epsilon\mathbb{B}^*(h)Y + N(Y) \tag{29}$$

for $Y = \mathbb{X}_2(\rho(x;h)/\epsilon)V$. By (18), we have

$$(h_t - \epsilon^3 H_2)\partial_h(\mathbb{S}(h) + \epsilon\Psi_1) + h_t Y_h + Y_t = \epsilon^3\Psi_3 + \mathbb{L}(h)Y + \epsilon\mathbb{B}^*(h)Y + N(Y) \quad (30)$$

Let X^ω be the fractional space with the norm $\|\cdot\|_\omega$ of $X := L^\infty(\Omega)$ with the norm $\|\cdot\|_\infty$ for $1/2 < \omega < 1$ such that $|\nabla U| \le c\|U\|_\omega$. Define the set $V(D_1) := \{V \in X^\omega;\ \|V\|_\omega \le D_1\epsilon^3\}$. Suppose $Y = Y(t;h)(x) = \mathbb{X}_2(\rho(x;h)/\epsilon)V(x;h) \in E^\perp(h)$ for $V \in V(D_1)$. Then taking the inner product of (30) with $\Phi^*(h)$, we have

$$\begin{aligned}(h_t - \epsilon^3 H_2)(1+O(\epsilon)) + h_t < Y_h, \Phi^*(h) >_\Omega + < Y_t, \Phi^*(h) >_\Omega & \qquad (31)\\ = \quad \epsilon^3 < \Psi_3, \Phi^*(h) >_\Omega + < \mathbb{L}(h)Y + \epsilon\mathbb{B}^*(h)Y + N(Y), \Phi^*(h) >_\Omega &\\ = \quad O(\epsilon^4) + < Y, \lambda_0(h)\Phi^*(h) >_\Omega + \epsilon O(\|V\|_\infty) + O(\|V\|_\infty^2) &\\ = \quad O(\epsilon^4 + \epsilon\|V\|_\infty + \|V\|_\infty^2). &\end{aligned}$$

Here we note that

$$< \Psi_3, \Phi^*(h) >_\Omega = < \Psi_3, \Phi_0^*(h) >_\Omega + O(\epsilon) = \int_\Omega \psi_3(x;h)\phi_0^*(x;h)dx + O(\epsilon) = O(\epsilon)$$

holds because $|\psi_3(x;h)|,\ |\phi_0^*(x;h)| \le c\chi_2(\rho(x;h)/\epsilon)$.

On the other hand, $Y(t,h) \in E^\perp(h)$ implies that $< Y_t, \Phi^*(h) >_\Omega = 0$ and $< Y_h, \Phi^*(h) >_\Omega = - < Y, \Phi_h^*(h) >_\Omega = O(\|V\|_\infty)$. Hence (31) is

$$h_t(1 + O(\epsilon + \|V\|_\infty)) = \epsilon^3 H_2(1 + O(\epsilon)) + O(\epsilon^4 + \epsilon\|V\|_\infty + \|V\|_\infty^2).$$

Let $h_t = J_1 = J_1(h, V; \epsilon)$. Then the above shows

$$J_1(h, V; \epsilon) = \epsilon^3 H_2(h) + O(\epsilon^4 + \epsilon\|V\|_\infty + \|V\|_\infty^2) \quad (32)$$

holds for $V \in V(D_1)$.

Next, operating $R(h)$ to (30), we have

$$(h_t - \epsilon^3 H_2)O(\epsilon) + h_t R(h)Y_h + Y_t = \epsilon^3 R(h)\Psi_3 + R(h)\mathbb{L}(h)Y + \epsilon\mathbb{B}^*(h)Y + R(h)N(Y)$$

by using $R(h)\partial_h\mathbb{S}(h) = O(\epsilon)$. Let $Y_t = R(h)\mathbb{L}(h)Y + \mathbb{J}_2(h, V; \epsilon)$. Then

$$\|\mathbb{J}_2(h, V; \epsilon)\|_{\mathbb{X}(h)} \le O(\epsilon^3 + \epsilon\|V\|_\infty + \|V\|_\infty^2 + |J_1(h, V; \epsilon)| \cdot \|V_h\|_\infty) \quad (33)$$

holds for $Y = \mathbb{X}_2(\rho(x;h)/\epsilon)V$ and $V \in V(D_1)$.

Let $\widehat{E}^\perp(h) := \{V \in X^\omega;\ \mathbb{X}_2(\rho(x;h)/\epsilon)V \in E^\perp(h)\}$ and fix h_0.

Lemma 4.4. *There exist a map $\Pi(h)$ such that $\Pi(h) : \widehat{E}^\perp(h_0) \to \widehat{E}^\perp(h)$ and $\|\Pi_h(h)W\|_\infty \le C\|W\|_\omega$ for $W \in \widehat{E}^\perp(h_0)$.*

Proof. In $\Omega_1' = \Omega(\delta)$, $U = {}^t(u, \xi) \in X^\omega$ is represented by $u = u(\sigma, z)$. First we define a map $\widehat{\Pi}(h)$ by

$$(\widetilde{\Pi}(h)U)(\sigma, z) := {}^t(u(\sigma - h, z), \xi)$$

in Ω_1'. In $\Omega_0 = \Omega\backslash\Omega_1'$, we define

$$(\widetilde{\Pi}(h)U)(x) := U(x) + {}^t(v(h)(x), \xi),$$

where $v(h)(x)$ is a function satisfying

$$\mathcal{A}_0 v = 0,\ x \in \Omega_0, v(x) = u(\sigma - h, z) - u(\sigma, z), x \in \partial\Omega_0 = \partial\Omega_1'. \quad (34)$$

We construct the map $\widehat{\Pi}(h)\ : E^\perp(h_0) \to E^\perp(h)$ by $\widehat{\Pi}(h) := R(h)\widetilde{\Pi}(h)$. Then the map $\Pi(h) : \widehat{E}^\perp(h_0) \to \widehat{E}^\perp(h)$ is given by $\Pi(h) := \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)\widehat{\Pi}(h)\mathbb{X}_2(\rho(x;h)/\epsilon)$. □

For $Y = \mathbb{X}_2(\rho(x;h)/\epsilon)\Pi(h)W$,

$$Y_t = h_t\mathbb{X}_2(\rho(x;h)/\epsilon)\{-c^*(\rho(x;h)/\epsilon)\frac{\rho_h(x;h)}{\epsilon}Id_0\Pi(h) + \Pi_h(h)\}W \\ + \mathbb{X}_2(\rho(x;h)/\epsilon)\Pi(h)W_t$$

holds since we may assume $\chi_2'(\rho) = -c^*(\rho)\chi_2(\rho)$ and $\mathbb{X}_2'(\rho) = -c^*(\rho)\mathbb{X}_2(\rho)Id_0$ for a bounded positive function $c^*(\rho)$ and a matrix $Id_0 := diag(1,0)$. Then the equation $Y_t = R(h)\mathbb{L}(h)Y + \mathbb{J}_2(h,V;\epsilon)$ becomes

$$\begin{aligned} & J_1(h,V;\epsilon)\{-c^*(\rho(x;h)/\epsilon)\frac{\rho_h(x;h)}{\epsilon}Id_0\Pi(h) + \Pi_h(h)\}W + \Pi(h)W_t \\ = \quad & \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)R(h)\mathbb{L}(h)\mathbb{X}_2(\rho(x;h)/\epsilon)\Pi(h)W + \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)\mathbb{J}_2(h,V;\epsilon) \end{aligned}$$

and

$$W_t = \widehat{\mathbb{L}}(h) + \widehat{\mathbb{J}}_2(h,W;\epsilon), \tag{35}$$

where

$$\widehat{\mathbb{L}}(h) = \Pi^{-1}(h)\mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)R(h)\mathbb{L}(h)\mathbb{X}_2(\rho(x;h)/\epsilon)\Pi(h)W,$$

$$\widehat{\mathbb{J}}_2(h,W;\epsilon) = -J_1(h,\Pi(h)W;\epsilon)\Pi^{-1}(h)\{-c^*(\rho(x;h)/\epsilon)\frac{\rho_h(x;h)}{\epsilon}Id_0\Pi(h) + \Pi_h(h)\}W \\ + \Pi^{-1}(h)\mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)\mathbb{J}_2(h,\Pi(h)W;\epsilon).$$

By (33), $\widehat{\mathbb{J}}_2(h,W;\epsilon)$ is estimated by

$$\begin{aligned} \|\widehat{\mathbb{J}}_2(h,W;\epsilon)\|_\infty &\le O(\frac{1}{\epsilon}|J_1(h,\Pi(h)W;\epsilon)|\cdot\|W\|_\infty + \epsilon^3 + \epsilon\|W\|_\infty + \|W\|_\infty^2) \\ &\le O(\frac{1}{\epsilon}\|W\|_\omega^3 + \epsilon^3 + \epsilon\|W\|_\infty + \|W\|_\infty^2). \end{aligned} \tag{36}$$

Let $\widehat{W}(D_1) := \{W \in X^\omega;\ \|W\|_\omega \le D_1\epsilon^3\}$ again instead of $V(D_1)$ and $\widehat{W}(D_1,D_2) := \{W \in C([0,\sigma_0];\widehat{E}^\perp(h_0));\ \|W(h)\|_\omega \le D_1\epsilon^3,\ \|W(h)-W(k)\|_\omega \le D_2\epsilon|h-k|\}$. Then,

$$\|\widehat{\mathbb{J}}_2(h,W;\epsilon) - \widehat{\mathbb{J}}_2(h',W';\epsilon)\|_\infty \le c_1\epsilon(|h-h'| + \|W-W'\|_\omega) \tag{37}$$

holds for $0 \le h,\ h' \le \sigma_0$ and $W,\ W' \in \widehat{W}(D_1,D_2)$, where $c_1 = c_1(D_1,D_2)$ is a positive constant depending on D_1, D_2.

Now, $\widehat{\mathbb{L}}(h)$ is

$$\widehat{\mathbb{L}}(h) = \Pi^{-1}(h)\widehat{R}(h)\widetilde{\mathbb{L}}(h)\Pi(h)W,$$

where $\widehat{R}(h) := \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)R(h)\mathbb{X}_2(\rho(x;h)/\epsilon)$ and $\widetilde{\mathbb{L}}(h) := \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)\mathbb{L}(h)$ $\mathbb{X}_2(\rho(x;h)/\epsilon)$. $\widetilde{\mathbb{L}}(h)$ has the same properties as $\mathbb{L}(h)$, that is, the spectral set $\widetilde{I}(h)$ of $\widetilde{\mathbb{L}}(h)$ consists of $\widetilde{I}_1(h)$ and $\widetilde{I}_2(h)$ such that $\widetilde{I}_1(h) \subset \{|\lambda| \le c\sqrt{\delta}\}$ and $\widetilde{I}_2(h) \subset \{Re\lambda < -\alpha_1\}$ for a positive constant α_1 as in Lemma 4.1. Projections and eigenfunctions are given by $\widetilde{Q}(h) := \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)Q(h)\mathbb{X}_2(\rho(x;h)/\epsilon)$, $\widetilde{\Phi}(h) = \mathbb{X}_2^{-1}(\rho(x;h)/\epsilon)\Phi(h)$ and so on.

The following lemma is proved in a similar manner to [7].

Lemma 4.5.

$$\|(\widehat{\mathbb{L}}(h_1) - \widehat{\mathbb{L}}(h_2))W\|_\infty \le \frac{c}{\epsilon}|h_1 - h_2|\|W\|_\infty$$

holds for $W \in \widehat{W}(D_1)$.

Then, quite a similar manner to [7], we can construct an exponentially attractive invariant manifold $\Lambda := \{(h, \Lambda(h));\ 0 \leq h \leq \sigma_0\}$ of

$$\left\{ \begin{array}{lcl} h_t & = & J_1(h, V; \epsilon), \\ W_t & = & \widehat{\mathbb{L}}(h) + \widehat{\mathbb{J}}_2(h, W; \epsilon) \end{array} \right. \tag{38}$$

with $W = \Lambda(h) \in \widehat{W}(D_1, D_2)$ by taking appropriate positive constants D_1, D_2 and sufficiently small, but $O(1)$ attractive region in (38). Thus, the solution U of (4) is given by $U(t,x) = \Psi(x; h(t)) + \mathbb{X}_2(\rho(x;h)/\epsilon)\Pi(h)\ \Lambda(h(t))(x)$.

5. **Proof of Theorem 3.2.** In this section we prove Theorem 3.2. Only in this section we denote the resolvent set (resp. the spectral set) of an operator by $\rho(\,\cdot\,)$ (resp. $\sigma(\,\cdot\,)$). Let $\mathbb{S} := (S, \zeta)$ be the solution stated in Theorem 2.1, i.e., a boundary one-spike layer in the stretched domain $\mathbb{R}^2_+$. Let $\mathcal{L} := F'(\mathbb{S})$, i.e.,

$$\mathcal{L} := \left(\begin{array}{cc} L - \dfrac{qr}{\tau(p-1)} \left\langle S^{r-1}, \,\cdot\, \right\rangle \zeta^{\gamma} S & -\dfrac{q\gamma}{\tau(p-1)} \left\langle S^{r}, 1 \right\rangle \zeta^{\gamma-1} S \\ \dfrac{r}{\tau} \left\langle S^{r-1}, \,\cdot\, \right\rangle \zeta^{\gamma+1} & \dfrac{\gamma}{\tau} \end{array} \right)$$

with the Neumann boundary condition, where

$$L := \Delta - 1 + (p-1)S^{p-1}$$

with the Neumann boundary condition. In this section we assume that $p = r - 1$. The spectra of $\mathcal{L}$ may not consist only of eigenvalues, since the underlying set $\mathbb{R}^2_+$ is not bounded. In this case studying the resolvent set seems to be easier than studying the spectral set. In order to study the resolvent set of $\mathcal{L}$ we will find the set of $\lambda \in \mathbb{C}$ where the following problem has the unique solution (ϕ, η):

$$(\mathcal{L} - \lambda) \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} \Phi \\ Y \end{pmatrix} \quad \text{in} \quad \mathbb{R}^2_+, \qquad \partial_\nu \phi = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+. \tag{39}$$

From the second equation of (39) we have

$$\eta = \frac{\gamma \left\langle S^{r-1}, \phi \right\rangle}{\tau\lambda - \gamma} - \frac{\tau Y}{\tau\lambda - \gamma}, \tag{40}$$

provided that $\lambda \neq \gamma/\tau$. Substituting (40) into the first equation of (39), we have

$$(L-\lambda)\phi - \frac{qr \left\langle S^{r-1}, \phi \right\rangle \zeta^{\gamma} S}{\tau(p-1)} - \frac{q\gamma \left\langle S^{r}, 1 \right\rangle \zeta^{\gamma-1} S}{\tau(p-1)} \left(\frac{r \left\langle S^{r-1}, \phi \right\rangle}{\tau\lambda - \gamma} \zeta^{\gamma+1} - \frac{\tau Y}{\tau\lambda - \gamma} \right) = \Phi.$$

Substituting $\zeta^{-\gamma} = \left\langle S^{r-1}, 1 \right\rangle$ into this equation, we have

$$(L + B_\lambda - \lambda)\phi = \Phi - \frac{q\gamma Y}{(p-1)(\tau\lambda - \gamma)\zeta} S, \tag{41}$$

where

$$B_\lambda[\phi] := -\frac{qr\lambda \left\langle S^{r-1}, \phi \right\rangle}{(p-1)(\tau\lambda - \gamma) \left\langle S^{r}, 1 \right\rangle} S. \tag{42}$$

If $L + B_\lambda - \lambda$ is invertible, then it follows from (40) and (41) that (39) has the unique solution, hence $\lambda \in \rho(\mathcal{L})$. Therefore, we have obtained the following:

Lemma 5.1. *Suppose that $\lambda \neq \gamma/\tau$. If $L + B_\lambda - \lambda$ is invertible, then $\lambda \in \rho(\mathcal{L})$.*

Here we recall the Sherman-Morrison formula [23] which is useful for the analysis of the spectra of $\mathcal{L}$. Let A be an invertible linear operator on $L^2(\mathbb{R}^2_+)$, and let B be a rank-one operator on $L^2(\mathbb{R}^2_+)$ defined by $B[\ \cdot\ ] := \langle \Psi_1,\ \cdot\ \rangle\, \Psi_2$, where Ψ_1, $\Psi_2 \in L^2(\mathbb{R}^2_+)$. Then the Sherman-Morrison formula is

$$(A+B)^{-1} = \left(I - \frac{A^{-1}B}{k} \right) A^{-1},$$

where $k = 1 + \left\langle \Psi_1, A^{-1}\Psi_2 \right\rangle (\in \mathbb{R})$. Hence,

$$\text{if } k \neq 0, \text{ then } A+B \text{ is invertible.} \tag{43}$$

Before going to the next lemma, we recall a known result about the spectra of L.

Proposition 2 ([20, Lemma C]). *The problem*

$$L\varphi = \lambda\varphi \quad \text{in} \quad \mathbb{R}^2_+, \qquad \partial_\nu \varphi = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+ \tag{44}$$

has the following set of eigenvalues: $\lambda_1 > 0$, $\lambda_2 = 0$, and other spectra are (real) negative and bounded away from 0*. Moreover, λ_1 and λ_2 are simple, and* $\ker(L - \lambda_2) = \operatorname{span}\{S_l\}$*. Here l denote the first argument of the coordinate of $\mathbb{R}^2_+$.*

In this section we do not use μ for the second argument of the coordinate. We use μ for eigenvalues of (44). Hereafter, by (λ_1, ψ_1) (resp. (λ_2, ψ_2)) we denote the first (resp. the second) eigenpair of (44). Since we can take $(0, S_l/\left\|S_l\right\|_{L^2})$ as (λ_2, ψ_2), we see $\left\langle \psi_2, S^{r-1} \right\rangle = \langle \psi_2, S \rangle = 0$.

We study the invertibility of $L + B_\lambda - \lambda$ in order to study the resolvent set of $\mathcal{L}$. Assume that $\lambda \notin \sigma(L) \cup \{\gamma/\tau\}$. Since B_λ is a rank-one operator, it follows from the Sherman-Morrison formula that

$$(L + B_\lambda - \lambda)^{-1} = \left(I - \frac{(L-\lambda)^{-1}B_\lambda}{k(\lambda)} \right) (L-\lambda)^{-1}, \tag{45}$$

where

$$k(\lambda) := 1 - \frac{qr\lambda \left\langle S^{r-1}, (L-\lambda)^{-1}[S] \right\rangle}{(p-1)(\tau\lambda - \gamma)\left\langle S^r, 1 \right\rangle}. \tag{46}$$

From (43) we see that $L + B_\lambda - \lambda$ is invertible, if $k(\lambda) \neq 0$ and if $\lambda \notin \sigma(L) \cup \{\gamma/\tau\}$. Using

$$LS = (p-1)S^p = (p-1)S^{r-1}, \tag{47}$$

we have

$$\begin{aligned} \left\langle S^{r-1}, (L-\lambda)^{-1}[\lambda S] \right\rangle &= \left\langle S^{r-1}, (L-\lambda)^{-1}[(\lambda - L)S + LS] \right\rangle \\ &= -\left\langle S^{r-1}, S \right\rangle + \left\langle S^{r-1}, (L-\lambda)^{-1}[LS] \right\rangle \\ &= -\left\langle S^r, 1 \right\rangle + (p-1)\left\langle S^{r-1}, (L-\lambda)^{-1}\left[S^{r-1}\right] \right\rangle. \end{aligned} \tag{48}$$

Substituting (48) into (46), we have

$$k(\lambda) = \frac{\tau\lambda + s + 1 - k_0(\lambda)}{\tau\lambda - \gamma}, \quad \text{where} \quad k_0(\lambda) = qr\frac{\left\langle S^{r-1}, (L-\lambda)^{-1}[S^{r-1}] \right\rangle}{\left\langle S^r, 1 \right\rangle}. \tag{49}$$

Hence we see that $k(\lambda) = 0$, if

$$\tau\lambda + s + 1 = k_0(\lambda). \tag{50}$$

Therefore, we see by Lemma 5.1 that

$$\lambda \in \rho(\mathcal{L}), \text{ if (50) does not hold and if } \lambda \notin \sigma(L) \cup \{\gamma/\tau\}. \tag{51}$$

When we check the invertibility of $\mathcal{L}-\lambda$ for $\lambda \in \{\lambda_1, \gamma/\tau\}$, we cannot use (51) and use other methods.

First we study the non-real spectra of $\mathcal{L}$.

Lemma 5.2. $\sup\{\mathrm{Re}\lambda;\ \lambda \in \sigma(\mathcal{L})\backslash\mathbb{R}\} \le \lambda_1/2-(s+1)/(2\tau)$.

Proof. Let $\lambda := \lambda_R + i\lambda_I$ ($\lambda_I \neq 0$) be a spectrum of $\mathcal{L}$. Then $\lambda \notin \sigma(L)$. From the spectral decomposition we have

$$(L-\lambda_R-i\lambda_I)^{-1}\left[S^{r-1}\right] = \int_{\sigma(L)} \frac{dE_\mu\left[S^{r-1}\right]}{\mu-\lambda_R-i\lambda_I}. \tag{52}$$

Since λ is a spectrum, λ should satisfy (50), otherwise $\lambda \in \rho(\mathcal{L})$. Substituting (52) into (50), and taking the real and the imaginary parts of it, we have

$$\tau\lambda_R+s+1 = \int_{\sigma(L)} \frac{(\mu-\lambda_R)d\left\langle E_\mu\left[S^{r-1}\right], S^{r-1}\right\rangle}{(\mu-\lambda_R)^2+\lambda_I^2}, \tag{53}$$

$$\tau\lambda_I = \int_{\sigma(L)} \frac{\lambda_I d\left\langle E_\mu\left[S^{r-1}\right], S^{r-1}\right\rangle}{(\mu-\lambda_R)^2+\lambda_I^2}. \tag{54}$$

Since $\lambda_I \neq 0$, we multiply (54) by $(\lambda_1-\lambda_R)/\lambda_I$ and subtract it from (53). Then

$$2\tau\lambda_R-\tau\lambda_1+s+1 = \int_{\sigma(L)} \frac{(\mu-\lambda_1)d\left\langle E_\mu\left[S^{r-1}\right], S^{r-1}\right\rangle}{(\mu-\lambda_R)^2+\lambda_I^2}. \tag{55}$$

Since $\sup\{\mathrm{Re}\lambda;\ \lambda \in \sigma(L)\} \le \lambda_1$, the right-hand side of (55) is non-positive, i.e., $2\tau\lambda_R-\tau\lambda_1+s+1 \le 0$. This inequality proves the lemma. □

The similar argument to the proof above appears in [29].

Lemma 5.3. *Let $\Omega_{\delta',R}$ be as in Theorem 3.1. There are $\delta'>0$ and $R>0$ such that $\Omega_{\delta',R} \subset \rho(\mathcal{L})$.*

Proof. We see that there are $c>0$, $\delta'>0$, $R>0$ such that $\Omega_{\delta',R} \subset \rho(\mathcal{L})$ and $\left\|(L-\lambda I)^{-1}\right\| \le C/|\lambda|$ for $\lambda \in \Omega_{\delta',R}$.

Because of (51), it is enough to show that $k(\lambda) \neq 0$ for $\lambda \in \Omega_{\delta',R}$. Since

$$\begin{aligned}|\left\langle S^{r-1}, (L-\lambda)^{-1}[\lambda S]\right\rangle &\le \left\|S^{r-1}\right\| \left\|(L-\lambda)^{-1}\right\| \|\lambda S\| \\ &\le (C/|\lambda|)|\lambda| \left\|S^{r-1}\right\| \|S\| \le C,\end{aligned}$$

we have

$$|k(\lambda)| \ge 1-\left|\frac{qr\left\langle S^{r-1}, (L-\lambda)^{-1}[\lambda S]\right\rangle}{(p-1)(\tau\lambda-\gamma)\left\langle S^r, 1\right\rangle}\right| \ge 1-\frac{C}{|\tau\lambda-\gamma|}.$$

Thus, if R is large, then

$$|k(\lambda)| \ge 1/2 \quad \text{for} \quad \lambda \in \Omega_{\delta',R}, \tag{56}$$

hence $k(\lambda) \neq 0$ if $\lambda \in \Omega_{\delta',R}$. The proof is complete. □

From now on we study the real spectra of $\mathcal{L}$.

Lemma 5.4. *For small $\tau>0$, there is $\delta>0$ such that $(-\delta,+\infty)\backslash\{0,\lambda_1,\gamma/\tau\} \subset \rho(\mathcal{L})$.*

Proof. We use (50). Specifically, we will show that, if $\tau > 0$ is small, there is $\delta > 0$ such that $\tau\lambda + s + 1 \neq k_0(\lambda)$ for $\lambda \in (-\delta, +\infty)\backslash\{0, \lambda_1, \gamma/\tau\}(\subset \rho(L))$. From the spectral decomposition we have

$$k_0(\lambda) = \frac{qr}{\langle S^r, 1\rangle}\left(\frac{\langle \psi_1, S^{r-1}\rangle^2}{\lambda_1 - \lambda} + \frac{\langle \psi_2, S^{r-1}\rangle^2}{\lambda_2 - \lambda} \right. \\ \left. + \int_{\sigma(L)\backslash\{\lambda_1,\lambda_2\}} \frac{d\langle E_\mu[S^{r-1}], S^{r-1}\rangle}{\mu - \lambda}\right).$$

It follows from Proposition 2 that $\langle \psi_2, S^{r-1}\rangle = 0$ and there is $\delta_0 > 0$ such that $\sup\{\lambda;\ \lambda \in \sigma(L)\backslash\{\lambda_1, \lambda_2\}\} < -\delta_0$. Hence, $k_0(\lambda) \in C^0((-\delta_0, +\infty)\backslash\{\lambda_1\})$. Moreover,

$$\lim_{\lambda\to\lambda_1-0} k_0(\lambda) = +\infty, \qquad \lim_{\lambda\to\lambda_1+0} k_0(\lambda) = -\infty. \tag{57}$$

Differentiating $k_0(\lambda)$ with respect to λ, we have

$$\frac{d}{d\lambda}k_0(\lambda) = \frac{qr}{\langle S^r, 1\rangle}\left(\frac{\langle \psi_1, S^{r-1}\rangle^2}{(\lambda_1 - \lambda)^2} + \int_{\sigma(L)\backslash\{\lambda_1,\lambda_2\}} \frac{d\langle E_\mu[S^{r-1}], S^{r-1}\rangle}{(\mu - \lambda)^2}\right) > 0 \tag{58}$$

for $\lambda \in (-\delta_0, +\infty)\backslash\{\lambda_1\}$. Since L is invertible in span $\{S_l\}^\perp$, we see by (47) that $L^{-1}[S^{r-1}] = S/(p-1)$. Therefore

$$k_0(0) = \frac{qr\langle S^{r-1}, L^{-1}[S]\rangle}{\langle S^r, 1\rangle} = \frac{qr}{p-1}. \tag{59}$$

Combining (57), (58), and (59), we see that there is $\delta > 0$ such that $k_0(\lambda) \neq \tau\lambda + s + 1$ for $\lambda \in (-\delta, +\infty)\backslash\{\lambda_1, \gamma/\tau\}$, if τ is small. □

The similar proof to the above appears in [21].

Lemma 5.5. *For small $\tau > 0$, $\lambda_1 \in \rho(\mathcal{L})$.*

Proof. Because of Lemma 5.1, it is enough to show that $L + B_{\lambda_1} - \lambda_1$ is invertible. We consider the problem

$$(L + B_{\lambda_1} - \lambda_1)\phi = \Phi_0. \tag{60}$$

Let $\phi := \alpha\psi_1 + \phi^\perp$ ($\langle \psi_1, \phi^\perp\rangle = 0$). Let P be the projection operator onto span $\{\psi_1\}^\perp$, i.e., $P := I - \langle \psi_1, \cdot\,\rangle\psi_1$. Then the equation on span $\{\psi_1\}$ and the equation on span $\{\psi_1\}^\perp$ become

$$\alpha\langle \psi_1, B_{\lambda_1}[\psi_1]\rangle + \langle \psi_1, B_{\lambda_1}[\phi^\perp]\rangle = \langle \psi_1, \Phi_0\rangle, \tag{61}$$

$$(L - \lambda_1)\phi^\perp + \alpha P B_{\lambda_1}[\psi_1] + P B_{\lambda_1}[\phi^\perp] = P\Phi_0, \tag{62}$$

respectively. Since

$$\langle \psi_1, B_{\lambda_1}[\psi_1]\rangle = \frac{q\lambda\lambda_1\langle S^{r-1}, \psi_1\rangle}{(p-1)(\tau\lambda_1 - \gamma)\langle S^r, 1\rangle}\langle \psi_1, S\rangle \neq 0,$$

we can solve (61) with respect to α. Substituting it into (62), we have

$$(L - \lambda_1)\phi^\perp = P\Phi_0 - \frac{\langle \psi_1, \Phi_0\rangle}{\langle \psi_1, S\rangle}PS. \tag{63}$$

Note that $\langle \psi_1, S\rangle \neq 0$, because $\psi_1 > 0$. The operator $L - \lambda_1$ is invertible in $\mathrm{span}\{\psi_1\}^\perp$, and the right-hand side of (63) is in $\mathrm{span}\{\psi_1\}^\perp$. Thus (63) can be solved with respect to $\phi^\perp$. Let $\phi_0^\perp$ be the solution of (63), namely,

$$\phi_0^\perp := (L-\lambda_1)^{-1}\left[P\Phi_0 - \frac{\langle \psi_1, \Phi_0\rangle}{\langle \psi_1, S\rangle} PS\right].$$

Substituting $\phi_0^\perp$ into (61), we obtain the solution α_0 of (61), namely

$$\alpha_0 := \frac{\langle \psi_1, \Phi_0\rangle - \left\langle \psi_1, B_{\lambda_1}[\phi_0^\perp]\right\rangle}{\langle \psi_1, B_{\lambda_1}[\psi_1]\rangle}.$$

Then $\phi_0 := \alpha_0\psi_1 + \phi_0^\perp$ is the unique solution of (60). □

Lemma 5.6. *For small $\tau > 0$, $\gamma/\tau \in \rho(\mathcal{L})$.*

Proof. Let $\lambda = \gamma/\tau$. By D we define

$$D\phi := -\frac{qr\left\langle S^{r-1}, \phi\right\rangle}{\tau(p-1)\left\langle S^r, 1\right\rangle} S. \tag{64}$$

It is enough to show that (39) has the unique solution. The first equation of (39) is

$$(L + D - \lambda)\phi = \Phi + \frac{qr\eta}{\tau(p-1)\zeta} S.$$

It follows from (43) that $L + D - \lambda$ is invertible, if

$$(k_1(\lambda) :=)\ 1 - \frac{qr\left\langle S^{r-1}, (L-\lambda)^{-1}[S]\right\rangle}{\tau(p-1)\left\langle S^r, 1\right\rangle} \neq 0$$

and if $\lambda \notin \sigma(L)$. Using (48), we have

$$k_1(\lambda) = \frac{1}{\tau\lambda}\left(\tau\lambda + \frac{qr}{p-1} - k_0(\lambda)\right).$$

We see by the graph of $k_0(\lambda)$ that $k_0(\lambda) \neq \tau\lambda + qr/(p-1)$, when τ is small. Thus $k_1(\lambda) \neq 0$, and $L + D - \lambda$ is invertible. Let $K := (L + D - \lambda)^{-1}$. We have

$$\phi = K\Phi + \frac{qr\eta}{\tau(p-1)\zeta} KS. \tag{65}$$

Substituting (65) into the second equation of (39), we have

$$\frac{r}{\tau}\left\langle S^{r-1}, K\Phi\right\rangle + \frac{qr^2\eta}{\tau^2(p-1)\zeta}\left\langle S^{r-1}, KS\right\rangle = Y.$$

If $\left\langle S^{r-1}, KS\right\rangle \neq 0$, then this equation has the unique solution with respect to η, hence the pair ϕ, which obtained by (65), and η is the unique solution of (39) and $\lambda \in \rho(\mathcal{L})$. We will show that $\left\langle S^{r-1}, KS\right\rangle \neq 0$. We see by a direct calculation that $KS = (L-\lambda)^{-1}[S]/k_1(\lambda)$, where we use $k_1(\lambda) \neq 0$. Using (48), we have

$$\left\langle S^{r-1}, KS\right\rangle = \frac{\left\langle S^{r-1}, (L-\lambda)^{-1}[S]\right\rangle}{k_1(\lambda)} = \frac{-\left\langle S^r, 1\right\rangle}{\lambda k_1(\lambda) qr}\left(qr - k_0(\lambda)\right).$$

We see by the graph of $k_0(\lambda)$ that $k_0(\lambda) \neq qr$. Thus $\left\langle S^{r-1}, KS\right\rangle \neq 0$. □

Lemma 5.7. *For small $\tau > 0$, 0 is a simple eigenvalue of $\mathcal{L}$.*

Proof. It follows from a direct calculation that 0 is an eigenvalue of $\mathcal{L}$ and that $(S_l, 0)$ is a corresponding eigenvector.

We will show that $\dim\ker\mathcal{L} = 1$. It is enough to show that there is no eigenvector corresponding to 0 perpendicular to $(S_l, 0)$ in the L^2 sense. Let $(\phi, \eta) \in \mathrm{span}\{(S_l, 0)\}^\perp$. Specifically, $\phi \in \mathrm{span}\{S_l\}^\perp$. We consider the problem

$$\mathcal{L}\begin{pmatrix}\phi\\ \eta\end{pmatrix} = 0. \tag{66}$$

Solving the second equation of (66) with respect to η, and substituting it into the first equation of (66), we have $(L + D)\phi = 0$, where D is defined by (64). Since L is invertible in span $\{S_l\}^\perp$, we see by (43) that $L + D$ is invertible, if

$$1 - \frac{qr\left\langle S^{r-1}, L^{-1}[S]\right\rangle}{\tau(p-1)\left\langle S^r, 1\right\rangle} \neq 0. \tag{67}$$

Since L is self-adjoint, we have

$$\left\langle S^{r-1}, L^{-1}[S]\right\rangle = \left\langle LL^{-1}[S^{r-1}], L^{-1}[S]\right\rangle = \left\langle L^{-1}[S^{r-1}], S\right\rangle = \langle S, S\rangle/(p-1).$$

The second term of the left-hand side of (67) goes to $-\infty$ as $\tau \to 0$, and it is not 0 when τ is small. Therefore, (67) holds provided that τ is small. 0 is the unique solution in span $\{S_l\}^\perp$ of $(L + D)\phi = 0$. It follows from the second equation of (66) that $\eta = 0$, hence $(\phi, \eta) = (0, 0)$. We have shown that $\dim\ker\mathcal{L} = 1$.

Next, we will show that $\ker\mathcal{L}^2 = \ker\mathcal{L}$. We consider the problem

$$\mathcal{L}\begin{pmatrix}\phi\\ \eta\end{pmatrix} = \begin{pmatrix}S_l\\ 0\end{pmatrix}.$$

Solving the second equation with respect to η, and substituting it into the first equation, we have

$$(L + D)\phi = S_l. \tag{68}$$

Calculating $\langle(68), S_l\rangle$, we have

$$0 = \langle\phi, LS_l\rangle + \langle D\phi, S_l\rangle = \langle L\phi + D\phi, S_l\rangle = \langle S_l, S_l\rangle \neq 0,$$

which is a contradiction. The proof of the lemma is complete. □

Proof of Theorem 3.2. It follows from Lemmas 5.2 and 5.3 that $(\Omega_{\delta',R} \cup \Omega_\delta)\backslash\mathbb{R} \subset \rho(\mathcal{L})$. Combining Lemmas 5.4, 5.5 and 5.6, we see that $(-\delta, +\infty)\backslash\{0\} \subset \rho(\mathcal{L})$. Lemma 5.7 says that 0 is a simple eigenvalue. We have proven (i).

We will prove (ii). Hereafter we assume that $\lambda \in \Omega_{\delta',R}$. By (42) we have

$$\|B_\lambda[\phi]\| \leq \frac{C|\lambda|}{|\tau\lambda - \gamma|}\|\phi\| \leq C\|\phi\|. \tag{69}$$

By (56) we see that $(L + B_\lambda - \lambda)$ is invertible if $\lambda \in \Omega_{\delta',R}$. Using (45), (69) and (56), we have

$$\begin{aligned}\left\|(L + B_\lambda - \lambda)^{-1}[\phi]\right\| &\leq \left\|(L-\lambda)^{-1}[\phi]\right\| + \frac{1}{|k(\lambda)|}\left\|(L-\lambda)^{-1}B_\lambda\left[(L-\lambda)^{-1}[\phi]\right]\right\|\\ &\leq \frac{C}{|\lambda|}\|\phi\| + \frac{C}{|\lambda|}\left\|B_\lambda\left[(L-\lambda)^{-1}[\phi]\right]\right\|\\ &\leq \frac{C}{|\lambda|}\|\phi\| + \frac{C}{|\lambda|}\left\|(L-\lambda)^{-1}[\phi]\right\|\\ &\leq \frac{C}{|\lambda|}\left(1 + \frac{1}{|\lambda|}\right)\|\phi\|. \end{aligned} \tag{70}$$

We consider the solution (ϕ, η) of (39). By (41) and (70), we have

$$\begin{aligned}\|\phi\| &\leq \left\|(L+B_\lambda-\lambda)^{-1}[\Phi]\right\| + \frac{C}{|\tau\lambda-\gamma|}|Y| \left\|(L+B_\lambda-\lambda)^{-1}[S]\right\| \\ &\leq \frac{C}{|\lambda|}\|\Phi\| + \frac{C}{|\lambda||\tau\lambda-\gamma|}|Y|. \end{aligned} \tag{71}$$

Using (40), we have

$$|\eta| \leq \frac{C}{|\tau\lambda-\gamma|}(\|\phi\|+|Y|) \leq \frac{C}{|\tau\lambda-\gamma|}(\|\Phi\|+|Y|) + \frac{C}{|\tau\lambda-\gamma|}|Y|. \tag{72}$$

We obtain the conclusion by (71) and (72). □

Lemma 5.8. *Let $\mu > 0$ be a small and let ϕ be a solution of*

$$\lambda\phi - L\phi = ge^{-\mu|x|} \quad \text{in} \quad \mathbb{R}^2. \tag{73}$$

Then there is $\theta \in (\pi/2, \pi)$ such that

$$|\phi(x)| \leq \frac{C}{|\lambda|}\|g\|_{L^\infty} e^{-\mu|x|} \quad \text{for} \quad x \in \mathbb{R}^2 \text{ and } \lambda \in S_\theta, \tag{74}$$

where $S_\theta := \{\lambda \in \mathbb{C};\ |\arg\lambda| \leq \theta\}$.

Proof. SInce $ge^{-\mu|x|} \in L^2(\mathbb{R}^2)$, (73) has a solution of $\phi \in H^2(\mathbb{R}^2)$ such that

$$\|\phi\|_{L^\infty} \leq c\|\phi\|_{H^2} \leq \frac{c}{|\lambda|}\left\|ge^{-\mu|x|}\right\|_{L^2} \leq \frac{c}{|\lambda|}\|g\|_{L^\infty}. \tag{75}$$

We choose large $R > 0$. Then (74) holds for $|x| \leq 2R$.

We will show that (74) holds for $|x| \geq 2R$. Let δ_{x_0} be the Dirac delta function, and let $G_\lambda(x, x_0)$ be the Green function of

$$-\Delta\varphi + (\lambda+1)\varphi = \delta_{x_0} \quad \text{in} \quad \mathbb{R}^2.$$

Then ϕ satisfies

$$\begin{aligned}\phi(x) &= \int_{\mathbb{R}^2} G_\lambda(x,y)\left\{g(y)e^{-\mu|y|} + pS^{p-1}(y)\phi(y)\right\}dy \\ &= \int_{\mathbb{R}^2} G_\lambda(0,y)\left\{g(x-y)e^{-\mu|x-y|} + pS^{p-1}(x-y)\phi(x-y)\right\}dy. \end{aligned} \tag{76}$$

The Green function has an explicit which satisfies

$$z^2\frac{d^2K}{dz^2} + z\frac{dK}{dz} - z^2K = 0.$$

Let $z \in \mathbb{C}$. K satisfies

$$K(z) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{\pi}{2}}e^{-z}\int_0^{+\infty} e^{-t}e^{-\frac{1}{2}}\left(z+\frac{t}{2}\right)^{-\frac{1}{2}}dt.$$

See [10, Appendix C] for more details. Then

$$G_\lambda(0,y) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sqrt{\frac{\pi}{2}}e^{-\sqrt{1+\lambda}r}\int_0^{+\infty} e^{-t}t^{-\frac{1}{2}}\left(\sqrt{1+\lambda}r+\frac{t}{2}\right)^{-\frac{1}{2}}dt,$$

where $r = |y|$. We consider the case where r is large. Since

$$\left|\left(\sqrt{1+\lambda}r+\frac{t}{2}\right)^{-\frac{1}{2}}\right| \leq \frac{1}{\sqrt{r}\,|1+\lambda|^{\frac{1}{4}}} \quad \text{and} \quad \left|e^{-\sqrt{1+\lambda}r}\right| \leq e^{-\mathrm{Re}\sqrt{1+\lambda}r},$$

we have

$$|G_\lambda(0,y)| \le ce^{-\mathrm{Re}\sqrt{1+\lambda}r}. \tag{77}$$

We consider the case where r is small. Let $r_0>0$ be fixed. Then

$$\begin{aligned}&\int_0^{r_0}\left|e^{-\sqrt{1+\lambda}r}\right|\int_0^{+\infty}e^{-t}t^{-\frac12}\left|\left(\sqrt{1+\lambda}r+\frac{t}{2}\right)^{-\frac12}\right|dtrdr\\ &=\int_0^{+\infty}e^{-t}t^{-\frac12}\int_0^{r_0}\left|e^{-\sqrt{1+\lambda}r}\left(\sqrt{1+\lambda}r+\frac{t}{2}\right)^{-\frac12}\right|rdrdt.\end{aligned}$$

Since

$$\int_0^{r_0}\left|e^{-\sqrt{1+\lambda}r}\left(\sqrt{1+\lambda}r+\frac{t}{2}\right)^{-\frac12}\right|rdr \le \frac{1}{|1+\lambda|^{\frac14}}\int_0^{r_0}e^{-\mathrm{Re}\sqrt{1+\lambda}r}\sqrt{r}dr,$$

we have

$$\int_0^{r_0}\left|K\left(\sqrt{1+\lambda}r\right)\right|rdr \le \frac{c}{|1+\lambda|^{\frac14}}\int_0^{r_0}e^{-\mathrm{Re}\sqrt{1+\lambda}r}\sqrt{\lambda}dr\int_0^{+\infty}e^{-t}t^{-\frac12}dt \le c. \tag{78}$$

We will estimate (76). If μ is small, then $\mathrm{Re}\sqrt{1+\lambda}>\mu$ for $\lambda\in S_\theta$. Using (77), (78) and $-|y-x|+|x|\le|y|$, we have

$$\left|\int_{\mathbb{R}^2}G_\lambda(0,y)g(x-y)e^{-\mu|x-y|}dy\right| \le c\left\|g\right\|_{L^\infty}.$$

We divide $\mathbb{R}^2$ into three regions:

$$I_1:=\{y\in\mathbb{R}^2;\ |y-x|\le R\},\quad I_2:=\{y\in\mathbb{R}^2;\ |y|\le R\}\quad\text{and}\quad I_3:=\mathbb{R}^2\backslash(I_1\cup I_2).$$

Because of (75) and the boundedness of S, we have

$$\left|\int_{I_1}G_\lambda(0,y)S^{p-1}(x-y)\phi(x-y)dy\right|e^{\mu|x|} \le \frac{c\left\|g\right\|_{L^\infty}}{|\lambda|},$$

where we use (77) and $-|y-x|+|x|\le|y|$.

We will estimate (76) on I_2. There is $\alpha>0$ such that

$$S^{p-1}(x-y)\le ce^{-\alpha|x-y|}\le ce^{-\alpha|x|+\alpha|y|}. \tag{79}$$

Using (75) and $|y|\le R$, we have

$$\left|\int_{I_2}G_\lambda(0,y)S^{p-1}(x-y)\phi(x-y)dy\right|e^{\mu|x|} \le \frac{c\left\|g\right\|_{L^\infty}}{|\lambda|}\int_{|y|\le R}|G_\lambda(0,y)|\,dy=\frac{c\left\|g\right\|_{l^\infty}}{|\lambda|},$$

where we choose μ such that $\alpha>\mu>0$.

We will estimate (76) on I_3. We can choose $\alpha>0$ such that $\mathrm{Re}\sqrt{1+\lambda}>\alpha$. Combining (77) and (79), we have

$$\begin{aligned}\left|\int_{I_2}G_\lambda(0,y)S^{p-1}(x-y)\phi(x-y)dy\right|e^{\mu|x|} &\le \frac{c\left\|g\right\|_{L^\infty}}{|\lambda|}\int_{\mathbb{R}^2}e^{-\mathrm{Re}\sqrt{1+\lambda}|y|+\alpha|y|}dy\\ &=\frac{c\left\|g\right\|_{L^\infty}}{|\lambda|}.\end{aligned}$$

Summing the estimates on I_1, I_2 and I_3, we obtain the conclusion of the lemma. $\square$

6. **Proof of Theorem 3.3.** Denote the eigenfunction of (16) corresponding to λ_1 by ψ. It is well-known that $\psi = \psi(y)$ is radially symmetric and decays exponentially as $|y| \to \infty$. In the proof of Theorem 3.3, we construct an eigenpair (λ, ϕ, η) which satisfies

$$\phi \sim \psi_1 + \psi_2, \qquad \eta \sim 0, \qquad \lambda \sim \lambda_1 \tag{80}$$

as $\epsilon \to 0$, where

$$\psi_1(x) = \begin{cases} \psi(\dfrac{|x-h_1|}{\epsilon}), & h_1 \in \Omega, \\ 2\psi(\dfrac{|x-h_1|}{\epsilon}), & h_1 \in \partial\Omega \end{cases} \qquad \psi_2(x) = \begin{cases} -\psi(\dfrac{|x-h_2|}{\epsilon}), & h_2 \in \Omega, \\ -2\psi(\dfrac{|x-h_2|}{\epsilon}), & h_2 \in \partial\Omega. \end{cases}$$

If there is an eigenpair (λ, ϕ, η) and η is close to 0, we rewrite (15) into

$$\begin{cases} \lambda\phi \sim \epsilon^2\Delta\phi - \phi + p\dfrac{u^{p-1}}{\xi^q}\phi, \\ \displaystyle\int_\Omega u^{r-1}\phi dx \sim 0. \end{cases}$$

From the first relation, the pair of the eigenvalue λ and the eigenfunction ϕ is close to (λ_1, ψ). Since u is close to the positive spiky solution $\zeta^{q/(p-1)}S$ in neighborhoods of h_1 and h_2, ϕ needs to change the sign because of the second relation. This is why we construct the pair of the eigenvalue and the eigenfunction (λ, ϕ, η) such as (80), and ψ_1 and ψ_2 have the opposite sign. In fact, it seems that there is no pair of the eigenvalue and the eigenfunction (λ, ϕ, η) such that ϕ is positive and λ tends to λ_1 as $\epsilon \to 0$. Namely, if ϕ is positive, λ is away from λ_1, which is not shown in this paper rigorously.

In the proof, we suppose that $\lambda = \lambda_1$ is not an eigenvalue of (15) without loss of generality. This assumption shall be used in the last part of the proof of Theorem 3.3.

Now we define $T = (T_1, T_2)$ by

$$\begin{cases} T_1(\phi, \eta, \lambda, \epsilon) = L(\psi_1 + \psi_2 + \phi) - q\dfrac{u^p}{\xi^{q+1}}\eta - (\lambda_1 + \lambda)(\psi_1 + \psi_2 + \phi), \\ T_2(\phi, \eta, \lambda, \epsilon) = -\eta + \dfrac{r}{\epsilon^n \xi^s |\Omega|}\displaystyle\int_\Omega u^{r-1}(\psi_1 + \psi_2 + \phi)dx \\ \qquad\qquad - \dfrac{s}{\epsilon^n \xi^{s+1}|\Omega|}\displaystyle\int_\Omega u^r dx - \tau(\lambda_1 + \lambda)\eta, \end{cases}$$

where $L = \epsilon^2\Delta - 1 + pu^{p-1}/\xi^q$. This nonlinear functional T operates from $H^2_{N,\epsilon}(\Omega) \times \mathbb{R} \times \mathbb{R} \times (0, \epsilon_0) \to L^2_\epsilon(\Omega) \times \mathbb{R}$, where $L^2_\epsilon(\Omega)$ and $H^2_{N.\epsilon}(\Omega)$ are defined by

$$L^2_\epsilon(\Omega) = \left\{ \varphi \in L^2(\Omega) \;\middle|\; \|\varphi\|^2_{L^2_\epsilon} \equiv \frac{1}{\epsilon^n}\int_\Omega |\varphi|^2 dx < \infty \right\},$$

$$H^2_\epsilon(\Omega) = \{\varphi \in H^2(\Omega) \mid \|\varphi\|^2_{H^2_\epsilon} \equiv \|\varphi\|^2_{L^2_\epsilon} + \epsilon^2\|\nabla\varphi\|^2_{L^2_\epsilon} + \epsilon^4\|\nabla^2\varphi\|^2_{L^2_\epsilon} < \infty\},$$

$$H^2_{N,\epsilon}(\Omega) = \left\{ \varphi \in H^2_\epsilon(\Omega) \;\middle|\; \int_\Omega (\psi_1 + \psi_2)\phi dx = 0, \quad \partial_\nu\phi = 0 \text{ on } \partial\Omega \right\}.$$

Here $\nabla\phi$ and $\nabla^2\phi$ represent the gradient and the Hessian matrix, respectively. We also define $\|\cdot\|_{H^1_\epsilon}$ by $\|\varphi\|^2_{H^1_\epsilon} \equiv \|\varphi\|^2_{L^2_\epsilon} + \epsilon^2\|\nabla\varphi\|^2_{L^2_\epsilon}$. When we fix ϵ, it is clear that $L^2_\epsilon(\Omega)$ and $H^2_\epsilon(\Omega)$ are Hilbert spaces and each norms are equivalent to ones for the usual spaces $L^2(\Omega)$ and $H^2(\Omega)$.

We shall find a solution (ϕ, η, λ) of $T = 0$ by two facts. At first, we have the following proposition.

Proposition 3. *$T(0, 0, 0, \epsilon)$ tends to 0 in $L^2_\epsilon(\Omega) \times \mathbb{R}$ as $\epsilon \to 0$.*

Proof. Straightforward calculation gives

$$T_1(0,0,0,\epsilon) = p\left\{\frac{u^{p-1}}{\xi^q} - S(\frac{x-h_1}{\epsilon})^{p-1}\right\}\psi_1 + p\left\{\frac{u^{p-1}}{\xi^q} - S(\frac{x-h_2}{\epsilon})^{p-1}\right\}\psi_2.$$

Since u is close to $\zeta^{q/(p-1)}S$ in neighborhoods of h_1 and h_2, and ψ_1, ψ_2 are exponentially small with respect to ϵ outside the neighborhoods, $T_1(0,0,0,\epsilon) \to 0$ in $L^2_\epsilon(\Omega)$ as $\epsilon \to 0$.

We suppose that $h_1 \in \partial\Omega$ and $h_2 \in \Omega$. Taking the limit of $\epsilon \to 0$ for $T_2(0,0,0,\epsilon)$, we have

$$T_2(0,0,0,0) = \frac{r\zeta^{\frac{q(r-1)-s}{p-1}}}{|\Omega|}\left(2\int_{\mathbb{R}^n_+} S^{r-1}\psi dy - \int_{\mathbb{R}^n} S^{r-1}\psi dy\right) = 0$$

because S and ψ are radially symmetric, where $\mathbb{R}^n_+ = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$. In the other cases, we can prove $T_2(0,0,0,0) = 0$ similarly. Hence we complete the proof. $\square$

Next we study the invertibility of the linearized operator of T with respect to (ϕ, η, λ) at $(\phi, \eta, \lambda) = (0, 0, 0)$.

Lemma 6.1. *Suppose that $\lambda = \lambda_1$ is not an eigenvalue of (15). Then $T_{(\phi,\eta,\lambda)}(0,0,0,\epsilon)$ is invertible. Moreover, $\|T^{-1}_{(\phi,\eta,\lambda)}(0,0,0,\epsilon)\| \le M$ for a constant $M > 0$ independent of $\epsilon > 0$, where $\|\cdot\|$ is the usual norm for operators.*

Proof. Put $T_{(\phi,\eta,\lambda)}(0,0,0,\epsilon)[\phi_\epsilon, \eta_\epsilon, \lambda_\epsilon] = 0$. Then,

$$\begin{cases} L\phi_\epsilon - q\dfrac{u^p}{\xi^{q+1}}\eta_\epsilon - \lambda_1\phi_\epsilon - \lambda_\epsilon(\psi_1 + \psi_2) = 0, \\ -\eta_\epsilon + \dfrac{r}{\epsilon^n\xi^s|\Omega|}\displaystyle\int_\Omega u^{r-1}\phi_\epsilon dx - \dfrac{s\eta_\epsilon}{\epsilon^n\xi^{s+1}|\Omega|}\displaystyle\int_\Omega u^r dx - \tau\lambda_1\eta_\epsilon = 0. \end{cases} \tag{81}$$

It follows from the second equation that

$$\eta_\epsilon = \frac{r}{\epsilon^n\xi^s|\Omega|(\tau\lambda_1 + 1 + \frac{s}{\epsilon^n\xi^{s+1}|\Omega|}\int_\Omega u^r dx)}\int_\Omega u^{r-1}\phi_\epsilon dx. \tag{82}$$

Without loss of generality, we suppose that $|\lambda_\epsilon| + \|\phi_\epsilon\|_{L^2_\epsilon} = 1$. Then, it is easy to see from (81) that $\|\phi_\epsilon\|_{H^2_\epsilon} \le c$ and $|\eta_\epsilon| \le c$ for a constant c independent of ϵ. Therefore, there are $\lambda_0, \eta_0 \in \mathbb{R}$ such that $\lambda_\epsilon \to \lambda_0$ and $\eta_\epsilon \to \eta_0$ as $\epsilon \to 0$. Although we may need to take a subsequence of ϵ, we use the same notation.

Next we study the behavior of ϕ_ϵ as $\epsilon \to 0$, which will determine η_0 and λ_0. Let χ be a smooth cut-off function defined by

$$\chi(r) = \begin{cases} 1, & 0 \le r \le \dfrac{1}{2}, \\ 0, & r \ge 1. \end{cases}$$

Set $\phi_{\epsilon,R,i}(x) = \chi_i(x + h_i)\phi_\epsilon(x + h_i)$ for $i = 1, 2$, and $\phi_{\epsilon,R,0}(x) = (1 - \chi_1(x) - \chi_2(x))\phi_\epsilon(x)$, where $\chi_i(x) = \chi(|x - h_i|/R\epsilon)$ for $i = 1, 2$ for R sufficiently large independently of ϵ. Note that $\phi_\epsilon(x) = \phi_{\epsilon,R,0}(x) + \phi_{\epsilon,R,1}(x+h_1) + \phi_{\epsilon,R,2}(x+h_2)$, and

$$|\nabla\chi_i| \leq \frac{c}{\epsilon R}, \quad |\nabla^2\chi_i| \leq \frac{c}{\epsilon^2 R^2} \tag{83}$$

hold true. The constant R will be determined later.

In the following, we just consider the case $h_1 \in \partial\Omega$ and $h_2 \in \Omega$. Other cases are shown by the same argument as this case. Let c be a constant independent of ϵ and R. This constant shall appear several times in the proof and we use the same notation unless readers are confused.

We estimate $\phi_{\epsilon,R,0}$ in $L^2_\epsilon(\Omega)$. It follows from simple calculations that

$$\begin{aligned} &\epsilon^2\phi_{\epsilon,R,0} - (1+\lambda_1)\phi_{\epsilon,R,0} \\ &= -p\frac{u^{p-1}}{\xi^q}\phi_{\epsilon,R,0} + \lambda_\epsilon(\psi_1+\psi_2)(1-\chi_1-\chi_2) - 2\epsilon^2\nabla(\chi_1+\chi_2)\cdot\nabla\phi_\epsilon \\ &\quad - \epsilon^2\Delta(\chi_1+\chi_2)\phi_\epsilon + (1-\chi_1-\chi_2)q\frac{u^p}{\xi^{q+1}}\eta_\epsilon. \end{aligned} \tag{84}$$

We multiply $\phi_{\epsilon,R,0}$ to the both sides of this equality and integrate it by parts. Then, the integral over $\partial\Omega\cap B_{\epsilon R}(h_1)$ naturally appears because $\phi_{\epsilon,R,0}$ may not satisfy the homogeneous Neumann boundary condition on $\partial\Omega\cap B_{\epsilon R}(h_1)$. By the similar argument to the proof of Trace Theorem (see [9]), we readily see that

$$\|\varphi\|^2_{L^2(\partial\Omega\cap B_{\epsilon R}(h))} \leq c(\|\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}) \tag{85}$$

for $\varphi \in H^1(\Omega)$ and $h \in \partial\Omega$ and a constant c independent of ϵ and R. Since $h_1 \in \partial\Omega$ and $h_2 \in \Omega$, we have

$$\int_\Omega -\Delta\phi_{\epsilon,R,0}\phi_{\epsilon,R,0}dx = \int_\Omega |\nabla\phi_{\epsilon,R,0}|^2 dx - \int_{\partial\Omega\cap B_{\epsilon R}(h_1)} \frac{\partial\chi_1}{\partial\nu}\chi_1|\phi_\epsilon|^2 dS_\sigma$$

and

$$\begin{aligned} \epsilon^2\left|\int_{\partial\Omega\cap B_{\epsilon R}(h_1)} \frac{\partial\chi_1}{\partial\nu}\chi_1|\phi_\epsilon|^2 dS_\sigma\right| &\leq \epsilon^2\int_{\partial\Omega\cap B_{\epsilon R}(h_1)} |\nabla\chi_1||\phi_\epsilon|^2 dS_\sigma \\ &\leq \frac{c\epsilon}{R}(\|\phi_\epsilon\|^2_{L^2(\Omega)} + \|\nabla\phi_\epsilon\|_{L^2(\Omega)}\|\phi_\epsilon\|_{L^2(\Omega)}), \end{aligned}$$

where S_σ represents the $(n-1)$-dimensional surface measure. Since $\|\phi_\epsilon\|_{H^2_\epsilon(\Omega)} \leq c$ and $\|\phi_{\epsilon,R,0}\|_{L^2_\epsilon(\Omega)} \leq 1$, and ψ_1, ψ_2 and u is small outside $B_{\epsilon R}(h_1) \cup B_{\epsilon R}(h_2)$, it follows from (84) that

$$\epsilon^2\|\nabla\phi_{\epsilon,R,0}\|^2_{L^2_\epsilon} + (1+\lambda_1)\|\phi_{\epsilon,R,0}\|^2_{L^2_\epsilon} \leq \delta$$

for small $\delta > 0$.

Next we estimate $\phi_{\epsilon,R,1}$. Without loss of generality, we suppose that $h_1 = 0$ and the tangent space of $\partial\Omega$ at the origin corresponds to a $(n-1)$-dimensional hyper plane $\{x \in \mathbb{R}^n \mid x_n = 0\}$. Let $z = \Phi(x)$ be a diffeomorphism defined in a neighborhood N of the origin such that

$$\Phi(\Omega\cap N) \subset \mathbb{R}^n_+, \quad \Phi(\partial\Omega\cap N) \subset \partial\mathbb{R}^n_+$$

and $\Phi(0) = 0$, $D\Phi(0) = I$, where $\mathbb{R}^n_+ \equiv \{z \in \mathbb{R}^n \mid z_n > 0\}$, I is the unit matrix on $\mathbb{R}^n$ and $D\Phi$ represents the Jacobian matrix of Φ. Since $\partial\Omega$ belongs to C^2, Φ also belongs to C^2-class. We take the neighborhood N independently of ϵ and R.

Roughly speaking, the mapping $\Phi(x)$ straightens out $\partial\Omega$ around the origin. Put $\Psi(z) = \Phi^{-1}(z)$ and $\tilde{\phi}_{\epsilon,R,1}(y) = \phi_{\epsilon,R,1}(\Psi(\epsilon y))$. Then,

$$\begin{aligned}&\Delta\tilde{\phi}_{\epsilon,R,1} - \tilde{\phi}_{\epsilon,R,1} + p\frac{\tilde{u}^{p-1}}{\xi^q}\tilde{\phi}_{\epsilon,R,1} - \lambda_1\tilde{\phi}_{\epsilon,R,1}\\ &= \lambda_\epsilon\tilde{\chi}_1\tilde{\psi}_1 + q\tilde{\chi}_1\frac{\tilde{u}^p}{\xi^{q+1}}\eta_\epsilon + 2\nabla\tilde{\chi}_1\cdot\nabla\tilde{\phi}_\epsilon + \Delta\tilde{\chi}_1\tilde{\phi}_\epsilon + O(\epsilon)\end{aligned} \tag{86}$$

where $\tilde{\chi}_1(y) = \chi_1(\Psi(\epsilon y))$, $\tilde{u}(y) = u(\Psi(\epsilon y))$ and $\tilde{\psi}_1(y) = \psi_1(\Psi(\epsilon y))$. Here we use the Landau's symbol $O(\epsilon)$, which means that $\|O(\epsilon)\|_{L^2(K)} \le c\epsilon$ for any compact subset $K \subset \mathbb{R}^n_+$ independent of ϵ. Since $\tilde{\phi}_{\epsilon,R,1}$ has a compact support in $\mathbb{R}^n_+$, we can extend $\tilde{\phi}_{\epsilon,R,1}$ to the half space by a natural way, i.e., set $\tilde{\phi}_{\epsilon,R,1} \equiv 0$ outside the support. From $\|\phi_{\epsilon,R,1}\|_{H^1_\epsilon} \le c$, we obtain $\|\tilde{\phi}_{\epsilon,R,1}\|_{H^1(\mathbb{R}^n_+)} \le c$. Then, there is $\phi_{0,R,1} \in H^1(\mathbb{R}^n_+)$ such that

$$\tilde{\phi}_{\epsilon,R,1} \to \phi_{0,R,1} \quad \text{weakly in } H^1(\mathbb{R}^n_+) \text{ as } \epsilon \to 0.$$

Direct calculations give us

$$\tilde{\chi}_1(y) = \chi_1(\Psi(\epsilon y)) = \chi(|\Psi(\epsilon y)|/\epsilon R) \to \chi(|y|/R) \equiv \chi_R(y)$$

as $\epsilon \to 0$ because $\Psi(0) = 0$ and $D\Psi(0) = I$. By (86), $\phi_{0,R,1}$ belongs to $H^2(\mathbb{R}^n_+)$ and satisfies

$$\begin{aligned}&\Delta\phi_{0,R,1} - \phi_{0,R,1} + pS^{p-1}\phi_{0,R,1} - \lambda_1\phi_{0,R,1}\\ &= 2\lambda_0\chi_R\psi - q\chi_R\zeta^{\frac{q+p-1}{p-1}}S^p\eta_0 + O(1/R).\end{aligned} \tag{87}$$

Since $\|\phi_{0,R,1}\|_{H^1(\mathbb{R}^n_+)} \le \liminf_{\epsilon\to 0} \|\tilde{\phi}_{\epsilon,R,1}\|_{H^1(\mathbb{R}^n_+)} \le c$, there is a function $\phi_1 \in H^1(\mathbb{R}^n_+)$ such that

$$\phi_{0,R,1} \to \phi_1 \quad \text{weakly in } H^1(\mathbb{R}^n_+) \text{ as } R \to \infty.$$

Then ϕ_1 belongs to $H^2(\mathbb{R}^n_+)$ and satisfies

$$\Delta\phi_1 - \phi_1 + pS^{p-1}\phi_1 = \lambda_1\phi_1 + 2\lambda_0\psi - q\zeta^{\frac{q+p-1}{p-1}}S^p\eta_0 \tag{88}$$

because of (83).

Now we see that $\partial\phi_1/\partial y_n = 0$ on $y_n = 0$. Since ϕ_ϵ satisfies the homogeneous Neumann boundary condition, direct calculation gives us

$$-\int_{B_K(0)\cap\mathbb{R}^n_+} \Delta\phi_{0,R,1}\varphi dy = \int_{B_K(0)\cap\mathbb{R}^n_+} \nabla\phi_{0,R,1}\cdot\nabla\varphi dy,$$

where $\varphi \in C^1_0(B_K(0))$ with $0 < K < R/2$ fixed independently of ϵ and R. It follows from (83) and the above equality that

$$-\int_{B_K(0)\cap\mathbb{R}^n_+} \Delta\phi_1\varphi dy = \int_{B_K(0)\cap\mathbb{R}^n_+} \nabla\phi_1\cdot\nabla\varphi dy.$$

Since K is arbitrarily fixed, the above equality holds true for any $\varphi \in C^1_0(\mathbb{R}^n)$, which implies that $\partial\phi_1/\partial y_n = 0$ on $y_n = 0$.

We can extend ϕ_1 to the whole space by setting

$$\overline{\phi}_1(y_1,\dots,y_{n-1},y_n) = \begin{cases}\phi_1(y_1,\dots,y_{n-1},y_n), & y_n > 0,\\ \phi_1(y_1,\dots,y_{n-1},-y_n), & y_n < 0.\end{cases}$$

In the following, we do not distinguish the original function ϕ_1 from the extended one $\overline{\phi}_1$, and use the same notation ϕ_1. Then, the equation (88) holds in the whole space. Similarly, for $\phi_{\epsilon,R,2}$, there is a function $\phi_2 \in H^2(\mathbb{R}^n)$ such that

$$\Delta\phi_2 - \phi_2 + pS^{p-1}\phi_2 = \lambda_1\phi_2 - \lambda_0\psi - q\zeta^{\frac{q+p-1}{p-1}}S^p\eta_0.$$

We set $\varphi_+ = \phi_1 + 2\phi_2$ and $\varphi_- = \phi_1 - \phi_2$. Then, φ_+ and φ_- satisfy

$$\begin{cases} \Delta\varphi_+ - \varphi_+ + pS^{p-1}\varphi_+ = \lambda_1\varphi_+ - 3q\zeta^{\frac{q+p-1}{p-1}}S^p\eta_0, \\ \Delta\varphi_- - \varphi_- + pS^{p-1}\varphi_- = \lambda_1\varphi_- + 3\lambda_0\psi. \end{cases} \tag{89}$$

Since ϕ_ϵ is orthogonal to $\psi_1 + \psi_2$, we have

$$\int_{\mathbb{R}^n} \psi\varphi_- dy = 0. \tag{90}$$

Multiplying ψ to the both sides of the first and second equations of (89) and integrating by parts, we have $\lambda_0 = 0$ and $\eta_0 = 0$ because ψ is the eigenfunction of (16), and ψ and S are positive functions. Since the eigenfunction of (16) is unique up to multiplicities of constants, φ_+ and φ_- must be multiplicities of ψ. Then, (90) implies $\varphi_- = 0$. The second equation (81) implies that

$$\int_{\mathbb{R}^n} S^{r-1}\varphi_+ = 0,$$

from which we deduces that $\phi_1 = \phi_2 = 0$.

Now we show that $\phi_{\epsilon,R,1}$ tends to 0 strongly in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$ and $R \to \infty$. We set $\varphi(y) = \sqrt{1+|y|^2}\tilde{\phi}_{\epsilon,R,1}(y)$. Then φ satisfies

$$\begin{aligned} &-\Delta\varphi + (1+\lambda_1)\varphi \\ &= p\frac{\tilde{u}^{p-1}}{\xi^q}\sqrt{1+|y|^2}\tilde{\phi}_{\epsilon,R,1} - 2\frac{y}{\sqrt{1+|y|^2}}\cdot\nabla\tilde{\phi}_{\epsilon,R,1} - \frac{1}{(1+|y|^2)^{\frac{3}{2}}}\tilde{\phi}_{\epsilon,R,1} \\ &\quad - \sqrt{1+|y|^2}(\lambda_\epsilon\tilde{\chi}_1\tilde{\psi}_1 + q\frac{\tilde{u}^p}{\xi^{q+1}}\tilde{\chi}_1\eta_\epsilon + 2\nabla\tilde{\chi}_1\cdot\nabla\tilde{\phi}_\epsilon + \Delta\tilde{\chi}_1\tilde{\phi}_\epsilon. \end{aligned}$$

We multiplying φ to the both sides and integrating it over $\mathbb{R}^n_+$ by parts. Calculating

$$\int_{\mathbb{R}^n_+} -\Delta\varphi\varphi dx = \int_{\mathbb{R}^n_+} |\nabla\varphi|^2 dx + \int_{\partial\mathbb{R}^n_+} \frac{\partial\varphi}{\partial y_n}\varphi dS_\sigma,$$

we estimate the integral on the boundary of the right-hand side. Direct calculations give us

$$\begin{aligned} \int_{\partial\mathbb{R}^n_+} \frac{\partial\varphi}{\partial y_n}\varphi dS_\sigma &= \epsilon\sum_{k=1}^n \int_{\partial\mathbb{R}^n_+} \frac{\partial\Psi_k}{\partial z_n}(\epsilon y)\frac{\partial\phi_{\epsilon,R,1}}{\partial x_k}(\Psi(\epsilon y))\sqrt{1+|y|^2}\varphi(y)dS_\sigma(y) \\ &= \epsilon\sum_{k=1}^n \int_{\partial\mathbb{R}^n_+} \frac{\partial\Psi_k}{\partial z_n}(\epsilon y)\sqrt{1+|y|^2}\varphi(y) \\ &\qquad \cdot\left(\chi_1(\Psi(\epsilon y))\frac{\partial\phi_\epsilon}{\partial x_k}(\Psi(\epsilon y)) + \frac{\partial\chi_1}{\partial x_k}(\Psi(\epsilon y))\phi_\epsilon(\Psi(\epsilon y))\right) dS_\sigma(y), \end{aligned}$$

where Ψ_k stands for the k-th component of Ψ. For $k \neq n$, we have $|\partial \Psi_k/\partial z_n| \leq c\epsilon$ in the support of $\tilde{\phi}_{\epsilon,R,1}$ so that

$$\begin{aligned}
&\left|\epsilon \int_{\partial \mathbb{R}^n_+} \frac{\partial \Psi_k}{\partial z_n}(\epsilon y)\chi_1(\Psi(\epsilon y))\frac{\partial \phi_\epsilon}{\partial x_k}(\Psi(\epsilon y))\sqrt{1+|y|^2}\varphi(y)dS_\sigma(y)\right| \\
&\leq cR^2\epsilon^{3-n}\|\nabla\phi_\epsilon\|_{L^2(\partial\Omega)}\|\phi_\epsilon\|_{L^2(\partial\Omega)} \leq cR^2\epsilon
\end{aligned}$$

by (85) and $\|\phi_\epsilon\|_{H^2_\epsilon} \leq c$. Similarly,

$$\left|\epsilon \int_{\partial \mathbb{R}^n_+} \frac{\partial \Psi_k}{\partial z_n}(\epsilon y)\phi_\epsilon(\Psi(\epsilon y))\frac{\partial \chi_1}{\partial x_k}(\Psi(\epsilon y))\sqrt{1+|y|^2}\varphi(y)dS_\sigma(y)\right| \leq cR^2\epsilon$$

by (83), (85) and $\|\phi_\epsilon\|_{H^2_\epsilon} \leq c$.

Next we estimate the boundary integral for $k = n$. Since $\partial\phi_\epsilon/\partial\nu = 0$, and $|\nu_k| \leq c\epsilon$, $|\nu_n - 1| \leq c\epsilon$ as $\epsilon \to 0$ on $B_{\epsilon R}(0) \cap \partial\Omega$, it holds that $\nu\partial\phi_\epsilon/\partial x_n = -\sum_{k=1}^{n-1}\nu_k\partial\phi_\epsilon/\partial x_k$, and then

$$\left|\frac{\partial\phi_\epsilon}{\partial x_n}\right| \leq c\epsilon\left|\sum_{k=1}^{n-1}\frac{\partial\phi_\epsilon}{\partial x_k}\right| \leq c\epsilon|\nabla\phi_\epsilon|,$$

where we denote the normal outer vector by $\nu = (\nu_1, \ldots, \nu_n)$. On the other hand, thanks to $|x_n| \leq c\epsilon^2R^2$ and $|x| \geq cR\epsilon$ for $x \in (B_{\epsilon R}(0)\backslash B_{\epsilon R/2}(0)) \cap \partial\Omega$,

$$\left|\frac{\partial\chi_1}{\partial x_n}\right| = \frac{1}{R\epsilon}\frac{|x_n|}{|x|}|\chi'| \leq c.$$

From these estimates, we know that the boundary integral for $k = n$ tends to 0 as $\epsilon \to 0$ by the similar argument to the previous paragraph.

Noting that $|\tilde{u}|^{p-1}(1+|y|^2) \leq c$ and $\|\tilde{\phi}_{\epsilon,R,1}\|_{L^2(\mathbb{R}^n_+)} \leq 1$, we have

$$\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n_+)} + (1+\lambda_1)\|\varphi\|^2_{L^2(\mathbb{R}^n_+)} \leq c.$$

For $K > 0$ arbitrarily fixed, we obtain

$$\begin{aligned}
\frac{1}{\epsilon^n}\int_\Omega |\phi_{\epsilon,R,1}|^2dx \leq c\int_{\mathbb{R}^n_+}|\tilde{\phi}_{\epsilon,R,1}|^2dy& \\
\leq c\int_{B_K(0)}|\tilde{\phi}_{\epsilon,R,1}|^2dy &+ \frac{c}{K^2}\int_{\mathbb{R}^n_+\backslash B_K(0)}(1+|y|^2)|\tilde{\phi}_{\epsilon,R,1}|^2dy \\
\leq c\int_{B_K(0)}|\tilde{\phi}_{\epsilon,R,1}|^2dy &+ \frac{c}{K^2}.
\end{aligned}$$

From Rellich's theorem, $\tilde{\phi}_{\epsilon,R,1}$ tends to $\phi_{0,R,1}$ strongly in $L^2(B_K(0))$ as $\epsilon \to 0$. Similarly, $\phi_{0,R,1}$ tends to 0 strongly in $L^2(B_K(0))$ as $R \to \infty$. Then,

$$\limsup_{R\to\infty}\limsup_{\epsilon\to 0}\frac{1}{\epsilon^n}\int_\Omega|\phi_{\epsilon,R,1}|^2dx \leq \frac{c}{K^2} \to 0$$

as $K \to \infty$. Similarly, $\|\phi_{\epsilon,R,2}\|_{L^2_\epsilon} \to 0$ as $\epsilon \to 0$ and $R \to \infty$. In the end, $\|\phi_\epsilon\|_{L^2_\epsilon} \to 0$ and $\lambda_\epsilon \to 0$ as $\epsilon \to 0$, which contradicts $\|\phi_\epsilon\|_{L^2_\epsilon} + |\lambda_\epsilon| = 1$. Hence $T_t(0,0,0,\epsilon)$ is injective.

Define $\mathcal{L}$ between $H^2_{N,\epsilon}(\Omega)\times\mathbb{R}\to L^2_\epsilon(\Omega)\times\mathbb{R}$ by

$$\mathcal{L}\begin{pmatrix}\phi\\ \eta\end{pmatrix}\equiv\begin{pmatrix} L\phi - q\dfrac{u^p}{\xi^{q+1}}\eta-\lambda_1\phi \\ -\eta+\dfrac{r}{\xi^s\epsilon^n}\displaystyle\int_\Omega u^{r-1}\phi dx-\dfrac{s\eta}{\xi^{s+1}\epsilon^n}\int_\Omega u^r dx-\tau\lambda_1\eta\end{pmatrix}.$$

Since $T_t(0,0,0,\epsilon)$ is injective and $\psi_1+\psi_2\in L^2_\epsilon(\Omega)$, the following equation

$$\mathcal{L}\begin{pmatrix}\phi\\ \eta\end{pmatrix}=\begin{pmatrix}\psi_1+\psi_2\\ 0\end{pmatrix}$$

does not have a solution $(\phi,\eta)\in H^2_{N,\epsilon}(\Omega)\times\mathbb{R}$. This implies that

$$R(\mathcal{L})\subset L^2_{N,\epsilon}(\Omega),$$

where $L^2_{N,\epsilon}(\Omega)$ is defined by

$$L^2_{N,\epsilon}(\Omega)=\left\{\varphi\in L^2_\epsilon(\Omega)\;\middle|\;\int_\Omega(\psi_1+\psi_2)\varphi dx=0\right\}.$$

So we regard $\mathcal{L}$ as an operator between $H^2_{N,\epsilon}(\Omega)\times\mathbb{R}\to L^2_{N,\epsilon}(\Omega)\times\mathbb{R}$. If $\mathcal{L}$ is not surjective, there is a solution $(\phi,\eta)\in H^2_{N,\epsilon}(\Omega)\times\mathbb{R}$ of $\mathcal{L}(\phi,\eta)^t=0$, where the superscript ^{t} stands for transpose. However, this contradicts the assumption that $\lambda=\lambda_1$ is not an eigenvalue of (15). Hence, by taking any $(f,\zeta)\in L^2_\epsilon(\Omega)\times\mathbb{R}$ and putting

$$\lambda=\frac{1}{\|\psi_1+\psi_2\|^2_{L^2_\epsilon}\epsilon^n}\int_\Omega f(\psi_1+\psi_2)dx,\quad F=f-\lambda(\psi_1+\psi_2),$$

there is a unique solution $(\phi,\eta)\in H^2_{N,\epsilon}(\Omega)\times\mathbb{R}$ such that

$$\mathcal{L}\begin{pmatrix}\phi\\ \eta\end{pmatrix}=\begin{pmatrix}F\\ \zeta\end{pmatrix}.$$

Therefore $T_t(0,0,0,\epsilon)$ is invertible.

Also, the last part of the lemma can be shown by the same argument as above. Hence we complete the proof. □

Now we are in position to prove Theorem 3.3. Define a map between $H^2_{N,\epsilon}(\Omega_\epsilon)\times\mathbb{R}\times\mathbb{R}\times(0,\epsilon_0)$ by

$$G(\phi,\eta,\lambda,\epsilon)=-T^{-1}_{(\phi,\eta,\lambda)}(0,0,0,\epsilon)\{T(0,0,0,\epsilon)+R(\phi,\eta,\lambda,\epsilon)\},$$

where

$$R(\phi,\eta,\lambda,\epsilon)=T(\phi,\eta,\lambda,\epsilon)-T(0,0,0,\epsilon)-T_{(\phi,\eta,\lambda)}(0,0,0,\epsilon)[(\phi,\eta,\lambda)]=-\begin{pmatrix}\lambda\phi\\ \tau\lambda\eta\end{pmatrix}.$$

From Proposition 3 and Lemma 6.1, it holds true that

$$\|G(0,0,0,\epsilon)\|\le\delta,\quad\|G(\phi,\eta,\lambda,\epsilon)-G(\tilde\phi,\tilde\eta,\tilde\lambda,\epsilon)\|\le\delta$$

for sufficiently small δ independent of ϵ and any $(\phi,\eta,\lambda),(\tilde\phi,\tilde\eta,\tilde\lambda)\in H^2_{N,\epsilon}(\Omega)\times\mathbb{R}\times\mathbb{R}$ satisfying $\|\phi\|_{H^2_\epsilon(\Omega)}+|\eta|+|\lambda|\le\delta$ and $\|\tilde\phi\|_{H^2_\epsilon(\Omega)}+|\tilde\eta|+|\tilde\lambda|\le\delta$, respectively. Hence there is a fixed point of G, which complete the proof.

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E-mail address: ichiro@math.kyushu-u.ac.jp
E-mail address: ikeda@isc.meiji.ac.jp
E-mail address: miyamoto@math.titech.ac.jp