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Vertical modes of quasi-geostrophic flows in an ocean with bottom topography – evolution equation and energetics –

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Abstract

Quasi-geostrophic current is expanded in terms of vertical modes such as barotropic and baroclinic ones. Then the evolution of quasi-geostrophic motion is understood from the behavior of each vertical mode. There are some subtle issues, however, as regards vertical modes: boundary conditions, difference between a level model and a layer model, and so on. A comprehensive formulation is given of the expansion of the quasi-geostrophic flows in terms of vertical modes both for a level model and for a layer model. Vertical modes are defined in almost the same manner for a level model and a layer model. Evolution equation and energetics of each mode are derived and argued. In addition to the ordinary nonlinear advective three-mode interaction, a gentle relief of bottom topography allows inter-modal energy transfer via bottom topography; in particular the so-called JEBAR effect and topographic torque is classified into one of such two-mode interaction (coupling). Under the rigid lid surface condition, the barotropic mode does not make a triplet with two different baroclinic modes, in the three-mode nonlinear advective interaction. Detailed balance turns out to hold for inter-modal energy transfer with respect to three-mode nonlinear advective interaction and two-mode coupling via topography. Potential enstrophy transfer among vertical modes is formulated also. It is shown that detailed balance is assured for inter-modal enstrophy transfer as well, if potential enstrophy is chosen properly. On the other hand detailed balance is not expected for inter-modal enstrophy transfer by two-mode coupling via bottom topography.

Key words : *vertical modes, quasi-geostrophic dynamics, continuous and discrete formulation, inter-modal energy/enstrophy transfer, nonlinear advective interaction. two-mode coupling via bottom topography*

1. Introduction

Large-scale ocean currents at middle to high latitudes are described well as a quasi-geostrophic motion, where primary balance of momentum is geostophic in the horizontal direction and hydrostatic in the vertical direction. See Pedlosly (1977), Rhines (1977), and Treguier and Hua (1987), say, for fundamental formulation and applications.

Nowadays most numerical simulations are based on the primitive equations, so that quasi-geostrophy is not an essential ingredient of such *numerical* models. On the other hand, dynamics or mechanism tends to be hidden behind a vast amount of complicated data produced by a black box of computation. In this sense quasi-geostrophic formulation still remains useful in discussing and understanding the dynamics of large-scale oceanic current within a simple framework.

Meanwhile long-term baroclinic behavior of current becomes more and more important for understanding the mechanism of large-scale motion of the ocean. A way to resolve the baroclinic structure is to expand the current field in terms of vertical modes such as barotropic and baroclinic ones. There are somewhat subtle issues, however, in the expansion in terms of vertical modes. In this article therefore we want to present fairly comprehensive formulas necessary in discussing and analyzing the quasi-geostrophic motion in a multi-level model or a multi-layer model with a gently variable depth. This is the primary purpose of the paper.

The secondary purpose is to argue some features of energy transfer among vertical modes. Also inter-modal transfer of (potential) enstrophy is argued. We see that, in the nonlinear advective interaction among

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any triplet of vertical modes, the total energy transfer and enstrophy among them vanishes or closed within the triplet. The barotropic mode cannot make an interacting triplet with two *different* baroclinic modes under the rigid lid surface condition. Argument is addressed also to the inter-modal energy transfer via bottom topography. The so-called JEBAR effect is shown to be a particular kind of inter-modal energy transfer, in which only the barotropic mode has a special role. These effects are reduced to topographic stress, which is often related with the Neptune effect (Holloway 1992, Eby and Holloway 1994).

The next section prepares for the framework of the Q-G dynamics such as notations and situations. The third section gives a systematic formulation of the method of expansion of vertical modes for a level model (continuous model). Energetics of each mode and interaction among different vertical modes are investigated. Detailed balance of three-mode interaction and two-mode coupling via bottom topography is derived for inter-modal energy transfer. Detailed balance extends to inter-modal enstrophy transfer for the three-mode interaction, but not for the two-mode coupling via bottom topography. In the fourth section similar formulation is presented for a discrete version or a layer model, with some differences noted. Evolution equation and energetics go almost parallel to those in a level model. The final section gives a summary and discussion.

2. Notation and situation

In this section we (1) prepare for notation, (2) describe the ocean to be considered and (3) relate relevant variables with the quasi-geostrophic stream function ψ.

2.1 coordinates and notation

In this article, we adopt the GFDVN (geophysical fluid dynamics vector notation), a tool for a simple description of geophysical fluid dynamics. For details see Masuda (2010). However, its minimums are described here for consistency.

A horizontal vector is expressed in boldface, while a three-dimensional vector is in boldface with an underline. A vector expressed in boldface usually indicates a column vector with its components. A dash of a vector or a matrix expresses its transpose. For example we have

$$\begin{cases} \boldsymbol{x} = (x, y)' \\ \underline{\boldsymbol{x}} = (x, y, z)' \end{cases}, \qquad \begin{cases} \boldsymbol{u} = (u, v)' \\ \underline{\boldsymbol{w}} = (u, v, w)' \end{cases},$$

where x, y, z are the eastward, northward, and upward coordinate, respectively. Also (u, v, w) denote the corresponding velocity components. According to the above notation, we have such an expression as $\underline{\boldsymbol{x}} = (\boldsymbol{x}', z)'$.

A useful operator of GFDVN is the *strophe* operator $\neg$, which twists a horizontal vector clockwise at right angle, so that

$$\boldsymbol{u} = (u, v)' \quad \Rightarrow \quad \neg\boldsymbol{u} = (v, -u)'.$$

We have a few related operators and symbols as follows:

$$\begin{cases} \nabla \equiv \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)' & \text{(nabla)} \\ \lhd \equiv \neg\nabla = \left(\dfrac{\partial}{\partial y}, -\dfrac{\partial}{\partial x}\right)' & (\textit{blana}) \end{cases}$$

and

$$\delta_{i,j} \equiv \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\epsilon_{i,j} \equiv \begin{cases} 1 & \text{for } (i,j) = (1,2) \\ -1 & \text{for } (i,j) = (2,1) \\ 0 & \text{otherwise} \end{cases},$$

respectively, where suffix i (j) is 1 or 2, corresponding to x and y; $\delta_{i,j}$ is called Kronecker's delta.

In GFDVN, a vector product is a scalar, so that we have

$$\nabla a \times \nabla b \equiv J(a, b) \equiv \sum_{i,j} \epsilon_{i,j} \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial y_j}$$

where $J(a, b)$ is the Jacobian of scalar fields a and b. There are a few useful identities such as

$$\lhd \cdot \nabla a = \nabla \cdot \lhd a = \sum_{i,j} \epsilon_{i,j} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} a = 0, \qquad (1)$$

$$\nabla a \cdot \nabla b = \lhd a \cdot \lhd b, \qquad \nabla^2 a = \lhd^2 a, \qquad (2)$$

where (1) corresponds to the well-known identities

$$\operatorname{rot}\operatorname{grad} = 0 \quad \text{and} \quad \operatorname{div}\operatorname{rot} = 0$$

in the ordinary notation.

Often we write as "$A \models B$", which reads "A is a different representation of B or vice versa". An example is

$$\boldsymbol{u} \models u_i,$$

where the left-hand side is a two-dimensional vector $\boldsymbol{u}$ and the right-hand side is its i-th component u_i. This convention is used as a general rule.

For the convenience of later discussion, let $\boldsymbol{R}^N$ be the set of the N-dimensional column vectors of real components, N being a positive integer. We then write as $\boldsymbol{u} \in \boldsymbol{R}^2$ and $\underline{\boldsymbol{u}} \in \boldsymbol{R}^3$. Similarly $\boldsymbol{C}^N$ denotes the set of the N-dimensional column vectors of complex components. Likewise let $\boldsymbol{M}(k, l)$ be the set of the (k, l)-matrices of real (or complex) components, k and l being positive integers. Then the expression

$$(\ \boldsymbol{M}(k, l) \ni\ ) \quad M \models M_{i,j}$$

means that M is a (k,l)-matrix and that its (i,j)-component is $M_{i,j}$.

We put another convention for vectors and matrices as follows. Let $u_i \models \boldsymbol{u} \in \boldsymbol{R}^n$ and $M_{i,j} \models M \in \boldsymbol{M}(n,n)$. Vector $\boldsymbol{u}$ and matrix M are specified by their components with suffixes running from 1 to n. If a suffix of the component lies outside the intrinsic range from 1 to n, that component reads 0 automatically unless stated otherwise; for example, $u_0 = 0$ and $M_{n+1,2} = 0$.

2.2 ocean and stratification

Consider a mid-latitude ocean on a β-plane. The Coriolis parameter f is expressed as $f = f(y) = f_r + \beta y$, where f_r denotes the Coriolis parameter in the middle of the basin, $\beta > 0$ is the meridional gradient of f, and the northward coordinate y has its origin at the center of the basin.

We consider two kinds of oceans with respect to the side boundary condition. The first is a closed ocean of a finite area (closed ocean basin or COB), while the second is an ocean periodic both in x and in y. (periodic ocean basin or POB), The former is an ordinary model ocean, and the latter is used often for the study of the quasi-geostrophic turbulence (Rhines 1975, Treguier and Hua 1987, Takase and Masuda 1996, Okuno and Masuda 2003).

Let L be a characteristic length scale of motion and U a horizontal velocity scale of current. We assume $\dfrac{\beta L}{f_r} \sim Ro \ll 1$, where $Ro \equiv \dfrac{U}{f_r L}$ is the Rossby number. The time scale of the motion is assumed so small compared with f_r^{-1}. Then the large-scale dynamics of the ocean is governed by the quasi-geostrophic (abbreviated as Q-G) vorticity equation. We do not repeat the Q-G formulation; it is familiar enough and the reader can refer to Pedlosky (1987) for its systematic description. The focus here is not the Q-G formualtion itself, but the expansion of the Q-G motion in terms of vertical modes.

The ocean has a mean depth of $H > 0$ with a gentle bottom topography. The mean surface is located at $z = 0$ and the bottom at $z = z_b(\boldsymbol{x}) = -H + h(\boldsymbol{x}) < 0$; h represents the bottom relief. As is well known (Pedlosky 1987), the bottom topography should be gentle enough (so that $\|\nabla h\| \leq Ro$) for the Q-G formulation to be valid. We further assume that bottom relief h is so small that the values at $z = z_b = -H + h$ can be approximated by those at $z = -H$.

As regards viscosity, we assume a (constant) horizontal kinematic eddy viscosity ν, whereas we ignore the vertical eddy viscosity except at the surface and the bottom (Ekman suction).

Let ρ be the density of sea water, ρ_r its reference value, and g the acceleration of gravity. We adopt the Boussinesq approximation, which is valid when density stratification is weak enough. In the Boussinesq approximation, it is convenient to define buoyancy B as

$$B \equiv -\frac{\rho - \rho_r}{\rho_r} g$$

and $\overline{B}$ is defined as the horizontal average of B, where $\overline{\bullet}$ indicates the horizontal average of $\bullet$. Then we have $N^2 \equiv \dfrac{d\overline{B}}{dz} > 0$, where N is called the buoyancy frequency. Density diffusion is ignored both in the horizontal and vertical directions for the quasi-gestorophic motion in concern, though it plays a key role in maintaining the basic stratification of the ocean.

2.3 Q-G stream function and others

In the Q-G dynamics the only unknown variable to be solved is the quasi-geostrophic stream function ψ. The other related variables are expressed in terms of ψ.

Let p, b, and ψ be the Q-G pressure, Q-G buoyancy, and Q-G stream function defined respectively by

$$p \equiv \text{"pressure"} - \int_z^0 \overline{\rho} g \, dz, \tag{3}$$

$$b \equiv B - \overline{B} = -\frac{\rho - \overline{\rho}}{\rho_r} g, \tag{4}$$

$$\psi \equiv \frac{p}{\rho_r f_r}, \tag{5}$$

where the term "Q-G" is often omitted for simplicity.

Buoyancy b is related with ψ as

$$b = \frac{1}{\rho_r}\frac{\partial p}{\partial z} = f_r \frac{\partial \psi}{\partial z} \tag{6}$$

from the hydrostatic approximation. Geostrophic balance $f_r \neg \boldsymbol{u} = \dfrac{\nabla p}{\rho_r}$ is rewritten as

$$\begin{aligned} \boldsymbol{u} &= -\neg\neg\boldsymbol{u} = -\frac{\neg\nabla p}{\rho_r f_r} = -\frac{\lhd p}{\rho_r f_r} \\ &= -\lhd\psi = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right)' = -\nabla\times\psi. \end{aligned} \tag{7}$$

The substantial derivative in the Q-G dynamics is defined by

$$\begin{aligned} \frac{D}{Dt}\bullet &\equiv \frac{\partial\bullet}{\partial t} + \boldsymbol{u}\cdot\nabla\bullet = \frac{\partial\bullet}{\partial t} - \lhd\psi\cdot\nabla\bullet \\ &= \frac{\partial\bullet}{\partial t} + \nabla\psi\times\nabla\bullet = \frac{\partial\bullet}{\partial t} + J(\psi,\bullet) \end{aligned} \tag{8}$$

for a quantity $\bullet$, since the advection by such a small vertical velocity w is ignored in the Q-G framework, where $\nabla\cdot\boldsymbol{u} = -\nabla\cdot\lhd\psi = 0$ by virtue of (1).

Also we have

$$\begin{aligned} N^2 w &= -\frac{Db}{Dt} = -\frac{\partial b}{\partial t} - \boldsymbol{u}\cdot\nabla b \\ &= -f_r\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\frac{\partial\psi}{\partial z} \\ f_r\frac{\partial w}{\partial z} &= -\frac{\partial}{\partial z}\left[\frac{f_r^2}{N^2}\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\frac{\partial\psi}{\partial z}\right] \end{aligned} \tag{9}$$

$$
\begin{aligned}
&= -\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\left[\frac{\partial}{\partial z}\left(\frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] \\
&\quad -J\left(\frac{\partial \psi}{\partial z}, \frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z}\right) \\
&= -\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\left[\frac{\partial}{\partial z}\left(\frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right]. \qquad (10)
\end{aligned}
$$

Since the upward displacement $\eta(\boldsymbol{x}, z, t)$ of the isopycnal is related with w by

$$
\begin{aligned}
w = \frac{D\eta}{Dt} &= -\frac{f_r}{N^2}\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\frac{\partial \psi}{\partial z} \\
&= -\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\frac{f_r}{N^2}\frac{\partial \psi}{\partial z} \\
&= -\frac{D}{Dt}\left[\frac{f_r}{N^2}\frac{\partial \psi}{\partial z}\right], \qquad (11)
\end{aligned}
$$

we obtain

$$
\eta = -\frac{f_r}{N^2}\frac{\partial \psi}{\partial z}, \qquad (12)
$$

because $\eta = 0$ for the initial quiet condition. Also the elevation of the sea surface becomes

$$
\eta = \frac{p - p_a}{\rho_r g} = \frac{f_r \psi}{g} - \frac{p_a}{\rho_r g} \qquad (13)
$$

at $z = 0$, where p_a denotes the deviation of the atmospheric pressure from the standard value at the sea surface.

Vorticity ζ is expressed as

$$
\zeta \equiv \nabla \times \boldsymbol{u} = -\nabla \times \lhd\psi = \lhd^2\psi = \nabla^2\psi.
$$

We define potential vorticity q as the sum of relative vorticity and the thickness term with the β term omitted:

$$
\begin{aligned}
q &\equiv \zeta - f_r\frac{\partial \eta}{\partial z} \\
&= \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z}\right). \qquad (14)
\end{aligned}
$$

The kinetic energy ($K.E.$) and (available) potential energy ($P.E.$) of the ocean are defined by

$$
\begin{aligned}
K.E. &\equiv \frac{\rho_r}{2}\int\int_{-H}^{0} \boldsymbol{u}^2\, dz\, d\boldsymbol{x} \\
&= \frac{\rho_r}{2}\int\int_{-H}^{0} |\nabla\psi|^2\, dz\, d\boldsymbol{x}, \qquad (15) \\
P.E. &\equiv \frac{\rho_r}{2}\int\left(g\eta_{z=0}^2 + \int_{-H}^{0} N^2\eta^2\, dz\right) d\boldsymbol{x} \\
&= \frac{\rho_r}{2}\int\left(\frac{f_r^2}{g}\left(\psi - \frac{p_a}{\rho_r f_r}\right)_{z=0}^2 \right. \\
&\qquad \left. + \int_{-H}^{0}\frac{f_r^2}{N^2}\left(\frac{\partial \psi}{\partial z}\right)^2 dz\right) d\boldsymbol{x}, \qquad (16)
\end{aligned}
$$

respectively, where

$$
\int \bullet\, d\boldsymbol{x} \equiv \int\int \bullet\, dx\, dy
$$

denotes the horizontal integration of $\bullet$ over the ocean basin (POB or COB). If we assume a rigid lid on the sea surface, the potential energy due to the surface elevation vanishes in (16).

2.4 boundary conditions

At the bottom $z = z_b(x, h) = -H + h(x, y)$ we have

$$
w_b = \boldsymbol{u} \cdot \nabla h + w_{bE},
$$

where w_{bE} is the bottom Ekman upwelling, which is assumed to be $\sim \frac{H\lambda}{f_r}\nabla^2\psi$ with λ a positive constant related with the vertical eddy viscosity and f_r. Omitting all the details of Ekman layer over the sloping bottom we put λ as a positive constant for simplicity.

We have supposed that h is so small that the evaluation at the bottom is approximated by that at the mean depth $z = -H$. The above condition at the bottom is expressed in terms of ψ as

$$
w_b = -\lhd\psi \cdot \nabla h + \frac{H\lambda}{f_r}\nabla^2\psi = J(\psi, h) + \frac{H\lambda}{f_r}\nabla^2\psi
$$

or

$$
\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z} = -f_r\left[J(\psi, h) + w_{bE}\right] \qquad (17)
$$

at $z = -H$, where (11) has been used.

On the other hand at the surface $z = 0$, Ekman suction is imposed by the wind stress $\boldsymbol{\tau}$. Usually in the Q-G dynamics, the surface elevation is quite small, so that a rigid lid is assumed on the sea surface. That is, the surface elevation $\eta_{z=0}$ by (13) and the associated $w_{z=0} = (\boldsymbol{u} \cdot \eta)_{z=0}$ are neglected. We follow this convention, unless referring to otherwise.

Then we have

$$
w_s = w_{sE} = \nabla \times \frac{\boldsymbol{\tau}}{f_r \rho_r} \qquad (18)
$$

at $z = 0$. This is related with ψ by

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z} \\
&= -f_r w_{sE} = -f_r\nabla \times \frac{\boldsymbol{\tau}}{\rho_r} \qquad (19)
\end{aligned}
$$

at $z = 0$ (surface).

One may replace that surface condition by an external force $\boldsymbol{F}_s$ concentrated at the surface. Then we have

$$
V_s = \nabla \times \frac{\boldsymbol{F}_s}{\rho_r} = \delta(z)\nabla \times \frac{\boldsymbol{\tau}}{\rho_r}
$$

where $\delta(\bullet)$ denotes the delta function.

The side boundary conditions are as follows. For the periodic ocean basin (POB), all the relevant physical quantities like ψ should satisfy the periodic boundary condition. For a closed ocean basin (COB), we require the non-slip condition for $\nu > 0$:

$$
0 = \boldsymbol{u} = -\lhd\psi \quad \text{along} \quad \partial O, \qquad (20)
$$

where ∂O denotes the side boundary. If the ocean is inviscid ($\nu = 0$), the no-slip condition is replaced by

$$0 = \boldsymbol{n} \cdot \boldsymbol{u} = -\boldsymbol{n} \cdot \lhd \psi, \tag{21}$$

where $\boldsymbol{n}$ denotes the unit horizontal vector normal to the basin boundary.

We should note that the Q-G formulation poses an auxiliary constraint that supplements the side boundary condition. That rule is mass conservation, which is expressed as

$$\frac{\partial}{\partial t} \int \psi \, d\boldsymbol{x} = 0 \quad \text{for} \quad z \in (-H, 0). \tag{22}$$

See Appendix D for details.

3. Level model

This section defines vertical modes for a level model (model of continuous stratification) and discusses their fundamental properties. Then evolution equation of each vertical mode is derived. Energetics of each mode and energy transfer among vertical modes are formulated.

3.1 vertical modes

We first determine the vertical modes in terms of which to expand the quasi-geostrophic stream function ψ. Any complete set of functions does expand ψ in the vertical direction. Usually, however, we prefer a set of orthonomal eigenfunctions that reflects the dynamics of the system in concern. In our case, such a complete orthonormal set is obtained from the eigenvalue problem of the Sturm-Liouville type below.

The differential operation with respect to z coordinate, in the definition of q (14) should be replaced by a multiplication of an eigenvalue for the associated eigenfucntion. Thus it is natural to define the problem as

$$\begin{cases} \dfrac{d}{dz}\left[\dfrac{f_r^2}{N^2}\dfrac{d\hat{\psi}_k}{dz}\right] = -F_k \hat{\psi}_k \qquad (-H < z < 0) \\ 0 = \left(\dfrac{d\hat{\psi}_k}{dz}\right)_{z=0} = \left(\dfrac{d\hat{\psi}_k}{dz}\right)_{z=-H} \\ 1 = \dfrac{1}{H}\displaystyle\int_{-H}^{0} |\hat{\psi}_k|^2 \, dz \end{cases} \tag{23}$$

where F_k is the k-th eigenvalue and $\hat{\psi}_k = \hat{\psi}_k(z)$ is the associated eigenfunction. The first of the above equations defines the differential equation to be satisfied by $\hat{\psi}_k$. The second gives the boundary conditions at the surface (rigid lid) and the bottom (flat bottom, formally), which are the same as those of ψ. The third is the normalization condition.

This eigenvalue problem corresponds to that of the well-known Sturm Liouville type. For a uniform stratification $N^2(z) = \text{const} > 0$, we have

$$F_k = \frac{f_r^2}{N^2}\frac{k^2\pi^2}{H^2} \quad (0 \le k;\ k: \text{ integer})$$

$$\hat{\psi}_k = \begin{cases} 1 & (k = 0) \\ \sqrt{2}\,\cos\dfrac{\pi k\, z}{H} & (1 \le k) \end{cases}. \tag{24}$$

For smooth distribution of $N^2(z) > 0$, it is easy to obtain the following results:

$$(1): \delta_{m,n} = \frac{1}{H}\int_{-H}^{0} \hat{\psi}_m(z)\hat{\psi}_n(z)\, dz, \tag{25}$$

$$(2): F_m \delta_{m,n} = \frac{1}{H}\int_{-H}^{0} \frac{f_r^2}{N^2}\frac{d\hat{\psi}_m}{dz}\frac{d\hat{\psi}_n}{dz}\, dz, \tag{26}$$

$$(3): F_m \ge F_0 = 0, \quad \hat{\psi}_0(z) = 1. \tag{27}$$

Property (1) means the ordinary orthogonality of eigenfunctions. The second property shows another less familiar orthogonality. Property (3) means that eigenvalues are non-negative, and the lowest eigenvalue is 0 with the associated eigenfunction of $\hat{\psi}_0(z) = 1$. In oceanography we call $\hat{\psi}_0$ as the barotropic mode, and the others $\{\hat{\psi}_k \mid k \ge 1\}$ as the baroclinic modes. These results are well-known so that we admit them (see Section 4, where derivation of the above properties is given for a layer model).

1) Attention is to be paid that $F_0 = 0$ is due to the assumption of the rigid lid at the surface $z = 0$; if the free variable surface is allowed, F_0 becomes positive though it is quite small (see Appendix A).

2) Since the set of all the vertical modes $\{\hat{\psi}_m \mid 0 \le m\}$ is complete and orthonormal, any ordinary function $\phi = \phi(z)$ can be expanded in terms of those vertical modes as

$$\phi(z) = \sum_{m=0}^{\infty} \tilde{\phi}_m \hat{\psi}_m(z), \tag{28}$$

where the expansion coefficient $\tilde{\phi}_m$ becomes

$$\tilde{\phi}_m \equiv \frac{1}{H}\int_{-H}^{0} \phi(z)\hat{\psi}_m(z)\, dz. \tag{29}$$

Here the expansion coefficient of vertical modes is indicated by $\tilde{\bullet}_m$. We therefore have

$$\psi(\boldsymbol{x}, z, t) = \sum_{m=0}^{\infty} \tilde{\psi}_m(\boldsymbol{x}, t)\hat{\psi}_m(z), \tag{30}$$

where

$$\tilde{\psi}_m \equiv \frac{1}{H}\int_{-H}^{0} \psi\hat{\psi}_m\, dz = \tilde{\psi}_m(\boldsymbol{x}, t) \tag{31}$$

represents the k-th mode Q-G stream function. Likewise we may expand the horizontal current $\boldsymbol{u}$ as

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, z, t) &= \sum_m \tilde{\boldsymbol{u}}_m(\boldsymbol{x}, t)\hat{\psi}_m(z) \\ &= -\sum_m \lhd\tilde{\psi}_m(\boldsymbol{x}, t)\hat{\psi}_m(z) \end{aligned}$$

in terms of $\hat{\psi}_m$.

3) It is often convenient to divide the current into the barotropic and baroclinc component. We define

$$\begin{cases} \psi_{tro} \equiv \tilde{\psi}_0 \hat{\psi}_0 = \tilde{\psi}_0 \\ \boldsymbol{u}_{tro} \equiv - \lhd \psi_{tro} \\ p_{tro} \equiv \rho_r f_r \psi_{tro} \end{cases} \tag{32}$$

as the barotropic component of ψ, $\boldsymbol{u}$, and p. The remaining part

$$\begin{cases} \psi_{cli} \equiv \psi - \psi_{tro} = \sum_{m \geq 1} \tilde{\psi}_m(\boldsymbol{x}, t) \hat{\psi}_m(z) \\ \boldsymbol{u}_{cli} \equiv - \lhd \psi_{cli} = \boldsymbol{u} - \boldsymbol{u}_{tro} \\ p_{cli} \equiv \rho_r f_r \psi_{cli} = p - p_{tro} \end{cases} \tag{33}$$

are the baroclinic component of ψ, $\boldsymbol{u}$, and p, respectively. Of course, the baroclinic component is the sum over all the baroclinic modes.

4) Property (2) shows the orthonormal condition of the set of $\left\{ \frac{f_r}{N\sqrt{F_m}} \frac{d\hat{\psi}_m}{dz} \mid 1 \leq m \right\}$. For a uniform stratification $N^2(z) = \text{const} > 0$, we have

$$\frac{f_r}{N\sqrt{F_m}} \frac{d\hat{\psi}_m}{dz} = -\sqrt{2} \sin \frac{\pi m z}{H}. \quad (1 \leq m)$$

just as (24).

One may expand any smooth function $\phi = \phi(z)$ as

$$\phi = \sum_{m=1}^{\infty} \frac{f_r}{N\sqrt{F_m}} \frac{d\hat{\psi}_m}{dz} \left(\int_{-H}^{0} \frac{f_r}{N} \frac{\phi}{\sqrt{F_m}} \frac{d\hat{\psi}_m}{dz} dz \right).$$

This type of expansion is better for w, b, and η, which are expressed through $\frac{\partial \psi}{\partial z}$.

It is convenient, however, to use the basis functions and inner product in a slightly modified way. For instance, the basis function of the isopycnal elevation is defined by

$$\hat{\eta}_m \equiv - \frac{f_r}{N^2} \frac{d\hat{\psi}_m}{dz} \tag{34}$$

and the orthonormal condition (26) has a form of

$$F_m \delta_{m,n} = \frac{1}{H} \int_{-H}^{0} N^2 \hat{\eta}_m \hat{\eta}_n \, dz.$$

Then, the expansion of η in terms of $\{\hat{\eta}_m\}$ is written as

$$\eta = \sum_m \tilde{\eta}_m \hat{\eta}_m(z),$$

where the coefficient $\tilde{\eta}_m$ is obtained from

$$F_m \tilde{\eta}_m \equiv \frac{1}{H} \int_{-H}^{0} N^2(z) \eta \cdot \hat{\eta}_m(z) \, dz.$$

We should note that the inner product of $\{\hat{\eta}_m\}$ differs from the ordinary one (25) and that $\{\hat{\eta}_m\}$ is not a normal basis even with this inner product. The advantage of this representation is that we may expand η simply as

$$\eta = \sum_m \tilde{\psi}_m \hat{\eta}_m(z)$$

in terms of $\{\tilde{\psi}_m\}$; ψ is the principal variable in the Q-G dynamics.

5) It appears natural to call

$$\check{q}_m \equiv (\nabla^2 - F_m)\tilde{\psi}_m \tag{35}$$

as the m-th mode of potential vorticity. Attention must be paid to it, however, that, by our definition,

$$\begin{aligned} \tilde{q}_m &= \frac{1}{H} \int_{-H}^{0} \hat{\psi}_m \left(\nabla^2 + \frac{\partial}{\partial z} \left[\frac{f_r^2}{N^2} \frac{\partial}{\partial z} \right] \right) \psi \, dz \\ &= (\nabla^2 - F_m)\tilde{\psi}_m + \frac{1}{H} \left[\frac{f_r^2}{N^2} \frac{\partial \psi}{\partial z} \hat{\psi}_m \right]_{-H}^{0} \\ &= \check{q}_m + \frac{1}{H} \left[\frac{f_r^2}{N^2} \frac{\partial \psi}{\partial z} \hat{\psi}_m \right]_{-H}^{0} \neq \check{q}_m. \end{aligned}$$

Thus $\check{q}_m$ differs from the original definition in (29), but let us call $\check{q}_m$ simply as "the m-th mode potential vorticity".

6) As will be shown later, three-mode interaction or interaction among a triplet of vertical modes is the most important in the evolution and energetics of each vertical mode. We here define a few quantities related with three-mode interaction for later convenience. The first is the triplet interaction coefficient defined by

$$I_{ijk} \equiv \frac{1}{H} \int_{-H}^{0} \hat{\psi}_i \hat{\psi}_j \hat{\psi}_k \, dz, \tag{36}$$

which is symmetric with respect to indices.

In particular for $i = 0$ we find

$$\begin{aligned} I_{0jk} &= \frac{1}{H} \int_{-H}^{0} \hat{\psi}_0 \, \hat{\psi}_j \hat{\psi}_k \, dz \\ &= \frac{1}{H} \int_{-H}^{0} \hat{\psi}_j \hat{\psi}_k \, dz = \delta_{j,k}. \end{aligned} \tag{37}$$

It implies that I_{ijk} vanishes, if one of the interacting triplet is the barotropic mode, unless the other vertical modes are the same. Note, however, that $I_{00i} \neq 0$ $(i \neq 0)$, though it may be small, for the free-surface condition (see Appendix A).

The second quantity is

$$\begin{aligned} I_{i,jk} \equiv I_{i,kj} &\equiv \frac{1}{H} \int_{-H}^{0} \hat{\psi}_i \frac{d\hat{\psi}_j}{dz} \frac{f_r^2}{N^2} \frac{d\hat{\psi}_k}{dz} dz \\ &= \left[\frac{\hat{\psi}_i \hat{\psi}_j}{H} \frac{f_r^2}{N^2} \frac{d\hat{\psi}_k}{dz} \right]_{-H}^{0} \\ &\quad - \frac{1}{H} \int_{-H}^{0} \hat{\psi}_j \frac{d\hat{\psi}_i}{dz} \frac{f_r^2}{N^2} \frac{d\hat{\psi}_k}{dz} dz \\ &\quad - \frac{1}{H} \int_{-H}^{0} \hat{\psi}_i \hat{\psi}_j \frac{d}{dz} \left(\frac{f_r^2}{N^2} \frac{d\hat{\psi}_k}{dz} \right) dz. \\ &= -I_{j,ik} + F_k \, I_{ijk}. \end{aligned} \tag{38}$$

3.2 evolution equation of each mode

The Q-G vorticity equation becomes

$$\frac{D\nabla^2\psi}{Dt} + \beta v = f_r \frac{\partial w}{\partial z} + \nu\nabla^4\psi + V$$
$$= -\frac{D}{Dt}\left(\frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \nu\nabla^4\psi + V$$

or

$$\left(\frac{\partial}{\partial t} + J(\psi,\cdot)\right)\left[\nabla^2\psi + \frac{\partial}{\partial z}\left[\frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right]\right] + \beta\frac{\partial\psi}{\partial x}$$
$$= \nu\nabla^4\psi + V, \tag{39}$$

where $V = V(\boldsymbol{x}, z, t)$ denotes the prescribed (external) volume input of vorticity, if ever. Usually we have

$$V = \nabla \times \frac{\boldsymbol{F}}{\rho_r},$$

where $\boldsymbol{F}$ expresses some external force. Eq. (39) governs the evolution of potential vorticity q at each level z.

If we know the evolution of $\{\tilde{\psi}_m \;|0 \leq m\}$, we know the evolution of $\psi = \psi(\boldsymbol{x}, z, t)$ by (30). We therefore convert (39) to the evolution equation of the m-th mode by taking the inner product of (39) and $\hat{\psi}_m$, where boundary conditions at the surface and bottom are to be taken into account in this calculation.

The resulting equation becomes

$$\frac{\partial}{\partial t}(\nabla^2 - F_m)\tilde{\psi}_m + \beta\frac{\partial\tilde{\psi}_m}{\partial x}$$
$$= \check{N}_m + \nu\nabla^4\tilde{\psi}_m + \tilde{V}_m + \check{W}_m, \tag{40}$$

where

$$\check{N}_m \equiv -\sum_{i,j} I_{ijm} J(\tilde{\psi}_i, \check{q}_j), \tag{41}$$

$$\tilde{V}_m \equiv \frac{1}{H}\int_{-H}^{0} V\hat{\psi}_m\, dz, \tag{42}$$

$$\check{W}_m \equiv \frac{f_r}{H}\left[w\hat{\psi}_m\right]_{-H}^{0}. \tag{43}$$

Along the side boundary ∂O, $\tilde{\psi}_m$ must be periodic for a POB or satisfy the no-slip condition $\nabla\tilde{\psi}_m = 0$ for a COB ($\nabla\cdot\lhd\tilde{\psi}_m = 0$ when $\nu = 0$). Also mass conservation constraint becomes

$$\frac{\partial}{\partial t}\int \tilde{\psi}_m\, d\boldsymbol{x} = 0.$$

Those side boundary conditions are obtained from the inner product of $\hat{\psi}_m$ with (20), (21), and (22).

The left-hand side of (40) describes the temporal evolution of $\check{q}_m = (\nabla^2 - F_m)\tilde{\psi}_m$ and the β-effect. If $\nu = 0$, $\lambda = 0$, $h = 0$, $V = 0$, $\boldsymbol{\tau} = 0$ and current is small enough, only the left-hand side of (40) is significant. Then the left-hand expresses the propagation of the m-th mode linear Rossby waves. The first term on the right-hand side of (40) describes the nonlinear advection. The second term shows the diffusion of vorticity by horizontal viscosity and the third the external input of vorticity. The last term indicates the forcing due to stretching of water column at the surface ($z = 0$) and bottom ($z = -H$).

The derivation of (40) to (43) is as follows. The terms $\nu\nabla^4\tilde{\psi}_m$ and $\tilde{V}_m$ on the right-hand side are straightforward. Somewhat cumbersome is the manipulation of the stretching term in q and the nonlinear term.

We first show

$$\frac{1}{H}\int_{-H}^{0}\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]\hat{\psi}_m\, dz$$
$$= \frac{\partial\nabla^2\tilde{\psi}_m}{\partial t} + \left[\frac{f_r^2}{H\,N^2}\frac{\partial^2\psi}{\partial t\partial z}\hat{\psi}_m\right]_{-H}^{0}$$
$$-\frac{1}{H}\int_{-H}^{0}\frac{f_r^2}{H\,N^2}\frac{\partial^2\psi}{\partial t\partial z}\frac{d\hat{\psi}_m}{dz}\, dz$$
$$= \frac{\partial\nabla^2\tilde{\psi}_m}{\partial t} + \left[\frac{f_r^2}{H\,N^2}\frac{\partial^2\psi}{\partial t\partial z}\hat{\psi}_m\right]_{-H}^{0} - F_m\frac{\partial\tilde{\psi}_m}{\partial t}$$
$$= \frac{\partial}{\partial t}(\nabla^2 - F_m)\tilde{\psi}_m + \left[\frac{f_r^2}{H\,N^2}\frac{\partial^2\psi}{\partial t\partial z}\hat{\psi}_m\right]_{-H}^{0}. \tag{44}$$

That is, integration by parts yields a boundary part in addition to the mode term, just as $\tilde{q}_m$ has not only $\check{q}_m$ but also a boundary part.

Next we have

$$\check{N}_m \equiv -\frac{1}{H}\int_{-H}^{0} J\left(\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left[\frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right]\right)\hat{\psi}_m\, dz$$
$$= -\frac{1}{H}\int_{-H}^{0} J\left(\psi, \nabla^2\psi\right)\hat{\psi}_m\, dz$$
$$-\frac{1}{H}\int_{-H}^{0} J\left(\psi, \frac{\partial}{\partial z}\left[\frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right]\right)\psi_m\, dz$$
$$= -\frac{1}{H}\int_{-H}^{0} J\left(\psi, \nabla^2\psi\right)\hat{\psi}_m\, dz - \left[\frac{1}{H}J\left(\psi, \frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\hat{\psi}_m\right]_{-H}^{0}$$
$$+\frac{1}{H}\int_{-H}^{0} J\left(\psi, \frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\frac{d\hat{\psi}_m}{dz}\, dz$$
$$= \check{N}_m - \left[\frac{1}{H}J\left(\psi, \frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\hat{\psi}_m\right]_{-H}^{0}, \tag{45}$$

where we have used

$$\frac{1}{H}\int_{-H}^{0} J\left(\frac{\partial\psi}{\partial z}, \frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\hat{\psi}_m\, dz = 0$$

and defined $\check{N}_m$ by

$$\check{N}_m \equiv -\frac{1}{H}\int_{-H}^{0} J(\psi, \nabla^2\psi)\hat{\psi}_m\, dz$$
$$+\frac{1}{H}\int_{-H}^{0} J\left(\psi, \frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right)\frac{d\hat{\psi}_m}{dz}\, dz.$$

The above definition of $\check{N}_m$ is further rewritten as

$$\begin{aligned}
\check{N}_m &= -\sum_{i,j} J(\tilde{\psi}_i, \nabla^2\tilde{\psi}_j) I_{ijm} \\
&\quad -\sum_{i\neq j} J(\tilde{\psi}_i, \tilde{\psi}_j)(F_i - F_j) I_{ijm} \\
&= -\frac{1}{2}\sum_{i,j}\left[J(\tilde{\psi}_i, \nabla^2\tilde{\psi}_j)\right. \\
&\quad \left.+J(\tilde{\psi}_j, \nabla^2\tilde{\psi}_i) + J(\tilde{\psi}_i, \tilde{\psi}_j)(F_i - F_j)\right] I_{ijm} \\
&= -\frac{1}{2}\sum_{i,j}\left[J(\tilde{\psi}_i, (\nabla^2 - F_j)\tilde{\psi}_j)\right. \\
&\quad \left.+J(\tilde{\psi}_j, (\nabla^2 - F_i)\tilde{\psi}_i)\right] I_{ijm} \\
&= -\sum_{i,j} J(\tilde{\psi}_i, (\nabla^2 - F_j)\tilde{\psi}_j) I_{ijm} \\
&= -\sum_{i,j} J(\tilde{\psi}_i, \tilde{q}_j) I_{ijm}
\end{aligned}$$

in agreement with (41), where we have used (38).

The boundary parts of (44) and (45) yields $\check{W}_m$

$$\begin{aligned}
&\frac{1}{H}\left[\left\{\left(\frac{\partial}{\partial t} + J(\psi, \cdot)\right)\frac{f_r^2}{N^2}\frac{\partial\psi}{\partial z}\right\}\hat{\psi}_m\right]_{-H}^{0} \\
&= -\frac{f_r}{H}\left[w\hat{\psi}_k\right]_{-H}^{0} = -\check{W}_m,
\end{aligned}$$

by use of (11). The derivation of (40) is over.

To close the equation in terms of $\{\tilde{\psi}_m\}$, however, we have to express w at the surface and bottom in terms of $\{\tilde{\psi}_m\}$. For that purpose we use the boundary condition there. We find

$$\begin{cases}
w(0) = w_s = \nabla\times\dfrac{\boldsymbol{\tau}}{f_r\rho_r} \\
w(-H) \equiv w_b = \left(J(\psi, h) + \dfrac{H\lambda}{f_r}\nabla^2\psi\right)_{z=-H} \\
\quad = \displaystyle\sum_j\left[J(\tilde{\psi}_j, h) + \frac{H\lambda}{f_r}\nabla^2\tilde{\psi}_j\right]\hat{\psi}_j(-H).
\end{cases}$$

Then $\check{W}_m$ becomes

$$\begin{aligned}
\check{W}_m &= \frac{f_r}{H}\left[\hat{\psi}_m(0)\nabla\times\frac{\boldsymbol{\tau}}{\rho_r f_r}\right] \\
&\quad -\sum_j\left[J\left(\tilde{\psi}_j, \frac{f_r}{H}h\right) + \lambda\nabla^2\tilde{\psi}_j\right]\left(\hat{\psi}_j\hat{\psi}_m\right)_{z=-H}. \quad (46)
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
&\frac{f_r}{H}\left(\hat{\psi}_m(0)\nabla\times\frac{\boldsymbol{\tau}}{\rho_r f_r} - \hat{\psi}_m(-H)J(\psi, h)\right) \\
&= \frac{1}{H}\int_{-H}^{0} dz\hat{\psi}_m(z) \\
&\qquad \nabla\times\left[\frac{\boldsymbol{\tau}}{\rho_r}\delta(z) - \frac{p\nabla h}{\rho_r}\delta(z+H)\right],
\end{aligned}$$

where we have used

$$f_r J(\psi, h) = \nabla\frac{p}{\rho_r}\times\nabla h = \nabla\times\left(\frac{p}{\rho_r}\nabla h\right).$$

This formula implies that the terms of $\check{W}_m$, except that due to the bottom Ekman suction, represent none other than the torque of the horizontal force concentrated at the surface and bottom. The term of $-p\nabla h$ is often referred to as the *topographic stress*. Accordingly its curl means the *bottom pressure torque*. In some literature the Neptune effect was related with the *topographic stress* (Holloway 1992), though the Neptune effect is not necessarily due to the topographic stress. Also $-p\nabla h$ is associated with the JEBAR effect (see later discussion).

In (40) one may include the advective term due to the m-th mode itself to the left-hand side as

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - I_{mmm}\lhd\tilde{\psi}_m\cdot\nabla\right)\tilde{q}_m + \beta\frac{\partial\tilde{\psi}_m}{\partial x} \\
&= -\sum_{(i,j)\neq(m,m)} I_{ijm}J(\tilde{\psi}_i, \tilde{q}_j) + \nu_h\nabla^4\tilde{\psi}_m + \check{V}_m + \check{W}_m.
\end{aligned}$$

In particular for the barotropic mode we obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \lhd\tilde{\psi}_0\cdot\nabla\right)\nabla^2\tilde{\psi}_0 + \beta\frac{\partial\tilde{\psi}_0}{\partial x} \\
&= -J\left(\tilde{\psi}_0, \frac{f_r}{H}h\right) - \lambda\nabla^2\tilde{\psi}_0 + \nu\nabla^4\tilde{\psi}_0 \\
&\quad -\sum_{j>0}\left[J\left(\tilde{\psi}_j, \tilde{q}_j\right)\right] + \check{V}_0 + \frac{f_r}{H}\left[\nabla\times\frac{\boldsymbol{\tau}}{\rho_r f_r}\right] \\
&\quad -\sum_{j>0}\left[J\left(\tilde{\psi}_j, \frac{f_r}{H}h\right) + \lambda\nabla^2\tilde{\psi}_j\right]\left(\hat{\psi}_j\right)_{z=-H}.
\end{aligned}$$

The m-th vertical mode interacts with the others through $\check{N}_m$ (three-mode interaction; see (41)) and $\check{W}_m$ (two-mode coupling; see(46)); note that two-mode coupling is not a nonlinear interaction but a linear coupling of two vertical modes.

Within $\check{W}$ there are two sources of interaction. One arises from the bottom Ekman suction, which is essentially linear. We consider it rather insubstantial; that conversion from one mode to another arises from the inviscid definition of vertical modes, which do not take into account the bottom Ekamn suction properly.

The other kind reflects the coupling of two vertical modes via topography ∇h. In particular, for the barotropic mode $\tilde{\psi}_0$, we may write this term in different ways as follows.

$$\begin{aligned}
&-\frac{f_r}{H}\sum_{j>0}\left[J(\tilde{\psi}_j, h)\right]\hat{\psi}_j(-H) = -\frac{f_r}{H}J(\psi_{cli}, h(\boldsymbol{x}))_{z=-H} \\
&= -\frac{f_r}{H}(\boldsymbol{u}_{cli}\cdot\nabla h(\boldsymbol{x}))_{z=-H} \qquad (47) \\
&= -\frac{f_r}{H}\nabla(\psi_{cli}\times\nabla\boldsymbol{h})_{z=-H} = -\frac{f_r}{H}\nabla\times(\psi_{cli}\nabla\boldsymbol{h}))_{z=-H} \\
&= -\frac{1}{\rho_r H}\nabla\times(p_{cli}\nabla\boldsymbol{h})_{z=-H}. \qquad (48)
\end{aligned}$$

Formula (47) means that bottom flow $\boldsymbol{u}_{cli}$ not parallel to the contours of h would yield vertical velocity, which produces the barotropic component of relative vorticity

$\nabla^2\tilde{\psi}_0$. This mechanism of conversion from the baroclinic modes to the bartoropic mode is often called as the *joint effect of baroclinicity and relief* (JEBAR). On the other hand (48) is referred to as the torque of bottom pressure, since its right-hand side expresses the torque of the force by bottom $-p_{cli}\nabla h$, as was mentioned before.

In any case, the present expression (46) describes various kinds of conversion from one vertical mode to another; JEBAR is only one of such modal inter-modal conversion via bottom topography.

3.3 energetics of each mode and inter-modal energy transfer

1) Before energetics, a few kinds of notation and useful properties are described.

Let a, b, and c be scalar fields and $\boldsymbol{u}$ and $\boldsymbol{v}$ be vector fields of the basin, i.e., they are functions of (x, y). We define some integration over the basin as

$$\begin{cases} (a| \cdot |b) \equiv \displaystyle\int a\, b\, d\boldsymbol{x} \\ \|a\|^2 \equiv (a| \cdot |a) \\ (\boldsymbol{u}| \cdot |\boldsymbol{v}) \equiv \displaystyle\int \boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ (\boldsymbol{u}| \times |\boldsymbol{v}) \equiv \displaystyle\int \boldsymbol{u} \times \boldsymbol{v}\, d\boldsymbol{x} \end{cases}.$$

We then have

$$\begin{aligned} \int J(a,b)\, d\boldsymbol{x} &= (\nabla a\,| \times |\nabla b) = \int \nabla a \times \nabla b\, d\boldsymbol{x} \\ &= \int \nabla a \cdot \lhd b\, d\boldsymbol{x} \quad \left(= -\int \nabla b \cdot \lhd a\, d\boldsymbol{x}\right) \\ &= \begin{cases} \displaystyle\int (+\nabla \cdot [a \lhd b])\, d\boldsymbol{x} = + \oint a\boldsymbol{n} \cdot \lhd b\, ds \\ \displaystyle\int (-\nabla \cdot [b \lhd a])\, d\boldsymbol{x} = - \oint b\boldsymbol{n} \cdot \lhd a\, ds \end{cases} \end{aligned} \tag{49}$$

by virtue of Gauss's theorem and (1). Thus we find

$$\int J(a,b)\, d\boldsymbol{x} = (\nabla a| \times |\nabla b) = 0, \tag{50}$$

if a and b sastisfy one of the conditions below:

$$\begin{cases} (i) \text{ both } a \text{ and } b \text{ are periodic in x, y} \\ (ii)\ \boldsymbol{n} \cdot \lhd a = 0 \text{ along the side boundary} \\ (iii)\ \boldsymbol{n} \cdot \lhd b = 0 \text{ along the side boundary} \end{cases}. \tag{51}$$

Under the same boundary condition, (50) is generalized to

$$\begin{aligned} \int \frac{dh_a}{da}\frac{dh_b}{db} J(a,b)\, d\boldsymbol{x} &= \int J(h_a(a), h_b(b))\, d\boldsymbol{x} \\ &= (\nabla h_a(a)| \times |\nabla h_b(b)) = 0. \end{aligned} \tag{52}$$

for arbitrary smooth functions of h_a and h_b. In particular, for $m \geq 0$ and $n \geq 0$, we obtain

$$\begin{aligned} \int a^m b^n J(a,b)\, d\boldsymbol{x} &= (a^m \nabla a| \times |b^n \nabla b^n) \\ &= \left(\frac{\nabla a^{m+1}}{m+1}| \times |\frac{\nabla b^{n+1}}{n+1}\right) = 0. \end{aligned} \tag{53}$$

Obviously, the geostrophic stream function ψ (and its veritcal mode $\tilde{\psi}_m$) is periodic for the case of a periodic ocean basin (POB), or $\boldsymbol{n} \cdot (\lhd\psi) = -\boldsymbol{n} \cdot \boldsymbol{u} = 0$ for the case of a closed ocean basin (COB).

Then for scalar fields a, b, and c, we have

$$\begin{aligned} (c| \cdot |J(a,b)) &= \begin{cases} (c\nabla a| \times |\nabla b) \\ (\nabla a| \times |c\nabla b) \end{cases} \\ &= \begin{cases} (\nabla[ca]| \times |\nabla b) - (a\nabla c| \times |\nabla b) \\ (\nabla a| \times |\nabla[cb]) - (\nabla a| \times |b\nabla c) \end{cases} \\ &= \begin{cases} -(a\nabla c| \times |\nabla b) \\ -(\nabla a| \times |b\nabla c) \end{cases} \end{aligned}$$

respectively, if

$$\begin{cases} \text{(i) or (iii) of (51)} \\ \text{(i) or (ii) of (51)} \end{cases} \tag{54}$$

is satisfied. Further we may rewrite them as

$$\begin{aligned} &(c| \cdot |J(a,b)) \\ &= \begin{cases} (\nabla b| \times |a\nabla c) = (a\,| \cdot |J(b,c)) \\ (b\nabla c| \times |\nabla a) = (b\,| \cdot |J(c,a)) \end{cases}, \end{aligned} \tag{55}$$

which allows the cyclic permutation of (a, b, c) when (54) is satisfied.

2) Now energetics of each mode is formulated as follows. From the definition of (15) and (16) we have

$$T.E. + P.K. = \rho_r H \sum_m \check{E}_m, \tag{56}$$

where

$$\check{E}_m \equiv \frac{1}{2}\int [(\nabla\tilde{\psi}_m)^2 + F_k\tilde{\psi}_m^2]\, d\boldsymbol{x}$$

denotes the energy of the m-th vertical mode (divided by $\rho_r H$).

For the periodic basin (POB), the energy of a vertical mode is divided into components characterized by horizontal wavenumbers $\boldsymbol{k}$. This decomposition allows a further analysis of energetics such as inverse energy cascade (Batchelor 1969) or Rhines effect (Rhines 1975). This makes a main subject of Q-G turbulence. That problem is not pursued, however, in this paper, which discusses only inter-modal energy transfer.

Multiplying (40) by $-\tilde{\psi}_m$ and integrating over the basin, we obtain the evolution equation of $\check{E}_m$:

$$\begin{aligned} \frac{\partial}{\partial t}\check{E}_m &= \check{S}_{dsh:m} + \check{S}_{in:m} \\ &+ \check{S}_{nl:m} + \check{S}_{sf:m} + \check{S}_{bt:m} + \check{S}_{dsb:m}, \end{aligned} \tag{57}$$

where the right-hand side expresses energy input to the m-th mode by various mechanisms, which are described below.

(1) : $\check{S}_{dsh:m} \equiv -(\tilde{\psi}_m| \cdot |\nu\nabla^4\tilde{\psi}_m|)$

$$= -\nu\|\nabla^2\tilde{\psi}_m\|^2 - \nu\int \nabla\cdot\left(\tilde{\psi}_m\nabla\tilde{\zeta}_m\right)\,d\boldsymbol{x},$$

(integral term vanishes for a POB)

$$(2) : \check{S}_{in:m} \equiv -(\tilde{\psi}_m|\cdot|\tilde{V}_m) = -\left(\tilde{\psi}_m|\cdot|\nabla\times\tilde{\boldsymbol{F}}_m\right)$$

$$= \left(\tilde{\psi}_m|\cdot|\lhd\cdot\tilde{\boldsymbol{F}}_m\right) = -(\lhd\tilde{\psi}_m|\cdot|\tilde{\boldsymbol{F}}_m)$$

$$= (\tilde{\boldsymbol{u}}_m|\cdot|\tilde{\boldsymbol{F}}_m),$$

$$(3) : \check{S}_{nl:m} \equiv \sum_{i,j}\check{S}_{nl:i,j\to m},$$

$$\check{S}_{nl:i,j\to m} \equiv -\frac{I_{ijm}}{2}$$

$$\times\left(\tilde{\psi}_m|\cdot|J(\tilde{\psi}_i,\tilde{q}_j)) + J(\tilde{\psi}_j,\tilde{q}_i))\right),$$

$$(4) : \check{S}_{sf:m} \equiv -\frac{f_r}{H}\hat{\psi}_m(0)\left(\tilde{\psi}_m|\cdot|\nabla\times\frac{\boldsymbol{\tau}}{f_r\rho_r}\right)$$

$$= -\frac{\hat{\psi}_m(0)}{\rho_r H}(\lhd\tilde{\psi}_m|\cdot|\boldsymbol{\tau}) = \frac{\hat{\psi}_m(0)}{\rho_r H}(\tilde{\boldsymbol{u}}_m|\cdot|\boldsymbol{\tau}),$$

$$(5) : \check{S}_{dsb:m} \equiv -\lambda\sum_j\left(\hat{\psi}_m\hat{\psi}_j\right)_{z=-H}\left(\nabla\tilde{\psi}_m|\cdot|\nabla\tilde{\psi}_j\right)$$

$$= -\lambda\sum_j\left(\hat{\psi}_m\hat{\psi}_j\right)_{z=-H}(\tilde{\boldsymbol{u}}_m|\cdot|\tilde{\boldsymbol{u}}_j),$$

$$(6) : \check{S}_{bt:m} \equiv \sum_j\check{S}_{bt:j\to m}$$

$$= \frac{f_r}{H}\sum_j\left(\hat{\psi}_m\hat{\psi}_j\right)_{z=-H}\left(\tilde{\psi}_m|\cdot|J(\tilde{\psi}_j,h)\right).$$

The first term $S_{dsh:m}$ expresses the energy dissipation of the m-th mode by horizontal viscosity. It is always negative, except for the side boundary term. The second is the input of energy by external forcing V, which may be either positive or negative.

The third term $S_{nl:m}$ is the sum of $S_{nl:i,j\to m}$ over (i,j), which expresses the nonlinear energy transfer from the pair of (i,j)-th vertical modes to the m-th mode. Generally speaking energy transfer from higher modes to lower modes means the conversion of potential to kinetic energy, since higher modes have higher partition of potential energy in general. The baroclinic instability is a typical mechanism that transfers the potential energy of baroclinic modes to the kinetic energy of the barotropic mode.

The fourth term means the energy transfer from the surface wind to the m-th mode. It may be considered as one of $S_{in:m}$ as mentioned before. The fifth term means the energy transfer to the m-th mode via bottom friction (Ekman suction). Note that this term is not necessarily negative. Of course, the summation over all the modes are negative, since

$$\sum_m\check{S}_{dsb:m} = -\lambda\sum_{j,m}\left(\hat{\psi}_m\hat{\psi}_j\right)_{z=-H}(\tilde{\boldsymbol{u}}_m|\cdot|\tilde{\boldsymbol{u}}_j)$$

$$= -\lambda\left\|\sum_m\tilde{\boldsymbol{u}}_m\hat{\psi}_m(-H)\right\|^2$$

$$= -\lambda\|\boldsymbol{u}(x,y,-H,t)\|^2 \le 0.$$

The sixth term means the energy input to the m-th mode by the coupling with the j-th mode via bottom topography. If $\check{S}_{bt:1\to 0} > 0$, the first baroclinic mode gives energy to the barotropci mode. The expression of $\sum_{j>0} S_{bt:j\to 0}$ reflects the JEBAR effect in energetics.

It is to be born in mind that even within the same vertical mode, different horizontal scales (different wavenumbers in a POB) interact with one another. In other words, intra-modal energy transfer occurs among different horizontal scales of motion, as mentioned before, within the same vertical mode.

3.4 detailed balance of inter-modal energy transfer

As has been stressed so far, inter-modal energy transfer occurs mainly through nonlinear advective interaction $S_{nl:i,j\to m}$.

In the case of the COB, we see

$$\frac{1}{2}\boldsymbol{n}\cdot\lhd\psi^2 = \psi\boldsymbol{n}\cdot\lhd\psi = -\boldsymbol{u}\cdot\boldsymbol{n}\psi = 0 \qquad (58)$$

on ∂O. For the case of the POB, ψ^2 and q are periodic. In either case therefore we have from (55)

$$\iint\psi J(\psi,q)\,dz\,d\boldsymbol{x} = \int(q|\cdot|J(\psi,\psi))\,dz = 0,$$

which means the advective term neither produces nor dissipates energy. Therefore the total nonlinear energy transfer among vertical modes should disappear. As a consequence we have

$$\sum_{i,j,m} S_{nl:i,j\to m} = 0. \qquad (59)$$

We have a still stronger property of detailed balance. That is, for any triplet of vertical modes (i,j,m), nonlinear energy transfer is closed:

$$0 = \check{S}_{nl:i,j\to m} + \check{S}_{nl:j,m\to i} + \check{S}_{nl:m,i\to j}. \qquad (60)$$

It is derived as follows. By virtue of (55) we find

$$S_{nl:i,j\to m} = -\frac{I_{ijm}}{2}\left[\left(\tilde{\psi}_m|\cdot|J(\tilde{\psi}_i,\tilde{q}_j) + J(\tilde{\psi}_j,\tilde{q}_i)\right)\right]$$

$$= \frac{I_{ijm}}{2}\left[\left(\tilde{q}_i|\cdot|J(\tilde{\psi}_j,\tilde{\psi}_m)\right) - \left(\tilde{q}_j|\cdot|J(\tilde{\psi}_m,\tilde{\psi}_i)\right)\right].$$

Since I_{ijm} is symmetric with respect to indices, cyclic summation of indices in the above equation leads to (60), i.e., detailed balance of inter-modal energy transfer within any triplet of vertical modes.

Also bottom topography causes inter-modal energy transfer through $S_{bt:j\to m}$. The total sum vanishes

$$\sum_{j,m}\check{S}_{bt:j\to m} = 0, \qquad (61)$$

because

$$\int \psi \boldsymbol{u} \cdot \nabla h \, d\boldsymbol{x} = - \int h \nabla \cdot [\psi \boldsymbol{u}] \, d\boldsymbol{x}$$
$$= \int h \nabla \cdot \lhd \frac{\psi^2}{2} \, d\boldsymbol{x} = 0$$

at $z = -H$, where we have used $\boldsymbol{u} = - \lhd \psi$ and (1).

A stronger property (reciprocal relation) holds for any pair (j, m) of vertical modes,

$$0 = S_{bt:j \to m} + S_{bt:m \to j}. \tag{62}$$

This is because

$$S_{bt:j \to m} = \frac{f_r}{H} \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H} \left(\tilde{\psi}_m \mid \cdot \mid J(\tilde{\psi}_j, h) \right)$$
$$= \frac{f_r}{H} \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H} (h| \cdot |J(\tilde{\psi}_m, \tilde{\psi}_j))$$
$$= -\frac{f_r}{H} \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H} (h| \cdot |J(\tilde{\psi}_j, \tilde{\psi}_m))$$
$$= -S_{bt:m \to j}.$$

3.5 inter-modal enstrophy transfer

Energetic discussion so far is extended to enstrophy of each mode, which is essential in understanding the evolution of two-dimensional turbulence and Q-G turbulence. Enstrophy of each mode may be defined in a few ways, however:

$$(1): \ \tilde{G}_m \equiv \int \frac{\tilde{\zeta}_m^2}{2} \, d\boldsymbol{x} = \int \frac{(\nabla^2 \tilde{\psi}_m)^2}{2} \, d\boldsymbol{x},$$
$$(2): \ \acute{G}_m \equiv \int \frac{\zeta_m \check{q}_m}{2} \, d\boldsymbol{x}$$
$$= \tilde{G}_m + \frac{F_m}{2} \int |\nabla \tilde{\psi}_m|^2 \, d\boldsymbol{x} \geq \tilde{G}_m,$$
$$(3): \ \check{G}_m \equiv \int \frac{(\check{q}_m)^2}{2} \, d\boldsymbol{x} = \acute{G}_m + F_m \check{E}_m \geq \acute{G}_m,$$
$$(4): \ \check{G}_m \equiv \int \frac{(\check{q}_m + \beta y)^2}{2} \, d\boldsymbol{x}$$
$$= \check{G}_m + \int \beta y \check{q}_m \, d\boldsymbol{x} + \int \frac{\beta^2 y^2}{2} \, d\boldsymbol{x}.$$

The simplest is the first definition, which is usually called *relative* enstrophy and used in the study of two-dimensional turbulence (Batchelor 1969). The others have been used in Q-G turbulence. The second definition was adopted in the study of two-dimensional turbulence governed by the Charney-Hasegaw-Mima equation (Iwayama et al., 2002, Masuda and Okuno 2002). The fourth represents the squared potential vorticity without the planetary β term. It is to be noted that

$$\frac{\partial \check{G}}{\partial t} = \frac{\partial \check{G}}{\partial t} + \frac{\partial}{\partial t} \int \beta y \check{q}_m \, d\boldsymbol{x}, \tag{63}$$

of which the second term may take either sign.

We adopt the third definition $\check{G}_m$ here. It corresponds to

$$G \equiv \int \int \frac{q^2}{2} \, d\boldsymbol{x} \, dz$$
$$= \frac{1}{2} \int \int \left(\nabla^2 \psi + \frac{\partial}{\partial z} \left[\frac{f_r^2}{N^2} \frac{\partial \psi}{\partial z} \right] \right)^2 d\boldsymbol{x} \, dz.$$

Multiplying (40) by $\check{q}_m = \tilde{\zeta}_m - F_m \tilde{\psi}_m$ and integrating over the basin, we obtain the evolution of $\check{G}_m$:

$$\frac{\partial}{\partial t} \check{G}_m = \check{T}_{dsh:m} + \check{T}_{in:m}$$
$$+ \check{T}_{nl:m} + \check{T}_{sf:m} + \check{T}_{dsb:m} + \check{T}_{bt:m}, \tag{64}$$

where each term on the right-hand side contributes to the enstrophy of the m-th mode. The mechanism representing each term is described below.

$$(1): \ \check{T}_{dsh:m} \equiv (\check{q}_k | \cdot | \nu \nabla^4 \tilde{\psi}_m |)$$
$$= -\nu \| \nabla \tilde{\zeta}_m \|^2 + \nu \int \nabla \cdot (\tilde{\zeta}_m \nabla \tilde{\zeta}_m),$$
$$+ F_m \check{S}_{dsh:m}$$

(integral term vanishes for a POB)

$$(2): \ \check{T}_{in:m} \equiv (\check{q}_m | \cdot | \tilde{V}_m) = (\tilde{\zeta}_m | \cdot | \tilde{V}_m) + F_m \check{S}_{in:m}$$
$$(3): \ \check{T}_{nl:m} \equiv \sum_{i,j} \check{T}_{nl:i,j \to m}$$
$$\check{T}_{nl:i,j \to m} \equiv -\frac{I_{ijm}}{2} \left(\check{q}_m \mid \cdot \mid J(\tilde{\psi}_i, \check{q}_j) + J(\tilde{\psi}_j, \check{q}_i) \right)$$
$$= -\frac{I_{ijm}}{2} \times \left(\tilde{\zeta}_m \mid \cdot \mid J(\tilde{\psi}_i, \check{q}_j) + J(\tilde{\psi}_j, \check{q}_i) \right)$$
$$+ F_m \check{S}_{nl:i,j \to m}$$
$$(4): \ \check{T}_{sf:m} \equiv \frac{f_r}{H} \hat{\psi}_m(0) \left(\check{q}_m \mid \cdot \mid \nabla \times \frac{\boldsymbol{\tau}}{f_r \rho_r} \right)$$
$$= \frac{f_r}{H} \hat{\psi}_m(0) \left(\tilde{\zeta}_m \mid \cdot \mid \nabla \times \frac{\boldsymbol{\tau}}{f_r \rho_r} \right)$$
$$+ F_m \check{S}_{sf:m}$$
$$(5): \ \check{T}_{dsb:m} \equiv -\lambda \sum_j \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H}$$
$$\times \left[\left(\tilde{\zeta}_m \mid \cdot \mid \tilde{\zeta}_j \right) + F_m \left(\tilde{\boldsymbol{u}}_m \mid \cdot \mid \tilde{\boldsymbol{u}}_j \right) \right]$$
$$= -\lambda \sum_j \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H} \left(\tilde{\zeta}_m \mid \cdot \mid \tilde{\zeta}_j \right)$$
$$+ F_m \check{T}_{dsb:m}$$
$$(6): \ \check{T}_{bt:m} \equiv \sum_j \check{T}_{bt:j \to m}$$
$$\equiv \frac{f_r}{H} \sum_j \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H} \left(\check{q}_m \mid \cdot \mid J(\tilde{\psi}_j, h) \right)$$
$$= \frac{f_r}{H} \sum_j \left(\hat{\psi}_m \hat{\psi}_j \right)_{z=-H} \left(\tilde{\zeta}_m \mid \cdot \mid J(\tilde{\psi}_j, h) \right)$$
$$+ F_m \check{S}_{bt:m}.$$

We see the β term formally does not contribute to the inter-modal enstrophy transfer at all, though it plays a key role in controlling the ψ field.

As well as for the inter-modal energy transfer, detailed balance of inter-modal enstrophy transfer holds for $T_{nl;i,j\to m}$, only for this form of $\check{G}$. If we rewrite $\check{T}_{nl:i,j\to m}$ as

$$\check{T}_{nl:i,j\to m} = -\frac{I_{ijm}}{2} \times \left[\left(\check{q}_m \mid \cdot \mid J(\tilde{\psi}_i, \check{q}_j) \right) - \left(\check{q}_i \mid \cdot \mid J(\tilde{\psi}_j, \check{q}_m) \right) \right]$$

and take cyclic permutation of indices (i,j,m), summation yields the required detailed balance of potential entstrophy

$$\check{T}_{nl:i,j\to m} + \check{T}_{nl:j,m\to i} + \check{T}_{nl:m,i\to j} = 0. \qquad (65)$$

Consequently we have

$$0 = \sum_m \check{T}_{nl:m} = \sum_{m,i,j} \check{T}_{nl:i,j\to m}. \qquad (66)$$

This result is based on that our choice of potential enstrophy assures inviscid conservation of potential enstrophy.

Detailed balance, however, is not assured for three-mode interaction for enstrophy of the form of $\tilde{G}_m$ or for two-mode coupling via topography $T_{bt;j\to m}$. In general, we have

$$\sum_m \check{T}_{bt:m} \neq 0,$$

so that detailed balance cannot be expected from the beginning.

This result may be viewed from a more generalized form as follows. For $m \geq 0$ and $n \geq 0$, (53) yields

$$\int\int \psi^m q^n J(\psi, q)\, dz\, d\boldsymbol{x} = \int\int J\left(\frac{\psi^{m+1}}{m+1}, \frac{q^{n+1}}{n+1} \right) dz\, d\boldsymbol{x} = 0. \qquad (67)$$

This formula (67) implies that nonlinear advection conserves potential vorticity, energy, and potential enstrophy for $(m,n) = (0,0)$, $(m,n) = (1,0)$, and $(m,n) = (0,1)$, respectively. Of course, total $h_q(q)$ is conserved by nonlinear advective interaction for any smooth function of h_q. However, even when total $h_q(q)$ is conserved over all the vertical modes, detailed balance is not necessarily guaranteed.

It is also worthwhile to remark that $\check{T}_{:m}$ is the sum of a vorticity-pertinent term, which is related with the evolution of relative enstrophy $\frac{\partial \check{G}_m}{\partial t}$, and a term of the form of $F_m \check{S}_{:m}$ corresponding to energy transfer multiplied by F_m. All the above formulas are obtained also from

$$\tilde{\zeta}_m \times \text{“r.h.s. of (40)”} = \tilde{\zeta}_m \frac{\partial \check{q}_m}{\partial t} = (\check{q}_m + F_m \tilde{\psi}) \frac{\partial \check{q}_m}{\partial t}.$$

Then we have

$$\int (\tilde{\zeta}_m \times \text{“r.h.s. of (40)”})\, d\boldsymbol{x} = \frac{\partial}{\partial t}(\check{G}_m - F_m E_m)$$

4. Layer model

This section deals with a layer model or a discrete model, which fits right in numerical simulation. The layer model is shown to be an approximation to the level model in a sense. Hence the methodology and principal formulation remain essentially the same. There are a few noticeable differences from the level model though.

4.1 vertical modes in a layer model

1) The situation of the layer model here is almost the same as in the level model. The layer model differs from the level model only in the background stratification. Vertically the ocean is composed of K layers, each of which is filled with water of a uniform density $\overline{\rho}_k$ or a uniform buoyancy $\overline{B}_k$, so that $\overline{B}$ is a step function of z where the suffix k refers to the value of the k-th layer from the surface layer ($k = 1$). The k-th layer has mean thickness of H_k ($1 \leq k \leq K$). The total mean depth of the ocean H is

$$H \equiv \sum_{k=1}^{K} H_k.$$

The bottom relief h is small enough to assure $|h| < H_K$.

Then we define the buoyancy difference between the adjacent layers as

$$\begin{cases} B_{ku} \equiv \overline{B}_{k-1} - \overline{B}_k = g\dfrac{\overline{\rho}_k - \overline{\rho}_{k-1}}{\rho_r} > 0 \\ B_{kd} \equiv \overline{B}_k - \overline{B}_{k+1} = g\dfrac{\overline{\rho}_{k+1} - \overline{\rho}_k}{\rho_r} > 0 \\ \quad \text{so that} \quad B_{(k-1)d} = B_{ku}, \end{cases}$$

where the suffix "u" of B suggests that the quantitiy is related with the upper layer, whereas "l" the lower layer. For convenience we define

$$B_{1u} = \infty$$

for the rigid lid approximation to the sea surface. If a free surface is considered, we should put $B_{1u} = g$, which is finite but much larger than B_{1d}. We may put $B_{Kd} = \infty$ without loss of generality, because the bottom is rigid.

From the hydrostatic relation the Q-G pressure of the k-th layer becomes

$$\frac{p_k}{\rho_r} = f_r \psi_k = f_r \psi_{k-1} + (\overline{B}_{k-1} - \overline{B}_k)\eta_{ku}, \qquad (68)$$

where ψ_k denotes the Q-G stream function of the k-th layer and η_{ku} is the vertical displacement of the interface between the $(k-1)$-th and k-th layers, which is rewritten as

$$\eta_k \equiv \eta_{ku} \equiv \eta_{(k-1)d} \equiv -f_r \frac{\psi_{k-1} - \psi_k}{B_{k-1} - B_k} \approx -\frac{f_r}{\frac{dB}{dz}} \frac{\partial \psi}{\partial z} = -\frac{f_r}{N^2} \frac{\partial \psi}{\partial z}. \tag{69}$$

The latter shows that η_k is an approximation of (12), the vertical displacement of the isopycnal surface in the continuous model.

By a matrix $Y \in \boldsymbol{M}(K,K)$ defined by

$$Y \models Y_{k,l} = \frac{f_r}{B_{ku}}(\delta_{k,l} - \delta_{k,l+1}),$$

(69) is expressed in a vector form as

$$\eta = Y\psi, \tag{70}$$

where we use such a notation as

$$\boldsymbol{R}^K \ni \psi \models \psi_k, \quad \boldsymbol{R}^K \ni \eta \models \eta_k,$$

which means that the K-dimensional vectors ψ and η have ψ_k and η_k as their k-th component, respectively. Obviously η_k are the interfacial displacement for $2 \leq k \leq K$, while η_1 denotes the sea surface elevation, which vanishes under the rigid lid condition ($B_{1u} = \infty$).

2) Now let us derive the vorticity equation of the quasi-geostrophic flow. The potential vorticity of the k-th layer (without the β term) is defined by

$$\begin{aligned} q_k &\equiv \zeta_k - \frac{f_r}{H_k}(\eta_{ku} - \eta_{kd}) \equiv \nabla^2\psi_k - \frac{f_r}{H_k}(\eta_{ku} - \eta_{kd}) \\ &\left(\approx \nabla^2\psi_k - f_r\frac{\partial \eta}{\partial z} \approx \nabla^2\psi_k + \frac{\partial}{\partial z}\left[\frac{f_r^2}{N^2}\frac{\partial \psi}{\partial z}\right]\right) \\ &= \nabla^2\psi_k + F_{ku}(\psi_{ku} - \psi_k) + F_{kd}(\psi_{kd} - \psi_k) \\ &= \nabla^2\psi_k + \sum_l M_{k,l}\psi_l, \end{aligned} \tag{71}$$

where

$$\zeta_k \equiv \nabla \times \boldsymbol{u}_k = \nabla^2\psi_k, \qquad \boldsymbol{u}_k \equiv -\lhd\,\psi_k \tag{72}$$

$$F_{ku} \equiv \frac{f_r^2}{H_k B_{ku}}, \qquad F_{kd} \equiv \frac{f_r^2}{H_k B_{kd}}, \tag{73}$$

$$M_{k,l} \equiv (\delta_{k-1,l} - \delta_{k,l})F_{ku} + (\delta_{k+1,l} - \delta_{k,l})F_{kd}. \tag{74}$$

We have

$$H_k F_{ku} = H_{k-1} F_{(k-1)d}, \tag{75}$$

$$H_k M_{k,k+1} = \frac{f_r^2}{B_{kd}} = \frac{f_r^2}{B_{k+1,u}} = H_{k+1} M_{k+1,k}, \tag{76}$$

$$H_k M_{k,k-1} = \frac{f_r^2}{B_{ku}} = \frac{f_r^2}{B_{k-1,l}} = H_{k-1} M_{k,k-1}, \tag{77}$$

$$\forall k, \ \forall l: \ H_k M_{k,l} = H_l M_{l,k}. \tag{78}$$

The ocean is driven by the surface and bottom forcing through the stretching of water columns. To express these forcing it is convenient to define

$$W_k \equiv \frac{f_r}{H_k}\left[w_s \delta_{1,k} - w_b \delta_{k,K}\right] \tag{79}$$

which vanishes except for the surface layer ($k = 1$) or the bottom layer ($k = K$). Obviously W_k corresponds to the boundary part in the level model. In (79) we put

$$\begin{cases} w_s = w_{sE} = \nabla \times \dfrac{\boldsymbol{\tau}}{f_r \rho_r} \\ w_b = J(\psi_K, h) + w_{bE} = J(\psi_K, h) + \dfrac{H\lambda}{f_r}\nabla^2\psi_K \end{cases}$$

just as in the level model. Note that w_b is evaluated in terms of ψ_K.

As is well known, the potential vorticity of the k-th layer is governed by

$$\left\{\frac{\partial}{\partial t} - \lhd\psi_k \cdot \nabla\right\}\left\{\nabla^2\psi_k + \sum_l M_{kl}\psi_l\right\} + \beta\frac{\partial \psi_k}{\partial x} = \nu\nabla^4\psi_k + V_k + W_k, \tag{80}$$

where V_k is the rate of the external input of vorticity to the k-th layer.

If we define M as a matrix that has components $M_{k,l}$: $\boldsymbol{M}(K,K) \ni M \models M_{k,l}$, (80) becomes a vector form as

$$\left\{\frac{\partial}{\partial t} - \lhd\psi \cdot \nabla\right\}\left\{\nabla^2\psi + M\psi\right\} + \beta\frac{\partial \psi}{\partial x} = \nu\nabla^4\psi + V + W, \tag{81}$$

where $\boldsymbol{R}^K \ni V \models V_k$ and $\boldsymbol{R}^K \ni W \models W_k$. This equation is almost the same as (39), if M is replaced by $\frac{\partial}{\partial z}\left(\frac{f_r^2}{N^2}\frac{\partial}{\partial z}\right)$; exception is the presence of W, which corresponds to the surface and bottom boundary condition of (39). Note that M for $k = 1$ and $k = K$ (the surface layer and bottom layer) expresses homogeneous boundary conditions of the form of $^\exists\alpha\frac{\partial \psi}{\partial z} + {^\exists}\gamma\psi = 0$, not the differential equation of the second-order, so that the surface and bottom boundary conditions are taken into account explicitly by W.

3) Then we define our vertical modes as the solution of the eigenvalue problem for the matrix M by

$$M\hat{\psi}^{(m)} = -F^{(m)}\hat{\psi}^{(m)}, \tag{82}$$

where $F^{(m)}$ denotes the m-th eigenvalue and $\boldsymbol{R}^K \ni \hat{\psi}^{(m)} \models \hat{\psi}_k^{(m)}$ the (K-dimensional) eigenvector associated with $F^{(m)}$. The symbol on the shoulder $\bullet^{(m)}$ indicates the mode number m ($1 \leq m \leq K$).

Matrix M has K complex eigenvalues and associated complex eigenvectors. The eigenvalues of M may not necessarily be real, however, since M is not a symmetric matrix. We therefore start our investigation with confirming that M has only real eigenvalues $F^{(m)}$ and that we may choose real eigenvectors for any $F^{(m)}$.

For this purpose we define another inner product and the corresponding norm by

$$\langle \psi \mid \varphi \rangle \equiv \sum_k \psi_k \frac{H_k}{H}\varphi_k^*, \tag{83}$$

$$\langle\!\langle \psi \rangle\!\rangle^2 \equiv \langle \psi | \psi \rangle, \tag{84}$$

respectively, for vectors $\{\psi, \varphi\} \subset \boldsymbol{C}^K$, where $\bullet^*$ denotes the complex conjugate of $\bullet$. We should note

$$\langle \psi \mid \varphi \rangle = \sum_k \psi_k \frac{H_k}{H} \varphi_k^* \approx \frac{1}{H} \int_{-H}^{0} \psi(z) \varphi^*(z)\, dz.$$

That is, (83) is an extension of the ordinary integral form of inner product in the level model. See Appendix B for a different method by which to assure M to be a self-adjoint operator.

This metric (inner product) enables us to transform the eigenvalue problem into that for a self-adjoint operator similar to the level model.

Summing up, we have the following properties:

$(0):$ M is self-adjoint by $\langle \cdot | \cdot \rangle$,

$(1): 0 \leq F^{(m)}$, and $\hat{\psi}^{(m)}$ can be real,

$$(2): \delta_{m,n} = \langle \hat{\psi}^{(m)} \mid \hat{\psi}^{(n)} \rangle = \sum_k \hat{\psi}_k^{(m)} \hat{\psi}_k^{(n)} \frac{H_k}{H}, \quad (85)$$

$$-F^{(m)} \delta_{m,n} = \langle \hat{\psi}^{(m)} | M \hat{\psi}^{(n)} \rangle, \quad (86)$$

$$\delta_{j,k} = \sum_m \hat{\psi}_j^{(m)} \hat{\psi}_k^{(m)} \frac{\sqrt{H_j H_k}}{H}, \quad (87)$$

$(3):$ (if $B_{1u} = \infty$; rigid lid surface)

$$0 = F^{(1)} \leq F^{(2)} \leq F^{(3)} \leq \cdots F^{(K)},$$

$$\hat{\psi}_k^{(1)} = 1,$$

where $F^{(m)}$ are ordered from the least value. For the layer model, we call the first mode $m = 1$ as the barotropic mode and those of $m \geq 2$ as the baroclinic ones.

The above properties are proved easily as follows. For any complex vectors ψ and φ, we have

$$\langle M\psi \mid \varphi \rangle = \sum_{l,k} M_{lk} \psi_k \frac{H_l}{H} \varphi_l^* = \sum_{l,k} \psi_k \frac{H_k}{H} M_{kl} \varphi_l^*$$
$$= \langle \psi \mid M\varphi \rangle \quad (88)$$

by virtue of (78). That is, M is self-adjoint with respect to the inner product (83), so that we may write

$$\langle \psi | M | \phi \rangle \equiv \langle \psi | M \phi \rangle = \langle M\psi | \phi \rangle.$$

From the self-adjointness of M the other properties are readily derived.

First, suppose that

$$M \hat{\psi}^{(m)} = -F^{(m)} \hat{\psi}^{(m)}, \quad \langle\!\langle \hat{\psi}^{(m)} \rangle\!\rangle = 1.$$

Then we have

$$\begin{cases} \langle \hat{\psi}^{(m)} | M \hat{\psi}^{(m)} \rangle = -F^{(m)*} \langle\!\langle \hat{\psi}^{(m)} \rangle\!\rangle^2 \\ \langle M \hat{\psi}^{(m)} | \hat{\psi}^{(m)} \rangle = -F^{(m)} \langle\!\langle \hat{\psi}^{(m)} \rangle\!\rangle^2 \end{cases}.$$

Subtracting one from the other and using (88), we get

$$0 = (F^{(m)} - F^{(m)*}) \langle\!\langle \hat{\psi}^{(m)} \rangle\!\rangle^2,$$

which shows that $F^{(m)} = F^{(m)*}$. Thus $F^{(m)}$ must be real. Since M and $F^{(m)}$ are real, we may choose real eigenvectors $\hat{\psi}^{(m)}$.

Next, for any $\{\psi, \phi\} \subset \boldsymbol{R}^K$, we have

$$\frac{H}{f_r^2} \langle \psi | M \phi \rangle = \frac{H}{f_r^2} \langle M\psi | \phi \rangle = \frac{1}{f_r^2} \sum_{k,l} M_{k,l} \psi_l H_k \phi_k$$
$$= \sum_{k=1}^{K} \left[\frac{(\psi_{ku} - \psi_k)\phi_k}{B_{ku}} + \frac{(\psi_{kd} - \psi_k)\phi_k}{B_{kd}} \right]$$
$$= \sum_{k=1}^{K} \left[\frac{(\psi_{k-1} - \psi_k)(\phi_{k-1} + \phi_k)}{2B_{ku}} - \frac{(\psi_k - \psi_{k+1})(\phi_k + \phi_{k+1})}{2B_{kd}} \right]$$
$$- \sum_{k=1}^{K} \left[\frac{(\psi_{ku} - \psi_k)(\phi_{ku} - \phi_k)}{2B_{ku}} + \frac{(\psi_{kd} - \psi_k)(\phi_{kd} - \phi_k)}{2B_{kd}} \right]$$
$$= -\frac{\psi_1 \phi_1}{B_{1u}} - \sum_{k=2}^{K} \frac{(\psi_{k-1} - \psi_k)(\phi_{k-1} - \phi_k)}{B_{ku}},$$

since $B_{Kd} = \infty$, $\psi_{K+1} = \phi_{K+1} = 0 = \psi_0 = \phi_0$, and $B_{kd} = B_{(k+1)u}$. Arranging further we obtain

$$-\langle \psi | M \phi \rangle = \frac{1}{H} \sum_{k=1}^{K} (Y\psi)_k B_{ku} (Y\phi)_k$$
$$\sim \frac{f_r^2}{gH} \psi(0)\phi(0) + \frac{1}{H} \int_{-H}^{0} \left(\frac{f_r}{N^2} \frac{\partial \psi}{\partial z} \right) N^2 \left(\frac{f_r}{N^2} \frac{\partial \phi}{\partial z} \right) dz$$
$$= \frac{1}{H} \int_{-H}^{0} \left(\frac{f_r}{N^2} \frac{\partial \psi}{\partial z} \right) N^2 \left(\frac{f_r}{N^2} \frac{\partial \phi}{\partial z} \right) dz. \quad \text{(rigid lid)}$$

The formula is similar to that in the level model; note that the boundary contribution disappears in the layer model.

If $\psi = \phi$, we have

$$\langle \psi | M \psi \rangle = -\frac{f_r^2}{H} \sum_{k=1}^{K} \frac{(\psi_{ku} - \psi_k)^2}{B_{ku}} \leq 0.$$

In particular for $\psi = \hat{\psi}^{(m)}$, it follows that

$$\langle \hat{\psi}^{(m)} | M \hat{\psi}^{(m)} \rangle = -F^{(m)} \langle\!\langle \hat{\psi}^{(m)} \rangle\!\rangle^2 \leq 0$$

Thus we see $F^{(m)} \geq 0$, since $\langle\!\langle \hat{\psi}^{(m)} \rangle\!\rangle = 1$.

When $F^{(m)}$ differs from $F^{(n)}$, we have

$$\langle \hat{\psi}^{(m)} | M \hat{\psi}^{(n)} \rangle = \begin{cases} -F^{(m)} \langle \hat{\psi}^{(m)} | \hat{\psi}^{(n)} \rangle \\ -F^{(n)} \langle \hat{\psi}^{(m)} | \hat{\psi}^{(n)} \rangle \end{cases}.$$

Subtraction yields

$$0 = (F^{(m)} - F^{(n)}) \langle \hat{\psi}^{(m)} \mid \hat{\psi}^{(n)} \rangle$$
$$\Rightarrow \langle \hat{\psi}^{(m)} \mid \hat{\psi}^{(n)} \rangle = 0.$$

That is, eigenvectors associated with different eigenvalues are perpendicular to one another with respect to $\langle \cdot | \cdot \rangle$. Even when there occurs degeneration of eigenvalues, we may choose orthogonal eigenvectors for the symmetric operator M; for example use the method of maximizing (or minimizing) sequences (Courant and Hilbert 1953, Masuda 1993). In fact, the eigenvalue problem is formulated by seeking a sequence of functions ψ that minimize the available potential energy $\int \frac{f_r^2}{N^2}\left|\frac{\partial \psi}{\partial z}\right|^2 dz$, among functions that satisfy the boundary condtions, the normalization condition, and orthogonality conditions (see Appendix C).

Thus adequate normalization procedure with respect to the inner product of $\langle\cdot|\cdot\rangle$ gives the orthonormal properties (85) and (86). The third relation (87) expresses the transpose of (85).

These orthonormal relations show that the set of $\{\hat{\psi}^{(m)} \mid 1 \leq m \leq K\}$ is an orthonormal basis of $\boldsymbol{R}^K$ with respect to the inner product of $\langle\cdot|\cdot\rangle$. Thus any $\phi \in \boldsymbol{R}^K$ can be expanded as

$$\phi_k \models \phi = \sum_m \hat{\psi}^{(m)}\tilde{\phi}^{(m)} \models \sum_m \hat{\psi}_k^{(m)}\tilde{\phi}^{(m)},$$

where the expansion coefficient $\tilde{\phi}^{(m)}$ is obtained from

$$\tilde{\phi}^{(m)} \equiv \langle\phi|\hat{\psi}^{(m)}\rangle = \sum_k \phi_k \frac{H_k}{H}\hat{\psi}_k^{(m)}.$$

Finally we prove that for a rigid lid, $F^{(1)} = 0$ and $\hat{\psi}_k^{(1)} = 1$ as follows. Because $B_{1u} = \infty$ we have

$$0 = M_{k,k-1} + M_{k,k} + M_{k,k+1} = \sum_l M_{k,l} \tag{89}$$

for any k. Then, for $F^{(1)} = 0$ and $\hat{\psi}_k^{(1)} = 1$, it is easy to confirm the eigenvalue equation

$$\sum_l M_{k,l}\hat{\psi}_l^{(0)} = \sum_l M_{k,l} \cdot 1 = 0 = -F^{(0)} \cdot \hat{\psi}_k^{(0)}.$$

When $B_{1u} \neq \infty$, or when the surface is not a rigid lid, $F^{(1)}$ becomes positive (see Appendix).

4) Let us define a matrix $\Lambda \in \boldsymbol{M}(K,K)$ by

$$\Lambda \models \Lambda_{m,k} \equiv \frac{H_k}{H}\hat{\psi}_k^{(m)}.$$

Since

$$\delta_{m,n} = \langle\hat{\psi}^{(m)}|\hat{\psi}^{(n)}\rangle = \sum_k \hat{\psi}_k^{(m)}\frac{H_k}{H}\hat{\psi}_k^{(n)} = \sum_k \Lambda_{m,k}\hat{\psi}_k^{(n)},$$

we obtain

$$\Lambda^{-1} \models (\Lambda^{-1})_{k,m} = \hat{\psi}_k^{(m)}.$$

It is obvious that Λ converts a vector with components at each layer to another vector with componets of the coefficient of each mode $\tilde{\psi}^{(m)}$:

$$\Lambda\psi = \tilde{\psi} \models \tilde{\psi}_m = \sum_k \hat{\psi}_k^{(m)}\frac{H_k}{H}\psi_k.$$

Conversely $\Lambda^{-1} \models \psi_i^{(j)}$ is used when we want to get the value at each layer from the coefficients of modes:

$$\Lambda^{-1}\tilde{\psi} = \psi \models \psi_k.$$

5) The three-mode interaction coefficient becomes

$$I^{(ijm)} \equiv \sum_k \frac{H_k}{H}\hat{\psi}_k^{(i)}\hat{\psi}_k^{(j)}\hat{\psi}_k^{(m)}$$
$$\left(\approx \frac{1}{H}\int_{-H}^{0} \hat{\psi}^{(i)}\hat{\psi}^{(j)}\hat{\psi}^{(m)}\, dz\right),$$
$$(B_{1u} = \infty\ ) \quad I^{(1ij)} \equiv \sum_k \frac{H_k}{H}\hat{\psi}_k^{(i)}\hat{\psi}_k^{(j)} = \delta_{ij}.$$

4.2 evolution equation of each mode in a layer model

Now we consider such an expansion of the Q-G stream function at each layer as

$$\psi_k(\boldsymbol{x},t) = \sum_m \tilde{\psi}^m(\boldsymbol{x},t)\hat{\psi}_k^{(m)}.$$

Multiplying the evolution equation of the vorticity at each level (80) or (81) by Λ, we have the same form of evolution equation of each mode as for a level model:

$$\frac{\partial}{\partial t}(\nabla^2 - F^{(m)})\tilde{\psi}^{(m)} + \beta\frac{\partial\tilde{\psi}^{(m)}}{\partial x}$$
$$= \tilde{N}^{(m)} + \nu\nabla^4\tilde{\psi}^{(m)} + \tilde{V}^{(m)} + \tilde{W}^{(m)} \tag{90}$$

where

$$\tilde{N}^{(m)} = -\sum_{ij} J(\tilde{\psi}^{(i)}, \tilde{q}^{(j)})\tilde{\psi}^{(m)}I^{(ijm)},$$
$$(\tilde{q}^{(j)} \equiv \nabla^2 - F^{(j)})$$
$$\tilde{V}^{(m)} = \langle V|\hat{\psi}^{(m)}\rangle = \sum_k \frac{H_k}{H}\hat{\psi}_k^{(m)}\, V_k,$$
$$\begin{aligned}\tilde{W}_k^{(m)} &= \langle W|\hat{\psi}^{(m)}\rangle \\ &= \sum_k \frac{H_k}{H}\frac{f_r}{H_k}\hat{\psi}_k^{(m)}(w_s\delta_{k1} - w_b\delta_{kK}) \\ &= \frac{f_r}{H}(w_s\hat{\psi}_1^{(m)} - w_b\hat{\psi}_K^{(m)}) \\ &= \frac{1}{H}\left(\nabla \times \frac{\boldsymbol{\tau}}{\rho_r}\right)\hat{\psi}_1^{(m)} \\ &\quad - \left[J\left(\psi_K, \frac{f_r}{H}h\right) + \lambda\nabla^2\psi_K\right]\hat{\psi}_K^{(m)} \\ &= \frac{1}{H}\left(\nabla \times \frac{\boldsymbol{\tau}}{\rho_r}\right)\hat{\psi}_1^{(m)} \\ &\quad - \sum_j \left[J\left(\tilde{\psi}^{(j)}, \frac{f_r h}{H}\right) + \lambda\nabla^2\tilde{\psi}^{(j)}\right]\hat{\psi}_K^{(j)}\hat{\psi}_K^{(m)}.\end{aligned}$$

For the layer model, the JEBAR effect becomes

$$-\sum_{j>1} J\left(\frac{f_r}{H}\tilde{\psi}^{(j)}, h\right)\hat{\psi}_K^{(j)}.$$

The nonlinear advective term for the barotropic mode becomes

$$\tilde{N}^{(1)} = -\sum_j J(\tilde{\psi}^{(j)}, \nabla^2\tilde{\psi}^{(j)}).$$

We thus obtain the evolution equation for the layer-model similar to that for the level model.

4.3 energetics and inter-modal energy transfer

As in the level model it follows from the evolution equation of each mode that

$$\frac{\partial \tilde{E}^{(m)}}{\partial t} = S^{(m)}_{dsh} + S^{(m)}_{in} + S^{(m)}_{nl} + S^{(m)}_{bt} + S^{(m)}_{sf} + S^{(m)}_{dsb} + S^{(m)}_{bt},$$

where

$$\tilde{E}^{(m)} \equiv -\int \tilde{\psi}^{m}\tilde{q}^{(m)}\, d\boldsymbol{x} = \frac{1}{2}\int \left(\left|\nabla\tilde{\psi}^{(m)}\right|^2 + F_m\left|\tilde{\psi}^{(m)}\right|^2\right) d\boldsymbol{x},$$

$$(1) : \tilde{S}^{(m)}_{dsh} \equiv -(\tilde{\psi}^{(m)}|\cdot|\nu\nabla^4\tilde{\psi}^{(m)}) = -\nu\|\nabla^2\tilde{\psi}^{(m)}\|^2 - \nu\int \nabla\cdot(\tilde{\psi}^{(m)}\nabla\tilde{\zeta}^{(m)})\, d\boldsymbol{x},$$

$$(2) : \tilde{S}_{in:m} = (\tilde{\boldsymbol{u}}^{(m)}|\cdot|\tilde{\boldsymbol{F}}^{(m)}),$$

$$(3) : \tilde{S}_{nl:m} \equiv \sum_{i,j}\tilde{S}_{nl:i,j\to m},$$

$$\tilde{S}_{nl:i,j\to m} \equiv -\frac{I^{(ijm)}}{2} \times \left(\tilde{\psi}^{(m)}\,|\cdot|\, J(\tilde{\psi}^{(i)}, \tilde{q}^{(j)})) + J(\tilde{\psi}^{(j)}, \tilde{q}^{(i)})\right),$$

$$(4) : \tilde{S}_{sf:m} \equiv -\frac{f_r}{H}\hat{\psi}^{(m)}_1\left(\tilde{\psi}^{(m)}\,|\cdot|\,\nabla\times\frac{\tau}{f_r\rho_r}\right),$$

$$(5) : \tilde{S}_{dsb:m} \equiv -\lambda\sum_j \hat{\psi}^{(m)}_K\hat{\psi}^{(j)}_K\left(\nabla\tilde{\psi}^{(m)}\,|\cdot|\,\nabla\tilde{\psi}^{(j)}\right),$$

$$(6) : \tilde{S}_{bt:m} \equiv \sum_j \tilde{S}_{bt:j\to m} \equiv \frac{f_r}{H}\sum_j \hat{\psi}^{(m)}_K\hat{\psi}^{(j)}_K\left(\tilde{\psi}^{(m)}\,|\cdot|\,J(\tilde{\psi}^{(j)}, h)\right).$$

Note that boundary terms are absent. Detailed balance holds for three-mode advective interaction and two-mode coupling via bottom topography discussed in Section 3 as well in the layer model. Enstrophy arguments go parallel to those in the level model. We omit further formulas therefore.

5. Summary and discussion

A comprehensive formulation is presented for discussing the geostrophic current in terms of vertical modes. Both a level model (continuous model) and a layer model are dealt with in a similar manner, since the latter is considered as a finite-difference approximation to the former. In other words, most formulation goes parallel in both models. Of course, some differences are unavoidable; for instance, the (surface and bottom) boundary part in the former does not appear in the latter. The modification of free surface condition to the rigid-lid condition is straightforward in the latter.

For either kind of model, inter-modal energy transfer is discussed fairly in details. The barotropic mode does not make a triplet with two different baroclinic modes in the nonlinear advective interaction under the rigid lid surface condition. We find that detailed balance holds for nonlinear interaction among three vertical modes. Also topographic conversion from one vertical mode to another has a nature of detailed balance. Such a terminology as the JEBAR effect and bottom torque is argued in relation to the inter-modal energy transfer.

In Appendix, modifications that are necessary for the free surface condition are described. Also a variational principle and others are added to supplement the formulation in the text.

Though vertical modes are dealt with in this paper, we have not discussed a few subjects that are related with vertical modes. The first is about *stagnant-bottom vertical modes*, which are defined as vertical modes that satisfy the condition of $\psi = 0$ at bottom. The problem has been discussed for an ocean with bottom relief (Takase and Masuda 1996, Taileux and McWilliams 2001). This kind of vertical modes is a candidate for mechanisms that account for the abnormally fast propagation of suface disturbances observed by satellites (Chelton and Schlax 1996, Aoki et al. 2009).

Meanwhile Rhines effect (zonal bands of zonal flows) and Neptune effect (current with shallow area on the right-hand side in the northern hemisphere) are puzzling topics that are important and unsolved yet. So far they have been studied essentially for a single layer. There is little knowledge of how they are influenced by stratification or what are their vertical structure (Okuno and Masuda 2003). The framework of vertical modes presented here would be useful in investigating such phenomena.

The method of the expansion in terms of vertical modes applies as well to phenomena on a shorter time scale such as the response of the upper ocean to the passage of typhoons. In fact, some of the obseverved influence of typhoons on the upper ocean have been realized already by a three-dimensional numerical simulation (Hong and Yoon 2003, say). So far as I know, however, they have not been understood yet to a sastifactory degree; vertical modes may explain such observed and simulated features.

Those untouched topics will be addressed to in near future.

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Appendix

A1. Free surface condition

In Sections 3-4, we assumed a rigid lid on the sea surface, which greatly simplifies formulas for vertical modes. Let us give supplementary remarks about how the barotropic mode and F_0 are modified when the surface is allowed to be free.

1) Level model: The dynamic condition for the free sea surface becomes

$$\frac{(p-p_a)}{\rho_r} = g\eta = -\frac{gf_r}{N^2}\frac{\partial\psi}{\partial z} \quad \text{at} \quad z=0.$$

We must replace (18) by

$$\begin{aligned} w_s &= (\boldsymbol{u}\cdot\eta)_{z=0} + w_{sE} \\ &= -\left(\lhd\psi\cdot\nabla\left[\frac{f_r\psi}{g} - \frac{p_a}{g\rho_r}\right]\right)_{z=0} + w_{sE}. \end{aligned}$$

This change of w_s by the nonlinear term and p_a alters $\check{W}_m$, and consequently induces advective three-mode interaction and two-mode coupling via p_a like h, though small.

In determining vertical modes, p_a is discarded and the equations are linearized. From (5) it follows that

$$\left(\frac{g}{f_r}\eta = \right) \quad \psi = -\frac{g}{N^2}\frac{\partial\psi}{\partial z} \quad \text{at} \quad z=0. \quad \text{(A1)}$$

When a necessary condition for the Boussinesq fluid

$$1 \ll \frac{g}{N^2 H} \sim \frac{\rho_r}{\overline{\rho}(-H) - \overline{\rho}(0)}$$

is satisfied, (A1) is reduced to

$$0 = \frac{\partial \psi}{\partial z} = -\frac{N^2}{f_r}\eta \quad \text{at} \quad z = 0. \tag{A2}$$

which means $w = 0 = \eta$ at the sea surface, i.e., the rigid lid assumption.

If we go back to (A1), however, there arise some differences, even though the difference may be small. For instance, the second orthogonal condition (26) is modified to

$$\begin{aligned} F_m \delta_{m,n} &= \frac{f_r^2}{gH}\hat{\psi}_n(0)\hat{\psi}_m(0) + \frac{1}{H}\int_{-H}^{0} \frac{f_r^2}{N^2}\frac{d\hat{\psi}_m}{dz}\frac{d\hat{\psi}_n}{dz}\,dz \\ &= \frac{g}{H}\hat{\eta}_n(0)\hat{\eta}_m(0) + \frac{1}{H}\int_{-H}^{0} N^2 \hat{\eta}_m \hat{\eta}_n\,dz, \end{aligned} \tag{A3}$$

where the right-hand side means twice the potential energy of the m-th mode. In particular for the lowest mode $\hat{\psi}_0(z) \approx 1$, putting $m = n = 0$ in (A3), we have

$$F_0 \approx \frac{f_r^2}{gH}, \tag{A4}$$

which follows more directly from (A1), the bottom boundary condition, and the first of (23) as

$$\begin{aligned} -\frac{f_r^2}{g}\hat{\psi}_0(0) &= \frac{f_r^2}{N^2(0)}\frac{d\hat{\psi}_0}{dz}(0) \\ &= \int_{-H}^{0} \frac{\partial}{\partial z}\left[\frac{f_r^2}{N^2}\frac{d\hat{\psi}_0}{dz}\right] dz = -F_0 \int_{-H}^{0} \hat{\psi}_0\,dz \\ &\approx -F_0 H \hat{\psi}_0(0) \end{aligned}$$

in agreement with (A4). In the rigid lid assumption the barotropic mode has kinetic energy only, while it has potential energy as well under the free surface condition.

The free surface condition yields

$$\frac{1}{H} \gg \left|\frac{d\hat{\psi}_0}{dz}\right| \neq 0 \tag{A5}$$

so that a slight modification occurs for interaction coefficients I_{ijk} as well. In particular we have

$$\begin{aligned} &I_{0ij} \neq \delta_{i,j}, \\ &I_{i,jk} = -\frac{f_r^2}{gH}\hat{\psi}_i(0)\hat{\psi}_j(0)\hat{\psi}_k(0) - I_{j,ik} + F_k\, I_{ijk}. \end{aligned}$$

2) Layer model: The free surface condition is easily taken into account for the layer model. Instead of $B_{1u} = \infty$, we should put $B_{1u} = g$. Then we have a slight change of $M_{1,1}$ as

$$M_{1,1} = -M_{1,2} - \frac{f_r^2}{gH_1}.$$

This change causes a slight change of $F^{(m)}$ and $\hat{\psi}^{(m)}$. As mentioned before the difference becomes negligible if $g \to \infty$ with $\{\overline{B}_k\}$ fixed. The definition of inner product and orthogonality of eigenfunctions remain the same.

When $B_{1u} = g < \infty$, $F_{1u} \neq 0$ and $M_{1,1} + M_{1,2} \neq 0$, but still $M_{k-1,k} + M_{k,k} + M_{k,k+1} = 0$ for $k \geq 2$. In this case, $F^{(1)}$ does not vanish and $\hat{\psi}_k^{(1)}$ differs from one layer to another. In order to show it, suppose $F^{(1)} = 0$ is the lowest eigenvalue. Since $\hat{\psi}^{(1)}$ satisfies the eivenvalue equation as

$$\sum_l M_{kl}\hat{\psi}_l^{(1)} = F^{(1)}\hat{\psi}_k^{(1)} = 0, \quad (k \geq 2)$$

we find

$$\hat{\psi}_K^{(1)} = \hat{\psi}_{K-1}^{(1)} = \hat{\psi}_{K-2}^{(1)} = \cdots = \hat{\psi}_1^{(1)}. \tag{A6}$$

Again from the eigenvalue equation, we have

$$0 = 0 \cdot \psi_1^{(1)} = \sum_l M_{1l}\psi_l^{(1)} = -F_{1u}\psi_1^{(1)},$$

leading to $\hat{\psi}_1^{(1)} = 0$ because of $F_{1u} > 0$. Then (A6) yields $\hat{\psi}_k^{(1)} = 0$ $(1 \leq k \leq K)$, which contradicts with $\langle\langle\hat{\psi}^{(1)}\rangle\rangle \neq 0$.

Moreover we see that $\hat{\psi}^{(1)} \models \hat{\psi}_k^{(1)} = 1$ cannot be an eigenvector associated with $F^{(1)} > 0$. If it is the case, we have

$$1 = \hat{\psi}_k^{(1)} = -\frac{1}{F^{(1)}}\sum_l M_{k,l} \cdot 1 = 0$$

for $2 \leq k \leq K$, which leads to contradiction.

A2. Symmetrization of M

In Section 4 we introduced an inner product by which to make M self-adjoint. An alternative method is to modify M into a symmetric matrix for the ordinary inner product of $\boldsymbol{R}^K$.

We define $\breve{M} \models \breve{M}_{i,j}$ by

$$\breve{M}_{k,l} \equiv \sqrt{\frac{H_k}{H_l}}M_{k,l} = \frac{H_k M_{k,l}}{\sqrt{H_k H_l}} = \frac{H_l M_{l,k}}{\sqrt{H_k H_l}}, \tag{A7}$$

which is symmetric with respect to indices by virtue of (78). At the same time we put

$$\breve{\psi}_k \equiv \psi_k \sqrt{\frac{H_k}{H}}. \tag{A8}$$

The ordinary inner product of $\breve{\bullet}$ is the same as $\langle\cdot|\cdot\rangle$ in Section 4, since

$$\sum_k \breve{\psi}_k \breve{\phi}_k = \sum_k \psi_k \frac{H_k}{H}\phi_k = \langle\psi|\phi\rangle.$$

We may solve an alternative eigenvalue problem defined by

$$\breve{M}\breve{\psi} = -\breve{F}\breve{\psi}, \tag{A9}$$

where $\breve{F}$ is the eigenvalue and $(0 \neq)\ \breve{\psi} \in \boldsymbol{R}^K$ the associated eigenvector. This eigenvalue problem has well-known properties for a symmetric matrix $\breve{M}$. It is easy to show that the modified eigenvalue problem (A9) is essentially the same as the original problem of

$$M\psi = -F\psi \qquad \text{(A10)}$$

as follows.

Suppose that $\breve{F}$ and $\breve{\psi}$ satisfy (A9) and ψ is defined from $\breve{\psi}$ through (A8). Then we have

$$\begin{aligned} M\psi &\models (M\psi)_k = \sum_l M_{k,l}\psi_l = \sum_l \breve{M}_{k,l}\sqrt{\frac{H_l}{H_k}}\sqrt{\frac{H}{H_l}}\breve{\psi}_l \\ &= \sqrt{\frac{H}{H_k}}\sum_l \breve{M}_{k,l}\breve{\psi}_l = \sqrt{\frac{H}{H_k}}(\breve{M}\breve{\psi})_k = -\sqrt{\frac{H}{H_k}}\breve{F}\breve{\psi}_k \\ &= -\breve{F}\psi_k \models -\breve{F}\psi, \end{aligned}$$

so that ψ satisfies (A10) with $\breve{F}$ as an eigenvalue F.

A3. Vertical modes, variational principle, and minimizing sequence

In Section 3, we defined vertical modes as solutions of the differential equation (23), which is obtained also from a variational principle as follows.

First consider a level model and the rigid lid surface condition. Let $\psi = \psi(z)$ be a function that minimizes

$$Q \equiv \frac{1}{H}\int_{-H}^{0} \frac{f_r^2}{N^2}\left(\frac{d\psi}{dz}\right)^2 dz \qquad \text{(A11)}$$

under the conditions of

$$\frac{1}{H}\int_{-H}^{0} \psi^2\, dz = 1, \qquad \text{(A12)}$$

$$\frac{d\psi}{dz} = 0 \quad \text{at} \quad z = 0, \quad -H. \qquad \text{(A13)}$$

Roughly speaking, Q corresponds to the available potential energy of a stratified fluid, while the left-hand side of (A12) to kinetic energy, though the integrand is not $|\nabla\psi|^2$ but ψ^2.

Integration by pars with the boundary conditions (A13) yields

$$\begin{aligned} &\delta\left(Q - F\int_{-H}^{0}\psi^2\, dz\right) \\ &= -\frac{2}{H}\int_{-H}^{0}\left[\frac{d}{dz}\left(\frac{f_r^2}{N^2}\frac{d\psi}{dz}\right) + F\psi\right]\delta\psi\, dz, \end{aligned} \qquad \text{(A14)}$$

where $\delta\bullet$ denotes the variation of $\bullet$ and F is Lagrange's indeterminate coefficient. In order for (A14) to vanish for arbitrary $\delta\psi$, the minimizing function ψ satisfies

$$\frac{d}{dz}\left(\frac{f_r^2}{N^2}\frac{d\psi}{dz}\right) = -F\psi, \qquad \text{(A15)}$$

which agrees with (23). The minimizing function ψ gives

$$\begin{aligned} Q &= \frac{1}{H}\left[\psi\frac{f_r^2}{N^2}\frac{d\psi}{dz}\right]_{-H}^{0} - \int_{-H}^{0}\psi\frac{d}{dz}\left(\frac{f_r^2}{N^2}\frac{d\psi}{dz}\right)dz \\ &= \frac{F}{H}\int_{-H}^{0}\psi^2\, dz = F \end{aligned}$$

by integration by parts using (A12), (A13), and (A15), in accordance with (26).

The second lowest mode $\hat{\psi}_1$ is defined as such a function that minimizes Q under the same conditions (A12)–(A13) and another (orthogonality) constraint of

$$\frac{1}{H}\int_{-H}^{0}\hat{\psi}_1\hat{\psi}_0\, dz = 0.$$

The next mode is chosen from the functions that are perpendicular to $\hat{\psi}_0$ and $\hat{\psi}_1$. In this way, we have a minimizing sequence of $\{\hat{\psi}_0, \hat{\psi}_1, \cdots\}$ together with $0 \le F_0 \le F_1 \le \cdots$.

When the sea surface is not rigid but free, we must replace Q by

$$Q \equiv \frac{g}{H}\eta_{z=0}^2 + \frac{1}{H}\int_{-H}^{0}\frac{f_r^2}{N^2}\left(\frac{d\psi}{dz}\right)^2 dz$$

and the surface boundary condition by (A1). Then the variation of Q remains the same by virtue of (A1), leading to the same differential equation.

All the arguments above proceed as well for a layer model. Irrespective of whether the sea surface is rigid or free, we minimize

$$Q = \frac{1}{H}\sum_k (Y\psi)_k B_{ku}(Y\psi)_k = -\langle\psi|M|\psi\rangle$$

under the condition of

$$1 = \langle\!\langle\psi\rangle\!\rangle^2 = \sum_k \psi_k\frac{H_k}{H}\psi_k,$$

among the vectors $\psi \in \boldsymbol{R}^K$. It follows from the variational principle that the minimizing vector ψ satisfies

$$M\psi = -F\psi,$$

which agrees with (82).

A4. Mass conservation condition

The Q-G formulation poses an auxiliary or supplementary constraint, because the ordinary side boundary conditions on flows are insufficient to determine the evolution of the Q-G stream function ψ. Let us start with a simplest equation

$$\begin{cases} \dfrac{\partial q}{\partial t} = V(\boldsymbol{x}, z, t) \\ \quad q \equiv \nabla^2\psi + \dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial\psi}{\partial z}\right] \\ 0 = \boldsymbol{n}\cdot\triangleleft\psi \quad \text{on} \quad \partial O \\ 0 = \dfrac{\partial\psi}{\partial z} \quad \text{on} \quad z = 0, \ -H \end{cases}, \qquad \text{(A16)}$$

which represents the evolution of potential vorticity q in a flat closed ocean basin (COB) governed by inviscid linearized dynamics: $\beta = 0$, $\boldsymbol{\tau} = 0$, $\lambda = 0$, and $\nu = 0$. Note that the side boundary condition of (A16) requires ψ to take a certain boundary value $s(t, z)$ on ∂O, but does not pose any condition on s itself. Let $\phi_V(\boldsymbol{x}, z, t)$ and $\phi_b(\boldsymbol{x}, z, t)$ be functions that satisfy

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\nabla^2\phi_V + \dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \phi_V}{\partial z}\right]\right) = V(\boldsymbol{x}, z, t) \\ \phi_V = 0 \quad \text{on} \quad \partial O \\ 0 = \dfrac{\partial \phi_V}{\partial z} \quad \text{on} \quad z = 0,\ -H \end{cases},$$

$$\begin{cases} \nabla^2\phi_b + \dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \phi_b}{\partial z}\right] = 0 \\ \phi_b = s(z, t) \quad \text{on} \quad \partial O \\ 0 = \dfrac{\partial \phi_b}{\partial z} \quad \text{on} \quad z = 0,\ -H \end{cases},$$

respectively, where s is a certain function of z and t; we have a mixed boundary-value problem of an elliptic partial differential equation. It is easy to confirm that

$$\psi = \phi_V(\boldsymbol{x}, z, t) + \phi_b(\boldsymbol{x}, z, t)$$

satisfy (A16) for arbitrary $s(z, t)$. Physically it is stated as follows. Potential vorticity q increases or decreases according to the input of V. The evolution may be realized by ϕ_V, which vanishes on ∂O. But another field of ϕ_b with boundary values $s(z, t)$ may be added without changing q; for ϕ_b vorticity change is compensated exactly by the stretching of water columns.

Thus the side boundary condition of (A16) allows a degree of freedom of $s(z, t)$ for the evolution of ψ. We have to determine $s(z, t)$ for the unique evolution of ψ. Such a condition, which is independent of the Q-G vorticity equation, is derived from the conservation of mass as follows.

Because $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on the side boundary, no water flows into the basin from the side boundary. Mass conservation at each z requires

$$0 = \frac{\partial}{\partial z}\int w\, d\boldsymbol{x} = -\frac{\partial}{\partial t}\int \frac{\partial}{\partial z}\left(\frac{f_r}{N^2}\frac{\partial \psi}{\partial z}\right) d\boldsymbol{x}, \tag{A17}$$

where we have used (11). The above condition may be rewritten as

$$0 = \frac{\partial}{\partial t}\int \psi\, d\boldsymbol{x}. \tag{A18}$$

This is the required constraint by which to determine the boundary value $s(z, t)$.

As a second example, consider a similar evolution equation with viscosity:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla^2\psi + \dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \psi}{\partial z}\right]\right) \\ \qquad +\nu\nabla^2\dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \psi}{\partial z}\right] = V(\boldsymbol{x}, z, t) \\ \boldsymbol{n}\cdot\lhd\psi = 0, \quad \boldsymbol{n}\cdot\nabla\psi = 0 \quad \text{on} \quad \partial O \\ 0 = \dfrac{\partial \psi}{\partial z} \quad \text{on} \quad z = 0,\ -H \end{cases}. \tag{A19}$$

In this case we have one more boundary condition because of the higher order of the differential equation. Again let $\phi_V(\boldsymbol{x}, z, t)$ and $\phi_b(\boldsymbol{x}, z, t)$ be such functions that satisfy

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla^2\phi_V + \dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \phi_V}{\partial z}\right]\right) \\ \qquad +\nu\nabla^2\dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \phi_V}{\partial z}\right] = V(\boldsymbol{x}, z, t) \\ \phi_V = 0, \quad \boldsymbol{n}\cdot\nabla\phi_V = 0 \quad \text{on} \quad \partial O \\ 0 = \dfrac{\partial \phi_V}{\partial z} \quad \text{on} \quad z = 0,\ -H \end{cases},$$

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla^2 + \dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial}{\partial z}\right]\right)\phi_b \\ \qquad +\nu\nabla^2\dfrac{\partial}{\partial z}\left[\dfrac{f_r^2}{N^2}\dfrac{\partial \phi_b}{\partial z}\right] = 0 \\ \phi_b = s(z, t), \quad \boldsymbol{n}\cdot\nabla\phi_b = 0 \quad \text{on} \quad \partial O \\ 0 = \dfrac{\partial \phi_b}{\partial z} \quad \text{on} \quad z = 0,\ -H \end{cases},$$

respectively, where s is a function of z and t. Obviously $\psi \equiv \phi_V + \phi_b$ satisfies (A19) for arbitrary $s(z, t)$. Thus we should pose a constraint (A18) to determine $s(z, t)$.

A good illustration is given by the evolution equation of a single baroclinic mode $\tilde{\psi}_m$:

$$\begin{cases} (\nabla^2 - F_m)\dfrac{\partial \tilde{\psi}_m}{\partial t} = \tilde{V}_m(\boldsymbol{x}, t) \\ 0 = \boldsymbol{n}\cdot\lhd\tilde{\psi}_m \quad \text{on} \quad \partial O \end{cases}, \tag{A20}$$

In this case we choose $\phi_V(\boldsymbol{x}, t)$ and $\phi_b(\boldsymbol{x})$ satisfying

$$\begin{cases} (\nabla^2 - F_m)\phi_V = \displaystyle\int^t \tilde{V}_m(\boldsymbol{x}, t)\, dt \\ \phi_V = 0 \quad \text{on} \quad \partial O \end{cases},$$

$$\begin{cases} (\nabla^2 - F_m)\phi_b = 0 \\ ^{\exists}B \neq 0 : \ \phi_b = B \quad \text{on} \quad \partial O \\ \displaystyle\int \phi_b\, d\boldsymbol{x} = 1 \end{cases}, \tag{A21}$$

respectively. Taking the inner product of (A18) and $\hat{\psi}_m$, we obtain the auxiliary constraint

$$0 = \frac{\partial}{\partial t}\int \tilde{\psi}_m\, d\boldsymbol{x}. \tag{A22}$$

We confirm that

$$\tilde{\psi}_m(\boldsymbol{x}, t) = \phi_V(\boldsymbol{x}, t) - \left(\int \phi_V(\boldsymbol{x}, t)\, d\boldsymbol{x}\right)\phi_b(\boldsymbol{x})$$

is the unique solution of (A20) supplemented by (A22).

If there is water supply through some boundaries, we go back to (A17), the inner product of which with $\hat{\psi}_m$ yields a condition like

$$\frac{F_m}{f_r}\frac{\partial}{\partial t}\int \tilde{\psi}_m\, d\boldsymbol{x} = \text{supply}_m(t)$$

instead of (A22), where $\text{supply}_m(t)$ denotes the m-th mode of water inflow.

Even for a POB, mass conservation requires (A18) to be satisfied. However, it is guaranteed by the periodic side boundary conditions, when horizontal average of V equals to 0.