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Analytical formulas representing the idealized growth of wind-waves, duration-limited and fetch-limited

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Abstract

The SMB method is an historical wave-forecasting method, which is based on fetch-limited and duration-limited growth of wind-waves. However, wind-wave growth envisaged by the words fetch-limited and duration-limited has been ambiguous, and clear explanation seems to have been absent. A simplest conceptual model is proposed here for the idealized growth of wind-waves, where the model equation is based on several assumptions: (1) a well-established fetch relation for small to moderate fetches, (2) ubiquitous equilibrium governed by the 3/2-power law of Toba, (3) propagation of wind-waves as a whole at the group velocity of waves with the significant wave period, (4) presence of the upper bound of the significant wave period, and (5) tanh-type fetch dependence of wave energy. In particular, the fetch relation of (5) is a modified Wislon's formula I, where variables are normalized by a single scale of fectch and upper bounds of wind-waves. The model yields analytical formulas that clearly illustrate the growth of wind-waves, both fetch-limited and duration-limited. The result suggests that wave growth is better described by *substantial duration* t_{sd} than the conventional discrimination of *duration-limited* and *fetch-limited*; t_{sd} allows a unified description of wave-growth. Related problems are examined and discussed as well: fetch relations at large fetches, normalized and approximate formulas of fetch relations, momentum and energy input, and so on.

Key words : *wind-waves, fetch-limited growth, duration-limited growth, analytic solution, characteristic curves of wave-growth*

1. Introduction

Nowadays wind-waves are forecasted by spectral models, which compute the temporal and spatial evolution of wave energy for each wavenumber, or equivalently for each pair of frequency and wave-direction; see, say, Komen et al. (1994). Once, however, the Sverdrup-Munk-Bretshneider method (SMB method for brevity) was used widely for forecasting and hindcasting wind-waves (Sverdrup and Munk 1947; Bretschneider 1952, 1958). It is based on observation that the representative wave height and wave period of wind-waves are determined by three factors: wind speed, duration (time), and fetch. In other words, the growth of wind-waves is described in terms of fetch-limited or duration-limited growth if the wind velocity is prescribed.

Though simple and easy to deal with, the SMB method well grasps the growth of wind-waves. That view does hold even now and makes the most fundamental and essential part of the theory of the growth of wind-waves. The SMB method therefore is good to study first the growth of wind-waves and their forecast. However, the notion of duration-limited and fetch-limited growth, which underlies the SMB method, is not necessarily easy to explain, or somewhat ambiguous. For instance, Mitsuyasu and Rikiishi (1978) gave a description of the fetch-limited and duration-limited growth with illustration for short to moderate fetches. The resulting explanation does not appear so clear though. This is probably because explanation has been based on words; there has been absence of simple, adequate formulas representing well the duration-limited growth of wind-waves, though we have had many empirical formulas of fetch-relation such as Wilson (1965), Mitsuyasu (1968, 1969) or Hasselmann et al. (1973).

In this article, therefore, taking into account the recent development of the study of the growth of wind-waves, we want to present a model equation that gives formulas that represent well both the fetch-limited and duration-limited growth in a unified analytical manner. This is the primary purpose of the paper.

The secondary purpose is to review and clarify various fetch relations, the SMB method and Tohoku wave

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model (1985) from the viewpoint of the present model. In particular, arguments are addressed to the so far unsettled problem of the wave growth at large fetches.

The next section describes the situation to be considered and lists up the assumptions and empirical formulas to be used. The third section provides the model equation and its analytical solution, along with its approximation at small or large fetches. Both the fetch-limited and duration-limited growth of wind-waves are expressed by simple formulas and illustrated concisely. In the fourth section discussion is made of the related subjects of wind-wave growth on the basis of the proposed model; we also survey the SMB method and the Tohoku wave model from the viewpoint of the present model. The final section gives a summary.

2. Assumptions and formulation

Let us consider an idealized situation and simplify (or even modify) empirical laws as much as possible, since our primary purpose is to express the growth of wind-waves by simplest formulas.

2.1 Situation and notations

An infinitely wide ocean extends toward $x > 0$ with the shoreline along $x = 0$, where x denotes the offshore coordinate. The ocean is deep enough, so that wind waves are deep water waves everywhere.

Until $t = 0$, where t denotes time, there is no motion in the ocean, Suddenly at $t = 0$, wind begins to blow offshore uniformly over the ocean. Since then wind keeps its speed and direction; U is constant, where U is the wind speed at the height of 10 m above the mean sea surface. Then x (≥ 0) is identified with *fetch*, while t (≥ 0) with *duration* (time).

The situation set above appears unrealistic and too idealistic, but is possible conceptually at least. Now we have some simplifying assumptions, which may violate a little the properties of *real* wind-waves. Cautionary remarks are to be added therefore to each of the subsequent assumptions. Moreover explicit statement of assumptions or underlying postulates will make clear the limit of the present model and the physics of wind-waves so far unsettled and to be clarified in future.

(a) After the SMB method, we suppose that the overall *aspect* of the wind-sea is determined by two quantities of wind-waves: characteristic wave height and wave period. Usually they are significant wave height H and significant wave period T, respectively.

In our description, the growth of wind-waves means the spatial and temporal evolution of macroscopic characteristics averaged over hundreds of individual waves; H and T are such statistical quantities.

(b) Wind-waves propagate *as a whole*, preserving their spectral structure. In other words, irregular wind-waves are represented by only H and T. although wind-waves can have a variety of spectral features. Of course, H may be replaced by the power of surface displacement $E \equiv H^2/16$, and T by the phase velocity c of the linear waves with the significant wave period T.

This assumption may be validated partly, if the spectrum of wind-waves is self-similar everywhere. It is well known, however, that spectral form is not the same at different fetches even under the same wind velocity U. For instance the Pierson-Moskowitz spectrum differs from the standard spectrun of the JONSWAP type and from those observed in the laboratory (Pierson and Moskowitz 1964; Hasselmann et al. 1973; Kusaba and Masuda 1989). The peak enhancement factor γ of the spectrum of the JONSWAP type has a subtle fetch-dependence (Hasselmann et al. 1973; Bandou et al. 1986; Kusaba and Masuda 1989).

(c) Microscopic quantities such as molecular viscosity or surface tension is ignored in order for wind-waves to be self-similar irrespective of the scale (wavelength) of wind-waves. Densities of water and air affect the dynamics of the growth or wind-waves, but they are constant effectively, so that we need not take them into account. Thus the only external parameter is g, the acceleration due to gravity. Meaningful independent parameters are x, t, g, and U, against which H (or E) and T (or c, the phase velocity of the linear waves with the period of T) are to be determined in our model.

Meanwhile the friction velocity of the air u_* is assumed to be constant for the given U. We here discard a possible dependence of the drag coefficient C_d of the sea surface on wind-waves (Masuda and Kusaba 1987; Jones and Toba 2001). Since energy input or wave growth is considered to be controlled dynamically by u_* rather than U, this assumption may affect the fetch relation somewhat. Also molecular quantities may affect the drag coefficient of the sea surface (Toba and Koga 1986; Masuda and Kusaba 1987; Jones and Toba 2001).

(d) Wind-waves as a whole propagate in the wind direction at $c/2$, the group velocity of the free linear waves corresponding to the significant wave period T. In this assumption, the coefficient need not to be $1/2$ exactly. That may be adjusted so as to represent best the propagation of wind waves as a whole; adjusted coefficient value will lead to more realistic result. Here, however, we simply set it at $1/2$, since we are discussing an idealized situation, rather than a realistic.

(e) Fetch-relations at short to moderate fetches have turned out quite reliable (Mitsuyasu 1968 1969; Hasselmann et al. 1973). One may adopt, say, the fetch-relation by Mitsuyasu, which is a standard formula based on laboratory experiment and field observation:

$$\begin{cases} \dfrac{g\sqrt{E}}{u_*^2} = 1.31 \times 10^{-2} \left(\dfrac{gx}{u_*^2}\right)^{0.504} \\ \dfrac{u_{*a} f_p}{g} = 1.00 \left(\dfrac{gx}{u_*^2}\right)^{-0.33} \end{cases}, \qquad (1)$$

where f_p denotes the spectral peak frequency.

For short to moderate fetches, Toba's 3/2-power law (Toba 1972) is obtained from the fetch-relation as follows. Approximating 0.504 by 1/2 and 0.33 by 1/3, eliminating x from (1), using an empirical relation $1.05 f_p = 1/T$ and the identity $E = H^2/16$, we obtain

$$\left(\frac{u_*}{gT}\right)^{3/2} \frac{gH}{u_*^2} = 0.056, \qquad (2)$$

which is almost the same as the 3/2-power law of Toba

$$\left(\frac{u_*}{gT}\right)^{3/2} \frac{gH}{u_*^2} = 0.062. \qquad (3)$$

If we put $C_d \equiv u_*^2/U^2 \approx 1.6 \times 10^{-3}$, Toba's form (3) in terms of u_* is converted to that in terms of U as

$$\frac{1}{(gU)^{1/2}} \frac{H}{T^{3/2}} = 1.24 \times 10^{-2} \qquad or$$
$$\frac{g}{U^{1/2}} \frac{H}{c^{3/2}} = 0.195. \qquad (4)$$

2.2 Assumptions at large fetches

When fetch increases to ∞, any empirical formulas are unlikely to be reliable. This is simply because the ocean is finite. Moreover wind-waves at large fetches are contaminated by many factors. First, wind is variable temporally and spatially, so that idealistic development rarely happens at large fetches. Second, at large fetches or far offshore, wind waves coexist with swell, which is also wind-waves of longer periods coming from other areas and generated under the action of different wind speed at older time. It is well known that interaction between wind-waves and swell much affects the evolution of wind-waves, but this process has remained as one of the least understood ones.

Thus observation of idealistic growth of wind-waves at long fetches are absent or poor. Such an idealized situation as was postulated in this paper rarely happens. We cannot expect a sufficient amount of data available at large fetches. In short, we have no reliable formula as regards the growth of wind-waves at large fetches.

Accordingly there remain unanswered questions about the growth of wind-waves at large fetches. In the present subsection let us discuss a bit in detail the growth of wind-waves on the basis of Wilson's formulas (Wilson 1965).

(f) Wave growth probably does not long forever, and we assume the presence of fully developed sea for the given wind speed U. Then the characteristic phase speed of wind-waves c approaches an upper bound, when fetch $x \to \infty$; equivalently fully developed sea has an upper bound of significant wave period T.

We are not sure whether there is such an upper bound or not. One may think that wind-waves continue to grow at a very slow pace at large fetches without a bound. The Tohoku wave model (Toba et al. 1985) adopts such a formulation allowing a boundless growth of wind-waves.

Obviously, however, wind-waves decay under the adverse wind. It seems more likely that wind-waves will decay under the wind slower than the propagation speed of wind-waves.

Observationally Wilson's formula I and formula IV anticipate the presence of those upper bounds. Also the Pierson Moskowitz spectrum is considered to be a spectrum of such saturated or fully developed sea characterized by a frequency and wave energy determined solely by the wind speed U. Carter (1982) adopts $H_\infty = 0.025U^2$ in MKS units, which has a nondimensional form of

$$\frac{gH_\infty}{U^2} = 0.245 \qquad (5)$$

as the upper bound, or the saturated value of the significant wave height for the given wind speed U.

(g) We suppose Toba's 3/2-power law holds everywhere and anytime. That is an empirical law, but is considered as one of the most reliable ones, supported by various observation and partly by Wilson's formula IV at large fetches. By virtue of this law, c is related with H or $E = H^2/16$, so that we need not calculate the evolution of c independently.

It is to be noted that some laboratory experiments show discrepancy from the 3/2-power law (Kusaba and Masuda 1989). Also enough evidence is absent for large fetches, for reasons mentioned in the beginning of this subsection.

(h) Finally we have to select or assume some form of fetch-relation that is consistent with the assumptions made so far. We require it to be simple enough to yield a simple expression of duration-limited growth. Of course it is desirable that the assumed fetch relation agrees with observation; at least it should not deviate much from observation.

Though there may be alternatives, we adopt the following form of fetch-relation:

$$\begin{cases} \dfrac{H}{H_\infty} = \tanh^{1/2} \dfrac{x}{x_s} \\ \dfrac{T}{T_\infty} = \dfrac{c}{c_\infty} = \tanh^{1/3} \dfrac{x}{x_s} \end{cases}, \qquad (6)$$

where H_∞, T_∞, and c_∞ are the upper bounds of significant wave height H, significant wave period T, and phase speed of waves with the significant wave period, respectively. We should note that H_∞, T_∞, and c_∞ are determined from U (and g) only, by our assumptions mentioned before. Here x_s denotes a scale of fetch, which is unknown now, but will be determined later (see Appendix too).

The choice of (6) might seem rather arbitrary, but there are some plausible reasons as follows.

Since we have assumed (i) the upper bounds of H_∞ (c_∞), (ii) a characteristic scale of fetch x_s, and (iii) Toba's 3/2-power at any fetch, it is natural to postulate that the fetch relation is expressed as

$$\begin{cases} \dfrac{H}{H_\infty} = f^{1/2}\left(\dfrac{x}{x_s}\right) \\ \dfrac{c}{c_\infty} = f^{1/3}\left(\dfrac{x}{x_s}\right) \end{cases} \tag{7}$$

by such a universal function f that

$$\begin{cases} f(z) \approx z & \text{for} \quad z \ll 1 \\ \dfrac{df}{dz} > 0 & \text{for} \quad 0 < z \\ f(z) \to 1 & \text{for} \quad z \to \infty\ . \end{cases}$$

Moreover $f(z)$ should be chosen so that it is close to empirical formulas based on observation.

Thus we are concerned with Wilson's formula I and formula IV, as representative fetch relations for large-fetches (Wilson 1965). Note that both of them anticipate the upper bound of wave growth. Wilson's formula I has a fairly simple form:

$$\begin{cases} \dfrac{gH}{U^2} = 0.26 \tanh\left[0.01\left(\dfrac{gx}{U^2}\right)^{\frac{1}{2}}\right] \\ \dfrac{c}{U} = 1.40 \tanh\left[0.044\left(\dfrac{gx}{U^2}\right)^{\frac{1}{3}}\right] \end{cases} . \tag{8}$$

For small fetches, they are approximated to be

$$\begin{cases} \dfrac{gH}{U^2} = 2.6 \times 10^{-3}\left(\dfrac{gx}{U^2}\right)^{\frac{1}{2}} \\ \dfrac{c}{U} = 6.16 \times 10^{-2}\left(\dfrac{gx}{U^2}\right)^{\frac{1}{3}} \end{cases} , \tag{9}$$

which are compared quite favorably with more recent empirical formulas, say, (1), if U is replaced by u_*. Eliminating x from (9), we have

$$\frac{g}{U^{1/2}}\frac{H}{c^{3/2}} = 0.17, \tag{10}$$

which is equivalent to Toba's law (3) or Mitsuyau's fetch-relation with only a slight difference of constants. Although Wilson's formula I, (8), does not satisfy Toba's 3/2-power law at moderate fetches, it gives a value close to (10) again when $x \to \infty$

$$\frac{g}{U^{1/2}}\frac{H}{c^{3/2}} = \frac{0.26}{1.40^{3/2}} = 0.16.$$

Wilson's formula I can be rewritten in a normalized form as

$$\begin{cases} \dfrac{H}{H_\infty} = \tanh\left(\dfrac{x}{x_{sh}}\right)^{1/2} \\ \dfrac{c}{c_\infty} = \tanh\left(\dfrac{x}{x_{sc}}\right)^{1/3} \end{cases} , \tag{11}$$

where

$$\frac{gH_\infty}{U^2} = 0.26, \qquad \frac{c_\infty}{U} = 1.40,$$
$$\frac{gx_{sh}}{U^2} \equiv 1.0 \times 10^4, \qquad \frac{gx_{sc}}{U^2} \equiv 1.18 \times 10^4.$$

Compare the value of gH_∞/U^2 with that of (5). Note also that there are two fetch scales: x_{sh} for H and x_{sc} for c.

Since $x_{sh} = 0.85x_{sc} \approx x_{xc}$, we may put $x_s \equiv x_{sh}$ and $x_{sc} = x_s$ for simplicity; x_s expresses a single fetch scale around which fetch-relation deviates from simple power laws of $H \approx x^{1/2}$ and $c \approx x^{1/3}$. If we use Wilson's formula I, the fetch scale of saturation is $gx_s/U^2 = 1.0 \times 10^4$; dimensionally $x_s \sim 100$ km when $U = 10$ m/s.

Another step of modification of (11) leads to (6). It is apparent that (6). follows (7) to satisfy the 3/2-power law everywhere. In addition it is equal to (8) both for $x \ll x_s$ and for $x \gg x_s$. For moderate values of x/x_s, however, (6) is not equal to (8) or (11), as will be shown later in Figs. 1-2.

In comparison with Wilson's formula I above, Wilson's formula IV has been used more frequently. It has a rather complicated form:

$$\begin{cases} \dfrac{gH}{U^2} = 0.30\left(1 - \left(1 + 0.004\left(\dfrac{gx}{U^2}\right)^{1/2}\right)^{-2}\right) \\ \dfrac{c}{U} = 1.37\left(1 - \left(1 + 0.008\left(\dfrac{gx}{U^2}\right)^{1/3}\right)^{-5}\right). \end{cases} \tag{12}$$

It is rewrittne also in a normalized form as

$$\begin{cases} \dfrac{H}{H_\infty} = 1 - \left(1 + \dfrac{1}{2}\left[\dfrac{x}{x_{sh}}\right]^{1/2}\right)^{-2} \\ \dfrac{c}{c_\infty} = 1 - \left(1 + \dfrac{1}{5}\left[\dfrac{x}{x_{sc}}\right]^{1/3}\right)^{-5} \end{cases} , \tag{13}$$

where

$$\frac{gH_\infty}{U^2} = 0.3, \qquad \frac{c_\infty}{U} = 1.37$$
$$\frac{gx_{sh}}{U^2} = 1.56 \times 10^4, \qquad \frac{gx_{sc}}{U^2} = 1.85 \times 10^4.$$

Here x_{sh}, x_{sc}, and H_∞ are larger, and c_∞ is smaller than those in Wilson's formula I.

The fetch scale x_{sh} is very close to x_{sc} ($x_{sh} = 0.84x_{sc}$), just as in Wilson's formula I. Thus we may use $x_s \equiv x_{sh}$ as a single characteristic scale of fetch; we put $x_{sc} = x_s$ again.

For the model fetch-relation satisfying (7), one may choose, for instance,

$$f^{1/2}(x) = 1 - \left(1 + \frac{x^{1/2}}{2}\right)^{-2}. \tag{14}$$

It yields Wilson's formula for H and satisfies the 3/2-power law exactly, but gives a bit different growth of c from that in Wilson's formula IV. The same is true for the choice of

$$f^{1/2}(x) = \tanh x^{1/2}. \tag{15}$$

Such cumbersome representations, however, complicate the final formulas we want; analytical representation of duration-limited growth will be difficult. We thus prefer (6), or

$$f(x) = \tanh x \tag{16}$$

for the sake of simplicity. This is a modified form of Wilson's formula I (15).

In this opportunity we examine the difference of various fetch-relations: simple power relations valid at small fetches, Wilson's formula I, formula IV, and ours.

For this purpose let us define three functions

$$\begin{cases} p_m(x) \equiv x^m \\ q_m(x) \equiv \tanh^m x \\ r_m(x) \equiv \tanh x^m \end{cases}$$

where $x \geq 0$, and $m = 1/2$ or $1/3$. Obviously p_m, q_m, and r_m express the fetch-relation at short fetches, (6), and (8), respectively. Also we define the fourth function as

$$\begin{cases} s_{1/2}(x) \equiv 1 - \left(1 + \dfrac{x^{1/2}}{2}\right)^{-2} \\ s_{1/3}(x) \equiv 1 - \left(1 + \dfrac{x^{1/3}}{5}\right)^{-5} \end{cases} \tag{17}$$

which corresponds to Wilson's formula IV.

Figs. 1 and 2 show the four functions for $0 \leq x \leq 2$ for $m = 1/2$ and $m = 1/3$, respectively, where $m = 1/2$ corresponds to the fetch-relation of H and $m = 1/3$ that of $c \sim 1/T$. The center of the abscissa $x = 1$ corresponds to the characteristic fetch scale x_s dimensionally in this scaling.

Note that in Figs. 1 and 2, another kind of functions are added together:

$$\begin{cases} t_{1/2}(x) \equiv \left(\tanh x^{2/9}\right)^{9/4} \\ t_{1/3}(x) \equiv \left(\tanh x^{2/9}\right)^{3/2} \end{cases}. \tag{18}$$

This kind of functions has a simple form similar to Wilson's formula I, and at the same time has values close to formula IV, in particular for $m = 1/2$. If H and c follow $t_{1/2}$ and $t_{1/3}$, respectively, the 3/2-power law is satisfied exactly.

We observe

$$\begin{cases} p_m(x) \approx q_m(x) \approx r_m(x) \approx s_m(x) \\ \qquad \text{for} \quad x \ll 1 \\ 1 \approx q_m(x) \approx r_m(x) \approx s_m(x) \\ \qquad \text{for} \quad x \gg 1 \\ p_m(x) > q_m(x) > r_m(x) > s_m(x) \\ \qquad \text{for} \quad 0 < x \end{cases}$$

That is, q_m, r_m, and s_m agree with one another at short fetches. They increase with x, and deviates from one another around $x \sim 1$. The power-law form p_m grows fastest, and s_m slowest. Mutual difference or relative magnitude shown in the figures is not very small, but may be considered insignificant on the logarithmic scale; note that even the discrepancy between Wilson's formula I and formula IV appears considerable in these figures on the linear scale.

Though not shown in Figs. 1 and 2, q_m, r_m, and s_m asymptote to unity as $x \to \infty$. The fetch-relation q_m of the present model gives a faster growth with the fetch than r_m and s_m. Overall properties of the four functions are similar between $m = 1/2$ and $m = 1/3$.

Fig. 3 examines the accuracy of the 3/2-power law of Toba for the three fetch-relations q_m, r_m, and s_m: $q_{1/2}/q_{1/3}^{3/2}$, $r_{1/2}/r_{1/3}^{3/2}$, and $s_{1/2}/s_{1/4}^{3/2}$. If Toba's law holds everywhere they should be unity irrespective of x. Of course we have $q_{1/2}/q_{1/3}^{3/2} = 1$ by definition. On logarithmic scale, the small deviation from unity for r_m and s_m is regarded negligible as mentioned in Toba (1972). Note, however, that the deviation from unity is the largest for s_m corresponding to Wilson's formula IV, on which Toba's law relies.

Recall that Toba's constant 0.062 in (3) is 1.1 times Mitsuyasu's 0.059 in (2); the former is for a wide range of x, while the latter for small fetches. Fig. 3 suggests that Toba's constant is chosen so as to fit both Mituyasu's fetch-relation at short fetches and Wilson's formula I or IV at moderate x. Note that the present scaling is made so that the fetch relation of H may be accurate at short fetches.

3. Model equation, its analytical solution and approximations

3.1 Model equation

We use $E = H/16$ instead of H and normalize the dependent and independent variables as

$$\begin{cases} \hat{E} \equiv \dfrac{E}{E_\infty} \\ \hat{c} \equiv \dfrac{c}{c_\infty} \end{cases}, \quad \begin{cases} \hat{x} \equiv \dfrac{x}{x_s} \\ \hat{t} \equiv \dfrac{c_\infty}{x_s} t \end{cases}. \tag{19}$$

For brevity $\hat{\bullet}$ is dropped henceforth.

According to the assumptions in the previous section, E propagate with the speed of $c/2 = E^{1/3}/2$. Then

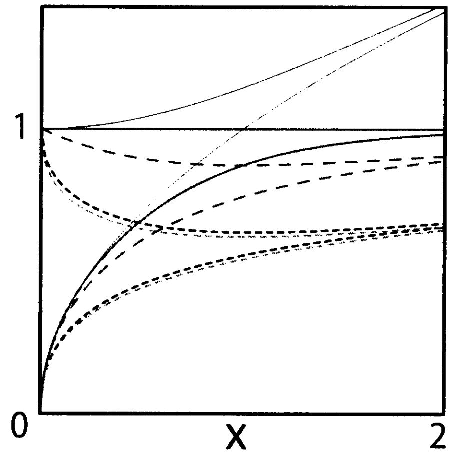

Fig. 1 Normalized fetch relations: five functions p_m to t_m for $m = 1/2$ corresponding to H. The five curves passing the lower left corner $(x,t) = (0,0)$ shows p_m (thin), q_m (thick), r_m (dashed), s_m (thick dotted), and t_m (thin dotted). They represent the fetch-relations of H expressed by: the power-form x^m valid at small fetches,, the present model (7), Wilson's formula I (11), Wilson's formula IV (13), and a newly devised formula (17). The five curves passing $(x,t) = (0,1)$ are p_m/q_m, $q_m/q_m = 1$, r_m/q_m, s_m/q_m and t_m/q_m, respectively. They express the magnitude of five functions relative to q_m.

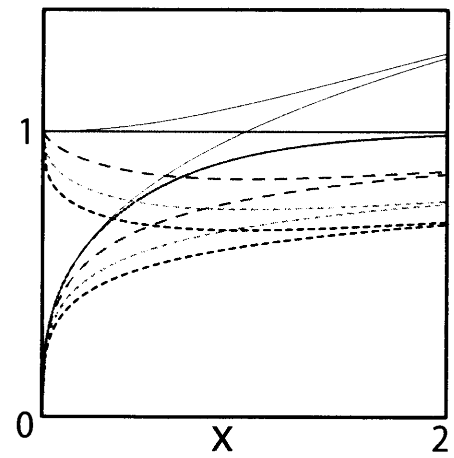

Fig. 2 The same as Fig. 1 except that $m = 1/3$ corresponding to c.

the model partial differential equation to be solved becomes

$$\frac{\partial E}{\partial t} + \frac{E^{1/3}}{2}\frac{\partial E}{\partial x} = S^{(E)}(E) \tag{20}$$

for a single unknown $E = E(x,t)$ with the boundary and initial conditions

$$\begin{cases} E(x,0) = 0 & (x \geq 0) \\ E(0,t) = 0 & (t \geq 0) \end{cases}, \tag{21}$$

where $S^{(E)}(E)$ is the net rate of energy input to wind-waves; $S^{(E)}$ must be defined as a function of E.

In the fetch-limited zone of the (x,t) plane, where $\dfrac{\partial E}{\partial t} = 0$, E must satisfy our model fetch relation

$$\begin{cases} E = E_f(x) \equiv \tanh x \\ c = c_f(x) = E_f^{1/3} = \tanh^{1/3} x \end{cases} \quad (0 \leq x) \tag{22}$$

so that $S^{(E)}$ should be

$$S^{(E)}(E) = \frac{E^{1/3}}{2}\tanh^2 x = \frac{E^{1/3}(1-E^2)}{2}, \tag{23}$$

a simple algebraic equation of E.

3.2 Distribution of E

The partial differential equation (20) has such a simple form as allows an analytical solution as follows. The method of characteristic curves resolves (20) into two odrinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{c}{2} = \dfrac{E^{1/3}}{2} \\ \dfrac{dE}{dt} = S^{(E)} = \dfrac{E^{1/3}(1-E^2)}{2} \end{cases} \tag{24}$$

where the former defines the characteristic curve on the (x,t) plane, and the latter describes the growth of E along the characteristic curve.

First we define a function $t = t_b(x)$ by

$$t_b(x) \equiv \int_0^x \frac{dx}{c_f(x)/2} = 2\int_0^x \frac{dx}{\tanh^{1/3} x}, \tag{25}$$

which represents the characteristic curve that starts $(x,t) = (0,0)$.

A new variable ξ is introduced here:

$$\xi \equiv \tanh^{2/3} x,$$
$$d\xi = \frac{2}{3}\frac{dx}{\tanh^{1/3} x}\mathrm{sech}^2 x = \frac{2}{3}\frac{dx}{\tanh^{1/3} x}(1-\xi^3).$$

Using ξ, we can integrate (24) analytically as

$$\begin{aligned} t_b(x) &= 3\int_0^\xi \frac{d\xi}{1-\xi^3} \\ &= \sqrt{3}\left(\arctan\frac{2\xi+1}{\sqrt{3}} - \frac{\pi}{6}\right) \\ &\quad -\frac{1}{2}\log\frac{(1-\xi)^2}{(1+\xi+\xi^2)}. \end{aligned} \tag{26}$$

Since $t_b(x)$ is a monotonic function, there exists its inverse, which is denoted as $x_b(t)$.

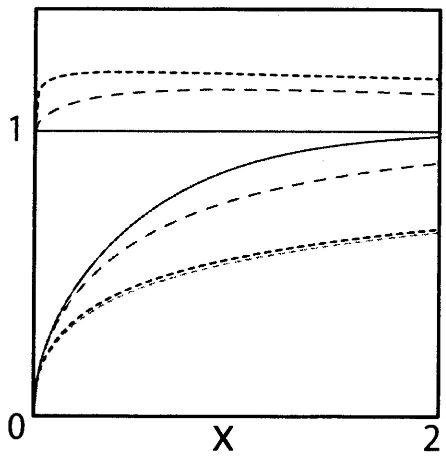

Fig. 3 Examination of the 3/2-power law of Toba for the three fetch relations q_m, r_m, and s_m. The three curves passing $(x, y) = (0, 1)$ are $q_{1/2}/q_{1/3}^{3/2}$ (thick), $r_{1/2}/r_{1/3}^{3/2}$ (dashed), and $s_{1/2}/s_{1/3}^{3/2}$ (dotted). The present model fetch-relation yields $q_{1/2}/q_{1/3}^{3/2} = 1$, whereas the other two formula gives values a little larger than 1 for moderate x. The four curves starting from $(x, t) = (0, 0)$ are the same as in Fig. 1 ($m = 1/2$).

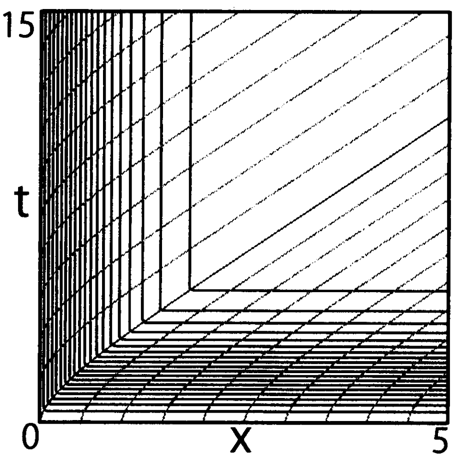

Fig. 4 Contours of E (thick lines) and characteristi curves (thin lines) on the (x, t) plane, where x denotes fetch and t duration; $0 \le x \le 5$ and $0 \le t \le 15$ in this figure. Contour interval is 0.05. Characteristic curves are drawn at an interval chosen adequately. The critical characteristic curve (CCC), which starts $(x, t) = (0, 0)$, separates the fetch-limited region (upper left) and and the duration-limited region (lower right). As x or t increases to ∞, E asymptotes to unity and characteristic curves become straight lines with a slope of $dx/dt = 1/2$.

Also the second of (24) is integrated if we use $\tau \equiv x_b(t)$ instead of t. It is easy to confirm that $E(t) = \tanh \tau(t) = \tanh x_b(t)$ satisfies (24) and $E(0) = 0$, since

$$\frac{dE}{dt} = \frac{d\tau}{dt}\frac{dE}{d\tau} = \frac{\tanh^{1/3}\tau}{2}\mathrm{sech}^2\,\tau = \frac{E^{1/3}}{2}(1 - E^2).$$

Of course, without soloving, we know it from the fetch-relation by virtue of the method of characteristic curves. The analytical solution of (20) thus becomes

$$E(x,t) = \tanh\left(\min\{x, x_b(t)\}\right) = \begin{cases} \tanh x & \text{if} \quad t > t_b(x) \\ \tanh x_b(t) & \text{if} \quad t \le t_b(x) \end{cases}. \tag{27}$$

The solution shows that the (x, t) plane is divided into two parts: zone of fetch-limited growth of wind-waves ($t > t_b(x)$; upper left) and that of duration-limited growth ($t < t_b(x)$; lower right); see Figs. 4-5. Also there are two corresponding families of characteristic curves:

$$\begin{cases} t = {}^{\exists}t_0 + t_b(x) & \text{if} \quad t > t_b(x) \\ t = t_b(x - {}^{\exists}x_0) & \text{if} \quad t \le t_b(x) \end{cases} \tag{28}$$

or

$$\begin{cases} x = x_b(t - {}^{\exists}t_0) & \text{if} \quad t > t_b(x) \\ x = {}^{\exists}x_0 + x_b(t) & \text{if} \quad t \le t_b(x) \end{cases}, \tag{29}$$

where $x_0 > 0$, $t_0 > 0$, and $t_b(x)$ is given by (26). The former family of characteristic curves start the shore $x = 0$ at time $t_0 > 0$ and fill the zone of fetch-limited growth of wind-waves. Likewise the latter family start from somewhere offshore $x_0 > 0$ at time $t = 0$ and fill the zone of duration-limited growth of wind-waves.

Fig. 4 shows the global distribution of E on the (x, t) plane ($0 \le x \le 5$, $0 \le t \le 15$). A cross section along $t =$ const yields the spatial distribution of E at that moment (duration), while a cross section along $x =$ const the time sequence of E at that offshore position (fetch).

Characteristic curves are drawn as well. In particular, $t = t_b(x)$ or $x = x_b(t)$ gives the boundary between the zones of fetch-limited growth and duration-limited growth. We call this particular curve the *critical characteristic curve* (CCC for brevity). It separates the zone of fetch-limited growth and that of duration-limited growth. We observe contours of E tend to its upper bound 1 as x or t increases to ∞. Also, in the same limit, characteristic curves tend to have a constant slope of $1/2$, showing the saturation of the propagation speed of wind-waves.

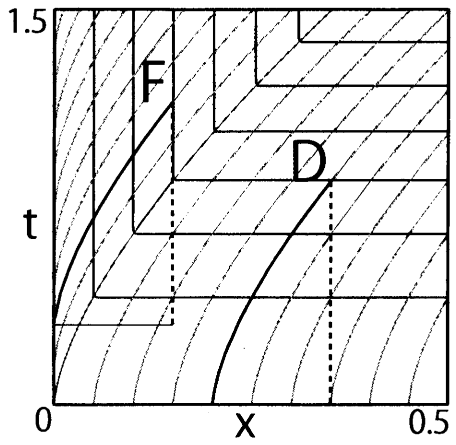

Fig. 5 The same as Fig.2 except that it enlarges the area of short to moderate fetches (duration) so that $0 \le x \le 0.5$ and $0 \le t \le 1.5$. Point F (D) belongs to the zone of fetch-imited (duration-limited) growth. Thick lines express the characterstic curves passing F (D). The substantial duration t_{sd} is denoted by the length of the dotted line. Because t_{sd} is the same for F and D, so is E in this example.

Fig. 5 shows the same as Fig. 4, but blows it up ($0 \le x \le 0.5$ and $0 \le t \le 1.5$). In this figure we observe that E increases with x almost linearly at small x. In either figure, contours of E are parallel either to the t axis (zone of fetch-limited growth) or to the x axis (zone of duration-limited growth).

Characteristic curves are parallel to one another; the fetch-limited family are obtained by translation of the CCC in the t direction, while the duration-limited family in the x direction.

In terms of t_{sd} defined by

$$t_{sd}(x,t) \equiv \min\{t, t_b(x)\} \tag{30}$$

we may rewrite (27) simply as

$$E(x,t) = \tanh x_b(t_{sd}(x,t)). \tag{31}$$

In other words. the degree of wave-growth, or E, observed at point (x,t) is determined by $t_{sd}(x,t)$. Let us call t_{sd} as the *substantial duration time*, which is the time for the characteristic curve passing (x,t) to reach (x,t) after it starts ($x = 0, {}^{\exists}t_0 > 0$) or (${}^{\exists}x_0 > 0, t = 0$). Fig. 5 shows that two thick lines express the characterstic curves passing F and D, respectively. The substantial duration t_{sd} of each point is denoted by the length of the dotted line. Because t_{sd} is the same for F and D, so is E in this example.

3.3 Approximation

Approximation helps us to understand better the global growth of wind-waves. We are concerned with (i) approximate growth at small fetches, (ii) approximate growth at large fetches, and (iii) global approximation of the CCC.

[1] Small fetch approximation

The short-fetch approximation is quite illustrating as follows. When $x \ll 1$, the fetch relation and the CCC are written explicitly as

$$\begin{cases} E_f(x) = \tanh x = x, \quad c_f(x) = x^{1/3} \\ t_b(x) = \displaystyle\int_0^x \frac{dx}{c_f(x)/2} = \int_0^x \frac{2\,dx}{x^{1/3}} = 3x^{2/3} \\ x_b(t) = \left(\dfrac{t}{3}\right)^{3/2} \end{cases} . \tag{32}$$

See Appendix to confirm that (32) is also a direct approximation to the exact global formula (26) for $x \ll 1$. The corresponding distribution of $E(x,t)$ becomes

$$\begin{aligned} E(x,t) &= \tanh(\min\{x, x_b(t)\}) \approx \min\{x, x_b(t)\} \\ &= \begin{cases} x & \text{if} \quad t \ge 3x^{2/3} \\ (t/3)^{3/2} & \text{if} \quad t \le 3x^{2/3} \end{cases} \end{aligned}$$

The characteristic curves are

$$\begin{cases} t = t_0 + t_b(x) = t_0 + 3x^{2/3} \\ \qquad \text{fetch-limited growth} \\ .t = t_b(x - x_0) = 3(x - x_0)^{2/3} \\ \qquad \text{duration-limited grown} \end{cases} .$$

Alternatively they are expressed as

$$\begin{cases} x = x_b(t - t_0) = ((t - t_0)/3)^{3/2} \\ \qquad \text{fetch-limited growth} \\ x = x_0 + x_b(t) = x_0 + (t/3)^{3/2} \\ \qquad \text{duration-limited zone} \end{cases} .$$

We observe how these approximations are valid in Fig. 5.

[2] Large fetch approximation

We first approximate $t_b(x)$ and $x_b(t)$ at large fetches $x \gg 1$ as is shown in Appendix:

$$\begin{cases} t_b(x) = 2x + \dfrac{\sqrt{3}\pi}{6} + \log\dfrac{3\sqrt{3}}{4} \\ x_b(t) = \dfrac{t}{2} - \dfrac{\sqrt{3}\pi}{12} + \dfrac{1}{2}\log\dfrac{3\sqrt{3}}{4} \end{cases} . \tag{33}$$

The characteristic curves are straight lines with a slope $dx/dt = c/2 = 1/2$, at large duration. Then we have

$$\begin{aligned} E(x,t) &= \tanh(\min\{x, x_b(t)\}) \\ &= \begin{cases} \tanh x & \text{if } t \ge t_b(x) \\ \tanh\left[\dfrac{t}{2} - \dfrac{\sqrt{3}\pi}{12} + \dfrac{1}{2}\log\dfrac{3\sqrt{3}}{4}\right] & \text{if } t \le t_b(x) \end{cases} . \end{aligned}$$

Fig. 6 examines the degree of approximation of the CCC (t_b or $x_b(t)$) for short fetches (32) and long fetches (33) with the exact one (26). The figure shows that they are good approximations in their proper ranges of validity of x or t.

[3] Approximation of $x_b(t)$ as an explicit function

While $t_b(x)$ is expressed as an explicit analytical function of x, the exact $x_b(t)$ is not; $x_b(t)$ is defined simply as the inverse of $t_b(x)$. It is convenient therefore if there is an explicit expression of $x_b(t)$.

At short and long duration, $x_b(t)$ is approximated by explicit functions of t. Though each of these approximations is valid either for short or for long duration, they are used to make a global approximation to $x_b(t)$ as an explicit function of t as follows.

$$x_b(t) \approx (t/3)^{3/2}\mathrm{sech}^2(t/3)^3 + \left(\frac{t}{2} - \frac{\sqrt{3}\pi}{12} + \frac{1}{2}\log\frac{3\sqrt{3}}{4}\right)\tanh^2(t/3)^3. \quad (34)$$

It is an average of short-duration approximation and long-duration approximation with the weight of $\mathrm{sech}^2(t/3)^3$ and $\tanh^2(t/3)^3 = 1 - \mathrm{sech}^2\,(t/3)^3$.

If necessary or desirable, one may use a similar approximation to $t_b(x)$ as

$$t_b(x) \approx 3x^{2/3}\mathrm{sech}^2 x^2 + \left(2x + \frac{\sqrt{3}\pi}{6} + \log\frac{3\sqrt{3}}{4}\right)\tanh^2 x^2. \quad (35)$$

Fig. 7 shows comparison of the global approximate $x_b(t)$ (34) and $t_b(x)$ (35) with the exact $x_b(t)$ as the inverse of $t_b(x)$. Difference among them is almost invisible in this illustration. That is, (34) and (35) are quite good global approximations to (26).

4. Discussion

Main results of the paper have been described in the previous two sections. Before conclusion, several miscellaneous topics are discussed that would supplement our subject of the idealized growth of wind-waves.

[1] Flux form of model equation

The advective form of (20) is rewritten in a flux form:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left(\frac{E^{4/3}}{2}\right) = \tilde{S}^{(E)} = \frac{E^{1/3}(1-E^2)}{3/2}, \quad (36)$$

where $\tilde{S}^E(E)$ is the source function for the flux form. We see $\tilde{S}^{(E)} = \frac{4}{3}S^{(E)}$; the difference of the factor is due to the growth of $c = E^{1/3}$ with x. Such different source terms between advective and flux formulations do not occur for ordinary spectral models of deep water, but becomes important even for spectral models for coastal areas, where the group velocity of each wavenumber component depends on the (finite) depth.

[2] Tohoku wave model

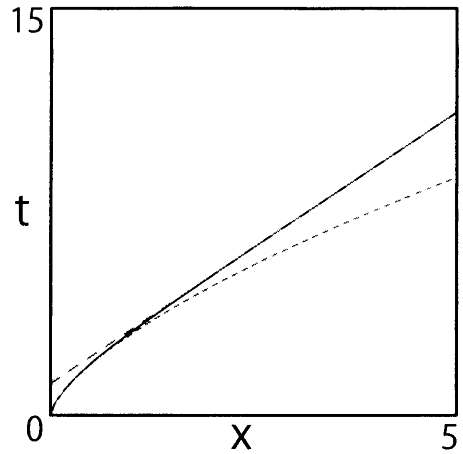

Fig. 6 The CCC (critical characteristic curve) $t_b(x)$ (solid line) compared with its two approximations: dotted line for short fetches and dashed line for large fetches. For the functional forms see text. The abscissa is $0 \leq x \leq 5$ and ordinate $0 \leq t \leq 15$.

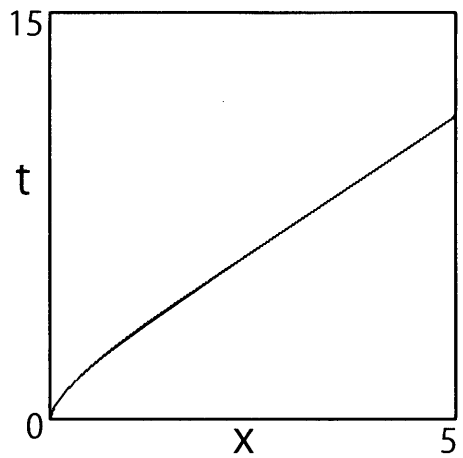

Fig. 7 The same as Fig. 5 except that the approximations cover the whole range of x, t: solid line for exact $t_b(x)$, dashed for $t_b(x)$, and dotted for $x_b(t)$. They are almost indistiuguishable in this illustration. For the functional forms see text.

The Tohoku wave model was developed by Toba et al. (1985) and was used as a routine of JMA forecast for a while. In a sense, it is a numerical version of the SMB method that is equipped with the 3/2-power law of Toba as its essential module. In this model the equation to be solved is

$$\frac{\partial E_*^{2/3}}{\partial t_*} + \frac{E_*^{1/3}}{0.74}\frac{\partial E_*^{2/3}}{\partial x_*} = 2.4 \times 10^{-4}\mathrm{cerf}(0.12E_*^{1/3}), \qquad (37)$$

where

$$\begin{cases} E_* \equiv \dfrac{g^2 H_s^2}{16u_*^4} = \dfrac{g^2 E}{u_*^4} \\ t_* \equiv \dfrac{gt}{u_*}, \qquad x_* \equiv \dfrac{gx}{u_*^2} \end{cases}$$

and

$$\mathrm{cerf}(q) \equiv \frac{2}{\sqrt{\pi}}\int_q^\infty \mathrm{e}^{-q^2/2}\,dq.$$

Conceptually (37) is almost the same as the present one (20). There are a few apparent differences though. First, the unknown is $E^{2/3}$, not E. Second, it is nondimensionalized by u_* and g, while ours is normalized by the fully developed values. Third, its source function is expressed by the error function, not an algebraic one.

The second difference above is related with the third as follows. The source function assumed in the Tohoku wave model allows a boundless growth of wind-waves for the given wind speed. Though in quite a slow pace, wind-waves grow infinitely, as is confirmed easily. Because there is no upper bounds, no normalization by saturation values is supposed. The third is the largest difference between the Tohoku wave model and ours (or Wilsons's formulas I and IV). As mentioned before, however, we have no reliable information about the growth at large fetches. It is a problem to be solved, though the presence of upper bounds is the prevailing opinion so far as the author knows.

[3] Evolution equation for wave momentum

The Tohoku wave model uses an equation for $E^{2/3}$. That may appear curious or artificial. It becomes plausible, however, if we consider wave-momentum M instead of wave energy E. As is well known the momentum contained by wind-waves as a whole is expressed as

$$M \equiv {}^{\exists}\alpha\frac{E}{c} = \alpha E^{2/3}$$

in our notation, where α is an adjustable factor by which to express the total wave momentum in terms of c and E. For convenience we put $\alpha = 1$, as has been assumed so far.

Then the equilibrium of wave momentum of wind-waves becomes

$$\frac{\partial M}{\partial t} + \frac{E^{1/3}}{2}\frac{\partial M}{\partial x} = S^{(M)},$$

where $S^{(M)}$ is the *net* input of wave momentum. That is, we have

$$\frac{\partial E^{2/3}}{\partial t} + \frac{E^{1/3}}{2}\frac{\partial E^{2/3}}{\partial x} = S^{(M)} \qquad (38)$$

which agrees with the Tohoku wave model formally with a suitable choice of adjustment factors. These equations for $M = E^{2/3}$ can be transformed to those for E or vice versa. For instance, the present equation (20) for E may be rewritten as the equation for $E^{2/3}$ as well.

In order for the equation of $M = E^{2/3}$ (38) to be consistent with (20) of E, we should put

$$S^{(M)} = \frac{1-E^2}{3}.$$

It is to be remarked that the net input of wave-momentum $S^{(M)}$ is almost constant for $x \ll 1$, irrespective of details of fetch relations so far argued.

[4] Momentum and energy input

Let $I^{(E)}$ and $I^{(M)}$ be the energy and momentum gain of wind-waves from wind, respectively. Then $L^{(E)} \equiv I^{(E)} - S^{(E)}$ and $L^{(M)} \equiv I^{(M)} - S^{(M)}$ mean the energy and momentum loss from wind-waves to the other forms through breaking, say.

Since energy and momentum transfer is a central problem of air-sea interction, let us investigate how I and L vary with fetch under some assumptions: (1) $S^{(E)} \sim cS^{(M)}$, $I^{(E)} \sim cI^{(M)}$, $L^{(E)} \sim cL^{(M)}$, (2) $S^{(E)} \sim E^{1/3}(1-E^2)$ (present model), and (3) $I^{(E)}$ is expressed as either

$$I^{(E)} \propto \omega \cdot \left(\frac{U^2}{c^2}\right) E \sim \frac{E}{c^3} \qquad (39)$$

or

$$I^{(E)} \propto \omega \cdot \left(\frac{U}{c}\right) E \sim \frac{E}{c^2}, \qquad (40)$$

where $\omega = 2\pi/T = g/c$, and variables except for c and E are omitted in the right hand side. As regards $I^{(E)}$, two types of empirical formual have been propsed on the basis of dimensional argument and observation. One is like (39) due to, say, Mitsuyasu and Honda (1982) and the other is like (40), which is similar to that by Komen et al. (1984).

If we use our fetch relation (22) we have

$$\begin{cases} I^{(E)} \sim \dfrac{E}{c^3} = {}^{\exists}i_1, \quad L^{(E)} \sim i_1 - E^{1/3}(1-E^2) \\ I^{(M)} \sim \dfrac{E}{c^4} \sim \dfrac{{}^{\exists}i_2}{E^{1/3}}, \quad L^{(M)} \sim \dfrac{i_2}{E^{1/3}} - (1-E^2) \end{cases}$$

for (39), where i_1 and i_2 are constants. In this case both momentum gain and loss become singular for $x \to 0$. On the other hand for (40) we have

$$\begin{cases} I^{(E)} \sim \dfrac{E}{c^2} = {}^{\exists}i_3 E^{1/3}, \quad L^{(E)} \sim E^{1/3}(i_3 - (1-E^2)) \\ I^{(M)} \sim \dfrac{E}{c^3} = {}^{\exists}i_4, \quad L^{(M)} \sim i_4 - (1-E^2) \end{cases},$$

i_3 and i_4 being constants. It is notable that in this case $I^{(M)}$, $S^{(M)}$, and $L^{(M)}$ are constant for $x \ll 1$ and for $x \gg 1$. That is, a fixed portion of momentum transfer from wind to sea is partitioned by wind-waves and the others. It should be born in mind, however, that underlying assumptions are rather dubious and one must be careful about their consequences.

[5] $E(x,t)$ for a general fetch relation

We have been concerned with a simple, concrete form of fetch relation which allows analytic and explicit representation of $t_b(x)$ and $E(x,t)$ in this paper. If analytical integration is set aside, a further general solution is obtained as follows.

Let $E_f(x) \geq 0$ be any form of fetch relation such that $E_f(0) = 0$ and $\frac{dE_f}{dx} > 0$. Then defining

$$\begin{cases} t_b(x) \equiv 2\int_0^x \frac{dx}{E_f^{1/3}(x)} \\ x_b(t) \equiv \text{ inverse of } \ t_b(x) \\ t_{sd}(x,t) \equiv \min\{t, t_b(x)\} \end{cases},$$

we easily confirm that

$$E(x,t) = E_f(x_b(t_{sd}(x,t))) \tag{41}$$

satisfies (20). Alternatively we may rewrite it as

$$E(x,t) = \min\{E_f(x), E_f(x_b(t))\}. \tag{42}$$

We observe that wave growth is determined by the substantial duration $t_{sd}(x,t)$ in (41) and *fetch* or *duration* according as (x,t) in (42).

5. Summary

A model of idealized growth of wind-waves is proposed under several conditions and assumptions. In particular, a fetch relation is assumed with a form of modified Wislon's formula I, where variables are normalized by a single scale of fectch and upper bounds of wind-waves. The model allows analytical formulas that describe explicitly the growth of wind-waves, both duration-limited and fetch-limited. The illustration based on the analytical formulas explain the fetch-limited and duration-limited growth clearly.

The terminology of *duration-limited* and *fetch-limited* growth has been conventional. However, the degree of wave growth seems to be better understood in terms of the *substantial duration* t_{sd}, whick yields a unified description of wave-growth.

In this paper, related topics are discussed as well: fetch relations at large fetches, normalized and approximate formulas of fetch relations, momentum and energy input, and so on. These subjects are not fully explored here, awaiting further research.

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Appendix

A1. Determination of E_∞, c_∞, x_s

In our model of fetch relation (7), variables are normalized by H_∞, T_∞, and x_s. They are determined naturally for Wilson's formulas I or IV, as we saw in Section 2. Generally speaking, however, more recent observation may give refined or more accurate empirical formulas. Then we have to choose adequate H_∞, T_∞, and x_s for new formulas.

So let us summarize how to determine three unknowns H_∞, T_∞, and x_s. in different forms of fetch-relation (7) with a prescirbed functional form f.

The fetch relations of H and T at $x \ll x_s$ give two conditions. The upper bound of T yields one condition of T_∞ (equivalent to c_∞); the last condition may be replaced by H_{infty}.

In any case these three conditions are sufficient for determining the unknowns. We first require a short-fetch relation

$$\frac{gH}{U^2} = a\left(\frac{gx}{U^2}\right)^{1/2}, \qquad \frac{gT}{U} = b\left(\frac{gx}{U^2}\right)^{1/3} \tag{A1}$$

like Mitsuyasu's formulas, where a and b are constants. Then Toba's constant is determined as

$$\frac{1}{(gU)^{1/2}}\frac{H}{T^{3/2}} = \frac{a}{b^{3/2}}. \tag{A2}$$

Further a certain upper bound of wave growth should be given. Let it be, say,

$$\frac{gH_\infty}{U^2} = {}^{\exists}h \tag{A3}$$

like (5), where h is a constant. It follows that

$$\frac{gT_\infty}{U} = \frac{h^{2/3}b}{a^{2/3}} \tag{A4}$$

from (A2) and (A4). Rewriting the first of (A1) as

$$\frac{H}{H_\infty} = \frac{gH}{U^2}\frac{U^2}{gH_\infty} = \frac{a}{h}\left(\frac{gx}{U^2}\right)^{1/2} = \left(\frac{gx}{U^2}\frac{U^2}{gx_s}\right)^{1/2} \tag{A5}$$

we obtain the nondiminesional fetch scale as

$$\frac{gx_s}{U^2} = \frac{h^2}{a^2}. \tag{A6}$$

A2. Approximation of $t_b(x)$ at $x \ll 1$ and $x \gg 1$

Here the exact formula (26) for

$$t_b(x) = \sqrt{3}\left(\arctan\frac{2\xi+1}{\sqrt{3}} - \frac{\pi}{6}\right) - \frac{1}{2}\log\frac{(1-\xi)^2}{(1+\xi+\xi^2)}.$$

is approximated for $x \ll 1$ and $x \gg 1$.

First let $x \ll 1$. Since $\xi = \tanh^{2/3}x \approx x^{2/3}$, each term of the exact formula is approximated to be

$$\begin{aligned}(1): \sqrt{3}\left(\arctan\frac{2\xi+1}{\sqrt{3}} - \frac{\pi}{6}\right) &= \sqrt{3}\left[\arctan\frac{2\xi+1}{\sqrt{3}}\right]_0^\xi \\ &\approx \sqrt{3}\xi\cdot\left[\frac{d}{d\xi}\arctan\frac{2\xi+1}{\sqrt{3}}\right]_{\xi=0} = \frac{3\xi}{2} \\ &\approx \frac{3x^{2/3}}{2} \qquad \left(\frac{d(\arctan y)}{dx} = \frac{1}{1+y^2}\right)\end{aligned}$$

$$(2): -\frac{1}{2}\log(1-\xi)^2 \approx \xi \approx x^{2/3}$$

$$(3): \frac{1}{2}\log(1+\xi+\xi^2) \approx \frac{\xi}{2} \approx \frac{x^{2/3}}{2}.$$

Summation of (1) to (3) yields

$$t_b(x) \approx 3x^{2/3}. \tag{A7}$$

Next let $x \gg 1$, where $\xi = \tanh^{2/3}x \approx 1$. Since

$$\frac{d\xi}{dx} = \frac{2}{3}\frac{\operatorname{sech}^2 x}{\tanh^{1/3}x} \approx \frac{8}{3}e^{-2x},$$
$$1-\xi \approx \frac{4}{3}e^{-2x},$$

we have

$$(1): \sqrt{3}\left(\arctan\frac{2\xi+1}{\sqrt{3}} - \frac{\pi}{6}\right) \approx \frac{\sqrt{3}\pi}{6}$$

$$(2): -\frac{1}{2}\log(1-\xi)^2 = -\log(1-\xi) \approx 2x - \log\frac{4}{3}$$

$$(3): \frac{\log(1+\xi+\xi^2)}{2} \approx \log\sqrt{3}.$$

Summing up (1) to (3) we find

$$t_b(x) \approx 2x + \frac{\sqrt{3}\pi}{6} + \log\frac{3\sqrt{3}}{4}. \tag{A8}$$