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A supplementary note to GFDVN: Complex representation of two-dimensional real vectors

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Abstract

In a previous article, GFDVN was proposed, which is a convenient way of vector notation for geophysical fluid dynamics. This short note supplements the previous article of GFDVN with a systematic method for the complex representation of two-dimensional real vectors. First, intimate relationships and correspondence rules are summarized between complex numbers and two-dimensional real vectors on a plane. Most of useful GFDVN are expressed by arithmetic of complex numbers as well. Examples are presented to show that complex representation is often easier to handle with than GFDVN or traditional ones. In particular, *strophe* operator $\neg$, which rotates a vector clockwise at a right angle, is expressed simply as the multiplication of $-i$, the imaginary unit. Likewise two-dimensional Lagrange's formula for triple vector product is proved by a straightforward arithmetic way of complex numbers by virtue of correspondence rules. In addition, complex representation turns out to give a concise expression of vector operations and trigonometric identities.

Key words : GFDVN *(geophysical fluid dynamics vector notation), two-dimensional real vector, complex representation, Lagrange's formula*

1. Introduction

In the ocean or atmosphere, the horizontal dimensions are quite different from the vertical one. The horizontal current velocity is much larger than the vertical velocity, indeed. Consequently most vector operations are made on the horizontal plane. In a previous article (Masuda 2010)[1] therefore a special vector notation was proposed that is designed for geophysical fluid dynamics. It was called *geophysical fluid dynamics vector notation*, or GFDVN for brevity. This notation allows us a much simpler description of geophysical fluid dynamics than traditional ones. It is not only convenient, but also provides vivid images of vector properties.

In Masuda (2010)[1], however, a few subjects were not discussed at all that are related intimately with GFDVN and quite useful in practice. One of them is the complex representation of two-dimensional real vectors. This short note is intended to supplement the previous article of GFDVN by enumerating and discussing formulas that are relevant to the complex representation of two-dimensional real vectors.

Complex representation of two-dimensional real vectors itself is not necessarily new, though. It appears widely either in ordinary fluid mechanics or in geophysical fluid dynamics. Almost every textbook of fluid mechanics instructs the potential theory of irrotational and incompressible flow based on complex functions (Lamb 1932, Landau and Lifshits 1959, Batchelor 1967, e.g.) [2][3][4]; The method has been used, however, separately or independently, in an *ad hoc* manner as in Masuda (2007)[5], where harmonic analysis of tidal currents and Ekman spirals are discussed by complex representation. The aim of this note is to present this powerful method in a fairly systematic form, starting with Euler's formula and extending to a wider area of applications.

Next section describes the correspondence between complex numbers and two-dimensional real vectors. During this process, several trigonometric identities are derived visually. Then, we translate most of useful GFDVN to their complex expression. In the third section, examples and applications are presented to show the utility of complex representation. In particular Lagrange's formula for triple vector product is derived by a purely arithmetic way of complex numbers. Complex representation sheds light on mutual relations between complementary quantities (such as $\boldsymbol{u}$ vs. $\neg\boldsymbol{u}$) or operations (such as divergence vs. rotation). Containing both of complementary components as the real and imaginary parts, complex representation turns out to provide

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a concise expression. The last section gives a summary.

2. Two-dimensional real vectors and complex numbers

As is well known, two-dimensional *real* vectors have one-to-one correspondence with complex numbers. In this section we list up and summarize their correspondence rules. Also simple and visual derivations of several trigonometric identities are presented in relation to Euler's formula.

First of all, fundamental notations of GFDVN are reviewed; see Masuda (2010)[1)] for details. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the set of all the real numbers and the set of all the complex numbers, respectively. Also $\boldsymbol{R}^2$ means the set of all the two-dimensional real vectors or a plane. We express a two-dimensional real vector $\in \boldsymbol{R}^2$ by a boldface as $\boldsymbol{u}$, a three-dimensional real vector $\in \boldsymbol{R}^3$ by a boldface with an underline as $\underline{\boldsymbol{u}}$, and a complex number $\in \boldsymbol{C}$ (or a real number $\in \boldsymbol{R}$) by an italic as u. A vector usually means a column vector of corresponding components. For a vector or a matrix denoted as A, A' denotes the transpose of A. There are two characteristic operators in GFDVN: one is $\neg$ (*strophe* or *turn*), which rotates a vector $\in \boldsymbol{R}^2$ clockwise at a right angle. The other is $\lhd \equiv \neg\nabla$, which is called *blana* or *alongent*.

2.1 Complex numbers, Euler's formula and trigonometric identities

A vector $\boldsymbol{a} \in \boldsymbol{R}^2$ is designated by two real numbers a_x and a_y as $\boldsymbol{a} = (a_x, a_y)'$, where a_x and a_y are called the x and y component, respectively. To $\boldsymbol{a}$ there corresponds a complex number $a \in \boldsymbol{C}$ with $\Re[a] = a_x$ and $\Im[a] = a_y$, where $\Re[a]$ and $\Im[a]$ indicate the real and imaginary part of a, respectively. Symbolically we may write these relations as

$$\boldsymbol{R}^2 \ni \boldsymbol{a} = (a_x, a_y)' \models a = a_x + ia_y \in \boldsymbol{C}, \tag{1}$$

where $i \equiv \sqrt{-1}$ is the imaginary unit. In this expression "$A \models B$" means "A is an alternative representation of B or vice versa". In the case of (1), the left-hand side denotes a two-dimensional real vector $\boldsymbol{a}$, while the right-hand side is a complex number a corresponding to $\boldsymbol{a}$. In other word, $a \in \boldsymbol{C}$ is another expression of $\boldsymbol{a} \in \boldsymbol{R}^2$. Using this convention we have

$$\begin{cases} \boldsymbol{C} \ni 1 \models \hat{\boldsymbol{x}} \in \boldsymbol{R}^2 \\ \boldsymbol{C} \ni i \models \hat{\boldsymbol{y}} \in \boldsymbol{R}^2 \end{cases},$$

where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ denote the unit vectors parallel to the x-axis and y-axis of the horizontal plane, respectively. Thus the plane $\boldsymbol{R}^2$ is identified with the so-called complex plane $\boldsymbol{C}$.

The most useful in this note is the well-known Euler's formula

$$\mathrm{e}^{it} = \cos t + i \sin t \qquad (t \in \boldsymbol{R}), \tag{2}$$

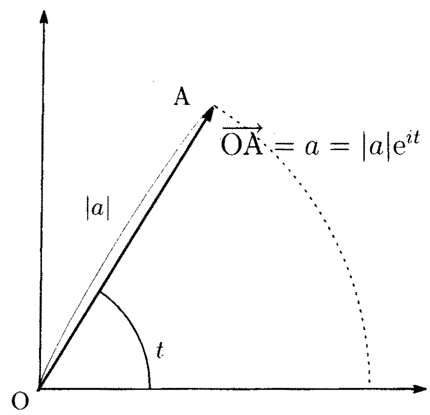

Fig. 1 Polar representation of a complex number a: $|a|$ is the absolute value and $t = \arg(a)$ the argument or the angle of the segment OA to the real axis. Euler's formula yields $a = |a|\mathrm{e}^{it} = |a|\mathrm{e}^{i\arg(a)}$.

which will be elaborated on soon. This formula yields a polar representation of a complex number $a \in \boldsymbol{C}$ as

$$\begin{aligned} a &= |a|\mathrm{e}^{i\arg(a)} \\ &= |a|\cos[\arg(a)] + i|a|\sin[\arg(a)], \end{aligned} \tag{3}$$

where $|a| \in \boldsymbol{R}$ is the absolute value of a and $\arg(a) \in \boldsymbol{R}$ is the argument of a (Fig. 1). It shows that a complex number is characterized by its magnitude (or length) and direction, just as a two-dimensional real vector. Then

$$\bar{a} \equiv a_x - ia_y = |a|\mathrm{e}^{-i\arg(a)} \in \boldsymbol{C} \tag{4}$$

is called the complex conjugate of a.

If the polar representation of complex numbers is combined with vector operations, several useful and familiar formulas of trigonometric functions are derived easily. We may summarize the equations relevant to the subject in concern as

$$(a.1): \ \frac{d}{dt}\mathrm{e}^{it} = i\mathrm{e}^{it} \tag{5}$$

$$(a.2): \ \mathrm{e}^{it} = \lim_{n\to\infty}\left(1 + i\frac{t}{n}\right)^n \tag{6}$$

$$(b): \ \mathrm{e}^{i(\alpha\pm\beta)} = \mathrm{e}^{i\alpha}\cdot\mathrm{e}^{\pm i\beta} \tag{7}$$

$$(c.1): \ \mathrm{e}^{i\alpha} + \mathrm{e}^{i\beta} = 2\cos\frac{\alpha-\beta}{2}\,\mathrm{e}^{i\frac{\alpha+\beta}{2}} \tag{8}$$

$$(c.2): \ \mathrm{e}^{i\alpha} - \mathrm{e}^{i\beta} = 2\sin\frac{\alpha-\beta}{2}\,i\,\mathrm{e}^{i\frac{\alpha+\beta}{2}}, \tag{9}$$

where t, α, and β are real numbers (arguments).

Figure 2 illustrates (a) showing that the derivative of e^{it} with respect to t is $i\mathrm{e}^{it}$. At this stage we need no knowledge about the exponential function e^{it}; (2) is considered as a definition of e^{it} by the right-hand side.

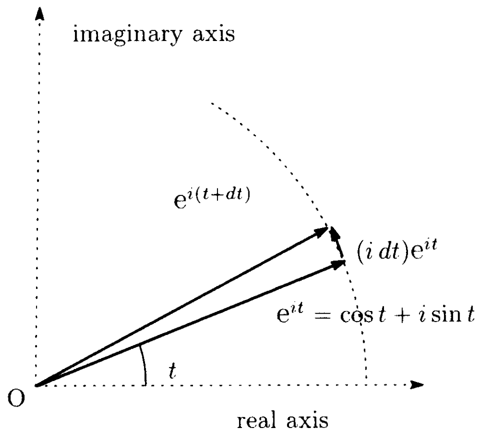

Fig. 2 Diagram of the derivative of e^{it}.

If we put $de^{it} \equiv e^{i(t+dt)} - e^{it}$,

$$\begin{cases} \left|de^{it}\right| \approx |dt| \\ \arg\left(de^{it}\right) \approx t + \dfrac{\pi}{2}, \end{cases}$$

as is evident from Fig. 2. Then we have

$$d(e^{it}) \approx dt\, e^{i\left(t+\frac{\pi}{2}\right)} = idt\, e^{it}$$

from (3). It follows immediately that

$$\frac{d(e^{it})}{dt} \approx ie^{it} \Rightarrow \frac{d}{dt}e^{it} = ie^{it},$$

namely (a.1).

From (2) and (a.1) we have

$$\begin{aligned} \frac{d(\cos t)}{dt} + i\frac{d(\sin t)}{dt} &= \frac{d}{dt}e^{it} = ie^{it} \\ &= i(\cos t + i\sin t) = -\sin t + i\cos t, \end{aligned}$$

so that

$$\begin{cases} \dfrac{d}{dt}\cos t = -\sin t \\ \dfrac{d}{dt}\sin t = +\cos t \end{cases} . \qquad (10)$$

See Appendix for a more intuitive derivation.

Then (a.2) is derived as follows. Let n be a large integer and k be an integer such that $0 \le k \le n$, so that $t = \dfrac{kt}{n} + \dfrac{n-k}{n}t$. Figure 2 together with (a.1) shows that $e^{i(t+dt)} = \left(1 + idt + O(dt^2)\right)e^{it}$, where $O(\bullet)$ denotes the order. Putting $dt = \dfrac{t}{n}$, we have

$$\begin{aligned} e^{it} &= \left(1 + i\frac{t}{n} + O\left(\frac{1}{n^2}\right)\right)e^{i\frac{(n-1)t}{n}} \\ &= \left(1 + i\frac{t}{n} + O\left(\frac{1}{n^2}\right)\right)^2 e^{i\frac{(n-2)t}{n}} \\ &= \cdots = \left(1 + i\frac{t}{n} + O\left(\frac{1}{n^2}\right)\right)^n e^{i0} \\ &= \left(1 + i\frac{t}{n} + O\left(\frac{1}{n^2}\right)\right)^n \to \left(1 + i\frac{t}{n}\right)^n, \end{aligned}$$

which yields (a.2) in accordance with the ordinary definition of the exponential function for a real argument.

We may admit (b) as a natural consequence of the *exponential* function. It is derived easily, however, from (a.2) as follows:

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= \lim_{n\to\infty}\left(1 + i\frac{\alpha}{n}\right)^n\left(1 + i\frac{\beta}{n}\right)^n \\ &= \lim_{n\to\infty}\left(1 + i\frac{\alpha+\beta}{n} + O\left(\frac{1}{n^2}\right)\right)^n \\ &= \lim_{n\to\infty}\left(1 + i\frac{\alpha+\beta}{n}\right)^n \\ &= e^{i(\alpha+\beta)}, \end{aligned}$$

where we have omitted details of limiting procedures.

Then it follows from (2) and (b) that

$$\begin{aligned} &\cos(\alpha \pm \beta) + i\sin(\alpha \pm \beta) \\ &\quad = e^{i(\alpha\pm\beta)} = e^{i\alpha}e^{i\pm\beta} \\ &\quad = (\cos\alpha + i\sin\alpha)(\cos\beta \pm i\sin\beta) \\ &\quad = (\cos\alpha\cos\beta \mp \sin\alpha\sin\beta) \\ &\qquad + i(\sin\alpha\cos\beta \pm \cos\alpha\sin\beta). \end{aligned}$$

Equating the real or imaginary part of both sides, we obtain the addition theorem of $\cos t$ or $\sin t$, respectively. See also Appendix for a geometrical derivation without using complex numbers.

The relations (c.1) and (c.2) are illustrated by Fig. 3. The left-hand side of (c.1) and (c.2) is the complex representation of $\overrightarrow{OC}$ and $\overrightarrow{BA}$, respectively. See Appendix for details and the resulting trigonometric identities.

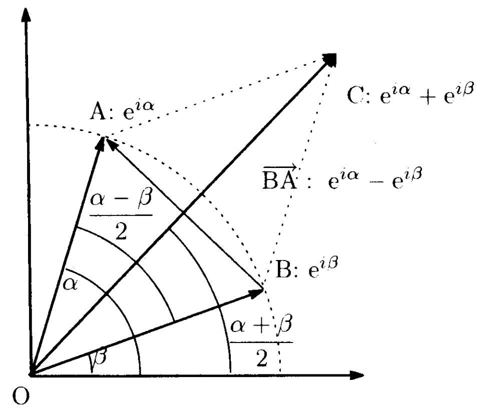

Fig. 3 Addition or subtraction of two unit vectors $e^{i\alpha}$ and $e^{i\beta}$ is expressed as (c.1) or (c.2), respectively.

2.2 Fundamental GFDVN in complex representation

The previous subsection provides only the one-to-one correspondence between $\boldsymbol{R}^2$ and $\boldsymbol{C}$. In this subsection we describe the correspondence (translation) of fundamental *operations* in $\boldsymbol{R}^2$ and those in $\boldsymbol{C}$. We want to clarify how fundamental operations in $\boldsymbol{R}^2$ such as scalar multiplication, vector addition, and inner/outer products are expressed in $\boldsymbol{C}$.

Let

$$\begin{cases} (\boldsymbol{R}^2 \ni)\ \boldsymbol{a} \models a\ (\in \boldsymbol{C}) \\ (\boldsymbol{R}^2 \ni)\ \boldsymbol{b} \models b\ (\in \boldsymbol{C}) \end{cases}.$$

That is, a (b) is the complex representation of the real two-dimensional vector $\boldsymbol{a}$ ($\boldsymbol{b}$).

We now write down the formulas that guarantee the correspondence of fundamental operations between $\boldsymbol{R}^2$ and $\boldsymbol{C}$:

$$(a):\ \begin{cases} \boldsymbol{a}+\boldsymbol{b} \models a+b \\ \lambda\boldsymbol{a} \models \lambda a \qquad (\lambda \in \boldsymbol{R}) \end{cases}, \tag{11}$$

$$(b):\ \begin{cases} \boldsymbol{a}\cdot\boldsymbol{b} = \Re[\bar{a}\,b] = \Re[a\,\bar{b}] = \dfrac{a\bar{b}+\bar{a}b}{2} \\ \boldsymbol{a}\times\boldsymbol{b} = \Im[\bar{a}\,b] = -\Im[a\,\bar{b}] = i\dfrac{a\bar{b}-\bar{a}b}{2} \end{cases}, \tag{12}$$

$$\Rightarrow \quad \bar{a}b = (\boldsymbol{a}\cdot\boldsymbol{b}) + i(\boldsymbol{a}\times\boldsymbol{b}) \in \boldsymbol{C} \tag{13}$$

$$= \boldsymbol{a}\cdot(\boldsymbol{b}+i\neg\boldsymbol{b}), \tag{14}$$

$$(c):\ \|\boldsymbol{a}\| = |a|, \tag{15}$$

$$(d):\ \boldsymbol{a}\cdot\hat{\boldsymbol{x}} = \Re[a], \quad \boldsymbol{a}\cdot\hat{\boldsymbol{y}} = \Im[a], \tag{16}$$

$$(e):\ 2(\boldsymbol{a}\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} - \boldsymbol{a} \models \bar{a}. \tag{17}$$

The meaning is straightforward. Formula (a) indicates that addition of two vectors corresponds to addition of two corresponding complex numbers and that scalar multiplication of a vector corresponds to the product of the real number with the complex number.

Property (b) shows that inner and outer products of two vectors are expressed by the multiplication of the corresponding complex numbers, one of them being the complex conjugate. It follows, in particular, that

$$\begin{cases} \boldsymbol{a}\perp\boldsymbol{b} \Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=0 \Leftrightarrow \Re(\bar{a}b)=0 \\ \boldsymbol{a}\parallel\boldsymbol{b} \Leftrightarrow \boldsymbol{a}\times\boldsymbol{b}=0 \Leftrightarrow \Im(\bar{a}b)=0 \end{cases}. \tag{18}$$

2.3 $\neg$, $\lhd$, and scalar fields

In GFDVN, emphasis is put on two characteristic operators: "strophe" $\neg$ and "blana" $\lhd = \neg\nabla$. Their utility is apparent, say, in the geostrophic balance, which is expressed simply by GFDVN as

$$0 = \neg f\boldsymbol{u} - \frac{\nabla p}{\rho} \quad \text{or} \quad f\boldsymbol{u} = -\frac{\lhd p}{\rho}, \tag{19}$$

where f is the Coriolis parameter, $\boldsymbol{u}$ horizontal current vector, p pressure, and ρ density. These vector operations should be translated to complex arithmetic.

Obviously we should put

$$\neg\boldsymbol{a} \models -ia \tag{20}$$

for any $\boldsymbol{R}^2 \ni \boldsymbol{a} \models a \in \boldsymbol{C}$. Just as $\neg$ turns a vector clockwise at right angle in $\boldsymbol{R}^2$, multiplication of $-i$ does the same in $\boldsymbol{C}$.

Operating $\neg$ twice reverses the direction of a vector. So does the double multiplication of $-i$ in $\boldsymbol{C}$:

$$-1 = (-\neg)^2 \models (-i)^2 = -1, \tag{21}$$

where the left-hand side is a vector operator in $\boldsymbol{R}^2$ and the right-hand side is a complex number multiplication in $\boldsymbol{C}$. It is interesting and mnemonic that the operator "strophe" $\neg$ accidentally looks like $-i$.

Then we turn to $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)'$ and $\lhd = \left(\dfrac{\partial}{\partial y}, -\dfrac{\partial}{\partial y}\right)'$. Let $p = p(x,y)$ be a real scalar field on an x-y plane. We should put

$$\begin{cases} \nabla p \models \dfrac{\partial p}{\partial x} + i\dfrac{\partial p}{\partial y} \\ \lhd p \models \dfrac{\partial p}{\partial y} - i\dfrac{\partial p}{\partial x} \\ \quad = -i\left(\dfrac{\partial p}{\partial x} + i\dfrac{\partial p}{\partial y}\right) \models \neg\nabla p \end{cases}, \tag{22}$$

which shows that correspondence rules work consistently.

The relation above can be expressed in a somewhat different manner as follows. We first introduce a complex number z for the coordinate $\boldsymbol{x}$ of the plane through

$$\boldsymbol{R}^2 \ni \boldsymbol{x} = (x,y)' \models z = x+iy \in \boldsymbol{C}.$$

Next we define $\tilde{p}(z,\bar{z})$, which is the same as p, but as a function of z and $\bar{z}$ rather than x and y:

$$\tilde{p}(z,\bar{z}) \equiv p(x,y) = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right). \tag{23}$$

Using these convention for the real scalar field $p = p(x,y) = \tilde{p}(z,\bar{z})$ and the transformation rules

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} \\ \dfrac{\partial}{\partial y} = i\dfrac{\partial}{\partial z} - i\dfrac{\partial}{\partial \bar{z}} \end{cases}, \quad \begin{cases} 2\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} \\ 2\dfrac{\partial}{\partial \bar{z}} = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} \end{cases} \tag{24}$$

we have

$$\begin{cases} \nabla p \models \dfrac{\partial p}{\partial x} + i\dfrac{\partial p}{\partial y} = 2\dfrac{\partial \tilde{p}}{\partial \bar{z}} \\ \lhd p \models \dfrac{\partial p}{\partial y} - i\dfrac{\partial p}{\partial x} = -2i\dfrac{\partial \tilde{p}}{\partial \bar{z}} \end{cases} \tag{25}$$

Similarly one may define

$$\begin{cases} \overline{\nabla p} \models \dfrac{\partial p}{\partial x} - i\dfrac{\partial p}{\partial y} = 2\dfrac{\partial \tilde{p}}{\partial z} \\ \overline{\lhd p} \models \dfrac{\partial p}{\partial y} + i\dfrac{\partial p}{\partial x} = 2i\dfrac{\partial \tilde{p}}{\partial z} \end{cases}.$$

We should note that the overbar of ∇p or $\lhd p$ does not mean the complex conjugate, but a kind of *real conjugate* field.

As a derivative of p along a vector $\boldsymbol{u}$, we have

$$\begin{aligned} 2\left(\overline{u}\frac{\partial \tilde{p}}{\partial \overline{z}}\right) &= (u - iv)\left(\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}\right) \\ &= \boldsymbol{u}\cdot\nabla p + i(\boldsymbol{u}\times\nabla p) \qquad (26)\\ &= \boldsymbol{u}\cdot(\nabla p + i \lhd p), \end{aligned}$$

where $\boldsymbol{C} \ni u \models \boldsymbol{u} \in \boldsymbol{R}^2$

Finally we have a formula for the laplacian of p as

$$4\frac{\partial}{\partial z}\frac{\partial \tilde{p}}{\partial \overline{z}} = \nabla^2 p \qquad (27)$$

using (24). Also one may deduce it from (25) and (33) in Section 2.4, as

$$\nabla^2 p = \nabla\cdot(\nabla p) + i\nabla\times(\nabla p) = 4\frac{\partial}{\partial z}\frac{\partial \tilde{p}}{\partial \overline{z}}. \qquad (28)$$

2.4 Divergence, rotation, and vector fields

Let $\boldsymbol{u} = (u_x, u_y)'$ be a real vector field on an x-y plane and u be a complex scalar field on the same x-y plane such that

$$\boldsymbol{C} \ni u(x,y) \equiv u_x(x,y) + iu_y(x,y). \qquad (29)$$

Then let $\tilde{u}$ be a map from $\boldsymbol{C}$ to $\boldsymbol{C}$ such that

$$\begin{cases} z = x + iy, \quad \overline{z} = x - iy \\ \tilde{u}(z,\overline{z}) \equiv u(x,y) = u\left(\dfrac{z+\overline{z}}{2}, \dfrac{z-\overline{z}}{2i}\right) \end{cases}. \qquad (30)$$

In other words, $\tilde{u}(z,\overline{z})$ is the same complex function as $u(x,y)$ except that the former is a function of $(z,\overline{z})$ rather than (x,y); situation and notation is the same as in the previous subsection for a real scalar field of p.

Next we define a related *real* vector field

$$\overline{\boldsymbol{u}} \equiv (u_x, -u_y)' \models \overline{u} \qquad (31)$$

where $\overline{\boldsymbol{u}}$ does not mean the complex conjugate of $\boldsymbol{u}$, but $\overline{u}$ is the complex conjugate of u.

If (24) is used, it is easy to confirm

$$\begin{aligned} 2\frac{\partial \tilde{u}}{\partial z} &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + i\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \qquad (32)\\ &= \nabla\cdot\boldsymbol{u} + i\nabla\times\boldsymbol{u}, \qquad (33)\\ 2\frac{\partial \tilde{u}}{\partial \overline{z}} &= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} \\ &= \nabla\cdot\overline{\boldsymbol{u}} - i\nabla\times\overline{\boldsymbol{u}}. \qquad (34) \end{aligned}$$

Thus we have seen that fundamental operations in GFDVN are expressed by operations in complex numbers and vice versa.

3. Applications

3.1 Thermal-wind relation and veering

As the first application, let us see how thermal wind relation and vertical shear is expressed in complex representation.

Let $\boldsymbol{u} \in \boldsymbol{R}^2$ be a geostrophic current field and u its complex representation. Also we assume the hydrostatic approximation. Then we have

$$\begin{cases} \boldsymbol{C} \ni fu \models f\boldsymbol{u} = -\lhd p \in \boldsymbol{R}^2 \\ 0 = -\dfrac{\partial p}{\partial z} + b \end{cases}, \qquad (35)$$

where f denotes the (constant) Coriolis parameter, z the upward coordinate, b the buoyancy and p the pressure divided by water density (which is assumed almost constant for the Boussinesq approximation to be valid). Differentiating with respect to z we obtain

$$f\frac{\partial u}{\partial z} \models f\frac{\partial \boldsymbol{u}}{\partial z} = -\frac{\partial \lhd p}{\partial z} = -\lhd\frac{\partial p}{\partial z} = -\lhd b,$$

which yields the thermal-wind relation as

$$\nabla b = \neg(-\lhd b) \models -if\frac{\partial u}{\partial z}. \qquad (36)$$

with the use of (20).

In general, vertical change of $\|\boldsymbol{u}\|$ or $\arg(\boldsymbol{u}) \equiv \arg(u)$ becomes

$$\begin{cases} \dfrac{1}{|u|}\dfrac{\partial |u|}{\partial z} = \dfrac{1}{2|u|^2}\dfrac{\partial |u|^2}{\partial z} = \Re\left[\dfrac{\overline{u}}{|u|^2}\dfrac{\partial u}{\partial z}\right] \\ \dfrac{\partial[\arg(u)]}{\partial z} = \Im\left[\dfrac{\overline{u}}{|u|^2}\dfrac{\partial u}{\partial z}\right] \end{cases},$$

respectively. These formulas are further reduced to a single formula

$$\frac{\partial[\log u]}{\partial z} = \frac{1}{|u|}\frac{\partial |u|}{\partial z} + i\frac{\partial[\arg(u)]}{\partial z}, \qquad (37)$$

where

$$\begin{cases} \Re\left[\dfrac{\partial[\log u]}{\partial z}\right] \models \dfrac{\boldsymbol{u}}{\boldsymbol{u}^2}\cdot\dfrac{\partial \boldsymbol{u}}{\partial z} = -\dfrac{\boldsymbol{u}\times\nabla b}{f\boldsymbol{u}^2} \\ \Im\left[\dfrac{\partial[\log u]}{\partial z}\right] \models \dfrac{\partial[\arg(\boldsymbol{u})]}{\partial z} = +\dfrac{\boldsymbol{u}\cdot\nabla b}{f\boldsymbol{u}^2} \end{cases}. \qquad (38)$$

Thus complex representation represents a concise expression of the thermal-wind relation and vertical shear. The veering of ocean current with depth discussed in Masuda (2010)[1] or β-spiral in Stommel and Schott (1977) and Pedlosky (1996) [6) 7)] could have been derived more easily by complex representation.

3.2 Ekman spiral

Probably one of the most familiar examples of complex representation in geophysical fluid dynamics is the

Ekman spiral beneath the sea surface, which is governed by

$$\begin{cases} -f\neg\boldsymbol{u} = \nu\dfrac{\partial^2\boldsymbol{u}}{\partial z^2} & \text{for} \quad z \in (-\infty, 0) \\ \nu\dfrac{\partial\boldsymbol{u}}{\partial z} = \boldsymbol{\tau}_0 = \text{const} & \text{at} \quad z = 0 \\ \boldsymbol{u} \to 0 & \text{as} \quad z \to -\infty \end{cases}$$

where ν is a (constant) kinematic viscosity and $\boldsymbol{\tau}_0$ denotes the wind stress (divided by ρ_0) on the sea surface. Let

$$\boldsymbol{R}^2 \ni \boldsymbol{u} \models u \in \boldsymbol{C}$$
$$\boldsymbol{R}^2 \ni \boldsymbol{\tau}_0 \models \tau_0 \in \boldsymbol{C}.$$

Through the correspondence rules in Section 2, the governing equations are rewritten as

$$\begin{cases} ifu = \nu\dfrac{\partial^2 u}{\partial z^2} & \text{for} \quad z \in (-\infty, 0) \\ \nu\dfrac{\partial u}{\partial z} = \tau_0 \models \boldsymbol{\tau}_0 & \text{at} \quad z = 0 \\ u \to 0 & \text{as} \quad z \to -\infty \end{cases}$$

which is easily solved to yield the Ekman spiral

$$u = \sqrt{\frac{1}{2f\nu}}(1-i)\tau_0 \mathrm{e}^{\sqrt{\frac{f}{2\nu}}(1+i)z},$$

whence

$$\frac{\partial[\log u]}{\partial z} = \sqrt{\frac{f}{2\nu}}(1+i).$$

Using (37) and (38) we obtain

$$\left.\begin{cases} \dfrac{1}{\|\boldsymbol{u}\|}\dfrac{\partial[\|\boldsymbol{u}\|]}{\partial z} = \Re\left[\dfrac{\partial[\log u]}{\partial z}\right] \\ \dfrac{\partial[\arg(\boldsymbol{u})]}{\partial z} = \Im\left[\dfrac{\partial[\log u]}{\partial z}\right] \end{cases}\right\} = \sqrt{\frac{f}{2\nu}} > 0.$$

3.3 Gauss' theorem and Stokes' theorem

Guass' theorem and Stokes' theorem are written as

$$\begin{cases} \displaystyle\int_\Omega \nabla\cdot\boldsymbol{u}\,d\boldsymbol{x} = \oint_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}\,ds = \oint_{\partial\Omega} \boldsymbol{u}\times d\boldsymbol{s} \\ \displaystyle\int_\Omega \nabla\times\boldsymbol{u}\,d\boldsymbol{x} = \oint_{\partial\Omega} \boldsymbol{u}\cdot d\boldsymbol{s} \end{cases}, \tag{39}$$

respectively, where $d\boldsymbol{x} \equiv dx\,dy$ denotes the area element (not a vector, but a scalar), $d\boldsymbol{s}$ the line element vector, $ds \equiv \|d\boldsymbol{s}\|$ is the length of the line element, Ω a region in $\boldsymbol{R}^2$, and $\partial\Omega$ its boundary (Fig. 4). The line integral goes around Ω counterclockwise, and $\boldsymbol{n} = \neg\dfrac{d\boldsymbol{s}}{ds}$ is the outward normal unit vector on $\partial\Omega$.

As before we define

$$\begin{cases} \boldsymbol{R}^2 \ni \boldsymbol{x} = (x,y)' \models z = x + iy \in \boldsymbol{C} & \text{coordinates} \\ \boldsymbol{R}^2 \ni \boldsymbol{u} = (u_x, u_y)' \models u = u_x + iu_y & \text{current field} \\ \boldsymbol{R}^2 \ni \overline{\boldsymbol{u}} \equiv (u_x, -u_y)' \models \overline{u} \in \boldsymbol{C} \end{cases}.$$

Then, by virtue of the correspondence rules, we obtain

$$\oint_{\partial\Omega} \overline{u}\cdot dz = \oint_{\partial\Omega} (\boldsymbol{u}\cdot d\boldsymbol{s} + i\boldsymbol{u}\times d\boldsymbol{s}) \tag{40}$$

$$= \int_\Omega (\nabla\times\boldsymbol{u} + i\nabla\cdot\boldsymbol{u})\,d\boldsymbol{x} \tag{41}$$

$$= 2i\int_\Omega \frac{\partial\bar{\overline{u}}}{\partial\bar{z}}\,d\boldsymbol{x} \tag{42}$$

where we have used (13), the complex conjugate of (33), and (39) above.

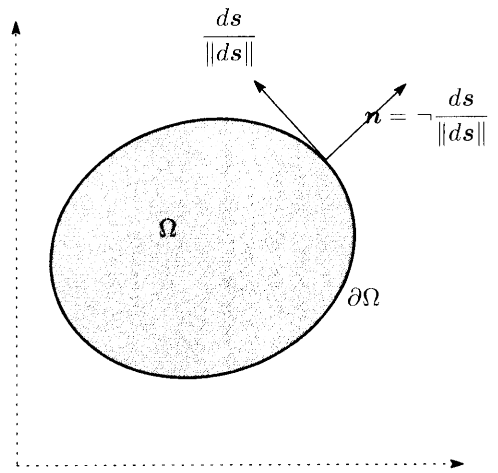

Fig. 4 Gauss' theorem and Stokes theorem for region $\Omega \subset \boldsymbol{R}^2$ bounded by $\partial\Omega$. The line integral goes along $\partial\Omega$ anticlockwise; $d\boldsymbol{s}$ is a small line element vector along $\partial\Omega$ and $\boldsymbol{n}$ is the unit vector outward normal to Ω, so that $\|d\boldsymbol{s}\|\,\boldsymbol{n} = \neg d\boldsymbol{s}$.

Thus Gauss' theorem and Stokes' theorem are expressed concisely by a *single* equation based on complex representation. We see that, if $\boldsymbol{u}$ is a nondivergent and irrotational field, or $\dfrac{\partial\bar{\overline{u}}}{\partial\bar{z}}$ vanishes identically, so that $\bar{\overline{u}}$ is a function of z only, which will be discussed again later in relation to velocity potential.

3.4 Derivation of the equation of motion in a rotating system

The frame in concern is rotating with a constant angular frequency Ω around a fixed axis of rotation like the earth. Let $\check{\bullet}$ denote a quantity viewed from the rotating frame of reference. We may consider the direction of the rotating axis as vertical without loss of generality. Then the equation of motion suffers no change in the vertical direction, so that we may confine our concern to a horizontal plane. Let

$$\begin{cases} \text{coordinates} \quad x + iy \equiv z = \check{z}\mathrm{e}^{i\Omega t}, \\ \text{force} \quad F_x + iF_y \equiv F = \check{F}\mathrm{e}^{i\Omega t} \in \boldsymbol{C} \end{cases}. \tag{43}$$

The equations of motion of a particle with mass m in an inertial frame of reference is governed by

$$\frac{d^2 z}{dt^2} = \frac{F}{m}. \qquad \text{(Newton's law)}$$

Since

$$\frac{d\check{z}}{dt} = \left(\frac{dz}{dt} - i\Omega z\right) \mathrm{e}^{-i\Omega t},$$

we have

$$\begin{aligned} \frac{d^2\check{z}}{dt^2} &= \left(\frac{d^2 z}{dt^2} - 2i\Omega\frac{dz}{dt} - \Omega^2 z\right) \mathrm{e}^{-i\Omega t} \\ &= \frac{F}{m}\mathrm{e}^{-i\Omega t} - 2i\Omega\left(\frac{dz}{dt} - i\Omega z\right)\mathrm{e}^{-i\Omega t} \\ &\qquad t + \Omega^2 z\mathrm{e}^{-i\Omega t} \\ &= \frac{\check{F}}{m} - 2i\Omega\frac{d\check{z}}{dt} + \Omega^2\check{z}. \end{aligned} \tag{44}$$

The second term expresses the Coriolis force and the last the centrifugal force.

Rewriting back using GFDVN with respect to a plane perpendicular to the axis of rotation, we have

$$\frac{d\check{\boldsymbol{u}}}{d\check{t}} = \frac{d^2\check{\boldsymbol{x}}}{d\check{t}^2} = \frac{\check{\boldsymbol{F}}}{m} + 2\Omega\neg\check{\boldsymbol{u}} + \Omega^2\check{\boldsymbol{x}}. \tag{45}$$

Adding the component of the axis of rotation

$$\frac{d\check{w}}{d\check{t}} = \frac{dw}{dt} = \frac{F_3}{m} = \frac{\check{F}_3}{m} \tag{46}$$

we obtain, in GFDVN,

$$\frac{d}{d\check{t}}(\check{\boldsymbol{u}} + \check{w}) = \frac{\check{\boldsymbol{F}} + \check{F}_3}{m} + 2\Omega\neg\check{\boldsymbol{u}} + \Omega^2\check{\boldsymbol{x}}, \tag{47}$$

or, in ordinary three-dimensional notation,

$$\frac{d\underline{\check{\boldsymbol{u}}}}{d\check{t}} = \frac{\underline{\check{\boldsymbol{F}}}}{m} - 2\underline{\boldsymbol{\Omega}} \times \underline{\check{\boldsymbol{u}}} - \underline{\boldsymbol{\Omega}} \times (\underline{\boldsymbol{\Omega}} \times \underline{\check{\boldsymbol{x}}}),$$

where we have used Lagange's formula for triple vector product (Section 3.6); the underline of a boldface indicates it is a three-dimensional vector.

3.5 Complex velocity potential

In this subsection we deal with the well-known velocity potential for irrotational and incompressible flows from a viewpoint of systematic usage of complex representation.

Let ϕ and ψ be real scalar fields on an x-y plane, and $\boldsymbol{\phi} \equiv (\phi, \psi)'$ and $\overline{\boldsymbol{\phi}} \equiv (\phi, -\psi)'$ be two related real vector fields. As before, Φ denotes a complex scalar field corresponding to the real vector field $\boldsymbol{\phi}$ such that

$$\begin{cases} \boldsymbol{\phi}(x,y) \models \Phi(x,y) = \phi(x,y) + i\psi(x,y) \in \boldsymbol{C} \\ \overline{\boldsymbol{\phi}}(x,y) \models \overline{\Phi}(x,y) = \phi(x,y) - i\psi(x,y) \in \boldsymbol{C} \end{cases} \tag{48}$$

Also let $\tilde{\Phi}$ be a map from $\boldsymbol{C}$ to $\boldsymbol{C}$ such that

$$\begin{cases} z = x + iy, \quad \overline{z} = x - iy \\ \tilde{\Phi}(z, \overline{z}) \equiv \Phi(x,y) = \Phi\left(\dfrac{z+\overline{z}}{2}, \dfrac{z-\overline{z}}{2i}\right) \end{cases}. \tag{49}$$

Then, just in the same way for a real vector field $\boldsymbol{u}$ in Section 2.4, we have

$$\begin{aligned} 2\frac{\partial\tilde{\Phi}}{\partial z} &= \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y} = \nabla\cdot\boldsymbol{\phi} + i\nabla\times\boldsymbol{\phi} \\ &= \left(\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}\right) + i\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) \end{aligned} \tag{50}$$

$$\begin{aligned} 2\frac{\partial\tilde{\Phi}}{\partial\overline{z}} &= \frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y} = \nabla\cdot\overline{\boldsymbol{\phi}} - i\nabla\times\overline{\boldsymbol{\phi}} \\ &= \left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right) + i\left(\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\right). \end{aligned} \tag{51}$$

Now, in order that $\tilde{\Phi}$ is a function of z only, $\tilde{\Phi}$ must satisfy

$$\frac{\partial\tilde{\Phi}}{\partial\overline{z}} = 0, \tag{52}$$

which is identical with Cauchy-Riemann's relation for $\Phi = \phi + i\psi$:

$$\begin{cases} \dfrac{\partial\phi}{\partial x} = +\dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x} \end{cases}, \tag{53}$$

from (51). That is, (52) is a necessary and sufficient condition for Φ to be an analytic function of z.

When (53) is satisfied, we may define two scalar fields

$$u \equiv \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v \equiv \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}, \tag{54}$$

which are expressed also as a real vector field

$$\boldsymbol{u} \equiv (u, v)' = \nabla\phi = \lhd\psi$$

by GFDVN. From (50), (53), and (54) we have

$$\frac{\partial\tilde{\Phi}}{\partial z} = \begin{cases} \dfrac{\partial\phi}{\partial x} - i\dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial\psi}{\partial y} + i\dfrac{\partial\psi}{\partial x} \end{cases} = u - iv. \tag{55}$$

Moreover it follows from (34) and (55) that

$$\begin{aligned} 0 &= 2\frac{\partial}{\partial\overline{z}}\frac{\partial\tilde{\Phi}}{\partial z} = 2\frac{\partial}{\partial\overline{z}}[u - iv] \\ &= \nabla\cdot\overline{\overline{\boldsymbol{u}}} - i\nabla\times\overline{\overline{\boldsymbol{u}}}. = \nabla\cdot\boldsymbol{u} - i\nabla\times\boldsymbol{u}. \end{aligned} \tag{56}$$

Thus $\boldsymbol{u}$ derived from the analytic function Φ through (55) should be irrotational and nondivergent. Also from (27) and (56) we have

$$\nabla^2\phi + i\nabla^2\psi = 0,$$

which shows that the *velocity potential function* ϕ and *stream function* ψ must be harmonic.

Thus a complex potential $\tilde{\Phi}$ represents irrotational nondivergent flow $\boldsymbol{u} = (u, v)'$ by

$$\overline{\boldsymbol{u}} \models u - iv = \frac{d\Phi}{dz}, \tag{57}$$

where the left-hand side is not $\boldsymbol{u} \models u + iv$, but its conjugate $u - iv$. According to the argument in Section 3.3, the complex velocity potential Φ satisfies

$$0 = \oint_{\partial\Omega} (u - iv)\, dz = \oint_{\partial\Omega} \frac{d\Phi}{dz}\, dz \tag{58}$$

for any closed curve $\partial\Omega$.

3.6 Lagrange's formula

Lagrange's formula for triple vector product is expressed as

$$\underline{\boldsymbol{a}} \times (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}}) = (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{c}})\underline{\boldsymbol{b}} - (\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}})\underline{\boldsymbol{c}} \tag{59}$$

for three-dimensional vectors $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, and $\underline{\boldsymbol{c}}$. If all the three vectors $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, and $\underline{\boldsymbol{c}}$ lie on a plane, it becomes "two-dimensional Lagrange's formula"

$$(\boldsymbol{b} \times \boldsymbol{c})\neg\boldsymbol{a} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}, \tag{60}$$

because vector product is a scalar in GFDVN. Replacing $\boldsymbol{a}$ by $-\neg\boldsymbol{a}$ in (60) and rearranging, we obtain a symmetric identity

$$(\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b} \times \boldsymbol{c})\boldsymbol{a} + (\boldsymbol{c} \times \boldsymbol{a})\boldsymbol{b} = 0, \tag{61}$$

where we have used $-\neg\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ and $\neg\boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

Now Lagrnge's formula in two-dimension is proved quite easily by an arithmetic of the complex representation as follows. From (13) and other corresponding rules we have

$$\begin{aligned}
&(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \models \frac{\overline{a}c + a\overline{c}}{2}b - \frac{\overline{a}b + a\overline{b}}{2}c \\
&= \frac{b\overline{c} - c\overline{b}}{2}a = \frac{\overline{b}c - b\overline{c}}{2i}(-ia) \\
&\models (\boldsymbol{b} \times \boldsymbol{c})\neg\boldsymbol{a},
\end{aligned}$$

namely, (60).

The symmetric form (61) is easier:

$$\begin{aligned}
&(\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b} \times \boldsymbol{c})\boldsymbol{b} + (\boldsymbol{c} \times \boldsymbol{a})\boldsymbol{c} \\
&\models \frac{\overline{a}b - a\overline{b}}{2i}c + \frac{\overline{b}c - b\overline{c}}{2i}a + \frac{\overline{c}a - c\overline{a}}{2i}b \\
&= 0.
\end{aligned}$$

Derivation of Lagrange's formula in three-dimension is straightforward from (60) or (61); see Masuda (2010)[1].

3.7 Hamiltonian equations and GFDVN

As the last example, we observe an interesting relation among Hamiltonian equations, GFDVN, and complex representation, though it is rather formal.

For simplicity consider a one-dimensional dynamical system with Hamiltonian $H = H(q,p)$, where q is a generalized coordinate and p is the corresponding momentum; H, q, and p are real, of course.

We consider a plane $\boldsymbol{R}^2$, where the orthogonal coordinates are q and p, instead of x and y, and let $\boldsymbol{q} \equiv (q,p)'$. On this plane, ∇ and $\lhd$ are defined as usual. Then we see that the canonical equations

$$\left\{\begin{aligned} \frac{dq}{dt} &= +\frac{\partial H}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned}\right. \tag{62}$$

are written simply as

$$\frac{d\boldsymbol{q}}{dt} = \lhd H \tag{63}$$

in a concise form by GFDVN, where t denotes time. This representation directly shows that the orbit of $\boldsymbol{q}$, or $d\boldsymbol{q}$, in the phase space $\boldsymbol{R}^2$ is parallel to $\lhd H$, namely along an isoline of H; recall that $\lhd$ has an alias of *alongent*. Extension to higher-dimensions are straightforward, formally at least.

Next we introduce a complex coordinate z and a corresponding Hamiltonian $\check{H}$ by

$$\begin{aligned}
&\boldsymbol{C} \ni z \equiv q + ip, \quad \overline{z} \equiv q - ip \\
&\check{H}(z,\overline{z}) \equiv \frac{i}{2}\tilde{H}(z,\overline{z}) \equiv \frac{i}{2}H(q,p) \\
&= \frac{i}{2}H\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)
\end{aligned}$$

where $\tilde{H}$ is as usual as in the text. Note that $\check{H}$ is pure imaginary, while $\tilde{H}$ is real.

It follows from (25) and (63) that

$$\frac{dz}{dt} \models \frac{d\boldsymbol{q}}{dt} = \lhd H \models -2i\frac{\partial \tilde{H}}{\partial \overline{z}} = \frac{\partial \check{H}}{\partial \overline{z}}.$$

That is, the canonical equations are expressed by a single complex equation. Moreover, we have

$$\left\{\begin{aligned} \frac{dz}{dt} &= +\frac{\partial \check{H}}{\partial \overline{z}} \\ \frac{d\overline{z}}{dt} &= -\frac{\partial \check{H}}{\partial z} \end{aligned}\right. , \tag{64}$$

which has just the same form as (62) if we replace H, q, and p by $\check{H}$, z, and $\overline{z}$, respectively. It must be noted that (64) is redundant, since the latter equation is the complex conjugate of the former, giving no information.

4. Summary and discussion

In this short note we have enumerated useful formulas in complex representation of two-dimensional real vectors, supplementing the previous article of GFDVN. Euler's formula and consequent trigonometric identities have been elaborated on in rather details, though not necessarily new.

Most of useful GFDVN were represented by complex numbers as well. Examples were presented to show that complex representation is easier to handle with than ordinary ones. Operator "turn" or "strophe" $\neg$, which rotates a vector clockwise at a right angle, is expressed simply as the multiplication of $-i$, the imaginary unit, in complex numbers. Likewise two-dimensional Lagrange's formula was proved in a purely arithmetic way of complex numbers via correspondence rules. Concise expressions were obtained by complex representation, where the real and imaginary parts respectively express the complementary quantities or operations such as $\boldsymbol{u}$ vs. $\neg\boldsymbol{u}$ or divergence vs. rotation. Indeed intimate relation between such complements manifests itself clearly through complex representation. These results shed some light on the relationship between arithmetic of complex numbers and vector operations on a plane.

It is to be noted finally the complex representation argued here is restricted to real two-dimensional vectors. This is contrasted with the original GFDVN, which applies to complex vector fields as well. For instance, consider a horizontal current field which has a harmonic temporal oscillation. One may use an expression

$$\boldsymbol{u} = \hat{\boldsymbol{u}}\mathrm{e}^{-i\omega t},$$

where $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ are two-dimensional *complex* vector fields, and ω the frequency. The real part of $\boldsymbol{u}$ is a vector with two components, not the x-component at all. In those cases, one should refrain from applying complex representation of two-dimensional real vectors. Or else, we should use it carefully by confining the real expression of temporal variation; for example always with a temporal variation expressed as $\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}$ or $i(\mathrm{e}^{i\omega t} - \mathrm{e}^{-i\omega t})$.

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References

1) Masuda, A. (2010): Vector notations suitable for geophysical fluid dynamics with examples and application. *Reports of Research Institute for Applied Mechanics*, No.138, 1-12.

2) Lamb, H. (1932): *Hydrodynamics*. 6-th ed., Cambridge University Press, Cambridge, 738 pp.

3) Landau, L. D. and Lifshits, E. M.: Fluid Mechanics (English edn.), translated by Sykes, J. B. and Reid, W. H. Pergamon Press, (1959) 536 pp.

4) Batchelor, G. K.: An Introduction to Fluid Dynamics, Cambridge University Press, (1967) 615 pp.

5) Masuda, A. (2007): A note on some aspects of surface currents and tides in relation to the measurement with the ocean radar. *Reports of Research Institute for Applied Mechanics*, No.132, 55-74(in Japanese).

6) Stommel, H. and Schott, F. (1977): The beta spiral and the determination of the absolute velocity field from hydrographic data. *Deep-Sea Research*, **24**, 325-329.

7) Pedlosky, J. (1996): *Ocean Circulation Theory*. Springer-Verlag, Berlin Hyderberg, 453 pp.

Appendix

A1. Derivative of $\sin t$ and $\cos t$

In Section 2.1, Euler's formula is used for the derivation of the derivatives of sinusoidal functions. Here we show a more elementary way of derivation. Figure A.1 illustrates how to derive the derivatives of $\sin t$, $\cos t$, and $\tan t$, where the radius of the circle is unity, namely $\overline{\mathrm{OA}} = 1$. We first note that $\mathrm{R} \models (x, y) = (\cos t, \sin t)$ with argument t and $\mathrm{R}' \models (x + dx, y + dy) = (\cos(t + dt), \sin(t + dt))$ with argument $t + dt$, where dt is small enough. It is obvious that $\triangle$ RSR'$\sim$ $\triangle$ EE'F $\sim$ $\triangle$ OAE. Enlarged $\triangle$ RSR'and $\triangle$ EE'F are added for visual convenience. From $\triangle$ RSR', we see

$$\begin{cases} -d(\cos t) = \overline{\mathrm{R'S}} = \overline{\mathrm{RR'}}\sin t = dt\sin t \\ +d(\sin t) = \overline{\mathrm{RS}} = \overline{\mathrm{RR'}}\cos t = dt\cos t \end{cases},$$

where we have used $\overline{\mathrm{RR'}} = dt$; the negative sign of $-d(\cos t)$ indicates that $x = \cos t$ decreases with t. It follows directly from these equations

$$\begin{cases} \dfrac{d(\cos t)}{dt} = -\sin t \\ \dfrac{d(\sin t)}{dt} = +\cos t \end{cases}. \tag{A1}$$

Likewise from $\triangle$ EE'F, we see

$$\overline{\mathrm{EE'}} = \overline{\mathrm{OE}}\,dt = \frac{\overline{\mathrm{OA}}}{\cos t}dt = \frac{dt}{\cos t}.$$

Thus we have

$$d(\tan t) = \overline{\mathrm{EF}} = \frac{\overline{\mathrm{EE'}}}{\cos t} = \frac{1}{\cos t}\frac{dt}{\cos t} = \frac{dt}{\cos^2 t},$$

or

$$\frac{d(\tan t)}{dt} = \frac{1}{\cos^2 t} \tag{A2}$$

The derivative of $\cot t$ is obtained in a similar graphical way, though omitted here.

A2. Addition theorem of $\cos t$ and $\sin t$

In Section 2.1 we started with Euler's formula and thence derived the addition theorem of $\cos t$ and $\sin t$. Conversely the addition theorem yields $\mathrm{e}^{i(\alpha+\beta)} = \mathrm{e}^{i\alpha}\cdot$

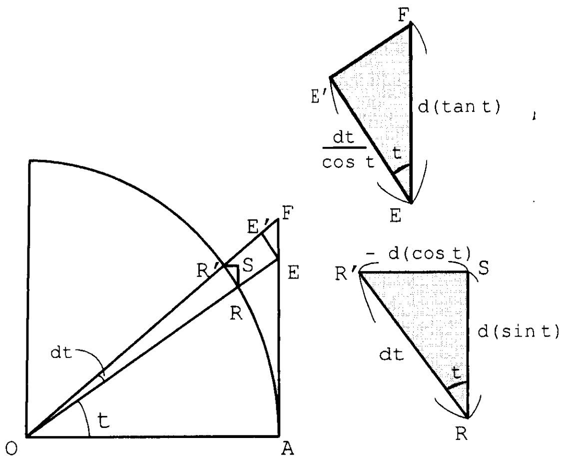

Fig. A1 Illustration of the derivative of $\cos t$, $\cos t$, and $\tan t$.

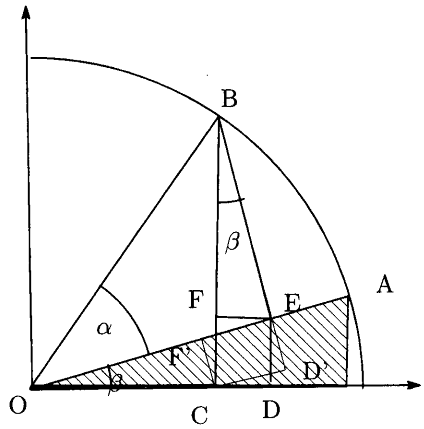

Fig. A2 Elementary explanation of the addition theorems of $\cos t$ and $\sin t$.

$e^{i\beta}$. For the sake of consistency, a graphical proof is presented for the addition theorem as follows. See Fig. 2 for α, β, and other marks of points; the radius of the circle is unity. It is easy to confirm

$$
\begin{aligned}
\cos(\alpha+\beta) &= \overline{\mathrm{OB}}\cos(\alpha+\beta) = \overline{\mathrm{OC}} \\
&= \overline{\mathrm{OD}} - \overline{\mathrm{CD}}(=\overline{\mathrm{FE}}) = \overline{\mathrm{OE}}\cos\beta - \overline{\mathrm{BE}}\sin\beta \\
&= \overline{\mathrm{OB}}\cos\alpha\cos\beta - \overline{\mathrm{OB}}\sin\alpha\sin\beta \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}
$$

and

$$
\begin{aligned}
\sin(\alpha+\beta) &= \overline{\mathrm{OB}}\sin(\alpha+\beta) = \overline{\mathrm{BC}} \\
&= \overline{\mathrm{BF}} + \overline{\mathrm{FC}}(=\overline{\mathrm{ED}}) = \overline{\mathrm{BE}}\cos\beta + \overline{\mathrm{OE}}\sin\beta \\
&= \overline{\mathrm{OB}}\sin\alpha\cos\beta + \overline{\mathrm{OB}}\cos\alpha\sin\beta \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta.
\end{aligned}
$$

A3. Trigonometric identities from (c.1) and (c.2) in Section 2.2

From Fig. A.3, which is the same as Fi. 3 but is repeated for the convenience of explanation, we have

$$
\begin{cases}
\overline{\mathrm{OC}} = 2\cos\dfrac{\alpha-\beta}{2}, & \arg(\overrightarrow{\mathrm{OC}}) = \dfrac{\alpha+\beta}{2} \\
\overline{\mathrm{BA}} = 2\sin\dfrac{\alpha-\beta}{2}, & \arg(\overrightarrow{\mathrm{BA}}) = \dfrac{\alpha+\beta}{2} + \dfrac{\pi}{2}.
\end{cases}
$$

This gives the right-hand side of formulas (c.1) or (c.2)

$$
\begin{cases}
(\mathrm{c.1}):\ e^{i\alpha} + e^{i\beta} = 2\cos\dfrac{\alpha-\beta}{2}\, e^{i\frac{\alpha+\beta}{2}} \\
(\mathrm{c.2}):\ e^{i\alpha} - e^{i\beta} = 2\sin\dfrac{\alpha-\beta}{2}\, i\, e^{i\frac{\alpha+\beta}{2}},
\end{cases}
\tag{A3}
$$

Taking the real and imaginary part of (c.1) or (c.2), we obtain

$$
\begin{cases}
\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha-\beta}{2}\cos\dfrac{\alpha+\beta}{2} \\
\sin\alpha + \sin\beta = 2\cos\dfrac{\alpha-\beta}{2}\sin\dfrac{\alpha+\beta}{2}
\end{cases}
$$

$$
\begin{cases}
\cos\alpha - \cos\beta = 2\sin\dfrac{\alpha-\beta}{2}\cos\dfrac{\alpha+\beta}{2} \\
\sin\alpha - \sin\beta = 2\sin\dfrac{\alpha-\beta}{2}\sin\dfrac{\alpha+\beta}{2}.
\end{cases}
\tag{A4}
$$

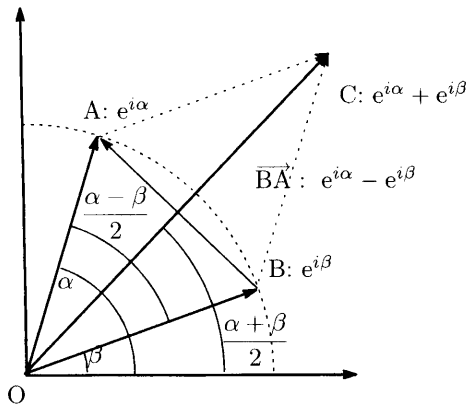

Fig. A3 The same as Fig. 3, which is repeated here for the convenience of explanation here. Addition and subtraction of two unit vectors $e^{i\alpha}$ and $e^{i\beta}$ are expressed as (c.1) and (c.2), respectively.