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Vector notations suitable for geophysical fluid dynamics with examples and applications

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Abstract

In geophysical fluids like the ocean and atmosphere, the horizontal plane and vertical direction have much different properties from each other, partly because of a small aspect ratio and partly because of stable density stratification and rotation of the earth. A useful system of vector notations is proposed for describing geophysical fluid dynamics in a concise manner. We first introduce "$\neg$", a vector operator to be called *strophe* or *turn*. It rotates a horizontal vector clockwise at right angle. Likewise "$\lhd \equiv \neg\nabla$" (*blana* or *alongent*) is defined. When operated on a scalar field ψ, blana yields a vector that is as large as $\|\nabla\psi\|$, but parallel to the isoline of ψ with higher ψ on the left-hand side. A three-dimensional vector is written as $\underline{\boldsymbol{u}} = \boldsymbol{u} + w$, where $\boldsymbol{u}$ in boldface indicates the horizontal components and w is the vertical component of three-dimensional vector $\underline{\boldsymbol{u}}$. Examples and applications are presented to show the utility of the present system of notations called GFDVN (Geophysical Fluid Dynamics Vector Notation), which not only simplifies the description, but also gives a clear geometrical image of vectors in oceanography or meteorology. We observe quite symmterical relations between the inner and outer product of two vectors, divergence and rotation of a vector field, through $\neg$ or $\lhd$. As a focused application, an investigatios is made of the geometrical implication of Lagrange's formula, or the formula of triple vector product. By using GFDVN we find that the formula is none other than a representation of skew coordinates on a plane. Also that formula in two-dimension turned out to have a simple relation with the mixing ratio of three water types on a T-S diagram used in oceanography.

Key words : *geophysical fluid dynamics, vector notation, horizontal two-dimensional plane, strophe operator, blana operator, Lagrange's foumula, skew coordinates, T-S diagram*

1. Introduction

In ordinary three-dimensional vector analysis we deal with all the directions of space equally since the physical space is isotropic in principle. But, in *geophysical fluid dynamics*, the vertical direction has a distinct characteristics different from the horizontal; vertical velocity is far weaker than horizontal, because of a small aspect ratio of the ocean or atmosphere. The momentum equation in the vertical direction therefore is approximated well by the hydrostatic equilibrium, as is usual in the primitive equations. Consequently most vector operations are made on the horizontal plane in oceanography and meteorology.

In geophysical fluid dynamics therefore the ordinary three-dimensional notation often becomes cumbersome than in the isotropic physical space. For instance, horizontal Coriolis force $\boldsymbol{F}$ acting on horizontal current $\boldsymbol{u}$ should be written as

$$\boldsymbol{F} = f\hat{\boldsymbol{k}} \times \boldsymbol{u}, \tag{1}$$

where f is the vertical component of the planetary vorticity and $\hat{\boldsymbol{k}}$ is the upward unit vector. If we want to denote quasi-geostrophic horizontal current in terms of the geostrophic stream function ψ, we need to write it as

$$\boldsymbol{u} = -\nabla \times (\psi\hat{\boldsymbol{k}}) \tag{2}$$

by the ordinary vector notation. Also the vertical component of the relative vorticity ζ is expressed in terms of the quasi-geostrophic stream function ψ as

$$\zeta = -\hat{\boldsymbol{k}} \cdot (\nabla \times \nabla \times \psi\hat{\boldsymbol{k}}).$$

Of'course an alternative way is to write out all the components of vectors, giving up vector notations. For instance, (1) and (2) are written as

$$\begin{cases} F_x = fv \\ F_y = -fu \end{cases} \quad \text{and} \quad \begin{cases} u = -\dfrac{\partial\psi}{\partial y} \\ v = \dfrac{\partial\psi}{\partial x} \end{cases} ,$$

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where u (v) and F_x (F_y) are the eastward (northward) component of $\boldsymbol{u}$ and $\boldsymbol{F}$, respectively.

These inconvenient notations have been traditional even in standard textbooks and monographs for geophysical fluid dynamics (Pedlosky 1987, say)[1]. In a recent paper of mine, I began to use slightly different notations for two-dimensional vectors, which proved quite useful (Masuda 2008)[2]. This short note is intended to give a more comprehensive system of notations for two-dimensional vector description that is suitable especially for geophysical fluid dynamics. Let us call it GFDVN (Geophysical Fluid Dynamics Vector Notation). This system allows us a much simpler description of geophysical fluid dynamics than traditional ones. It is not only convenient, but also provides vivid images of vector properties.

This short note therefore aims at presenting: (1) a comprehensive summary of GFDVN based on $\neg$ and $\lhd$ and extended to the three-dimensional space, (2) examples and applications illustrating the parallel or symmetric relations between ∇ and $\lhd$ or between divergence and rotation, and (3) Lagrange's formula of triple vector product, which is given a simple meaning from a viewpoint of GFDVN.

Next section introduces two operators $\neg$ and $\lhd \equiv \neg\nabla$ and gives their fundamental properties and forumulas. In the third section, geophysical examples and applications are presented to show the utility of GFDVN. Then GFDVN is applied to argue the meaning of Lagrange's formula or triple vector product. We see that it is a representation of a vector in skew coordinates on a plane. The last section gives a summary and discussion.

2. Geophysical fluid dynamics vector notation

A horizontal vector is expressed in boldface as $\boldsymbol{u}$, and the horizontal gradient operator by ∇. Such a horizontal two-dimensional notation is convenient in discussing geophysical fluid dynamics, where the vertical direction is special in that it is parallel to the gravity of the earth. In many situations of oceanography, however, one may want to express three-dimensional vectors with these two-dimensional vector notations hold. For extension to such cases, we make it a rule to add an underline to show that the vector is a three-dimensional one. The ordinary three-dimensional velocity vector and gradient operator are written as $\underline{\boldsymbol{u}}$ and $\underline{\nabla}$, respectively.

Let us denote the horizontal plane as $\boldsymbol{R}^2$ (real two-dimensional vectors), and the three-dimensional space as $\boldsymbol{R}^3$, which is the direct product of $\boldsymbol{R}^2$, horizontal plane, and $\boldsymbol{R}$, the vertical axis. For brevity "2-D" (2D) and "3-D" (3D) often stand for "two-dimension(al)" and "three-dimension(al)", respectively.

In $\boldsymbol{R}^3$, x, y and z denote the eastward, northward and upward coordinate, respectively, A vector in either two or three dimensional space, is understood as a column vector of corresponding components. For a vector or matrix denoted as A, A' means the transpose of A.

Then two-dimensional and three-dimensiona coordinates or position vectors are expressed as

$$\begin{cases} \boldsymbol{x} \equiv (x, y)' \\ \underline{\boldsymbol{x}} \equiv (x, y, z)' = (\boldsymbol{x}', z)' \end{cases}$$

respectively. Likewise we define the gradient operator by

$$\begin{cases} \nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)' \\ \underline{\nabla} = \left(\nabla', \dfrac{\partial}{\partial z}\right)' = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)' \end{cases},$$

on the horizontal plane and in the three-dimensional space, respectively. In the three-dimensional context we should read symbolically two-dimensional vectors as

$$\boldsymbol{x} = (x, y, 0)', \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)'.$$

This system of notations is called "GFDVN" or "two-dimensional vector notation (2DVN)", while the conventional notation is to be referred to as "3DVN". Unless stated otherwise we assume GFDVN henceforth.

2.1 Unit tensors

On a plane, the unit tensor $\delta_{i,j}$ and antisymmetric unit tensor $\epsilon_{i,j}$ are defined by

$$\delta_{i,j} \equiv \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \tag{3}$$

$$\epsilon_{i,j} \equiv \begin{cases} 1 & \text{for } (i,j) = (1,2) \\ -1 & \text{for } (i,j) = (2,1) \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

respectively, where suffix i (j) is 1 or 2. In contrast to the three-dimensional case, $\epsilon_{i,j}$ is expressed as the components of an antisymmetric square matrix of order 2. Throughout the paper, we observe that $\delta_{i,j}$ and $\epsilon_{i,j}$ play a symmetric role in the operation of two-dimensional vectors.

Properties of $\delta_{i,j}$ and $\epsilon_{i,j}$ are listed up first. Obviously $\delta_{i,j}$ and $\epsilon_{i,j}$ are tensors. The unit tensor $\delta_{i,j}$ (Kronecker's delta) has the following identities:

$$(a): \sum_{j=1}^{2} \delta_{j,j} = 2 \tag{5}$$

$$(b): \boldsymbol{\delta}_p \cdot \boldsymbol{\delta}_q = \sum_i \delta_{i,p}\delta_{i,q} = \delta_{p,q}, \tag{6}$$

where $\boldsymbol{\delta}_p \equiv (\delta_{1,p}, \delta_{2,p})' \in \boldsymbol{R}^2$, and $\boldsymbol{a} \cdot \boldsymbol{b}$ is the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$ (inner product will be defined later).

The antisymmetric unit tensor $\epsilon_{i,j}$ has different properties from $\delta_{i,j}$. If A is a matrix of (2,2)-order with components $A_{i,j}$, we have

$$\sum_{p,q} A_{p,i} A_{q,j} \epsilon_{p,q} = \det(A)\epsilon_{i,j}, \tag{7}$$

where $\det(A)$ is the determinant of A. This formula is derived as follows. By definition the left-hand side of (7) is the determinant of a matrix whose first column is $(A_{1,i}, A_{2,i})'$ and second $(A_{1,j}, A_{2,j})'$. It is $\det(A)\epsilon_{i,j}$, just the right-hand side of (7).

From (7) we have the following identities combining $\epsilon_{i,j}$ with $\delta_{i,j}$:

$$\begin{aligned}(c):\ \epsilon_{i,j} &= \sum_{p,q} \epsilon_{p,q}\delta_{p,i}\delta_{q,j} \\ &= \begin{vmatrix} \delta_{1,i} & \delta_{1,j} \\ \delta_{2,i} & \delta_{2,j} \end{vmatrix} = \delta_{1,i}\delta_{2,j} - \delta_{1,j}\delta_{2,i} \\ &= \begin{vmatrix} \delta_{1,i} & \delta_{2,i} \\ \delta_{1,j} & \delta_{2,j} \end{vmatrix}\end{aligned} \tag{8}$$

$$\begin{aligned}(d):\ \epsilon_{i,j}\epsilon_{p,q} &= \begin{vmatrix} \delta_{1,i} & \delta_{2,i} \\ \delta_{1,j} & \delta_{2,j} \end{vmatrix} \begin{vmatrix} \delta_{1,p} & \delta_{1,q} \\ \delta_{2,p} & \delta_{2,q} \end{vmatrix} \\ &= \begin{vmatrix} \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_p & \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_q \\ \boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_p & \boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_q \end{vmatrix} = \begin{vmatrix} \delta_{i,p} & \delta_{i,q} \\ \delta_{j,p} & \delta_{j,q} \end{vmatrix} \\ &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\end{aligned} \tag{9}$$

$$(e):\ \sum_i \epsilon_{i,j}\epsilon_{i,k} = \sum_i \delta_{i,i}(\delta_{j,k} - \delta_{i,k}\delta_{j,i}) = \delta_{j,k} \tag{10}$$

$$(f):\ \sum_{i,j} \epsilon_{i,j}\epsilon_{i,j} = \sum_j \delta_{j,j} = 2. \tag{11}$$

Here (c) is obtained by putting $A_{i,j} = \delta_{i,j}$ in (7), (d) from (c) with the use of (6) and a property of determinants, (e) by putting $p = i$ and $q = k$ in (d) and (f) by putting $k = j$ in (e). One will find formulas similar to (a)–(f) for $\delta_{i,j}$ and $\epsilon_{i,j,k}$ (antisymmetric unit tensor 3-D space) in the three-dimensional case.

2.2 Operators *strophe* ¬ and *blana* ◁

The primary purpose of this short note is to introduce two symbols or operators in a two-dimensional vector space.

First we define an operator "¬", which is related with $\epsilon_{i,j}$. For $(u,v)' = (u_1,u_2)' = \boldsymbol{u} \in \boldsymbol{R}^2$, we may write as

$$\begin{cases} \boldsymbol{u} = (u,v)' \models u_i = \sum_j \delta_{i,j}u_j \\ \neg\boldsymbol{u} \equiv (v,-u)' \models \sum_j \epsilon_{i,j}u_j \end{cases}, \tag{12}$$

where "$A \models B$" means "A is a different representation of B or vice versa". In the above example, the left-hand side of the former denotes a two dimensional vector $\boldsymbol{u}$, while the right-hand side indicates that it is expressed by or represented by its i-th component u_i. This convention is used as a general rule throughtout the paper.

Compare $\boldsymbol{u}$ with $\neg\boldsymbol{u}$ in (12); the former is based on $\delta_{i,j}$ while the latter on $\epsilon_{i,j}$.

As is obvious, $\neg\boldsymbol{u}$ is obtained when $\boldsymbol{u}$ is rotated clockwise at right angle (Fig. 1). The form of the symbol ¬ suggests its operation; a right-directed bar is twisted clockwise at right angle to yield a down-directed bar. One may thus call "¬" as *strophe* or *turn*.

Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be the eastward and northward unit vector, respectively. It is easy to see that

$$\begin{cases} \neg\boldsymbol{\delta}_p = \boldsymbol{\epsilon}_p \quad (p = 1,2) \\ \hat{\boldsymbol{i}} = \boldsymbol{\delta}_1 = +\boldsymbol{\epsilon}_2 \\ \hat{\boldsymbol{j}} = \boldsymbol{\delta}_2 = -\boldsymbol{\epsilon}_1 \end{cases}, \tag{13}$$

where $\boldsymbol{\epsilon}_p = (\epsilon_{1,p}, \epsilon_{2,p})'$.

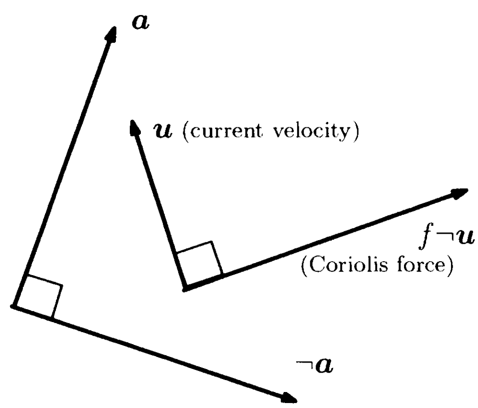

Fig. 1 Operator ¬ rotates a vector $\boldsymbol{a}$ clockwise at right angle to yield $\neg\boldsymbol{a}$. This operator is used conveniently in geophysical fluid dynamics. For instance Coriolis force on horizontal current of velocity $\boldsymbol{u}$ is expressed as $f\neg\boldsymbol{u}$, f being the vertical component of planetary vorticity.

For a scalar field ψ the most basic differential operator ∇ is represented in terms of $\delta_{i,j}$ as

$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)' \models \frac{\partial\psi}{\partial x_i} = \sum_j \delta_{i,j}\frac{\partial\psi}{\partial x_j}. \tag{14}$$

In parallel with ∇ related with $\delta_{i,j}$, a differential operator related with $\epsilon_{i,j}$ can be defined. That is, we have

$$\lhd\psi \equiv \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)' \models \sum_j \epsilon_{ij}\frac{\partial\psi}{\partial x_j}. \tag{15}$$

The operator ∇ is called "nabla" or *gradient* as usual, which gives a vector that has a magnitude of the largest gradient of ψ and is pointing toward higher value of ψ. On the other hand the operator ◁ yields a vector that is parallel to the isolines of ψ with higher ψ on the left-hand side and has the same norm as $\nabla\psi$ (Fig. 2).

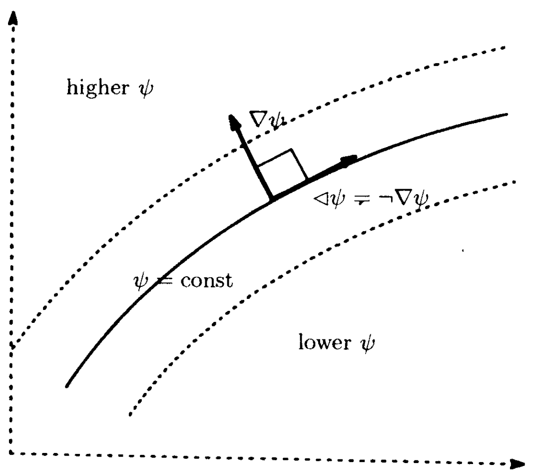

Fig. 2 Operators *nabla* ∇ and *blana* $\lhd$. For a scalar field $\psi = \psi(\boldsymbol{x})$, $\nabla\psi$ gives a vector pointing toward higher ψ, while $\lhd\psi$ gives a vector along the isoline of ψ with higher ψ on the left-hand side, where $\| \lhd \psi\| = \|\nabla\psi\|$. We may call $\lhd$ *alongent*, in analogy to *gradient*.

We call it "blana" in analogy to "nabla", or *alongent* operator.

An example of $\lhd$ is as follows. Let ψ be a stream function in geophysical fluid dynamics and $\boldsymbol{u}$ the horizontal current velocity vector. Then we have

$$\lhd\psi = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)' = -\boldsymbol{u} = -(u,v)' \qquad \text{or}$$
$$\boldsymbol{u} = - \lhd \psi,$$

which is much shorter and easier to have an image than the traditional expression like $\underline{\boldsymbol{u}} = -\nabla \times \psi\underline{\boldsymbol{k}}$. Moreover it directly shows that $\boldsymbol{u}$ is parallel to the contours of ψ with higher ψ on the right-hand side.

The identity

$$\neg^2 \equiv \neg\neg = -1 \tag{16}$$

means that the inverse of $\neg$ is $-\neg$, while

$$\begin{cases} \lhd = \neg\nabla \\ \nabla = -\neg\lhd \end{cases} \tag{17}$$

shows that $\neg$ converts ∇ to $\lhd$ and vice versa. Note that the symbol $\lhd$ has a form obtained when ∇ is turned clockwise at right angle, as is suggested by (17).

2.3 Inner and outer product

On a horizontal plane, inner and outer product are defined as

$$\begin{cases} \boldsymbol{R} \ni \boldsymbol{u}\cdot\boldsymbol{v} \equiv \sum_{i,j} \delta_{i,j} u_i v_j \\ \quad = \sum_i u_i v_i = \boldsymbol{v}\cdot\boldsymbol{u} \\ \boldsymbol{R} \ni \boldsymbol{u}\times\boldsymbol{v} \equiv \sum_{ij} \epsilon_{ij} u_i v_j = \boldsymbol{u}\cdot\neg\boldsymbol{v} \\ \quad = -\boldsymbol{v}\cdot\neg\boldsymbol{u} = -\boldsymbol{v}\times\boldsymbol{u} \end{cases}, \tag{18}$$

again, in a symmetric way in terms of $\delta_{i,j}$ and $\epsilon_{i,j}$. We should note that outer product yields a scalar in GFDVN, whereas it gives a vector or an antisymmetric tensor in 3DVN. Also note that $\boldsymbol{u}\times\boldsymbol{v}$ means the area of the rhomboid of apexes **o**, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{a}+\boldsymbol{b}$; it is also double the area of the triangle of apexes **o**, $\boldsymbol{a}$, and $\boldsymbol{b}$. Of course, the sign may be opposite according as the configuration of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Combining the inner/outer product with nabla/blana operator on a vector field $\boldsymbol{u}$, we have two scalar fields: divergence and rotation of $\boldsymbol{u}$

$$\begin{cases} \nabla\cdot\boldsymbol{u} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = +\sum_{i,j} \delta_{i,j} \dfrac{\partial u_i}{\partial x_j} \\ \nabla\times\boldsymbol{u} = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \equiv -\sum_{i,j} \epsilon_{ij} \dfrac{\partial u_i}{\partial x_j} \end{cases}, \tag{19}$$

respectively. We observe a symmetric relationship between $\lhd$ and ∇ as

$$\begin{cases} \nabla\times\boldsymbol{u} = -\lhd\cdot\boldsymbol{u} \\ \lhd\times\boldsymbol{u} = +\nabla\cdot\boldsymbol{u} \end{cases}, \tag{20}$$

which yields an alternative expression of divergence or rotation in terms of $\lhd$ instead of ∇.

2.4 3D-vector representation in GFDVN

Three-dimensional vectors should be written as

$$(u,v,w)' = \underline{\boldsymbol{u}} = (\boldsymbol{u}',0)' + w\underline{\hat{\boldsymbol{k}}} \tag{21}$$

in a legitimate way of 3DVN, where $\underline{\hat{\boldsymbol{k}}}$ expresses the unit upward vector.

That is quite cumbersome, however. So, if the situation is not confusing, we may write the vertical vector simply by a scalar corresponding to the vertical component. By virtue of this convention of GFDVN, we may write

$$\underline{\boldsymbol{u}} = \boldsymbol{u} + w. \tag{22}$$

Since the left-hand side expresses a 3-D vector, one should interpret the right-hand side as

$$\begin{cases} \boldsymbol{u} = (u,v,0)' \\ w = (0,0,w)' \end{cases}. \tag{23}$$

Le us apply this convention to a formula in Introduction. When ψ is a scalar field in $\boldsymbol{R}^3$, $\nabla \times \psi$ is meaningless in the ordinary three-dimensional notation. In GFDVN, however, we may interpret it as

$$\begin{cases} \psi = \psi \hat{\underline{k}} = (0, 0, \psi) \\ \nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, 0 \right)' \end{cases} . \tag{24}$$

If so, we find

$$\begin{aligned} \nabla \times \psi &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)' \times (0, 0, \psi)' \\ &= \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right)', = \left((\lhd \psi)', 0 \right)', \end{aligned}$$

which can be interpreted to be a horizontal vector either on the horizontal plane or in the three-dimensional space. Thus we have $\nabla \times \psi = \lhd \psi$.

In order to avoid confusion, in this convention of GFDVN, we have to distinguish scalar multiplication with inner/outer product. In scalar multiplication, one should not use "$\cdot$" or "$\times$" at all; "$\cdot$" and "$\times$" are preserved only for inner product and outer product, respectively. When w is a scalar (or the vertical component of a vector) and $\boldsymbol{p} = (p, q)'$ a horizontal vector, we have three different scalar multiplication or product as

$$\begin{aligned} &\boldsymbol{R}^2 \ni w\boldsymbol{p} \equiv (wp, wq)' = (pw, qw)' = \boldsymbol{p}w, \\ &\boldsymbol{R} \ni w \cdot \boldsymbol{p} \equiv w\hat{\underline{k}} \cdot \boldsymbol{p} = 0 = \boldsymbol{p} \cdot w, \\ &\boldsymbol{R}^3 \ni w \times \boldsymbol{p} \equiv w\hat{\underline{k}} \times \boldsymbol{p} = -w\neg\boldsymbol{p} = -\boldsymbol{p} \times w. \end{aligned}$$

When dealing with three-dimensional vectors under this convention of GFDVN, we should pay attention to inner and outer product. For two 3-D vectors

$$\begin{cases} \underline{\boldsymbol{u}} = \boldsymbol{u} + w \\ \underline{\boldsymbol{p}} = \boldsymbol{p} + r \end{cases} ,$$

inner product becomes

$$\begin{aligned} \underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{p}} &= (\boldsymbol{u} + w) \cdot (\boldsymbol{p} + r) \\ &= \boldsymbol{u} \cdot \boldsymbol{p} + w \cdot r \in \boldsymbol{R} \end{aligned}$$

and outer product

$$\begin{aligned} \underline{\boldsymbol{u}} \times \underline{\boldsymbol{p}} &= (\boldsymbol{u} + w) \times (\boldsymbol{p} + r) \\ &= \boldsymbol{u} \times \boldsymbol{p} + (\boldsymbol{u} \times r + w \times \boldsymbol{p}) + w \times r \\ &= \boldsymbol{u} \times \boldsymbol{p} + r\neg\boldsymbol{u} - w\neg\boldsymbol{p} \\ &= (\boldsymbol{u} \times \boldsymbol{p})\hat{\underline{k}} + r\neg\boldsymbol{u} - w\neg\boldsymbol{p} \in \boldsymbol{R}^3, \end{aligned}$$

where we have used

$$\begin{cases} w \cdot r = r \cdot w = wr = rw \in \boldsymbol{R} \\ w \cdot \boldsymbol{p} = 0 \\ w \times \boldsymbol{p} = w\hat{\underline{k}} \times \boldsymbol{p} = -w\neg\boldsymbol{p} \\ w \times r = w\hat{\boldsymbol{k}} \times r\hat{\boldsymbol{k}} = 0 \end{cases} .$$

Likewise we have 3-D divergence as

$$\begin{aligned} \underline{\nabla} \cdot \underline{\boldsymbol{u}} &= \left(\nabla + \frac{\partial}{\partial z} \right) \cdot (\boldsymbol{u} + w) \\ &= \nabla \cdot \boldsymbol{u} + \frac{\partial w}{\partial z} \in \boldsymbol{R}, \end{aligned}$$

where we have used

$$\begin{cases} \nabla \cdot w = 0 \\ \dfrac{\partial}{\partial z} \cdot \boldsymbol{u} = \hat{\underline{k}} \dfrac{\partial}{\partial z} \cdot \boldsymbol{u} = 0 \end{cases} .$$

In the same way, the 3-D rotation becomes

$$\begin{aligned} \underline{\nabla} \times \underline{\boldsymbol{u}} &= \left(\nabla + \hat{\underline{k}} \frac{\partial}{\partial z} \right) \times \left(\boldsymbol{u} + w\hat{\underline{k}} \right) \\ &= \left(\nabla + \frac{\partial}{\partial z} \right) \times (\boldsymbol{u} + w) \\ &= \nabla \times \boldsymbol{u} + \nabla \times w \\ &\quad + \frac{\partial}{\partial z} \times \boldsymbol{u} + \frac{\partial}{\partial z} \times w \\ &= \nabla \times \boldsymbol{u} + \lhd w - \neg \frac{\partial \boldsymbol{u}}{\partial z} \\ &= \nabla \times \boldsymbol{u} + \neg\nabla w - \neg \frac{\partial \boldsymbol{u}}{\partial z}. \end{aligned}$$

These notations simplify the description of vector operations even in the three-dimensional space. By GFDVN, the primitive eqaution of motion is written simply as

$$\left(\frac{\partial}{\partial t} + \underline{\boldsymbol{u}} \cdot \underline{\nabla} - f\neg \right) \boldsymbol{u} = -\frac{\nabla p}{\rho} - g$$

for inviscid flow on a β-plane, where t denotes time, $\underline{\boldsymbol{u}}$ 3-D current vector with horizontal components of $\boldsymbol{u}$ and vertical component of w, p pressure, ρ density, and f the Coriolis paramter.

2.5 Formulas for strophe $\neg$ and blana $\lhd$

Now useful formulas are listed up with respect to $\neg$, $\lhd$, $\cdot$, and $\times$. Let λ and μ be constant scalars, $\boldsymbol{a}$ and $\boldsymbol{b}$ constant vectors, $p = p(\boldsymbol{x})$ and $q = q(\boldsymbol{x})$ saclar fields, and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ vector fields.

It is obvious that

$$\neg^2 = -1 \tag{25}$$

$$\begin{cases} \times = \cdot\neg \\ \cdot = -\times\neg \end{cases} . \tag{26}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \cdot \neg\boldsymbol{b} \tag{27}$$

$$\neg\lambda\boldsymbol{u} \equiv \neg(\lambda\boldsymbol{u}) = \lambda\neg\boldsymbol{u} \tag{28}$$

$$\begin{cases} \lhd p = +\neg\nabla p \\ \nabla p = -\neg\lhd p \end{cases} . \tag{29}$$

We have identities as

$$\begin{cases} (\neg\boldsymbol{a}) \cdot (\neg\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} \\ (\neg\boldsymbol{a}) \times (\neg\boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b} \end{cases} \tag{30}$$

$$\begin{cases} \neg\nabla \cdot \neg\boldsymbol{u} = \nabla \cdot \boldsymbol{u} \\ \neg\nabla \times \neg\boldsymbol{u} = \nabla \times \boldsymbol{u} \end{cases} . \tag{31}$$

The first two formulas show that the inner or outer product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to each of $\neg\boldsymbol{a}$ and $\neg\boldsymbol{b}$. In other words, we have the same scalar by operating $\neg$ before operating "$\cdot$" or "$\times$".

Divergence and rotation are expressed in terms of either ∇ or $\lhd$:

$$\begin{cases} -\lhd \times \boldsymbol{u} = \nabla \times \neg\boldsymbol{u} = -\nabla\cdot\boldsymbol{u} \\ -\lhd \cdot\boldsymbol{u} = \nabla\cdot\neg\boldsymbol{u} = +\nabla\times\boldsymbol{u} \end{cases} . \quad (32)'$$

$$\begin{cases} \nabla\cdot(p\boldsymbol{u}) = \nabla p\cdot\boldsymbol{u} + p\nabla\cdot\boldsymbol{u} \\ \lhd\cdot(p\boldsymbol{u}) = \lhd p\cdot\boldsymbol{u} + p\lhd\cdot\boldsymbol{u} = \nabla\times(p\boldsymbol{u}) \end{cases} \quad (33)$$

$$\begin{cases} \nabla\times(p\boldsymbol{u}) = \nabla p\times\boldsymbol{u} + p\nabla\times\boldsymbol{u} \\ \lhd\times(p\boldsymbol{u}) = \lhd p\times\boldsymbol{u} + p\lhd\times\boldsymbol{u} = \nabla\cdot(p\boldsymbol{u}) \end{cases} \quad (34)$$

Orthogonal and parallel relationships are expressed in terms of either inner or outer product as

$$\begin{cases} \boldsymbol{u}\perp\boldsymbol{v} \Leftrightarrow 0 = \boldsymbol{u}\cdot\boldsymbol{v} = \neg\boldsymbol{u}\cdot\neg\boldsymbol{v} = \neg\boldsymbol{u}\times\boldsymbol{v} \\ \boldsymbol{u}\parallel\boldsymbol{v} \Leftrightarrow 0 = \boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{u}\cdot\neg\boldsymbol{v} \end{cases} . \quad (35)$$

Laplacian is expressed two ways as

$$\nabla^2\psi = (\neg\nabla)\cdot(\neg\nabla)\psi = \lhd^2\psi. \quad (36)$$

By applying the formula, we have vorticity ζ expressed two ways as

$$\zeta \equiv \nabla\times\boldsymbol{u} = \nabla\cdot\neg(-\lhd\psi) = \nabla\cdot(\nabla\psi) = \nabla^2\psi = \lhd^2\psi.$$

Jacobian $J(p,q)$ becomes

$$J(p,q) \equiv \sum_{i,j}\epsilon_{i,j}\frac{\partial p}{\partial x_i}\frac{\partial q}{\partial x_j} = \nabla p\cdot\lhd q = \nabla p\cdot\neg\nabla q = \nabla p\times\nabla q = \lhd p\times\lhd q. \quad (37)$$

In this expression it is easy to see that vanishing $J(p,q)$ everywhere implies that the contours of p and q are parallel to each other and that q is a function of p.

The following identities are trivial.

$$\|\lhd p\| = \|\nabla p\| \quad (38)$$

$$\lhd p\cdot\nabla p = \neg\nabla p\cdot\nabla p = 0, \quad (39)$$

$$\nabla p\cdot\nabla q = \lhd p\cdot\lhd q \quad (40)$$

$$\nabla p\times\nabla q = \lhd p\times\lhd q \quad (41)$$

$$\nabla p\times\lhd q = \nabla p\cdot\neg\neg\nabla q = -\nabla p\cdot\nabla q \quad (42)$$

$$\nabla\times\nabla p = \nabla\cdot\lhd p = \sum_{i,j}\frac{\partial}{\partial x_i}\epsilon_{i,j}\frac{\partial p}{\partial x_j} = 0, \quad (43)$$

the last of which is known as "rot grad p=0".

For three-dimensional vector notations in GFDVN, it suffices to list up the following formulas:

$$\begin{cases} \underline{\boldsymbol{u}} = \boldsymbol{u} + w\hat{\boldsymbol{k}} = \boldsymbol{u} + w \\ \underline{\boldsymbol{p}} = \boldsymbol{p} + r\hat{\boldsymbol{k}} = \boldsymbol{p} + r \\ \underline{\nabla} = \nabla + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} = \nabla + \frac{\partial}{\partial z} \end{cases} \quad (44)$$

$$w\cdot r = r\cdot w = wr = rw \in \boldsymbol{R} \quad (45)$$

$$w\cdot\boldsymbol{p} = 0 = \boldsymbol{p}\cdot w \in \boldsymbol{R} \quad (46)$$

$$w\times\boldsymbol{p} = -w\neg\boldsymbol{p} = -\boldsymbol{p}\times w \in \boldsymbol{R}^2 \quad (47)$$

$$\underline{\boldsymbol{u}}\cdot\underline{\boldsymbol{p}} = \boldsymbol{u}\cdot\boldsymbol{p} + w\cdot r = \boldsymbol{u}\cdot\boldsymbol{p} + wr \quad (48)$$

$$\underline{\boldsymbol{u}}\times\underline{\boldsymbol{p}} = \boldsymbol{u}\times\boldsymbol{p} + r\neg\boldsymbol{u} - w\neg\boldsymbol{p} \quad (49)$$

$$\underline{\nabla}\cdot\underline{\boldsymbol{u}} = \nabla\cdot\boldsymbol{u} + \frac{\partial w}{\partial z} \quad (50)$$

$$\underline{\nabla}\times\underline{\boldsymbol{u}} = \nabla\times\boldsymbol{u} + \lhd w - \neg\frac{\partial\boldsymbol{u}}{\partial z} \quad (51)$$

3. Examples and applications

A few examples and applications of $\neg$ and $\lhd$ are presented to show their utility in geophysical fluid dynamics. Adequate formulas in the preceding section are used in the argument of the following.

3.1 Coriols force and geostrophic current

A good example of both $\neg$ and $\lhd$ is the representation of the Coriolis force and geostrophic current. The Coriolis force acting on the current with velocity $\boldsymbol{u}$ is expressed as $f\neg\boldsymbol{u}$, f being the Corilos parameter (Fig. 1). Then the geostrophic equilibrium becomes

$$0 = f\neg\boldsymbol{u} - \frac{\nabla p}{\rho}, \quad (52)$$

where p is pressure and ρ density of water. Operating $\neg$ on (52) and using (16) and (17), we obtain

$$f\boldsymbol{u} = -\frac{\neg\nabla p}{\rho} = -\frac{\lhd p}{\rho}.$$

This formula directly implies that the geostrophic current $\boldsymbol{u}$ is parallel to the isoline of p with higher p on the right-hand side, if $f > 0$ as in the northern hemisphere.

Symmetry about $\neg$ or $\lhd$ is found for steady currents by two mechanisms: one is that equilibrated by Rayleigh friction without Coriolis force and the other by Coriolis force without Rayleigh friction. The momentum balance becomes

$$\begin{cases} 0 = -R\boldsymbol{u} - \frac{\nabla p}{\rho} \\ 0 = f\neg\boldsymbol{u} - \frac{\nabla p}{\rho} \end{cases} ,$$

respectively, where R is the coefficient of Rayleigh friction, its dimension being the same as the Coriolis parameter f. Then it follows that

$$\begin{cases} R\boldsymbol{u} = -\frac{\nabla p}{\rho} \\ f\boldsymbol{u} = -\frac{\lhd p}{\rho} \end{cases} ; \quad (53)$$

$\boldsymbol{u}$ is along pressure gradient if equilibrated by Rayleigh friction and along pressure contours if equilibrated by Coriolis force, respectively.

When both kinds of force work, the momentum balance becomes

$$\begin{cases} R\boldsymbol{u} - f\neg\boldsymbol{u} = -\frac{\nabla p}{\rho} \\ f\boldsymbol{u} + R\neg\boldsymbol{u} = -\frac{\lhd p}{\rho} \end{cases} ,$$

where the latter is obtained simply by operating $\neg$ on the former. From this expression follows a useful formula of $\boldsymbol{u}$ expressed in terms of p as

$$\boldsymbol{u} = -\frac{R\nabla p + f \lhd p}{\rho(R^2 + f^2)}, \tag{54}$$

which is reduced to (53) for the limiting cases of $R^2 \gg f^2$ and $R^2 \ll f^2$.

For a harmonic oscillation $\boldsymbol{u} \propto \Re[\mathrm{e}^{-i\omega t}]$ and $p \propto \Re[\mathrm{e}^{-i\omega t}]$ of frequency ω, (54) becomes

$$\boldsymbol{u} = -\Re\left[\frac{(-i\omega)\nabla p + f \lhd p}{\rho(f^2 - \omega^2)}\right], \tag{55}$$

where $i \equiv \sqrt{-1}$ is the imaginary unit and $\Re[A]$ means the real part of A. The formula will be derived in a different way in section 3.4.

3.2 Gauss' theorem and Stokes' theorem

Let Ω be a region in $\boldsymbol{R}^2$ and $\partial\Omega$ be its boundary. Guass' theorem and Stokes' theorem are written as

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \nabla \cdot \boldsymbol{u}\, d\boldsymbol{x} = \oint_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n}\, ds \\ \displaystyle\int_\Omega \nabla \times \boldsymbol{u}\, d\boldsymbol{x} = \oint_{\partial\Omega} \boldsymbol{u} \cdot d\boldsymbol{s} \end{array}\right. , \tag{56}$$

respectively, where $d\boldsymbol{x} \equiv dx\,dy$ is the area element (not a vector, but a scalar), $d\boldsymbol{s}$ is the line element vector, $ds \equiv \|d\boldsymbol{s}\|$ is the length of the line element, the closed line integral goes around Ω counterclockwise, and $\boldsymbol{n} = \neg\frac{d\boldsymbol{s}}{ds}$ is the outward normal unit vector on $\partial\Omega$ (Fig. 3). The right-hand side of Gauss' theorem is rewritten as

$$\oint_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n}\, ds = \oint_{\partial\Omega} \boldsymbol{u} \cdot \neg d\boldsymbol{s} = \oint_{\partial\Omega} \boldsymbol{u} \times d\boldsymbol{s}.$$

Hence (56) becomes

$$\left\{\begin{array}{ll} \displaystyle\int_\Omega \nabla \cdot \boldsymbol{u}\, d\boldsymbol{x} = \oint_{\partial\Omega} \boldsymbol{u} \times d\boldsymbol{s} & \text{(Gauss)} \\ \displaystyle\int_\Omega \nabla \times \boldsymbol{u}\, d\boldsymbol{x} = \oint_{\partial\Omega} \boldsymbol{u} \cdot d\boldsymbol{s} & \text{(Stokes)} \end{array}\right. , \tag{57}$$

which provides more symmetric expressions of the two theorems via " $\cdot$ " and " $\times$ ".

Now if we replace $\boldsymbol{u}$ by $\neg\boldsymbol{u}$ in Gauss' theorem, the left-hand side of (57) becomes

$$\int_\Omega \nabla \cdot (\neg\boldsymbol{u})\, d\boldsymbol{x} = \int_\Omega \nabla \times \boldsymbol{u}\, d\boldsymbol{x}, \tag{58}$$

while the right-hand of (57) becomes

$$\begin{aligned} \oint_{\partial\Omega} (\neg\boldsymbol{u}) \times d\boldsymbol{s} &= \oint_{\partial\Omega} (\neg\boldsymbol{u}) \cdot (\neg d\boldsymbol{s}) \\ &= \oint_{\partial\Omega} \boldsymbol{u} \cdot d\boldsymbol{s}. \end{aligned} \tag{59}$$

Equating the right-hand side of (58) with that of (59) we obatin Stokes' theorem.

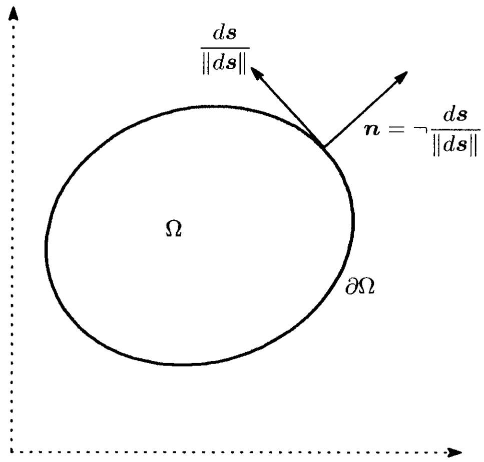

Fig. 3 Gauss' theorem and Stokes theorem for region $\Omega \subset \boldsymbol{R}^2$ bounded by $\partial\Omega$. The line integral goes along $\partial\Omega$ anticlockwise; $d\boldsymbol{s}$ is a small line element vector along $\partial\Omega$ and $\boldsymbol{n}$ is the unit vector outward normal to Ω, so that $\|d\boldsymbol{s}\|\,\boldsymbol{n} = \neg d\boldsymbol{s}$.

In the same way, if we replace $\boldsymbol{u}$ by $-\neg\boldsymbol{u}$ in Stokes' theorem, the left-hand side becomes

$$\begin{aligned} \int_\Omega \nabla \times (-\neg\boldsymbol{u})\, d\boldsymbol{x} &= \int_\Omega \nabla \cdot \neg(-\neg\boldsymbol{u})\, d\boldsymbol{x} \\ &= \int_\Omega \nabla \cdot \boldsymbol{u}\, d\boldsymbol{x}, \qquad (\because \quad -\neg^2 = 1) \end{aligned}$$

whlie the right-hand becomes

$$\begin{aligned} \oint_{\partial\Omega} (-\neg\boldsymbol{u}) \cdot d\boldsymbol{s} &= \oint_{\partial\Omega} (-\neg^2\boldsymbol{u}) \cdot (\neg d\boldsymbol{s}) \\ &= \oint_{\partial\Omega} \boldsymbol{u} \times d\boldsymbol{s}. \qquad (\because \quad \cdot\neg = \times) \end{aligned}$$

The result means Gauss' theorem.

Thus Gauss' theorem and Stokes' theorem are converted to each other easily through the operation of $\neg$ on the vector field $\boldsymbol{u}$.

3.3 Vorticity equation and divergence equation

If $\phi = \phi(\boldsymbol{x})$ is a scalar filed and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ is a vector field, then we have

$$\left\{\begin{array}{l} \nabla \cdot \neg(\phi\boldsymbol{u}) = -\neg \lhd \cdot \neg(\phi\boldsymbol{u}) = - \lhd \cdot (\phi\boldsymbol{u}) \\ \quad = -\phi \lhd \cdot \boldsymbol{u} - \lhd\phi \cdot \boldsymbol{u} = \phi\nabla \times \boldsymbol{u} + \nabla\phi \times \boldsymbol{u} \\ \nabla \times \neg(\phi\boldsymbol{u}) = \nabla \cdot \neg\neg(\phi\boldsymbol{u}) \\ \quad = -\nabla \cdot (\phi\boldsymbol{u}) = -\phi\nabla \cdot \boldsymbol{u} - \nabla\phi \cdot \boldsymbol{u}. \end{array}\right.$$

When $\phi = f = f_0 + \beta y$ is the Coriolis parameter on a β-plane and $\boldsymbol{u}$ denotes the horizontal current, the former and latter correspond to the divergence and rotation of the Coriolis force $f\neg\boldsymbol{u}$ acting on the current $\boldsymbol{u}$, respectively. Neglecting forces other than the Coriolis force we obtain symmetric expressions as

$$\begin{cases} \dfrac{\partial(\nabla\cdot\boldsymbol{u})}{\partial t} \sim f\nabla\times\boldsymbol{u} + \nabla f\times\boldsymbol{u} = f\zeta - \beta u \\ \dfrac{\partial(\nabla\times\boldsymbol{u})}{\partial t} \sim -f\nabla\cdot\boldsymbol{u} - \nabla f\cdot\boldsymbol{u} = -f\nabla\cdot\boldsymbol{u} - \beta v \end{cases},$$

where $\zeta \equiv \nabla\times\boldsymbol{u}$ is vorticity.

Let us consider the meaning of each term, assuming f is positive. The term $f\zeta$ means that Coriolis force acting on positive vorticitiy enhances (horizontal) divergence, while βu weakens horizontal divergence for eastward flow, as is obvious from the physics of Coriolis force.

On the other hand as regards vorticity, $-f\nabla\cdot\boldsymbol{u}$ shows that Coriolis force on convergin current ($-\nabla\cdot\boldsymbol{u} > 0$) twists the fluid particle anticlockwise and enhances vorticity. If we rewrite it as

$$-f\nabla\cdot\boldsymbol{u} = f\frac{\partial w}{\partial z} \tag{60}$$

for a three-dimensional incompressible flow satisfying $\nabla\cdot\boldsymbol{u} + \dfrac{\partial w}{\partial z} = 0$, this term is interpreted to be vorticity enhancement by vortex stretching. Finally the term $-\beta v$ on the northward flow ($v > 0$) twists the fluid particle clockwise to decrease vorticity.

3.4 Poincaré waves and deformation radius

Consider an infinitesimal amplitude of motion in a bartropic flat ocean on an f-plane, where density is homogeneous ($\rho = 1$). We start with

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \neg f\right)\boldsymbol{u} = -\nabla p & \text{(momentum eq.)} \\ \dfrac{1}{gH}\dfrac{\partial p}{\partial t} + \nabla\cdot\boldsymbol{u} = 0 & \text{(eq. of continuity)} \end{cases},$$

where $\boldsymbol{u}$ denotes the horizontal current velocity, p pressure, g the gravitational acceleration and H depth of the ocean; p corresponds $g\eta$, η being the displacement of the sea surface.

First let us express $\boldsymbol{u}$ through p. It follows from the momentum equation

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\boldsymbol{u} = \left(\frac{\partial}{\partial t} + \neg f\right)\left(\frac{\partial}{\partial t} - \neg f\right)\boldsymbol{u}$$
$$= -\left(\frac{\partial}{\partial t} + f\neg\right)\nabla p, \tag{61}$$

which is no other than (55). This expression (61) yields geostrophic balance

$$f\boldsymbol{u} \approx -\lhd p, \qquad \text{if} \quad \left|\frac{\partial}{\partial t}\right| \ll |f|.$$

In the opposite limit pressure gradient is balanced by acceleration

$$\frac{\partial\boldsymbol{u}}{\partial t} \approx -\nabla p \qquad \text{for} \quad \left|\frac{\partial}{\partial t}\right| \gg |f|.$$

Operating $\nabla\cdot$ on (61) and eliminating $\nabla\cdot\boldsymbol{u}$ by use of continuity equation, we obtain

$$\frac{1}{gH}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial p}{\partial t}$$
$$= \frac{\partial}{\partial t}\nabla^2 p + f(\nabla\cdot\lhd)p = \frac{\partial}{\partial t}\nabla^2 p,$$

because $\nabla\cdot\lhd p = \nabla\times\nabla p = 0$. Ignoring the trivial common operator $\dfrac{\partial}{\partial t}$, we find the equation of Poincaré waves.

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)p = (gH)\nabla^2 p. \tag{62}$$

In the limiting case of $\left|\dfrac{\partial}{\partial t}\right| \gg |f|$, (62) becomes the wave equation in two-dimensional space. In the opposite limit of $\left|\dfrac{\partial}{\partial t}\right| \ll |f|$, (62) expresses the conservation of potential vorticity

$$\frac{f^2}{gH}p = \nabla^2 p, \tag{63}$$

where $\dfrac{\sqrt{gH}}{|f|}$ is called the barotropic radius of deformation.

3.5 Thermal wind and vertical sheer

The *argument* of a horizontal vector $\boldsymbol{u}$, denoted by $\arg(\boldsymbol{u})$, is defined as the direction of $\boldsymbol{u}$, which is measured anticlockwise from east.

For a Boussinesq fluid, buoyancy b is defined by

$$b \equiv -g\frac{(\rho - \rho_0)}{\rho_0},$$

where g is gravity, ρ_0 a (constant) reference density. Putting $\rho_0 = 1$ we may write hydrostatic relation and geostrophic balance as

$$\begin{cases} b = \dfrac{\partial p}{\partial z} \\ f\neg\boldsymbol{u} = \nabla p \quad \text{or} \quad f\boldsymbol{u} = -\lhd p \end{cases}.$$

Eliminating p we have

$$f\frac{\partial\boldsymbol{u}}{\partial z} = -\frac{\partial\lhd p}{\partial z} = -\lhd\frac{\partial p}{\partial z} = -\lhd b \tag{64}$$

which indicates the relation of thermal wind. Alternatively it is written as

$$\nabla b = f\neg\frac{\partial\boldsymbol{u}}{\partial z}. \tag{65}$$

The vertical sheer of $\boldsymbol{u}$ may be expressed alternatively by vertical change of $\|\boldsymbol{u}\|$ and $\arg(\boldsymbol{u})$. We have

$$\frac{1}{\|\boldsymbol{u}\|}\frac{\partial\|\boldsymbol{u}\|}{\partial z} = \frac{1}{2\boldsymbol{u}^2}\frac{\partial\boldsymbol{u}^2}{\partial z} = \frac{\boldsymbol{u}}{\boldsymbol{u}^2}\cdot\frac{\partial\boldsymbol{u}}{\partial z}$$
$$= -\frac{\boldsymbol{u}\cdot\lhd b}{f\boldsymbol{u}^2} = -\frac{\boldsymbol{u}\times\nabla b}{f\boldsymbol{u}^2}, \tag{66}$$

and

$$\begin{aligned}\frac{\partial[\arg(\boldsymbol{u})]}{\partial z} &= \lim_{\delta z\to 0}\frac{1}{\delta z}\frac{\boldsymbol{u}\times(\boldsymbol{u}+\frac{\partial \boldsymbol{u}}{\partial z}\delta z)}{\|\boldsymbol{u}\|^2}\\ &= \frac{\boldsymbol{u}\times\frac{\partial \boldsymbol{u}}{\partial z}}{\|\boldsymbol{u}\|^2} = \frac{\boldsymbol{u}\cdot\neg\frac{\partial \boldsymbol{u}}{\partial z}}{\|\boldsymbol{u}\|^2}\\ &= \frac{\boldsymbol{u}\cdot\nabla b}{f\|\boldsymbol{u}\|^2}. = -\frac{\boldsymbol{u}\times\lhd b}{f\|\boldsymbol{u}\|^2}. \end{aligned} \tag{67}$$

Comparing (66) and (67), we again confirm symmetry between the vertical change of norm and argument of $\boldsymbol{u}$ through the exchange of "$\cdot$" and "$\times$".

If we further assume the conservation of b in the steady state

$$\boldsymbol{u}\cdot\nabla b + w\frac{\partial b}{\partial z} = 0, \tag{68}$$

we may rewrite (67) as

$$\frac{\partial[\arg(\boldsymbol{u})]}{\partial z} = -\frac{\partial b}{\partial z}\frac{w}{f\boldsymbol{u}^2}. \tag{69}$$

Usually oceans are stably stratified and $\frac{db}{dz} > 0$. The argument of horizontal current vector $\boldsymbol{u}$ rotates with depth according as the sign of w. This veering of ocean current with depth in association with density field is called β-spiral (Stommel and Schott 1977)[3]. In subtropic oceans, $w < 0$ in subsurface layers due to Ekman pumping, so that the horizontal current vector changes its direction with depth in the same way as the Ekmann spiral. The β-spiral has been argued in the theory of ventilated thermocline (Luyten et al. 1983, Pedlosky 1996) [4)5)].

4. Lagrange's formula

The subject of this section is an application of GFDVN to Lagrange's formula

$$\underline{\boldsymbol{a}}\times(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}}) = (\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{c}})\underline{\boldsymbol{b}} - (\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{b}})\underline{\boldsymbol{c}} \tag{70}$$

for three-dimensional vectors $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, and $\underline{\boldsymbol{c}}$. The formula yields

$$\begin{aligned}\nabla(\underline{\boldsymbol{u}}\cdot\underline{\boldsymbol{v}}) &= (\underline{\boldsymbol{u}}\cdot\nabla)\underline{\boldsymbol{v}} + (\underline{\boldsymbol{v}}\cdot\nabla)\underline{\boldsymbol{u}}\\ &+\underline{\boldsymbol{u}}\times(\nabla\times\underline{\boldsymbol{v}}) + \underline{\boldsymbol{v}}\times(\nabla\times\underline{\boldsymbol{u}}).\end{aligned} \tag{71}$$

when extended to $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, and $\underline{\nabla}$ instead of $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, and $\underline{\boldsymbol{c}}$. Equation (71) is used frequently in fluid dynamics to derive Bernoulli's theorem or vorticity equation (Batchelor 1967, e.g.)[6)].

So far as I know, however, Lagrange's formula is difficult to understand as it stands. This section aims at revealing its meaning. We first give a simple geometrical image in two-dimensions and shows that it is extended to three-dimensional space or to higher-dimensional ones. Also we refer to a somewhat unexpected fact that Lagrange's formula is related with the mixing ratio of three water types on a T-S diagram used in oceanography.

4.1 2-D Lagrange's formula

If all the three vectors $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, and $\underline{\boldsymbol{c}}$ lie on a plane. Lagrange's formula becomes

$$(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c})\neg\boldsymbol{a}, \tag{72}$$

which we call "2-D Lagrange's formula" for brevity.

It is derived by calculus for each component of the vector on the left-hand side. Also a direct tensor calculation gives its derivation as

$$\begin{aligned}(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} &\models \sum_k (a_k c_k b_i - a_k b_k c_i)\\ &= \sum_k a_k(c_k b_i - b_k c_i)\\ &= \sum_{k,l,m} a_k(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})c_l b_m\\ &= \sum_{k,l,m} a_k\epsilon_{ki}\epsilon_{lm}c_l b_m = \sum_{k,l,m}\epsilon_{ik}a_k\epsilon_{ml}b_m c_l\\ &= \sum_{l,m}(\epsilon_{ml}b_m c_l)\sum_k \epsilon_{ik}a_k\\ &\models (\boldsymbol{b}\times\boldsymbol{c})\neg\boldsymbol{a},\end{aligned}$$

where we have used formulas of sections 2.1 and 2.2.

4.2 2-D Lagrange's formula and skew coordinates

Another view of 2-D Lagrange's formula is provided by skew coordinates. When $\boldsymbol{b}$ is not parallel to $\boldsymbol{c}$, the two vectors make a basis of the plane $\boldsymbol{R}^2$. Then there are $\lambda\in\boldsymbol{R}$ and $\mu\in\boldsymbol{R}$ such that

$$\neg\boldsymbol{a} = \lambda\boldsymbol{b} + \mu\boldsymbol{c}.$$

The numbers λ and μ are called components of the skew coordinate system with the basis $\{\boldsymbol{b},\boldsymbol{c}\}$. Taking the outer product with $\boldsymbol{c}$ from right, we have

$$\neg\boldsymbol{a}\times\boldsymbol{c} = \lambda\boldsymbol{b}\times\boldsymbol{c}, \qquad (\because\ \ \boldsymbol{c}\times\boldsymbol{c} = 0)$$

which yields

$$\lambda = \frac{\neg\boldsymbol{a}\times\boldsymbol{c}}{\boldsymbol{b}\times\boldsymbol{c}} = \frac{\neg\boldsymbol{a}\cdot\neg\boldsymbol{c}}{\boldsymbol{b}\times\boldsymbol{c}} = \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{\boldsymbol{b}\times\boldsymbol{c}}.$$

The same augument gives μ, leadeing to

$$\neg\boldsymbol{a} = \frac{(\boldsymbol{a}\cdot\boldsymbol{c})}{\boldsymbol{b}\times\boldsymbol{c}}\boldsymbol{b} + \frac{(\boldsymbol{a}\cdot\boldsymbol{b})}{\boldsymbol{c}\times\boldsymbol{b}}\boldsymbol{c},$$

which is rewritten easily as (72).

This derivation is expressed in a somewhat different way. First we define

$$\boldsymbol{b}_\perp \equiv \frac{\neg\boldsymbol{c}}{\boldsymbol{b}\cdot\neg\boldsymbol{c}}, \quad \boldsymbol{c}_\perp \equiv \frac{\neg\boldsymbol{b}}{\boldsymbol{c}\cdot\neg\boldsymbol{b}}$$

so that

$$\begin{cases}\boldsymbol{b}_\perp\cdot\boldsymbol{b} = 1, & \boldsymbol{b}_\perp\cdot\boldsymbol{c} = 0\\ \boldsymbol{c}_\perp\cdot\boldsymbol{c} = 1, & \boldsymbol{c}_\perp\cdot\boldsymbol{b} = 0\end{cases}.$$

That is, $\boldsymbol{b}_\perp$ is chosen so that it is perpendicular to the subspace spanned by $\boldsymbol{c}$ and $\boldsymbol{b}\cdot\boldsymbol{b}_\perp = 1$. The same is true for $\boldsymbol{c}_\perp$.

If we take $\{\boldsymbol{b}_\perp,\ \boldsymbol{c}_\perp\}$ as another basis of $\boldsymbol{R}^2$, we have

$$\boldsymbol{a} = {}^{\exists}\beta\boldsymbol{b}_\perp + {}^{\exists}\gamma\boldsymbol{c}_\perp.$$

Taking the inner product with $\boldsymbol{b}$ or $\boldsymbol{c}$, we have

$$\begin{cases} (\boldsymbol{a}\cdot\boldsymbol{b}) = \beta(\boldsymbol{b}_\perp\cdot\boldsymbol{b}) = \beta \\ (\boldsymbol{a}\cdot\boldsymbol{c}) = \gamma(\boldsymbol{c}_\perp\cdot\boldsymbol{c}) = \gamma \end{cases},$$

from which follows

$$\begin{aligned} \boldsymbol{a} &= (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}_\perp + (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{c}_\perp \\ &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\neg\boldsymbol{c}}\neg\boldsymbol{c} + \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{\boldsymbol{c}\cdot\neg\boldsymbol{b}}\neg\boldsymbol{b} \\ &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\times\boldsymbol{c}}\neg\boldsymbol{c} - \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{\boldsymbol{b}\times\boldsymbol{c}}\neg\boldsymbol{b}. \end{aligned}$$

Operating $\neg$, we obtain

$$(\boldsymbol{b}\times\boldsymbol{c})\neg\boldsymbol{a} = -(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} + (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}.$$

It is worth while to note that in the skew coordinates with the basis $\{\boldsymbol{b},\boldsymbol{c}\}$, the following reciprocal expressions do hold:

$$\boldsymbol{a} = \begin{cases} (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}_\perp + (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{c}_\perp \\ (\boldsymbol{a}\cdot\boldsymbol{b}_\perp)\boldsymbol{b} + (\boldsymbol{a}\cdot\boldsymbol{c}_\perp)\boldsymbol{c} \end{cases}. \qquad (73)$$

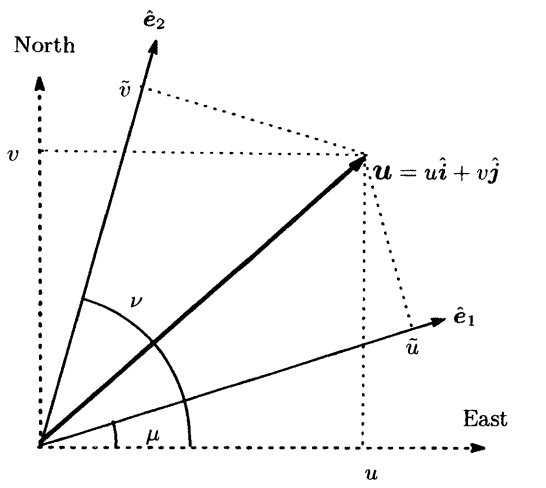

Fig. 4 Radar observes the radial componet of the current vector. In the figure two components $\tilde{u} = \boldsymbol{u}\cdot\hat{\mathbf{e}}_1$ and $\tilde{v} = \boldsymbol{u}\cdot\hat{\mathbf{e}}_2$ are measured by two radars in the direction of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, respectively. Those components are used to synthesize the current vector $\boldsymbol{u} = \tilde{u}\boldsymbol{e}_{1\perp} + \tilde{v}\boldsymbol{e}_{2\perp}$, where $\boldsymbol{e}_{1\perp} \equiv \neg\boldsymbol{e}_2/(\boldsymbol{e}_1\times\boldsymbol{e}_2)$ and $\boldsymbol{e}_{2\perp} \equiv \neg\boldsymbol{e}_1/(\boldsymbol{e}_2\times\boldsymbol{e}_1)$, as is explained in the text.

An oceanographic application of skew coordinates are found in radar observation of surface current $\boldsymbol{u}$ (Yamashita et al. 2004, Yoshikawa et al. 2006, Masuda 2007, Yoshikawa and Masuda 2009) [9)10)11)12)]. Figure 4 shows the radar coordinates. Radar measures only the radial components of current $\tilde{u} \equiv \boldsymbol{u}\cdot\hat{\mathbf{e}}_1$ and $\tilde{v} \equiv \boldsymbol{u}\cdot\hat{\mathbf{e}}_2$, where $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are unit vectors pointing the radial direction of the first and second radar. In order to apply the former formula of (73) we replace $\boldsymbol{a}$ by $\boldsymbol{u}$, and $\boldsymbol{b}$ ($\boldsymbol{c}$) by $\hat{\mathbf{e}}_1$ ($\hat{\mathbf{e}}_2$). It then follows that

$$\begin{aligned} \boldsymbol{u} &= (\boldsymbol{u}\cdot\hat{\mathbf{e}}_1)\boldsymbol{e}_{1\perp} + (\boldsymbol{u}\cdot\hat{\mathbf{e}}_2)\boldsymbol{e}_{2\perp} \\ &= \tilde{u}\boldsymbol{e}_{1\perp} + \tilde{v}\boldsymbol{e}_{2\perp}, \end{aligned}$$

where

$$\boldsymbol{e}_{1\perp} \equiv \frac{\neg\hat{\mathbf{e}}_2}{\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2}, \qquad \boldsymbol{e}_{2\perp} \equiv \frac{\neg\hat{\mathbf{e}}_1}{\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1}.$$

Note that $\|\boldsymbol{e}_{j\perp}\| \neq 1$, whereas $\|\hat{\mathbf{e}}_j\| = 1$ $(j = 1,2)$. When $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2$ is close to 0 (i.e. when $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ is almost papallel to each other), vectors $\boldsymbol{e}_{j\perp}$ is so large that a smallest error of measured $\tilde{u}_j$ leads to an enormous error in the estimation of $\boldsymbol{u}$.

4.3 Symmetric form of 2-D Lagrange's formula and *T*-*S* diagram

More symmetrically (72) is rewritten as

$$(\neg\boldsymbol{a}\times\boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b}\times\boldsymbol{c})\neg\boldsymbol{a} + (\boldsymbol{c}\times\neg\boldsymbol{a})\boldsymbol{b} = 0. \qquad (74)$$

Replacing $\neg\boldsymbol{a}$ by $\boldsymbol{a}$ yields a perfectly symmetric identity

$$(\boldsymbol{a}\times\boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{a} + (\boldsymbol{c}\times\boldsymbol{a})\boldsymbol{b} = 0. \qquad (75)$$

It is worth while to note that (75) is related with T-S analysis of sea water (Picard ad Emery 1985, Oceanography course team of open university 1989)[7) 8)].

Figure 5 shows a T-S diagram, where T and S denote temperature and salinity, respectively. Let P_j be the j-th water type $(1 \le j \le 3)$, whose coordinates on the diagram are their T and S values. Mixing of the three water types yields an intermediate type of water designated by point Q on the diagram. Then the position vector of Q ($\mathrm{Q} \models \boldsymbol{Q}$) is expressed as

$$\boldsymbol{Q} = \sum_{j=1}^{3} r_j\boldsymbol{P}_j,$$

where $\mathrm{P}_j \models \boldsymbol{P}_j$ is the position vector of the j-th water type, and r_j is the mixing ratio of the j-th water type satisfying

$$\sum_{j=1}^{3} r_j = 0. \qquad (76)$$

From geometry it follows easily that

$$r_3 = \frac{\triangle QP_1P_2}{\triangle P_1P_2P_3},$$

where $\triangle$ ABC means the area of triangle ABC. Similar expressions are obtained for r_2 and r_3 by cycling the index. Putting the origin of the coordinates at Q (Q $\models$ 0) and defining

$$\boldsymbol{a} \equiv \overrightarrow{\mathrm{QP}_1}, \quad \boldsymbol{b} \equiv \overrightarrow{\mathrm{QP}_2}, \quad \boldsymbol{c} \equiv \overrightarrow{\mathrm{QP}_3},$$

we find

$$0 = r_1\boldsymbol{a} + r_2\boldsymbol{b} + r_3\boldsymbol{c},$$

or

$$0 = (\triangle\, QP_2P_3)\boldsymbol{a} + (\triangle\, QP_3P_1)\boldsymbol{b} + (\triangle\, QP_1P_2)\boldsymbol{c}. \tag{77}$$

If we notice

$$\begin{cases} 2\triangle\, QP_1P_2 = \boldsymbol{a}\times\boldsymbol{b} \\ 2\triangle\, QP_2P_3 = \boldsymbol{b}\times\boldsymbol{c} \\ 2\triangle\, QP_3P_1 = \boldsymbol{c}\times\boldsymbol{a} \end{cases}, \tag{78}$$

it is easy to see that (77) expresses the symmetric form of 2-D Lagrange's formula (75).

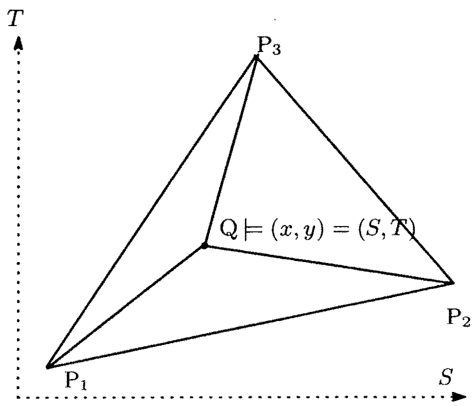

Fig. 5 Temperature-Salinity dagram (T-S diagram) and Lagrange's formula. Three water types designated by points P_j ($1 \le j \le 3$) is mixed to produce an intermediate water type designated by point Q. This geometry of T-S diagram is related with a symmetric form of Lagrnage's formula.

4.4 Derivation of the 3-D Lagrange's formula from the 2-D one

Lagrange's formula in the three-dimensional space is derived from the two-dimensional formula as follows.

When $\underline{\boldsymbol{b}}$ is parallel to $\underline{\boldsymbol{c}}$, either side of (70) vanishes, so that the formula is valid. When $\underline{\boldsymbol{b}}$ is independent of $\underline{\boldsymbol{c}}$, $\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$ span a plane, which is considered to be a horizontal plane without loss of generality. Then using the convention of GFDVN we may write

$$\underline{\boldsymbol{b}} = \boldsymbol{b}, \quad \underline{\boldsymbol{c}} = \boldsymbol{c}, \quad \underline{\boldsymbol{a}} = \boldsymbol{a} + a_z. \tag{79}$$

where a_z is the vertical component of $\underline{\boldsymbol{a}}$.

By GFDVN we may write as

$$\begin{aligned} \underline{\boldsymbol{a}} \times (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}}) &= (\boldsymbol{a} + a_z) \times (\boldsymbol{b} \times \boldsymbol{c}) \\ &= \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + a_z \times (\boldsymbol{b} \times \boldsymbol{c}) \\ &= (\boldsymbol{b} \times \boldsymbol{c})\neg\boldsymbol{a} \\ &= (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \end{aligned}$$

where the two-dimensional result (72) has been used. The right-hand side above further becomes

$$\begin{aligned} &((\boldsymbol{a} + a_z)\cdot\boldsymbol{c})\boldsymbol{b} - ((\boldsymbol{a} + a_z)\cdot\boldsymbol{b})\boldsymbol{c} \\ &= (\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{c}})\underline{\boldsymbol{b}} - (\underline{\boldsymbol{a}}\cdot\underline{\boldsymbol{b}})\underline{\boldsymbol{c}} \end{aligned}$$

because of trivial identities

$$\begin{cases} a_z\cdot\boldsymbol{b} = 0 \\ a_z\cdot\boldsymbol{c} = 0 \end{cases}$$

and by virtue of the convention of GFDVN. Thus we have derived three-dimensional Lagrange's formula from 2-D Lagrange's formula.

5. Summary and discussion

Two-dimensional vectors on a plane $\boldsymbol{R}^2$ are visible and appeal to intuition. They are easy to handle with and help us understand vector properties in even higher dimensions. This article presents a system of notations called GFDVN. That is adapted to two-dimensional vectors and intended for use especially in geophysical fluid dynamics.

Of course we can do without $\neg$, $\lhd$, or GFDVN; one may continue using traditional notations (3DVN). As has been fully exemplified and illustrated in the paper, however, GFDVN surely provides, I believe, an efficient way for describing geophysical fluid dynamics in a vivid and concise manner.

Although we have dealt with real vectors so far, operators $\neg$ and $\lhd$ may be applied to complex vectors too, by the definition through $\epsilon_{i,j}$. We must keep it in mind that $\neg$ and $\lhd$ have a meaning only for two-dimensional vectors, whereas ∇ applies to space of any dimensions.

In the last application we have seen that Lagrange's formula is understandable as a representation of a vector in skew coordinates. Unexpectedly it is related with the mixing ratio of three water types on a T-S diagram in oceanography.

Extension to higher dimension of Lagrange's formula is not difficult, but it is omitted here. This short note has not referred to another subject that is intimately related with GFDVN. That is, *complex* representation of *real* two-dimensional vectors. Complex representation allows an arithmetic means for vector operation. For example, $\neg$ corresponds to the multiplication of $-i = -\sqrt{-1}$ for complex numbers. Also we see that

2-D Lagrange's formula is derived quite easily by the complex arithmetic.

If there is an opportunity, I would like to revisit those subjects in future.

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Appendix

A1. supplement to 4.3

In section 4.3 we have seen that symmetric 2-D Lagrange's formual (75) is proved from the geometrical consideration of a triangle, when point Q is inside the triangle and $\boldsymbol{a} = \boldsymbol{P}_1 - \boldsymbol{Q}$, $\boldsymbol{b} = \boldsymbol{P}_2 - \boldsymbol{Q}$, $\boldsymbol{c} = \boldsymbol{P}_3 - \boldsymbol{Q}$. Is (75) still valid in different situatios such as $\boldsymbol{a} = \boldsymbol{P}_1$, $\boldsymbol{b} = \boldsymbol{P}_2$, and $\boldsymbol{c} = \boldsymbol{P}_3$? That is the subject here.

We start with putting

$$\begin{cases} \tilde{\boldsymbol{a}} \equiv \boldsymbol{a} - \boldsymbol{Q} \\ \tilde{\boldsymbol{a}} \equiv \boldsymbol{a} - \boldsymbol{Q} \\ \tilde{\boldsymbol{a}} \equiv \boldsymbol{a} - \boldsymbol{Q} \end{cases} . \tag{A1}$$

Provided $Q \models \boldsymbol{Q}$ is inside the triangle $\mathrm{P_1P_2P_3}$, (75) does hold, so that

$$0 = (\tilde{\boldsymbol{a}} \times \tilde{\boldsymbol{b}})\tilde{\boldsymbol{c}} + (\tilde{\boldsymbol{b}} \times \tilde{\boldsymbol{c}})\tilde{\boldsymbol{a}} + (\tilde{\boldsymbol{c}} \times \tilde{\boldsymbol{a}})\tilde{\boldsymbol{b}}. \tag{A2}$$

When $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{b}}$, and $\tilde{\boldsymbol{c}}$ are ordered clockwise (counter to the order in Fig. 5), the areas of small triangles become

$$\begin{cases} 2 \triangle QP_1P_2 = -\tilde{\boldsymbol{a}} \times \tilde{\boldsymbol{b}} \\ 2 \triangle QP_2P_3 = -\tilde{\boldsymbol{b}} \times \tilde{\boldsymbol{c}} \\ 2 \triangle QP_3P_1 = -\tilde{\boldsymbol{c}} \times \tilde{\boldsymbol{a}} \end{cases} .$$

Obviously (A2) still does hold in this case, too.

Direct calculus of the right-hand side of (A2) will yield (75), but an indirect approach will be simpler. The first term of (A2) is written as

$$\begin{aligned} &(\tilde{\boldsymbol{a}} \times \tilde{\boldsymbol{b}})\tilde{\boldsymbol{c}} \\ &\quad = (\boldsymbol{a} \times \boldsymbol{b} + [\boldsymbol{Q} \times (\boldsymbol{a} - \boldsymbol{b})])[\boldsymbol{c} - \boldsymbol{Q}] \\ &\quad = (\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{c} - (\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{Q} \\ &\qquad +[\boldsymbol{Q} \times (\boldsymbol{a} - \boldsymbol{b})])\boldsymbol{c} - [\boldsymbol{Q} \times (\boldsymbol{a} - \boldsymbol{b})])\boldsymbol{Q}. \end{aligned}$$

The same is true for the second or third term. Consequently the right-hand side of (A2) is considered quadratic polynomials of $Q_i \models \boldsymbol{Q}$. We note that (A2) holds for arbitrary Q_i. Accordingly the sum of the terms of the zeroth order with respect to Q_i must vanish. That sum of the zeroth order terms are no other than the right-hand side of (75). Therefore (75) does hold for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ on a plane.