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Abstract

The initial aim of the present paper is to provide a complete description for the eigenvalue problem of the non-commutative harmonic oscillator (NcHO), which is given by a two-by-two system of parity-preserving ordinary differential operator [19], in terms of Heun's ordinary differential equations, the second order Fuchsian differential equations with four regular singularities in a complex domain. This description has been achieved for odd eigenfunctions in Ochiai [16] nicely but missing for even eigenfunctions up to now. As a by-product of this study, using the monodromy representation of Heun's equation, we prove that the multiplicity of the eigenvalue of the NcHO is at most two. Moreover, we give a condition for the existence of a finite-type eigenfunction (essentially, given by a finite sum of Hermite functions) of the eigenvalue problem and an explicit example of such eigenvalues, from which one finds that doubly degenerate eigenstates of the NcHO actually exist even in the same parity. In the final section, as the second main purpose of this paper, we discuss a connection between the quantum Rabi model [2, 13, 28] and the operator naturally arising from the NcHO through the oscillator representation of the Lie algebra $\mathfrak{sl}_2$, by the general confluence procedure for Heun's equation and Langlands' quotient realization of a different representation of $\mathfrak{sl}_2$.

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1 Introduction

In recent years, special attention has been paid to studying the spectrum of self-adjoint operators with non-commutative coefficients, in other words, interacting quantum systems, like the quantum Rabi model, the Jaynes-Cumming (JC) model, etc., not only in mathematics [4, 5] but also in theoretical physics [13, 2, 14, 26] and experimental physics (see e.g. [3, 29]). For instance, the quantum Rabi model [22] is known to be the simplest model used in quantum optics to describe interaction of light and matter and the JC model is the widely studied rotating-wave approximation of the Rabi model [13, 5]. The non-commutative harmonic oscillator (NcHO) Q defined below has been expected similarly to provide one of these Hamiltonians describing such quantum interacting systems.

The purposes of this paper are providing explicit descriptions of i) the eigenvalue problem of NcHO in terms of Heun's operators, ii) degeneration of eigenvalues and explicit examples, and iii) connection between NcHO and the quantum Rabi model through the confluence process of Heun's ODE using representation theory of Lie algebra $\mathfrak{sl}_2$.

The normal form Q of NcHO ([20, 21, 19]) is given by

$$Q = Q_{(\alpha,\beta)} = A\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) + J\left(x\frac{d}{dx} + \frac{1}{2}\right),$$

where the mutually non-commuted (in general) coefficients A and J are given by

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

From the definition, Q is obviously a parity-preserving differential operator. We assume that $\alpha, \beta > 0$ and $\alpha\beta > 1$ throughout the paper. The former requirement comes from the formal self-adjointness of the operator Q relative to the natural inner product on $L^2(\mathbb{R}, \mathbb{C}^2)(= \mathbb{C}^2 \otimes L^2(\mathbb{R}))$. The latter guarantees that the eigenvalues of the eigenvalue problem $Q\varphi = \lambda\varphi$ ($\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$) are all positive and form a discrete set with finite multiplicity. It should be first noted that, Q is unitarily equivalent to a couple of quantum harmonic oscillators when $[A, J] = 0$, i.e. $\alpha = \beta$ holds, whence the eigenvalues are explicitly calculated as $\{\sqrt{\alpha^2 - 1}(n + \frac{1}{2}) \mid n \in \mathbb{Z}_{\geq 0}\}$ having multiplicity 2 ([21], I). Actually, when $\alpha = \beta$, there exists a structure behind Q corresponding to the tensor product of the two dimensional trivial representation and the oscillator representation [8] of the Lie algebra $\mathfrak{sl}_2$ ([21]). However, the clarification of the spectrum in the general case where $\alpha \neq \beta$ is considered to be highly non-trivial (see [10, 18, 6], also references in [19]). It is, nevertheless, worth noticing that the spectral zeta function of Q [9] (which is essentially given by the Riemann zeta function if $\alpha = \beta$) yields a new number theoretic study including the subjects such as elliptic curves, modular forms, Eichler integrals and their natural generalization (see [11, 12] and references therein).

We have constructed in the eigenfunctions and eigenvalues [21] in terms of continued fractions determined by a certain three terms recurrence relation, which can be derived from the expansion of eigenfunctions relative to a basis constructed by suitably twisting the classical Hermite functions. We call the eigenfunction $\varphi(x)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ is of a finite type if $\varphi(x)$ can be expanded by a finite number of this Hermite basis. The eigenvalue corresponding to the finite-type eigenfunction is called a finite type. Otherwise, we call the eigenvalues/eigenfunction an infinite type. We denote Σ_0 (reps. Σ_∞) the set of eigenvalues corresponding to eigenfunctions of finite (reps. infinite) type. Since the operator Q preserves the parity, we define $\Sigma^\pm$ to be the set of eigenvalues whose eigenfunctions are even/odd, that is, those satisfying $\varphi(-x) = \pm\varphi(x)$. Then there is a classification of eigenvalues; $\Sigma_0^\pm = \Sigma_0 \cap \Sigma^\pm$ (resp. $\Sigma_\infty^\pm = \Sigma_\infty \cap \Sigma^\pm$) corresponding to even/odd eigenfunctions of finite (resp. infinite) type. In [21], it is shown that $\Sigma_0^\pm \subset \Sigma_\infty^\pm$ and the multiplicity of each $\lambda \in \Sigma_\infty^+$ (resp. Σ_∞^-) is at most 2. This means that once the eigenvalue degenerates in the same parity, one of the eigenfunction is of the form $p(x) \times e^{-ax^2}$, $p(x)$ being a polynomial and a constant $a = a(\sqrt{\alpha\beta - 1}) > 0$, and this resembles the situation for the case of the Rabi model [13, 2] (see also [25, 28]). The spectral analysis of the Rabi model seems to be much simpler than the one of NcHO, while the latter seems to share certain interesting properties the former has. Actually, one finds that essentially the NcHO gives the Rabi model through the confluence limit procedure at the stage of Heun equations' picture (see §5).

It is known [15] that the eigenvalues of NcHO build a continuous curve with arguments α and β. It comes as an important problem to analyze the behavior of eigenvalue curves, in particular, a main issue of present day research, especially in mathematical physics, addresses the characterization of crossing/avoided crossing of eigenvalue curves (see e.g. [24, 4, 5]). From the observation in [15], since the eigenvalue curves are continuous, one can observe that $\Sigma_\infty^+ \cap \Sigma_\infty^- \neq \emptyset$ (see Figure 1 in [15], p.648; the graph of eigenvalue curves is drawn with respect to the variable $s = \beta/\alpha$ with a fixed α; $\alpha = 3.0$). However, one does not know whether $\Sigma_0^+ \cap \Sigma_\infty^-$ (resp. $\Sigma_0^- \cap \Sigma_\infty^+$) is empty or not, while it is shown in [21] that $\Sigma_0^+ \cap \Sigma_0^- = \emptyset$. Therefore, the multiplicity of eigenvalue is at most 3 and may reach 3. In this paper, we will solve one of the longstanding problems for the eigenstate degeneration of NcHO, in particular, prove that $\Sigma_0^+ \cap \Sigma_\infty^- = \Sigma_0^- \cap \Sigma_\infty^+ = \emptyset$ (Theorem 1.2).

Generally, in harmonic analysis on the real line, even/odd eigenspaces have completely analogues structures. Also, in view of the description of the lowest eigenvalue, the study on even eigenstates seems rather important. Moreover, we could not see any difference between the even/odd eigenspaces in the study [20, 21]. However, in the complex domain picture drawn in [16], the odd part Σ^- corresponds to the second order equation given by Heun's ordinary differential equation whereas the even part Σ^+ corresponds to the third-order equation (constructed by Heun's operator). Therefore, working out a solution to this asymmetry has been desirable for a long period. In this paper, we prove that there exists a completely parallel structure of even eigenfunctions with that of the odd eigenfunctions. For readers' convenience we state the results for the odd cases obtained in [16] in a parallel way. In fact, one of the main techniques to derive this correspondence is based on a brilliant idea developed in [16] but employing a modified Laplace transform different from that in [16] as an intertwiner (see Proposition 2.2).

Theorem 1.1. *There exist linear bijections:*

$$\text{Even}: \quad \{\varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid Q\varphi = \lambda\varphi,\ \varphi(-x) = \varphi(x)\} \xrightarrow{\sim} \{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\},$$
$$\text{Odd}: \quad \{\varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid Q\varphi = \lambda\varphi,\ \varphi(-x) = -\varphi(x)\} \xrightarrow{\sim} \{f \in \mathcal{O}(\Omega) \mid H_\lambda^- f = 0\},$$

where Ω is a simply-connected domain in $\mathbb{C}$ (w-space) such that $0, 1 \in \Omega$ while $\alpha\beta \notin \Omega$, $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions on Ω, and $H_\lambda^\pm = H_\lambda^+(w, \partial_w)$ are the Heun ordinary differential operators given respectively by

$$H_\lambda^+(w, \partial_w) := \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2} - p}{w} + \frac{-\frac{1}{2} - p}{w - 1} + \frac{p + 1}{w - \alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{1}{2}(p + \frac{1}{2})w - q^+}{w(w - 1)(w - \alpha\beta)} \tag{1.1}$$

and

$$H_\lambda^-(w, \partial_w) := \frac{d^2}{dw^2} + \left(\frac{1 - p}{w} + \frac{-p}{w - 1} + \frac{p + \frac{3}{2}}{w - \alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{3}{2}pw - q^-}{w(w - 1)(w - \alpha\beta)}. \tag{1.2}$$

Here the numbers $p = p(\lambda)$ and $\nu = \nu(\lambda)$ are defined thorough the following relation:

$$p = \frac{2\nu - 3}{4}, \quad \lambda = 2\nu \frac{\sqrt{\alpha\beta(\alpha\beta - 1)}}{\alpha + \beta}. \tag{1.3}$$

The accessory parameters $q^{\pm} = q^{\pm}(\lambda)$ in these Heun's operators are expressed by the parameters α, β and eigenvalue λ as

$$\begin{aligned} q^{+} &= \Big\{ \Big(p + \frac{1}{2}\Big)^2 - \Big(p + \frac{3}{4}\Big)^2 \Big(\frac{\beta - \alpha}{\beta + \alpha}\Big)^2 \Big\} (\alpha\beta - 1) - \frac{1}{2}\Big(p + \frac{1}{2}\Big), \\ q^{-} &= \Big\{ p^2 - \Big(p + \frac{3}{4}\Big)^2 \Big(\frac{\beta - \alpha}{\beta + \alpha}\Big)^2 \Big\} (\alpha\beta - 1) - \frac{3}{2} p. \end{aligned}$$

$\square$

We note that these Heun operators $H_\lambda^{\pm}(w, \partial_w)$ has four regular singular points, $w = 0, 1, \alpha\beta$ and ∞. Respective Riemann's scheme of the operators $H_\lambda^{\pm}(w, \partial_w)$ are expressed as

$$\mathrm{H}_\lambda^{+} : \begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ;\, w \;\; q^{+} \\ 0 & 0 & 0 & \frac{1}{2} & \\ p + \frac{1}{2} & p + \frac{3}{2} & -p & -\big(p + \frac{1}{2}\big) & \end{pmatrix} \tag{1.4}$$

and

$$\mathrm{H}_\lambda^{-} : \begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ;\, w \;\; q^{-} \\ 0 & 0 & 0 & \frac{3}{2} & \\ p & p + 1 & -p - \frac{1}{2} & -p & \end{pmatrix}, \tag{1.5}$$

where each element of the first row indicates a regular singular point of $H_\lambda^{\pm}$ and those in the second and third rows are expressing the corresponding exponents.

As a corollary of this theorem we may actually provide examples of finite-type eigenvalues (and corresponding solutions). In other words, we have an even (odd) polynomial solution when the parameter $p + \frac{1}{2} \in \mathbb{N}$ (resp. $p \in \mathbb{N}$) satisfies a certain algebraic equation obtained by the determinant of Jacobi's (i.e. tri-diagonal) matrix (Proposition 4.1). It is also worth noticing that the ground state of the NcHO consists of only even functions [7], whence its simplicity follows from the result in [27]. The criterion for this simplicity (Theorem 1.1 in [27]) can be proved also by Theorem 1.1 above together with an upper bound estimate of the lowest eigenvalue given in [19] (Theorem 8.2.1) (see [27]).

Furthermore, combining results in Theorem 1.1 for even and odd eigenfunctions and using the monodromy representation of the corresponding Heun differential equations, we will show the following.

Theorem 1.2. *The multiplicity m_λ of the eigenvalue λ for the non-commutative harmonic oscillators Q is at most 2. Moreover, when $\alpha \neq \beta$, $m_\lambda = 2$ holds if and only if either of the following two cases holds:*

1. *$\lambda \in \Sigma_0^{+}$ (resp. Σ_0^{-}); in this case one has a unique (up to scalar multiples) finite type solution, i.e., a finite linear combination of the even (resp. odd) twisted Hermite functions. Moreover, λ is of the form $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L + \frac{1}{2})$ (resp. $2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L + \frac{3}{2})$) for $L \in \mathbb{N}$.*
2. *$\lambda \in \Sigma_\infty^{+} \cap \Sigma_\infty^{-}$.*

$\square$

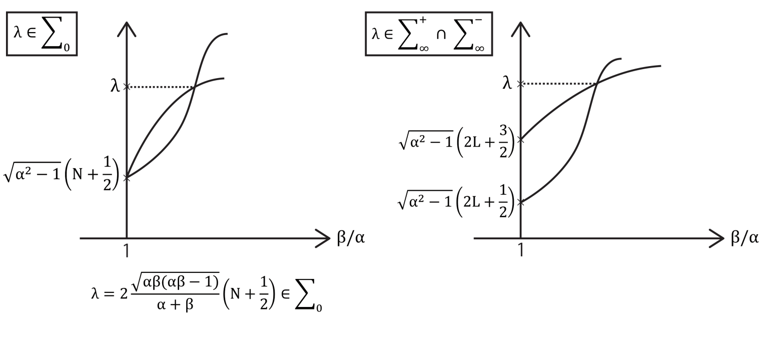

Figure 1: Examples of configuration for doubly degenerations

Remark 1.1. As to the twisted Hermite functions, see [21].

In the final section, we will discuss a connection between the operator R_{NcHO} (a degree 2 element of the universal enveloping algebra $U(\mathfrak{sl}_2)$) obtained naturally from NcHO through the oscillator representation of $\mathfrak{sl}_2$ (see Lemma 2.1) and the quantum Rabi model by taking the confluence procedure of Heun's equation (see [24, 23]). Actually, employing a certain modification of the oscillator representation ϖ_a of $\mathfrak{sl}_2$ (see Lemma 5.1), which is inequivalent to the oscillator representation when $a \neq 1, 2$ that gives the NcHO, one constructs the confluent Heun differential operator corresponding to the Rabi model from R_{NcHO}. In particular, the eigenstates degeneration of the Rabi model is obtained through the Langlands quotient of the representation on the space of even/odd Laurent polynomials. The result in §5 amounts to saying that the Rabi model is considered to be a sort of confluent version of the NcHO.

2 Representation theoretic setting

2.1 Oscillator representation of $\mathfrak{sl}_2$

Although there is no exact symmetry on Q ($\alpha \neq \beta$) described by the Lie algebra $\mathfrak{sl}_2$, like the quantum harmonic oscillator possesses, there seems to exist still a vague hidden (or modified) $\mathfrak{sl}_2$-symmetry behind it. Thus a formulation of the problem by the language of $\mathfrak{sl}_2$ is useful as we have observed in [20, 21, 16]. Moreover, as we will see in §5, in order to observe the relation between the NcHO and the Rabi model, a viewpoint employing Lie algebra representation of $\mathfrak{sl}_2$ is important.

Let H, E and F be the standard generators of the Lie algebra $\mathfrak{sl}_2$ defined by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

These satisfy the commutation relations

$$[H,\, E] = 2E,\ [H,\, F] = -2F,\ [E,\, F] = H.$$

We define the oscillator representation π of $\mathfrak{sl}_2$ by

$$\pi(H) = x\partial_x + 1/2,\ \pi(E) = x^2/2,\ \pi(F) = -\partial_x^2/2,$$

where $\partial_x = d/dx$. We will also denote the algebra homomorphism from the universal enveloping algebra $U(\mathfrak{sl}_2)$ to the ring $\mathbb{C}[x, \partial_x]$ of differential operators by the same letter π. By this realization the eigenvalue problem $Q\varphi(x) = \lambda\varphi(x)\,(u \in L^2(\mathbb{R}, \mathbb{C}^2))$ turns to be solving the following equation.

$$[A\pi(E+F) + J\pi(H) - \lambda I]\varphi(x) = 0. \tag{2.1}$$

As we have shown in [21] (see also [16]) this equation can be rewritten as

$$[\pi(E+F) + \frac{1}{\sqrt{\alpha\beta}} J\pi(H) - \lambda A^{-1}]\tilde{\varphi}(x) = 0, \tag{2.2}$$

where $\tilde{\varphi}(x) = A^{\frac{1}{2}}\varphi(x)$.

Now, as usual, let us realize the oscillator representation on the polynomial ring $\mathbb{C}[y]$ in place of $L^2(\mathbb{R})$ using the Cayley transform $C := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Define the annihilation operator $\psi = (x + \partial_x)/\sqrt{2}$ and creation operator $\psi^\dagger = (x - \partial_x)/\sqrt{2}$. Then one has $[\psi, \psi^\dagger] = 1$. Put $\varphi_0(x) := e^{-x^2/2} \in L^2(\mathbb{R})$. Then φ_0 is the ground state, that is, $\psi\varphi_0 = 0$. We define in general $\varphi_n := (\psi^\dagger)^n \varphi_0$, the Hermite functions. Then the set $\{\varphi_n \,|\, n = 0, 1, 2, \ldots\}$ forms an orthogonal basis with $(\varphi_n, \varphi_n) = \sqrt{\pi}n!$, $(\ ,\ )$ being the standard inner product of $L^2(\mathbb{R})$ (see e.g. [8]). We denote the set of all finite linear combinations of the Hermite functions φ_n by $L^2(\mathbb{R})_{\mathrm{fin}}$.

Let

$$T_C : L^2(\mathbb{R})_{\mathrm{fin}} \to \mathbb{C}[y]$$

be a linear map defined by the property $T_C(\varphi_n) = y^n$. Then one immediately sees that $T_C(\psi^\dagger\varphi) = yT_C(\varphi)$ and $T_C(\psi\varphi) = \partial_y T_C(\varphi)$. Then, if we define the representation $(\pi', \mathbb{C}[y])$ of $\mathfrak{sl}_2$ by

$$\pi'(H) = y\partial_y + 1/2,\ \pi'(E) = y^2/2,\ \pi'(F) = -\partial_y^2/2,$$

one may easily show that $\pi'(CXC^{-1})T_C = T_C\pi(X)\,(X \in \mathfrak{sl}_2)$. Moreover, if we put a Fisher inner product on $\mathbb{C}[y]$ by $(f, g)_F = \sqrt{\pi}(f(\partial_y)\bar{g}(y))|_{y=0}\,(f, g \in \mathbb{C}[y])$, one finds that $(y^m, y^n)_F = \delta_{m,n}\sqrt{\pi}n!$, whence T_C gives an isometry. If we denote the completion of $\mathbb{C}[y]$ with respect to this inner product by $\overline{\mathbb{C}[y]}$, then it follows that the map T_C can be extended to the isometry between two Hilbert spaces $L^2(\mathbb{R})$ and $\overline{\mathbb{C}[y]}$.

We now recall a basic formula in [16] (Corollary 9 with Lemma 8), which translates our eigenvalue problem (the system of differential equations) into a single differential equation.

Lemma 2.1. *Assume $\alpha \neq \beta$. Then the eigenvalue problem $Q\varphi = \lambda\varphi$ ($\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$) is equivalent to the equation $\pi'(R)u = 0$ ($u \in \mathbb{C}[y]$), where the operator $R = R_{\mathrm{NcHO}} \in U(\mathfrak{sl}_2)$ is defined by*

$$R := 2\Big[(E-F) - \coth 2\kappa H + \frac{\nu}{\sinh 2\kappa}\Big](H-\nu) + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}. \tag{2.3}$$

Here

$$\nu = \frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda,\ \ \varepsilon = \Big|\frac{\alpha-\beta}{\alpha+\beta}\Big|,\ \ \coth\kappa = \sqrt{\alpha\beta}\ \ \Big(\cosh\kappa = \sqrt{\frac{\alpha\beta}{\alpha\beta-1}},\ \ \sinh\kappa = \frac{1}{\sqrt{\alpha\beta-1}}\Big). \quad \square$$

Remark 2.1. Notice that $\pi'(R)$ is the 3rd order differential operator. Also, the correspondence $\varphi \leftrightarrow u$ in the lemma above can be given explicitly. For readers' convenience, we summarize the correspondence given in [16] briefly:

Put

$$S_\pm := E + F \pm \frac{i}{\sqrt{\alpha\beta}}H \in \mathfrak{sl}_2$$

and

$$\tilde{\varphi} := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{bmatrix}\varphi.$$

Define (invertible) transformations T and T' by

$$(Tf)(x) = e^{i(\sinh\kappa)x^2/2}(\cosh\kappa)^{1/4}f(\sqrt{\cosh\kappa}\,x) \quad \text{and} \quad (T'g)(y) = g\Big(\sqrt{\frac{\cosh\kappa}{i-\sinh\kappa}}\,y\Big).$$

Set

$$\begin{bmatrix} u \\ \tilde{u} \end{bmatrix} := T'T_CT\tilde{\varphi}.$$

Then one knows that whenever $\alpha \neq \beta$ the eigenvalue problem $Q\varphi = \lambda\varphi$ can be written as

$$\begin{bmatrix} \frac{1}{\cosh\kappa}\pi'(H) - \frac{\alpha+\beta}{2\alpha\beta}\lambda & -\frac{\alpha+\beta}{2\alpha\beta}\varepsilon\lambda \\ -\frac{\alpha+\beta}{2\alpha\beta}\varepsilon\lambda & \frac{1}{\cosh\kappa}\pi'((\cosh 2\kappa)H - (\sinh 2\kappa)(E-F)) - \frac{\alpha+\beta}{2\alpha\beta}\lambda \end{bmatrix}\begin{bmatrix} u \\ \tilde{u} \end{bmatrix} = 0.$$

Hence $Q\varphi = \lambda\varphi$ is equivalent to the equation $\pi'(R)u = 0$, and $\tilde{u} = \frac{2\alpha\beta}{(\alpha+\beta)\varepsilon}\Big[\sqrt{1-\frac{1}{\alpha\beta}}\,\pi'(H) - \frac{\alpha+\beta}{2\alpha\beta}\lambda\Big]u$. (Lemma 8 and Corollary 9 in [16]).

2.2 Intertwiners arising from Laplace transforms

In order to obtain a complex analytic picture of the equation $\pi'(R)u = 0$ in Lemma 2.1, we introduce two new realizations of the $\mathfrak{sl}_2$-triple as

$$\varpi_1(H) = z\partial_z + \frac{1}{2},\ \varpi_1(E) = \frac{1}{2}z^2(z\partial_z + 1),\ \varpi_1(F) = -\frac{1}{2z}\partial_z,$$

and

$$\varpi_2(H) = z\partial_z + \frac{1}{2},\ \varpi_2(E) = z^2(\frac{1}{2}z\partial_z + 1),\ \varpi_2(F) = -\frac{1}{2z}\partial_z + \frac{1}{2z^2}.$$

The representation ϖ_2 is the one given in [16]. Similarly to the discussion in [16] we observe the following correspondence between the representations above. We skip the proof.

Proposition 2.2. *Let $a \geq 1$. Define a modified Laplace transform $\mathcal{L}_a$ by*

$$(\mathcal{L}_a u)(z) = \int_0^\infty u(yz)e^{-\frac{y^2}{2}}y^{a-1}dy.$$

Then $\mathcal{L}_1$ possesses the following quasi-intertwining property:

$$\begin{aligned} \mathcal{L}_1\pi'(H) &= \varpi_1(H)\mathcal{L}_1, \\ \mathcal{L}_1\pi'(E) &= \varpi_1(E)\mathcal{L}_1, \\ (\mathcal{L}_1\pi'(F)u)(z) &= \varpi_1(F)(\mathcal{L}_1 u)(z) + u'(0)/(2z). \end{aligned}$$

In particular, the restriction of $\mathcal{L}_1$ to the space of even functions turns out to be an intertwiner between two representation π' and ϖ_1.

If $u(y) = \sum_{n=0}^N u_n y^n \in \mathbb{C}[y]$ then $(\mathcal{L}_1 u)(z) = \frac{1}{\sqrt{2}}\sum_{n=0}^N u_n\Gamma(\frac{n+1}{2})(\sqrt{2}z)^n$. Moreover, if one defines the inner product in z-space such that $\{z^n \mid n \in \mathbb{N}\}$ forms an orthogonal basis and $(z^n, z^n)_1 = \frac{2\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})}$, then $\mathcal{L}_1$ is an isometry. $\square$

Remark 2.2. One notes that asymptotically $(z^n, z^n)_1 \sim \sqrt{n/2}$ for large n.

Let $\mathbb{C}[z, z^{-1}]$ be the set of all Laurent polynomials in z. Since $u'(0) = 0$ for an even polynomial $u(y)$, the equivalence $(\pi', \mathbb{C}[y^2]) \cong (\varpi, \mathbb{C}[z^2])$ via $\mathcal{L}_1$ as irreducible representations of $\mathfrak{sl}_2$ is obvious. Moreover, by the quasi-intertwiner $\mathcal{L}_1$, we obtain the following equivalence between the odd part of the (oscillator) representation $(\pi', y\mathbb{C}[y^2])$ and the Langlands quotient of the representation $(\varpi, z\mathbb{C}[z^2, z^{-2}])$ of $\mathfrak{sl}_2$.

Corollary 2.3. *As irreducible representations of $\mathfrak{sl}_2$, $\mathcal{L}_1$ gives the equivalence*

$$(\pi', y\mathbb{C}[y^2]) \cong (\varpi_1, z\mathbb{C}[z^2, z^{-2}]/z^{-1}\mathbb{C}[z^{-2}]). \quad \square$$

Not only for readers' convenience but for the later use for the proof of Corollary 1.2, the corresponding quasi-intertwining property for $\mathcal{L}_2$ is referred from [16] as follows.

Lemma 2.4. *The operator $\mathcal{L}_2$ almost intertwines two representations π' and ϖ_2.*

$$\begin{aligned} \mathcal{L}_2\pi'(H) &= \varpi_2(H)\mathcal{L}_2, \\ \mathcal{L}_2\pi'(E) &= \varpi_2(E)\mathcal{L}_2, \\ (\mathcal{L}_2\pi'(F)u)(z) &= \varpi_2(F)(\mathcal{L}_2u)(z) - u(0)/(2z^2). \end{aligned}$$

In particular, the restriction of $\mathcal{L}_2$ on the space of odd functions defines an intertwiner between π' and ϖ_2. $\square$

The following corollary results from Proposition 2.2 and Lemma 2.4, and is a key refinement of the Theorem in [21]. For the case of $\mathcal{L}_2$, it has been established in [16].

Corollary 2.5. *The operator R satisfies the following equations:*

$$\begin{aligned} (\mathcal{L}_1\pi'(R)u)(z) &= \varpi_1(R)(\mathcal{L}_1u)(z) + (\nu - \tfrac{3}{2})u'(0)z^{-1}, \\ (\mathcal{L}_2\pi'(R)u)(z) &= \varpi_2(R)(\mathcal{L}_2u)(z) - (\nu - \tfrac{1}{2})u(0)z^{-2}. \end{aligned}$$

In particular, the eigenvalue problem $Q\varphi = \lambda\varphi$ for the even and odd case is respectivly equivalent to the equation

$$\varpi_1(R)(\mathcal{L}_1u)(z) = 0 \text{ (the even case)} \quad \text{and} \quad \varpi_2(R)(\mathcal{L}_2u)(z) = 0 \text{ (the odd case)}.$$

Here

$$\begin{aligned} \varpi_1(R) &= \Big\{(z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \tfrac{1}{2}) + \tfrac{1}{2}(z^2 - z^{-2}) + \tfrac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z + \tfrac{1}{2} - \nu) + \tfrac{2(\varepsilon\nu)^2}{\sinh 2\kappa}, \\ \varpi_2(R) &= \Big\{(z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \tfrac{1}{2}) + \tfrac{3}{2}(z^2 - z^{-2}) + \tfrac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z + \tfrac{1}{2} - \nu) + \tfrac{2(\varepsilon\nu)^2}{\sinh 2\kappa}, \end{aligned}$$

where we put $\theta_z = z\frac{\partial}{\partial z}$.

Proof. It follows from the intertwining property of the operator $\mathcal{L}_j$ and the realization of the representation ϖ_j by simple computation. $\square$

Remark 2.3. These differential operators $\varpi_1(R)$ and $\varpi_2(R)$ are of both 2nd order. Also, each differential equation above corresponds to the recurrence equation given in [21], which yields a continued fraction corresponding to an eigenfunction/eigenvalue of Q. Moreover, representation theoretically, the corollary above means that the 3rd order operator exists in the oscillator representation, while the 2nd operator exists in the flat picture of the principal series representation.

Remark 2.4. Conjugating by z of $\varpi_2(R)$ one can see ([16])

$$z^{-1}\varpi_2(R)z = \Big\{(z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \frac{3}{2}) + \frac{3}{2}(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z + \frac{3}{2} - \nu) + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

3 Heun's equations for even eigenfunctions

In this section, we first describe the Heun differential operator obtained from the differential operator $\varpi_1(R)$. Since the operators $\varpi_1(H), \varpi_1(E)$ and $\varpi_1(F)$ are invariant under the symmetry $z \to -z$, the operator ϖ_1 is invariant under this symmetry. This shows that the $\varpi_1(R)$ can be expressed in terms of the variable z^2. We therefore put $w := z^2\coth\kappa$. Note that $\coth^2\kappa = \alpha\beta (= \det A)$. Then, using $z\partial_z = 2w\partial_w$ and the relations

$$\begin{aligned} &z^2 + z^{-2} - 2\coth 2\kappa = (\tanh\kappa)w^{-1}(w-1)(w-\alpha\beta), \\ &z^2 - z^{-2} = (\tanh\kappa)w^{-1}(w^2 - \alpha\beta), \\ &2/\sinh 2\kappa = (\tanh\kappa)(\alpha\beta - 1), \end{aligned}$$

factoring out the leading coefficient of $\varpi_1(R)$ in the expression given in Corollary 2.5, by somewhat tedious computation one gets

$$\varpi_1(R) = 4(\tanh\kappa)\, w(w-1)(w-\alpha\beta)H_\lambda^+(w,\partial_w),$$

where $H_\lambda^+(w,\partial_w)$ is the Heun differential operator given by (1.1) in the Introduction:

$$H_\lambda^+(w,\partial_w) = \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2}-p}{w} + \frac{-\frac{1}{2}-p}{w-1} + \frac{p+1}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{1}{2}(p+\frac{1}{2})w - q^+}{w(w-1)(w-\alpha\beta)},$$

where $p = \frac{2\nu-3}{4}$ ($\nu = \frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda$) and the accessory parameter q^+ is given by

$$q^+ = \Big\{\Big(p+\frac{1}{2}\Big)^2 - \Big(p+\frac{3}{4}\Big)^2\Big(\frac{\beta-\alpha}{\beta+\alpha}\Big)^2\Big\}(\alpha\beta-1) - \frac{1}{2}\Big(p+\frac{1}{2}\Big).$$

By its expression, $H_\lambda^+(w,\partial_w)$ is a second-order linear differential operator with four regular singular points $0, 1, \alpha\beta$ and ∞ on $\mathbb{P}^1(\mathbb{C})$. Notice that the parameter ν designates the exponents. From these observation, we may summarize the properties of the operator $\varpi_1(R)$ as follows.

Proposition 3.1. *The second-order linear differential operator $\varpi_1(R)$ with rational coefficients in z has six singular points $z = 0, \pm1/\sqrt{\alpha\beta}, \pm\sqrt{\alpha\beta}, \infty$. Here, all these six points are of regular singular. The exponents of those singularities can be read from the following Riemann scheme:*

$$\varpi_1(R) : \begin{pmatrix} 0 & 1/\sqrt{\alpha\beta} & -1/\sqrt{\alpha\beta} & \sqrt{\alpha\beta} & -\sqrt{\alpha\beta} & \infty & ;\, z \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 2p+1 & p+\frac{3}{2} & p+\frac{3}{2} & -p & -p & -2p-1 & \end{pmatrix}. \tag{3.1}$$

□

Remark 3.1. For readers' convenience, we refer the Riemann scheme of the operator $\varpi_2(R)$ from [16]:

$$\varpi_2(R) : \begin{pmatrix} 0 & 1/\sqrt{\alpha\beta} & -1/\sqrt{\alpha\beta} & \sqrt{\alpha\beta} & -\sqrt{\alpha\beta} & \infty & ;\, z \\ 1 & 0 & 0 & 0 & 0 & 2 & \\ 2p+1 & p+1 & p+1 & -(p+\frac{1}{2}) & -(p+\frac{1}{2}) & -2p-1 & \end{pmatrix}. \tag{3.2}$$

Notice that since the operator $\varpi_1(R)$ is regular singular at the origin, any formal series solution of $f \in \mathbb{C}[[z]]$ of the equation $\varpi_1(R)f = 0$ coverges to a holomorphic function near the origin. Therefore, using a similar discussion in [16] about L^2-conditions on $\mathbb{R}$ (or convergence conditions in $\overline{\mathbb{C}[y]}$) and analytic continuation to a simply-connected region Ω' containing $0, \pm1/\sqrt{\alpha\beta}$ in the z-space, one can conclude that the spectral problem for the non-commutative harmonic oscillator Q is equivalent to that of finding all the holomorphic solutions $f(z) \in \mathcal{O}(\Omega')$ of the differential equation $\varpi_1(R)f = 0$. This obviously gives the assertion in Theorem 1.1.

4 Degeneration of eigenstates

In this section, we discuss degeneration of eigenstates, that is, focus on eigenvalues of finite-type and multiplicities. We give an example of finite-type eigenvalues and the proof of Theorem 1.2, which claims that the multiplicity of any eigenvalue of Q is at most 2 and actually reaches 2 in the identical parity.

4.1 Polynomial solutions of $\varpi_1(R)f = 0$

Recall first Theorem 1.1 in [21] that the finite-type eigenvalues are of the form

$$\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(N+\frac{1}{2}) \quad (N \in \mathbb{Z}_{\geq 0}).$$

This implies that $\nu = \lambda\delta\cosh\kappa = N + \frac{1}{2}$ if we have a polynomial solution of $\varpi_1(R)f = 0$. Suppose that $p(z) = \sum_{n=0}^{L} a_n z^{2n}$ ($a_L \neq 0$) is a polynomial solution of the equation $\varpi_1(R)f = 0$ with $\nu = N + \frac{1}{2}$. Since

$$\begin{aligned}
&\varpi_1(R)z^{2n} \\
=&\Big\{(z^2+z^{-2}-2\coth 2\kappa)(2n+\frac{1}{2}) + \frac{1}{2}(z^2-z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(2n+\frac{1}{2}-\nu)z^{2n} + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}z^{2n} \\
=&(2n+1)(2n-N)z^{2n+2} + \Big[\Big\{-2\coth 2\kappa(2n+\frac{1}{2}) + \frac{2N+1}{\sinh 2\kappa}\Big\}(2n-N) + \frac{\varepsilon^2(2N+1)^2}{2\sinh 2\kappa}\Big]z^{2n} + 2n(2n-N)z^{2n-2},
\end{aligned}$$

one observes

$$\begin{aligned}\varpi_1(R)p(z) &= \sum_{n=1}^{L+1} a_{n-1}(2n-1)(2n-2-N)z^{2n}\\ &+\sum_{n=0}^{L} a_n\Big[\Big\{-2(2n+\frac{1}{2})\coth 2\kappa+\frac{2N+1}{\sinh 2\kappa}\Big\}(2n-N)+\frac{\varepsilon^2(2N+1)^2}{2\sinh 2\kappa}\Big]z^{2n}\\ &+\sum_{n=0}^{L-1} a_{n+1}2(n+1)(2n+2-N)z^{2n}=0.\end{aligned}$$

If we look at the coefficient of z^{2L+2} then $a_L(2L+1)(2L-N)=0$, whence necessarily $N=2L$ if $p\not\equiv 0$. Therefore the condition N to be even is necessary for having an even polynomial solution of $\varpi_1(R)f=0$, i.e., a finite type eigenfunction of Q by Corollary 2.5.

Now we assume that $N=2L$. Then we have

$$\begin{cases} -2a_{L-1}(2L-1)+a_L\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}=0,\\ -4a_{L-2}(2L-3)+a_{L-1}\Big[-2\Big\{-2(2L-\frac{3}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\Big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big]=0,\\ a_{n-1}(2n-1)(2n-2-2L)+a_n\Big[2\Big\{-2(2n+\frac{1}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\Big\}(n-L)+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big]\\ \qquad\qquad +4a_{n+1}(n+1)(n+1-L)=0\quad (1\le n\le L-2),\\ a_0\Big[-2L\Big\{-\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\Big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big]+4a_1(1-L)=0.\end{cases} \tag{4.1}$$

Notice that there are no monomial solutions, especially, no constant function solution for $\varpi_1(R)f=0$, whenever $\varepsilon\neq 0$, i.e. $\alpha\neq\beta$. Let us hence consider the simplest case, that is, $L=1$. Then $N=2$ and the equations above reduce to the following

$$\begin{cases} -2a_0+a_1\frac{5^2\varepsilon^2}{2\sinh 2\kappa}=0,\\ a_0\Big[-2\Big\{-\coth 2\kappa+\frac{5}{\sinh 2\kappa}\Big\}+\frac{25\varepsilon^2}{2\sinh 2\kappa}\Big]=0.\end{cases} \tag{4.2}$$

Hence, if

$$-4(-\cosh 2\kappa+5)+25\varepsilon^2=0 \tag{4.3}$$

holds, then the polynomial $p(z)=a_0+a_1z^2=a_0\Big(1+\frac{4\sinh 2\kappa}{25\varepsilon^2}z^2\Big)$ is a non-trivial solution of $\varpi_1(R)p=0$. We now observe the existence of solutions for (4.3). For simplicity we put $\alpha=1$ and $\beta>1$. Since $\varepsilon^2=\Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2$ and $\cosh 2\kappa=\frac{\alpha\beta+1}{\alpha\beta-1}$, the equation (4.3) turns to be

$$25\Big(\frac{\beta-1}{\beta+1}\Big)^2+4\frac{\beta+1}{\beta-1}-20=0.$$

Define then the cubic polynomial by

$$f(\beta)=25(\beta-1)^3+4(\beta+1)^3-20(\beta+1)^2(\beta-1).$$

Then, one observes $f(1)>0$, $f(2)<0$, $f(8)<0$, $f(9)>0$. It follows immediately that we have 2 solutions of (4.3), one is in the intervals $(1,2)$ and another is in $(8,9)$. This shows that there exists a pair (α,β) such that $Q\varphi=5\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}\varphi$ and $\varphi(-x)=\varphi(x)$.

The general theorem in [21] implies that the multiplicity of the eigenvalue $5\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}$ is 2 for this $Q=Q_{(\alpha,\beta)}$. Hence, the eigenvalue curves can be actually crossing as the numerical graph in [15] (see Figure 1 on p.648) has indicated.

In general, we define the tri-diagonal $(L+1)\times(L+1)$-matrix $B_{2L}(\alpha,\beta)=(B_{ij})_{0\le i,j\le L}$ by

$$\begin{cases} B_{0,0}\Big[-2L\Big\{-\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\Big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big],\quad B_{0,1}=4(1-L),\\ B_{n-1,n}=(2n-1)(2n-2-2L),\quad B_{n,n}=\Big[2\Big\{-2(2n+\frac{1}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\Big\}(n-L)+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big],\\ \qquad B_{n+1,n}=4(n+1)(n+1-L)=0\quad (n=1,2,\dots L-2),\\ B_{L-2,L-1}=-4(2L-3),\quad B_{L-1,L-1}=\Big[-2\Big\{-2(2L-\frac{3}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\Big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big],\\ B_{L-1,L}=-2(2L-1),\quad B_{L,L}=\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}.\end{cases} \tag{4.4}$$

Note that $B_{ij}=0$ if $|i-j|>1$.

Since there can not be existing two independent polynomial solutions of $\varpi_1(R)f=0$, we notice that the rank of the matrix satisfies $L\le \operatorname{rank}(B_{2L}(\alpha,\beta))\le L+1$. Clearly one has the following.

Proposition 4.1. *Let* $L \in \mathbb{N}$. *If* $\alpha, \beta\, (\alpha \neq \beta)$ *satisfy the algebraic equation* $\det(B_{2L}(\alpha,\beta)) = 0$, *then* $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L+\frac{1}{2}) \in \Sigma_0^+$. □

Remark 4.1. Since if $\alpha \neq \beta$ one has the coefficient $B_{L,L} \neq 0$. Thus, if we set $\tilde{B}_{2L}(\alpha,\beta) = (B_{ij})_{0\le i,j\le L-1}$ we may consider the equation $\det(\tilde{B}_{2L}(\alpha,\beta)) = 0$ in place of $\det(B_{2L}(\alpha,\beta)) = 0$.

Remark 4.2. The odd cases corresponding to Proposition 4.1 can be established in the same way.

4.2 Proof of Theorem 1.2

We prove the multiplicity m_λ of the eigenvalue λ of Q is at most 2. Since Q is unitarily equivalent to a couple of quantum harmonic oscillators when $\alpha = \beta$ ([20, 21]), one may assume that $\alpha \neq \beta$. Since one knows the multiplicity of each eigenvalue is at most 3 in [21] it is enough to show that $m_\lambda \neq 3$ for every λ.

Suppose $m_\lambda = 3$. Then we have either $\lambda \in \Sigma_0^+ \cap \Sigma_\infty^-$ or $\lambda \in \Sigma_0^- \cap \Sigma_\infty^+$. We now assume $\lambda \in \Sigma_0^+ \cap \Sigma_\infty^-$. This implies that we have $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \,|\, H_\lambda^+ f = 0\} = 2$ and λ is of the form $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L+\frac{1}{2})$ for some $L \in \mathbb{N}$. Recall the relation $p = L - \frac{1}{2}$. Then, it follows that the parameter p in the Riemann's scheme of the Heun operator H_λ^+ satisfies $p + \frac{1}{2} \in \mathbb{N}$.

Let $f_1(w)$ and $f_2(w)$ be a polynomial and another holomorphic solutions of $H_\lambda^+(w, \partial_w) f = 0$, respectively. Let $\tilde{f}_j(z)\, (j = 1, 2)$ be the respective solutions of $\varpi_1(R)\tilde{f}(z) = 0$, that is, $f_j(w) = \tilde{f}_j(z)\, (w = z^2 \coth\kappa)$. Put $u_j = \mathcal{L}_1^{-1}\tilde{f}_j$. Note that u_1 is an even polynomial in $\mathbb{C}[y]$. We may construct a constant term free even solution $u^+ \in \mathbb{C}[y]$ of $\pi'(R)u^+ = 0$ by suitably chosen linear combination of u_1 and u_2. Then, by Corollary 2.5, one verifies that $\varpi_2(R)(\mathcal{L}_2 u^+)(z) = 0$. If we put $\tilde{g}^+(z) = (\mathcal{L}_2 u^+)(z)$ and define $g^+(w)$ by the equation $g^+(w) = \tilde{g}^+(z)$, then $g^+(w) \in \mathcal{O}(\Omega)$ is a solution of $H_\lambda^- g^+ = 0$. Therefore, by assumption we conclude that $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \,|\, H_\lambda^- g = 0\} = 2$.

Now we recall the Riemann scheme of H_λ^-

$$\begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ;w\;\; q^- \\ 0 & 0 & 0 & \frac{3}{2} & \\ p & p+1 & -p-\frac{1}{2} & -p & \end{pmatrix}$$

and the monodromy representation of the differential equation $H_\lambda^- g = 0$. We consider a base point near the origin. Take a basis of local solutions at this point and denote the monodromy matrix around the singular points $0, 1, \alpha\beta$ and ∞ by A_0, A_1, A_2 and A_3, respectively. Note that $A_0A_1A_2A_3 = I$. The existence of two dimentional holomorphic solutions on Ω implies that $A_0 = A_1 = I$, that is, both 0 and 1 are apparent singular points. The monodromy matrices A_2 and A_3 have two distinct eigenvalues, 1 and -1, and thus are semisimple. Then, since $A_3 = A_2^{-1}$, the monodromy representation factors through the cyclic group of order two. We then choose a basis such that $A_2 = A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, the solution corresponding to the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is invariant under the monodromy representation. It follows that this solution is meromorphic on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, whence it is actually a rational function of w. However, since $p \in \mathbb{N} - \frac{1}{2}$, from the Riemann scheme of H_λ^-, one finds that no such rational solution can exist. This contradicts to the claim $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \,|\, H_\lambda^- g = 0\} = 2$, hence to the assumption. Therefore we have $\Sigma_0^+ \cap \Sigma_\infty^- = \emptyset$. Simlarilly one can show $\Sigma_0^- \cap \Sigma_\infty^+ = \emptyset$. This completes the proof of the theorem. □

4.3 Heun polynomials for $H_\lambda^+ f = 0$

Assume again that $\alpha \neq \beta$ in this subsection. As in the odd case $H_\lambda^- f = 0$ studied in [17] one can determine a shape of the solution corresponding to the eigenvalue Σ_0^+. In the terminology in [23] (see, p. 41) these solutions are given by the Heun polynomials. We first recall the Riemann scheme of the Heun equation $H_\lambda^+ f = 0$.

$$\begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ;w\;\; q^+ \\ 0 & 0 & 0 & \frac{1}{2} & \\ p+\frac{1}{2} & p+\frac{3}{2} & -p & -(p+\frac{1}{2}) & \end{pmatrix}.$$

The Heun polynomial, which we denote as Hp (see [23]) is, by definition, a solution of the Heun equation given by the form

$$Hp(w) = w^{\sigma_1}(w-1)^{\sigma_2}(w-\alpha\beta)^{\sigma_3}p(w),$$

where $p(w)$ is a polynomial in w, and σ_1, σ_2 and σ_3 are each one of the two possible exponents at the corresponding singularity. Notice that $p + \frac{1}{2} \in \mathbb{N}$ if the corresponding λ is in Σ_0^+.

Theorem 4.2. *Suppose that* $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = 2$. *Then, there exist Heun polynomials* $Hp_1(w)$ *and* $Hp_2(w)$ *such that* $\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = \mathbb{C}Hp_1 \oplus \mathbb{C}Hp_2$. *More precisely,* $Hp_1(w)$ *is equal to a polynomial* $p_1(w)$ *of degree at most* $p + \frac{1}{2}$ *and* $Hp_2(w) = (w - \alpha\beta)^{-p} p_2(w)$, $p_2(w)$ *being a polynomial of degree at most* $p - \frac{1}{2}$, *and these polynomials* $p_j(w)\,(j = 1, 2)$ *are unique up to scalar multiples.* □

The proof of this theorem can be done in a similar way taken in [17]. We thus omit the proof. Furthermore, as in [17], we have two converse statements of Theorem 4.2. The first one is the following.

Theorem 4.3. *Suppose that the Heun equation* $H_\lambda^+ f = 0$ *has a solution of the form* $q(w)(w - \alpha\beta)^{\frac{1}{2}}$ *at the origin, where* $q(w)$ *is a non-zero rational function. Then, one has* $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = 2$. □

The second is the case where one has a rational solution of the equation $H_\lambda^+ f = 0$.

Theorem 4.4. *Assume* $p + \frac{1}{2} \in \mathbb{N}$. *Suppose that the Heun equation* $H_\lambda^+ f = 0$ *has a non-zero rational solution of* w *at the origin. Then, one has* $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = 2$. □

The proofs of these theorems are similar to that of in [17] and left to the readers. Note that once the condition of Theorem 4.3/4.4 holds, all the assertions in Theorem 4.2 follow.

5 Connection with the Rabi model via confluence process

In this section we will observe the relation between the NcHO, more correctly, the operator R in Lemma 2.1 arising from Q and the quantum Rabi model (see [13, 2, 5, 28]). The quantum Rabi model is define by the Hamiltonian

$$H_{\text{Rabi}}/\hbar = \omega\psi^\dagger\psi + \Delta\sigma_z + g\sigma_x(\psi^\dagger + \psi).$$

Here $\psi = (x + \partial_x)/\sqrt{2}$ (resp. $\psi^\dagger = (x - \partial_x)/\sqrt{2}$) is the annihilation (ress. creation) operator for a bosonic mode of frequency ω, $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices for the two-level system, 2Δ is the energy difference between the two levels, and g denotes the coupling strength between the two-level system and the bosonic mode. For simplicity and without loss of generality we may set $\hbar = 1$ and $\omega = 1$.

In order to observe a relation between the NcHO to the Rabi model, we will consider the confluent Heun differential equation which is obtained by confluence procedure from the Heun differential equation defined by $R = R_{\text{NcHO}}$ in Lemma 2.1 via some new representation of $\mathfrak{sl}_2$. Roughly speaking, our observation shows that the quantum Rabi model can be obtained by a confluence process by R_{NcHO} through their respective Heun's pictures:

$$\text{NcHO} \underset{\pi'}{\longleftarrow} \text{R}_{\text{NcHO}} \in \text{U}(\mathfrak{sl}_2) \underset{\pi''}{\xrightarrow{\mathcal{L}_a}} \text{Heun ODE} \overset{\text{confluence process}}{\longrightarrow} \text{Confluent Heun ODE} \sim \text{Rabi model}. \tag{5.1}$$

In this picture, under the action defined by the representation π'' on $\mathbb{C}[y, y^{-1}]$ (and ϖ_a) of $\mathfrak{sl}_2$ (see §5.1 below), which is not equivalent in general to the oscillator representation π', R_{NcHO} provides a target Heun operator for obtaining the confluent Heun operator corresponding the Rabi model through the Laplace transform $\mathcal{L}_a$. Especially, one can find that the eigenstates degeneration is obtained through the Langlands quotient of the representation on the space of even/odd Laurent polynomials.

5.1 Intertwiners $\mathcal{L}_a$

In this subsection, we first derive a general Heun operator arising from the 3rd order differential operator $R = R_{\text{NcHO}}$ using the modified Laplace operator $\mathcal{L}_a$.

Let $a \in \mathbb{N}$. Define the operator T_a acting on the space of Laurent polynomials $\mathbb{C}[y, y^{-1}]$ (or $y^2\mathbb{C}[y]$) by

$$T_a := -\frac{1}{2}\partial_y^2 + \frac{(a-1)(a-2)}{2} \cdot \frac{1}{y^2}$$

Recall the modified Laplace transform $\mathcal{L}_a$ in Proposition 2.2. Then, one finds that

$$(\mathcal{L}_a T_a u)(z) = \big(-\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}\big)(\mathcal{L}_a u)(z) + \frac{1}{2z}u'(0)\delta_{a,1} - \frac{a-1}{2z^2}u(0)\delta_{a,2},$$

where $\delta_{a,k} = 1$ when $k = a$ and 0 otherwise. This can be true whenever $u(0), u'(0)$ and $(\mathcal{L}_a u)(z)$ exist.

Let $a > 2$. We now define a new representation π'' of $\mathfrak{sl}_2$ on $y^{a-1}\mathbb{C}[y]$, which depends on the deformation parameter a, by

$$\pi''(H) = \pi'(H),\ \pi''(E) = \pi'(E),\ \pi''(F) = T_a = \pi'(F) + \frac{(a-1)(a-2)}{2} \cdot \frac{1}{y^2}. \tag{5.2}$$

For $a \neq 1, 2$, one easily verifies that

$$\mathcal{L}_a \pi''(X) = \varpi_a(X)\mathcal{L}_a \ (X \in \mathfrak{sl}_2). \tag{5.3}$$

Here ϖ_a denotes the representation of $\mathfrak{sl}_2$ defined by

$$\varpi_a(H) = z\partial_z + \frac{1}{2}, \ \varpi_a(E) = \frac{1}{2}z^2(z\partial_z + a), \ \varpi_a(F) = -\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}. \tag{5.4}$$

Since $\varpi_a(E)z^{-a} = 0$, one has the following second equivalence: the representation $(\pi'', y^{2-a}\mathbb{C}[y^2])$ can be considered as the Langlans quotient of the representations $(\varpi_a, \mathbb{C}[z^2, z^{-2}])$ or $(\varpi_a, z\mathbb{C}[z^2, z^{-2}])$ depending on the parity of a (see as in Corollary 2.3).

Lemma 5.1. *The modified Laplace operator $\mathcal{L}_a$ gives the following equivalence of irreducible modules of $\mathfrak{sl}_2$.*

$$(\pi'', y^{a-1}\mathbb{C}[y^2]) \cong (\varpi_a, z^{a-1}\mathbb{C}[z^2]), \tag{5.5}$$

$$(\pi'', y^{2-a}\mathbb{C}[y^2]) \cong (\varpi_a, z^{a}\mathbb{C}[z^2, z^{-2}]/z^{-a}\mathbb{C}[z^{-2}]). \tag{5.6}$$

Moreover, the Casimir operator $Z_C := 4EF + H^2 - 2H \in U(\mathfrak{sl}_2)$ takes the value $(a-1)(a-2) - \frac{3}{4}$ in both representations $(\pi'', y^{a-1}\mathbb{C}[y^2])$ and $(\pi'', y^{2-a}\mathbb{C}[y^2])$.

Remark 5.1. Suppose $a > 0$. Then the linear map A defined by

$$A : \mathbb{C}[y] \ni u \mapsto y^{a-1}u \in y^{a-1}\mathbb{C}[y]$$

is obviously a one to one map. Using this, one may define a representation π'_a of $\mathfrak{sl}_2$ that the map A defines the intertwiner between $(\pi', \mathbb{C}[y])$ and $(\pi'_a, y^{a-1}\mathbb{C}[y])$, i.e. $A\pi'(X) = \pi'_a(X)A\ (X \in \mathfrak{sl}_2)$. One remarks that this π'_a is different from π'' introduced above, that is, π'' is not equivalent to any sub-representation of the oscillator representation. Actually, the explicit form of the action π'_a on the space $y^{a-1}\mathbb{C}[y]$ is given as

$$\pi'_a(H) = y\partial_y - a + \frac{3}{2}, \ \pi'_a(E) = \frac{1}{2}y^2, \ \pi'_a(F) = -\frac{1}{2}\partial_y^2 + \frac{a-1}{y}\partial_y - \frac{a(a-1)}{2}\cdot\frac{1}{y^2}.$$

This representation π'_a actually defines two lowest weight modules $y^a\mathbb{C}[y]$ and $y^{a-1}\mathbb{C}[y]$. The Casimir element Z_C takes $-3/4$, whence obviously the representations in Lemma 5.1 are different from any sub-representation of the oscillator representation.

Recall the operator $R = R_{\mathrm{NcHO}}$ in (2.3).

$$R = \frac{2}{\sinh 2\kappa}\Big\{\Big[(\sinh 2\kappa)(E-F) - (\cosh 2\kappa)H + \nu\Big](H-\nu) + (\varepsilon\nu)^2\Big\} \in U(\mathfrak{sl}_2).$$

Then, as before, one observes

$$\varpi_a(R) = \Big\{(z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \frac{1}{2}) + (a - \frac{1}{2})(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z + \frac{1}{2} - \nu) + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

Hence, conjugating by z^{a-1} one obtains the following Lemma. Notice that since there is a symmetry $a \leftrightarrow 3-a$, we may take $a \in \mathbb{Z}$ also to be non-positive, i.e. $a \in \mathbb{Z}\,(a \neq 1, 2)$.

Lemma 5.2. *For a general $a \in \mathbb{Z}$ one has*

$$z^{-a+1}\varpi_a(R)z^{a-1} = \Big\{(z^2+z^{-2}-2\coth 2\kappa)(\theta_z+a-\frac{1}{2})+(a-\frac{1}{2})(z^2-z^{-2})+\frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z+a-\frac{1}{2}-\nu)+\frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

□

Furthermore, since the differential operator $z^{-a+1}\varpi_a(R)z^{a-1}$ preserves a parity, if we keep setting $w = z^2\coth 2\kappa$, one has the following.

Proposition 5.3. *Let $a \in \mathbb{Z}$. One has*

$$z^{-a+1}\varpi_a(R)z^{a-1} = 4(\tanh\kappa)\, w(w-1)(w-\alpha\beta)H^a_\lambda(w, \partial_w),$$

where $H^a_\lambda(w, \partial_w)$ is the Heun differential operator given as follows.

$$H^a_\lambda(w, \partial_w) = \frac{d^2}{dw^2} + \left(\frac{3-2\nu+2a}{4w} + \frac{-1-2\nu+2a}{4(w-1)} + \frac{-1+2\nu+2a}{4(w-\alpha\beta)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a-\frac{1}{2}-\nu)w - q_a}{w(w-1)(w-\alpha\beta)}, \tag{5.7}$$

where $\nu = \frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda$, λ being the eigenvalue of Q, and the accessory parameter q_a is given by

$$q_a = -(a-\frac{1}{2})(a-\frac{1}{2}-\nu)(1+\alpha\beta) + (a-\frac{1}{2}-\nu)(\alpha\beta-1)\nu + (\varepsilon\nu)^2(\alpha\beta-1). \tag{5.8}$$

Remark 5.2. Once we have the operator $H^a_\lambda(w, \partial_w)$, it is not necessarily assuming $a \in \mathbb{Z}$.

5.2 Confluence process of the Heun equation

Put $t=\alpha\beta(>1)$ and consider the equation defined by (5.7) in the preceding subsection.

$$\frac{d^2}{dw^2}+\left(\frac{3-2\nu+2a}{4w}+\frac{-1-2\nu+2a}{4(w-1)}+\frac{-1+2\nu+2a}{4(w-t)}\right)\frac{d}{dw}+\frac{(a-\frac{1}{2})bw-q_a}{w(w-1)(w-t)}=0, \tag{5.9}$$

where we put $b:=\frac{1}{2}(a-\frac{1}{2}-\nu)$. The corresponding generalized Riemann scheme ([24]) is expressed as

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & \\ 0 & 1 & t & \infty & ;\, w \;\; q_a \\ 0 & 0 & 0 & a-\frac{1}{2} & \\ \frac{1+2\nu-2a}{4} & \frac{5+2\nu-2a}{4} & \frac{5-2\nu-2a}{4} & b & \end{array}\right). \tag{5.10}$$

Here the first line indicates the s-rank of each singularity (see [24]).

Now let us consider a confluence process of the singular points at $w=t$ and $w=\infty$ at the equation (5.9) ([24] p.100, Table 3.1.2). The corresponding process is given as $t:=\epsilon^{-1}, b:=r\epsilon^{-1}$ and $\epsilon\to 0$ and write $r:=t$. Then one has the confluent Heun equation

$$\frac{d^2}{dw^2}+\left(-t+\frac{3-2\nu+2a}{4w}+\frac{-1-2\nu+2a}{4(w-1)}\right)\frac{d}{dw}+\frac{-(a-\frac{1}{2})tw+q}{w(w-1)}=0, \tag{5.11}$$

whose generalized Riemann's scheme is given as

$$\left(\begin{array}{cccc} 1 & 1 & 2 & \\ 0 & 1 & \infty & ;\, w \;\; -q \\ 0 & 0 & a-\frac{1}{2} & \\ \frac{1+2\nu-2a}{4} & \frac{5+2\nu-2a}{4} & -\nu & \\ & & 0 & \\ & & t & \end{array}\right). \tag{5.12}$$

Here

$$\begin{aligned} q &= \lim_{\epsilon\to 0}\epsilon q_a \\ &= \lim_{\epsilon\to 0}\Big[-(a-\frac{1}{2})(a-\frac{1}{2}-\nu)(\epsilon+1)+(a-\frac{1}{2}-\nu)(1-\epsilon)\nu+(\varepsilon\nu)^2(1-\epsilon)\Big] \\ &= -(a-\frac{1}{2}-\nu)^2+(\varepsilon\nu)^2. \end{aligned}$$

Notice that $w=\infty$ is an irregular singularity with s-rank 2 (see e.g. [24], p.33).

5.3 Confluent Heun's equation derived from Rabi's model

We recall the recent result obtained in [28]. The Schrödinger equation $H_{\mathrm{Rabi}}\varphi=E\varphi$ defined by the Hamiltonian

$$H_{\mathrm{Rabi}}=\psi^\dagger\psi+\Delta\sigma_z+g\sigma_x(\psi^\dagger+\psi)$$

of the quantum Rabi model is reduced to the following 2nd order differential equation:

$$\frac{d^2f}{dw^2}+p(w)\frac{df}{dw}+q(w)f=0, \tag{5.13}$$

where

$$\begin{aligned} p(w) &= \frac{(1-2E-2g^2)w-g}{w^2-g^2}, \\ q(w) &= \frac{-g^2w^2+gw+E^2-g^2-\Delta^2}{w^2-g^2}. \end{aligned}$$

Write $f(w)=e^{gw}\phi(x)$, where $x=(g+w)/2g$. Substituting f into the equation (5.13), one finds the function ϕ satisfies the following confluent Heun equation (Type II solutions in [28]).

$$\frac{d^2\phi}{dx^2}+\left(4g^2+\frac{1-(E+g^2)}{x}+\frac{1-(E+g^2+1)}{x-1}\right)\frac{d\phi}{dx}+\frac{-4g^2(E+g^2+\frac{3}{2})x+\mu}{x(x-1)}\phi=0, \tag{5.14}$$

where the accessory parameter μ is given by

$$\mu = -3g^4/2 - (3+2E)g^2/2 + (E^2 + E - 2\Delta^2 + 1)/2 - (E+g^2)/2 + (E+5g^2+1)(E+g^2-1)/2.$$

Let us now compare two equations (5.11) and (5.14). Then the following system of equations

$$\begin{cases} -t = 4g^2 \\ \frac{3-2\nu+2a}{4} = 1-(E+g^2) \\ \frac{-1-2\nu+2a}{4} = 1-(E+g^2+1) \\ -(a-\frac{1}{2}) = E+g^2+\frac{3}{2} \end{cases} \tag{5.15}$$

can be obtained and is actually compatible. Obviously this provides the following simple relation.

$$\begin{cases} a = -(E+g^2+1), \\ \nu = E+g^2-\frac{3}{2}. \end{cases} \tag{5.16}$$

Notice that the Rabi model may have a polynomial solution when $E+g^2$ is a non-negative integer. This fact has been first shown that in [13] by employing the Bargmann realization of the boson model [1] (see also [28]). Also, we have seen in §4 that the condition $\nu \in \mathbb{N}+\frac{1}{2}$ is necessary for the existence of a finite-type eigenfunction of Q. Moreover, the case $a = -(E+g^2+1) < 0\,(a \in \mathbb{Z})$ corresponds to the Langlands quotient realization ϖ_a of $(\pi'', y^{2-a}\mathbb{C}[y^2])$ in Lemma 5.1. From the observation, one expects the Heun element R_{NcHO}, under a suitable representation of $\mathfrak{sl}_2$, could give a universal model describing some sort of quantum interactions.

In this sense, the equation (5.7) with the accessary parameter (5.8) for $a = -(E+g^2+1)$ (and Δ should be defined compatible with others appropriately) can be considered as a universal Rabi model. Clarification of the reason of this similarity for doubly degeneration with detailed study of any connection between NcHO and several quantum models would be desirable.

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