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A remark on monotonicity for the Glauber dynamics on finite graphs

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Abstract: We show that under the heat-bath Glauber dynamics for the ferromagnetic Ising model on a finite graph, the single spin expectation at a fixed time starting at the all-up configuration is not necessarily an increasing function of coupling constants.

Key words: Glauber dynamics; spectral gap; spin expectation; monotonicity conjecture.

1. Introduction. The Glauber dynamics for the ferromagnetic Ising model on a finite graph has been widely studied. One of the most interesting open problems is whether the spectral gap, the difference between the first and second largest eigenvalues of the transition matrix of the Glauber dynamics is monotone decreasing in each coupling constant in the Hamiltonian, which is conjectured by Yuval Peres (cf. [6, 7]). For this conjecture, there has been a result in [7] which shows that the monotonicity holds for cycles of any length. The proof is based on two facts that the linear subspace spanned by single spins is invariant under the Glauber dynamics and the expectation of a spin is equal to the survival probability of a certain random walk with killing. Several correlation inequalities for spin systems such as GKS and GHS had been intensively investigated from the late 1960s to early 1980s (cf. [1–5]). They have many important implications among which is that the expectation of the product of spins in the equilibrium is monotone increasing for cycles of any length. The proof is based on two facts that the linear subspace spanned by single spins is invariant under the Glauber dynamics and the expectation of a spin is equal to the survival probability of a certain random walk with killing. Several correlation inequalities for spin systems such as GKS and GHS had been intensively investigated from the late 1960s to early 1980s (cf. [1–5]). They have many important implications among which is that the expectation of the product of spins in the equilibrium is monotone increasing for cycles of any length. The proof is based on two facts that the linear subspace spanned by single spins is invariant under the Glauber dynamics and the expectation of a spin is equal to the survival probability of a certain random walk with killing.

2. Result. Let $G = (V, E)$ be a finite connected graph. We define a probability measure $\pi$ on $S = \{-1, 1\}^V$ by

$$\pi(\sigma) = Z^{-1} \exp \left( \sum_{xy \in E} J_{xy} \sigma_x \sigma_y \right),$$

where $\sigma_x \in \{-1, 1\}$ is the spin value at $x$, $Z$ is the normalization constant and each coupling constant $J_{xy}$ is non-negative. Since the involution $\sigma \mapsto -\sigma$ leaves $\pi$ invariant, it follows that $E_\pi[\sigma_x] = 0$ for any $x \in V$. The transition matrix of the Glauber dynamics on $S$ is then defined by

$$A(\sigma, \eta) = \begin{cases} \frac{1}{|V|} \frac{\pi(\sigma)}{\pi(\sigma^r)}, & \eta = \sigma^r, x \in V \\ 1 - \sum_{x \in V} A(\sigma, \sigma^r), & \eta = \sigma \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma^r \in S$ is the spin configuration obtained from $\sigma \in S$ by flipping the spin at $x$ and leaving all other spins unchanged. The probability measure $\pi$ is reversible under this dynamics in which a vertex is chosen from $V$ uniformly at random and then the spin on it is flipped with probability $\frac{\pi(\sigma^r)}{\pi(\sigma) + \pi(\sigma^r)}$. The transition matrix $A$ acts on $\ell^2(S)$ as a bounded self-adjoint operator in a natural manner. Indeed, it is easy to see that $0 \leq A \leq I$ since

$$\langle (I - A) f, f \rangle = \frac{1}{2|V|} \sum_{\sigma \in S} \sum_{x \in V} \frac{|f(\sigma^r) - f(\sigma)|^2}{\pi(\sigma)^{-1} + \pi(\sigma^r)^{-1}} \geq 0,$$

and

$$\langle Af, f \rangle = \frac{1}{2|V|} \sum_{\sigma \in S} \sum_{x \in V} \frac{|\pi(\sigma)f(\sigma) + \pi(\sigma^r)f(\sigma^r)|^2}{\pi(\sigma) + \pi(\sigma^r)} \geq 0,$$
where \( \langle f, g \rangle = \sum_{\sigma \in S} f(\sigma)g(\sigma)\pi(\sigma) \). Hence we can enumerate the eigenvalues of \( A \) as
\[
1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{M-1} > \lambda_M = 0,
\]
where \( M = 2|V| \). Note that the eigenvector corresponding to 0 is given by \( \phi_0(\sigma) = (-1)^{n(\sigma)}\pi(\sigma)^{-1} \) where \( n(\sigma) \) is the number of 1’s in \( \sigma \). The original conjecture on the spectral gap mentioned in the introduction is then phrased as “the second largest eigenvalue \( \lambda_2 \) of the Glauber dynamics for the ferromagnetic Ising model on any finite graph is monotone increasing in each coupling constant \( J_{xy} \)”.

Now let us consider the case where \( G \) is the cycle \( C_n \) of length \( n \), that is, \( V = \{1, 2, \ldots, n\} \) and \( E = \{12, 23, \ldots, n1\} \). In this case, it is easy to see that for \( x \in V \) (\( V \) being considered in modulo \( n \) as \( \mathbb{Z}_n \))
\[
(2.1) \quad A e_x = \left( 1 - \frac{1}{n} \right) e_x + \frac{1}{n} (\alpha_x e_{x-1} + \beta_x e_{x+1} + \beta_x e_x),
\]
where \( e_x(\sigma) = \sigma_x \) for \( \sigma \in S \) and
\[
\alpha_x = \frac{s_{x-1}}{c_{x-1} + c_x}, \quad \beta_x = \frac{s_x}{c_{x-1} + c_x}
\]
with \( s_x = \sinh(2J_{x-1}, x+1) \) and \( c_x = \cosh(2J_{x-1}, x+1) \) (cf. [11]). We remark that the operator \( A \) in (2.1) leaves the subspace \( H_1 \) of \( \ell^2(S) \) spanned by \( \{e_x, x \in V\} \) invariant. We define \( Q^{\text{lazy}} \) and \( Q \) as follows:
\[
(2.2) \quad Q^{\text{lazy}} = \left( 1 - \frac{1}{n} \right) I + \frac{1}{n} Q
\]
with
\[
(2.3) \quad Q(x, y) = \begin{cases} 
\alpha_x, & y = x - 1, \\
\beta_x, & y = x + 1, \\
0, & \text{otherwise}.
\end{cases}
\]
The operator \( Q^{\text{lazy}} \) and \( Q \) are sub-stochastic and \( Q^{\text{lazy}} \) is nothing but the restriction of \( A \) on the invariant subspace \( H_1 \). Denote by \( \{X_t, Q_x, \zeta\} \) and \( \{X_t, Q_x^{\text{lazy}}, \zeta\} \) the discrete-time Markov chains on \( V \) with life time \( \zeta \) associated with \( Q \) and \( Q^{\text{lazy}} \), respectively. Let \( 1 \in S \) be the all-up spin configuration, i.e., \( 1_x = 1 \) for any \( x \in V \), and \( \sigma_x(t) \) the spin at \( x \in V \) and at time \( t \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\} \). Our main concern of this paper is the monotonicity of the spin expectation \( E_1[\sigma_x(t)] \) as a function of coupling constants \( J = (J_{xy}, xy \in E) \). Here \( E_1 \) stands for the expectation with respect to the Glauber dynamics starting at the all-up configuration 1. It is easy to see that the quantity \( E_1[\sigma_x(t)] \) is equal to the survival probability \( Q_x^{\text{lazy}}(\zeta > t) \) up to time \( t \) of the lazy \( Q \)-Markov chain starting at \( x \) since both functions of \( x \in V \) and \( t \) satisfy the same linear equation with the same boundary condition:
\[
(2.4) \quad \begin{cases} 
\phi(x, t+1) = Q^{\text{lazy}} \phi(x, t), & t \in \mathbb{Z}_{\geq 0} \\
\phi(x, 0) = 1
\end{cases}
\]
where \( Q^{\text{lazy}} \) acts on the \( x \)-variable as \( A|_H_1 \) in (2.1).

More directly,
\[
(2.5) \quad E_1[\sigma_x(t)] = Q_x^{\text{lazy}}(\zeta > t)
\]
\[
= \sum_{y \in V} (Q_x^{\text{lazy}})^j(x, y).
\]
The monotonicity problem is then reduced to the problem of whether \( Q_x^{\text{lazy}}(\zeta > t) \) is increasing in each coupling constant \( J_{xy} \).

The main result of this paper is the following theorem. See also Figures 1 and 2.

**Theorem.** Suppose a finite graph \( G = (V, E) \) has a path of length 3 as a subgraph. Then there exist \( a \in V \), \( pq \in E \), \( t \in \mathbb{Z}_{\geq 0} \) and positive coupling constants \( j = (j_{xy}, xy \in E) \) so that \( E_1[\sigma_a(t)] \) is decreasing near at \( J = j \) in the coupling constant \( J_{pq} \).

In order to prove this theorem, it suffices to consider the case of \( G = P_3 \), the path of length 3, and find \( a \in V \), \( pq \in E \), \( t \in \mathbb{Z}_{\geq 0} \) and coupling constants \( j \) at which the derivative of \( Q_x^{\text{lazy}}(\zeta > t) \) with respect to the coupling constant \( J_{pq} \) is negative because of the following observations: (i) if \( G \) has a path of length 3, say \( V = \{1, 2, 3, 4\} \) and \( E = \{12, 23, 34\} \), as a subgraph, we can compute the quantities on \( P_3 \) from those on \( G \) by setting \( J_{34} = 0 \) except \( J_{12}, J_{23} \) and \( J_{34} \), (ii) the derivative of \( Q_x^{\text{lazy}}(\zeta > t) \) with respect to a coupling constant is jointly continuous in the coupling constants \( J \).

Before showing the theorem, as a warm-up, we discuss the monotonicity for a smaller graph \( P_2 \), for which the conjecture is true.

**3. The case of \( P_2 \).** We first consider the case where \( G = (V, E) \) is the path \( P_2 \) of length 2, that is, \( V = \{1, 2, 3\} \) and \( E = \{12, 23\} \). We claim the following

**Proposition 1.** Let \( G = P_2 \). Then, for every \( x \in V \) and \( t \in \mathbb{Z}_{\geq 0} \), the expectation \( E_1[\sigma_x(t)] \) is increasing in both of coupling constants \( J_{12} \) and \( J_{23} \).

**Proof.** Since we can regard this example as the case where \( G = C_4 \) and the coupling constant \( J_{34} = 0 \), by putting \( x_1 = 2J_{12} \) and \( x_2 = 2J_{23} \), it follows from (2.3) that
where \( q_{xy} = Q(x, y) \). Set \( \phi_x(t) = Q_x(\zeta > t) \), the survival probability up to time \( t \) of the \( Q \)-Markov chain starting at \( x \). We observe that

\[
\phi_2(0) = 1, \\
\phi_2(1) = q_{21} + q_{23} = \tanh \left( \frac{x_1 + x_2}{2} \right), \\
\phi_2(2) = q_{21}q_{12} + q_{23}q_{32} = 1 - \frac{2}{\cosh x_1 + \cosh x_2},
\]

and these are obviously increasing functions of \( x_1 \) and \( x_2 \). It follows from the \( Q \)-version of (2.4) for \( n = 3 \) and \( J_{31} \), we obtain

\[
\begin{aligned}
\phi_1(t + 1) &= q_{12}\phi_2(t) \\
\phi_2(t + 1) &= q_{21}\phi_1(t) + q_{23}\phi_3(t) \\
\phi_3(t + 1) &= q_{32}\phi_2(t)
\end{aligned}
\]

and

\[
\phi_2(t + 2) = \phi_2(2)\phi_2(t), \quad t \in \mathbb{Z}_{\geq 0},
\]

from which together with the observation above, we conclude that \( \phi_2(t) \) is non-decreasing both in \( x_1 \) and \( x_2 \) for any \( t \in \mathbb{Z}_{\geq 0} \), and hence so are \( \phi_1(t) \) and \( \phi_3(t) \) because of (3.1) and the fact that \( q_{23} \) and \( q_{32} \) are increasing functions of \( x_1 \) and \( x_2 \), respectively. From (2.2) and (2.5), we see that

\[
Q^{[xy]}_x(\zeta > t) = \sum_{k=0}^{t} \binom{t}{k} \left( \frac{2}{3} \right)^{t-k} \left( \frac{1}{3} \right)^k Q_x(\zeta > k),
\]

and hence \( Q^{[xy]}_x(\zeta > t) \) is a positive linear combination of increasing functions.

**Remark 2.** We do not know whether the same claim holds for the case of \( C_3 \) instead of \( P_2 \).

4. **Counter example: the case of \( P_3 \).** Now we consider the case where \( G = (V, E) \) is the path \( P_3 \) of length 3, that is, \( V = \{1, 2, 3, 4\} \) and \( E = \{12, 23, 34\} \). In the same way as before, by regarding this example as the case where \( G = C_4 \) and the coupling constant \( J_{41} \) = 0, it follows from (2.3) that

\[
q_{12} = \frac{\sinh x_1}{1 + \cosh x_1}, \quad q_{21} = \frac{\sinh x_1}{\cosh x_1 + \cosh x_2},
\]

\[
q_{23} = \frac{\sinh x_2}{\cosh x_1 + \cosh x_2}, \quad q_{32} = \frac{\sinh x_2}{1 + \cosh x_2},
\]

and hence \( Q^{[xy]}_x(\zeta > t) \) is a positive linear combination of increasing functions.

**Proposition 3.** As \( s \to \infty \),

\[
\frac{\partial Q_x(\zeta > k)}{\partial x_2} \bigg|_{(x_1, x_2, x_3) = (2s, s, s)} = O(e^{-3s}) \quad (k = 1, 2, 3)
\]

\[
\frac{\partial Q_x(\zeta > 3)}{\partial x_2} \bigg|_{(x_1, x_2, x_3) = (2s, s, s)} = -\frac{e^{-3s}}{3} + O(e^{-3s})
\]

**Proof.** In this proof, \( f' \) means \( \frac{\partial f}{\partial x_2} \). We easily see that

\[
q_{21}' = -q_{21}q_{23}, \quad q_{23}' = \frac{1}{2}(1 - q_{21}^2 - q_{23}^2),
\]

\[
q_{32}' = -\frac{1}{2}(1 - q_{32}^2 - q_{34}^2), \quad q_{34}' = -q_{32}q_{34},
\]

and so

\[
\phi_2'(1) = q_{21}' + q_{23}' = \frac{1}{2}(1 - \phi_2(1)^2)
\]

(4.2) \( (q_{23}q_{34})' = \{\phi_2'(1) + q_{23}(q_{21} - q_{23})\}q_{34} \).

We also rewrite

\[
\phi_2(2) = \phi_2(1)q_{12} + q_{23}(q_{32} + q_{34} - q_{12}),
\]

(4.3) \( \phi_2(3) = (\phi_2(2) - q_{23}q_{34})(\phi_2(1) - q_{43}) + \phi_2(2)q_{43} \).

Now we set \( (x_1, x_2, x_3) = (2s, s, s) \) and regard functions like \( f(2s, s, s) \) and \( f'(2s, s, s) \) as those of the variable \( u = e^{-s} \). Since
(4.5) \( q_{21} = 1 - u + u^2 + O(u^3), \)

(4.6) \( q_{23} = u - u^2 + O(u^3), \)

we have

(4.7) \( \phi_2(1) = q_{21} + q_{23} = 1 + O(u^3), \)

and then it follows from (4.1) that

(4.8) \( \phi'_2(1) = \frac{1}{2}(1 - \phi_2(1)^2) = O(u^3). \)

Next we note that

(4.9) \( q_{32} = q_{34} = \frac{1}{2}q_{22} = \frac{1}{2} - u^2 + O(u^4), \)

(4.10) \( q'_{32} \equiv 0, q'_{34} + q'_{34} = 2u^2 + O(u^4). \)

Then, from (4.3), and (4.6) through (4.10), we obtain

(4.11) \( \phi'_2(2) = q_{23}(q'_{34} + q'_{34}) + O(u^3) = O(u^3). \)

From (4.4), (4.7), (4.8), (4.11), and

\( q_{43} = 1 - 2u(1 - u) + O(u^3), \)

we have

\( \phi'_2(3) = -2u(1 - u)(q_{23}q_{34}) + O(u^3). \)

On the other hand, from (4.2), (4.5), (4.6), (4.8) and (4.9), we see that

\( (q_{23}q_{34})' = (\phi'_2(1) + q_{23}(q_{21} - q_{32}))q_{34} \)

\( = \frac{u}{4} + O(u^2). \)

Consequently, we obtain

\( \phi'_2(3) = -\frac{u^2}{2} + O(u^3). \)

\[ \square \]

**Remark 4.** By similar computation, we can also show that for \( a > 1 \) and \( b \in \mathbb{R} \)

\[
\frac{\partial Q_2(\zeta > 3)}{\partial x_2} \bigg|_{(x_1, x_2, x_3) = (ax, s+b, s)} = -\frac{e^{-as}}{1 + \cosh(b)} + O(e^{-s \min(a+1, 2a-1)})
\]

as \( s \to \infty. \)

**Corollary 5.** There exists \( s_0 > 0 \) such that for any \( s \geq s_0, \) \( Q_{2_{d}}^{a_{2y}}(\zeta > 3) \bigg|_{(x_1, x_2, x_3) = (2s, x_2, s)} \) is decreasing in \( x_2 \) near at \( x_2 = s. \)

**Proof.** We have already shown the same assertion for \( Q_2(\zeta > 3). \) For a lazy version of the \( Q \)-Markov chain, \( Q_{2_{d}}^{a_{2y}}(\zeta > 3) \) is a positive linear combination of \( Q_2(\zeta > k), k = 0, 1, 2, 3. \) Indeed, from (2.2) and (2.5), it is easy to see that

\[
Q_{2_{d}}^{a_{2y}}(\zeta > 3) = \sum_{k=0}^{3} \binom{3}{k} \left( -\frac{3}{4} \right)^{3-k} \frac{1}{4} Q_2(\zeta > k).
\]

Therefore, from Proposition 3, we obtain

\[
\frac{\partial Q_{2_{d}}^{a_{2y}}(\zeta > 3)}{\partial x_2} \bigg|_{(x_1, x_2, x_3) = (2s, x_2, s)} = -\frac{1}{128} e^{-2s} + O(e^{-3s})
\]

as \( s \) tends to \( \infty. \) This implies the assertion. \[ \square \]

In Figs. 1, 2 one can see that \( Q_2(\zeta > 3) \) and \( E_1(a_{2y}(3)) \) is decreasing in \( x_2 \) near at \( (x_1, x_2, x_3) = (28, 14, 14). \)

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