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Abstract

We consider maxminimizations (minmaximizations) of functions of several variables defined on a Cartesian product of finite sets under some dependence conditions between variables. The effects of the combinations of dependence conditions between variables and permutations of optimizations on equilibrium values are rather complicated. We show an example to catch a glimpse of the intricacies.

Key Words and Phrases: Equilibrium, Order.

1. Introduction

Under some conditions a sequence of operations either maximization or minimization subject to variables applied to a real valued function of several variables gives a real number—an equilibrium value. For a natural number n let $C(I^n)$ be the set of real valued continuous functions on I^n where I is the unit interval of the real line and let Σ_n be the set of functionals that maps each element of $C(I^n)$ to such an equilibrium value. For example if $n = 2$,

$$\Sigma_2 = \{\max_{z_1}\max_{z_2},\ \max_{z_1}\min_{z_2},\ \max_{z_2}\min_{z_1},\ \min_{z_1}\max_{z_2},\ \min_{z_2}\max_{z_1},\ \min_{z_1}\min_{z_2}\}.$$

Here, for example, $\max_{z_1}\min_{z_2}$ is the functional given by $\max_{z_1}\min_{z_2}(F) = \max_{z_1}\min_{z_2} F(z_1, z_2)$ for $F \in C(I^2)$ where maximization and minimization are taken over all elements in I. Σ_n is an ordered set by the pointwise order. In Hisano and Maruyama (1989) we studied this order. There in Theorem 4.6 we proved that for $\sigma_1, \sigma_2, \ldots, \sigma_m \in \Sigma_n$, if an arbitrary given order $\sigma_1 < \sigma_2 < \cdots < \sigma_m$ does not contradict the order on Σ_n, there exists $F \in C(I^n)$ such that $\sigma_1(F) < \sigma_2(F) < \cdots < \sigma_m(F)$. This means that on Σ_n there exists no logical structure except the trivial one.

In the present paper, we treat the problem of maxminimization (minmaximization) by a sequence of variable functions. That is, to a function of several variables, when each variable is a function—a variable function—of some variables, we apply a sequence of optimal operations by these variable functions. In the case of maxminimization by a sequence of variables, the role—either maximization or minimization—of each variable and the dependence between variables determine the equilibrium value. The operation order of variables and the dependence between variables are united so a change in order of operations changes the dependence. On the other hand, in the case of maxminimization

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by a sequence of variable functions, there exists a change in order of operations that holds the dependence between variables but causes difference in the equilibrium values. It is easy to find such an example.

Let D be the set consists of 0 and 1 and let G the mapping from D^4 into D such that $G^{-1}(1) = \{(0,0,0,0),\ (0,0,1,1),\ (0,1,0,*),\ (1,1,1,*)\}$ where $*$ is either 0 or 1. In the following maxminimization let f_1, f_2 and f_4 be taken from all mappings from D into D and let z_3 be taken from D. Then

$$\max_{f_1}\min_{f_2}\min_{f_4}\max_{z_3} G(f_1(f_2(z_3)), f_2(z_3), z_3, f_4(f_1(f_2(z_3)))) = 0,$$
$$\min_{f_2}\max_{f_1}\min_{f_4}\max_{z_3} G(f_1(f_2(z_3)), f_2(z_3), z_3, f_4(f_1(f_2(z_3)))) = 1.$$

To verify the first equality set f_2 and f_4 according to f_1 as follows. If $f_1(0) = 1$ then $f_2(0) = f_2(1) = 0$. Otherwise $f_2(0) = f_1(1)$ and $f_2(1) = f_4(0) \neq f_1(1)$. To verify the second, set f_1 and z_3 according to f_2 as follows. If there exists some k such that $f_2(k) = 1$ then $f_1(1) = z_3 = k$. Otherwise $f_1(0) = 0$ and $z_3 = f_4(0)$. For the functionals of this sort, we use a shorthand notation like $x_1(y_2)y_2(x_3)y_4(x_1)x_3,\ y_2(x_3)x_1(y_2)y_4(x_1)x_3$ for the above for example.

In maxminimization by a sequence of variable functions, the elements that determine the equilibrium value are the roles of variables, the dependence between variables and the operation order of variable functions. The power of the first two elements is clear. But the power of operation order of variable functions to equilibrium value is obscure. Varing other factors that affect the equilibrium value, how much power, in the ultimate, does an exchange of operation order of two variable functions have ? Before a systematic investigation, let us observe it by a concrete example of moderate complexity. In this paper we give a limit example of three maximizers and three minimizers case.

For a positive integer N, let $\mathcal{D}_N$ be the set of all Cartesian products of N-finite sets and let $\mathcal{F}_N$ be the set of all real valued functions with domain in $\mathcal{D}_N$. An N-length sequence of optimizations by variable functions maps each element of $\mathcal{F}_N$ to an equilibrium value. We call such a functional an *equilibrium functional* of N-variables. In Section 2 we give a precise definition of an equilibrium functional and in Example 2.1 we give a pair of equilibrium functionals on $\mathcal{F}_6$ that shows the power of an order exchange of two variable functions. Varing other factors that affect the equilibrium value, we verify that this pair is a limit pair that shows the power limit of an order exchange of two variable functions. In Section 3 we construct a function—a separating function—in $\mathcal{F}_6$ which separates the values of the two functionals of Example 2.1 by the effect of the order exchange of two variable functions and provide a proof.

2. Equilibrium functionals

Definition 2.1. Let N be a positive integer let π be a permutation of $[1, N]$ (where for integers $m, n,\ [m, n]$ denotes $\{m, m+1, \ldots, n\}$) and let θ be a mapping from $[1, N]$ into $[0, 1]$. Let $\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_N)$ be a sequence of subsets of $[1, N]$ such that no $m, n_1, n_2, \ldots, n_m \in [1, N]$ satisfy $n_1 \in \Upsilon_{n_2}, n_2 \in \Upsilon_{n_3}, \ldots, n_{m-1} \in \Upsilon_{n_m}, n_m \in \Upsilon_{n_1}$. A triple (π, θ, Υ) is called an *equilibrium functional* of N-variables. The set of all equilibrium functionals of N-variables is denoted by $\mathcal{E}_N$.

Definition 2.2. Let $\sigma = (\pi, \theta, \Upsilon) \in \mathcal{E}_{\mathcal{N}}$ and let $F \in \mathcal{F}_{\mathcal{N}}$ with domain $\prod_{n\in[1,N]} Z_n$. For each $n \in [1,N]$ let h_n be a function from $\prod_{u\in\Upsilon_n} Z_u$ into Z_n. We define $\sigma^{(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(n)})}(F)$ $(n \in [1,N])$ and $\sigma(F)$ inductively as follows.

$$\sigma^{(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(N)})}(F) = F\Big(h_1\big((z_u)_{u\in\Upsilon_1}\big), h_2\big((z_u)_{u\in\Upsilon_2}\big),\ldots,h_N\big((z_u)_{u\in\Upsilon_N}\big)\Big)$$
$$\big(\text{where } z_n = h_n\big((z_u)_{u\in\Upsilon_n}\big)\ (n\in[1,N])\big),$$

$$\sigma^{(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(n-1)})}(F) = \begin{cases} \max\limits_{h_{\pi(n)}} \sigma^{(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(n)})}(F) & (\theta(\pi(n))=1) \\ \min\limits_{h_{\pi(n)}} \sigma^{(h_{\pi(1)},h_{\pi(2)},\ldots,h_{\pi(n)})}(F) & (\theta(\pi(n))=0) \end{cases}$$
$$(n\in[2,N]),$$
$$\sigma(F) = \begin{cases} \max\limits_{h_{\pi(1)}} \sigma^{h_{\pi(1)}}(F) & \big(\theta\big(\pi(1)\big)=1\big) \\ \min\limits_{h_{\pi(1)}} \sigma^{h_{\pi(1)}}(F) & \big(\theta\big(\pi(1)\big)=0\big). \end{cases}$$

We define an order on $\mathcal{E}_N$ by

$$\sigma \leqslant \sigma' \ \text{ if and only if } \ \sigma(F)\leqslant\sigma'(F)\ (F\in\mathcal{F}_N).$$

The following theorem provides a sufficient condition for an alteration in dependence between variables to extinguish the effect of an order exchange of two variable functions.

Theorem 2.1. *Let $n\in[1,N-1]$ and let $\sigma=(\pi,\theta,\Upsilon)$, $\sigma'=(\pi',\theta,\Upsilon')\in\mathcal{E}_N$ satisfy the following conditions.*

(i) $\pi'(m)\neq\pi(m)$ *if and only if* $m\in\{n,n+1\}$.

(ii) $\theta\big(\pi(n)\big)=0$ *and* $\theta\big(\pi(n+1)\big)=1$.

(iii) $\Upsilon_{\pi(n+1)}=\varnothing$, $\{\pi(n)\}\cup\Upsilon_{\pi(n)}\subset\Upsilon'_{\pi(n+1)}$ *and* $\Upsilon'_{\pi(m)}=\Upsilon_{\pi(m)}$ *for all* $m\in[1,N]\setminus\{n+1\}$.

(iv) $\pi(n+1)\in\Upsilon_{\pi(m)}$ *for all* $m>n+1$ *with* $\theta\big(\pi(m)\big)=1$.

Then $\sigma\leqslant\sigma'$.

Proof. No generality is lost by assuming that π is the transposition of 1 and 2, π' is the identity and that $\Upsilon'_1=\{2\}\cup\Upsilon_2$. Let $Z_1,Z_2,\ldots,Z_N$ be finite sets and let F be a function from $\prod_{n\in[1,N]} Z_n$ into $[0,1]$. It is enough to prove that $\sigma'(F)=1$ when $\sigma(F)=1$.

It follows from $\sigma(F)=1$ that, for all $g\in^{(\prod_{u\in\Upsilon_2}Z_u)}Z_2$ (the set of all mappings from $\prod_{u\in\Upsilon_2} Z_u$ into Z_2), $\sigma^g(F)=1$ and therefore $\{z_1\in Z_1\mid\sigma^{(g,z_1)}(F)=1\}\neq\varnothing$. We may suppose Z_1 is well ordered and define $f_1^g\in Z_1$ as $\min\{z_1\in Z_1\mid\sigma^{(g,z_1)}(F)=1\}$. Since $\sigma^{(g,f_1^g)}(F)=1$, there exist $f_n^{(g,f_1^g,h_3,\ldots,h_{n-1})}\in^{(\prod_{u\in\Upsilon_n}Z_u)}Z_n$ $(3\leqslant n\in\theta^{-1}(1))$ depending on $h_1 = f_1^g,\ h_2 = g,\ h_3 \in^{(\prod_{u\in\Upsilon_3} Z_u)} Z_3, \ldots, h_{n-1} \in^{(\prod_{u\in\Upsilon_{n-1}} Z_u)} Z_{n-1}$ such that

$h_n = f_n^{(g,f_1^g,h_3,\ldots,h_{n-1})}$ $\big(3 \leqslant n \in \theta^{-1}(1)\big)$ implies that

$$F\Big(f_1^g, g\big((z_u)_{u\in\Upsilon_2}\big), h_3\big((z_u)_{u\in\Upsilon_3}\big), \ldots, h_N\big((z_u)_{u\in\Upsilon_N}\big)\Big) = 1 \tag{1}$$

$$\big(\text{where } z_m = h_m\big((z_u)_{u\in\Upsilon_m}\big)\ (1 \leqslant m \leqslant N)\big).$$

Let $\prod_{u\in\Upsilon_2} Z_u = \{w_1, w_2, \ldots, w_M\}$ and let $f \in^{(Z_2\times\prod_{u\in\Upsilon_2} Z_u)} Z_1$ be defined by, for $v \in Z_2$ and w_m $(1 \leqslant m \leqslant M)$,

$$f(v, w_m) = \max_{v_1,v_2,\ldots,v_{m-1}\in Z_2} \min\{f_1^g \mid g \in^{(\prod_{u\in\Upsilon_2} Z_u)} Z_2,\ g(w_1) = v_1,$$
$$g(w_2) = v_2, \ldots, g(w_{m-1}) = v_{m-1}, g(w_m) = v\}.$$

¿From the definitions of f_1^g and f, for $g \in^{(\prod_{u\in\Upsilon_2} Z_u)} Z_2$ and $z_1 \in \Big\{f\big(g(w), w\big) \mid w \in \prod_{u\in\Upsilon_2} Z_u\Big\}$, there exists $G(g, z_1) \in^{(\prod_{u\in\Upsilon_2} Z_u)} Z_2$ such that

$$f_1^{G(g,z_1)} = z_1 \text{ and } G(g, z_1)(w) = g(w)\ \big(w \in \prod_{u\in\Upsilon_2} Z_u,\ f\big(g(w), w\big) = z_1\big) \tag{2}$$

If $3 \leqslant n \in \theta^{-1}(1)$ and if $z_1 = f\big(g\big((z_u)_{u\in\Upsilon_2}\big), (z_u)_{u\in\Upsilon_2}\big)$, since $1 \in \Upsilon'_n$ by (iv), the maximizer in σ' can take $h_n = f_n^{(G(g,z_1),f_1^{G(g,z_1)},h_3,\ldots,h_{n-1})}$ and hence by (2) and (1)

$$F\Big(f\Big(g\big((z_u)_{u\in\Upsilon_2}\big), (z_u)_{u\in\Upsilon_2}\Big), g\big((z_u)_{u\in\Upsilon_2}\big), h_3\big((z_u)_{u\in\Upsilon_3}\big), \ldots, h_N\big((z_u)_{u\in\Upsilon_N}\big)\Big)$$
$$= F\Big(f_1^{G(g,z_1)}, G(g, z_1)\big((z_u)_{u\in\Upsilon_2}\big), h_3\big((z_u)_{u\in\Upsilon_3}\big), \ldots, h_N\big((z_u)_{u\in\Upsilon_N}\big)\Big) = 1.$$

Therefore ${\sigma'}^f(F) = 1$, hence $\sigma'(F) = 1$. □

Example 2.1. Let $\tau, \mu \in \mathcal{E}_6$ be, in the shorthand notation, such that

$$\tau = x_1(y_2, y_4)y_2(y_0)x_3(x_1, x_5, y_2, y_0)y_0(y_4)x_5(x_1)y_4,$$
$$\mu = y_2(y_4, y_0)x_1x_3x_5y_4y_0.$$

Then $\mu \nleqslant \tau$.

Let τ' and μ' be modifications of τ and μ respectively such that $\tau < \tau'$ and $\mu' < \mu$. Then $\mu < \tau'$ and $\mu' < \tau$. In this sence the pair (τ, μ) shows the power limit of an order exchange of two variable functions. We prove $\mu < \tau'$ and $\mu' < \tau$ for several examples of τ' and μ'. The proofs for the other cases are similar. Let

$$\tau_1 = x_1(y_2, y_4)y_2(y_0)y_0(y_4)x_3(x_1, x_5, y_2, y_0)x_5(x_1)y_4$$
$$\tau_2 = x_1(y_2, y_4)y_2(y_0)x_3(x_1, x_5, y_2, y_4, y_0)y_0(y_4)x_5(x_1)y_4,$$
$$\tau_3 = x_1(y_2, y_4)y_2(y_0)x_3(x_1, x_5, y_2, y_0)y_0(y_4)x_5(x_1, y_2)y_4,$$
$$\tau_4 = x_1(y_2, y_4, y_0)y_2(y_0)x_3(x_1, x_5, y_2, y_0)y_0(y_4)x_5(x_1)y_4,$$
$$\tau_5 = x_1(y_2, x_4)y_2(y_0)x_3(x_1, x_5, y_2, y_0)y_0(x_4)x_5(x_1)x_4,$$
$$\mu_1 = y_2(x_1, y_4, y_0)x_1x_3x_5y_4y_0,$$
$$\mu_2 = y_2(x_3, y_4, y_0)x_1x_3x_5y_4y_0,$$
$$\mu_3 = y_2(y_4, y_0)x_1x_3y_5y_4y_0.$$

Then

$$
\begin{aligned}
\tau_1 &> x_1(y_2,y_4)y_2(y_4)y_0(y_4)x_3(x_1,x_5,y_2,y_0)x_5(x_1)y_4 \\
&> y_2(y_4)x_1y_0(y_4)x_3(x_5,y_2,y_0)x_5y_4 \quad \text{(by Theorem 2.1)} \\
&> \mu, \\
\tau_2 &= x_1(y_2,y_4)y_2(y_0)y_0(y_4)x_5(x_1)y_4x_3 \\
&> x_1(y_2,y_4)y_2(y_4)y_0(y_4)x_5(x_1)y_4x_3 \\
&> y_2(y_4)x_1y_0(y_4)x_5y_4x_3 \quad \text{(by Theorem 2.1)} \\
&> \mu, \\
\tau_3 &= y_2x_1(y_4)x_3(x_1,x_5,y_0)y_0(y_4)x_5(x_1)y_4 > \mu, \\
\tau_4 &> y_2(y_0)x_1x_3(x_5,y_2,y_0)y_0(y_4)x_5y_4 \quad \text{(by Theorem 2.1)} \\
&> \mu, \\
\tau_5 &> x_4x_1(y_2)y_2(y_0)x_3(y_0)y_0x_5 \\
&= x_4y_2x_1y_0x_3x_5 \\
&> y_4y_2x_1y_0x_3x_5 > \mu, \\
\mu_1 &= x_1y_2(y_4,y_0)x_3x_5y_4y_0 < \tau, \\
\mu_2 &= x_3y_2(y_4,y_0)x_1x_5y_4y_0 \\
&< x_3y_2(y_4)x_1x_5y_4y_0 \\
&< x_3x_1(y_2,y_4)y_2(y_4)x_5(x_1)y_4y_0 \quad \text{(by Theorem 2.1)} \\
&< x_3x_1(y_2,y_4)y_2(y_4)y_0(y_4)x_5(x_1)y_4 \\
&< x_3x_1(y_2,y_4)y_2(y_0)y_0(y_4)x_5(x_1)y_4 < \tau, \\
\mu_3 &< y_2(y_4)x_1y_0x_3y_5y_4 \\
&< x_1(y_2,y_4)y_2(y_4)y_0x_3(x_1)y_5y_4 \quad \text{(by Theorem 2.1)} \\
&= x_1(y_2,y_4)y_2(y_4,y_0)y_0x_3(x_1,y_0)y_5y_4 \\
&< x_1(y_2,y_4)y_2(y_0)x_3(x_1,y_0)y_0y_5y_4 \\
&= x_1(y_2,y_4)y_2(y_0)x_3(x_1,y_0)y_0(y_4)y_5y_4 \\
&< x_1(y_2,y_4)y_2(y_0)x_3(x_1,y_0)y_0(y_4)x_5y_4 < \tau.
\end{aligned}
$$

3. Proof of $\mu \nleq \tau$ in Example 2.1

In Subsection 3.1, by a series of definitions, we construct a Cartesian product Z of six finite sets with internal structure. On the structure we define a separating function $F_0 \in^Z [0,1]$ in Definition 3.12 and prove $\mu(F_0) = 1$. In Subsection 3.2 we prove $\tau(F_0) = 0$.

3.1. Construction of a separating function

Let U be a set and let n be a non-negative integer. We denote by $\sharp U$ the cardinality of U, and by $\binom{U}{n}$ the set of all subsets of U with cardinality n. We use i, j, k, l for elements of $[0,1]$ and p, p' for elements of $[0,4]$. Calculations concerning only i, j, k, l will be carried out modulo 2, and calculations concerning p, p' will be carried out modulo 5.

Definition 3.1. We define $X_{1,0}, X_{1,1}, X_{5,0}, X_{5,1}, Y_{2,0}, Y_{2,1}, X_1, X_3, X_5, Y_2, Y_4, Y_0$ and Z as follows. $X_{1,0} = [0,4]$, $X_{1,1} = [5, 8046090004]$, $X_{5,0} = [0,2]$, $X_{5,1} = [3, 362]$, $Y_{2,0} = [0, 59]$, $Y_{2,1} = [60, 299]$, $X_1 = X_{1,0} \cup X_{1,1}$, $X_3 = Y_4 = Y_0 = [0,1]$, $X_5 = X_{5,0} \cup X_{5,1}$, $Y_2 = Y_{2,0} \cup Y_{2,1}$ and $Z = X_1 \times X_3 \times X_5 \times Y_2 \times Y_4 \times Y_0$.

We denote by $*$ an arbitrary element of a factor of Z.

Definition 3.2. Let α_0, α_1 be injections from $\left\{ (C_0, C_1) \in \binom{X_{1,0}}{3}^2 \mid \sharp C_0 \cap C_1 = 1 \right\}$ into $Y_{2,0}$ with disjoint images and let $\beta_{0,0}, \beta_{0,1}$, $\beta_{1,0}, \beta_{1,1}$ be injections from $\left\{ (C_0, C_1) \in \binom{X_{1,0}}{3}^2 \mid \sharp C_0 \cap C_1 = 2 \right\}$ into $Y_{2,1}$ with disjoint images. If $a = \alpha_i(C, C')$ then we set $\bar{a} = \alpha_{i+1}(C, C')$ $(i \in [0,1])$.

Definition 3.3. For $i, j \in [0,1]$ and $C \in \binom{X_{1,0}}{3}$, we define $S'_{i,j}(C), S''_{i,j}(C), S'_i(C)$, $S''_i(C), S_i(C), S', S''$ as follows.

$$S'_{0,j}(C) = \left\{ \alpha_j(C, C') \mid C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' = 1 \right\},$$
$$S'_{1,j}(C) = \left\{ \alpha_j(C', C) \mid C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' = 1 \right\},$$
$$S''_{0,j}(C) = \left\{ \beta_{k,j}(C, C') \mid C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' = 2,\ k \in [0,1] \right\},$$
$$S''_{1,j}(C) = \left\{ \beta_{k,j}(C', C) \mid C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' = 2,\ k \in [0,1] \right\},$$
$$S'_i(C) = S'_{i,0}(C) \cup S'_{i,1}(C),\ S''_i(C) = S''_{i,0}(C) \cup S''_{i,1}(C),\ S_i(C) = S'_i(C) \cup S''_i(C),$$
$$S' = \bigcup \left\{ S'_0(C) \mid C \in \binom{X_{1,0}}{3} \right\} = \bigcup \left\{ S'_1(C) \mid C \in \binom{X_{1,0}}{3} \right\},$$
$$S'' = \bigcup \left\{ S''_0(C) \mid C \in \binom{X_{1,0}}{3} \right\} = \bigcup \left\{ S''_1(C) \mid C \in \binom{X_{1,0}}{3} \right\}.$$

Definition 3.4. For $p \in X_{1,0}$ let γ_p be a fixed bijection from $\left\{ (c, i) \mid i \in Y_4,\ p \in C \in \binom{X_{1,0}}{3}, c \in S_i(C) \right\}$ onto $X_{5,1}$, let $A_p = \{p, p+1, p+2\}$, $B_p = \{p, p+1, p+3\}$ and let $A_p(0) = B_p(0) = p$, $A_p(1) = B_p(1) = p+1$, $A_p(2) = p+2$, $B_p(2) = p+3$.

Definition 3.5. We define mappings ρ from Y_2 into $X_{1,0} \cup \binom{X_{1,0}}{2}$ and ρ_0, ρ_1 from $Y_{2,1}$ into $X_{1,0}$ as follows. For $i \in [0,1]$ and $C, C' \in \binom{X_{1,0}}{3}$ with $C \cap C' = \{p\}$ let $\rho(\alpha_i(C, C')) = p$. For $i, j, k \in [0,1]$ and $C, C' \in \binom{X_{1,0}}{3}$ with $C \cap C' = \{p_0, p_1\}$ where $p_1 \in \{p_0 + 1, p_0 + 2\}$ let $\rho\big(\beta_{i,j}(C, C')\big) = \{p_0, p_1\}$ and let $\rho_k\big(\beta_{i,j}(C, C')\big) = p_k$.

Definition 3.6. Let $i, j \in [0,1]$ and $p \in X_{1,0}$. We define $\lambda_i \in^{Y_{2,0}} X_{1,0}$ as follows.

$$\begin{aligned}
\lambda_i\big(\alpha_j(A_p, A_{p+3})\big) = \lambda_i\big(\alpha_j(B_{p+4}, B_p)\big) = p+3-i,\\
\lambda_i\big(\alpha_j(B_p, B_{p+4})\big) = \lambda_i\big(\alpha_j(B_{p+2}, A_{p+4})\big) = p+4-i,\\
\lambda_i\big(\alpha_j(A_{p+4}, B_{p+2})\big) = \lambda_i\big(\alpha_j(A_{p+3}, A_p)\big) = p+2+2i.
\end{aligned}$$

Definition 3.7. Let $i, j, k \in [0,1]$ and $C \in \binom{X_{1,0}}{3}$. We define $\iota_{i,j} \in^{Y_2} X_{1,0}$ and $R_i(C, j) \subset S''_i(C)$ as follows. For $a \in Y_{2,0}$ if $a = \alpha_j(C_0, C_1)$ then $\iota_{i,j}(a) \in X_{1,0} \setminus (C_i \cup \{\lambda_i(a)\})$ and $\iota_{i,j+1}(a) = \lambda_i(a)$. For $b \in Y_{2,1}$ if $b = \beta_{j,k}(C_0, C_1)$ then $\iota_{i,j}(b) \in X_{1,0} \setminus (C_0 \cup C_1)$ and $\iota_{i,j+1}(b) \in C_{i+1} \setminus C_i$.

$$R_i(C, j) = \{r \in S''_i(C) \mid \iota_{i,i+j}(r) = C(2) + 1\}.$$

Definition 3.8. Let $\mathcal{G} = \big\{g \in^{(Y_4 \times Y_0)} Y_2 \mid g(i,0) \neq g(i,1)\ (i \in Y_4)\big\}$ and let h be a fixed bijection from $\mathcal{G}$ onto $X_{1,1}$. We define a subset T_- of Z by

$$T_- = \{(h(g), 0, 0, g(i,j), i, j) \mid g \in \mathcal{G},\ i \in Y_4,\ j \in Y_0\}.$$

Definition 3.9. Let $i, j \in [0,1]$ and let $C \in \binom{X_{1,0}}{3}$. For each $a \in S'_{i,j}(C)$ and each $p \in C$, we define a subset $T^0_p\big(\gamma_p(a,i)\big)$ of Z as follows.

$$\begin{aligned}
&T^1_p\big(\gamma_p(a,i)\big) = \big\{(p, 0, \gamma_p(a,i), a, i, *)\big\},\\
&T^2_p\big(\gamma_p(a,i)\big) = \big\{(p, 0, \gamma_p(a,i), r, i+1, k) \in Z \mid \iota_{i+1,k+1}(r) = p \text{ or}\\
&\qquad\qquad r \in S'' \text{ with } \iota_{i+1,k}(r) \in C \setminus \{p\},\ k \in [0,1]\big\},\\
&T^3_p\big(\gamma_p(a,i)\big) = \Bigg\{(p, 0, \gamma_p(a,i), r, i+1, *) \in Z \mid r \in \bigcup \Big\{S'_{i+1}(C') \mid p \in C' \in \binom{X_{1,0}}{3}\Big\}\\
&\qquad \setminus \{\bar{a}\} \text{ or } r \in \bigcup \Big\{S''_{i+1,k}(C') \mid C' \in \binom{X_{1,0}}{3} \text{ is such that } C \cap C' = \{p, p'\}\\
&\qquad\qquad \text{where } p' \in \{p+1+2k, p+2+2k\},\ k \in [0,1]\Big\} \text{ or } r \in S''_{i+1}(C)\Bigg\},\\
&T^4_{\lambda_{i+1}(a)}\big(\gamma_{\lambda_{i+1}(a)}(a,i)\big) = \{(\lambda_{i+1}(a), 0, \gamma_{\lambda_{i+1}(a)}(a,i), r, i+1, j) \in Z \mid\\
&\qquad\qquad \iota_{i+1,j}(r) \neq \lambda_{i+1}(a),\ \iota_{i+1,j+1}(r) \neq \rho(a)\},\\
&T^4_{\rho(a)}\big(\gamma_{\rho(a)}(a,i)\big) = \{(\rho(a), 0, \gamma_{\rho(a)}(a,i), \bar{a}, i+1, j)\},\\
&T^4_p\big(\gamma_p(a,i)\big) = \varnothing\ (p \in C \setminus \{\lambda_{i+1}(a), \rho(a)\}),\\
&T^0_p\big(\gamma_p(a,i)\big) = \bigcup_{n \in [1,4]} T^n_p\big(\gamma_p(a,i)\big).
\end{aligned}$$

Definition 3.10. Let $i, j, k \in [0,1]$ and let $C \in \binom{X_{1,0}}{3}$. For each $b \in S''_{i,j}(C) \cap$

$R_i(C,k)$ and each $p \in C$, we define a subset $T_p^0\bigl(\gamma_p(b,i)\bigr)$ of Z as follows.

$$
\begin{aligned}
&T_p^1\bigl(\gamma_p(b,i)\bigr) = \{(p,0,\gamma_p(b,i),b,i,*)\},\\
&T_p^2\bigl(\gamma_p(b,i)\bigr) = \{(p,0,\gamma_p(b,i),r,i+1,l) \in Z \mid \iota_{i+1,l}(r) \in C\setminus\{p\} \text{ or } \iota_{i+1,l+1}(r) = p\},\\
&T_p^3\bigl(\gamma_p(b,i)\bigr) = \Bigl\{(p,0,\gamma_p(b,i),r,i+1,*) \in Z \mid r \in S'_{i+1}(C) \text{ or}\\
&\qquad r \in \bigcup\Bigl\{S''_{i+1,j+l}(C') \mid C' \in \binom{X_{1,0}}{3} \text{ is such that } C\cap C' = \{p,p'\}\\
&\qquad \text{where } p' \in \{p+1+2l, p+2+2l\},\ l\in[0,1]\Bigr\}\setminus\{b\} \text{ or } r \in \bigcup\bigl\{S'_{i+1}(C') \mid\\
&\qquad\quad C' \text{ is such that } C\cap C' = \{p,p'\} \text{ where } p' \in \{p+1+2j, p+2+2j\}\bigr\}\Bigr\}\\
&T^4_{\rho_j(b)}\bigl(\gamma_{\rho_j(b)}(b,i)\bigr) = \{(\rho_j(b),0,\gamma_{\rho_j(b)}(b,i),b,i+1,*)\},\\
&T_p^4\bigl(\gamma_p(b,i)\bigr) = \varnothing \quad (p \in C\setminus\{\rho_j(b)\}),\\
&T^5_{C(0)}\bigl(\gamma_{C(0)}(b,i)\bigr) = \{(C(0),0,\gamma_{C(0)}(b,i),r,i+1,i) \in Z \mid r \in R_{i+1}(C,k), \rho(r) \neq \rho(b)\},\\
&T^5_{C(1)}\bigl(\gamma_{C(1)}(b,i)\bigr) = \{(C(1),0,\gamma_{C(1)}(b,i),r,i+1,i) \in Z \mid r \in R_{i+1}(C,k)\},\\
&T^5_{C(2)}\bigl(\gamma_{C(2)}(b,i)\bigr) = \{(C(2),0,\gamma_{C(2)}(b,i),r,i+1,i) \in Z \mid r \in R_{i+1}(C,k), \rho(r) = \rho(b)\},\\
&T^6_{C(0)}\bigl(\gamma_{C(0)}(b,i)\bigr) = \{(C(0),0,\gamma_{C(0)}(b,i),r,i+1,i+1) \in Z \mid\\
&\qquad\qquad r \in R_{i+1}(C,k), \rho(r) = \rho(b)\},\\
&T^6_{C(1)}\bigl(\gamma_{C(1)}(b,i)\bigr) = \{(C(1),0,\gamma_{C(1)}(b,i),r,i+1,i+1) \in Z \mid\\
&\qquad\qquad r \in R_{i+1}(C,k), \rho(r) \neq \rho(b)\},\\
&T^6_{C(2)}\bigl(\gamma_{C(2)}(b,i)\bigr) = \{(C(2),0,\gamma_{C(2)}(b,i),r,i+1,i+1) \in Z \mid r \in R_{i+1}(C,k)\},\\
&T^7_{C(0)}\bigl(\gamma_{C(0)}(b,i)\bigr) = \{(C(0),0,\gamma_{C(0)}(b,i),r,i+1,*) \in Z \mid r \in R_{i+1}(C,k+1)\},\\
&T^7_{C(1)}\bigl(\gamma_{C(1)}(b,i)\bigr) = \{(C(1),0,\gamma_{C(1)}(b,i),r,i+1,k) \in Z \mid r \in R_{i+1}(C,k+1)\},\\
&T^7_{C(2)}\bigl(\gamma_{C(2)}(b,i)\bigr) = \{(C(2),0,\gamma_{C(2)}(b,i),r,i+1,k+1) \in Z \mid r \in R_{i+1}(C,k+1)\},\\
&T_p^0\bigl(\gamma_p(b,i)\bigr) = \bigcup_{n\in[1,7]} T_p^n\bigl(\gamma_p(b,i)\bigr).
\end{aligned}
$$

Definition 3.11. For each $p \in X_{1,0}$ and each $n \in [0,2]$, we define a subset $T_p^0(n)$ of Z as follows.

$$
\begin{aligned}
&T_p^1(0) = \{(p,i+j,0,r,i,j) \in Z \mid r \in \{\alpha_j(A_p,A_{p+3}), \alpha_j(B_{p+4},B_p)\},\ i,j\in[0,1]\},\\
&T_p^1(1) = \{(p,i+j,1,r,i,j) \in Z \mid r \in \{\alpha_j(B_p,B_{p+4}), \alpha_j(B_{p+2},A_{p+4})\},\ i,j\in[0,1]\},\\
&T_p^1(2) = \{(p,i+j,2,r,i,j) \in Z \mid r \in \{\alpha_j(A_{p+4},B_{p+2}), \alpha_j(A_{p+3},A_p)\},\ i,j\in[0,1]\},\\
&T_p^2(0) = \{(p,*,0,r,i,j) \in Z \mid \iota_{i,j}(r) = p+3-i \text{ or } \iota_{i,j+1}(r) = p,\ i,j\in[0,1]\},\\
&T_p^2(1) = \{(p,*,1,r,i,j) \in Z \mid \iota_{i,j}(r) = p+4-i \text{ or } \iota_{i,j+1}(r) = p,\ i,j\in[0,1]\},\\
&T_p^2(2) = \{(p,*,2,r,i,j) \in Z \mid \iota_{i,j}(r) = p+2+2i \text{ or } \iota_{i,j+1}(r) = p,\ i,j\in[0,1]\},\\
&T_p^0(n) = T_p^1(n) \cup T_p^2(n).
\end{aligned}
$$

Definition 3.12.

$$T_p = \bigcup \left\{ T_p^0\big(\gamma_p(c,i)\big) \mid p \in C \in \binom{X_{1,0}}{3},\ i \in Y_4,\ c \in S_i(C) \right\} \cup \bigcup_{n \in [0,2]} T_p^0(n) \quad (p \in X_{1,0}),$$

$$T = \bigcup_{p \in X_{1,0}} T_p \cup T_-.$$

We define $F_0 \in^Z [0,1]$ by $F_0^{-1}(1) = T$.

Definition 3.13.

$$W_{p,q} = \big\{ \big((r,i,j),(r',i+1,j')\big) \in (Y_2 \times Y_4 \times Y_0)^2 \mid \text{ there exist } k,k' \in X_3 \text{ such that } (p,k,q,r,i,j),(p,k',q,r',i+1,j') \in T_p^0(q) \big\} \quad (p \in X_{1,0},\ q \in X_5),$$

$$W_p = \bigcup_{q \in X_5} W_{p,q} \quad (p \in X_{1,0}).$$

The following two lemmas are direct consequences of the definitions.

Lemma 3.1. *Let* $i,j \in [0,1]$ *and* $C \in \binom{X_{1,0}}{3}$.

(1) $(p,0,\gamma_p(c,i),c,i,*) \in T_p\big(\gamma_p(c,i)\big)$ *for all* $p \in C$ *and* $c \in S_i(C)$.

(2) *If* $(c,r,k) \in \big(S_i(C) \times Y_2 \times Y_0\big) \setminus \bigcup_{a \in S'_{i,j}(C)} \{(a,\bar{a},j+1)\}$, *then there exists* $P \in \binom{C}{2}$ *such that for all* $p \in P$ $(p,0,\gamma_p(c,i),r,i+1,k) \in T_p^0\big(\gamma_p(c,i)\big)$.

(3) *If* $a \in S'_{i,j}(C)$ *then there exists* $q \in X_{5,0}$ *such that* $(\rho(a),i+j,q,a,i,*) \in T^0_{\rho(a)}(q)$ *and* $(\rho(a),i+j,q,\bar{a},i+1,j+1) \in T^0_{\rho(a)}(q)$ *and if* $\iota_{i+1,j}(r) = \lambda_{i+1}(a)$ *or* $\iota_{i+1,j+1}(r) = \rho(a)$ *then* $(\rho(a),*,q,r,i+1,j) \in T^0_{\rho(a)}(q)$.

(4) *If* $a \in S'_{i,j}(C)$ *then* $(\lambda_{i+1}(a),0,\gamma_{\lambda_{i+1}(a)}(a,i),\bar{a},i+1,j+1) \in T^0_{\lambda_{i+1}(a)}\big(\gamma_{\lambda_{i+1}(a)}(a,i)\big)$ *and if* $\iota_{i+1,j}(r) \neq \lambda_{i+1}(a)$ *and* $\iota_{i+1,j+1}(r) \neq \rho(a)$ *then* $(\lambda_{i+1}(a),0,\gamma_{\lambda_{i+1}(a)}(a,i),r,i+1,j) \in T^0_{\lambda_{i+1}(a)}\big(\gamma_{\lambda_{i+1}(a)}(a,i)\big)$.

Proof. (1) Clear from the definition of $T_p^1(\gamma_p(c,i))$.

(2) In case $c \in S'_i(C)$ the claim follows from the definitions of $T_p^2(\gamma_p(c,i))$ and $T_p^3(\gamma_p(c,i))$, in case $(c,r) \in S''_i(C) \times (Y_2 \setminus S''_{i+1}(C))$ it follows from the definitions of $T_p^2(\gamma_p(c,i)), T_p^3(\gamma_p(c,i))$ and $T^4_{\rho_j(C)}(\gamma_{\rho_j(C)}(c,i))$, and in case $(c,r) \in S''_i(C) \times S''_{i+1}(C)$ it follows from the definition of $T^m_{C(n)}(\gamma_{C(n)}(b,i))$ $(m \in [5,7], n \in [0,2])$.

(3) Clear from Definition 3.11.

(4) Clear from $\iota_{i+1,j}(\bar{a}) = \lambda_{i+1}(a)$ and the definitions of $T^2_{\lambda_{i+1}(a)}(\gamma_{\lambda_{i+1}(a)}(a,i))$ and $T^4_{\lambda_{i+1}(a)}(\gamma_{\lambda_{i+1}(a)}(a,i))$. □

Lemma 3.2.

(1) *Let* $i, j, k, l \in [0,1]$ *let* $C, C' \in \binom{X_{1,0}}{3}$ *with* $C \cap C' = \{p, p'\}$ *where* $p \in \{p' + 1 + 2k, p' + 2 + 2k\}$*. Then for* $b \in S''_{i,j}(C)$ *and* $b' \in S''_{i+1,j+k}(C')$*, if* $\iota_{i,l}(b), \iota_{i+1,l+1}(b') \in X \setminus (C \cup C')$*, then* $\big((b, i, l), (b', i+1, l+1)\big) \notin W_p$*.*

(2) *Let* $i, j, k \in [0,1]$ *let* $C, C' \in \binom{X_{1,0}}{3}$ *with* $\sharp C \cap C' = 2$ *and let* $b = \beta_{i,j}(C, C')$*. Then* $\big((b, k, i), (b, k+1, i)\big) \notin W_{\rho_{j+1}}(b)$*.*

(3) *Let* $i, j, k \in [0,1]$*, let* $C \in \binom{X_{1,0}}{3}$ *and let* $b \in R_k(C, i)$ *and* $b' \in R_{k+1}(C, j)$*. If* $i = j$*, then there exists* $l \in Y_0$ *such that* $\big((b, k, l), (b', k+1, l+1)\big) \notin W_{C(0)}$*. If* $i = j$ *and* $\rho(b) = \rho(b')$ *or* $i \neq j$*, then there exists* $l \in Y_0$ *such that* $\big((b, k, l), (b', k+1, l+1)\big) \notin W_{C(1)}$*. If* $i = j$ *and* $\rho(b) \neq \rho(b')$ *or* $i \neq j$*, then there exists* $l \in Y_0$ *such that* $\big((b, k, l), (b', k+1, l+1)\big) \notin W_{C(2)}$*.*

(4) *Let* $i, j, k \in [0,1]$ *and let* $C, C' \in \binom{X_{1,0}}{3}$ *with* $C \cap C' = \{p, p'\}$ *where* $p \in \{p' + 1 + 2k, p' + 2 + 2k\}$*. Then for* $a \in S'_{i,j}(C)$ *and* $b \in S''_{i+1,k}(C')$*, if* $\lambda_i(a) \in C'$ *and* $\iota_{i+1,j+1}(b) \in X_1 \setminus (C \cup C')$*, then* $\big((a, i, j), (b, i+1, j+1)\big) \notin W_p$*.*

(5) *Let* $C, C' \in \binom{X_{1,0}}{3}$ *with* $C \cap C' = \{p\}$ *and let* $a_0 = \alpha_0(C, C')$ *and* $a_1 = \alpha_1(C, C')$*. Then there exists* $q \in X_{5,0}$ *such that for all* $i \in [0,1]$ *and* $q' \in X_5 \setminus \{q\}$*,* $\big((a_0, i, 0), (a_1, i+1, 1)\big) \in W_{p,q}$*,* $\big((a_0, i, 0), (a_1, i+1, 1)\big) \notin W_{p,q'}$ *and* $(p, i, q, a_0, i+1, 0) \notin T_p$*.*

Proof. (1) Clear from $(p, 0, \gamma_p(b, i), b', i+1, l+1) \notin T^n_p(\gamma_p(b, i))$ and $(p, 0, \gamma_p(b', i+1), b, i, l) \notin T^n_p(\gamma_p(b', i+1))$ $(n \in [2,3])$.

(2) Clear from $(p, 0, \gamma_p(b, l), b, l+1, i) \notin T^2_p(\gamma_p(b, l))$ $(l \in [0,1])$.

(3) If $i \neq j$, since there do not exist $l, l' \in [0,1]$ and $n \in [0,2]$ such that $(C(1), l, n, b, k, i), (C(1), l', n, b', k+1, j) \in T^0_{C(1)}(n)$ and since $(C(1), 0, \gamma_{C(1)}(b, k), b', k+1, j) \notin T^0_{C(1)}(\gamma_{C(1)}(b, k))$ and $(C(1), 0, \gamma_{C(1)}(b', k+1), b, k, i) \notin T^0_{C(1)}(\gamma_{C(1)}(b', k+1))$, it follows that $((b, k, i), (b', k+1, j)) \notin W_{C(1)}$. Similarly we have $((b, k, j), (b', k+1, i)) \notin W_{C(2)}$. Similarly if $i = j$ and $\rho(b) = \rho(b')$ we have $((b, k, k+1), (b', k+1, k)) \notin W_{C(0)}$ and $((b, k, k), (b', k+1, k+1)) \notin W_{C(1)}$, if $i = j$ and $\rho(b) \neq \rho(b')$ we have $((b, k, k), (b', k+1, k+1)) \notin W_{C(0)}$ and $((b, k, k+1), (b', k+1, k)) \notin W_{C(2)}$.

(4) Clear from $(p, 0, \gamma_p(b, i+1), a, i, j) \notin T^n_p(\gamma_p(b, i+1))$ and $(p, 0, \gamma_p(a, i), b, i+1, j+1) \notin T^n_p(\gamma_p(a, i))$ $(n \in [2,3])$.

(5) If $((a_0, i, 0), (a_1, i+1, 1)) \in W_{p,q}$, then $q \in X_{5,0}$. From Definition 3.11 it follows that there exists a unique $q \in X_{5,0}$ such that $((a_0, i, 0), (a_1, i+1, 1)) \in W_{p,q}$, and for this q, $(p, i, q, a_0, i+1, 0) \notin T_p$. □

As an immediate consequence of Lemma 3.1, we can prove $\mu(F_0) = 1$. Let $r, r' \in Y_2$ and let $i \in Y_4$. Let $C \in \binom{X_{1,0}}{3}$ and let $c \in S_i(C)$. Lemma 3.1 implies that there exist $p \in C,\ k \in X_3,\ q \in X_5$ such that $(p, k, q, c, i, *), (p, k, q, r, i+1, 0), (p, k, q, r', i+1, 1) \in$

$T^0_p(q)$, and so for $g \in^{(Y_4 \times Y_0)} Y_2 \setminus \mathcal{G}$, there exist $p \in C,\ k \in X_3,\ q \in X_5$ such that for all $i \in Y_4$ and $j \in Y_0$ $(p, k, q, g(i,j), i, j) \in T^0_p(q)$. If $g \in \mathcal{G}$ then $(h(g), 0, 0, g(i,j), i, j) \in T_-$ for all $i \in Y_4$ and $j \in Y_0$. Therefore, for all $g \in^{(Y_4 \times Y_0)} Y_2$, there exist $p \in X_1,\ k \in X_3$ and $q \in X_5$ such that $(p, k, q, g(i,j), i, j) \in T$ for all $i \in Y_4$ and $j \in Y_0$. Hence $\mu(F_0) = 1$.

3.2. Proof of $\tau(F_0) = 0$

Lemma 3.3. *Let $f \in^{(Y_2 \times Y_4)} X_1$ and $i, j \in [0,1]$.*

(1) *For all $C \in \binom{X_{1,0}}{3}$ and $c \in S_i(C)$, if $f(c,i) \notin C$, then $\tau^f(F_0) = 0$.*

(2) *For all $c_0, c_1 \in Y_2$ and $p \in X_{1,0}$, if $\big((c_i, i, j), (c_{i+1}, i+1, j+1)\big) \notin W_p$ and $f(c_0, 0) = f(c_1, 1) = p$, then $\tau^f(F_0) = 0$.*

(3) *For all $c \in Y_2$ and $p \in X_{1,0}$, if $\big((c, i, j), (c, i+1, j)\big) \notin W_p$ and $f(c, 0) = f(c, 1) = p$, then $\tau^f(F_0) = 0$.*

Proof. Let $f_3 \in^{(X_1 \times X_5 \times Y_2 \times Y_0)} X_3,\ f_5 \in^{X_1} X_5,\ g_2 \in^{Y_0} Y_2,\ g_0 \in^{Y_4} Y_2$ and let $x_1 = f_1(y_2, y_4),\ x_3 = f_3(x_1, x_5, y_2, y_0),\ x_5 = f_5(x_1),\ y_2 = g_2(y_0)$ and $y_0 = g_0(y_4)$.

(1) If $p = f(c,i) \notin C$, then there exists $k \in Y_0$ such that $(p, *, *, c, i, k) \notin T$. So if we let $y_4 = i$ and g_2, g_0 be such that $g_2(k) = c,\ g_0(i) = k$, and set $q = f_5(p)$ and $l = f_3(p, q, c, k)$, then

$$\begin{aligned}(x_1, x_3, x_5, y_2, y_4, y_0) &= \big(f(y_2, y_4), f_3(x_1, x_5, y_2, y_0), f_5(x_1), g_2(y_0), y_4, g_0(y_4)\big)\\ &= (p, l, q, c, i, k) \notin T.\end{aligned}$$

Hence $\tau^f(F_0) = 0$.

(2) Let g_2, g_0 be such that $g_2(j) = c_i,\ g_2(j+1) = c_{i+1},\ g_0(i) = j,\ g_0(i+1) = j+1$, and set $q = f_5(p),\ k_i = f_3(p, q, c_i, j)$ and $k_{i+1} = f_3(p, q, c_{i+1}, j+1)$. The condition $\big((c_i, i, j), (c_{i+1}, i+1, j+1)\big) \notin W_p$ ensures that there exists $l \in Y_4$ such that $(p, k_l, q, c_l, l, i+j+l) \notin T_p$. So if we let $y_4 = l$, then

$$\big(f(y_2, y_4), f_3(x_1, x_5, y_2, y_0), f_5(x_1), g_2(y_0), y_4, g_0(y_4)\big) = (p, k_l, q, c_l, l, i+j+l) \notin T_p.$$

Hence $\tau^f(F_0) = 0$.

(3) Let g_2, g_0 be such that $g_2(j) = c,\ g_0(0) = g_0(1) = j$, and set $q = f_5(p)$ and $k = f_3(p, q, c, j)$. The condition $\big((c, i, j), (c, i+1, j)\big) \notin W_p$ ensures that there exists $l \in Y_4$ such that $(p, k, q, c, l, j) \notin T_p$. So if we let $y_4 = l$, then

$$\big(f(y_2, y_4), f_3(x_1, x_5, y_2, y_0), f_5(x_1), g_2(y_0), y_4, g_0(y_4)\big) = (p, k, q, c, l, j) \notin T_p.$$

Hence $\tau^f(F_0) = 0$. □

Lemma 3.4. *Let $C \in \binom{X_{1,0}}{3}$ and $f \in^{(Y_2 \times Y_4)} X_1$. If $\tau^f(F_0) = 1$ then*

$$f(S''_{i+1}(C) \times \{i+1\}) \not\subset f(S''_i(C) \times \{i\}) \subset C \quad (i \in [0,1])$$

and hence

$$\sharp f(S''_i(C) \times \{i\}) \leqslant 2 \ (i \in [0,1]) \text{ and } \sharp \bigcap_{i \in [0,1]} f(S''_i(C) \times \{i\}) \leqslant 1.$$

Proof. Let $C \in \binom{X_{1,0}}{3}$, $f \in^{(Y_2 \times Y_4)} X_1$ and let $\tau^f(F_0) = 1$. For each $i \in [0,1]$, let $Q_i = \{r_{i,0}, r_{i,1}, r_{i,2}, r_{i,3}, r_{i,4}, r_{i,5}\} \subset S''_i(C)$ satisfies the following two conditions.

- $r_{i,2m} \in R_i(C,0),\ r_{i,2m+1} \in R_i(C,1)\ \ (m \in [0,2])$.
- $\rho(r_{i,m}) \neq \rho(r_{i,n})\ \ (m,n \in [0,5],\ m \equiv n \pmod 2,\ m \neq n)$.

Then, since $\tau^f(F_0) = 1$, it follows from Lemma 3.3(1) that $f(Q_i \times \{i\}) \subset C$ $(i \in [0,1])$ and from Lemma 3.2(3), Lemma 3.3(2) that

- if $f(r_{0,m},0) = f(r_{1,n},1) = C(0)$ then $m \not\equiv n \pmod 2$,
- if $f(r_{0,m},0) = f(r_{1,n},1) = C(1)$ then $m \equiv n \pmod 2$ and $\rho(r_{0,m}) \neq \rho(r_{1,n})$, (1)
- if $f(r_{0,m},0) = f(r_{1,n},1) = C(2)$ then $m \equiv n \pmod 2$ and $\rho(r_{0,m}) = \rho(r_{1,n})$ $(m,n \in [0,5])$.

Therefore

$$\text{if } C(m) \in f(Q_i \times \{i\}),\ \text{then } \sharp\{r \in Q_{i+1} \mid f(r,i{+}1) = C(m)\} \leqslant 3 - m \quad (i \in [0,1],\ m \in [0,2]). \tag{2}$$

Hence, if $\sharp f(Q_i \times \{i\}) = 3$, then $\sharp f(Q_{i+1} \times \{i+1\}) = 3$ and so

$$\sharp\{r \in Q_0 \mid f(r,0) = C(1)\} = \sharp\{r \in Q_1 \mid f(r,1) = C(1)\} = 2.$$

But this contradicts (1). Therefore $\sharp f(Q_i \times \{i\}) \leqslant 2$. Then it follows from (2) that

$$f(Q_{i+1} \times \{i+1\}) \not\subset f(Q_i \times \{i\}) \quad (i \in [0,1]). \tag{3}$$

Since (3) holds for any Q_0, Q_1 satisfying the above conditions, we have

$$f(S''_{i+1}(C) \times \{i+1\}) \not\subset f(S''_i(C) \times \{i\}) \subset C \quad (i \in [0,1])$$

and hence

$$\sharp f(S''_i(C) \times \{i\}) \leqslant 2 \quad (i \in [0,1]) \text{ and } \sharp \bigcap_{i \in [0,1]} f(S''_i(C) \times \{i\}) \leqslant 1.$$

□

Lemma 3.5. *Let* $p \in C \in \binom{X_{1,0}}{3}$, $i \in [0,1]$ *and* $f \in^{(Y_2 \times Y_4)} X_1$. *If* $\tau^f(F_0) = 1$ *and* $\sharp\{r \in S''_i(C) \mid f(r,i) = p\} > 12$, *then*

(1) $\{r \in S''_{i+1}(C) \mid f(r,i+1) = p\} = \varnothing$,

(2) $\sharp\{r \in S''_{i+1}(C') \mid f(r,i+1) = p\} \leqslant 6 \quad \left(C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' = 2\right)$.

Proof. (1) The result follows immediately from Lemma 3.2(3) and Lemma 3.3(2).

(2) Let $l \in [0,1]$ and let $C' \in \binom{X_{1,0}}{3}$ with $C \cap C' = \{p, p'\}$ where $p' \in \{p+1+2l, p+2+2l\}$. The condition $\sharp\{r \in S''_i(C) \mid f(r,i) = p\} > 12$ ensures that there exist $j, k \in [0,1]$, $r_0 \in S''_{i,0}(C)$, $r_1 \in S''_{i,1}(C)$ and $r_2 \in S''_{i,k}(C)$ such that $\iota_{i,j}(r_0), \iota_{i,j}(r_1), \iota_{i,j+1}(r_2) \in X_{1,0} \setminus (C \cup C')$ and $f(r_0,i) = f(r_1,i) = f(r_2,i) = p$. Then, since $\tau^f(F_0) = 1$, it follows from Lemma 3.2(1) and Lemma 3.3(2) that

$$\{r \in S''_{i+1}(C') \mid f(r,i+1) = p\} \subset \{r \in S''_{i+1,k+l}(C') \mid \iota_{i+1,j}(r) \in X_{1,0} \setminus (C \cup C')\}.$$

Hence $\sharp\{r \in S''_{i+1}(C') \mid f(r,i+1) = p\} \leqslant 6$. □

Lemma 3.6. *Let $p \in C \in \binom{X_{1,0}}{3}$, $i \in [0,1]$ and $f \in^{(Y_2 \times Y_4)} X_1$. If $\tau^f(F_0) = 1$ and $\{r \in S''_i(C) \mid f(r,i) = p\} = S''_i(C)$ then*

$$\{r \in S''_{i+1}(C') \mid f(r,i+1) = p\} = \varnothing \quad \left(C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' \geqslant 2\right).$$

Proof. Let $C' \in \binom{X_{1,0}}{3}$. In case $\sharp C \cap C' = 3$, the result follows from Lemma 3.5(1). In case $\sharp C \cap C' = 2$, let $j \in [0,1]$ and let $C \cap C' = \{p, p'\}$ where $p \in \{p'+1+2j, p'+2+2j\}$. Assume to the contrary that $\{r \in S''_{i+1}(C') \mid f(r,i+1) = p\} \neq \varnothing$. Let $k, l \in [0,1]$ and let $r' \in S''_{i+1,k}(C')$ be such that $f(r',i+1) = p$ and $\iota_{i+1,l}(r') \in X_{1,0} \setminus (C \cup C')$. Then if $r \in S''_{i,k+j}(C)$ and $\iota_{i,l+1}(r) \in X \setminus (C \cup C')$ it follows from Lemma 3.2(1) that $\big((r,i,l+1),(r',i+1,l)\big) \notin W_p$. So by Lemma 3.3(2), since $f(r,i) = f(r',i+1) = p$, we get $\tau^f(F_0) = 0$, contradicting $\tau^f(F_0) = 1$. □

Lemma 3.7. *Let $p \in C \in \binom{X_{1,0}}{3}$, $i \in [0,1]$ and $f \in^{(Y_2 \times Y_4)} X_1$. If $\tau^f(F_0) = 1$ and $\sharp\{r \in S''_i(C) \mid f(r,i) = p\} = 12$, then $\{r \in S''_{i+1}(C) \mid f(r,i+1) = p\} = \varnothing$ or $\sharp\{r \in S''_{i+1}(C') \mid f(r,i+1) = p\} < 12 \quad \left(C' \in \binom{X_{1,0}}{3},\ \sharp C \cap C' = 2\right)$.*

Proof. ¿From $\sharp\{r \in S''_i(C) \mid f(r,i) = p\} = 12$, we have

$$\{r \in S''_i(C) \mid f(r,i) = p\} \cap R_i(C,j) \neq \varnothing \ (j \in [0,1])$$

or

$$\{r \in S''_i(C) \mid f(r,i) = p\} = R_i(C,i_0) \text{ for some } i_0 \in [0,1].$$

In the first case it follows immediately from Lemma 3.2(3) and Lemma 3.3(2) that $\{r \in S''_{i+1}(C) \mid f(r,i+1) = p\} = \varnothing$.

In the second case, $\{r \in S''_i(C) \mid f(r,i) = p\} \cap S''_{i,j}(C) \neq \varnothing$ $(j \in [0,1])$ and for $C' \in \binom{X_{1,0}}{3}$ with $\sharp C \cap C' = 2$, there exists $k \in [0,1]$ such that

$$\{r \in S''_i(C) \mid f(r,i) = p\} = \{r \in S''_i(C) \mid \iota_{i,k}(r) \in X_{1,0} \setminus (C \cup C')\}.$$

Assume to obtain a contradiction that $\sharp\{r \in S''_{i+1}(C') \mid f(r,i+1)=p\} \geqslant 12$. Then

$$\{r \in S''_{i+1}(C') \mid f(r,i+1)=p\} \cap \{r \in S''_{i+1}(C') \mid \iota_{i+1,k+1}(r) \in X_{1,0} \setminus (C \cup C')\} \neq \varnothing$$

or

$$\{r \in S''_{i+1}(C') \mid f(r,i+1)=p\} = \{r \in S''_{i+1}(C') \mid \iota_{i+1,k}(r) \in X_{1,0} \setminus (C \cup C')\}.$$

In the former case it follows from Lemma 3.2(1) and Lemma 3.3(2) that $\tau^f(F_0) = 0$. In the latter case if we let $C = C_i$ and $C' = C_{i+1}$, then $f(\beta_{k,j}(C_0,C_1),0) = f(\beta_{k,j}(C_0,C_1),1) = p$ $(j \in [0,1])$. Then it follows from Lemma 3.2(2) and Lemma 3.3(3) that $\tau^f(F_0)=0$. Since $\tau^f(F_0)=1$, we obtain a contradiction in both cases. □

Lemma 3.8. *Let* $C_0, C_1 \in \binom{X_{1,0}}{3}$ *and* $f \in^{(Y_2\times Y_4)} X_1$. *If* $\tau^f(F_0)=1$ *then*

$$\sharp \bigcap_{i\in[0,1]} f(S''_i(C_i)\times\{i\}) \leqslant 1.$$

Proof. If $\sharp C_0 \cap C_1 = 1$ or $C_0 = C_1$, the result is clear from Lemma 3.3(1) or Lemma 3.4. Let $C_0 \cap C_1 = \{p_0,p_1\}$ where $p_1 \in \{p_0+1, p_0+2\}$. Since $\tau^f(F_0)=1$, Lemma 3.4 implies that $f(S''_i(C_i)\times\{i\}) \subset \binom{C_i}{2}$ $(i \in [0,1])$. We assume that $\bigcap_{i\in[0,1]} f(S''_i(C_i)\times\{i\}) = \{p_0,p_1\}$ and derive a contradiction to the assumption $\tau^f(F_0)=1$. There are two cases. Either there exist $i,j \in [0,1]$ such that

$$\{f(r,i) \mid r \in S''_i(C_i),\ \iota_{i,j}(r) \in X_{1,0} \setminus (C_0,C_1)\} = \{p_0,p_1\}$$

or

$$\sharp\{f(r,i) \mid r \in S''_i(C_i),\ \iota_{i,j}(r) \in X_{1,0} \setminus (C_0,C_1)\} = 1 \ (i,j \in [0,1]).$$

In the former case there exist $k \in [0,1]$, $r_0 \in S''_{i,k}(C_i)$, $r_1 \in S''_{i,k+1}(C_i)$ such that $\iota_{i,j}(r_l) \in X_{1,0} \setminus (C_0 \cup C_1)$ and $f(r_l,i) = p_l$ $(l \in [0,1])$. Let $r \in S''_{i+1,k+1}(C_{i+1})$ with $\iota_{i+1,j+1}(r) \in X_{1,0} \setminus (C_0 \cup C_1)$. Then by Lemma 3.2(1) we get $\big((r_l,i,j),(r,i+1,j+1)\big) \notin W_{p_l}$ $(l \in [0,1])$. Hence, if $f(r,i+1) \in \{p_0,p_1\}$, it follows from Lemma 3.3(2) that $\tau^f(F_0)=0$.

In the latter case, for each $i \in [0,1]$, let $k_i \in [0,1]$ be such that

$$\{f(r,i) \mid r \in S''_i(C_i),\ \iota_{i,0}(r) \in X_{1,0} \setminus (C_0 \cup C_1)\} = \{p_{k_i}\}.$$

Set $k = k_0$. If $k \neq k_1$ then, since $\bigcap_{i\in[0,1]} f(S''_i(C_i)\times\{i\}) = \{p_0,p_1\}$, we have

$$\begin{aligned}&\{f(r,0) \mid r \in S''_0(C_0),\ \iota_{0,0}(r) \in X_{1,0} \setminus (C_0 \cup C_1)\}\\ =&\{f(r,1) \mid r \in S''_1(C_1),\ \iota_{1,1}(r) \in X_{1,0} \setminus (C_0 \cup C_1)\} = \{p_k\}.\end{aligned}$$

Let $r_0 \in S''_{0,0}(C_0)$ with $\iota_{0,0}(r_0) \in X_{1,0} \setminus (C_0 \cup C_1)$ and let $r_1 \in S''_{1,k+1}(C_1)$ with $\iota_{1,1}(r_1) \in X_{1,0} \setminus (C_0 \cup C_1)$. Then it follows from Lemma 3.2(1) that $\big((r_0,0,0),(r_1,1,1)\big) \notin W_{p_k}$ and so by Lemma 3.3(2), $\tau^f(F_0)=0$. If $k=k_1$ then

$$\{f(r,0) \mid r \in S''_0(C_0),\ \iota_{0,0}(r) \in X_{1,0} \setminus (C_0 \cup C_1)\}$$

$$=\{f(r,1) \mid r \in S''_1(C_1),\ \iota_{1,0}(r) \in X_{1,0} \setminus (C_0 \cup C_1)\} = \{p_k\}.$$

Then if we let $b=\beta_{0,k+1}(C_0,C_1)$ then $f(b,0)=f(b,1)=p_k$ and $\rho_k(b)=p_k$. Hence, by Lemma 3.2(2) and Lemma 3.3(3), $\tau^f(F_0)=0$. □

Lemma 3.9. *Let $C_0, C_1 \in \begin{pmatrix} X_{1,0} \\ 3 \end{pmatrix}$ with $C_0 \cap C_1 = \{p\}$ and $f \in^{(Y_2 \times Y_4)} X_1$. If $f(\alpha_i(C_0,C_1),j)=p$ $(i,j \in [0,1])$ then $\tau^f(F_0)=0$.*

Proof. Let $f_3 \in^{(X_1\times X_5\times Y_2\times Y_0)} X_3$, $f_5 \in^{X_1} X_3$, $g_2 \in^{Y_0} Y_2$ and $g_0 \in^{Y_4} Y_0$. Let $x_1 = f(y_2,y_4)$, $x_3 = f_3(x_1,x_5,y_2,y_0)$, $x_5 = f_5(x_1)$, $y_2 = g_2(y_0)$ and $y_0 = g_0(y_4)$. Set $a_i = \alpha_i(C_0,C_1)$ $(i \in [0,1])$. By Lemma 3.2(5), there exists a unique $q \in X_5$ such that $\big((a_0,i,0),(a_1,i+1,1)\big) \in W_{p,q}$ $(i \in [0,1])$. Let $g_2(0)=a_0$ and $g_2(1)=a_1$. Set $k=f_3(p,q,a_0,0)$ and let $g_0(k)=1$ and $g_0(k+1)=0$.

In case $f_5(p)=q$, let $y_4=k+1$, then by Lemma 3.2(5),

$$\begin{aligned}(x_1,x_3,x_5,y_2,y_4,y_0) &= \big(f(y_2,y_4), f_3(x_1,x_5,y_2,y_0), f_5(x_1), g_2(y_0), y_4, g_0(y_4)\big)\\ &= (p,k,q,a_0,k+1,0) \notin T_p.\end{aligned}$$

In case $f_5(p)=q' \neq q$, if $(p,*,q',a_0,k+1,0) \notin T_p$ then let $y_4=k+1$ and set $l=f_3(p,q',a_0,0)$. Then

$$\big(f(y_2,y_4), f_3(x_1,x_5,y_2,y_0), f_5(x_1), g_2(y_0), y_4, g_0(y_4)\big) = (p,l,q',a_0,k+1,0) \notin T_p.$$

If there exists $j \in [0,1]$ such that $(p,j,q',a_0,k+1,0) \in T_p$, then since $\big((a_0,k+1,0),(a_1,k,1)\big) \notin W_{p,q'}$, if we let $y_4=k$ and set $l=f_3(p,q',a_1,1)$ we have

$$\big(f(y_2,y_4), f_3(x_1,x_5,y_2,y_0), f_5(x_1), g_2(y_0), y_4, g_0(y_4)\big) = (p,l,q',a_1,k,1) \notin T_p.$$

Therefore $\tau^f(F_0)=0$. □

Proposition 3.1. *Let $s>0$. For each $k \in [0,1]$, let M_k be a nonnegative 4×4 matrix such that $M_k(m,m)=0$ and $\sum_{n=0}^{3} M_k(m,n)=2s$ for all $m \in [0,3]$. Furthermore we assume the following properties.*

(i) *For all $k \in [0,1]$ and $m,m',n \in [0,3]$, if $M_k(m,n)>s$ then $M_{k+1}(m,n)=0$ and $M_{k+1}(m',n)<s$.*

(ii) *For all $k \in [0,1]$ and $m,n \in [0,3]$, if $M_k(m,n)=s$ then $M_{k+1}(m,n)<s$ or $M_{k+1}(m',n)<s$ $(m' \in [0,3]\setminus\{m\})$.*

(iii) *$\sharp\{n \in [0,3] \mid M_0(m,n), M_1(m',n)>0\} \leqslant 1$ $(m,m' \in [0,3])$.*

Then there exist $\mathfrak{M}_0, \mathfrak{M}_1 \in \binom{[0,3]}{2}$ *with* $\mathfrak{M}_0 \cap \mathfrak{M}_1 = \varnothing$ *such that for each* $k \in [0,1]$, *if* $\{m,n\} = \mathfrak{M}_k$ *then* $M_k(m,n) > s$.

Proof. It follows immediately from (i), (iii) that

(iv) $\sharp\{n \in [0,3] \mid M_k(m,n) > 0\} \leqslant 2 \quad (k \in [0,1],\ m \in [0,3])$,

and from (ii) that

(v) $\sum_{k=0}^{1} \sharp\{m \in [0,3] \mid M_k(m,n) = s\} \leqslant 3 \quad (n \in [0,3])$.

Suppose that $\max_{k,m,n} M_k(m,n) \leqslant s$. Then by (iv), $M_k(m,n) = 0$ or $M_k(m,n) = s$ $(k \in [0,1],\ m,n \in [0,3])$. Therefore it follows from (v) that

$$\sum_{k,m,n} M_k(m,n) = \sum_{n} \sum_{k,m} M_k(m,n) \leqslant 4 \times 3s = 12s.$$

On the other hand

$$\sum_{k,m,n} M_k(m,n) = \sum_{k,m} \sum_{n} M_k(m,n) = 8 \times 2s = 16s.$$

This contradiction proves that there exist $i \in [0,1]$ and $m_i \in [0,3]$ such that $\max_n M_i(n, m_i) > s$. By (iv) there exists $m_{i+1} \neq m_i$ such that $M_i(m_i, m_{i+1}) = 0$. Let $m'_i, m'_{i+1} \in [0,3]$ be such that $\{m'_i, m'_{i+1}\} = [0,3] \setminus \{m_i, m_{i+1}\}$ and $M_i(m_i, m'_{i+1}) \leqslant M_i(m_i, m'_i)$. Suppose that $M_i(m_i, m'_i) \leqslant s$. Then $M_i(m_i, m'_{i+1}) = M_i(m_i, m'_i) = s$ and so by (i) $M_{i+1}(m_{i+1}, m'_{i+1}), M_{i+1}(m_{i+1}, m'_i) \leqslant s$. Since $\max_n M_i(n, m_i) > s$, by (i) $M_{i+1}(m_{i+1}, m_i) < s$. Then it follows from (iv) that $M_{i+1}(m_{i+1}, m'_{i+1}) = M_{i+1}(m_{i+1}, m'_i) = s$. Consequently $M_i(m_i, m'_{i+1}), M_i(m_i, m'_i), M_{i+1}(m_{i+1}, m'_{i+1}), M_{i+1}(m_{i+1}, m'_i) > 0$ which contradicts (iii). Hence $M_i(m_i, m'_i) > s$. Set $\mathfrak{M}_k = \{m_k, m'_k\}$ $(k \in [0,1])$. Since $\max_n M_i(n, m_i), M_i(m_i, m'_i) > s$, it follows from (i) that $M_{i+1}(m,n) < s$ $(m \in \mathfrak{M}_{i+1}, n \in \mathfrak{M}_i)$. Therefore by (iv) $M_{i+1}(m,n) > s$ $(\{m,n\} = \mathfrak{M}_{i+1})$. Then by (i) $M_i(m,n) < s$ $(m \in \mathfrak{M}_i, n \in \mathfrak{M}_{i+1})$. Hence by (iv) $M_i(m,n) > s$ $(\{m,n\} = \mathfrak{M}_i)$. □

Proposition 3.2. *Let* $s > 0$. *For each* $k \in [0,1]$ *let* M_k *be a nonnegative* 10×5 *matrix such that* $M_k(\mathfrak{M}, m) = 0$, $\sum_{n=0}^{4} M_k(\mathfrak{M}, n) = 2s$ *for all* $\mathfrak{M} \in \binom{[0,4]}{2}$ *and* $m \in \mathfrak{M}$. *Furthermore we assume the following properties.*

(i) *For all* $k \in [0,1]$, $n \in [0,4]$ *and* $\mathfrak{M}, \mathfrak{M}' \in \binom{[0,4]}{2}$ *with* $\mathfrak{M} \cap \mathfrak{M}' \neq \varnothing$, *if* $M_k(\mathfrak{M}, n) > s$, *then* $M_{k+1}(\mathfrak{M}, n) = 0$ *and* $M_{k+1}(\mathfrak{M}', n) < s$.

(ii) *For all* $k \in [0,1]$, $n \in [0,4]$ *and* $\mathfrak{M} \in \binom{[0,4]}{2}$, *if* $M_k(\mathfrak{M}, n) = s$, *then*

$M_{k+1}(\mathfrak{M}, n) < s$ *or* $M_{k+1}(\mathfrak{M}', n) < s$ $\left(\mathfrak{M}' \in \binom{[0,4]}{2} \text{ with } \mathfrak{M} \cap \mathfrak{M}' \neq \varnothing\right)$.

(iii) $\sharp\{n \in [0,4] \mid M_0(\mathfrak{M}, n), M_1(\mathfrak{M}', n) > 0\} \leqslant 1 \ \left(\mathfrak{M}, \mathfrak{M}' \in \left(\begin{array}{c}[0,4]\\2\end{array}\right)\right)$.

(iv) *For all* $k \in [0,1]$, $n \in [0,4]$ *and* $\mathfrak{M}, \mathfrak{M}' \in \left(\begin{array}{c}[0,4]\\2\end{array}\right)$ *with* $\mathfrak{M} \cap \mathfrak{M}' \neq \varnothing$, *if* $M_k(\mathfrak{M}, n) = 2s$, *then* $M_{k+1}(\mathfrak{M}', n) = 0$.

Then there exist $\mathfrak{M}_0, \mathfrak{M}_1 \in \left(\begin{array}{c}[0,4]\\2\end{array}\right)$ *with* $\mathfrak{M}_0 \cap \mathfrak{M}_1 = \varnothing$ *such that* $M_0(\mathfrak{M}_0, p) = M_1(\mathfrak{M}_1, p) = 2s$ *where* p *is the unique element of* $[0,4] \setminus (\mathfrak{M}_0 \cup \mathfrak{M}_1)$ *and* $M_k(\mathfrak{M}, n) = 2s$ *for all* $k \in [0,1]$, $\mathfrak{M} \in \left(\begin{array}{c}[0,4]\\2\end{array}\right)$ *and* $n \in [0,4]$ *such that* $\{n\} = \mathfrak{M}_k \setminus \mathfrak{M}$.

Proof. For each $i \in [0,1]$ and each $m \in [0,4]$ define a 4×4 matrix $M_i(m)$ by

$$M_i(m)(n, n') = M_i(\{m, n\}, n') \quad (n, n' \in [0,4] \setminus \{m\}).$$

Then, when we identify the set $[0,4] \setminus \{m\}$ with $[0,3]$, the matrices $M_0(m)$ and $M_1(m)$ have all properties required by Proposition 3.1. So the proposition implies that there exist all distinct $n_0, n_1, n_2, n_3 \in [0,4] \setminus \{m\}$ such that

$$M_0(m)(n_0, n_1), M_0(m)(n_1, n_0), M_1(m)(n_2, n_3), M_1(m)(n_3, n_2) > s.$$

Since m was arbitrary, we have

$$\sharp\{n \in [0,4] \mid \max_{\mathfrak{M}} M_i(\mathfrak{M}, n) > s\} \geqslant 3 \quad (i \in [0,1]).$$

Hence there exist $p \in [0,4]$ and $\mathfrak{M}_0, \mathfrak{M}_1 \in \left(\begin{array}{c}[0,4]\\2\end{array}\right)$ such that $M_0(\mathfrak{M}_0, p), M_1(\mathfrak{M}_1, p) > s$. Clearly $p \notin \mathfrak{M}_0 \cup \mathfrak{M}_1$ and by (i), $\mathfrak{M}_0 \cap \mathfrak{M}_1 = \varnothing$. Let $\mathfrak{M}_i = \{m_{i,0}, m_{i,1}\}$ $(i \in [0,1])$.

For each $i, j \in [0,1]$, since $M_0(\mathfrak{M}_0, p), M_1(\mathfrak{M}_1, p) > s$, it follows from (i) that $M_i(m_{i+1,j})(m, p) < s$ $(m \neq m_{i+1,j})$ and $M_{i+1}(m_{i+1,j})(m, p) < s$ $(m \notin \mathfrak{M}_{i+1})$. Therefore by Proposition 3.1

$$M_{i+1}(m_{i+1,j})(p, m_{i+1,j+1}), M_i(m_{i+1,j})(m_{i,k}, m_{i,k+1}) > s \quad (k \in [0,1]).$$

Then, since i, j was arbitrary, (i) implies that

$$M_i(\mathfrak{M}, n) > s, M_{i+1}(\mathfrak{M}, n) = 0 \left(i \in [0,1], \mathfrak{M} \in \left(\begin{array}{c}[0,4]\\2\end{array}\right), \{n\} = \mathfrak{M}_i \setminus \mathfrak{M}\right). \tag{1}$$

Let $i, j, k \in [0,1]$. Suppose that $M_i(\mathfrak{M}_i, m_{i+1,j}) > 0$. Then by (1), (iv)

$$s < M_{i+1}(\{m_{i,k}, m_{i+1,j+1}\}, m_{i+1,j}) < 2s, \quad M_{i+1}(\{m_{i,k}, m_{i+1,j+1}\}, m_{i,k+1}) = 0$$

and so $M_{i+1}(\{m_{i,k}, m_{i+1,j+1}\}, p) > 0$. But $M_i(\mathfrak{M}_i, m_{i+1,j}), M_i(\mathfrak{M}_i, p)$, $M_{i+1}(\{m_{i,k}, m_{i+1,j+1}\}, m_{i+1,j}), M_{i+1}(\{m_{i,k}, m_{i+1,j+1}\}, p) > 0$ contradicts (iii). Therefore $M_i(\mathfrak{M}_i, m_{i+1,j}) = 0$. Since j was arbitrary, $M_i(\mathfrak{M}_i, p) = 2s$. Then by (iv) $M_{i+1}(\{m_{i,j}, m_{i+1,k}\}, p) = 0$, and since $M_{i+1}(\{m_{i,j}, m_{i+1,k}\}, m_{i,j+1}) = 0$ by (1), we have

$$M_{i+1}(\{m_{i,j}, m_{i+1,k}\}, m_{i+1,k+1}) = 2s. \tag{2}$$

Since k was arbitrary it follows from (2) and (iv) that

$$M_i(\{p, m_{i,j}\}, m_{i+1,0}) = M_i(\{p, m_{i,j}\}, m_{i+1,1}) = 0.$$

Hence

$$M_i(\{p, m_{i,j}\}, m_{i,j+1}) = 2s. \tag{3}$$

Since i, j, k was arbitrary it follows from (2) and (3) that

$$M_k(\mathfrak{M}, n) = 2s \quad \left(k \in [0,1],\ \mathfrak{M} \in \binom{[0,4]}{2},\ \{n\} = \mathfrak{M}_k \setminus \mathfrak{M}\right).$$

□

Lemma 3.10. *Let $f \in^{(Y_2 \times Y_4)} X_1$. If $\tau^f(F_0) = 1$ then there exist $C_0, C_1 \in \binom{X_{1,0}}{3}$ with $\sharp C_0 \cap C_1 = 1$ such that $\bigcup_{i,j \in [0,1]} \{f(\alpha_i(C_0, C_1), j)\} = C_0 \cap C_1$.*

Proof. For each $i \in [0,1]$, define a 10×5 matrix M_i by

$$M_i(\mathfrak{M}, n) = \sharp\{r \in S''_i(X_{1,0} \setminus \mathfrak{M}) \mid f(r, i) = n\} \quad \left(\mathfrak{M} \in \binom{X_{1,0}}{2},\ n \in X_{1,0}\right).$$

Then, since $\tau^f(F_0) = 1$, it follows from Lemma 3.3(1), 3.5, 3.6, 3.7 and 3.8 that M_0 and M_1 satisfy the conditions of Proposition 3.2 with $s = 12$. And so by the proposition, there exist $\mathfrak{M}_0, \mathfrak{M}_1 \in \binom{X_{1,0}}{2}$ with $\mathfrak{M}_0 \cap \mathfrak{M}_1 = \varnothing$ such that $M_i(\mathfrak{M}, n) = 24$ for all $i \in [0,1]$, $\mathfrak{M} \in \binom{X_{1,0}}{2}$ and $n \in X_{1,0}$ such that $\{n\} = \mathfrak{M}_i \setminus \mathfrak{M}$.

Set $C_0 = X_{1,0} \setminus \mathfrak{M}_0$, $C_1 = X_{1,0} \setminus \mathfrak{M}_1$. Then $\sharp C_0 \cap C_1 = 1$, and by the definition of M_0 and M_1, for each $i \in [0,1]$, if $C \in \binom{X_{1,0}}{3}$ and $n \in C$ are such that $\{n\} = C \setminus C_i$ then $f(S''_i(C) \times \{i\}) = \{n\}$.

Set $a_i = \alpha_i(C_0, C_1)$ $(i \in [0,1])$ and let $\{p\} = C_0 \cap C_1$. Assume to the contrary that there exist $i, j \in [0,1]$ such that $f(a_i, j) \in C_j \setminus \{p\}$, let $f(a_i, j) = n$ and let $C = \{p, n, \lambda_j(a_i)\}$. Clearly $\{n\} = C \setminus C_{j+1}$ and so $f(S''_{j+1}(C) \times \{j+1\}) = \{n\}$. Let $k \in [0,1]$ and $b \in S''_{j+1,k}(C)$ be such that $n \in \{p+1+2k, p+2+2k\}$ and $\iota_{j+1,i+1}(b) \in X_{1,0} \setminus (C \cup C_j)$. Then $f(a_i, j) = f(b, j+1) = n$ and by Lemma 3.2(4), $\big((a_i, j, i), (b, j+1, i+1)\big) \notin W_n$. Hence, by Lemma 3.3(2), $\tau^f(F_0) = 0$, which contradicts $\tau^f(F_0) = 1$. □

Proof of $\tau(F_0) = 0$: Assume to the contrary that $\tau(F_0) = 1$. Then there exists $f \in^{(Y_2 \times Y_4)} X_1$ such that $\tau^f(F_0) = 1$. Then by Lemma 3.10 there exist $C_0, C_1 \in \binom{X_{1,0}}{3}$ with $\sharp C_0 \cap C_1 = 1$ such that $\bigcup_{i,j \in [0,1]} \{f(\alpha_i(C_0, C_1), j)\} = C_0 \cap C_1$. Hence it follows from Lemma 3.9 that $\tau^f(F_0) = 0$. This contradiction proves $\tau(F_0) = 0$.

Since $\mu(F_0) = 1$, this completes the proof of $\mu \not\leq \tau$ in Example 2.1.

References

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