Geometric properties of singular points on extended spacelike CMC surfaces in L^3

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Geometric properties of singular points on extended spacelike CMC surfaces in $\mathbb{L}^3$

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Contents

Chapter 1

Introduction

In the Euclidean 3-space $\mathbb{E}^3$, *minimal surfaces* with zero mean curvature and more generally *constant mean curvature surfaces* (*CMC surfaces*, for short) have been studied for a long time. A minimal surface is a critical point of the surface area, while a CMC surface is a critical point of the surface area under volume-preserving deformations. This is why these surfaces are sometimes called mathematical models of soap films and soap bubbles, respectively. These surfaces are applied not only in mathematics, but also in architecture, physics, and other fields of science and technology. Therefore, it is very important to study such surfaces.

On the other hand, in the Lorentz-Minkowski 3-space $\mathbb{L}^3$, a spacelike surface with zero mean curvature is called a *maximal surface*. In this sense, maximal surfaces in $\mathbb{L}^3$ have many properties similar to minimal surfaces in $\mathbb{E}^3$, and there are a lot of studies about such a surfaces. For example, Kobayashi introduced the Weierstrass-Enneper type representation formula for maximal surfaces for the first time (see [9]). In contrast, CMC surfaces in $\mathbb{L}^3$ have fewer studies and more open problems than maximal surfaces. This is why we consider the shape and singularities of CMC surfaces in $\mathbb{L}^3$ in this paper. CMC surfaces in $\mathbb{L}^3$ is important because certain singularities which do not exist in $\mathbb{E}^3$ appear.

A surface in $\mathbb{L}^3$ is called *spacelike, timelike* or *lightlike* at a point if its induced metric is Riemannian, Lorenzian or degenerate at the point, respectively. In particular, we study spacelike CMC surfaces.

In Chapter 2, we introduce the behavior of the *singular curvature* and the *limiting normal curvature* at singular points of the first kind for extended spacelike CMC surfaces, which is a class of "surfaces with constant mean curvature" that allows certain kinds of singularities. (cf. Definition 1.0.1)

A Kenmotsu-type representation formula for spacelike surfaces with prescribed mean curvature in $\mathbb{L}^3$ was given by Akutagawa and Nishikawa [1]. Applying this formula, Umeda introduced the notion of *extended spacelike constant mean curvature (CMC) surfaces* in $\mathbb{L}^3$, and investigated singularities of them [16]. Locally, extended spacelike CMC surfaces are constructed as follows:

Definition 1.0.1. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let $\Sigma \subset \mathbb{C}$ be a simply-connected domain and $g \colon \Sigma \to \hat{\mathbb{C}}$ be a smooth map.

(1) g is said to be a ***regular extended harmonic map*** if

(a) $g_{z\bar{z}} + 2(1 - |g|^2)\bar{g} g_z \overline{\hat{\omega}} = 0$ holds, where

$$\hat{\omega} = \frac{\bar{g}_z}{(1 - |g|^2)^2} \tag{1.0.1}$$

and z is a complex coordinate of Σ, and

(b) $\omega = \hat{\omega} dz$ can be extended to a 1-form of class C^1 across

$$\{p \in \Sigma \mid |g(p)| = 1\}.$$

(2) Let $g \colon \Sigma \to \hat{\mathbb{C}}$ be a regular extended harmonic map and H a nonzero constant. Then a map $f \colon \Sigma \to \mathbb{L}^3$ given by

$$f = \frac{2}{H} \mathrm{Re}\left(\int (-2g, 1 + g^2, i(1 - g^2)) \hat{\omega} dz \right) \tag{1.0.2}$$

is called a ***generalized spacelike constant mean curvature* (*CMC*) *surface*** with mean curvature H, where $\hat{\omega}$ is defined as in (1.0.1). Moreover, a generalized spacelike CMC surface f given as (1.0.2) is said to be an ***extended spacelike constant mean curvature* (*CMC*) *surface*** if g satisfies the following properties:

- $\hat{\omega}$ never vanishes on $\{p \in \Sigma \mid |g(p)| < \infty\}$, and
- $g^2 \hat{\omega}$ does not vanish on $\{p \in \Sigma \mid |g(p)| = \infty\}$.

In such a case, g is called an ***extended harmonic map***.

On the other hand, a cuspidal edge, a cuspidal cross cap and a cuspidal S_1^- singularity belong to the class of non-degenerate singular points of the first kind of *frontals* (see [12]). The singular set of such singularities consists of regular curves (called *singular curves*) on the source and the singular images are regular space curves. Along such curves, several geometric invariant are defined. In particular, the *singular curvature* κ_s and the *limiting normal curvature* κ_ν are representative because they satisfy $\kappa = \sqrt{\kappa_s^2 + \kappa_\nu^2}$, where κ is the curvature of a singular image as a space curve (see [12, 15]). Moreover, κ_s is an *intrinsic invariants* of a front, and its sign relates to the *convexity* and *concavity*. Further, κ_ν relates to the boundedness of the Gaussian curvature of a frontal near a singular point. A main theorem in chapter 2 is as follows (cf. Theorem 2.1):

Theorem. Let f be a map given by (1.0.2). Then the singular curvature κ_s (cf. Definition 2.1.5) is strictly negative at a non-degenerate singular point of the first kind, and the limiting normal curvature κ_ν (cf. Definition 2.1.5) vanishes at such a singular point.

The fact that the singular curvature κ_s is negative at such a point means that the shape of the surface is concave.

Chapter 3 is a preparation chapter to explain the results of Chapter 4. The classical Björning problem for minimal surfaces ($H = 0$) in $\mathbb{E}^3$ is to find minimal surfaces containing a given real analytic curve with prescribed tangent planes along the curve (see [4]). Kim and Yang proved that in $\mathbb{L}^3$there is also a solution when the initial curve is null, that is, the surface is not immersed there (see [10]). Moreover, Brander showed that the singular Björling problem can also be solved for non-maximal ($H \neq 0$) CMC surfaces. For more details on this, refer to [2].

In Chapter 4, We consider a criteria for a certain singularity. More precisely, we proved the following theorem:

Theorem. Let f be a non-degenerate generalized H-surface constructed from the Björling data in Theorem 3.1. Then:

f is locally diffeomorphic to a cuspidal butterfly at $z = 0$ if and only if

$$t(0) \neq 0, \quad s(0) = s'(0) = 0 \quad \text{and} \quad s''(0) \neq 0$$

For swallowtail, cuspidal edge and cuspidal cross caps, Brander already proved this kind of criteria in the paper [2] in 2011.

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Chapter 2

Singular curvature and limiting normal curvature for extended spacelike CMC surfaces

2.1 Lorentz-Minkowski space

We recall some properties of Lorentz-Minkowski 3-space. Let $\mathbb{R}^3 = \{\boldsymbol{x} = (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ be an affine 3-space. Then we set a bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{L}}$ on $\mathbb{R}^3$ as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathbb{L}} = -x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{2.1.1}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. We call $\langle \cdot, \cdot \rangle_{\mathbb{L}}$ the *Lorentzian inner product* on $\mathbb{R}^3$. Moreover, a space $\mathbb{L}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{L}})$ is called the ***Lorentz-Minkowski 3-space***. For two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}^3 \setminus \{0\}$, we say that $\boldsymbol{x}$ is ***pseudo-orthogonal*** to $\boldsymbol{y}$ if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathbb{L}} = 0$.

A vector $\boldsymbol{x} \in \mathbb{L}^3 \setminus \{0\}$ is said to be ***spacelike***, ***lightlike*** or ***timelike*** if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathbb{L}} > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathbb{L}} = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathbb{L}} < 0$, respectively. The *norm* $|\cdot|_{\mathbb{L}}$ of $\boldsymbol{x} \in \mathbb{L}^3$ is defined as $|\boldsymbol{x}|_{\mathbb{L}} = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathbb{L}}|}$. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the canonical pseudo-orthonormal basis of $\mathbb{L}^3$, that is, $\boldsymbol{e}_1 = (1, 0, 0), \boldsymbol{e}_2 = (0, 1, 0)$, and $\boldsymbol{e}_3 = (0, 0, 1)$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be non-zero vectors in $\mathbb{L}^3$. Then we set a *pseudo-vector product* $\boldsymbol{x} \wedge \boldsymbol{y}$ as

$$\boldsymbol{x} \wedge \boldsymbol{y} = -\det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} \boldsymbol{e}_1 - \det \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix} \boldsymbol{e}_2 + \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \boldsymbol{e}_3, \tag{2.1.2}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$. It is easy to see that $\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{z} \rangle_{\mathbb{L}} = \det(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. In particular, $\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} \rangle_{\mathbb{L}} = \langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{y} \rangle_{\mathbb{L}} = 0$.

Let U be a domain in $\mathbb{R}^2$. Let $f\colon U \to \mathbb{L}^3$ be a C^∞ immersion. Then a point $p \in U$ is said to be a ***spacelike point***, a ***lightlike point*** or a ***timelike point*** of f if the induced metric $f^*\langle\cdot,\cdot\rangle_{\mathbb{L}}$ is positive definite, null or indefinite at p, respectively. We call an immersion f a ***spacelike immersion*** if every point $p \in U$ is a spacelike point of f. For a spacelike immersion $f\colon U \to \mathbb{L}^3$, it is known that $f_u \wedge f_v$ is a timelike vector, where (u, v) is a local coordinate system on U, $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$. We set $\nu\colon U \to H^2$ as

$$\nu = \frac{f_u \wedge f_v}{|f_u \wedge f_v|_{\mathbb{L}}}, \tag{2.1.3}$$

where $H^2 = \{\boldsymbol{x} \in \mathbb{L}^3 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\mathbb{L}} = -1\}$ is the hyperbolic 2-space. We call ν the ***pseudo unit normal vector*** of f.

2.1.1 Frontals

We review some notions of frontals quickly. Let $f\colon \Sigma \to \mathbb{R}^3$ be a C^∞ map, where Σ is an open domain in $\mathbb{R}^2$ and $\mathbb{R}^3$ is the Euclidean 3-space with canonical inner product $\langle\cdot,\cdot\rangle$. Then f is called a ***frontal*** if there exists a C^∞ map $\boldsymbol{n}\colon \Sigma \to S^2$ such that $\left\langle df_q(X), \boldsymbol{n}(q)\right\rangle = 0$ holds for any $q \in \Sigma$ and $X \in T_q\Sigma$, where S^2 denotes the standard unit sphere in $\mathbb{R}^3$. We say that the map $\boldsymbol{n}$ is a ***unit normal vector*** or ***the Gauss map*** of f. A frontal f is called a *front* if the pair $(f, \boldsymbol{n})\colon \Sigma \to \mathbb{R}^3 \times S^2$ gives an immersion.

We fix a frontal f. A point $p \in \Sigma$ is a ***singular point*** of f if $\operatorname{rank} df_p < 2$ holds. We denote by $S(f)$ the set of singular points of f. On the other hand, we define a function $\lambda\colon \Sigma \to \mathbb{R}$ by

$$\lambda(u, v) = \det(f_u, f_v, \boldsymbol{n})(u, v), \tag{2.1.4}$$

where (u, v) is a coordinate on Σ.

This function λ is called the ***signed area density function***. For the function λ, it is known that there exist functions $\hat{\lambda}$ and μ such that $\lambda = \hat{\lambda}\cdot\mu$, $\hat{\lambda}^{-1}(0) = S(f)$ and $\mu > 0$ on Σ. We call $\hat{\lambda}$ the ***singularity identifier*** of f.

A singular point $p \in S(f)$ of a frontal f is ***non-degenerate*** if $(\hat{\lambda}_u(p), \hat{\lambda}_v(p)) \neq (0, 0)$. Take a non-degenerate singular point p. Then there exist a neighborhood $U(\subset \Sigma)$ of p and a regular curve $\gamma = \gamma(t)\colon (-\varepsilon, \varepsilon) \to U$ $(\varepsilon > 0)$ with $\gamma(0) = p$ such that $\hat{\lambda}(\gamma(t)) = 0$ on U by the implicit function theorem. We call the curve γ a ***singular curve***. Moreover, since a non-degenerate singular point p satisfies $\operatorname{rank} df_p = 1$, there exists a non-zero vector field η on U such that $df_q(\eta_q) = 0$ for any $q \in S(f) \cap U$. This vector field η is called a ***null vector field***. Further, one can take a vector field ξ on U so that ξ is tangent to γ on $S(f) \cap U$. We call the direction of ξ along γ the ***singular direction***. A non-degenerate singular point p is of the ***first kind*** if ξ and η are linearly independent at p. Otherwise, it is said to be of the ***second kind***.

Definition 2.1.1. (1) Let $f, g\colon (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ be C^∞ map-germs. Then f and g are $\mathcal{A}$-***equivalent*** if there exist diffeomorphism-germs $\varphi\colon (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ on the source and $\Phi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ on the target such that $\Phi \circ f \circ \varphi^{-1} = g$ holds.

(2) Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a C^∞ map-germ. Then

- f at 0 is a ***cuspidal edge*** (or an *ordinary cuspidal edge*) if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ at 0.
- f at 0 is a ***swallowtail*** if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at 0.
- f at 0 is a ***cuspidal butterfly*** if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 4v^5 + uv^2, 5v^4 + 2uv)$ at 0.
- f at 0 is a ***cuspidal cross cap*** if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, uv^3)$ at 0.
- f at 0 is a ***cuspidal $S_k^\pm$ singularity*** ($k \geq 0$) if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3(u^{k+1} \pm v^2))$ at 0.
- f at 0 is a ***5/2-cuspidal edge*** (or a *rhamphoid cuspidal edge*) if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^5)$ at 0.

We note that these singularities are all non-degenerate frontal singularities. Moreover, a cuspidal edge, a cuspidal cross cap, a cuspidal $S_k^\pm$ singularity and a $5/2$-cuspidal edge are of the first kind, but a swallowtail and a cuspidal butterfly are of the second kind.

We next consider the geometric invariants of a frontal at a singular point of the first kind. As the above discussion, one can take ξ and η around a singular point of the first kind. Using these vector fields, we define two geometric invariant as follows:

$$\kappa_s(\gamma(t)) = \varepsilon_\gamma \frac{\det(\xi f, \xi\xi f, \boldsymbol{n})}{|\xi f|^3}\bigg|_{(u,v)=\gamma(t)}, \quad \kappa_\nu(\gamma(t)) = \frac{\langle \xi\xi f, \boldsymbol{n}\rangle}{|\xi f|^2}\bigg|_{(u,v)=\gamma(t)}, \tag{2.1.5}$$

where $\varepsilon_\gamma = \operatorname{sgn}(\det(\xi, \eta) \cdot \eta\lambda)$ along the singular curve γ. The invariants κ_s and κ_ν are called the ***singular curvature*** and the ***limiting normal curvature***, respectively. We remark that κ_s is an intrinsic invariant and its sign has a geometrical meaning [15]. Moreover, κ_ν relates to the behavior of the Gaussian curvature [12]. For more details and other invariants, see [15, 12, 7].

We suppose that any singular point is non-degenerate in the following. Then there exists a singular curve $\gamma(t)$ such that $\hat{\lambda}(\gamma(t)) = 0$. Differentiating this, we see that

$$\frac{d}{dt}(\hat{\lambda}(\gamma(t))) = \hat{\lambda}_z(\gamma(t))\gamma'(t) + \hat{\lambda}_{\bar{z}}(\gamma(t))\overline{\gamma'(t)}$$
$$= g_z\bar{g}\gamma' + g\overline{g_z}\overline{\gamma'} = \mathrm{Re}\left(\frac{g_z}{g}\gamma'\right) = 0,$$

where $' = d/dt$. This implies that γ' is perpendicular to $\overline{(g_z/g)}$. Thus we may take

$$\xi = ig\overline{g_z}\partial_z - ig_z\bar{g}\partial_{\bar{z}} \quad (\xi_\gamma = i\overline{(g_z/g)}\partial_z - i(g_z/g)\partial_{\bar{z}}) \tag{2.1.6}$$

near p. Here we used the following identification:

$$\zeta = a + ib \in \mathbb{C} \leftrightarrow (a,b) \in \mathbb{R}^2 \leftrightarrow a\partial_u + b\partial_v \leftrightarrow \zeta\partial_z + \bar{\zeta}\partial_{\bar{z}}. \tag{2.1.7}$$

We sometimes use the following relation:

$$\overline{\left(\frac{g_z}{g}\right)} = \frac{g}{g_z}\left|\frac{g_z}{g}\right|^2 \tag{2.1.8}$$

near p.

We next consider the null vector field η. Setting $\eta = \ell\partial_z + \bar{\ell}\partial_{\bar{z}}$, we have

$$\begin{aligned}\eta f &= \ell f_z + \bar{\ell} f_{\bar{z}} \\ &= \frac{\ell}{2}\left(-2, \frac{1}{g} + g, i\left(\frac{1}{g} - g\right)\right)g\hat{\omega} + \frac{\bar{\ell}}{2}\left(-2, \frac{1}{\bar{g}} + \bar{g}, -i\left(\frac{1}{\bar{g}} - \bar{g}\right)\right)\bar{g}\overline{\hat{\omega}} \\ &= \frac{\ell}{2}(-2, \bar{g} + g, i(\bar{g} - g))g\hat{\omega} + \frac{\bar{\ell}}{2}(-2, g + \bar{g}, -i(g - \bar{g}))\bar{g}\overline{\hat{\omega}} \\ &= (-1, \mathrm{Re}(g), \mathrm{Im}(g))(\ell g\hat{\omega} + \bar{\ell}\bar{g}\overline{\hat{\omega}})\end{aligned}$$

at a singular point. Thus one can take η as

$$\eta = \frac{i}{g\hat{\omega}}\partial_z - \frac{i}{\bar{g}\overline{\hat{\omega}}}\partial_{\bar{z}}. \tag{2.1.9}$$

Lemma 2.1.1. Let $f\colon \Sigma \to \mathbb{L}^3$ be an extended spacelike CMC surface given by (1.0.2), where Σ is a simply-connected domain in the complex plane $\mathbb{C}$ with complex coordinate $z = u + iv$. We now identify $\mathbb{L}^3$ with $\mathbb{R}^3$ to investigate singularities of f, that is, we regard f as a surface in $\mathbb{R}^3$. Moreover, we assume that $|g|$ takes finite values on Σ. Then the Euclidean unit normal vector $\boldsymbol{n}$ of f can be taken as

$$\boldsymbol{n} = \frac{1}{\sqrt{(1+|g|^2)^2 + 4|g|^2}}(1 + |g|^2, 2\,\mathrm{Re}(g), 2\,\mathrm{Im}(g)). \tag{2.1.10}$$

Proof. By (1.0.2), the first order differentials of f by z and $\bar{z}$ are

$$f_z = \frac{1}{H}(-2g, 1+g^2, i(1-g^2))\hat{\omega}, \quad f_{\bar{z}} = \frac{1}{H}(-2\bar{g}, 1+\bar{g}^2, -i(1-\bar{g}^2))\overline{\hat{\omega}}. \tag{2.1.11}$$

We note that f_z is a holomorphic map with respect to z. Since

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v), \tag{2.1.12}$$

we have

$$f_u \times f_v = -2i f_z \times f_{\bar{z}} = \frac{(|g|^2-1)|\hat{\omega}|^2}{H^2}(1+|g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)), \tag{2.1.13}$$

where $\times$ is the canonical vector product of $\mathbb{R}^3$. Thus the Euclidean unit normal vector $\boldsymbol{n}$ of f can be taken as

$$\boldsymbol{n} = \frac{1}{\sqrt{(1+|g|^2)^2 + 4|g|^2}}(1+|g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)) \tag{2.1.14}$$

□

Moreover, by using f_z, $f_{\bar{z}}$ and $\boldsymbol{n}$, the signed area density function of f is

$$\lambda = (|g|^2 - 1)|\hat{\omega}|^2 \frac{\sqrt{(1+|g|^2)^2 + 4|g|^2}}{H^2}.$$

Since $\sqrt{(1+|g|^2)^2 + 4|g|^2}/H^2 > 0$, the set of singular points $S(f)$ of f is the union $S(f) = S_1(f) \cup S_2(f)$, where

$$S_1(f) = \{p \in \Sigma \mid |g(p)| - 1 = 0\}, \quad S_2(f) = \{p \in \Sigma \mid |\hat{\omega}(p)| = 0\}.$$

If f is an extended spacelike CMC surface in $\mathbb{L}^3$, $\hat{\omega} \neq 0$ (see Definition 1.0.1). Thus $S_2(f) = \emptyset$ in such a case. Moreover, the singularity identifier $\hat{\lambda}$ is $\hat{\lambda}(z) = g(z)\overline{g(z)} - 1$.

Lemma 2.1.2 (cf. [16, Theorem 4.1])**.** A non-degenerate singular point p of an extended spacelike CMC surface f constructed by (1.0.2) is of the first kind if and only if $\operatorname{Im}(g_z/g^2\hat{\omega}) \neq 0$ at p.

Theorem 2.1. Let f be a map constructed by (1.0.2) in $\mathbb{E}^3$. Then the singular curvature κ_s is strictly negative at a non-degenerate singular point of the first kind, and the limiting normal curvature κ_ν vanishes at such a singular point.

Proof. We consider the first order directional derivative of f given by (1.0.2) in the direction ξ.

By (2.1.11) and the relation $\overline{g} = 1/g$ on the set of singular points $S_1(f)$, it follows that

$$\begin{aligned} \xi f &= i\overline{\left(\frac{g_z}{g}\right)} f_z - i\left(\frac{g_z}{g}\right) f_{\bar z} = \frac{2i}{H}\left(\overline{\frac{g_z}{g^2\hat\omega}} - \frac{g_z}{g^2\hat\omega}\right)|\hat\omega|^2(-1,\mathrm{Re}(g),\mathrm{Im}(g)) \\ &= \frac{4|\hat\omega|^2}{H}\,\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)(-1,\mathrm{Re}(g),\mathrm{Im}(g)) \end{aligned} \tag{2.1.15}$$

on $S_1(f)$. In particular, if $p \in S_1(f)$ is of the first kind, then ξf does not vanish at p by Lemma 2.1.2. By (2.1.11),

$$\begin{aligned} f_{zz} &= \frac{2g_z\hat\omega}{H}(-1, g, -ig) + \frac{2g\hat\omega_z}{H}(-1,\mathrm{Re}(g),\mathrm{Im}(g)), \\ f_{z\bar z} &= \frac{2g\hat\omega_{\bar z}}{H}(-1,\mathrm{Re}(g),\mathrm{Im}(g)), \quad f_{\bar z z} = \frac{2\overline{g}\overline{\hat\omega}_z}{H}(-1,\mathrm{Re}(g),\mathrm{Im}(g)), \\ f_{\bar z\bar z} &= \frac{2\overline{g}_{\bar z}\overline{\hat\omega}}{H}(-1,\overline{g}, i\overline{g}) + \frac{2\overline{g}\overline{\hat\omega}_{\bar z}}{H}(-1,\mathrm{Re}(g),\mathrm{Im}(g)) \end{aligned}$$

hold at $p \in S_1(f)$ since $g_{\bar z} = \overline{g}_z = 0$ at p. Therefore we have

$$\xi\xi f = Z(-1,\mathrm{Re}(g),\mathrm{Im}(g)) - \frac{2|g_z|^2|\hat\omega|^2}{H}(-\varphi - \overline{\varphi}, g\overline{\varphi} + \overline{g}\varphi, -i(g\overline{\varphi} - \overline{g}\varphi)) \tag{2.1.16}$$

holds on $S_1(f)$, where Z is a some function and $\varphi = g_z/g^2\hat\omega$.

We set $\hat{\boldsymbol{n}} = \boldsymbol{n} \circ \gamma$, where γ is a singular curve through $p \in S_1(f)$. This is expressed as

$$\hat{\boldsymbol{n}}(t) = \boldsymbol{n}(\gamma(t)) = \frac{1}{2\sqrt{2}}(2, 2\,\mathrm{Re}(g), 2\,\mathrm{Im}(g)) = \frac{1}{\sqrt{2}}(1,\mathrm{Re}(g),\mathrm{Im}(g)). \tag{2.1.17}$$

Since $(-1,\mathrm{Re}(g),\mathrm{Im}(g))$ is perpendicular to $\hat{\boldsymbol{n}}$, we have $\langle \xi\xi f, \hat{\boldsymbol{n}}\rangle = 0$. This implies that the limiting normal curvature κ_ν vanishes identically along γ. Moreover, by (2.1.15) and (2.1.17), we obtain

$$\hat{\boldsymbol{n}} \times \xi f = \frac{8}{\sqrt{2}H}\,\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)|\hat\omega|^2(0, -\mathrm{Im}(g), \mathrm{Re}(g)) \tag{2.1.18}$$

along γ. Thus by (2.1.16) and (2.1.18), it follows that

$$\det(\xi f, \xi\xi f, \hat{\boldsymbol{n}}) = \langle \hat{\boldsymbol{n}} \times \xi f, \xi\xi f\rangle = \frac{16}{\sqrt{2}H^2}\left(\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)\right)^2 |g_z|^2|\hat\omega|^4 (>0) \tag{2.1.19}$$

On the other hand, from (2.1.6) and (2.1.19), we have

$$\eta\hat\lambda = -2\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right), \quad \det(\xi,\eta) = 2\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)$$

at $p \in S_1(f)$ and $\varepsilon_\gamma = \mathrm{sgn}(\eta\lambda \cdot \det(\xi,\eta)) = \mathrm{sgn}(\eta\hat\lambda \cdot \det(\xi,\eta)) = -1$, where $\hat\lambda = |g|^2 - 1$.

Therefore we have

$$\kappa_s = -\frac{|H||g_z|^2}{16\left|\mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)\right||\hat{\omega}|^2} < 0$$

along γ. This completes the proof.

□

Chapter 3

Singular Björling problem

In this chapter we introduce the singular Björling problem. We give a summary of [2].

We consider singularities of spacelike CMC surfaces in the Lorentz-Minkowski 3-space $\mathbb{L}^3$. The singular Björling problem for such surfaces is stated as follows: given a real analytic null-curve $f_0(x)$, and a real analytic null vector field $v(x)$ parallel to the tangent field of f_0, find a conformally parameterized (generalized) spacelike CMC H surface in $\mathbb{L}^3$ which contains this curve as a singular set and such that the partial derivatives f_x and f_y are given by $\frac{\mathrm{d}f_0}{\mathrm{d}x}$ and v along the curve. Within the class of generalized surfaces considered, the solution is unique and we give a formula for the generalized Weierstrass data for this surface. This gives a framework for studying the singularities of non-maximal spacelike CMC surfaces in $\mathbb{L}^3$. We use this to find the Björling data – and holomorphic potentials – which characterize cuspidal edge, swallowtail and cross cap singularities. See Brander's paper [2] for details. In the next chapter we will give the criteria of A_4-singularities (Cuspidal butterfly) via the Björing data.

3.1 The loop group formulation

We use the basis

$$e_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

for the Lie algebra $\mathfrak{su}_{1,1}$. Then, $\mathbb{L}^3 = (\mathfrak{su}_{1,1}, \langle\, , \,\rangle)$ gives the Lorentz-Minkowski 3-space, where $\langle\, , \,\rangle$ is the Killing metric

$$\langle X, Y \rangle = \frac{1}{2}\mathrm{trace}(XY).$$

Moreover, these vectors are orthogonal and normalized as follows:

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = -\langle e_3, e_3\rangle = 1,$$

so we identify $\mathfrak{su}_{1,1}$ with the Lorentz-Minkowski space $\mathbb{L}^3 = \mathbb{R}^{2,1}$, and also use the notation

$$[a, b, c]^T = ae_1 + be_2 + ce_3$$

for a point in $\mathbb{L}^3$.

Let G be the subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of elements of either $\mathrm{SU}_{1,1}$ or of $ie_1 \cdot \mathrm{SU}_{1,1}$,

$$G = \left\{ \begin{pmatrix} a & b \\ \varepsilon\bar{b} & \varepsilon\bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C}, \quad \varepsilon(a\bar{a} - b\bar{b}) = 1, \quad \varepsilon = \pm 1 \right\}.$$

The Lie algebra of G is $\mathfrak{g} = \mathfrak{su}_{1,1}$.

Set ΛG_σ as

$$\Lambda G_\sigma := \Big\{ \xi : S^1 \to G; [\xi(\lambda)]_{11}, [\xi(\lambda)]_{22} : \text{even functions},$$

$$[\xi(\lambda)]_{12}, [\xi(\lambda)]_{21} : \text{odd functions} \Big\},$$

where $[\xi(\lambda)]_{ij}$ is the (i, j)-component of $\xi(\lambda)$. We call ΛG_σ the ***twisted loop group***, and denote it by $\mathcal{U}$. If we replace above G by $\mathrm{SL}(2,\mathbb{C})$ (resp. $G_\pm$), we have $\mathcal{U}^{\mathbb{C}} := \Lambda\,\mathrm{SL}(2,\mathbb{C})_\sigma$ (resp. $\mathcal{U}_\pm := \Lambda(G_\pm)_\sigma$). Here, $\mathcal{U}^{\mathbb{C}}$ is the complexification of $\mathcal{U}$. Then $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_{-1}$ holds.

We will also refer to the following subgroups of $\mathcal{U}^{\mathbb{C}}$,

$$\mathcal{U}^{\mathbb{C}}_\pm := \left\{ \hat{\mathrm{B}} \in \mathcal{U}^{\mathbb{C}} \middle| \mathrm{B} \text{ extends holomorphically to } \mathbb{D}_\pm \right\},$$

$$\hat{\mathcal{U}}^{\mathbb{C}}_+ := \left\{ \hat{B} \in \mathcal{U}^{\mathbb{C}}_+ \middle| \hat{B}|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \ \rho \in \mathbb{R}, \ \rho > 0 \right\},$$

where $\mathbb{D}_\pm := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid |\lambda|^{\pm 1} < 1\}$.

Then, $\mathcal{B}_{1,1} := \mathcal{U} \cdot \mathcal{U}^{\mathbb{C}}_+$ is called the ***big cell***. For $n \in \mathbb{N}$, the ***n-th small cell*** is defined as $\mathcal{P}_n := \mathcal{U}_1 \cdot \omega_n \cdot \mathcal{U}^{\mathbb{C}}_+$, where ω_n is given by

$$\omega_n = \begin{pmatrix} 1 & 0 \\ \lambda^{-n} & 0 \end{pmatrix} \quad \text{(if } n \text{ is odd)}, \qquad \omega_n = \begin{pmatrix} 1 & \lambda^{1-n} \\ 0 & 1 \end{pmatrix} \quad \text{(if } n \text{ is even)}.$$

Fact 1 ($\mathrm{SU}_{1,1}$-Iwasawa decompositon [3])**.**

(1) The group $\mathcal{U}^{\mathbb{C}}$ is a disjoint union

$$\mathcal{U}^{\mathbb{C}} = \mathcal{B}_{1,1} \sqcup \left(\bigsqcup_{m \in \mathbb{Z}^+} \mathcal{P}_m \right).$$

(2) Any loop $\varphi \in \mathcal{B}_{1,1}$ can be expressed as

$$\varphi = FB, \quad F \in \mathcal{U}, \quad B \in \mathcal{U}_+^{\mathbb{C}}.$$

The factor F is unique up to right multiplication by an element of $\mathcal{U}^0$.

The factors are unique if we require that $B \in \hat{\mathcal{U}}_+^{\mathbb{C}}$ and then the product map $\mathcal{U} \times \hat{\mathcal{U}}_+^{\mathbb{C}} \to \mathcal{B}_{1,1}$ is a real analytic diffeomorphism.

(3) The big cell, $\mathcal{B}_{1,1}$ is an open dense subset of $\mathcal{U}^{\mathbb{C}}$. The complement of $\mathcal{B}_{1,1}$ in $\mathcal{U}^{\mathbb{C}}$ is locally given as the zero set of a non-constant real analytic function $g : \mathcal{U}^{\mathbb{C}} \to \mathbb{C}$.

3.2 Generalized spacelike CMC surfaces.

Definition 3.2.1 ([3]). On a simply-connected Riemann surface Σ with local coordinate $z = x + iy$, we define a ***standard (holomorphic) potential*** as an $\mathfrak{sl}_2\mathbb{C}$-valued λ-dependent 1-form, the Fourier expansion of which begins at λ^{-1} as

$$\hat{\xi} = \begin{pmatrix} \sum_{j=0}^{\infty} c_{2j}\lambda^{2j} & \sum_{j=0}^{\infty} a_{2j-1}\lambda^{2j-1} \\ \sum_{j=0}^{\infty} b_{2j-1}\lambda^{2j-1} & -\sum_{j=0}^{\infty} c_{2j}\lambda^{2j} \end{pmatrix} dz,$$

where $a_j\,dz, b_j\,dz, c_j\,dz$ are all holomorphic 1-forms defined on Σ, and a_{-1} is never zero.

The map $\mathcal{S} : \mathcal{U} \to \mathrm{Lie}(\mathcal{U})$ given by

$$\mathcal{S}(\hat{F}) := -\frac{1}{2H}\left(\hat{F}e_3\hat{F}^{-1} + 2i\lambda\frac{\partial\hat{F}}{\partial\lambda}\hat{F}^{-1}\right)$$

is called the ***Sym-Bobenko formula***. We write $\mathcal{S}_\lambda : \mathcal{U} \to \mathbb{L}^3$ for the map given by evaluating this at $\lambda \in \mathbb{S}^1$.

Let $\pi : \mathcal{B}_{1,1} \to \mathcal{U}/\mathcal{U}^0$ denote the projection defined by taking the equivalence class of $\hat{F}$ (under right multiplication by elements of $\mathcal{U}^0$) in the Iwasawa factorization $\hat{\Phi} = \hat{F}\hat{B}$ of $\hat{\Phi} \in \mathcal{B}_{1,1}$. Since the Sym-Bobenko formula $\mathcal{S}$ is invariant under right multiplication by constant diagonal matrices, $\mathcal{S} : \mathcal{U}/\mathcal{U}^0 \to \mathrm{Lie}(\mathcal{U})$ is well defined (cf. [3], Lemma 4.7), and we can extend it to a map

$$\widetilde{\mathcal{S}} : \mathcal{B}_{1,1} \to \mathrm{Lie}(\mathcal{U}), \quad \widetilde{\mathcal{S}} = \mathcal{S} \circ \pi.$$

We call $\widetilde{\mathcal{S}}$ the ***extended Sym-Bobenko formula***. Denote by $\widetilde{\mathcal{S}}_{\lambda_0} : \mathcal{B}_{1,1} \to \mathbb{L}^3$ the map given by evaluating $\widetilde{\mathcal{S}}$ at $\lambda_0 \in \mathbb{S}^1$. The function $\widetilde{\mathcal{S}}$ is extended to a real analytic function $\mathcal{B}_{1,1} \sqcup \mathcal{P}_1 \to \mathrm{Lie}(\mathcal{U})$. (cf. [3], Lemma 4.8)

Definition 3.2.2. [cf. [2, Definition 3.1]] Let Σ be a simply-connected Riemann surface, $\hat{\xi}$ a standard potential, and $\hat{\Phi} : \Sigma \to \mathcal{U}^{\mathbb{C}}$ the map obtained by integrating $\hat{\Phi}^{-1}d\hat{\Phi} = \hat{\xi}$ with an initial condition $\hat{\Phi}(z_0) = \hat{\Phi}_0 \in \mathcal{U}^{\mathbb{C}}$. Assume that $\hat{\Phi}(w) \in \mathcal{B}_{1,1}$ for at least one point $w \in \Sigma$. Let $\Sigma_s \subset \Sigma$ be the open dense subset given by $\Sigma_s = \hat{\Phi}^{-1}(\mathcal{B}_{1,1} \cup \mathcal{P}_1)$, and define, for any $\lambda \in \mathbb{S}^1$,

$$f^\lambda : \Sigma_s \to \mathbb{L}^3, \quad f^\lambda(z) = \widetilde{\mathcal{S}}_\lambda(\hat{\Phi}(z)).$$

We call the map f^λ a ***generalized spacelike constant mean curvature surface,*** or *a* ***CMC H-surface***, in $\mathbb{L}^3$.

3.3 Singular Björling problem

We introduce the *weak non-degeneracy* for singular points, *singular holomorohic potential* and *singular holomorphic frame* in [2].

Definition 3.3.1 (cf. [2, Definition 3.5])**.** A point $p \in \mathbb{C}$ is called ***weakly non-degenerate***, if there exists a real analytic curve $\gamma : (-\delta, \delta) \to \Sigma$, for some $\delta > 0$, such that $\gamma(0) = p$ and $\gamma((-\delta, \delta)) \subset \mathbb{C}$ hold.

Definition 3.3.2. Let $\Sigma \subset \mathbb{C}$ be a simply connected domain such that $\{\mathrm{Re}(z) \mid z \in \Sigma\}$ is an open interval $J = \Sigma \cap \mathbb{R}$. Consider a holomorphic 1-form $\hat{\xi}_\omega$ on Σ expressed as

$$\hat{\xi}_\omega = \left\{ \begin{pmatrix} -a\lambda^{-2} & a\lambda^{-1} \\ b\lambda^{-1} - a\lambda^{-3} & a\lambda^{-2} \end{pmatrix} + \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} + \begin{pmatrix} \tilde{a}\lambda^2 & \tilde{b}\lambda - \tilde{a}\lambda^3 \\ \tilde{a}\lambda & -\tilde{a}\lambda^2 \end{pmatrix} \right\} \mathrm{d}z,$$

where

(i) a, b and r are holomorphic on Σ,

(ii) the restriction of r to J is real-valued, that is $\overline{r(\bar{z})} = r(z)$, and

(iii) $\tilde{a}$ and $\tilde{b}$ are holomorphic extensions of the restrictions $\tilde{a}|_{\mathbb{R}}$ and $\tilde{b}|_{\mathbb{R}}$, that is $\tilde{a}(z) = \overline{a(\bar{z})}, \tilde{b}(z) = \overline{b(\bar{z})}$.

Moreover, we assume the regularity condition:

$$a(z) \neq 0, \quad {}^\forall z \in \Sigma.$$

We call $\hat{\xi}_\omega$ a ***standard singular holomorphic potential*** on Σ.

Definition 3.3.3. The ***singular holomorphic frame*** $\hat{\Phi}_\omega$ corresponding to $\hat{\xi}_\omega$ to be the map $\hat{\Phi}_\omega : \Sigma \to \mathcal{U}^{\mathbb{C}}$ obtained by solving the equation

$$\hat{\Phi}_\omega^{-1} \mathrm{d}\hat{\Phi}_\omega = \hat{\xi}_\omega, \quad \hat{\Phi}_\omega(0) = I.$$

In [2], Brander showed that if $f : \Sigma_s \to \mathbb{L}^3$ is a generalized H-surface, and $z_0 \in \Sigma_s$ is a weakly non-degenerate singular point, then, f can be constructed locally from a singular frame $\hat{F}_\omega$ which satisfies the equations

$$\begin{cases} \hat{F}_\omega^{-1} f_x^\lambda \hat{F}_\omega = \dfrac{-4\mathrm{Re}(a\lambda^{-2})}{H} \begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix}, \\ \hat{F}_\omega^{-1} f_y^\lambda \hat{F}_\omega = \dfrac{4\mathrm{Im}(a\lambda^{-2})}{H} \begin{pmatrix} i & -i\lambda \\ i\lambda^{-1} & -i \end{pmatrix}, \end{cases}$$

which, at $\lambda = 1$, are:

$$F_\omega^{-1} f_x F_\omega = \frac{-4\mathrm{Re}(a)}{H}(-e_2 + e_3), \quad F_\omega^{-1} f_y F_\omega = \frac{4\mathrm{Im}(a)}{H}(-e_2 + e_3). \tag{3.3.1}$$

Suppose we have an open set $\Omega \subset \mathbb{C}$, with coordinates $z = x + yi$, and such that $J = \Omega \cap \mathbb{R}$ is a non-empty open interval containing the origin. Moreover, we assume that there exists a generalized H-surface $f : \Omega \to \mathbb{L}^3$, satisfying the Björling data along J, and with associated holomorphic extended frame Φ, such that $\hat{\Phi}(J) \subset \mathcal{P}_1$.
Since the vector fields f_x and f_y are both necessarily null and parallel along J, on this interval, and after an isometry of $\mathbb{L}^3$, we can write

$$f_x = s\begin{pmatrix} i & e^{i\theta} \\ e^{-i\theta} & -i \end{pmatrix}, \quad f_y = t\begin{pmatrix} i & e^{i\theta} \\ e^{-i\theta} & -i \end{pmatrix}, \quad \theta(0) = -\frac{\pi}{2},$$

where s, θ and t are all real analytic functions $J \to \mathbb{R}$. We assume that s and t never vanish at the same time, so that θ is well defined on J.

The equations (3.3.1) suggest that we choose a frame F_0 to be the rotation about the x_3-axis which rotates $[\cos\theta, \sin\theta, 0]^T \in \mathbb{L}^3$ so that it points in the $-e_2$ direction:

$$F_0 = \begin{pmatrix} e^{i\frac{2\theta+\pi}{4}} & 0 \\ 0 & e^{-i\frac{2\theta+\pi}{4}}. \end{pmatrix}$$

The normalization of θ means that $F_0(0) = I$. Then

$$F_0^{-1} f_x F_0 = s(-e_2 + e_3), \quad F_0^{-1} f_y F_0 = t(-e_2 + e_3). \tag{3.3.2}$$

To find λ dependence of the singular frame, we consider the following 1-form:

$$\hat{F}_\omega^{-1} \mathrm{d}\hat{F}_\omega = \left\{ \begin{pmatrix} -a\lambda^{-2} & a\lambda^{-1} \\ b\lambda^{-1} - a\lambda^{-3} & a\lambda^{-2} \end{pmatrix} + \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} + \begin{pmatrix} \bar{a}\lambda^2 & \bar{b}\lambda - \bar{a}\lambda^3 \\ \bar{a}\lambda & -\bar{a}\lambda^2 \end{pmatrix} \right\} \mathrm{d}x. \tag{3.3.3}$$

A smooth map $f_0 : J \to \mathbb{L}^3$ is called a *null curve* if $(f_0)_x = \mathrm{d}f_0(x)/\mathrm{d}x$ satisfies $\langle (f_0)_x, (f_0)_x \rangle = 0$ for each $x \in J$. The following theorem by Brander is a solution to the singular Björling problem.

Theorem 3.1 (cf. [2, Theorem 4.1])**.** Let $f_0 : J \to \mathbb{L}^3$ be a real analytic function on an interval $J \subset \mathbb{R}$ with a null vector field df_0/dx. Moreover, let $v(x)$ be a real analytic null vector field defined on J such that $v(x)$ is a scalar multiple of $(\mathrm{d}f_0/\mathrm{d}x)(x)$ for each $x \in J$. Suppose also that the vector fields $df_0/dx, v(x)$ do not vanish simultaneously at any point $x \in J$. Let s and t be the functions on J defined as above. Let $\hat{\Phi}_\omega$ be the singular holomorphic frame obtained by analytically extending the 1-form $\hat{F}_\omega^{-1}\mathrm{d}\hat{F}_\omega$ given by (3.3.3), with

$$a = \frac{H}{4}(-s + it), \quad b = \frac{1}{2}iHt, \quad r = \frac{1}{2}(\theta_x + Ht),$$

to some simply connected open set containing J, and integrating with initial condition $\hat{\Phi}_\omega(0) = I$. Suppose that $\hat{\Phi} = \hat{\Phi}_\omega \omega_1$ maps at least one point into $\mathcal{B}_{1,1}$. Then the surface

$$f(x, y) := \widetilde{\mathcal{S}_1}(\hat{\Phi}_\omega(x, y)) + \frac{1}{2H}e_3 + f_0(0),$$

is the unique weakly non-degenerate generalized H-surface such that f, f_x and f_y coincide respectively with $f_0, \mathrm{d}f_0/\mathrm{d}x$ and v along the real interval J.

Chapter 4

Identifying singularity types via the Björling construction

In this chapter, we consider the criteria for A_4 singularities (Cuspidal butterfly) via the Björling data (s, t).

Definition 4.0.1. A point $z_0 \in \hat{\Phi}^{-1}(\mathcal{P}_1)$ is called a non-degenerate singular point if the derivative map $\mathrm{d}h$ has rank 1 at z_0, and a degenerate singular point if $\mathrm{d}h = 0$. If, at a point $z_0 \in \hat{\Phi}^{-1}(\mathcal{P}_1)$ we have the milder condition that there exists a real analytic curve $\gamma : (-\delta, \delta) \to \Sigma$, for some $\delta > 0$, with $\gamma(0) = z_0$ and $\gamma((-\delta, \delta)) \subset \hat{\Phi}^{-1}(\mathcal{P}_1)$, then we say that z_0 is weakly non-degenerate. A generalized H-surface is non-degenerate or weakly non-degenerate if all singular points have the corresponding property.

4.1 Criteria for A_4-singularities (Cuspidal butterfly)

Lemma 4.1.1 (cf. [2, Lemma 5.5])**.** Let f be a non-degenerate generalized H-surface constructed from the Björling data in Theorem 3.1. Then f is a front on a neighborhood of $z = 0$ if and only if

$$t(0) \neq 0.$$

Definition 4.1.1. Suppose that $\gamma : (-\delta, \delta) \to U$ is a local parameterization of a singular curve, with parameter x and tangent vector γ', and $\gamma(0) = p$. Take the null vector field $\eta(x)$ on γ. Then we define a function of x by

$$\mu(x) = \det(\gamma'(x), \eta(x)).$$

Proposition 4.1.1. (cf. [8, Corollary A.9]) Let $f : U \to \mathbb{R}^3$ be a front, and p a non-degenerate singular point. The image of f in a neighborhood of p is diffeomorphic to a A_4-singularity (cuspidal butterfly) if and only if

$$\mu(0) = \mu'(0) = 0 \quad \text{and} \quad \mu''(0) = 0$$

From this Proposition, we can easily show the following Theorem.

Theorem 4.1. Let f be a non-degenerate generalized H-surface constructed from the Björling data in Theorem 3.1. Then:

f is locally diffeomorphic to a cuspidal butterfly at $z = 0$ if and only if

$$t(0) \neq 0, \quad s(0) = s'(0) = 0 \quad \text{and} \quad s''(0) \neq 0$$

Proof. By Lemma 4.1.1, f is a front at $z = 0$ if and only if $t(0) \neq 0$, so we can use the above Proposition 4.1.1 . We also have, along J,

$$f_x = sF_0(-e_2 + e_3)F_0^{-1}, \quad f_y = tF_0(-e_2 + e_3)F_0^{-1},$$

and the null direction is

$$\eta(x) = t(x)\frac{\partial}{\partial x} - s(x)\frac{\partial}{\partial y}.$$

Writing $x + iy = [x, y]^T$, the singular curve is given by $\gamma(x) = [x, 0]^T$ and the null direction by $\eta(x) = [t(x), -s(x)]^T$. Then we have

$$\mu(x) = \det\begin{pmatrix} 1 & t(x) \\ 0 & -s(x) \end{pmatrix} = -s(x)$$

Therefore, this theorem is proved by the previous Proposition 4.1.1. □

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