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# On the convergence rates for the compressible Navier- Stokes equations with potential force


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# On the convergence rates for the compressible Navier- Stokes equations with potential force

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**Abstract:** In this paper we are concerned with the convergence rates to the stationary solutions for the compressible Navier-Stokes equations with a potential external force $\nabla\Phi$ in the whole space $\mathbb{R}^n$ for $n \geq 2$. It is proved that the perturbation decays in $L^2$ norm in the same order as that of the $n$-dimensional heat kernel, if the initial perturbation is small in $H^{s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$ and the potential $\Phi$ is small in some Sobolev space. The results also hold for $n = 2$ when $\Phi = 0$.

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## 1 Introduction

This paper studies the initial value problem for the compressible Navier-Stokes equation with potential force in $\mathbb{R}^n$ :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \frac{\nabla P(\rho)}{\rho} = \frac{\mu}{\rho}\triangle u + \frac{\mu+\mu'}{\rho}\nabla(\nabla \cdot u) - \nabla\Phi(x), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \longrightarrow (\rho_\infty, 0) \qquad |x| \to \infty. \end{cases} \tag{1}$$

Here $t > 0$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ ($n \geq 2$); the unknown functions $\rho = \rho(t, x) > 0$ and $u = u(t, x) = (u_1(t, x), u_2(t, x), \cdots, u_n(t, x))$ denote the density and velocity, respectively; $P = P(\rho)$ is the pressure that are assumed to be a function of the density $\rho$; $-\nabla\Phi(x)$ is a time independent potential force; $\mu$ and $\mu'$ are the viscosity coefficients satisfying the conditions $\mu > 0$ and $\mu' + \frac{2}{n}\mu \geq 0$; $\rho_\infty$ is a given positive constant; and $\nabla\cdot$, $\nabla$ and $\triangle$ denote the usual divergence, gradient and Laplacian with respect to $x$, respectively.

We assume that $P(\rho)$ is smooth in a neighborhood of $\rho_\infty$ with $P'(\rho) > 0$ ($\rho \in [\frac{1}{2}\rho_\infty, \frac{3}{2}\rho_\infty]$).

When $\Phi$ is small, the Navier-Stokes equation $(1)_1-(1)_2$ with potential force has the stationary solution $(\rho_*, u_*) = (\rho_*(x), 0)$, where $\rho_*$ satisfies

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P'(s)}{s} ds + \Phi(x) = 0. \tag{2}$$

In this paper we derive the convergence rate of solution of problem (1) to the stationary solution $(\rho_*, 0)$ as $t \to \infty$ when the initial perturbation $(\rho_0 - \rho_*, u_0)$ is sufficiently small in $H^{s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$, $n \geq 2$.

When $\Phi = 0$, $(1)_1-(1)_2$ has a (constant) stationary solution $(\rho_*(x), u_*) = (\rho_\infty, 0)$. In this case Matsumura-Nishida [8] showed the global in time existence of solution of (1) for $n = 3$, provided that the initial perturbation $(\rho_0 - \rho_\infty, u_0)$ is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Furthermore, the following decay estimate was obtained in [8];

$$\|\nabla^k(\rho - \rho_\infty, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad k = 0, 1. \tag{3}$$

These results were proved by combining the energy method and the decay estimates of the semigroup $E(t)$ generated by the linearized operator $A$ at the constant state $(\rho_\infty, 0)$.

When $\Phi \neq 0$, Matsumura-Nishida [10] proved the global in time existence of solution of (1) for $n = 3$, provided that the initial perturbation $(\rho_0 - \rho_*, u_0)$ and $\Phi$ are sufficiently small. Moreover, Duan-Liu-Ukai-Yang [1] established the decay estimates :

$$\|\nabla^k(\rho - \rho_*, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad k = 0, 1$$

for initial perturbation $(\rho_0 - \rho_*, u_0)$ sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. The proof of these estimates was given by a suitable energy estimate for $\|\nabla(\rho - \rho_*, u)(t)\|_{H^2}$ and the decay estimates for $\nabla E(t)$.

On the other hand, Kawashita [7] showed the global existence of solution for initial perturbations sufficiently small in $H^{s_0}(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$, $n \geq 2$, when $\Phi = 0$. (Note that $s_0 = 2$ for $n = 3$). Wang-Tan [13] then considered the case $n = 3$ and $\Phi = 0$ when the initial perturbation $(\rho_0 - \rho_\infty, u_0)$ is sufficiently small in $H^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^3)$, and proved the decay estimates (3). The proof in [13] is similar to that in [1]; and the key in the proof of [13] is to use the bound $\int_0^\infty \|\nabla u(t)\|_{H^2}^2 \, dt \leq C\|(\rho_0 - \rho_\infty, u_0)\|_{H^2}^2$, which is obtained by the energy method, in the estimate of the nonlinearity to obtain the decay estimates under the less regularity assumption on $(\rho_0 - \rho_\infty, u_0)$.

In this paper we will extend the results in [1] and [13] in the following way by an approach different to [1, 13].

We show that if $n \geq 3$ then the following estimates hold true for the solution $(\rho, u)$ of (1) :

$$\|\nabla^k(\rho - \rho_*, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \quad k = 0, 1, \tag{4}$$

provided that $(\rho_0 - \rho_*, u_0)$ is sufficiently small in $H^{s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$ and that $\Phi$ is sufficiently small but $\Phi \neq 0$. Furthermore, if $\Phi = 0$, then the estimates (4) also hold for the case $n = 2$, provided that $(\rho_0 - \rho_*, u_0)$ is sufficiently small in $H^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. (Note that $s_0 = [\frac{n}{2}] + 1 = 2$ for $n = 2$.)

To prove (4), as in [6], we introduce a decomposition of the perturbation $U(t) = (\rho - \rho_*, u)(t)$ associated with the spectral properties of the linearized operator $A$ at the constant state $(\rho_\infty, 0)$. In the case of our problem, we simply decompose the perturbation $U(t)$ into low and high frequency parts. As for the low frequency part, we apply the decay estimates for the low frequency part of $E(t)$; while the high frequency part is estimated by using the energy method. One of the points of our approach is that by restricting the use of the decay estimates for $E(t)$ to its low frequency part, one can avoid the derivative loss due to the convective term of the transport equation $(1)_1$. On the other hand, the convective term of $(1)_1$ can be controlled by the energy method which we apply to the high frequency part. Another point is that in the high frequency part we have a Poincaré type inequality: $\|\nabla U_\infty\|_{L^2} \geq C\|U_\infty\|_{L^2}$, where $U_\infty$ is the high frequency part of the perturbation $U$. This yields the strict positivity inequality $(AU_\infty, U_\infty)_{L^2} + \gamma\|\nabla\sigma_\infty\|_{L^2}^2 \geq C_0\|U_\infty\|_{L^2}^2$ for some positive constants $C_0$ and $\gamma$, where $\sigma_\infty$ denotes the density component of $U_\infty$. Furthermore, the Poincaré type inequality makes the estimate of the nonlinearity a little bit simpler in the energy method. Using these properties we can deal with the time decay of $\|U(t)\|_{H^{s_0}}$ in contrast to the approach in [1, 13] which, roughly speaking, deals mainly with $\|\nabla U(t)\|_{H^{s_0-1}}$. In particular, by our approach, we can treat the case $n = 2$ if $\Phi = 0$.

The paper is organized as follows. In Section 2 we introduce the notation and auxiliary Lemmas used in this paper. In Section 3 we state the main result of this paper. In Section 4 we introduce a decomposition of the solution. In Section 5 we give the proof of the main result.

# 2 Preliminaries

In this section we first introduce the notation which will be used throughout this paper. We then introduce some auxiliary lemmas which will be useful in the proof of the main result.

Let $L^p (1 \leq p \leq \infty)$ denote the usual $L^p$-Lebesgue space on $\mathbb{R}^n$ with norm $\|\cdot\|_p$. For nonnegative integer $m$, we denote by $W^{m,p}(1 \leq p \leq \infty)$ the usual $L^p$-Sobolev space of order $m$ whose norm is denoted by $\|\cdot\|_{W^{m,p}}$. When $p = 2$, we define $H^m = W^{m,2}$. The inner-product of $L^2$ is denoted by $(\cdot, \cdot)$. We denote by $H^{-1}$ the dual space of $H^1$, and $\langle , \rangle$ denote the pairing between $H^{-1}$ and $H^1$.

We introduce the following notation for spatial derivatives. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, we denote

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{i=1}^n \alpha_i,$$

and for any integer $l \geq 0, \nabla^l f$ denotes all of $l$-th derivatives of $f$.

For a function $f$, we denote its Fourier transform by $\mathfrak{F}[f] = \hat{f}$:

$$\mathfrak{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx.$$

The inverse of $\mathfrak{F}$ is denoted by $\mathfrak{F}^{-1}[f] = \check{f}$,

$$\mathfrak{F}^{-1}[f](\xi) = \check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi) e^{i\xi\cdot x} d\xi.$$

For operators $A, B$, we denote the commutator of A and B by $[A, B]$:

$$[A, B]f = A(Bf) - B(Af).$$

$BC^k$ denotes the set of all functions such that $\nabla^l f$ is a bounded function for $l \leq k$.

We next state some basic Lemmas.

**Lemma 2.1** (Hardy's inequality )**.** *Assume that $n \geq 3$. Then there holds the inequality*

$$\|\frac{u}{|x|}\|_2 \leq C\|\nabla u\|_2$$

*for $u \in H^1$.*

See, e.g., [2], for the proof.

**Lemma 2.2.** *Assume that $n \geq 3$. Then there holds the inequality*

$$\|f\|_\infty \leq C\|\nabla f\|_{H^{s_0-1}}$$

*for $f \in H^{s_0}$.*

Lemma 2.2 is proved as follows. Let $p = \frac{2n}{n-2}$. Then, since $s_0 - 1 > \frac{n}{p}$, by the Sobolev inequalities, we have

$$\|f\|_\infty \leq C\|f\|_{W^{s_0-1,p}} \leq C\|\nabla f\|_{H^{s_0-1}}.$$

This proves Lemma 2.2.

**Lemma 2.3.** *Suppose $a(x) \in BC^1$. For $u \in L^2$ set*

$$[a(x)\frac{\partial}{\partial x_k}, \eta_\epsilon *]u(x) = a(x)\frac{\partial}{\partial x_k}(\eta_\epsilon * u)(x) - \big(\eta_\epsilon * (a\frac{\partial u}{\partial x_k})\big)(x).$$

*Here $\eta_\epsilon * u$ is standard Friedrichs mollifier. Then it holds that*

$$\|[a(x)\frac{\partial}{\partial x_k}, \eta_\epsilon *]u(x)\|_2 \leq C\|\nabla a\|_\infty \|u\|_2.$$

*and*

$$\|[a(x)\frac{\partial}{\partial x_k}, \eta_\epsilon *]u(x)\|_2 \longrightarrow 0 \quad (\epsilon \to 0).$$

See, e.g., [11], for the proof.

**Lemma 2.4.** *Suppose $u \in L^2(0,T;H^1)$ and $\frac{\partial}{\partial t}u \in L^2(0,T;H^{-1})$. Then, the mapping $t \mapsto \|u(t)\|_2^2$ is absolutely continuous, with*

$$\frac{d}{dt}\|u(t)\|_2^2 = 2 < u'(t), u(t) >$$

*in the sense of distribution.*

See, e.g., [2], for the proof.

**Lemma 2.5.** *If $0 \le s_j \; (j = 1,2,\cdots,l)$ satisfy $s_j \le \frac{n}{2} \; (j = 1,2,\cdots,l)$ and $s_1 + s_2 + \cdots + s_l > (\frac{n}{2})(l-1)$, then there holds*

$$\|f_1 \cdot f_2 \cdots f_l\|_2 \le C_{s_1 \cdots s_l} \prod_{j=1}^{l} \|f_j\|_{H^{s_j}}.$$

See, e.g., [7], for the proof.

By using Lemma 2.5 we have the following estimates.

**Lemma 2.6.** *(i) If $1 \le |\alpha| \le s_0$, $g \in H^{s_0}$ and $f \in H^{|\alpha|}$, then*

$$\|[\partial_x^\alpha, g]f\|_2 \le C \left\{ \begin{array}{l} \|\nabla g\|_{H^{s_0-1}}\|f\|_{H^{|\alpha|}} \\ \|\nabla g\|_{H^{s_0}}\|f\|_{H^{|\alpha|-1}}. \end{array} \right.$$

*(ii) Let $I$ be a compact interval of $\mathbb{R}$ and let $R(y,x) \in C^\infty(I \times \mathbb{R}^n)$. If $1 \le |\alpha| \le s_0$, then there holds*

$$\begin{aligned} \|\partial_x^\alpha\big(R(g(x),x)f\big)\|_2 \;\; \le \;\; & C\{R_0(g)\|f\|_2 + R_1(g)\|\nabla f\|_{H^{|\alpha|-1}} \\ & +R_2(g)(1+\|g\|_{H^{s_0}})^{|\alpha|-1}\|\nabla g\|_{H^{s_0-1}}\|f\|_{H^{|\alpha|}}\} \end{aligned}$$

*for $g \in H^{s_0}$ such that $g(x) \in I (x \in \mathbb{R}^n)$ and $f \in H^{|\alpha|}$. Here*

$$R_0(g) := \sup_{x\in\mathbb{R}^n} \big|(\partial_x^\alpha R)(g(x),x)\big|,$$

$$R_1(g) := \sup_{\beta<\alpha, x\in\mathbb{R}^n} \big|(\partial_x^\beta R)(g(x),x)\big|,$$

$$R_2(g) := \max_{k\ge1, k+|\beta|\le|\alpha|} \sup_{x\in\mathbb{R}^n} |(\partial_y^k \partial_x^\beta)R(g(x),x)|.$$

Lemma 2.6 can be proved in a similar way to the proof of [7, Lemma 3]. (See also [5, Lemma 4.3] and [4, Lemma A.2])

# 3 Main result

In this section, we first state the existence of stationary solution $(\rho_*, 0)$ and some estimates on $\rho_*$ which were obtained in Matsumura-Nishida [10]. We then state our main result on the convergence rate of solutions $(\rho(t), u(t))$ to $(\rho_*, 0)$ as $t \to \infty$.

**Proposition 3.1** (Matsumura-Nishida [10]). *There exist positive constants $\epsilon$ and $C$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|\big(1+|x|\big)\nabla\Phi\|_{L^2} \le \epsilon,$$

*the problem $(1)_1 - (1)_2$ has a stationary solution $\big(\rho_*, u\big) = \big(\rho_*(x), 0\big)$ in a small neighborhood of $(\rho_\infty, 0)$ ; and it satisfies*

$$\begin{aligned} & \|\rho_*(x) - \rho_\infty\|_{H^{s_0+1}} + \|\big(1+|x|\big)\nabla\rho_*(x)\|_{L^2} \\ \le\ & C\Big(\|\Phi\|_{H^{s_0+1}} + \|\big(1+|x|\big)\nabla\Phi\|_{L^2}\Big), \end{aligned}$$

$$|\rho_*(x) - \rho_\infty| < \frac{1}{2}\rho_\infty.$$

Let us rewrite the problem (1). By the change of variables,

$$\tilde\rho(t,x) = \rho(t,x) - \rho_*(x), \quad \tilde u(t,x) = u(t,x),$$

problem (1) is transformed into

$$\begin{cases} \partial_t\tilde\rho + \nabla\cdot(\rho_*\tilde u) = \tilde F_1, \\ \partial_t\tilde u - \frac{\mu}{\rho_*}\triangle\tilde u - \frac{\mu+\mu'}{\rho_*}\nabla\nabla\cdot\tilde u + \frac{P'(\rho_*)}{\rho_*}\nabla\tilde\rho + \big(\frac{P''(\rho_*)}{\rho_*} - \frac{P'(\rho_*)}{\rho_*^2}\big)\nabla\rho_*\tilde\rho = \tilde F_2, \\ (\tilde\rho, \tilde u)(0,x) = (\rho_0 - \rho_*, u_0)(x) \longrightarrow (0,0) \quad (|x|\to\infty), \end{cases}$$

where

$$\tilde F_1 = -\nabla\cdot(\tilde\rho\tilde u),$$

$$\begin{aligned} \tilde F_2 \ =\ & -(\tilde u\cdot\nabla)\tilde u - \mu\frac{\tilde\rho}{\rho_*(\tilde\rho+\rho_*)}\triangle\tilde u - (\mu+\mu')\frac{\tilde\rho}{\rho_*(\tilde\rho+\rho_*)}\nabla(\nabla\cdot\tilde u) \\ & + \Big(\frac{P'(\rho_*)}{\rho_*(\tilde\rho+\rho_*)} - \frac{1}{\tilde\rho+\rho_*}\int_0^1 P''(s\tilde\rho+\rho_*)ds\Big)\tilde\rho\nabla\tilde\rho \\ & + \Big(\frac{P''(\rho_*)}{\rho_*(\tilde\rho+\rho_*)}\nabla\rho_* - \frac{P'(\rho_*)}{\rho_*^2(\tilde\rho+\rho_*)}\nabla\rho_* - \frac{\nabla\rho_*}{\tilde\rho+\rho_*}\int_0^1(1-s)P'''(s\tilde\rho+\rho_*)ds\Big)\tilde\rho^2. \end{aligned}$$

Next, we define $\mu_1, \mu_2$ and $\gamma$ by

$$\mu_1 = \frac{\mu}{\rho_\infty},\ \mu_2 = \frac{\mu+\mu'}{\rho_\infty},\ \gamma = \sqrt{P'(\rho_\infty)}.$$

We also set

$$\bar\rho = \rho_*(x) - \rho_\infty.$$

By using the new unknown functions

$$\sigma(t,x)=\frac{1}{\rho_\infty}\tilde{\rho}(t,x),\ w(t,x)=\frac{1}{\sqrt{P'(\rho_\infty)}}\tilde{u}(t,x),$$

the initial value problem (1) is reformulated as

$$\begin{cases}\partial_t\sigma+\gamma\nabla\cdot w-B_1U=F_1(U),\\ \partial_t w-\mu_1\triangle w-\mu_2\nabla(\nabla\cdot w)+\gamma\nabla\sigma-B_2U=F_2(U),\\ (\sigma,w)(0,x)=(\sigma_0,w_0)(x),\end{cases}\tag{5}$$

where, $U=\begin{pmatrix}\sigma\\ w\end{pmatrix}$,

$$B_1U=-\frac{\gamma}{\rho_\infty}(w\cdot\nabla\bar{\rho}+\bar{\rho}\nabla\cdot w),$$

$$\begin{aligned}B_2U\ &=\ -\mu_1\frac{\bar{\rho}}{\rho_*}\triangle w-\mu_2\frac{\bar{\rho}}{\rho_*}\nabla(\nabla\cdot w)+\gamma\frac{\bar{\rho}}{\rho_*}\nabla\sigma\\ &\quad-\frac{\bar{\rho}\rho_\infty}{\gamma\rho_*}\nabla\sigma\int_0^1P''(s\bar{\rho}+\rho_\infty)ds-\frac{\rho_\infty\nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*}-\frac{P'(\rho_*)}{\rho_*^2}\Big)\sigma,\end{aligned}$$

$$F_1(U)=-\gamma(w\cdot\nabla\sigma+\sigma\nabla\cdot w),$$

$$\begin{aligned}F_2(U)\ &=\ -\gamma(w\cdot\nabla)w-\mu_1\frac{\rho_\infty^2}{\rho_*(\rho_\infty\sigma+\rho_*)}\sigma\triangle w-\mu_2\frac{\rho_\infty^2}{\rho_*(\rho_\infty\sigma+\rho_*)}\sigma\nabla(\nabla\cdot w)\\ &\quad+\frac{\rho_\infty^2}{\gamma}\Big(\frac{P'(\rho_*)}{\rho_*(\rho_\infty\sigma+\rho_*)}-\frac{1}{\rho_\infty\sigma+\rho_*}\int_0^1P''(s\rho_\infty\sigma+\rho_*)ds\Big)\sigma\nabla\sigma\\ &\quad+\frac{\rho_\infty^2\nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*(\rho_\infty\sigma+\rho_*)}-\frac{P'(\rho_*)}{\rho_*^2(\rho_\infty\sigma+\rho_*)}\\ &\qquad-\frac{1}{\rho_\infty\sigma+\rho_*}\int_0^1(1-s)P'''(s\rho_\infty\sigma+\rho_*)ds\Big)\sigma^2.\end{aligned}$$

**Remark 3.2.** *When $\Phi=0$, we have $\rho_*=\rho_\infty$, and thus, $\bar{\rho}=0$ and $\nabla\rho_*=0$. It then follows that $B_1U=0$ and $B_2U=0$ and $F(U)=-\gamma(w\cdot\nabla)w-\mu_1\frac{1}{\sigma+1}\sigma\triangle w-\mu_2\frac{1}{\sigma+1}\nabla(\nabla\cdot w)+\frac{\gamma}{\sigma+1}\sigma\nabla\sigma$.*

For problem (5), Kawashita [7] proved the following global existence result.

**Proposition 3.3** (Kawashita [7])**.** *Let $n\geq 2$ and let $U_0=(\rho_0,w_0)\in H^{s_0}$. There exist a positive constant $\epsilon_1$ such that if*

$$\|U_0\|_{H^{s_0}}\leq\epsilon_1,$$

$$\left\{ \begin{array}{ll} \|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon_1 & (n \ge 3), \\ \Phi = 0 & (n = 2), \end{array} \right.$$

*then problem* (5) *has a unique global solution* $U$*:*

$$U = (\sigma, w) \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{s_0-j}) \times C^j([0,\infty); H^{s_0-2j}),$$

$$w \in L^2(0,\infty; H^{s_0+1}) \cap H^1(0,\infty; H^{s_0-1}).$$

Proposition 3.3 were proved for the case $\Phi = 0$ in [7]. In a similar manner one can see that Proposition 3.3 holds for $\Phi \neq 0$ satisfying the smallness condition of Proposition 3.3 when $n \ge 3$.

We now state our main result of this paper.

**Theorem 3.4.** *Let* $U = (\sigma, w)$ *be a unique global solution of* (5) *with initial value* $U_0 = (\sigma_0, w_0)$ *obtained in Proposition* 3.3. *Assume that* $n \ge 3$. *Then there exist* $\epsilon_2 > 0$ *such that if* $U_0 \in H^{s_0} \cap L^1$ *and*

$$\|U_0\|_{H^{s_0} \cap L^1} \le \epsilon_2$$

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon_2$$

*then, the estimates*

$$\|\nabla^k U(t)\|_2 \le C_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \quad k = 0, 1, \tag{6}$$

*hold for* $t \ge 0$.

*The estimates* (6) *also hold for* $n = 2$ *if* $\Phi = 0$.

The proof of Theorem 3.4 will be given in section 5

**Remark 3.5.** *When* $\Phi = 0$, *one can also obtain the decay rates for the perturbation of higher-order spatial derivatives. In fact, one can prove the following estimates. Let* $U$ *and* $U_0$ *satisfy the assumption of Theorem* 3.4. *When* $\Phi = 0$ $(n \ge 2)$, *we have*

$$\|\nabla^k U(t)\|_2 \le C_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \quad k = 0, 1, \cdots, s_0$$

*for* $t \ge 0$.

# 4 Decomposition of solution

In this section we introduce a decomposition of solutions to prove Theorem 3.4.

We set

$$U = \begin{pmatrix} \sigma \\ w \end{pmatrix}, \; U_0 = \begin{pmatrix} \sigma_0 \\ w_0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & -\gamma\nabla\cdot \\ -\gamma\nabla & \mu_1\triangle + \mu_2\nabla\nabla\cdot \end{pmatrix}.$$

Then problem (5) is written as

$$\partial_t U - AU - BU = F(U), \quad U|_{t=0} = U_0, \tag{7}$$

where

$$BU = \begin{pmatrix} B_1 U \\ B_2 U \end{pmatrix}, \ F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \end{pmatrix}.$$

We next decompose a solution $U$ of (7) into low and high frequency parts. Let $\hat{\chi_1}$ be a cutoff function defined by

$$\hat{\chi_1}(\xi) = \begin{cases} 1 & \big(|\xi| < r\big) \\ 0 & \big(|\xi| \geq r\big) \end{cases}, \quad \hat{\chi_\infty}(\xi) = 1 - \hat{\chi_1}(\xi).$$

Here $r = \frac{\gamma}{\sqrt{\mu_1+\mu_2}}$. (As for the number $r$, see Lemma 5.1 below.)

We define operator $Q_j (j = 1, \infty)$ on $L^2$ by

$$Q_j u := \mathfrak{F}^{-1}(\hat{\chi_j}\hat{u}) \quad (j = 1, \infty), \quad u \in L^2.$$

The operators $Q_j (j = 1, \infty)$ have the following properties.

**Lemma 4.1.** *$Q_j (j = 1, \infty)$ satisfy the following relations.*

*(i)* $Q_1 + Q_\infty = I$.

*(ii)* $Q_j^2 = Q_j$.

*(iii)* $Q_1 Q_\infty = 0$.

*(iv)* $(Q_j u, v) = (u, Q_j v)$ *for* $u, v \in L^2$.

Lemma 4.1 can be easily verified; and we omit the proof.

We next state boundedness properties of $Q_j$ $(j = 1, \infty)$.

**Lemma 4.2.** *(i) For each nonnegative integer $k$, $Q_j (j = 1, \infty)$ are bounded linear operator on $H^k$.*

*(ii) For each nonnegative integer $k$, it holds that* $\|\nabla^k Q_1 u\|_2 \leq \|u\|_2 \quad (u \in L^2)$.

*(iii) For each nonnegative integer $k$, it holds that* $\|\nabla^k Q_1 u\|_\infty \leq C\|u\|_2 \quad (u \in L^2)$.

*(iv)* $Q_\infty$ *satisfies* $\|\nabla Q_\infty u\|_2 \geq C\|Q_\infty u\|_2 \quad (u \in H^1)$.

The assertions $(i), (ii), (iv)$ easily follow from the Plancherel theorem. The inequality $(iii)$ is obtained by $(ii)$ and the Sobolev inequality.

In terms of $Q_1$ and $Q_\infty$, we decompose a solution $U(t)$ of (7) as

$$U(t) = U_1(t) + U_\infty(t),\ U_j(t) = Q_j U(t) \quad (j = 1, \infty).$$

It then follows that $U_1(t)$ and $U_\infty(t)$ are governed by equations (13) and (14) given in Proposition 4.3 below.

To state Proposition 4.3 we introduce a semigroup associated with a low frequency part of $A$. We set

$$E_1(t)u := \mathfrak{F}^{-1}[\hat{\chi}_1 e^{\hat{A}(\xi)t}\hat{u}] \quad \text{for } u \in L^2,$$

where

$$\hat{A}(\xi) = \begin{pmatrix} 0 & -i\gamma\xi^{\mathrm{t}} \\ -i\gamma\xi & -\mu_1|\xi|^2 I_n - \mu_2\xi\xi^{\mathrm{t}} \end{pmatrix}.$$

Here and in what follows the superscript $\cdot^{\mathrm{t}}$ means the transposition.

**Proposition 4.3.** *Let $T > 0$ and let $U = (\sigma, w)^{\mathrm{t}}$ be a solution of problem (7) on $[0, T]$ such that*

$$U = (\sigma, w)^{\mathrm{t}} \in \bigcap_{j=0}^{1} C^j([0, T]; H^{s_0-j}) \times C^j([0, T]; H^{s_0-2j}), \tag{8}$$

$$w \in L^2(0, T; H^{s_0+1}) \cap H^1(0, T; H^{s_0-1}), \tag{9}$$

*and let*

$$U_j = Q_j U, \quad \sigma_j = Q_j\sigma,\ w_j = Q_j w\ (j = 1, \infty).$$

*Then,*

$$U_1 \in C^1([0, T]; H^k), \quad \forall k = 0, 1, 2, \cdots, \tag{10}$$

$$U_\infty \in \bigcap_{j=0}^{1} C^j([0, T]; H^{s_0-j}) \times C^j([0, T]; H^{s_0-2j}), \tag{11}$$

$$w_\infty \in L^2(0, T; H^{s_0+1}) \cap H^1(0, T; H^{s_0-1}). \tag{12}$$

*Furthermore $U_1(t)$ and $U_\infty(t)$ satisfy*

$$U_1(t) = E_1(t)U_{01} + \int_0^t E_1(t-s)Q_1\big(B(U_1 + U_\infty)(s) + F(U_1 + U_\infty)(s)\big)ds \tag{13}$$

*and*

$$\partial_t U_\infty - AU_\infty - Q_\infty B(U_1 + U_\infty) = Q_\infty F(U_1 + U_\infty), \tag{14}$$

$$U_\infty|_{t=0} = U_{0\infty}, \tag{15}$$

*where $U_{0j} = Q_j U_0\ (j = 1, \infty)$.*

**Proof.** Let $U(t) = (\sigma, w)^{\mathrm{t}}$ be a solution of (7) satisfying (8) and (9). It then follow from Lemma 4.2 that $U_1(t)$ and $U_\infty(t)$ satisfy (10), (11) and (12), respectively.

Since $Q_j AU = AQ_j U$ for $U \in H^{s_0}$ $(j = 1, \infty)$, applying $Q_j$ to (7), we obtain

$$\begin{cases} \partial_t U_1 - AU_1 - Q_1 B(U_1 + U_\infty) = Q_1 F(U_1 + U_\infty),\ U_1|_{t=0} = U_{01}, \\ \partial_t U_\infty - AU_\infty - Q_\infty B(U_1 + U_\infty) = Q_\infty F(U_1 + U_\infty),\ U_\infty|_{t=0} = U_{0\infty}. \end{cases} \tag{16}$$

Taking the Fourier transform of $(16)_1$, we have

$$\hat{\chi}_1 \partial_t \hat{U} = \hat{\chi}_1 \hat{A} \hat{U} + \hat{\chi}_1 \widehat{BU} + \hat{\chi}_1 \widehat{F(U)}. \tag{17}$$

It follows from (17) that

$$\hat{\chi}_1 \hat{U}(t) = e^{\hat{A}t} \hat{\chi}_1 \hat{U}(0) + \int_0^t e^{\hat{A}(t-s)} \big( \hat{\chi}_1 \widehat{BU} + \hat{\chi}_1 \widehat{F(U)} \big)(s) ds.$$

We thus obtain

$$U_1(t) = E_1(t) U_{01} + \int_0^t E_1(t-s) Q_1 \big( B(U_1 + U_\infty) + F(U_1 + U_\infty) \big)(s) ds.$$

This completes the proof. □

# 5 Proof of main result

In this section we prove Theorem 3.4. In subsections 5.1 and 5.2 we establish the necessary estimates for $U_1(t)$ and $U_\infty(t)$, respectively. In subsection 5.3 we derive the a priori estimate to complete the proof of Theorem 3.4.

Set

$$M_1(t) := \sup_{0 \le \tau \le t} \sum_{k=0}^{1} (1+\tau)^{\frac{n}{4} + \frac{k}{2}} \|\nabla^k U_1(\tau)\|_2,$$

$$M_\infty(t) := \sup_{0 \le \tau \le t} (1+\tau)^{\frac{n}{4} + \frac{1}{2}} \|U_\infty(\tau)\|_{H^{s_0}},$$

$$M(t) := M_1(t) + M_\infty(t).$$

We also set $\delta = \frac{\rho_\infty}{4C_s}$, where $C_s$ is constant such that $\|f\|_\infty \le C_s \|f\|_{H^{s_0}}$ for all $f \in H^{s_0}$. Hereafter, we assume that

$$\sup_{0 \le t \le T} \|\sigma(t)\|_{H^{s_0}} \le \delta.$$

Then we have

$$\|\sigma(t)\|_\infty \le C_s \|\sigma(t)\|_{H^{s_0}} \le \frac{\rho_\infty}{4}.$$

## 5.1 Estimate of $U_1(t)$

In this subsection we derive the estimate of $U_1(t)$, in other words, we estimate $M_1(t)$.

**Lemma 5.1** (Matsumura-Nishida [9]). *(i) The set of all eigenvalues of $\hat{A}(\xi)$ consists of $\lambda_i(\xi)$ $(i = 1, 2, 3)$, where*

$$\begin{cases} \lambda_1(\xi) = \frac{-(\mu_1+\mu_2)|\xi|^2 + i|\xi|\sqrt{4\gamma^2-(\mu_1+\mu_2)|\xi|^2}}{2}, \\ \lambda_2(\xi) = \overline{\lambda_1(\xi)}, \\ \lambda_3(\xi) = -\mu_1|\xi|^2, \end{cases}$$

*for $|\xi| \leq r$, where $r = \frac{\gamma}{\sqrt{\mu_1+\mu_2}}$. Here $\overline{\lambda_1(\xi)}$ denotes the complex conjugate of $\lambda_1(\xi)$.*

*(ii) $e^{t\hat{A}(\xi)}$ has the spectral resolution*

$$e^{t\hat{A}(\xi)} = \sum_{j=1}^{3} e^{t\lambda_j(\xi)} P_j(\xi),$$

*where $P_j(\xi)$ is the eigenprojection for $\lambda_j(\xi)$ and $P_j(\xi)$ satisfies*

$$\|P_j(\xi)\| \leq C \qquad \big(|\xi| \leq r\big).$$

*where $r = \frac{\gamma}{\sqrt{\mu_1+\mu_2}}$.*

$E_1(t)$ satisfies the following estimate:

**Lemma 5.2.** *Let $k$ be a nonnegative integer. Then there holds*

$$\|\nabla^k E_1(t)Q_1U_0\|_2 \leq C(1+t)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1$$

*for $t \geq 0$.*

**Proof.** By Lemma 5.1 $(i)$ we see that there exists a constant $\beta > 0$ such that

$$e^{2Re\lambda_j(\xi)t} \leq Ce^{-\beta|\xi|^2 t} \qquad (1 \leq j \leq 3).$$

Therefore, by Plancherel's theorem and Lemma 5.1 $(ii)$, we have

$$\begin{aligned} \|\nabla^k E_1(t)Q_1U_0(t)\|_2 &\leq C\Big(\int_{|\xi|\leq r} |\xi|^{2k}\big|e^{\hat{A}(\xi)t}\hat{U_0}\big|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq C\big(\int_{\mathbb{R}^n} |\xi|^{2k} e^{-\beta|\xi|^2 t}|\hat{U}_0(\xi)|^2 d\xi\big)^{\frac{1}{2}} \\ &\leq Ct^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1. \end{aligned} \tag{18}$$

We also find that

$$\begin{aligned}\|\nabla^k E_1(t)Q_1U_0\|_2 &\leq C\|\hat{U_0}\|_\infty\Big(\int_{|\xi|<1} e^{-\beta|\xi|^2 t}d\xi\Big)^{\frac{1}{2}}\\ &\leq C\|U_0\|_1. \end{aligned}\tag{19}$$

The estimate of Lemma 5.2 follows from (18) and (19). □

As for $M_1(t)$, we show the following estimate.

**Proposition 5.3.** *There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon,$$

$$\sup_{0\leq t\leq T} \|\sigma(t)\|_{H^{s_0}} \leq \delta,$$

*and*

$$M(t) \leq 1$$

*for $t \in [0,T]$, then there exists a constant $C > 0$ independent of $T$ such that*

$$M_1(t) \leq C\|U_0\|_1 + C\epsilon M(t) + CM^2(t)$$

*for $t \in [0,T]$*

To prove Proposition 5.3, we will use the following estimates on $B(U)$ and $F(U)$.

**Lemma 5.4.** *Let $n \geq 3$. There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon,$$

*and*

$$M(t) \leq 1$$

*for $t \in [0,T]$, then there exists a constant $C > 0$ independent of $T$ such that*

$$\|B(U_1(t) + U_\infty(t))\|_1 \leq C\epsilon(1+t)^{-\frac{n+2}{4}}M(t)$$

*for $t \in [0,T]$.*

**Lemma 5.5.** *Let $n \geq 2$. There exists a $\epsilon > 0$ such that if*

$$M(t) \leq 1$$

*and*

$$\begin{cases} \|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon & (n \geq 3), \\ \Phi = 0 & (n = 2) \end{cases}$$

*for $t \in [0,T]$, then there exists a constant $C > 0$ independent of $T$ such that*

$$\|F(U_1(t) + U_\infty(t))\|_1 \leq C(1+t)^{-\frac{n+1}{2}}M^2(t)$$

*for $t \in [0,T]$.*

We will prove Lemma 5.4 and Lemma 5.5 later. Now we prove Proposition 5.3.

**Proof of Proposition 5.3.** We first consider the case $n \geq 3$. By Lemma 5.2 and (13), we see that

$$\begin{aligned}
&\|\nabla^k U_1(\tau)\|_2 \leq \|\nabla^k E_1(\tau)U_{01}\|_2 \\
&+\int_0^\tau \|\nabla^k E_1(\tau-s)\big(Q_1 B(U_1(s)+U_\infty(s)) + Q_1 F(U_1(s)+U_\infty(s))\big)\|_2 ds \\
\leq\ & C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1 \\
&+\int_0^t (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}\big(\|B(U_1(s)+U_\infty(s))\|_1 \\
&\qquad\qquad +\|F(U_1(s)+U_\infty(s))\|_1\big)ds. \qquad (20)
\end{aligned}$$

Using Lemma 5.4 and Lemma 5.5, we have

$$\begin{aligned}
&\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}\big(\|B(U_1(s)+U_\infty(s))\|_1 + \|F(U_1(s)+U_\infty(s))\|_1\big)ds \\
&\leq C\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}\{\epsilon(1+s)^{-\frac{n+2}{4}}M(t) + (1+s)^{-\frac{n+1}{2}}M^2(t)\}ds \\
&\leq C\epsilon M(t)\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}(1+s)^{-\frac{n+2}{4}}ds \\
&+CM^2(t)\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}(1+s)^{-\frac{n+1}{2}}ds \\
&\leq C\epsilon(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M(t) + C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M^2(t). \qquad (21)
\end{aligned}$$

Here we used $\frac{n+2}{4} > 1$ for $n \geq 3$ to handle the term $\epsilon(1+s)^{-\frac{n+2}{4}}M(t)$. By (20) and (21), we obtain

$$\|\nabla^k U_1(\tau)\|_2 \leq C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1 + C\epsilon(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M(t) + C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M^2(t),$$

and hence,

$$(1+\tau)^{\frac{n}{4}+\frac{k}{2}}\|\nabla^k U_1(\tau)\|_2 \leq C\|U_0\|_1 + C\epsilon M(t) + CM^2(t).$$

Taking the supremum in $\tau \in [0,t]$, we obtain the desired estimate for $n \geq 3$.

When $n = 2$, we have $BU = 0$. Therefor the term $\epsilon(1+s)^{-\frac{n+2}{4}}M(t)$ in the computation above is missing; and one can obtain the desired estimate for $n = 2$. $\square$

It remains to prove Lemma 5.4 and Lemma5.5.

**Proof of Lemma 5.4.** By Lemma 2.1, we have

$$\begin{aligned}
\|w\cdot\nabla\bar{\rho}\|_1 &\leq \|(1+|x|)\nabla\bar{\rho}\|_2 \|\frac{1}{1+|x|}(w_1+w_\infty)\|_2 \\
&\leq \epsilon(\|\nabla w_1\|_2 + \|\nabla w_\infty\|_2),
\end{aligned}$$

$$
\begin{aligned}
& \| -\frac{\rho_\infty \nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*}+\frac{P'(\rho_*)}{\rho_*^2}\Big)\sigma\|_1 \\
\le\ & C\|\frac{P''(\rho_*)}{\rho_*}+\frac{P'(\rho_*)}{\rho_*^2}\|_\infty \|(1+|x|)\nabla\rho_*\|_2 \|\frac{1}{1+|x|}\sigma\|_2 \\
\le\ & C\epsilon\big(\|\nabla\sigma_1\|_2+\|\nabla\sigma_\infty\|_2\big).
\end{aligned}
$$

By using the Hölder inequality and Lemma 4.2, one can see that the $L^1$ norms of the others terms are bounded by $C\epsilon\big(\|\nabla\sigma_1\|_2+\|\nabla\sigma_\infty\|_2\big)$. We thus conclude that

$$
\begin{aligned}
\|B(U_1+U_\infty)\|_1 &\le C\epsilon\big(\|\nabla U_1\|_2+\|\nabla U_\infty\|_{H^1}\big) \\
&\le C\epsilon(1+s)^{-\frac{n+2}{4}}M(t).
\end{aligned}
$$

This completes the proof. □

**Proof of Lemma 5.5.** When $n\ge 3$, we see from Lemma 2.2 that

$$
\begin{aligned}
& \|\nabla\rho_*\Big(\frac{P''(\rho_*)}{\rho_*(\rho_\infty\sigma+\rho_*)}-\frac{P'(\rho_*)}{\rho_*^2(\rho_\infty\sigma+\rho_*)} \\
& \qquad -\frac{1}{\rho_\infty\sigma+\rho_*}\int_0^1(1-s)P'''(s\rho_\infty\sigma+\rho_*)ds\Big)\sigma^2\|_1 \\
\le\ & C\|\nabla\rho_*\|_2\|\sigma\|_\infty\|\sigma\|_2 \\
\le\ & C\|\nabla\sigma\|_{H^{s_0-1}}\|\sigma\|_2 \\
\le\ & C(1+s)^{-\frac{n+1}{2}}M^2(t).
\end{aligned}
$$

Note that this term does not appear when $n=2$ since it is assumed that $\Phi=0$.

The $L^1$ norm of the other terms are estimated by using the Hölder inequality, and bounded by $C(1+s)^{-\frac{n+1}{2}}M^2(t)$. Hence, we have

$$
\|F(U_1+U_\infty)\|_1 \le C(1+s)^{-\frac{n+1}{2}}M^2(t).
$$

This completes the proof. □

## 5.2 Estimate of $U_\infty(t)$

We next derive estimates for $U_\infty$. The system (14) is written as

$$
\begin{cases}
\partial_t\sigma_\infty+\gamma\nabla\cdot w_\infty = Q_\infty(B_1U+F_1(U)), \\
\partial_t w_\infty-\mu_1\triangle w_\infty-\mu_2\nabla\cdot(\nabla w_\infty)+\gamma\nabla\sigma_\infty = Q_\infty(B_2U+F_2(U)).
\end{cases} \tag{22}
$$

**Proposition 5.6.** *There holds*

$$
\sum_{0\le|\alpha|\le s_0}\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha U_\infty(t)\|_2^2+\mu_1\|\nabla\partial_x^\alpha w_\infty\|_2^2+\mu_2\|\nabla\cdot\partial_x^\alpha w_\infty(t)\|_2^2=\sum_{i=1}^4 I_i \tag{23}
$$

*for a.e.* $t \in [0,T]$. *Here,*

$$I_1 := \sum_{0 \le |\alpha| \le s_0} (\partial_x^\alpha B_1 U, \partial_x^\alpha \sigma_\infty),$$

$$\begin{aligned} I_2 \quad := \quad & \sum_{0 \le |\alpha| \le s_0 - 1} (\partial_x^\alpha F_1(U), \partial_x^\alpha \sigma_\infty) \\ & - \sum_{|\alpha| = s_0} (\frac{\gamma}{\rho_\infty}[\partial_x^\alpha, w\cdot]\nabla\sigma, \partial_x^\alpha \sigma_\infty) - \sum_{|\alpha|=s_0} (\frac{\gamma}{\rho_\infty}\partial_x^\alpha(\sigma\nabla\cdot w), \partial_x^\alpha\sigma_\infty) \\ & -\frac{1}{2}\sum_{|\alpha|=s_0} \big(\nabla\cdot w, |\partial_x^\alpha \sigma_\infty|^2\big) + \sum_{|\alpha|=s_0} (w\cdot\nabla\partial_x^\alpha\sigma_1, \partial_x^\alpha\sigma_\infty), \end{aligned}$$

$$I_3 := -\sum_{|\alpha|=s_0}\sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} B_2 U, \partial_x^{\alpha+\gamma} w_\infty) + \sum_{0\le|\alpha|\le s_0-1} (\partial_x^\alpha B_2 U, \partial_x^\alpha w_\infty),$$

$$I_4 := -\sum_{|\alpha|=s_0}\sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} F_2(U), \partial_x^{\alpha+\gamma} w_\infty) + \sum_{0\le|\alpha|\le s_0-1} (\partial_x^\alpha F_2(U), \partial_x^\alpha w_\infty).$$

**Proof.** Let $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\eta \ge 0$, $\operatorname{supp}\eta \subset \{x; |x| \le 1\}$, $\eta(-x) = \eta(x)$ and $\int \eta(x)dx = 1$. Set $\eta_\epsilon(x) = \epsilon^{-n}\eta(\frac{x}{\epsilon})$. Note that due to $\eta(-x) = \eta(x)$ we have

$$(\eta_\epsilon * f, g) = (f, \eta_\epsilon * g).$$

Let $\varphi \in C_0^\infty$ and $|\alpha| = s_0$. We take the inner product of $(22)_1$ with $\partial_x^\alpha(\eta_\epsilon * \varphi)$ to obtain

$$\begin{aligned} & (\partial_t\sigma_\infty, \partial_x^\alpha\eta_\epsilon * \varphi) + (\gamma\nabla\cdot w_\infty, \partial_x^\alpha\eta_\epsilon*\varphi) \\ = \; & \big(Q_\infty(B_1U + F_1(U)), \partial_x^\alpha(\eta_\epsilon*\varphi)\big) \\ = \; & (Q_\infty B_1 U, \partial_x^\alpha\eta_\epsilon*\varphi) - \gamma\{\big(Q_\infty(\sigma\nabla\cdot w), \partial_x^\alpha\eta_\epsilon*\varphi\big) \\ & +\big(w\cdot\nabla\sigma, \partial_x^\alpha\eta_\epsilon*\varphi\big) - \big(Q_1(w\cdot\nabla\sigma), \partial_x^\alpha\eta_\epsilon*\varphi\big)\}. \end{aligned} \tag{24}$$

By integration by parts, we have

$$\begin{aligned} & (\partial_t(\eta_\epsilon*\partial_x^\alpha\sigma_\infty), \varphi) + \gamma(\eta_\epsilon*\nabla\cdot\partial_x^\alpha w_\infty, \varphi) \\ = \; & (\eta_\epsilon*\partial_x^\alpha Q_\infty B_1 U, \varphi) - \gamma\{\big(\eta_\epsilon*\partial_x^\alpha Q_\infty(\sigma\nabla\cdot w), \varphi\big) \\ & +\big(\eta_\epsilon*([\partial_x^\alpha, w]\nabla)\sigma, \varphi\big) - \big([\eta_\epsilon*, w\cdot\nabla]\partial_x^\alpha\sigma, \varphi)\big) \\ & -\big(w\cdot\nabla\eta_\epsilon*\partial_x^\alpha\sigma, \varphi\big) - \big(\eta_\epsilon*\partial_x^\alpha Q_1(w\cdot\nabla\sigma), \varphi\big)\}. \end{aligned} \tag{25}$$

Next, we multiply (25) by $h \in C_0^\infty(0,T)$ and take $\varphi = \eta_\epsilon * \partial_x^\alpha\sigma_\infty \in C^\infty \cap L^2$. Integrating the resulting equation over $[0,T]$, we obtain

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\eta_\epsilon * \partial_x^\alpha \sigma_\infty\|_2 \frac{d}{dt} h dt + \int_0^T \big(\eta_\epsilon * \partial_x^\alpha(\gamma \nabla\cdot w_\infty), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
= \int_0^T -\big(\eta_\epsilon * (\partial_x^\alpha Q_\infty B_1 U), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big([\eta_\epsilon *, w\cdot\nabla]\partial_x^\alpha \varphi, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(\eta_\epsilon * [\partial_x^\alpha, w], \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
-\gamma \int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_\infty(\sigma\nabla\cdot w), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_1(w\cdot\nabla\sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt.
\end{aligned}
$$

We rewrite this equality to let $\epsilon \to 0$. The second term on the right hand-side is written as

$$
\begin{aligned}
& \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
= & \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma_\infty), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) + \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
= & \frac{1}{2}\big(w, \nabla|\eta_\epsilon * \partial_x^\alpha \sigma_\infty|^2\big) + \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
= & -\frac{1}{2}\big(\nabla\cdot w, |\eta_\epsilon * \partial_x^\alpha \sigma_\infty|^2\big) + \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\eta_\epsilon * \partial_x^\alpha \sigma_\infty\|_2^2 \frac{d}{dt} h dt + \gamma\int_0^T (\eta_\epsilon * \nabla\cdot \partial_x^\alpha w_\infty, \eta_\epsilon * \partial_x^\alpha \sigma_\infty) h dt \\
= -\int_0^T (\eta_\epsilon * \partial_x^\alpha Q_\infty B_1 U, \eta_\epsilon * \partial_x^\alpha \sigma_\infty) h dt \\
-\gamma \int_0^T \frac{1}{2}\big(\nabla\cdot w, |\eta_\epsilon * \partial_x^\alpha \sigma_\infty|^2\big) h dt \\
+\gamma \int_0^T \big(w\cdot\nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big([\eta_\epsilon *, w\cdot\nabla]\partial_x^\alpha \varphi, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(\eta_\epsilon * [\partial_x^\alpha, w]\nabla\sigma, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
-\gamma \int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_\infty(\sigma\nabla\cdot w), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_1(w\cdot\nabla\sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \qquad (26)
\end{aligned}
$$

Letting $\epsilon \to 0$ in (26), we can obtain

$$
\begin{aligned}
& \frac{1}{2}\int_0^T \frac{d}{dt}\|\partial_x^\alpha \sigma_\infty\|_2^2 h dt + \gamma \int_0^T (\nabla \cdot \partial_x^\alpha w_\infty, \partial_x^\alpha \sigma_\infty) h dt \\
= & -\int_0^T (\partial_x^\alpha B_1 U, \partial_x^\alpha \sigma_\infty) h dt \\
& -\gamma \int_0^T \frac{1}{2}\big(\nabla \cdot w, |\partial_x^\alpha \sigma_\infty|^2\big) h dt \\
& +\gamma \int_0^T \big(w \cdot \nabla \partial_x^\alpha \sigma_1, \partial_x^\alpha \sigma_\infty\big) h dt \\
& +\gamma \int_0^T \big([\partial_x^\alpha, w]\nabla \sigma, \partial_x^\alpha \sigma_\infty\big) h dt \\
& -\gamma \int_0^T \big(\partial_x^\alpha(\sigma \nabla \cdot w), \partial_x^\alpha \sigma_\infty\big) h dt. \qquad (27)
\end{aligned}
$$

In fact, as for the third term on the right hand-side of (26), by Lemma 4.2, we see that

$$
\begin{aligned}
& \big(w \cdot \nabla(\eta_\epsilon * (\partial_x^\alpha \sigma_1)), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
\leq \; & \|w\|_\infty \|\nabla \partial_x^\alpha \sigma_1\|_2 \|\partial_x^\alpha \sigma_\infty\|_2 \\
\leq \; & \|w\|_{H^{s_0}} \|\nabla \sigma_1\|_2 \|\partial_x^\alpha \sigma_\infty\|_2 \; \in L^1(0,T).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& \int_0^T \big(w \cdot \nabla(\eta_\epsilon * (\partial_x^\alpha \sigma_1)), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) dt \\
\longrightarrow \; & \int_0^T \big(w \cdot \nabla(\partial_x^\alpha \sigma_1), \partial_x^\alpha \sigma_\infty\big) dt.
\end{aligned}
$$

The fourth term on the right hand-side of (26) can be shown to go to zero by using Lemma 2.3. In fact, since $\partial_x^\alpha \sigma \in C([0,T]; L^2), w \in L^2(0,T; H^{s_0+1}) \subset L^2(0,T; BC^1)$, applying Lemma 2.3, we have

$$
\|[\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \sigma\|_2 \longrightarrow 0 \quad (\epsilon \to 0),
$$

for a.e. $t \in (0,T)$. We thus obtain

$$
\begin{aligned}
& \big|\big([\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \sigma, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h\big| \\
\leq \; & C \left\{ \begin{array}{l} \|\nabla w(t)\|_\infty \|\partial_x^\alpha \sigma(t)\|_2^2 |h(t)| \\ \|[\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \sigma\|_2 \|\eta_\epsilon * \partial_x^\alpha \sigma_\infty\|_2 |h(t)| \longrightarrow 0 \quad (\epsilon \to 0). \end{array} \right.
\end{aligned}
$$

for a.e. $t \in (0,T)$. Since, $\|\nabla w(t)\|_\infty \leq C\|w(t)\|_{H^{s_0+1}} \in L^2(0,T)$, we see that

$$
\int_0^T \big([\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \sigma, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \longrightarrow 0.
$$

As for the seventh term on the right hand-side of (26), by Lemma 4.2 and the dominated convergence theorem, we have

$$\begin{aligned}&\int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_1(w\cdot\nabla\sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big)hdt\\ \to\ &\int_0^T \big(\partial_x^\alpha Q_1(w\cdot\nabla\sigma), \partial_x^\alpha\sigma_\infty\big)hdt = 0.\end{aligned}$$

Here we have used $\big(\partial_x^\alpha Q_1(w\cdot\nabla\sigma), \partial_x^\alpha\sigma_\infty\big) = \big(Q_1\partial_x^\alpha Q_1(w\cdot\nabla\sigma), \partial_x^\alpha\sigma_\infty\big) = \big(\partial_x^\alpha Q_1(w\cdot\nabla\sigma), \partial_x^\alpha Q_1\sigma_\infty\big) = 0$.

For the other terms of (26), one can apply the dominated convergence theorem to pass the limit and we obtain (27). It then follows from (27) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma_\infty(t)\|_2^2 + \gamma(\nabla\cdot\partial_x^\alpha w_\infty, \partial_x^\alpha\sigma_\infty) \;=\;& (\partial_x^\alpha B_1U, \partial_x^\alpha\sigma_\infty) - \gamma\frac{1}{2}(\nabla\cdot w, |\partial_x^\alpha\sigma|^2)\\ &+\gamma\big(w\cdot\nabla\partial_x^\alpha\sigma_1, \partial_x^\alpha\sigma_\infty\big)\\ &+\gamma\big([\partial_x^\alpha, w]\nabla\sigma, \partial_x^\alpha\sigma_\infty\big)\\ &-\gamma\big(\partial_x^\alpha(\sigma\nabla\cdot w), \partial_x^\alpha\sigma_\infty\big)\end{aligned} \tag{28}$$

for a.e. $t\in(0,T)$ and $|\alpha| = s_0$.

When $|\alpha|\le s_0-1$, by simply taking the inner product of $\partial_x^\alpha(22)_1$ with $\partial_x^\alpha\sigma_\infty$, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma_\infty(t)\|_2^2 + \gamma(\nabla\cdot\partial_x^\alpha w_\infty, \partial_x^\alpha\sigma_\infty) = (\partial_x^\alpha B_1U, \partial_x^\alpha\sigma_\infty) + (\partial_x^\alpha F_1U, \partial_x^\alpha\sigma_\infty). \tag{29}$$

We see from (28) and (29) that

$$\begin{aligned}&\sum_{0\le|\alpha|\le s_0}\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma_\infty(t)\|_2^2 + \gamma(\nabla\cdot\partial_x^\alpha w_\infty, \partial_x^\alpha\sigma_\infty)\\ =\;&\sum_{0\le|\alpha|\le s_0}(\partial_x^\alpha B_1U, \partial_x^\alpha\sigma_\infty) + \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha F_1(U), \partial_x^\alpha\sigma_\infty)\\ &+\sum_{|\alpha|=s_0}(-\frac{\gamma}{\rho_\infty}[\partial_x^\alpha, w\cdot]\nabla\sigma, \partial_x^\alpha\sigma_\infty) + \sum_{|\alpha|=s_0}(-\frac{\gamma}{\rho_\infty}\partial_x^\alpha(\sigma\nabla\cdot w)\partial_x^\alpha\sigma_\infty)\\ &-\frac{1}{2}\sum_{|\alpha|=s_0}\big(\nabla\cdot w, |\partial_x^\alpha\sigma_\infty|^2\big) + \sum_{|\alpha|=s_0}(w\cdot\nabla\partial_x^\alpha\sigma_1, \partial_x^\alpha\sigma_\infty)\end{aligned} \tag{30}$$

for a.e. $t\in(0,T)$.

We next consider $(22)_2$. Let $\varphi\in C_0^\infty$ and let $|\alpha| = s_0$. We take the inner-product of $(22)_2$ with $\partial_x^\alpha\varphi$ to obtain

$$\begin{aligned}&(\partial_t w_\infty, \partial_x^\alpha\varphi) - \mu_1(\triangle w_\infty, \partial_x^\alpha\varphi) - \mu_2\big(\nabla(\nabla\cdot w_\infty), \partial_x^\alpha\varphi\big)\\ &+\gamma(\nabla\sigma_\infty, \partial_x^\alpha\varphi) = (Q_\infty B_2U, \partial_x^\alpha\varphi) + (Q_\infty F_2(U), \partial_x^\alpha\varphi).\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\left\langle \partial_x^\alpha \partial_t w_\infty, \varphi \right\rangle + \mu_1(\nabla \partial_x^\alpha w_\infty, \nabla \varphi) \\
&+\mu_2(\nabla \cdot \partial_x^\alpha w_\infty, \nabla \cdot \varphi) - \gamma(\partial_x^\alpha \sigma_\infty, \nabla \cdot \varphi) \\
= &\sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} Q_\infty B_2 U, \partial_x^\gamma \varphi) + \sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} Q_\infty F_2(U), \partial_x^\gamma \varphi).
\end{aligned}$$

Here we have used the fact that $\sum_{|\gamma|=1} \partial_x^{\alpha-\gamma} Q_\infty F_2 \in L^2$, which can be seen from the proof of Lemma 5.7 below. By density, we can set $\varphi = \partial_x^\alpha w_\infty$. So we obtain by Lemma 2.4,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w_\infty\|_2^2 + \mu_1\|\nabla \partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla \cdot \partial_x^\alpha w_\infty\|_2^2 - \gamma(\partial_x^\alpha \sigma_\infty, \nabla \cdot \partial_x^\alpha w_\infty) \\
= &-\sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} B_2 U, \partial_x^\gamma \partial_x^\alpha w_\infty) - \sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} F_2(U), \partial_x^\gamma \partial_x^\alpha w_\infty). \qquad (31)
\end{aligned}$$

for a.e. $t \in (0, T)$.

When $|\alpha| \le s_0 - 1$, in a similar way as above we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w_\infty\|_2^2 + \mu_1\|\nabla \partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla \cdot \partial_x^\alpha w_\infty\|_2^2 - \gamma(\partial_x^\alpha \sigma_\infty, \nabla \cdot \partial_x^\alpha w_\infty) \\
= &\sum_{0\le|\alpha|\le s_0-1} (\partial_x^\alpha B_2 U, \partial_x^\alpha w_\infty) + (\partial_x^\alpha F_2(U), \partial_x^\alpha w_\infty). \qquad (32)
\end{aligned}$$

We see from (31) and (32) that

$$\begin{aligned}
&\sum_{0\le|\alpha|\le s_0} \frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w_\infty\|_2^2 + \mu_1\|\partial_x^\alpha \nabla w_\infty\|_2^2 + \mu_2\|\nabla \cdot \partial_x^\alpha w_\infty\|_2^2 \\
&-\gamma(\partial_x^\alpha \sigma_\infty, \nabla \cdot \partial_x^\alpha w_\infty) \\
= &-\sum_{|\alpha|=s_0}\sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} B_2 U, \partial_x^\gamma \partial_x^\alpha w_\infty) - \sum_{|\alpha|=s_0}\sum_{|\gamma|=1} (\partial_x^{\alpha-\gamma} F_2(U), \partial_x^\gamma \partial_x^\alpha w_\infty). \\
&+\sum_{0\le|\alpha|\le s_0-1} (\partial_x^\alpha B_2 U, \partial_x^\alpha w_\infty) + (\partial_x^\alpha F_2(U), \partial_x^\alpha w_\infty). \qquad (33)
\end{aligned}$$

A linear combination of (30) and (33) yields the desired result. □

We next estimate $I_1$ and $I_3$.

**Proposition 5.7.** *Let $n \ge 3$. There exists a constant $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon,$$

$$\sup_{0\le t\le T} \|\sigma(t)\|_{H^{s_0}} \le \delta,$$

*and*

$$M(t) \le 1$$

*for $t \in [0, T]$, then*

$$|I_1| + |I_3| \le C\epsilon\{(1+t)^{-(\frac{n+1}{2})} M^2(t) + \|\nabla^{s_0+1} w_\infty(t)\|_2^2\}$$

*for $t \in [0, T]$. Here $C > 0$ is a constant independent of $T$.*

**Proof.** First we show the estimate of $I_1$. We have

$$\begin{aligned} |I_1| &= |\sum_{0\le|\alpha|\le s_0} (\partial_x^\alpha(w\cdot\nabla\bar\rho), \partial_x^\alpha\sigma_\infty) + (\partial_x^\alpha(\bar\rho\cdot\nabla w), \partial_x^\alpha\sigma_\infty)\}| \\ &\le \sum_{0\le|\alpha|\le s_0} (\|\partial_x^\alpha(w\cdot\nabla\bar\rho)\|_2 + \|\partial_x^\alpha(\bar\rho\nabla\cdot w)\|_2)\|\partial_x^\alpha\sigma_\infty\|_2. \end{aligned} \tag{34}$$

By Lemma 2.2 and Lemma 2.6, the terms on the right-hand side of (34) is estimated as

$$\begin{aligned} \|\partial_x^\alpha(w\cdot\nabla\bar\rho)\|_2 &\le C\{\|w\|_\infty\|\nabla\bar\rho\|_{H^{s_0}} + \|\nabla w\|_{s_0-1}\|\nabla\bar\rho\|_{s_0}\} \\ &\le C\epsilon(1+t)^{-(\frac{n}{4}+\frac{1}{2})}M(t), \end{aligned}$$

$$\begin{aligned} \|\partial_x^\alpha(\bar\rho\nabla\cdot w)\|_2 &\le C\|\bar\rho\|_{H^{s_0}}\|\nabla w\|_{H^{s_0}} \\ &\le C\epsilon(1+t)^{-(\frac{n}{4}+\frac{1}{2})}M(t) + C\epsilon\|\nabla^{s_0+1}w_\infty\|_2. \end{aligned}$$

Hence, we obtain the estimate of $I_1$.

Let us next consider $I_3$:

$$\begin{aligned} |I_3| &= |-\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}B_2U, \partial_x^{\alpha+\gamma}w_\infty) + \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha B_2U, \partial_x^\alpha w_\infty)| \\ &\le C\big(\sum_{|\alpha|\le s_0-1}\|\partial_x^\alpha B_2U\|_2\big)\|\nabla w_\infty\|_{H^{s_0}} \end{aligned}$$

We estimate $\|\partial_x^\alpha B_2U\|_2$ ($|\alpha|\le s_0-1$). We write $B_2U$ as

$$B_2U = G_1(\bar\rho,x)\triangle w + G_2(\bar\rho,x)\nabla(\nabla\cdot w) + G_3(\bar\rho,x)\nabla\sigma + G_4(x)\sigma,$$

where

$$\begin{aligned} G_1(\bar\rho,x) &= -\mu_1\frac{\bar\rho}{\rho_*} \\ G_2(\bar\rho,x) &= -\mu_2\frac{\bar\rho}{\rho_*} \\ G_3(\bar\rho,x) &= -\gamma\frac{\bar\rho}{\rho_*} + \frac{\bar\rho\rho_\infty}{\gamma\rho_*}\int_0^1 P''(s\bar\rho+\rho_\infty)ds, \\ G_4(x) &= -\frac{\rho_\infty\nabla\rho_*}{\gamma}\big(\frac{P''(\rho_*)}{\rho_*} - \frac{P'(\rho_*)}{\rho_*^2}\big). \end{aligned}$$

We thus obtain by Lemma 2.2 and Lemma 2.6

$$\begin{aligned} \|\partial_x^\alpha B_2U\|_2 &\le C\{\|\bar\rho\|_{H^{s_0}}\|\nabla^2 w\|_{H^{s_0-1}} + \|\bar\rho\|_{H^{s_0+1}}\|\sigma\|_{H^{s_0}} + \|\partial_x^\alpha G_4(x)\|_2\|\sigma\|_\infty\} \\ &\le C\epsilon\{(1+t)^{-\frac{n}{4}-\frac{1}{2}}M(t) + \|\nabla^{s_0+1}w_\infty\|_2\}. \end{aligned}$$

Hence, we have

$$|I_3| \le C\epsilon(1+t)^{-(\frac{n}{2}+1)}M^2(t) + C\epsilon\|\nabla^{s_0+1}w_\infty\|_2^2.$$

This completes the proof. $\square$

**Proposition 5.8.** *Let $n \geq 2$. There exists a constant $\epsilon > 0$ such that if*

$$\begin{cases} \|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon & (n \geq 3), \\ \Phi = 0 & (n = 2), \end{cases}$$

$$\sup_{0 \leq t \leq T} \|\sigma(t)\|_{H^{s_0}} \leq \delta,$$

*and*

$$M(t) \leq 1$$

*for $t \in [0, T]$, then*

$$|I_2| + |I_4| \leq C(1+t)^{-\frac{n}{4}} M(t)\{(1+t)^{-(\frac{n}{2}+1)} M^2(t) + \|\nabla^{s_0+1} w_\infty(t)\|_2^2\}$$

*for $t \in [0, T]$. Here $C > 0$ is a constant independent of $T$.*

**Proof.** We first consider the case $n \geq 3$. Let us estimate $I_2$. For the first term of $I_2$, by Lemma 2.6 and Lemma 4.2 we have

$$\begin{aligned} & \Big| \sum_{0 \leq |\alpha| \leq s_0 - 1} (\partial_x^\alpha F_1(U), \partial_x^\alpha \sigma_\infty) \Big| \\ \leq \;& C(\|\nabla\sigma\|_{H^{s_0-1}} \|w\|_{H^{s_0}} + \|\sigma\|_{H^{s_0}} \|\nabla w\|_{H^{s_0-1}}) \|\sigma_\infty\|_{H^{s_0}} \\ \leq \;& C\{(1+t)^{-(\frac{3n}{4}+1)} M^3(t) + (1+t)^{-\frac{n}{4}} M(t) \|\nabla^{s_0+1} w_\infty\|_2\}. \end{aligned}$$

By Lemma 2.6, the second term of $I_2$ is estimated as

$$\Big| \sum_{|\alpha| = s_0} (-\frac{\gamma}{\rho_\infty} [\partial_x^\alpha, w\cdot]\nabla\sigma, \partial_x^\alpha \sigma_\infty) \Big| \leq C \|\nabla w\|_{H^{s_0}} \|\nabla\sigma\|_{H^{s_0-1}} \|\sigma_\infty\|_{H^{s_0}}$$

We finally, we consider $I_4$ :

$$\begin{aligned} |I_4| &= \Big| \sum_{|\alpha| = s_0} \sum_{|\gamma| = 1} (\partial_x^{\alpha-\gamma} F_2(U), \partial_x^{\alpha+\gamma} w_\infty) + \sum_{0 \leq |\alpha| \leq s_0 - 1} (\partial_x^\alpha F_2(U), \partial_x^\alpha w_\infty) \Big| \\ &\leq \sum_{|\alpha| = s_0} \sum_{|\gamma| = 1} \|\partial_x^{\alpha-\gamma} F_2(U)\|_2 \|\partial_x^{\alpha+\gamma} w_\infty\|_2 + \sum_{0 \leq |\alpha| \leq s_0 - 1} \|\partial_x^\alpha F_2(U)\|_2 \|\partial_x^\alpha w_\infty\|_2 \\ &\leq \Big( \sum_{0 \leq |\alpha| \leq s_0 - 1} \|\partial_x^\alpha F_2(U)\|_2 \Big) \|\nabla w_\infty\|_{H^{s_0}} \end{aligned}$$

Let us estimate $\|\partial_x^\alpha F_2(U)\|_2$ for $|\alpha| \leq s_0 - 1$. $F_2(U)$ is written as

$$\begin{aligned} F_2(U) &= R_0(w) \cdot \nabla w + R_1(\sigma, x) \triangle w + R_2(\sigma, x) \nabla(\nabla \cdot w) \\ &\quad + R_3(\sigma, x)\sigma + R_4(\sigma, x) \nabla\sigma, \end{aligned}$$

where

$$R_0(w) = -\gamma w$$

$$R_1(\sigma, x) = -\mu_1 \frac{\rho_\infty^2}{\rho_*(\rho_\infty \sigma + \rho_*)} \sigma, \quad R_2(\sigma, x) = -\mu_2 \frac{\rho_\infty^2}{\rho_*(\rho_\infty \sigma + \rho_*)} \sigma,$$

$$
\begin{aligned}
R_3(\sigma,x) &= \frac{\rho_\infty^2\nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*(\rho_\infty\sigma+\rho_*)}-\frac{P'(\rho_*)}{\rho_*^2(\rho_\infty\sigma+\rho_*)}\\
&\quad -\frac{1}{\rho_\infty\sigma+\rho_*}\int_0^1(1-s)P'''(s\rho_\infty\sigma+\rho_*)ds\Big)\sigma,
\end{aligned}
$$

$$
R_4(\sigma,x) = \frac{\rho_\infty^2}{\gamma}\Big(\frac{P'(\rho_*)}{\rho_*(\rho_\infty\sigma+\rho_*)}-\frac{1}{\rho_\infty\sigma+\rho_*}\int_0^1 P''(s\rho_\infty\sigma+\rho_*)ds\Big)\sigma.
$$

From Lemma 2.2 and Lemma 2.6, we have

$$
\begin{gathered}
\|\partial_x^\alpha\big(R_0(w)\cdot\nabla w\big)\|_2 \le C\|\nabla w\|_{H^{s_0-1}}^2,\\
\|\partial_x^\alpha\big(R_1(\sigma,x)\triangle w\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\triangle w\|_{H^{s_0-1}},\\
\|\partial_x^\alpha\big(R_2(\sigma,x)\nabla(\nabla\cdot w)\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\nabla(\nabla\cdot w)\|_{H^{s_0-1}},\\
\|\partial_x^\alpha\big(R_3(\sigma,x)\sigma\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\sigma\|_{H^{s_0-1}},\\
\|\partial_x^\alpha\big(R_4(\sigma,x)\nabla\sigma\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\nabla\sigma\|_{H^{s_0-1}}.
\end{gathered}
$$

We thus obtain

$$
|I_4| \le C(1+t)^{-\frac{n}{4}}M(t)\{(1+t)^{-(\frac{n}{2}+1)}M^2(t)+\|\nabla^{s_0+1}w_\infty(t)\|_2^2\}.
$$

When $n=2$, by using the Hölder and Sobolev inequalities we see that

$$
|I_2|+|I_4| \le C(1+t)^{-\frac{n}{4}}M(t)\{(1+t)^{-(\frac{n}{2}+1)}M^2(t)+\|\nabla^{s_0+1}w_\infty(t)\|_2^2\}.
$$

This completes the proof. □

**Proposition 5.9.** *There holds the inequality*

$$
\sum_{0\le|\alpha|\le s_0-1}\frac{d}{dt}\big(\partial_x^\alpha w_\infty(t),\partial_x^\alpha\nabla\sigma_\infty(t)\big)+\frac{\gamma}{2}\|\partial_x^\alpha\nabla\sigma_\infty(t)\|_2^2 \le C\|\nabla w_\infty\|_{H^{s_0}}^2+\sum_{i=1}^4 J_i \quad (35)
$$

*for a.e.* $t\in(0,T)$, *where,*

$$
J_1=\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty B_1U,\partial_x^\alpha\nabla\cdot w_\infty)|,\quad J_2=\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty F_1(U),\partial_x^\alpha\nabla\cdot w_\infty)|,
$$

$$
J_3=C\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha B_2U,\partial_x^\alpha\nabla\sigma_\infty)|,\quad J_4=\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty F_2(U),\partial_x^\alpha\nabla\sigma_\infty)|.
$$

**Proof.** Let $|\alpha|\le s_0-1$. We take the inner-product of $\partial_x^\alpha(22)_2$ with $\partial_x^\alpha\nabla\sigma_\infty$ to obtain

$$
\begin{aligned}
&(\partial_x^\alpha\partial_t w_\infty,\partial_x^\alpha\nabla\sigma_\infty)+\gamma\|\nabla\partial_x^\alpha\sigma_\infty\|_2\\
=\ &\mu_1(\partial_x^\alpha\triangle w_\infty,\partial_x^\alpha\nabla\sigma_\infty)+\mu_2\big(\partial_x^\alpha\nabla\cdot(\nabla w_\infty),\partial_x^\alpha\nabla\sigma_\infty\big)\\
&+(\partial_x^\alpha Q_\infty F_2(U),\partial_x^\alpha\nabla\sigma_\infty)-(\partial_x^\alpha Q_\infty B_2U,\partial_x^\alpha\nabla\sigma_\infty). \quad (36)
\end{aligned}
$$

We next take the inner-product of $\partial_x^\alpha$ $(22)_1$ with $-\partial_x^\alpha \nabla\cdot w_\infty$ to obtain

$$\begin{aligned}-(\partial_x^\alpha\partial_t\sigma_\infty, \partial_x^\alpha\nabla\cdot w_\infty) = {}&+\gamma(\partial_x^\alpha(\nabla\cdot w_\infty), \partial_x^\alpha\nabla\cdot w_\infty)\\ &-(\partial_x^\alpha Q_\infty B_1 U, \partial_x^\alpha\nabla\cdot w_\infty) - (\partial_x^\alpha Q_\infty F_1(U), \partial_x^\alpha\nabla\cdot w_\infty),\end{aligned} \tag{37}$$

Since

$$\mu_1(\partial_x^\alpha\triangle w_\infty, \partial_x^\alpha\nabla\sigma_\infty) \le C\|\partial_x^\alpha\triangle w_\infty\|_2^2 + \frac{\gamma}{4}\|\partial_x^\alpha\nabla\sigma_\infty\|_2^2,$$

and

$$\mu_2\big(\partial_x^\alpha\nabla\cdot(\nabla w_\infty), \partial_x^\alpha\nabla\sigma_\infty\big) \le C\|\partial_x^\alpha\triangle w_\infty\|_2^2 + \frac{\gamma}{4}\|\partial_x^\alpha\nabla\sigma_\infty\|_2^2,$$

by adding (36) and (37), we obtain the desired inequality

$$\begin{aligned}&\sum_{0\le|\alpha|\le s_0-1}\frac{d}{dt}\big(\partial_x^\alpha w_\infty(t), \partial_x^\alpha\nabla\sigma_\infty(t)\big) + \frac{\gamma}{2}\|\nabla\partial_x^\alpha\sigma_\infty(t)\|_2^2\\ \le\;& C\Big(\|\nabla w\|_{H^{s_0}}^2 + \sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty B_1 U, \partial_x^\alpha\nabla\cdot w_\infty)| + |(\partial_x^\alpha Q_\infty F_1(U), \partial_x^\alpha\nabla\cdot w_\infty)|\\ &+|(\partial_x^\alpha B_2 U, \partial_x^\alpha\nabla\sigma_\infty)| + |(\partial_x^\alpha Q_\infty F_2(U), \partial_x^\alpha\nabla\sigma_\infty)|\Big)\end{aligned} \tag{38}$$

for a.e. $t\in[0,T]$. In fact, let $h\in C_0^\infty(0,T)$ and let $\eta_\epsilon$ is standard Friedrichs mollifier, as for the first term on the left hand side of (38)

$$\begin{aligned}&\int_0^T(\partial_x^\alpha\partial_t w_\infty, \partial_x^\alpha\nabla\eta_\epsilon*\sigma_\infty)h\,dt\\ =\;&\int_0^T\frac{d}{dt}(\partial_x^\alpha w_\infty, \partial_x^\alpha\nabla\eta_\epsilon*\sigma_\infty)h\,dt - \int_0^T(\partial_x^\alpha w_\infty, \partial_x^\alpha\partial_t(\nabla\eta_\epsilon*\sigma_\infty))h\,dt\\ =\;&-\int_0^T(\partial_x^\alpha w_\infty, \partial_x^\alpha\nabla(\eta_\epsilon*\sigma_\infty))\frac{d}{dt}h\,dt + \int_0^T(\partial_x^\alpha\nabla\cdot w_\infty, \eta_\epsilon*\partial_t\partial_x^\alpha\sigma_\infty)h\,dt.\end{aligned} \tag{39}$$

Since $\partial_x^\alpha w_\infty, \partial_x^\alpha\nabla\cdot w_\infty, \partial_x^\alpha\partial_t\sigma$ and $\partial_x^\alpha\partial_t\sigma \in C([0,T];L^2)$ for $|\alpha|\le s_0-1$, letting $\epsilon\to 0$ in (39) we can obtain by similar to proof of Lemma 5.6

$$(\partial_x^\alpha\partial_t w_\infty, \partial_x^\alpha\nabla\sigma_\infty) = \frac{d}{dt}(\partial_x^\alpha w_\infty, \partial_x^\alpha\nabla\sigma_\infty) + (\partial_x^\alpha\nabla\cdot w_\infty, \partial_x^\alpha\partial_t\sigma_\infty)$$

for a.e. $t\in[0,T]$.

This completes the proof. □

**Proposition 5.10.** *Let $n\ge 3$. There exists a $\epsilon>0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon,$$

$$\sup_{0\le t\le T}\|\sigma(t)\|_{H^{s_0}} \le \delta,$$

*and*

$$M(t)\le 1$$

*for $t \in [0,T]$, then there holds*

$$|J_1| + |J_3| \le C\epsilon\{(1+t)^{-\frac{n+2}{2}} M^2(t) + \|\nabla^{s_0+1} w_\infty(t)\|_2^2\}.$$

*for $t \in [0,T]$. Here $C > 0$ is a constant independent of $T$.*

The proof is similar to that of Proposition 5.7 . We omit it.

**Proposition 5.11.** *There exists a $\epsilon > 0$ such that if*

$$\begin{cases} \|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon & (n \ge 3), \\ \Phi = 0 & (n = 2), \end{cases}$$

$$\sup_{0 \le t \le T} \|\sigma(t)\|_{H^{s_0}} \le \delta,$$

*and*

$$M(t) \le 1$$

*for $t \in [0,T]$, then there hold*

$$|J_2| + |J_4| \le C(1+t)^{-\frac{n}{4}} M(t)\{(1+t)^{-\frac{n+2}{2}} M^2(t) + \|\nabla^{s_0+1} w_\infty(t)\|_2^2\}$$

*fir $t \in [0,T]$. Here $C > 0$ is a constant independent of $T$.*

The proof is similar to that of Proposition 5.8. We omit it.

**Proposition 5.12.** *There exists a $\epsilon > 0$ such that if*

$$\begin{cases} \|\Phi\|_{H^{s_0}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon & (n \ge 3), \\ \Phi = 0 & (n = 2), \end{cases}$$

$$\sup_{0 \le t \le T} \|\sigma(t)\|_{H^{s_0}} \le \delta,$$

*and*

$$M(t) \le 1$$

*for $t \in [0,T]$, then there holds*

$$\begin{aligned} &\frac{d}{dt} E_\infty(t) + C_1 E_\infty(t) + C_2 D_\infty(t) \le C\epsilon(1+t)^{-\frac{n}{2}-\frac{1}{2}} M^2(t) \\ &\quad + C(1+t)^{-\frac{3n+4}{4}} M^3(t) + C(1+t)^{-\frac{n}{4}} M(t) D_\infty(t) \end{aligned} \tag{40}$$

*for $t \in [0,T]$. Here, $E_\infty(t)$ and $D_\infty(t)$ are equivalent to $\|U_\infty(t)\|_{H^{s_0}}^2$ and $\|\nabla w_\infty(t)\|_{H^{s_0}}^2 + \|\nabla\sigma_\infty(t)\|_{H^{s_0-1}}^2$ respectively. That is, there exist $d_1, d_2 > 0$ such that*

$$\frac{1}{d_1} E_\infty(t) \le \|U_\infty(t)\|_{H^{s_0}}^2 \le d_1 E_\infty(t),$$

$$\frac{1}{d_2} D_\infty(t) \le \|\nabla w_\infty(t)\|_{H^{s_0}}^2 + \|\nabla\sigma_\infty(t)\|_{H^{s_0-1}}^2 \le d_2 D_\infty(t).$$

**Proof.** We add $\kappa \times$ (23) to (35) with a constant $\kappa > 0$ to be determined later. Then, by Proposition 5.7 and Proposition 5.11, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{\kappa}{2}\|U_\infty\|_{H^{s_0}}^2 + \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha w_\infty, \partial_x^\alpha \nabla\sigma_\infty)\Big) \\ &+\kappa\Big(\sum_{0\le|\alpha|\le s_0}\mu_1\|\nabla\partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty(t)\|_2^2\Big) + \frac{\gamma}{2}\|\nabla\sigma_\infty\|_{H^{s_0-1}}^2 \\ \le\ & C\sum_{0\le|\alpha|\le s_0-1}\|\partial_x^\alpha\nabla w_\infty\|_{H^{s_0-1}}^2 + C\epsilon\{(1+t)^{-\frac{n+1}{2}}M(t)^2 + \|\nabla^{s_0+1}w_\infty\|_2^2\} \\ &+C(1+t)^{-\frac{n}{4}}M(t)\big((1+t)^{-\frac{n+2}{2}}M(t)^2 + \|\nabla^{s_0+1}w_\infty\|_2^2\big). \end{aligned} \tag{41}$$

We set

$$E_\infty(t) = \frac{\kappa}{2}\|U_\infty(t)\|_{H^{s_0}}^2 + \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha w_\infty(t), \partial_x^\alpha\nabla\sigma_\infty(t)),$$

$$D_\infty(t) = \frac{\kappa}{2}\sum_{0\le|\alpha|\le s_0}(\mu_1\|\nabla\partial_x^\alpha w_\infty(t)\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty(t)\|_2^2) + \frac{\gamma}{2}\|\nabla\sigma_\infty(t)\|_{H^{s_0-1}}^2.$$

For each $\kappa > 0$, $D_\infty(t)$ and $\|\nabla w_\infty(t)\|_{H^{s_0}}^2 + \|\nabla\sigma_\infty(t)\|_{H^{s_0-1}}^2$ are equivalent. Since

$$\Big|\sum_{0\le|\alpha|\le s_0}(\partial_x^\alpha w_\infty(t), \partial_x^\alpha\nabla\sigma_\infty(t))\Big| \le C'\|U_\infty(t)\|_{H^{s_0}}^2,$$

if $\kappa$ is fixed in such a way that $\kappa > 2C'$, then one can see that $E_\infty(t)$ and $\|U_\infty(t)\|_{H^{s_0}}^2$ are equivalent. With this $\kappa > 0$, we see from (41) that

$$\begin{aligned} \frac{d}{dt}E_\infty(t) + 2C_2D_\infty &\le C\epsilon(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t) \\ &+C(1+t)^{-\frac{3n+4}{4}}M^3(t) + C(1+t)^{-\frac{n}{4}}M(t)D_\infty(t). \end{aligned} \tag{42}$$

By Lemma 4.2, we have

$$E_\infty(t) \le CD_\infty(t).$$

This, together with (42), gives the desired inequality (40). □

## 5.3 Proof of Theorem 3.4.

**Proposition 5.13.** *There exists a constant $\epsilon_2 > 0$ such that if*

$$\|U_0\|_{H^{s_0}\cap L^1}^2 \le \epsilon_2,$$

*then there holds*

$$M(t) \le C\|U_0\|_{H^{s_0}\cap L^1}$$

*for $0 \le t \le T$, where the constant $C$ does not depend on $T$.*

**Proof.** By (40) we have

$$
\begin{aligned}
& E_\infty(t) + C_2\int_0^t e^{-C_1(t-\tau)}D_\infty(\tau)d\tau \\
\leq\ & e^{-C_1 t}E_\infty(0) + C\epsilon M^2(t)\int_0^t e^{-C_1(t-\tau)}(1+\tau)^{-\frac{n+2}{2}}d\tau \\
& + C\big\{M^3(t)\int_0^t (1+\tau)^{-\frac{3n+4}{4}}e^{-C_1(t-\tau)}d\tau \\
& + M(t)\int_0^t (1+\tau)^{-\frac{n}{4}}e^{-C_1(t-\tau)}D_\infty(\tau)d\tau\big\} \\
\leq\ & e^{-C_1 t}E_\infty(0) + C\epsilon(1+t)^{-\frac{n+2}{2}}M^2(t) + C(1+t)^{-\frac{3n+4}{4}}M^3(t) \\
& + CM(t)\int_0^t e^{-C_1(t-\tau)}D_\infty(\tau)d\tau. \qquad (43)
\end{aligned}
$$

We set $\mathcal{D}_\infty(t) := (1+t)^{\frac{n+2}{2}}\int_0^t e^{-C_1(t-\tau)}D_\infty(\tau)d\tau$. Since $\frac{3n+4}{4} > \frac{n+2}{2}$, we see from (43) that

$$
M_\infty^2(t) + C_2\mathcal{D}_\infty(t) \leq C\big(E_\infty(0) + \epsilon M^2(t) + M^3(t) + CM(t)\mathcal{D}_\infty(t)\big).
$$

This, together with Proposition 5.3, gives

$$
M^2(t) + C_2\mathcal{D}_\infty(t) \leq C\big(E_\infty(0) + \|U_0\|_1^2 + M^4(t) + M^3(t) + M(t)\mathcal{D}_\infty(t) + \epsilon M^2(t)\big).
$$

By taking $\epsilon > 0$ suitable small, we obtain

$$
M^2(t) + C_2'\mathcal{D}_\infty(t) \leq C_3\big(\|U_0\|^2_{H^{s_0}\cap L^1} + M^3(t) + M(t)\mathcal{D}_\infty(t)\big). \qquad (44)
$$

We observe that there exists a constant $C_4 > 0$ such that

$$
M(0) \leq C_4\|U_0\|_{H^{s_0}\cap L^1}.
$$

Since $M(t)$ is continuous in $t$, there exists $t_0 > 0$ such that

$$
M(t) < 2C_4\|U_0\|_{H^{s_0}\cap L^1}
$$

for all $t \in [0, t_0]$. Moreover there exists constants $C_6 > 0$ and $C_7$ such that

$$
\|\sigma_0\|_{H^{s_0}} + \|w_0\|_{H^{s_0}} \leq C_6 M(0).
$$

We set $C_5 := \max\{\sqrt{\frac{C_3}{2}}, C_4\}$, and take $\epsilon_2$ in such a way that $0 < \epsilon_2 < \min\{\frac{1}{4C_5^2}, \frac{\delta}{4C_5^2}, \frac{\epsilon_1^2}{C_4^2C_6^2}, \frac{1}{16C_3^2C_5^2}, \frac{C_2'}{16C_3^2C_5^2}\}$. We will show $M(t) < 2C_5\|U_0\|_{H^{s_0}\cap L^1},\ 0 \leq t \leq T$.

Assume that there exists $t_1 \in (t_0, T]$ such that

$$
M(t) < 2C_5\|U_0\|_{H^{s_0}\cap L^1}
$$

for $0 \le t < t_1$ and

$$M(t_1) = 2C_5\|U_0\|_{H^{s_0}\cap L^1}.$$

It then follows from (44) that

$$\begin{aligned} M^2(t) + C_2'\mathcal{D}_\infty(t) &\le C_3\|U_0\|^2_{H^{s_0}\cap L^1} + C_3 M(t)\big(M^2(t) + \mathcal{D}_\infty(t)\big) \\ &< C_3\|U_0\|^2_{H^{s_0}\cap L^1} + \frac{1}{2}\big(M^2(t) + C_2'\mathcal{D}_\infty(t)\big) \end{aligned}$$

for $t \in [0, t_1]$, and hence,

$$\begin{aligned} M^2(t) + C_2'\mathcal{D}_\infty(t) &< 2C_3\big(E_\infty(0) + \|U_0\|^2_{L^1\cap L^2}\big) \\ &\le 4C_5^2\|U_0\|^2_{H^{s_0}\cap L^1} \end{aligned}$$

for $t \in [0, t_1]$. But this contradicts to $M(t_1) = 2C_5\|U_0\|_{H^{s_0}\cap L^1}$. We thus conclude that

$$M(t) < 2C_5\|U_0\|_{H^{s_0}\cap L^1}$$

for all $0 \le t \le T$. □

It follows from Theorem 3.3 and Proposition 5.13 that

$$M(t) \le C_0 \qquad \text{for all } t.$$

Hence we obtain the desired decay estimate in Theorem 3.4.

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