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ON SHAPES AND SINGULARITIES OF SPACELIKE CONSTANT MEAN CURVATURE SURFACES IN MINKOWSKI 3-SPACE

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Abstract. We investigate 'shapes' of spacelike constant mean curvature (CMC) surfaces near singular points of the first kind. In particular, we calculate the singular curvature and the limiting normal curvature of spacelike CMC surfaces along the set of singular points consisting of singular points of the first kind. Moreover, we study certain singularities of such surfaces.

1. Introduction

A *maximal surface* is a spacelike immersed surface in Minkowski 3-space $\boldsymbol{R}^3_1$ with vanishing mean curvature, and a *maxface* is a maximal surface with certain singularities introduced in [36]. There are several studies on maximal surfaces and maxfaces ([8–11, 13, 21–23, 25, 36]). A Weierstrass-type representation formula for maximal surfaces is given by Kobayashi [23], and Umehara and Yamada gave the following Weierstrass-type representation formula for maxfaces.

Fact 1.1 (Local version of Maxface [36])**.** *Any maxface $f : \Sigma \subset \boldsymbol{C} \to \boldsymbol{R}^3_1$ is represented as*

$$f = \operatorname{Re}\left(\int (-2g, 1 + g^2, i(1 - g^2))\omega\right) \tag{1.1}$$

for a simply-connected domain Σ, where g is meromorphic, and $\omega = \hat{\omega}\, dz$ is holomorphic 1-form such that $g^2\hat{\omega}$ is holomorphic on Σ, $(1 + |g|^2)^2|\omega|^2 \neq 0$ on Σ and $(1 - |g|^2)^2$ does not vanish identically. Moreover, the set of singular points of f is $S(f) = \{q \in \Sigma \mid |g(q)| = 1\}$.

We note that other generalization for maximal surfaces is known ([7]).

It is known that generic singularities of maxfaces are a cuspidal edge, a swallowtail and a cuspidal cross cap [13]. Moreover, there are maxfaces with a cuspidal butterfly and a cuspidal S_1^- singularity [6, 30]. Criteria for these singularities using the Weierstrass data (g, ω) are known [13, 30, 36]. Moreover, we show that there are no maxfaces with cuspidal S_k singularities for $k \geq 2$ (Theorem 2.5).

On the other hand, a Kenmotsu-type representation formula for spacelike surfaces with prescribed mean curvature in $\boldsymbol{R}^3_1$ is given by Akutagawa and Nishikawa [2] as an analogy of the Kenmotsu formula for prescribed mean curvature in $\boldsymbol{R}^3$ ([20]). Applying this formula, Umeda introduced the notion of *generalized spacelike constant mean curvature* (*CMC*) *surfaces* in $\boldsymbol{R}^3_1$ which are spacelike CMC surfaces with certain singularities, and investigated singularities of them ([35]). Locally, generalized spacelike CMC surfaces are constructed as follows:

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Definition 1.2. Let $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ be the Riemann sphere. Let $\Sigma \subset \boldsymbol{C}$ be a simply-connected domain and let $g\colon \Sigma \to \hat{\boldsymbol{C}}$ be a smooth map.

(1) Then g is said to be a *regular extended harmonic map* if
  (a) $g_{z\bar{z}} + 2(1-|g|^2)\bar{g}g_z\overline{\hat{\omega}} = 0$ holds,
  (b) $\omega = \hat{\omega}dz$ can be extended to a 1-form of class C^1 across $\{p \in \Sigma \mid |g(p)| = 1\}$,

  where

$$\hat{\omega} = \frac{\bar{g}_z}{(1-|g|^2)^2} \tag{1.2}$$

  and z is a complex coordinate of Σ.

(2) Let $g\colon \Sigma \to \hat{\boldsymbol{C}}$ be a regular extended harmonic map and H a non-zero constant. Then a map $f\colon \Sigma \to \boldsymbol{R}^3_1$ given by

$$f = \frac{2}{H}\operatorname{Re}\left(\int (-2g, 1+g^2, i(1-g^2))\hat{\omega}dz\right) \tag{1.3}$$

  is called a *generalized spacelike constant mean curvature* (*CMC*) *surface* with mean curvature H, where $\hat{\omega}$ is defined as in (1.2). Moreover, a generalized spacelike CMC surface f given as (1.3) is said to be an *extended spacelike constant mean curvature* (*CMC*) *surface* if g satisfies the following properties:
  - $\hat{\omega}$ never vanishes on $\{p \in \Sigma \mid |g(p)| < \infty\}$, and
  - $g^2\hat{\omega}$ does not vanish on $\{p \in \Sigma \mid |g(p)| = \infty\}$.

  In such a case, g is called an *extended harmonic map*.

It is known that criteria for a cuspidal edge, a swallowtail and a cuspidal cross cap on extended spacelike CMC surfaces are given in terms of g and $\hat{\omega}$ ([35]).

Singular points such as cuspidal edges, cuspidal cross caps and cuspidal S_k singularities belong to the class of *singular points of the first kind* of frontal surfaces (cf. [27]). The set of such singular points consists of regular curves (called *singular curves*) on the source and their images are regular space curves. Along such curves, several geometric invariants are introduced ([26, 27, 34]). In particular, the *singular curvature* κ_s and the *limiting normal curvature* κ_ν are representative because they satisfy $\kappa^2 = \kappa_s^2 + \kappa_\nu^2$, where κ is the curvature of a singular image as a space curve in $\boldsymbol{R}^3$ (cf. [26, 27]). Moreover, κ_s is an *intrinsic invariants* of a frontal, and its sign relates to the *convexity* and *concavity* (see Figure 1) ([14, 34]). Further, κ_ν relates to the boundedness of the Gaussian curvature of a frontal near a singular point ([27, 34]).

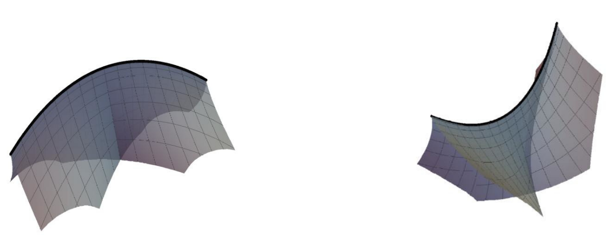

FIGURE 1. Cuspidal cross caps with positive κ_s (left) and negative κ_s (right).

In this paper, we study shapes of spacelike CMC surfaces near a singular point of the first kind. To do this, we regard spacelike CMC surfaces as a frontal surface in $\boldsymbol{R}^3$. In particular, we shall show the following.

Theorem A. *Let f be a maxface (resp. extended spacelike CMC surface) in $\boldsymbol{R}^3_1$. If we regard f as a frontal surface in $\boldsymbol{R}^3$, then the singular curvature κ_s of f is strictly negative at a singular point of the first kind, and the limiting normal curvature κ_ν vanishes at a non-degenerate singular point.*

This theorem says that the Gaussian curvature K_{E} in the Euclidean sense of a frontal surface given by (1.1) or (1.3) is bounded near non-degenerate singular points. Especially, we show that relationship between signs of the Euclidean Gaussian curvature and of the singular curvature near singular points of the first kind (Corollary 3.3). Moreover, the Euclidean Gauss map $\boldsymbol{n}$ of a maxface or an extended spacelike CMC surface has singularities along the singular curve. We investigate types of singularities of $\boldsymbol{n}$ (Proposition 3.5).

Further, we give a certain characterization for a fold singular point of a frontal surface (Theorem 4.2).

2. Preliminaries

2.1. **Minkowski space.** We recall some fundamental properties of Minkowski 3-space. Let $\boldsymbol{R}^3_1$ be Minkowski 3-space with the Lorentzian inner product $\langle\cdot,\cdot\rangle_L$ of signature $(-,+,+)$, that is,

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

where $\boldsymbol{x} = (x_1,x_2,x_3)$ and $\boldsymbol{y} = (y_1,y_2,y_3) \in \boldsymbol{R}^3_1$. For two vectors $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{R}^3_1 \setminus \{0\}$, we say that $\boldsymbol{x}$ is *pseudo-orthogonal* to $\boldsymbol{y}$ if $\langle \boldsymbol{x},\boldsymbol{y}\rangle_L = 0$. A vector $\boldsymbol{x} \in \boldsymbol{R}^3_1 \setminus \{0\}$ is said to be *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x},\boldsymbol{x}\rangle_L > 0$, $\langle \boldsymbol{x},\boldsymbol{x}\rangle_L = 0$ or $\langle \boldsymbol{x},\boldsymbol{x}\rangle_L < 0$, respectively. The *norm* $|\cdot|_L$ of $\boldsymbol{x} \in \boldsymbol{R}^3_1$ is defined as $|\boldsymbol{x}|_L = \sqrt{|\langle \boldsymbol{x},\boldsymbol{x}\rangle_L|}$. In particular, if $\boldsymbol{x}$ is a spacelike vector, then $|\boldsymbol{x}|_L = \sqrt{\langle \boldsymbol{x},\boldsymbol{x}\rangle_L}$.

Let $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ be a pseudo-orthonormal basis of $\boldsymbol{R}^3_1$ with $\langle \boldsymbol{e}_1,\boldsymbol{e}_1\rangle_L = -1$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be non-zero vectors in $\boldsymbol{R}^3_1$. Then we set a *pseudo-vector product* $\boldsymbol{x}\wedge\boldsymbol{y}$ as

$$\boldsymbol{x}\wedge\boldsymbol{y} = -\det\begin{pmatrix} x_2 & x_3 \\ y_2 & y_3\end{pmatrix}\boldsymbol{e}_1 - \det\begin{pmatrix} x_1 & x_3 \\ y_1 & y_3\end{pmatrix}\boldsymbol{e}_2 + \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2\end{pmatrix}\boldsymbol{e}_3, \tag{2.2}$$

where $\boldsymbol{x} = (x_1,x_2,x_3)$ and $\boldsymbol{y} = (y_1,y_2,y_3)$. It is easy to see that $\langle \boldsymbol{x}\wedge\boldsymbol{y},\boldsymbol{z}\rangle_L = \det(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$. In particular, $\langle \boldsymbol{x}\wedge\boldsymbol{y},\boldsymbol{x}\rangle_L = \langle \boldsymbol{x}\wedge\boldsymbol{y},\boldsymbol{y}\rangle_L = 0$.

Let U be an open set in $\boldsymbol{R}^2$. Let $f\colon U \to \boldsymbol{R}^3_1$ be a C^∞ immersion. Then a point $p \in U$ is said to be a *spacelike point*, a *lightlike point* or a *timelike point* of f if the induced metric $f^*\langle\cdot,\cdot\rangle_L$ is positive definite, null or indefinite at p, respectively. We call an immersion f a *spacelike immersion* if every point $p \in U$ is a spacelike point of f. For a spacelike immersion $f\colon U \to \boldsymbol{R}^3_1$, it is known that $f_u \wedge f_v$ is a timelike vector, where (u,v) is a local coordinate system on U, $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$. We set $\nu\colon U \to H^2$ as

$$\nu = \frac{f_u\wedge f_v}{|f_u\wedge f_v|_L}, \tag{2.3}$$

where $H^2 = \{\boldsymbol{x} \in \boldsymbol{R}^3_1 \mid \langle \boldsymbol{x},\boldsymbol{x}\rangle_L = -1\}$ is hyperbolic 2-space. We call ν a *pseudo unit normal vector* of f.

For a spacelike immersion $f\colon U \to \boldsymbol{R}^3_1$, we set the coefficients of the *first fundamental form* and the *second fundamental form* as follows:

$$E_M = \langle f_u, f_u\rangle_L, \qquad F_M = \langle f_u, f_v\rangle_L, \qquad G_M = \langle f_v, f_v\rangle_L,$$
$$L_M = -\langle f_u, \nu_u\rangle_L, \qquad M_M = -\langle f_u, \nu_v\rangle_L, \qquad N_M = -\langle f_v, \nu_v\rangle_L,$$

where ν is a pseudo unit normal vector of f. Using these functions, the *Gaussian curvature* K_M and the *mean curvature* H_M are given as

$$K_M = -\frac{L_M N_M - M_M^2}{E_M G_M - F_M^2}, \quad H_M = \frac{E_M N_M - 2F_M M_M + G_M L_M}{2(E_M G_M - F_M^2)}. \tag{2.4}$$

We say that a spacelike immersion $f\colon U \to \boldsymbol{R}^3_1$ is a *spacelike constant mean curvature* (*CMC*) *surface* if the mean curvature H_M as in (2.4) is constant. Moreover, a spacelike CMC surface f is said to be a *maximal surface* if H_M vanishes identically.

2.2. **Frontal.** We review some notions of frontal surfaces quickly. For details, see [3, 4, 18, 24, 26, 27, 34].

Let $f\colon \Sigma \to \boldsymbol{R}^3$ be a C^∞ map, where Σ is an open domain in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$ is Euclidean 3-space with canonical inner product $\langle\cdot,\cdot\rangle$. Then f is a *frontal surface* (or a *frontal* for short) if there exists a C^∞ map $\boldsymbol{n}\colon \Sigma \to S^2$ such that $\left\langle df_q(X), \boldsymbol{n}(q)\right\rangle = 0$ holds for any $q \in \Sigma$ and $X \in T_q\Sigma$, where S^2 denotes the standard unit sphere in $\boldsymbol{R}^3$. We say that the map $\boldsymbol{n}$ is a (*Euclidean*) *unit normal vector* or the (*Euclidean*) *Gauss map* of f. A frontal f is called a *front* if the pair $(f, \boldsymbol{n})\colon \Sigma \to \boldsymbol{R}^3 \times S^2$ gives an immersion.

We fix a frontal f. A point $p \in \Sigma$ is a *singular point* of f if $\operatorname{rank} df_p < 2$ holds. We denote by $S(f)$ the set of singular points of f. On the other hand, we define a function $\lambda\colon \Sigma \to \boldsymbol{R}$ by

$$\lambda(u, v) = \det(f_u, f_v, \boldsymbol{n})(u, v), \tag{2.5}$$

where (u, v) is a local coordinate system on Σ.This function λ is called the *signed area density function*. For the function λ, it is known that there exist functions $\hat\lambda$ and μ such that $\lambda = \hat\lambda\cdot\mu$, $\hat\lambda^{-1}(0) = S(f)$ and $\mu > 0$ on Σ. We call $\hat\lambda$ the *singularity identifier* of f.

A singular point $p \in S(f)$ of a frontal f is *non-degenerate* if $(\hat\lambda_u(p), \hat\lambda_v(p)) \neq (0, 0)$. Take a non-degenerate singular point p. Then there exist a neighborhood $U(\subset \Sigma)$ of p and a regular curve $\gamma = \gamma(t)\colon (-\varepsilon, \varepsilon) \to U$ $(\varepsilon > 0)$ with $\gamma(0) = p$ such that $\hat\lambda(\gamma(t)) = 0$ on U by the implicit function theorem. We call the curve γ a *singular curve*. Moreover, since a non-degenerate singular point p satisfies $\operatorname{rank} df_p = 1$, there exists a non-zero vector field η on U such that $df_q(\eta_q) = 0$ for any $q \in S(f) \cap U$. This vector field η is called a *null vector field*. Further, one can take a vector field ξ on U so that ξ is tangent to γ on $S(f) \cap U$. We call the direction of ξ along γ the *singular direction*. A non-degenerate singular point p is of the *first kind* if ξ and η are linearly independent at p. Otherwise, it is said to be of the *second kind*.

Definition 2.1. (1) Let $f, g\colon (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be C^∞ map-germs. Then f and g are $\mathcal{A}$-*equivalent* if there exist diffeomorphism-germs $\varphi\colon (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ on the source and $\Phi\colon (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ on the target such that $\Phi \circ f \circ \varphi^{-1} = g$ holds.

(2) Let $f\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a C^∞ map-germ. Then
- f at 0 is a *cuspidal edge* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ at 0.
- f at 0 is a *swallowtail* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at 0.
- f at 0 is a *cuspidal butterfly* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 4v^5 + uv^2, 5v^4 + 2uv)$ at 0.

- f at 0 is a *cuspidal cross cap* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, uv^3)$ at 0.
- f at 0 is a *cuspidal $S_k^\pm$ singularity* ($k \geq 1$) if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3(u^{k+1} \pm v^2))$ at 0.
- f at 0 is a $5/2$-*cuspidal edge* (or a *rhamphoid cuspidal edge*) if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^5)$ at 0.

We note that these singularities are all non-degenerate frontal singularities. Moreover, a cuspidal edge, a cuspidal cross cap, a cuspidal $S_k^\pm$ singularity and a $5/2$-cuspida edge are of the first kind, but a swallowtail and a cuspidal butterfly are of the second kind. Criteria for these singularities are known (see [16, 24, 32, 33]). We remark that certain dualities of singularities for maxfaces and generalized spacelike CMC surfaces are known ([13, 16, 35]).

We next consider the geometric invariants of a frontal at a singular point of the first kind. As discussions above, one can take ξ and η around a singular point of the first kind. Using these vector fields, we define two geometric invariants as follows:

$$\kappa_s(\gamma(t)) = \varepsilon_\gamma \frac{\det(\xi f, \xi\xi f, \boldsymbol{n})}{|\xi f|^3}\bigg|_{(u,v)=\gamma(t)}, \quad \kappa_\nu(\gamma(t)) = \frac{\langle \xi\xi f, \boldsymbol{n}\rangle}{|\xi f|^2}\bigg|_{(u,v)=\gamma(t)}, \tag{2.6}$$

where $\varepsilon_\gamma = \operatorname{sgn}(\det(\gamma', \eta) \cdot \eta\lambda) = \operatorname{sgn}(\det(\xi, \eta) \cdot \eta\hat{\lambda})$ along the singular curve γ. The invariants κ_s and κ_ν are called the *singular curvature* and the *limiting normal curvature*, respectively. We remark that κ_s is an intrinsic invariant and its sign has a geometrical meaning ([14, 34]). Moreover, κ_ν relates to the behavior of the Gaussian curvature ([27]). For more details and other invariants at singular points of the first kind, see [14, 26, 27, 29, 34].

2.3. **Maxfaces.** We recall singularities of maxfaces. To consider singularities, we identify $\boldsymbol{R}^3_1$ with $\boldsymbol{R}^3$ since types of singular points do not depend on the metric. Let f be a maxface with the Weierstrass data $(g, \omega = \hat{\omega} dz)$. We then regard f as a surface in $\boldsymbol{R}^3$. By (1.1), the differentials of f by z and $\bar{z}$ are

$$f_z = \frac{1}{2}(-2g, 1 + g^2, i(1 - g^2))\hat{\omega}, \quad f_{\bar{z}} = \frac{1}{2}(-2\bar{g}, 1 + \bar{g}^2, -i(1 - \bar{g}^2))\overline{\hat{\omega}}. \tag{2.7}$$

We note that f_z is a holomorphic map with respect to z. Since

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v), \tag{2.8}$$

we have

$$f_u \times f_v = -2i f_z \times f_{\bar{z}} = (|g|^2 - 1)|\hat{\omega}|^2(1 + |g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)), \tag{2.9}$$

where $\times$ is the canonical vector product of $\boldsymbol{R}^3$. Thus the Euclidean unit normal vector $\boldsymbol{n}$ of f can be taken as

$$\boldsymbol{n} = \frac{1}{\sqrt{(1 + |g|^2)^2 + 4|g|^2}}(1 + |g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)). \tag{2.10}$$

Moreover, by (2.9) and (2.10), the signed area density function λ of f is

$$\lambda = (|g|^2 - 1)|\hat{\omega}|^2 \sqrt{(1 + |g|^2)^2 + 4|g|^2}.$$

Since $|\hat{\omega}|^2 \sqrt{(1 + |g|^2)^2 + 4|g|^2} > 0$, the singularity identifier $\hat{\lambda}$ of f is

$$\hat{\lambda}(z, \bar{z}) = g(z)\overline{g(z)} - 1. \tag{2.11}$$

Lemma 2.2. *A singular point $p \in S(f)$ is non-degenerate if and only if $g_z(p) \neq 0$.*

Proof. By a direct calculation, we have the assertion (cf. [36]). □

We suppose that any singular point is non-degenerate in the following. Then there exists a singular curve $\gamma(t)$ such that $\hat{\lambda}(\gamma(t)) = 0$. Differentiating this, we see that

$$\begin{aligned}\frac{d}{dt}(\hat{\lambda}(\gamma(t))) &= \hat{\lambda}_z(\gamma(t))\gamma'(t) + \hat{\lambda}_{\bar{z}}(\gamma(t))\overline{\gamma'(t)}\\ &= g_z\bar{g}\gamma' + g\bar{g}_{\bar{z}}\overline{\gamma'} = 2\,\mathrm{Re}\left(\frac{g_z}{g}\gamma'\right) = 0,\end{aligned}$$

where $' = d/dt$. This implies that γ' is perpendicular to $\overline{(g_z/g)}$. Thus we may take

$$\xi = ig\bar{g}_{\bar{z}}\partial_z - ig_z\bar{g}\partial_{\bar{z}} \quad (\xi_\gamma = i\overline{(g_z/g)}\partial_z - i(g_z/g)\partial_{\bar{z}}) \tag{2.12}$$

near p. Here we used the following identification:

$$\zeta = a + ib \in \boldsymbol{C} \leftrightarrow (a, b) \in \boldsymbol{R}^2 \leftrightarrow a\partial_u + b\partial_v \leftrightarrow \zeta\partial_z + \bar{\zeta}\partial_{\bar{z}}. \tag{2.13}$$

We sometimes use the following relation:

$$\overline{\left(\frac{g_z}{g}\right)} = \frac{g}{g_z}\left|\frac{g_z}{g}\right|^2 \tag{2.14}$$

near p.

We next consider the null vector field η. Setting $\eta = \ell\partial_z + \bar{\ell}\partial_{\bar{z}}$, we have

$$\begin{aligned}\eta f &= \ell f_z + \bar{\ell} f_{\bar{z}}\\ &= \frac{\ell}{2}\left(-2, \frac{1}{g} + g, i\left(\frac{1}{g} - g\right)\right)g\hat{\omega} + \frac{\bar{\ell}}{2}\left(-2, \frac{1}{\bar{g}} + \bar{g}, -i\left(\frac{1}{\bar{g}} - \bar{g}\right)\right)\bar{g}\overline{\hat{\omega}}\\ &= \frac{\ell}{2}(-2, \bar{g} + g, i(\bar{g} - g))g\hat{\omega} + \frac{\bar{\ell}}{2}(-2, g + \bar{g}, -i(g - \bar{g}))\bar{g}\overline{\hat{\omega}}\\ &= (-1, \mathrm{Re}(g), \mathrm{Im}(g))(\ell g\hat{\omega} + \overline{\ell g\hat{\omega}})\end{aligned}$$

at a singular point. Thus one can take η as

$$\eta = \frac{i}{g\hat{\omega}}\partial_z - \frac{i}{\bar{g}\overline{\hat{\omega}}}\partial_{\bar{z}}. \tag{2.15}$$

Lemma 2.3 (cf. [36, Theorem 3.1])**.** *A non-degenerate singular point p of a maxface f constructed by (1.1) is of the first kind if and only if* $\mathrm{Im}(g_z/g^2\hat{\omega}) \neq 0$ *at p.*

Proof. By the identification (2.13), we identify ξ and η with $\xi = i\overline{(g_z/g)}$ and $\eta = i/g\hat{\omega}$, respectively. Then

$$\det(\xi, \eta)(p) = \mathrm{Im}(\bar{\xi}\eta)(p) = \mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)(p) \tag{2.16}$$

since $\bar{g} = 1/g$ at p. Thus we have the conclusion. □

For maxfaces, the following criteria for singularities of them are known.

Fact 2.4 ([13, 30, 36])**.** *Let f be a maxface constructed by the Weierstrass data $(g, \omega = \hat{\omega}dz)$. Let p be a non-degenerate singular point of f. Then the following assertions hold.*

(1) *f is a front at p if and only if* $\mathrm{Re}(g_z/g^2\hat{\omega}) \neq 0$ *at p.*
(2) *f at p is a cuspidal edge if and only if* $\mathrm{Re}(g_z/g^2\hat{\omega}) \neq 0$ *and* $\mathrm{Im}(g_z/g^2\hat{\omega}) \neq 0$ *at p.*

(3) *f at p is a swallowtail if and only if* $\operatorname{Im}(g_z/g^2\hat{\omega}) = 0$, $\operatorname{Re}((g_z/g^2\hat{\omega})) \neq 0$ *and*

$$\operatorname{Re}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) \neq 0$$

at p.

(4) *f at p is a cuspidal butterfly if and only if* $\operatorname{Im}(g_z/g^2\hat{\omega}) = 0$, $\operatorname{Re}((g_z/g^2\hat{\omega})) \neq 0$,

$$\operatorname{Re}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) = 0 \quad \textit{and} \quad \operatorname{Im}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right)_z\right) \neq 0$$

at p.

(5) *f at p is a cuspidal cross cap if and only if* $\operatorname{Im}(g_z/g^2\hat{\omega}) \neq 0$, $\operatorname{Re}((g_z/g^2\hat{\omega})) = 0$ *and*

$$\operatorname{Im}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) \neq 0$$

at p.

(6) *f at p is a cuspidal S_1^- singularity if and only if* $\operatorname{Im}(g_z/g^2\hat{\omega}) \neq 0$, $\operatorname{Re}((g_z/g^2\hat{\omega})) = 0$,

$$\operatorname{Im}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) = 0 \quad \textit{and} \quad \operatorname{Re}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right)_z\right) \neq 0$$

at p. Moreover, there are no maxfaces with cuspidal S_1^+ singularity.

Furthermore, we have a stronger result than the last statement of (6) in Fact 2.4.

Theorem 2.5. *For $k \geq 2$, there are no maxfaces with cuspidal $S_k^\pm$ singularities.*

Proof. We identify $\boldsymbol{R}^3_1$ with $\boldsymbol{R}^3$. Let f be a maxface as in (1.1) constructed by the Weierstrass data $(g, \omega = \hat{\omega}dz)$. Let p be a singular point of the first kind of f and $\gamma(t)$ ($t \in (-\varepsilon, \varepsilon)$) a singular curve through $p(= \gamma(0))$. Then we set a function $\psi\colon (-\varepsilon, \varepsilon) \to \boldsymbol{R}$ by

$$\psi(t) = \det(\hat{\gamma}'(t), \boldsymbol{n} \circ \gamma(t), d\boldsymbol{n}_{\gamma(t)}(\eta(t))),$$

where $\hat{\gamma} = f \circ \gamma$, $\boldsymbol{n}$ is the Euclidean Gauss map of f as in (2.10) and η is a null vector field as in (2.15). By a direct calculation, we see that

$$\psi = -|\hat{\omega}|^2 \operatorname{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)\operatorname{Re}\left(\frac{g_z}{g^2\hat{\omega}}\right)$$

holds at p. We assume that f is not a front at p, that is, $\operatorname{Re}(g_z/g^2\hat{\omega})(p) = 0$ (see Fact 2.4). If f has a cuspidal S_k singularity ($k \geq 2$), then $\psi = \psi' = \psi'' = \cdots = \psi^{(k)} = 0$ and $\psi^{(k+1)} \neq 0$ at p (see [32, Theorem 3.2]), where we take a parameter t satisfying $d/dt = i(\overline{(g_z/g)}\partial_z - (g_z/g)\partial_{\bar{z}})$ (cf. [13, 36]). In particular, $\psi = \psi' = \psi'' = 0$ at p is equivalent to

$$\operatorname{Re}\left(\frac{g_z}{g^2\hat{\omega}}\right) = \operatorname{Im}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) = \operatorname{Re}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right)_z\right) = 0 \tag{2.17}$$

at p by using the relation (2.14) (see [30, Page 124]).

On the other hand, the function a which is defined in condition (b) of [32, Theorem 3.2] can be written as

$$a = 32|\hat{\omega}|^2 \operatorname{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)^5 \operatorname{Re}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right)_z\right)$$

(see [30, Page 125]). Thus if f has a cuspidal $S_{k\geq 2}$ singularity at p, a vanishes at p by (2.17). This implies that f cannot have a cuspidal $S_{k\geq 2}$ singularity by [32, Theorem 3.2]. □

2.4. **Spacelike constant mean curvature surfaces.** We review some notions of spacelike (non-zero) constant mean curvature surfaces in $\boldsymbol{R}^3_1$. Let $f\colon \Sigma \to \boldsymbol{R}^3_1$ be an extended spacelike CMC surface given by (1.3) with an extended harmonic map g, where Σ is a simply-connected domain in the complex plane $\boldsymbol{C}$ with complex coordinate $z = u + iv$. We now identify $\boldsymbol{R}^3_1$ with $\boldsymbol{R}^3$ to investigate singularities of f, that is, we regard f as a surface in $\boldsymbol{R}^3$. Moreover, we assume that $|g|$ takes finite values on Σ to focus on non-degenerate singular points.

By (1.3), the first order differentials of f by z and $\bar{z}$ are

$$f_z = \frac{1}{H}(-2g, 1 + g^2, i(1 - g^2))\hat{\omega}, \quad f_{\bar{z}} = \frac{1}{H}(-2\bar{g}, 1 + \bar{g}^2, -i(1 - \bar{g}^2))\overline{\hat{\omega}}. \tag{2.18}$$

Similarly to the case of a maxface, we have

$$f_u \times f_v = -2i f_z \times f_{\bar{z}} = \frac{(|g|^2 - 1)|\hat{\omega}|^2}{H^2}(1 + |g|^2, 2\,\mathrm{Re}(g), 2\,\mathrm{Im}(g)) \tag{2.19}$$

by (2.8). Thus the Euclidean unit normal vector $\boldsymbol{n}$ of f can be taken as

$$\boldsymbol{n} = \frac{1}{\sqrt{(1 + |g|^2)^2 + 4|g|^2}}(1 + |g|^2, 2\,\mathrm{Re}(g), 2\,\mathrm{Im}(g)). \tag{2.20}$$

Using f_z, $f_{\bar{z}}$ and $\boldsymbol{n}$, the signed area density function of f is

$$\lambda = (|g|^2 - 1)|\hat{\omega}|^2 \frac{\sqrt{(1 + |g|^2)^2 + 4|g|^2}}{H^2}. \tag{2.21}$$

Since $\sqrt{(1 + |g|^2)^2 + 4|g|^2}/H^2 > 0$, the set of singular points $S(f)$ of f is the union $S(f) = S_1(f) \cup S_2(f)$, where

$$S_1(f) = \{p \in \Sigma \mid |g(p)| - 1 = 0\}, \quad S_2(f) = \{p \in \Sigma \mid |\hat{\omega}(p)| = 0\}.$$

If f is an extended spacelike CMC surface in $\boldsymbol{R}^3_1$, $\hat{\omega} \neq 0$ (see Definition 1.2). Thus $S_2(f) = \emptyset$ in such a case. Moreover, the singularity identifier $\hat{\lambda}$ is $\hat{\lambda}(z) = g(z)\overline{g(z)} - 1$.

Lemma 2.6 ([35])**.** *Under the above setting, a singular point p of f is non-degenerate if and only if $g_z(p) \neq 0$.*

Let us assume that a point $p \in S_1(f)$ is a non-degenerate singular point of f. Then by similar discussions for the case of maxfaces above, we can take vector fields ξ and η as

$$\xi = i\bar{g}_{\bar{z}} g \partial_z - i g_z \bar{g} \partial_{\bar{z}}, \quad \eta = \frac{i}{g\hat{\omega}}\partial_z - \frac{i}{\bar{g}\overline{\hat{\omega}}}\partial_{\bar{z}}, \tag{2.22}$$

which are the singular direction along the singular curve γ and a null vector field, respectively (cf. [35]). Thus we have the following.

Lemma 2.7 ([35, Theorem 4.1])**.** *A non-degenerate singular point p of an extended spacelike CMC surface f constructed by (1.3) is of the first kind if and only if $\mathrm{Im}(g_z/g^2\hat{\omega}) \neq 0$ at p.*

For extended spacelike CMC surfaces, the following characterizations of singularities are known.

Fact 2.8 ([35, Theorem 4.1])**.** *Let f be an extended spacelike CMC surface constructed by (1.3) with an extended harmonic map g. Let p be a non-degenerate singular point of f. Then the following assertions hold.*

(1) *f at p is a front if and only if $\mathrm{Re}(g_z/g^2\hat{\omega}) \neq 0$ at p.*
(2) *f at p is a cuspidal edge if and only if $\mathrm{Re}(g_z/g^2\hat{\omega}) \neq 0$ and $\mathrm{Im}(g_z/g^2\hat{\omega}) \neq 0$ at p.*

(3) *f at p is a swallowtail if and only if* $\mathrm{Re}(g_z/g^2\hat{\omega}) \neq 0$, $\mathrm{Im}(g_z/g^2\hat{\omega}) = 0$ *and*

$$\mathrm{Re}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) \neq \mathrm{Re}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}}\right)$$

at p.

(4) *f at p is a cuspidal cross cap if and only if* $\mathrm{Re}(g_z/g^2\hat{\omega}) = 0$, $\mathrm{Im}(g_z/g^2\hat{\omega}) \neq 0$ *and*

$$\mathrm{Im}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) \neq \mathrm{Im}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}}\right)$$

at p.

We shall extend this result under some additional assumption.

Theorem 2.9. *Let f be an extended spacelike CMC surface constructed by (1.3) with an extended harmonic map g. Let p be a non-degenerate singular point of f. Assume that $\hat{\omega}$ as in (1.2) can be extended to a function of at least class C^2 across $S_1(f) = \{p \in \Sigma \mid |g(p)| = 1\}$. Then f at p is a cuspidal butterfly if and only if* $\mathrm{Re}(g_z/g^2\hat{\omega}) \neq 0$, $\mathrm{Im}(g_z/g^2\hat{\omega}) = 0$,

$$\mathrm{Re}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right) = \mathrm{Re}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}}\right)$$

and

$$\mathrm{Im}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right)_z\right) + \mathrm{Im}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}}\right)_{\bar{z}}\right) \neq \frac{1}{|g_z|^2}\,\mathrm{Im}\left(\left(\frac{g_z}{g^2\hat{\omega}}\right)_{z\bar{z}}\right)$$

hold at p.

Proof. Let $\gamma(t)$ be a singular curve passing through p. Take a parameter t satisfying the relation $d/dt = i(\overline{(g_z/g)}\partial_z - (g_z/g)\partial_{\bar{z}})$ (cf. [13, 35, 36]). Then we set a function δ as

$$\delta(t) = \det(\gamma', \eta)(t),$$

where η is as in (2.22). In this case, δ can be written as

$$\delta = \mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right) = \mathrm{Im}(\varphi) = \frac{1}{2i}(\varphi - \overline{\varphi}),$$

where we set $\varphi = g_z/g^2\hat{\omega}$. By [19, Corollary A. 9], f at a non-degenerate singular point p is a cuspidal butterfly if and only if f at p is a front and $\delta(0) = \delta'(0) = 0$ but $\delta''(0) \neq 0$. Thus we calculate the first and the second order derivatives of δ by t. By the above expression and the relation, we have

$$\delta' = \left(\mathrm{Re}\left(\frac{g}{g_z}\varphi_z\right) - \mathrm{Re}\left(\overline{\left(\frac{g}{g_z}\right)}\varphi_{\bar{z}}\right)\right)\left|\frac{g_z}{g}\right|^2 = \tilde{\delta}\left|\frac{g_z}{g}\right|^2,$$

where we used the relation as in (2.14).

We now suppose that $\delta'(0) = 0$. Then $\delta''(0) \neq 0$ is equivalent to $\tilde{\delta}'(0) \neq 0$. Hence we calculate $\tilde{\delta}'$. By a direct computation, we see that

$$
\begin{aligned}
\tilde{\delta}' &= \frac{i}{2}\frac{g}{g_z}\left(\left(\frac{g}{g_z}\varphi_z\right)_z + \overline{\left(\frac{g}{g_z}\right)}\overline{\varphi}_{z\bar{z}} - \overline{\left(\frac{g}{g_z}\right)}\varphi_{z\bar{z}} - \left(\frac{g}{g_z}\overline{\varphi}_z\right)_z\right)\left|\frac{g_z}{g}\right|^2 \\
&\quad - \frac{i}{2}\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g}{g_z}\varphi_{z\bar{z}} + \overline{\left(\left(\frac{g}{g_z}\varphi_z\right)\right)_z} - \left(\overline{\left(\frac{g}{g_z}\right)}\varphi_{\bar{z}}\right)_{\bar{z}} - \frac{g}{g_z}\overline{\varphi}_{z\bar{z}}\right)\left|\frac{g_z}{g}\right|^2 \\
&= -\operatorname{Im}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\varphi_z\right)_z\right)\left|\frac{g_z}{g}\right|^2 - \operatorname{Im}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\overline{\left(\frac{g}{g_z}\right)}\varphi_{\bar{z}}\right)_{\bar{z}}\right)\left|\frac{g_z}{g}\right|^2 + \left|\frac{g}{g_z}\right|^2 \operatorname{Im}(\varphi_{z\bar{z}})\left|\frac{g_z}{g}\right|^2 \\
&= |g_z|^2\left(-\operatorname{Im}\left(\frac{g}{g_z}\left(\frac{g}{g_z}\varphi_z\right)_z\right) - \operatorname{Im}\left(\overline{\left(\frac{g}{g_z}\right)}\left(\overline{\left(\frac{g}{g_z}\right)}\varphi_{\bar{z}}\right)_{\bar{z}}\right) + \frac{1}{|g_z|^2}\operatorname{Im}(\varphi_{z\bar{z}})\right)
\end{aligned}
$$

holds at p. Therefore we have the conclusion. □

In [5], Brander gave the Björling formula for spacelike CMC surfaces and investigated singularities. We remark that criteria for a cuspidal edge, a swallowtail and a cuspidal cross cap are known in terms of the Björling data ([5]). Moreover, a criterion for a cuspidal butterfly by the Björling data is known ([28]).

3. Shapes of spacelike CMC surfaces near singular points

In this section, we study shapes of spacelike CMC surfaces near singular points of the first kind. First, we give a proof of Theorem A. To do this, we suppose that p is a singular point of the first kind, and γ is a singular curve passing through p which consists of singular points of the first kind.

Proof of Theorem A. We first give a proof for the case of a maxface. Let f be a map from Σ to $\boldsymbol{R}^3$ given by (1.1). Then we consider the first order directional derivative of f in the direction ξ. By (2.7) and (2.12), we have

$$
\begin{aligned}
\xi f &= i\overline{\left(\frac{g_z}{g}\right)}f_z - i\frac{g_z}{g}f_{\bar{z}} = i\left(\overline{\left(\frac{g_z}{g^2}\right)}\hat{\omega} - \frac{g_z}{g^2}\overline{\hat{\omega}}\right)(-1, \operatorname{Re}(g), \operatorname{Im}(g)) \\
&= i\left(\overline{\left(\frac{g_z}{g^2\hat{\omega}}\right)} - \frac{g_z}{g^2\hat{\omega}}\right)|\hat{\omega}|^2(-1, \operatorname{Re}(g), \operatorname{Im}(g)) = 2\operatorname{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)|\hat{\omega}|^2(-1, \operatorname{Re}(g), \operatorname{Im}(g))
\end{aligned} \tag{3.1}
$$

along γ. Thus we see that

$$
|\xi f| = 2\sqrt{2}\left|\operatorname{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)\right||\hat{\omega}|^2. \tag{3.2}
$$

Moreover, by (2.10) and (3.1), it holds that

$$
\hat{\boldsymbol{n}} \times \xi f = \frac{4}{\sqrt{2}}\operatorname{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)|\hat{\omega}|^2(0, -\operatorname{Im}(g), \operatorname{Re}(g)) \tag{3.3}
$$

along γ, where $\hat{\boldsymbol{n}}$ is

$$
\hat{\boldsymbol{n}}(t) = \boldsymbol{n}(\gamma(t)) = \frac{1}{2\sqrt{2}}(2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)) = \frac{1}{\sqrt{2}}(1, \operatorname{Re}(g), \operatorname{Im}(g)). \tag{3.4}
$$

We next consider the second order directional derivative $\xi\xi f$. Since $\xi = ig\overline{g_z}$, we see that

$$\begin{aligned}\xi\xi f &= -g\overline{g}_{\bar z}(g_z\overline{g}_{\bar z}f_z + g\overline{g}_{\bar z}f_{zz} - g_{zz}\overline{g}f_{\bar z}) + g_z\overline{g}(g\overline{g}_{\bar z\bar z}f_z - g_z\overline{g}_{\bar z}f_{\bar z} - g_z\overline{g}f_{\bar z\bar z}) \\ &= (g_z\overline{g}_{\bar z\bar z} - gg_z\overline{g}_{\bar z}^2)f_z + (\overline{g}_{\bar z}g_{zz} - \overline{g}\overline{g}_{\bar z}g_z^2)f_{\bar z} - (g^2\overline{g}_{\bar z}^2f_{zz} + \overline{g}^2g_z^2f_{\bar z\bar z})\end{aligned}$$

along γ. Setting $X = g_z\overline{g}_{\bar z\bar z} - gg_z\overline{g}_{\bar z}^2$, it follows that

$$Xf_z + \overline{X}f_{\bar z} = \mathrm{Re}(Xg\hat\omega)(-1, \mathrm{Re}(g), \mathrm{Im}(g)).$$

This is perpendicular to $\hat{\boldsymbol{n}}$ and parallel to ξf. We calculate $g^2\overline{g_z}^2f_{zz} + \overline{g}^2g_z^2f_{\bar z\bar z}$. By (2.7), we have

$$f_{zz} = (-1, g, -ig)g_z\hat\omega + \frac{1}{2}(-2g, 1+g^2, i(1-g^2))\hat\omega_z,$$
$$f_{\bar z\bar z} = (-1, \overline{g}, i\overline{g})\overline{g}_{\bar z}\overline{\hat\omega} + \frac{1}{2}(-2\overline{g}, 1+\overline{g}^2, -i(1-\overline{g}^2))\overline{\hat\omega}_{\bar z} \tag{3.5}$$

at p. Thus we have

$$g^2\overline{g}_{\bar z}^2f_{zz} + g_z^2\overline{g}^2f_{\bar z\bar z} = \mathrm{Re}(g^3\overline{g}_{\bar z}\hat\omega_z)\psi + |g_z|^2|\hat\omega|^2\,\mathrm{Re}\left(\overline{\left(\frac{g_z}{g^2\hat\omega}\right)}(-1, g, -ig)\right)$$

along γ. Here we set $\varphi = (g_z/g^2\hat\omega)$ and $\psi = (-1, \mathrm{Re}(g), \mathrm{Im}(g))$. Then

$$\xi\xi f = Y\psi - |g_z|^2|\hat\omega|^2(-\overline{\varphi} - \varphi, g\overline{\varphi} + \overline{g}\varphi, -i(g\overline{\varphi} - \overline{g}\varphi)) \tag{3.6}$$

holds, where Y is a some function. It is obvious that $\langle\psi, \hat{\boldsymbol{n}}\rangle = \det(\psi, \xi f, \hat{\boldsymbol{n}}) = 0$ at a singular point p. Therefore by (3.3) and (3.6), we have

$$\det(\xi f, \xi\xi f, \hat{\boldsymbol{n}}) = \langle\hat{\boldsymbol{n}} \times \xi f, \xi\xi f\rangle = \frac{8}{\sqrt{2}}\left(\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)\right)^2|g_z|^2|\hat\omega|^4 \tag{3.7}$$

along the singular curve γ.

On the other hand, by (2.11) and (2.15), we have

$$\eta\hat\lambda = \frac{i}{g\hat\omega}\hat\lambda_z - \frac{i}{\overline{g}\overline{\hat\omega}}\hat\lambda_{\bar z} = i\left(\frac{g_z}{g^2\hat\omega} - \overline{\left(\frac{g_z}{g^2\hat\omega}\right)}\right) = -2\,\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right) \tag{3.8}$$

at a singular point p, and hence we get

$$\varepsilon_\gamma = \mathrm{sgn}\left(-2\left(\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)\right)^2\right) = -1 \tag{3.9}$$

by (2.16) and (3.8). Thus we see that

$$\kappa_s(\gamma) = \varepsilon_\gamma\frac{\det(\xi f, \xi\xi f, \boldsymbol{n})}{|\xi f|^3}\bigg|_\gamma = -\frac{|g_z|^2}{4\left|\mathrm{Im}\left(\frac{g_z}{g^2\hat\omega}\right)\right||\hat\omega|^2}\Bigg|_\gamma \tag{3.10}$$

along γ by (3.2), (3.7) and (3.9). This implies that κ_s is strictly negative.

Further we consider the limiting normal curvature κ_ν. By (3.4) and (3.6),

$$\begin{aligned}\langle \xi\xi f, \hat{\boldsymbol{n}}\rangle &= -\frac{|\hat{\omega}|^2}{\sqrt{2}}(-\varphi - \overline{\varphi} + \mathrm{Re}(g)(g\overline{\varphi} + \overline{g}\varphi) - i\,\mathrm{Im}(g)(g\overline{\varphi} - \overline{g}\varphi)) \\ &= -\frac{|\hat{\omega}|^2}{\sqrt{2}}\left(-\varphi - \overline{\varphi} + \frac{1}{2}(g^2\overline{\varphi} + \varphi + \overline{\varphi} + \overline{g}^2\varphi) - \frac{1}{2}(g^2\overline{\varphi} - \varphi - \overline{\varphi} + \overline{g}^2\varphi)\right) \\ &= 0\end{aligned}$$

holds at a singular point p. This implies that $\kappa_\nu = 0$ along the singular curve γ (see (2.6)). Therefore we have the assertion for the case of a maxface.

We next consider the case for an extended spacelike CMC surface. Let f be a map in $\boldsymbol{R}^3$ given by (1.3). Assume that f is an extended spacelike CMC surface in $\boldsymbol{R}^3_1$. In this case, the Euclidean Gauss map $\boldsymbol{n}$ is given by (2.20). Then we calculate the first and the second order directional derivatives in the direction ξ as in (2.22). By (2.18) and the relation $\overline{g} = 1/g$ on the set of singular points $S_1(f)$, it follows that

$$\begin{aligned}\xi f &= i\overline{\left(\frac{g_z}{g}\right)} f_z - i\left(\frac{g_z}{g}\right) f_{\overline{z}} = \frac{2i}{H}\left(\overline{\left(\frac{g_z}{g^2\hat{\omega}}\right)} - \frac{g_z}{g^2\hat{\omega}}\right)|\hat{\omega}|^2(-1, \mathrm{Re}(g), \mathrm{Im}(g)) \\ &= \frac{4|\hat{\omega}|^2}{H}\,\mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)(-1, \mathrm{Re}(g), \mathrm{Im}(g))\end{aligned} \tag{3.11}$$

on $S_1(f)$. In particular, if $p \in S_1(f)$ is of the first kind, then ξf does not vanish at p by Lemma 2.7. By (2.18),

$$\begin{aligned} f_{zz} &= \frac{2g_z\hat{\omega}}{H}(-1, g, -ig) + \frac{2g\hat{\omega}_z}{H}(-1, \mathrm{Re}(g), \mathrm{Im}(g)), \\ f_{z\overline{z}} &= \frac{2g\hat{\omega}_{\overline{z}}}{H}(-1, \mathrm{Re}(g), \mathrm{Im}(g)), \qquad f_{\overline{z}z} = \frac{2\overline{g}\overline{\hat{\omega}}_z}{H}(-1, \mathrm{Re}(g), \mathrm{Im}(g)), \\ f_{\overline{z}\overline{z}} &= \frac{2\overline{g}_{\overline{z}}\overline{\hat{\omega}}}{H}(-1, \overline{g}, i\overline{g}) + \frac{2\overline{g}\overline{\hat{\omega}}_{\overline{z}}}{H}(-1, \mathrm{Re}(g), \mathrm{Im}(g))\end{aligned}$$

hold at $p \in S_1(f)$ since $g_{\overline{z}} = \overline{g}_z = 0$ at p. Therefore we have

$$\xi\xi f = Z(-1, \mathrm{Re}(g), \mathrm{Im}(g)) - \frac{2|g_z|^2|\hat{\omega}|^2}{H}(-\varphi - \overline{\varphi}, g\overline{\varphi} + \overline{g}\varphi, -i(g\overline{\varphi} - \overline{g}\varphi)) \tag{3.12}$$

holds on $S_1(f)$, where Z is a some function and $\varphi = g_z/g^2\hat{\omega}$.

We set $\hat{\boldsymbol{n}} = \boldsymbol{n} \circ \gamma$, where γ is a singular curve through $p \in S_1(f)$. This is expressed as (3.4). Since $(-1, \mathrm{Re}(g), \mathrm{Im}(g))$ is perpendicular to $\hat{\boldsymbol{n}}$, we have $\langle \xi\xi f, \hat{\boldsymbol{n}}\rangle = 0$. This implies that the limiting normal curvature κ_ν vanishes identically along γ. Moreover, by (3.11) and (3.4), we obtain

$$\hat{\boldsymbol{n}} \times \xi f = \frac{8}{\sqrt{2}H}\,\mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)|\hat{\omega}|^2(0, -\mathrm{Im}(g), \mathrm{Re}(g)) \tag{3.13}$$

along γ. Thus by (3.12) and (3.13), it follows that

$$\det(\xi f, \xi\xi f, \hat{\boldsymbol{n}}) = \langle \hat{\boldsymbol{n}} \times \xi f, \xi\xi f\rangle = \frac{16}{\sqrt{2}H^2}\left(\mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)\right)^2 |g_z|^2|\hat{\omega}|^4 (> 0) \tag{3.14}$$

On the other hand, the singular identifier $\hat{\lambda}$ of f is $\hat{\lambda} = g\overline{g} - 1$, and the null vector field η is $\eta = i/g\hat{\omega}$ (cf. (2.22)), and hence we have

$$\eta\hat{\lambda} = -2\,\mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right), \quad \det(\xi, \eta) = \mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)$$

at $p \in S_1(f)$. Thus it holds that $\varepsilon_\gamma = \mathrm{sgn}(\eta\hat{\lambda} \cdot \det(\xi, \eta)) = -1$. By (3.14), κ_s is given as

$$\kappa_s = -\frac{|H||g_z|^2}{16\left|\mathrm{Im}\left(\frac{g_z}{g^2\hat{\omega}}\right)\right||\hat{\omega}|^2}$$

along γ. This completes the proof. □

We remark that if p is singular point of the second kind and the singular curve $\gamma(t)$ through $p = \gamma(0)$ consists of singular points of the first kind for $t \neq 0$, then the singular curvature behaves $\lim_{t\to 0} \kappa_s(t) = -\infty$ ([34]).

Example 3.1. Let f be a maxface constructed by the data $(g, \omega) = (z, dz)$ on $\boldsymbol{C}$. This surface is known as the *Lorentzian Enneper surface* in $\boldsymbol{R}^3_1$ (see Figure 2). The set of singular points $S(f)$ is $S(f) = \{|z| = 1\}$. The points $z = \pm 1, \pm i$ are swallowtails and the points $z = e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}$ are cuspidal cross caps (cf. [36]). Thus the curve $\gamma(t) = e^{it}$ consists of singular points of the first kind for $t \in (0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2) \cup (3\pi/2, 2\pi)$. Regard f as a surface in $\boldsymbol{R}^3$. Then we have the singular curvature κ_s as

$$\kappa_s = \frac{-1}{4|\sin 2t|} < 0$$

along $\gamma(t)$ for $t \in (0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2) \cup (3\pi/2, 2\pi)$.

FIGURE 2. The Lorentzian Enneper surface. The thick curve is the image of the singular curve.

3.1. **Behavior of the Gaussian curvature.**

We consider behavior of the Gaussian curvature of f given by (1.1) or (1.3) as a surface in $\boldsymbol{R}^3$. First, we investigate the case of maxfaces. We consider behavior of the Gauss map $\boldsymbol{n}$ as in (2.10). The first order derivatives $\boldsymbol{n}_z$ and $\boldsymbol{n}_{\bar{z}}$ of $\boldsymbol{n}$ by z and $\bar{z}$ are

$$\text{(3.15)} \quad \begin{aligned} \boldsymbol{n}_z &= \frac{g_z}{\sqrt{(1+|g|^2)^2 + 4|g|^2}}\Big(\overline{g}(1 + \hat{\rho}(1 + |g|^2)), 2\hat{\rho}\overline{g}\,\mathrm{Re}(g) + 1, 2\hat{\rho}\overline{g}\,\mathrm{Im}(g) - i\Big), \\ \boldsymbol{n}_{\bar{z}} &= \frac{\overline{g_z}}{\sqrt{(1+|g|^2)^2 + 4|g|^2}}\Big(g(1 + \hat{\rho}(1 + |g|^2)), 2\hat{\rho}g\,\mathrm{Re}(g) + 1, 2\hat{\rho}g\,\mathrm{Im}(g) + i\Big), \end{aligned}$$

where

$$\hat{\rho} = -\frac{3+|g|^2}{(1+|g|^2)^2+4|g|^2}.$$

Thus the vector product $\boldsymbol{n}_z \times \boldsymbol{n}_{\bar{z}}$ is given as

$$\boldsymbol{n}_z \times \boldsymbol{n}_{\bar{z}} = \frac{2i|g_z|^2(1-|g|^2)}{((1+|g|^2)^2+4|g|^2)^2}(1+|g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)), \tag{3.16}$$

and hence we have

$$\Lambda = \det(\boldsymbol{n}_u, \boldsymbol{n}_v, \boldsymbol{n}) = \langle \boldsymbol{n}_u \times \boldsymbol{n}_v, \boldsymbol{n}\rangle = -2i\,\langle \boldsymbol{n}_z \times \boldsymbol{n}_{\bar{z}}, \boldsymbol{n}\rangle = -\frac{4|g_z|^2(|g|^2-1)}{((1+|g|^2)^2+4|g|^2)^{3/2}} \tag{3.17}$$

by (2.10) and (3.16). Therefore the Gaussian curvature K_{E} of a maxface f given by (1.1) as a surface in $\boldsymbol{R}^3$ is

$$K_{\mathrm{E}} = \frac{\Lambda}{\lambda} = -\frac{4|g_z|^2}{((1+|g|^2)^2+4|g|^2)^2|\hat{\omega}|^2} \tag{3.18}$$

by (2.5) and (3.17). This implies that K_{E} is strictly negative near a non-degenerate singular point p of f.

We next consider the case of extended spacelike CMC surfaces f. In this case, by similar calculations, we have

(3.19)

$$\boldsymbol{n}_z = \rho^3 \begin{pmatrix} X(\rho^{-2}-(3+|g|^2)(1+|g|^2)) \\ \rho^{-2}(g_z+\bar{g}_z) - 2X(3+|g|^2)\operatorname{Re}(g) \\ \dfrac{\rho^{-2}}{i}(g_z-\bar{g}_z) - 2X(3+|g|^2)\operatorname{Im}(g) \end{pmatrix}^T, \quad \boldsymbol{n}_{\bar{z}} = \rho^3 \begin{pmatrix} \overline{X}(\rho^{-2}-(3+|g|^2)(1+|g|^2)) \\ \rho^{-2}(g_{\bar{z}}+\bar{g}_{\bar{z}}) - 2\overline{X}(3+|g|^2)\operatorname{Re}(g) \\ \dfrac{\rho^{-2}}{i}(g_{\bar{z}}-\bar{g}_{\bar{z}}) - 2\overline{X}(3+|g|^2)\operatorname{Im}(g) \end{pmatrix}^T,$$

where we set

$$\rho = \frac{1}{\sqrt{(1+|g|^2)^2+4|g|^2}}, \quad X = g_z\bar{g} + g\bar{g}_z,$$

and $\boldsymbol{x}^T$ is a transposed vector of $\boldsymbol{x}$. Then by (3.19), the cross product of $\boldsymbol{n}_z$ and $\boldsymbol{n}_{\bar{z}}$ is

$$\boldsymbol{n}_z \times \boldsymbol{n}_{\bar{z}} = \frac{2(1-|g|^2)(|g_{\bar{z}}|^2-|g_z|^2)}{i((1+|g|^2)^2+4|g|^2)}(1+|g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)).$$

Thus the set of singular points of $\boldsymbol{n}$ is $S(\boldsymbol{n}) = \{p \in \Sigma \mid |g(p)| = 1\} \cup \{p \in \Sigma \mid |g_z(p)| = |g_{\bar{z}}(p)|\}$. Moreover, we have

$$\boldsymbol{n}_u \times \boldsymbol{n}_v = -2i\boldsymbol{n}_z \times \boldsymbol{n}_{\bar{z}} = \frac{-4(1-|g|^2)(|g_{\bar{z}}|^2-|g_z|^2)}{(1+|g|^2)^2+4|g|^2}(1+|g|^2, 2\operatorname{Re}(g), 2\operatorname{Im}(g)), \tag{3.20}$$

and hence the Euclidean Gaussian curvature K_E is

$$K_E = \frac{\det(\boldsymbol{n}_u, \boldsymbol{n}_v, \boldsymbol{n})}{\det(f_u, f_v, \boldsymbol{n})} = \frac{4(|g_{\bar{z}}|^2-|g_z|^2)H^2}{((1+|g|^2)^2+4|g|^2)^2|\hat{\omega}|^2} \tag{3.21}$$

by (2.21) and (3.20). This implies that K_E changes the sign across the set $\{p \in \Sigma \mid |g_z(p)| = |g_{\bar{z}}(p)|\}$. Further, when p is a non-degenerate singular point of f, then $g_z(p) \neq 0$ and $g_{\bar{z}}(p) = 0$. Thus K_E is strictly negative at p by (3.21). As a result of above discussions, we have the following.

Proposition 3.2. *Let $f\colon \Sigma \to \boldsymbol{R}^3_1$ be a maxface given by (1.1) (resp. extended spacelike CMC surface given by (1.3)). Let p be a non-degenerate singular point of f. When we regard f as a surface in $\boldsymbol{R}^3$, then its Gaussian curvature K_E is strictly negative at p.*

By Theorem A and Proposition 3.2, we have the following assertion immediately.

Corollary 3.3. *Under the same assumptions as in Proposition 3.2, the sign of the singular curvature κ_s at singular point of the first kind p of f coincides with the sign of the Gaussian curvature K_E of f at p.*

Remark 3.4. Let f be a maxface given by the Weierstrass data $(g, \hat{\omega} dz)$. Then the Gaussian curvature K_M (cf. (2.4)) of f is given as

$$K_M = \frac{|g_z|^2}{(1-|g|^2)^4|\hat{\omega}|^2}$$

on the set of regular points (cf. [36]). Thus K_M is non-negative on the set of regular points. On the other hand, let f be an extended spacelike CMC $H(\neq 0)$ surface with extended harmonic map g. Then the Gaussian curvature K_M of f is given by

$$K_M = -H^2\left(\left|\frac{g_z}{g_{\bar{z}}}\right|^2 - 1\right)$$

on the set of regular points (see [2, 35]). Thus there are possibilities that K_M takes positive or negative value. In particular, K_M is unbounded near a singular points in both cases.

For a front in $\boldsymbol{R}^3$ with a cuspidal edge p, it follows that if the Gaussian curvature K_E is non-negative near p, then the singular curvature is non-positive ([34, Theorem 3.1]). However, the inverse of this fact does not hold in general. Thus we can construct several examples of frontal surfaces in $\boldsymbol{R}^3$ with bounded negative Gaussian curvatures and negative singular curvatures along the singular curves by (1.1) and (1.3) .

We remark that Akamine [1] investigated relationships between signs of the singular curvature and of the (Lorentzian) Gaussian curvature for *timelike minfaces* which are timelike surfaces with vanishing mean curvature admitting certain singularities.

We turn to our attention to singularities of the Euclidean Gauss map $\boldsymbol{n}$. By (3.17) and (3.20), a singular point p of f is also a singular point of $\boldsymbol{n}$. Thus locally, we may consider $\hat{\lambda} = |g|^2 - 1$ and $\xi = i\overline{(g_z/g)}$ as a singular identifier and the singular direction of $\boldsymbol{n}$, respectively. Since $g_z(p) \neq 0$, $\boldsymbol{n}$ has a non-degenerate singular point at p.

Proposition 3.5. *Suppose that the Gauss map $\boldsymbol{n}$ of a maxface* (*resp. an extended spacelike CMC surface*) *f given by* (1.1) (*resp.* (1.3)) *as a surface in $\boldsymbol{R}^3$ has a non-degenerate singularity at p. Then p must be a fold of $\boldsymbol{n}$.*

Here a *fold* is a map germ $h\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ which is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2)$ at the origin.

Proof. We look for a null vector field $\eta^{\boldsymbol{n}}$ of $\boldsymbol{n}$. By (3.15) and (3.19), we see that

$$\boldsymbol{n}_z = \frac{i}{2\sqrt{2}}\left(\frac{g_z}{g}\right)(0, \mathrm{Im}(g), -\mathrm{Re}(g)), \qquad \boldsymbol{n}_{\bar{z}} = -\frac{i}{2\sqrt{2}}\overline{\left(\frac{g_z}{g}\right)}(0, \mathrm{Im}(g), -\mathrm{Re}(g))$$

hold at p. Here we used the relation $g_{\bar{z}} = 0$ at p for the case of extended CMC surfaces. Thus we can take $\eta^{\boldsymbol{n}}$ as

$$\eta^{\boldsymbol{n}} = \overline{\left(\frac{g_z}{g}\right)} \leftrightarrow \eta^{\boldsymbol{n}} = \overline{\left(\frac{g_z}{g}\right)}\partial_z + \left(\frac{g_z}{g}\right)\partial_{\bar{z}} \tag{3.22}$$

along the singular curve γ through p.

Using the singular direction ξ as in (2.12) and the null vector $\eta^{\boldsymbol{n}}$ as in (3.22), we have

$$\det(\xi, \eta^{\boldsymbol{n}}) = \mathrm{Im}(\bar{\xi}\eta^{\boldsymbol{n}}) = -\left|\frac{g_z}{g}\right|^2 = -|g_z|^2 \neq 0$$

along γ. This implies that $\boldsymbol{n}$ has a fold at p (see [37, Proposition 2.1]). □

By this proposition, the singular image of $\boldsymbol{n}$ forms regular spherical curve.

4. A characterization of a fold singularity

In [11], notions of a non-degenerate fold singularity and a fold singular point for maxfaces in $\boldsymbol{R}^3_1$ are introduced as follows.

Definition 4.1. Let f be a maxface in $\boldsymbol{R}^3_1$ constructed by the Weierstrass data $(g, \hat{\omega}dz)$. Let p be a non-degenerate singular point of f and $\gamma(t)$ $(t < |\varepsilon|)$ a singular curve through p. Then γ is a *non-degenerate fold singularity* if

$$\mathrm{Re}\left(\frac{g_z}{g^2\hat{\omega}}(\gamma(t))\right) = 0$$

holds. Each point on the non-degenerate fold singularity is called a *fold singular point*.

Moreover, a singular point of a C^∞ map $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3$ has a *fold singularity* at p if there exists a local coordinate system $(U; u, v)$ centered at p satisfying $f(u, v) = f(u, -v)$ ([11, page 182]). It is known that a non-degenerate fold singularity is actually a fold singularity ([11, Lemma 2.17]). Further, it is known that a non-degenerate fold singular point on a maxface corresponds to a *generalized cone-like singular point* on a conjugate maxface ([11, 12, 22]).

On the other hand, for a C^∞ map $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3$, we say that a singular point p is a *fold singular point* of f if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, 0)$ at the origin (cf. [16]). We note that both a fold singularity and a fold are frontal singularities but not fronts. In this section, we show the equivalence of fold singularities and folds. In particular, we prove the following.

Theorem 4.2. *Let $f\colon U \to \boldsymbol{R}^3$ be a frontal and p a singular point of the first kind. Then p is a fold singular point of f if and only if p has a fold singularity.*

Before giving a proof, we prepare some facts which we need.

Fact 4.3 (the division lemma)**.** *Let $h\colon U(\subset \boldsymbol{R}^2) \to \boldsymbol{R}$ be a C^∞ function. If h satisfies $h(u, 0) = 0$, then there exists a function b on U such that $h(u, v) = vb(u, v)$.*

As a corollary of Fact 4.3, we see the following.

Fact 4.4. *Let $h\colon U \to \boldsymbol{R}$ be a C^∞ function. Then there exist functions a and b such that $h(u, v) = a(u) + vb(u, v)$.*

On the other hand, the following is known.

Fact 4.5 (the Whitney lemma)**.** *Let $h\colon U \to \boldsymbol{R}$ be a C^∞ function. If h satisfies $h(u, v) = h(u, -v)$, then there exists a function b on U such that $h(u, v) = b(u, v^2)$.*

Proof of Theorem 4.2. If a frontal f has a fold singular point at p, then we may assume that f is given by $f(u, v) = (u, v^2, 0)$. This obviously satisfies the condition that $f(u, v) = f(u, -v)$, and hence a fold p is a fold singularity. Thus we show the opposite side in the following.

Let p has a fold singularity. Then there exists a local coordinate system (u, v) around p such that $f(u, v) = f(u, -v)$. Without loss of generality, we may assume that $p = (0, 0)$. First, we note that $S(f) = \{v = 0\}$ holds on U since $f_v(u, v) = -f_v(u, -v)$, and hence $f_v(u, 0) = 0$. Since $\operatorname{rank} df_p = 1$, we may assume that $(f_1)_u(p) \neq 0$, where $f = (f_1, f_2, f_3)$. Then the map $\varphi\colon (u, v) \mapsto (s, t) = (f_1(u, v), v)$ gives a coordinate change on the source around p. Thus one may have

$$g(s,t)(= f \circ \varphi^{-1}(s,t)) = (s, g_2(s,t), g_3(s,t)), \tag{4.1}$$

where g_i $(i = 2, 3)$ are some C^∞ functions of s, t. We note that the map g satisfies $g(s, t) = g(s, -t)$ by the construction.

On the other hand, by Fact 4.4, there exist functions $a_i(s)$ and $b_i(s, t)$ $(i = 2, 3)$ such that $g_i(s, t) = a_i(s) + t b_i(s, t)$. Moreover, since $(g_i)_t(s, 0) = 0$ $(i = 2, 3)$, there exist functions $\tilde{b}_i(s, t)$ such that $b_i(s, t) = t\tilde{b}_i(s, t)$ by Fact 4.3. Thus the map g as in (4.1) can be written as

$$g(s,t) = (s, a_2(s) + t^2\tilde{b}_2(s,t), a_3(s) + t^2\tilde{b}_3(s,t)). \tag{4.2}$$

Further, noticing that $g(s, t) = g(s, -t)$, there exist functions $\hat{b}_i$ $(i = 2, 3)$ such that $\tilde{b}_i(s, t) = \hat{b}_i(s, t^2)$ by Fact 4.5. Thus the map g as in (4.2) can be written as

$$g(s,t) = (s, a_2(s) + t^2\hat{b}_2(s,t^2), a_3(s) + t^2\hat{b}_3(s,t^2)). \tag{4.3}$$

We set $\Phi\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ as

$$\Phi(X, Y, Z) = (X, Y - a_2(X), Z - a_3(X)). \tag{4.4}$$

This gives a local diffeomorphism on $\boldsymbol{R}^3$. Composing g as in (4.3) and Φ as in (4.4), we have

$$h(s,t) = \Phi \circ g(s,t) = (s, t^2\hat{b}_2(s,t^2), t^2\hat{b}_3(s,t^2)).$$

Here we remark that either $\hat{b}_2(p)$ or $\hat{b}_3(p)$ does not vanish by non-degeneracy. Thus we may suppose that $\hat{b}_2(p) \neq 0$. In this case, a map $\tau\colon (s, t) \mapsto (x, y) = \left(s, t\sqrt{\left|\hat{b}_2(s, t^2)\right|}\right)$ gives a local coordinate change on the source. Thus by a coordinate change, we have

$$k(x,y)(= h \circ \tau^{-1}(x,y)) = (x, y^2, y^2 B(x, y^2)), \tag{4.5}$$

where B is a some C^∞ function of x, y. Setting a map $\Psi\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ as

$$\Psi(X, Y, Z) = (X, Y, Z - YB(X, Y)), \tag{4.6}$$

this map Ψ gives a local diffeomorphism. By (4.5) and (4.6), it holds that

$$\Psi \circ k(x,y) = (x, y^2, 0).$$

Therefore we have the conclusion. □

By this theorem, non-degenerate fold singularities of maxfaces are actually fold singular points. We remark that there are no generalized spacelike CMC surfaces with fold singular points ([16, Theorem 1.1]).

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