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The no-binding regime of the Pauli-Fierz model

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The Pauli-Fierz model $H(\alpha)$ in nonrelativistic quantum electrodynamics is considered. The external potential V is sufficiently shallow and the dipole approximation is assumed. It is proven that there exist constants $0 < \alpha_- < \alpha_+$ such that $H(\alpha)$ has no ground state for $|\alpha| < \alpha_-$, which complements an earlier result stating that there is a ground state for $|\alpha| > \alpha_+$. We develop a suitable extension of the Birman-Schwinger argument. Moreover, for any given $\delta > 0$ examples of potentials V are provided such that $\alpha_+ - \alpha_- < \delta$. © *2011 American Institute of Physics*. [doi:10.1063/1.3598465]

I. INTRODUCTION

Let us consider a quantum particle in an external potential described by the Schrödinger operator,

$$H_{\mathrm{p}}(m) = -\frac{1}{2m}\Delta + V(x), \tag{1.1}$$

acting on $L^2(\mathbb{R}^d)$. If the potential V is short ranged and attractive and if the dimension $d \geq 3$, then there is a transition from unbinding to binding as the mass m is increased. More precisely, there is some critical mass, m_{c}, such that $H_{\mathrm{p}}(m)$ has no ground state for $0 < m < m_{\mathrm{c}}$ and a unique ground state for $m_{\mathrm{c}} < m$. In fact, the critical mass is given by

$$\frac{1}{2m_{\mathrm{c}}} = \left\| |V|^{1/2} (-\Delta)^{-1} |V|^{1/2} \right\|,$$

see Lemma 3.3. We now couple $H_{\mathrm{p}}(m)$ to the quantized electromagnetic field with coupling strength α. (α is proportional to the charge of the particle and corresponds to the square root of the usual fine-structure constant). The corresponding Hamiltonian is denoted by $H(\alpha)$. On a heuristic level, through the dressing by photons the particle becomes effectively more heavy, which means that there is critical mass $m_{\mathrm{c}}(\alpha)$ for the existence of a ground state. By symmetry, $m_{\mathrm{c}}(\alpha) = m_{\mathrm{c}}(-\alpha)$ and, henceforth, we restrict ourselves to $\alpha \geq 0$. $m_{\mathrm{c}}(\alpha)$ is expected to be decreasing as a function of α with $m_{\mathrm{c}}(0) = m_{\mathrm{c}}$. In particular, for fixed $m < m_{\mathrm{c}}$, there should be an unbinding-binding transition as the coupling α is increased. This phenomenon has been baptized *enhanced binding* and has been studied for a variety of models by several authors.[4,6,7,11–14] In case $m > m_{\mathrm{c}}$, more general techniques are available and the existence of a unique ground state for the full Hamiltonian is proven in Refs. 3, 5, 9, 10, 17, and 20.

The heuristic picture also asserts that the full Hamiltonian has a regime of couplings with no ground state. This property is more difficult to establish and the only result we are aware of is proved by Benguria and Vougalter.[6] In essence, they establish that the line $m_{\mathrm{c}}(\alpha)$ is continuous as $\alpha \to 0$. (In fact, they use the strength of the potential as parameter). From this it follows that the no binding regime cannot be empty. In this paper, as in Ref. 14, we will use the dipole approximation for simplicity, but provide a fairly explicit bound on the critical mass. In the dipole approximation the effective mass $m_{\mathrm{eff}}(\alpha) = m + c_0\alpha^2$ with some explicitly computable coefficient c_0, see Eq. (2.10)

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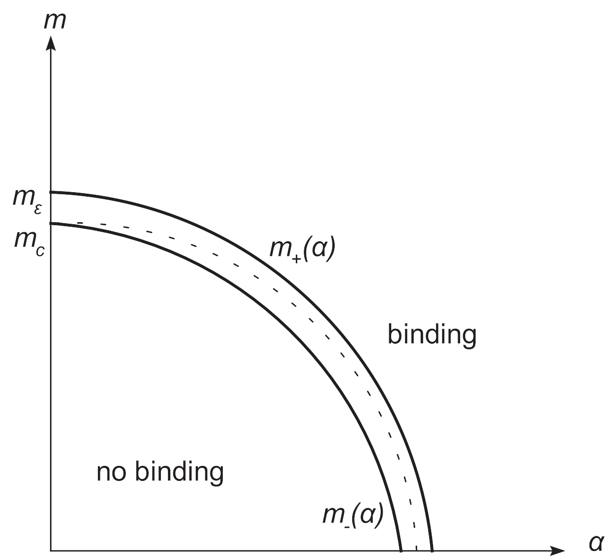

FIG. 1. Upper and lower bounds on the critical mass $m_{\mathrm{c}}(\alpha)$. The dashed line indicates $m_{\mathrm{c}}(\alpha)$.

below. Thus, the most basic guess for $m_{\mathrm{c}}(\alpha)$ would be $m_{\mathrm{c}}(\alpha) + c_0\alpha^2 = m_{\mathrm{c}}$. The corresponding curve is displayed in Fig. 1. In fact the guess turns out to be a lower bound on the true $m_{\mathrm{c}}(\alpha)$, which for small coupling has been proved already in Ref. 7 in a more general setting. We will complement our lower bound with an upper bound of the same qualitative form.

The unbinding for the Schrödinger operator $H_{\mathrm{p}}(m)$ is proven by the Birman-Schwinger principle. Formally one has

$$H_{\mathrm{p}}(m) = \frac{1}{2m}(-\Delta)^{1/2}\big(\mathbb{1} + 2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}\big)(-\Delta)^{1/2}.$$

If m is sufficiently small, then $2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}$ is a strict contraction. Hence, the operator $\mathbb{1} + 2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}$ has a bounded inverse and $H_{\mathrm{p}}(m)$ has no eigenvalue in $(-\infty, 0]$. More precisely the Birman-Schwinger principle states that

$$\dim \mathbb{1}_{[\frac{1}{2m},\infty)}(V^{1/2}(-\Delta)^{-1}V^{1/2}) \geq \dim \mathbb{1}_{(-\infty,0]}(H_{\mathrm{p}}(m)). \tag{1.2}$$

For small m the left-hand side equals 0 and, thus, $H_{\mathrm{p}}(m)$ has no eigenvalues in $(-\infty, 0]$.

Our approach will be to generalize (1.2) to the Pauli-Fierz model of non-relativistic quantum electrodynamics. The Pauli-Fierz Hamiltonian $H(\alpha)$ is defined on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}$, where $\mathscr{F}$ denotes the boson Fock space. Transforming $H(\alpha)$ unitarily by U one arrives at,

$$U^{-1}H(\alpha)U = H_0(\alpha) + W + g \tag{1.3}$$

as the sum of the free Hamiltonian

$$H_0(\alpha) = -\frac{1}{2m_{\mathrm{eff}}(\alpha)}\Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}, \tag{1.4}$$

involving the effective mass of the dressed particle and the Hamiltonian H_{f} of the free boson field, the transformed interaction

$$W = T^{-1}(V \otimes \mathbb{1})T, \tag{1.5}$$

and the global energy shift g. $m_{\mathrm{eff}}(\alpha)$ is an increasing function of α. We will show that (1.3) has no ground state for sufficiently small α by means of a Birman-Schwinger type argument such as (1.2). In combination with the results obtained in Ref. 14 we provide examples of external potentials V

such that for some given $\delta > 0$ there exist two constants $0 < \alpha_- < \alpha_+$ satisfying

$$\delta > \alpha_+ - \alpha_- > 0 \tag{1.6}$$

and $H(\alpha)$ has no ground state for $\alpha < \alpha_-$ but has a ground state for $\alpha > \alpha_+$.

The paper is organized as follows. In Sec. II, we define the Pauli-Fierz model and in Sec. III we prove the absence of ground states. Section IV lists examples of external potentials exhibiting the unbinding-binding transition.

II. THE PAULI-FIERZ HAMILTONIAN

We assume a space dimension $d \geq 3$ throughout and take the natural unit: the velocity of light $c = 1$ and the Planck constant divided by 2π, $\hbar = 1$. The Hilbert space $\mathscr{H}$ for the Pauli-Fierz Hamiltonian is given by

$$\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F},$$

where

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_s^n (\oplus^{d-1} L^2(\mathbb{R}^d))\right]$$

denotes the boson Fock space over the $(d-1)$-fold direct sum $\oplus^{d-1} L^2(\mathbb{R}^d)$. Let $\Omega = \{1, 0, 0, \ldots\} \in \mathscr{F}$ denote the Fock vacuum. The creation operator and the annihilation operator are denoted by $a^*(f, j)$ and $a(f, j)$, $j = 1, \ldots, d-1$, $f \in L^2(\mathbb{R}^d)$, respectively, and they satisfy the canonical commutation relations

$$[a(f, j), a^*(g, j')] = \delta_{jj'}(f, g)\mathbb{1}, \quad [a(f, j), a(g, j')] = 0 = [a^*(f, j), a^*(g, j')]$$

with (f, g) the scalar product on $L^2(\mathbb{R}^d)$. We write

$$a^\sharp(f, j) = \int a^\sharp(k, j) f(k) dk, \quad a^\sharp = a, a^*. \tag{2.1}$$

The energy of a single photon with momentum $k \in \mathbb{R}^d$ is

$$\omega(k) = |k|. \tag{2.2}$$

The free Hamiltonian on $\mathscr{F}$ is then given by

$$H_{\mathrm{f}} = \sum_{j=1}^{d-1} \int \omega(k) a^*(k, j) a(k, j) dk. \tag{2.3}$$

Note that $\sigma(H_{\mathrm{f}}) = [0, \infty)$, and $\sigma_{\mathrm{p}}(H_{\mathrm{f}}) = \{0\}$. $\{0\}$ is a simple eigenvalue of H_{f} and $H_{\mathrm{f}}\Omega = 0$.

Next we introduce the quantized radiation field. The d-dimensional polarization vectors are denoted by $e_j(k) \in \mathbb{R}^d$, $j = 1, \ldots, d-1$, which satisfy $e_i(k) \cdot e_j(k) = \delta_{ij}$ and $e_j(k) \cdot k = 0$ almost everywhere on $\mathbb{R}^d$. The quantized vector potential then reads as

$$A(x) = \sum_{j=1}^{d-1} \int \frac{1}{\sqrt{2\omega(k)}} e_j(k) \left(\hat{\varphi}(k) a^*(k, j) \mathrm{e}^{-ikx} + \hat{\varphi}(-k) a(k, j) \mathrm{e}^{ikx}\right) dk \tag{2.4}$$

for $x \in \mathbb{R}^d$ with ultraviolet cutoff $\hat{\varphi}$. Conditions imposed on $\hat{\varphi}$ will be supplied later. Assuming that V is centered, in the dipole approximation $A(x)$ is replaced by $A(0)$. We set $A = A(0)$. The Pauli-Fierz Hamiltonian $H(\alpha)$ in the dipole approximation is then given by

$$H(\alpha) = \frac{1}{2m} (p \otimes \mathbb{1} - \alpha \mathbb{1} \otimes A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}, \tag{2.5}$$

where $\alpha \in \mathbb{R}$ is the coupling constant, V is the external potential, and $p = (-i\partial_1, \ldots, -i\partial_d)$ is the momentum operator. For notational convenience we omit the tensor notation $\otimes$ in what follows.

Assumption 2.1: Suppose that V is relatively bounded with respect to $-\frac{1}{2m}\Delta$ with a relative bound strictly smaller than one, and

$$\hat{\varphi}/\omega \in L^2(\mathbb{R}^d), \quad \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^d). \tag{2.6}$$

By this assumption $H(\alpha)$ is self-adjoint on $D(-\Delta) \cap D(H_{\rm f})$ and bounded below for arbitrary $\alpha \in \mathbb{R}$.[1,2] We need, in addition, some technical assumptions on $\hat{\varphi}$ which are introduced in Definition 2.2 of Ref. 14. We list them as:

Assumption 2.2: The ultraviolet cutoff $\hat{\varphi}$ satisfies (1)-(4) *below.*

(1) $\hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^d)$;
(2) $\hat{\varphi}$ *is rotation invariant, i.e.,* $\hat{\varphi}(k) = \chi(|k|)$ *with some real-valued function* χ *on* $[0, \infty)$*; and* $\rho(s) = |\chi(\sqrt{s})|^2 s^{(d-2)/2} \in L^\epsilon([0, \infty), ds)$ *for some* $1 < \epsilon$*, and there exists* $0 < \beta < 1$ *such that* $|\rho(s+h) - \rho(s)| \le K|h|^\beta$ *for all* s *and* $0 < h \le 1$ *with some constant* K*;*
(3) $\|\hat{\varphi}\omega^{(d-1)/2}\|_\infty < \infty$*; and*
(4) $\hat{\varphi}(k) \neq 0$ *for* $k \neq 0$.

The Hamiltonian $H(\alpha)$ with $V = 0$ is quadratic and can, therefore, be diagonalized explicitly, which is carried out in Refs. 2 and 14. Assumption 2.2 ensures the existence of a unitary operator diagonalizing $H(\alpha)$.

Let

$$D_+(s) = m - \alpha^2 \frac{d-1}{d} \int \frac{|\hat{\varphi}(k)|^2}{s - \omega(k)^2 + i0} dk, \quad s \ge 0.$$

We see that $D_+(0) = m + \alpha^2 \frac{d-1}{d} \|\hat{\varphi}/\omega\|^2 > 0$ and the imaginary part of $D_+(s)$ is $\alpha^2 \frac{d-1}{d} \pi S_{d-1} \rho(s) \neq 0$ for $s \neq 0$, where ρ is defined in (2) of Assumption 2.2 and S_{d-1} is the volume of the $(d-1)$-dimensional unit sphere, and the real part of $D_+(s)$ satisfies that $\lim_{s\to\infty} \Re D_+(s) = m > 0$. These properties follows from Assumption 2.2. In particular,

$$\inf_{s\ge 0} |D_+(s)| > 0. \tag{2.7}$$

Define

$$\Lambda_j^\mu(k) = \frac{e_j^\mu(k)\hat{\varphi}(k)}{\omega^{3/2}(k) D_+(\omega^2(k))}. \tag{2.8}$$

Then $\|\Lambda_j^\mu\| \le C \|\hat{\varphi}/\omega^{3/2}\|$ for some constant C.

Proposition 2.3: Under the Assumptions 2.1 and 2.2, for each $\alpha \in \mathbb{R}$, there exist unitary operators U and T on $\mathscr{H}$ such that both map $D(-\Delta) \cap D(H_{\rm f})$ onto itself and

$$U^{-1} H(\alpha) U = -\frac{1}{2m_{\rm eff}(\alpha)} \Delta + H_{\rm f} + T^{-1} V T + g, \tag{2.9}$$

where $m_{\rm eff}(\alpha)$ and g are constants given by

$$m_{\rm eff}(\alpha) = m + \alpha^2 \left(\frac{d-1}{d}\right) \|\hat{\varphi}/\omega\|^2, \tag{2.10}$$

$$g = \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{t^2 \alpha^2 \left(\frac{d-1}{d}\right) \|\hat{\varphi}/(t^2 + \omega^2)\|^2}{m + \alpha^2 \left(\frac{d-1}{d}\right) \|\hat{\varphi}/\sqrt{t^2 + \omega^2}\|^2} dt. \tag{2.11}$$

Here, U is defined in (4.29) *of Ref.* 14 *and T by*

$$T = \exp\left(-i \frac{\alpha}{m_{\rm eff}(\alpha)} p \cdot \phi\right), \tag{2.12}$$

where $\phi = (\phi_1, \ldots, \phi_d)$ *is the vector field*

$$\phi_\mu = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int \left(\overline{\Lambda_j^\mu(k)} a^*(k, j) + \Lambda_j^\mu(k) a(k, j) \right) dk.$$

Proof: See Appendix of Ref. 14. □

III. THE BIRMAN-SCHWINGER PRINCIPLE

A. The case of Schrödinger operators

Let $h_0 = -\frac{1}{2}\Delta$. We assume that $V \in L^1_{\rm loc}(\mathbb{R}^d)$ and V is relatively form-bounded with respect to h_0 with relative bound $a < 1$, i.e., $D(|V|^{1/2}) \supset D(h_0^{1/2})$ and

$$||V|^{1/2}\varphi\|^2 \le a\|h_0^{1/2}\varphi\|^2 + b\|\varphi\|^2, \quad \varphi \in D(h_0^{1/2}), \tag{3.1}$$

with some $b > 0$. Then the operators

$$R_E = (h_0 - E)^{-1/2}\, |V|^{1/2}, \qquad E < 0, \tag{3.2}$$

are densely defined. From (3.1), it follows that $R_E^* = |V|^{1/2}(h_0 - E)^{-1/2}$ is bounded and thus R_E is closable. We denote its closure by the same symbol. Let

$$K_E = R_E^* R_E. \tag{3.3}$$

Then K_E ($E < 0$) is a bounded and positive self-adjoint operator and it holds

$$K_E f = |V|^{1/2}\,(h_0 - E)^{-1}\,|V|^{1/2} f, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Now let us consider the case $E = 0$. Let

$$R_0 = h_0^{-1/2}|V|^{1/2}. \tag{3.4}$$

The self-adjoint operator $h_0^{-1/2}$ has the integral kernel

$$h_0^{-1/2}(x, y) = \frac{a_d}{|x - y|^{d-1}}, \quad d \ge 3,$$

where $a_d = \sqrt{2}\pi^{(d-1)/2}/\,\Gamma((d-1)/2)$ and $\Gamma(\cdot)$ the Gamma function. It holds that

$$\left|(h_0^{-1/2} g, |V|^{1/2} f)\right| \le a_d \|g\|_2 \||V|^{1/2} f\|_{2d/(d+2)}$$

for $f, g \in C_0^\infty(\mathbb{R}^3)$ by the Hardy-Littlewood-Sobolev inequality. Since $f \in C_0^\infty(\mathbb{R}^3)$ and $V \in L^1_{\rm loc}(\mathbb{R}^3)$, one concludes $\||V|^{1/2} f\|_{2d/(d+2)} < \infty$. Thus, $|V|^{1/2} f \in D(h_0^{-1/2})$ and R_0 is densely defined. Since V is relatively form-bounded with respect to h_0, R_0^* is also densely defined, and R_0 is closable. We denote the closure by the same symbol. We define

$$K_0 = R_0^* R_0. \tag{3.5}$$

Next let us introduce assumptions on the external potential V.

Assumption 3.1: V satisfies that (1) $V \le 0$ *and* (2) R_0 *is compact.*

Lemma 3.2: Suppose Assumption 3.1. *Then*

(i) R_E, R_E^*, *and* K_E ($E \le 0$) *are compact.*
(ii) $\|K_E\|$ *is continuous and monotonously increasing in* $E \le 0$ *and it holds that*

$$\lim_{E \to -\infty} \|K_E\| = 0, \quad \lim_{E \uparrow 0} \|K_E\| = \|K_0\|. \tag{3.6}$$

Proof: Under (2) of Assumption 3.1, R_0^* and K_0 are compact. Since

$$(f, K_E f) \le (f, K_0 f), \quad f \in C_0^\infty(\mathbb{R}^d), \tag{3.7}$$

extends to $f \in L^2(\mathbb{R}^3)$, K_E, R_E, and R_E^* are also compact. Thus (i) is proven.

We will prove (ii). It is clear from (3.7) that K_E is monotonously increasing in E. Since R_0 is bounded, (3.7) holds on $L^2(\mathbb{R}^d)$ and

$$K_E = R_0^* \left((h_0 - E)^{-1} h_0\right) R_0, \quad E \le 0. \tag{3.8}$$

From (3.8) one concludes that

$$\|K_E - K_{E'}\| \le \|K_0\| \frac{|E - E'|}{|E'|}$$

for $E, E' < 0$. Hence, $\|K_E\|$ is continuous in $E < 0$. We have to prove the left continuity at $E = 0$. Since $\|K_E\| \le \|K_0\|$ $(E < 0)$, one has $\limsup_{E\uparrow 0} \|K_E\| \le \|K_0\|$. By (3.8) we see that $K_0 = \text{s-}\lim_{E\uparrow 0} K_E$ and

$$\|K_0 f\| = \lim_{E\uparrow 0} \|K_E f\| \le \left(\liminf_{E\uparrow 0} \|K_E\|\right) \|f\|, \quad f \in L^2(\mathbb{R}^d).$$

Hence, we have $\|K_0\| \le \liminf_{E\uparrow 0} \|K_E\|$ and $\lim_{E\uparrow 0} \|K_E\| = \|K_0\|$. It remains to prove that $\lim_{E\to -\infty} \|K_E\| = 0$. Since R_0^* is compact, for any $\epsilon > 0$, there exists a finite rank operator $T_\epsilon = \sum_{k=1}^n (\varphi_k, \cdot)\psi_k$ such that $n = n(\epsilon) < \infty$, $\varphi_k, \psi_k \in L^2(\mathbb{R}^d)$ and $\|R_0^* - T_\epsilon\| < \epsilon$. Then it holds that $\|K_E\| \le \left(\epsilon + \|T_\epsilon h_0 (h_0 - E)^{-1}\|\right) \|R_0\|$. For any $f \in L^2(\mathbb{R}^d)$, we have

$$\|T_\epsilon h_0 (h_0 - E)^{-1} f\| \le \left(\sum_{k=1}^n \|h_0 (h_0 - E)^{-1} \varphi_k\| \|\psi_k\|\right) \|f\|$$

and $\lim_{E\to -\infty} \|T_\epsilon h_0 (h_0 - E)^{-1}\| = 0$, which completes (ii). □

Let

$$H_\mathrm{p}(m) = -\frac{1}{2m}\Delta + V. \tag{3.9}$$

By (ii) of Lemma 3.2, we have $\lim_{E\to -\infty} \||V|^{1/2}(h_0 - E)^{-1/2}\| = 0$. Therefore, V is infinitesimally form bounded with respect to h_0 and $H_\mathrm{p}(m)$ is the self-adjoint operator associated with the quadratic form

$$f, g \mapsto \frac{1}{m}(h_0^{1/2} f, h_0^{1/2} g) + (|V|^{1/2} f, |V|^{1/2} g)$$

for $f, g \in D(h_0^{1/2})$. Note that the domain $D(H_\mathrm{p}(m))$ is independent of m.

Under (2) of Assumption 3.1, the essential spectrum of $H_\mathrm{p}(m)$ coincides with that of $-\frac{1}{2m}\Delta$, hence $\sigma_\mathrm{ess}(H_\mathrm{p}(m)) = [0, \infty)$. Next we will estimate the spectrum of $H_\mathrm{p}(m)$ contained in $(-\infty, 0]$. Let $\mathbb{1}_{(\mathcal{O})}(T)$, $\mathcal{O} \subset \mathbb{R}$, be the spectral resolution of self-adjoint operator T and set

$$N_\mathcal{O}(T) = \dim \operatorname{Ran} \mathbb{1}_\mathcal{O}(T). \tag{3.10}$$

The Birman-Schwinger principle[19] states that

$$\begin{aligned} &(E < 0)\ N_{(-\infty, \frac{E}{m}]}\left(H_\mathrm{p}(m)\right) = N_{[\frac{1}{m}, \infty)}(K_E), \\ &(E = 0)\ N_{(-\infty, 0]}\left(H_\mathrm{p}(m)\right) \le N_{[\frac{1}{m}, \infty)}(K_0). \end{aligned} \tag{3.11}$$

Now, let us define the constant m_c by the inverse of the operator norm of K_0,

$$m_\mathrm{c} = \|K_0\|^{-1}. \tag{3.12}$$

Lemma 3.3: Suppose Assumption 3.1.

(1) *If* $m < m_c$, *then* $N_{(-\infty,0]}(H_p(m)) = 0$.
(2) *If* $m > m_c$, *then* $N_{(-\infty,0]}(H_p(m)) \geq 1$.

Proof: It is immediate to see (1) by the Birman-Schwinger principle (3.11). Suppose $m > m_c$. Then, using the continuity and monotonicity of $E \to \|K\|$, see Lemma 3.2, there exists $\epsilon > 0$ such that $m_c < \|K_{-\epsilon}\|^{-1} \leq m$. Since $K_{-\epsilon}$ is positive and compact, $\|K_{-\epsilon}\| \in \sigma_p(K_{-\epsilon})$ follows and hence $N_{[\frac{1}{m},\infty)}(K_{-\epsilon}) \geq 1$. Therefore, (2) follows again from the Birman-Schwinger principle. □

Remark 3.4: By Lemma 3.3, *the critical mass at zero coupling* $m_c(0) = m_c$.

In the case $m > m_c$, by the proof of Lemma 3.3 one concludes that the bottom of the spectrum of $H_p(m)$ is strictly negative. For $\epsilon > 0$, we set

$$m_\epsilon = \|K_{-\epsilon}\|^{-1}. \tag{3.13}$$

Corollary 3.5: Suppose Assumption 3.1 *and* $m > m_\epsilon$. *Then*

$$\inf \sigma\left(H_p(m)\right) \leq \frac{-\epsilon}{m}. \tag{3.14}$$

Proof: The Birman-Schwinger principle states that $1 \leq N_{(-\infty,-\frac{\epsilon}{m}]}\left(H_p(m)\right)$, since $1/m < \|K_{-\epsilon}\|$, which implies the corollary. □

B. The case of the Pauli-Fierz model

In this subsection we extend the Birman-Schwinger type estimate to the Pauli-Fierz Hamiltonian.

Lemma 3.6: Suppose Assumption 3.1. *If* $m < m_c$, *then the zero coupling Hamiltonian* $H_p(m) + H_f$ *has no ground state.*

Proof: Since the Fock vacuum Ω is the ground state of H_f, $H_p(m) + H_f$ has a ground state if and only if $H_p(m)$ has a ground state. But $H_p(m)$ has no ground state by Lemma 3.3. Therefore, $H_p(m) + H_f$ has no ground state. □

From now on we discuss $U^{-1}H(\alpha)U$ with $\alpha \neq 0$. We set

$$U^{-1}H(\alpha)U = H_0(\alpha) + W + g, \tag{3.15}$$

where

$$\begin{aligned} H_0(\alpha) &= -\frac{1}{2m_{\mathrm{eff}}(\alpha)}\Delta + H_f, \\ W &= T^{-1}VT. \end{aligned} \tag{3.16}$$

Theorem 3.7: *Suppose Assumptions* 2.1, 2.2, *and* 3.1. *If* $m_{\mathrm{eff}}(\alpha) < m_c$, *then* $H_0(\alpha) + W + g$ *has no ground state.*

Proof: Since g is a constant, we prove the absence of ground state of $H_0(\alpha) + W$. Since V is negative, so is W. Hence $\inf \sigma(H_0(\alpha) + W) \leq \inf \sigma(H_0(\alpha)) = 0$. Then, it suffices to show that $H_0(\alpha) + W$ has no eigenvalues in $(-\infty, 0]$. Let $E \in (-\infty, 0]$ and set

$$\mathcal{K}_E = |W|^{1/2}(H_0(\alpha) - E)^{-1}|W|^{1/2}, \tag{3.17}$$

where $|W|^{1/2}$ is defined by the functional calculus. We shall prove now that if $H_0(\alpha) + W$ has eigenvalue $E \in (-\infty, 0]$, then $\mathcal{K}_E$ has eigenvalue 1. Suppose that $(H_0(\alpha) + W - E)\varphi = 0$ and $\varphi \neq 0$, then

$$\mathcal{K}_E|W|^{1/2}\varphi = |W|^{1/2}\varphi.$$

Moreover if $|W|^{1/2}\varphi = 0$, then $W\varphi = 0$ and hence $(H_0(\alpha) - E)\varphi = 0$, but $H_0(\alpha)$ has no eigenvalue by Lemma 3.6. Then $|W|^{1/2}\varphi \neq 0$ is concluded and $\mathcal{K}_E$ has eigenvalue 1. Then, it is sufficient to see $\|\mathcal{K}_E\| < 1$ to show that $H_0(\alpha) + W$ has no eigenvalues in $(-\infty, 0]$. Notice that $-\frac{1}{2m_{\rm eff}(\alpha)}\Delta$ and T commute, and

$$\left\|(-\Delta)^{1/2}\,(H_0(\alpha) - E)^{-1}\,(-\Delta)^{1/2}\right\| \leq 2m_{\rm eff}(\alpha).$$

Then, we have

$$\|\mathcal{K}_E\| \leq \left\||V|^{1/2}\left(-\frac{1}{2m_{\rm eff}(\alpha)}\Delta\right)^{-1/2}\right\|^2 = m_{\rm eff}(\alpha)\|K_0\| = \frac{m_{\rm eff}(\alpha)}{m_{\rm c}} < 1$$

and the proof is complete. □

IV. ABSENCE AND EXISTENCE OF A GROUND STATE

In this section we establish the absence, respectively, existence, of a ground state of the Pauli-Fierz Hamiltonian $H_0(\alpha) + W$. In order to construct examples of the Pauli-Fierz mode having such properties, we introduce a parameter $\kappa > 0$. As is seen below, it is a dummy and inessential. Let us define the Pauli-Fierz Hamiltonian with scaled external potential $V_\kappa(x) = V(x/\kappa)/\kappa^2$ by

$$H_\kappa = \frac{1}{2m}(p - \alpha A)^2 + V_\kappa + H_{\rm f}. \tag{4.1}$$

We also define K_κ by $H(\alpha)$ with $a^\sharp$ replaced by $\kappa a^\sharp$. Then

$$K_\kappa = \frac{1}{2m}(p - \kappa\alpha A)^2 + V + \kappa^2 H_{\rm f}. \tag{4.2}$$

H_κ and $\kappa^{-2}K_\kappa$ are unitarily equivalent,

$$H_\kappa \cong \kappa^{-2}K_\kappa. \tag{4.3}$$

Let $m < m_{\rm c}$ and $\epsilon > 0$. We define the function

$$\alpha_\epsilon = \left(\frac{d-1}{d}\|\hat{\varphi}/\omega\|^2\right)^{-1/2}\sqrt{m_\epsilon - m}, \quad \epsilon > 0 \tag{4.4}$$

$$\alpha_0 = \left(\frac{d-1}{d}\|\hat{\varphi}/\omega\|^2\right)^{-1/2}\sqrt{m_{\rm c} - m}, \tag{4.5}$$

where we recall that $m_\epsilon = \|K_{-\epsilon}\|^{-1}$ for $\epsilon \geq 0$.

In this section we prove that there exists $\kappa_\epsilon > 0$ such that for a fixed $\kappa > \kappa_\epsilon$, H_κ has a ground state for $\alpha > \alpha_\epsilon$ but no ground state for $\alpha < \alpha_0$. Thus, the external potential $\tilde{V}(x) = V_\kappa(x)$ gives an example of the existence and absence of ground state according to the value of parameter α.

Note that:

(1) $\alpha < \alpha_0$ if and only if $m_{\rm eff}(\alpha) < m_{\rm c}$;
(2) $\alpha > \alpha_\epsilon$ if and only if $m_{\rm eff}(\alpha) > m_\epsilon$.

Note that $\alpha_0 < \alpha_\epsilon$ because of $m_\epsilon > m_{\rm c}$. Since $\lim_{\epsilon\downarrow 0} m_\epsilon = m_{\rm c}$, it holds that $\lim_{\epsilon\downarrow 0}\alpha_\epsilon = \alpha_0$. We furthermore introduce assumptions on the external potential V and ultraviolet cutoff $\hat{\varphi}$.

Assumption 4.1: The external potential V and the ultraviolet cutoff $\hat{\varphi}$ satisfies:

(1) $V \in C^1(\mathbb{R}^d)$ *and* $\nabla V \in L^\infty(\mathbb{R}^d)$;
(2) $\hat{\varphi}/\omega^{5/2} \in L^2(\mathbb{R}^d)$.

Before going to the proof of the main theorem we will see the outline of our strategy developed in Ref. 14. We know that

$$H_0(\alpha)+W=-\frac{1}{2m_{\rm eff}(\alpha)}\Delta+V+H_{\rm f}+V\left(\cdot-\frac{\alpha}{m_{\rm eff}(\alpha)}\phi\right)-V.$$

The term on the right-hand side of the above expression,

$$H_{\rm int}=V\left(\cdot-\frac{\alpha}{m_{\rm eff}(\alpha)}\phi\right)-V, \tag{4.6}$$

is considered as the interaction, and

$$H_{\rm int}\sim\frac{\alpha}{m_{\rm eff}(\alpha)}\nabla V(\cdot)\cdot\phi. \tag{4.7}$$

Since $-\frac{1}{2m_{\rm eff}(\alpha)}\Delta+V$ has a ground state for sufficiently large α, we can show that $H_0(\alpha)+W$ with $\omega(k)$ replaced by $\omega(k)+\nu$ also has a ground state Ψ_ν for sufficiently large α. Here, $\nu>0$ denotes an artificial mass. In order to show the existence of a ground state for $\nu=0$, it is enough to show that the *non-zero* weak limit of Ψ_ν as $\nu\to0$ exists. A standard strategy is to show that

$$\|N^{1/2}\Psi_\nu\|\le c\|\Psi_\nu\| \tag{4.8}$$

uniformly in ν with some $c<1$, where $N=\sum_{j=1}^{d-1}\int a^*(k,j)a(k,j)dk$ is the number operator. By (4.7) we formally have the pull-through formula

$$\|a(k,j)\Psi_\nu\|\sim\frac{\alpha}{m_{\rm eff}(\alpha)}\left\|(H_0(\alpha)+W+\omega(k)-E)^{-1}(\nabla\cdot V)[\phi,a(k,j)]\Psi_\nu\right\|, \tag{4.9}$$

where $E=\inf\sigma(H_0(\alpha)+W)$. We introduced a scaling parameter κ and V was replaced by V_κ in (4.9). Hence, from (4.9) it follows that:

$$(\Psi_\nu,N\Psi_\nu)\le C\frac{1}{\kappa^6}\left(\sup_\alpha\left|\frac{\alpha}{m_{\rm eff}(\alpha)}\right|\right)^2\|\nabla V\|_\infty^2\|\Psi/\omega^{5/2}\|^2\|\Psi_\nu\|^2 \tag{4.10}$$

and for sufficiently large κ we can obtain (4.8). (4.10) is rigorously proven in Ref. 14. Then we conclude that H_κ with a sufficiently large κ has a ground state for $\alpha>\alpha_\epsilon$. This is outlined in the proof of (1) of Theorem 4.2 below.

On the other hand, we note that $\|K_0\|$ is invariant with respect to the scaling; $\||V_\kappa|^{1/2}(-\Delta)^{-1}|V_\kappa|^{1/2}\|=\||V|^{1/2}(-\Delta)^{-1}|V|^{1/2}\|$. Thus, the absence of ground state of H_κ for all κ and $\alpha<\alpha_0$ follows from Theorem 3.7.

Now we are in the position to state the main theorem.

Theorem 4.2: *Suppose Assumptions* 2.1, 2.2, 3.1, *and* 4.1. *Then* (1) *and* (2) *below hold.*

(1) *For any $\epsilon>0$, there exists κ_ϵ such that for all $\kappa>\kappa_\epsilon$, H_κ has a unique ground state for all α such that $\alpha>\alpha_\epsilon$,*
(2) *H_κ has no ground state for all $\kappa>0$ and all α such that $\alpha<\alpha_0$.*

Proof: Let U_κ (respectively T_κ) be defined by U (respectively T) with ω and $\hat\varphi$ replaced by $\kappa^2\omega$ and $\kappa\hat\varphi$. Then

$$U_\kappa^{-1}K_\kappa U_\kappa=H_{\rm p}(m_{\rm eff}(\alpha))+\kappa^2H_{\rm f}+\delta V_\kappa+\kappa^2g, \tag{4.11}$$

where $\delta V_\kappa=T_\kappa^{-1}VT_\kappa-V$. Since $U_\kappa^{-1}K_\kappa U_\kappa$ is unitary equivalent to κ^2H_κ, we prove the existence of a ground state for $U_\kappa^{-1}K_\kappa U_\kappa$. Since $H_{\rm p}(m_{\rm eff}(\alpha))$ has a ground state by the assumption $\alpha>\alpha_\epsilon$, i.e., $m_{\rm eff}(\alpha)>m_{\rm c}$, it can be shown that $U_\kappa^{-1}K_\kappa U_\kappa+\nu N$ with $\nu>0$ also has a ground state, see p.1168 of Ref. 14 for details. We denote its normalized ground state by $\Psi_\nu=\Psi_\nu(\kappa)$. Since the unit ball in a Hilbert space is weakly compact, there exists a subsequence of Ψ_ν such that the weak limit $\Psi=\lim_{\nu'\to0}\Psi_{\nu'}$ exists. If $\Psi\neq0$, then Ψ is a ground state.[3] Let

$$P=1\!\!1_{[\Sigma,\Sigma+a)}(H_{\rm p}(m_{\rm eff}(\alpha)))\otimes1\!\!1_{\{0\}}(H_{\rm f}) \tag{4.12}$$

and $\Sigma = \inf\sigma(H_{\rm p}(m_{\rm eff}(\alpha))) < 0$ be the bottom of the spectrum of $H_{\rm p}(m_{\rm eff}(\alpha))$. Let $a > 0$ be sufficiently small. Then P is a finite rank operator. We set $Q = \mathbb{1}_{[\Sigma+a,\infty)}(H_{\rm p}(m_{\rm eff}(\alpha))) \otimes \mathbb{1}_{\{0\}}(H_{\rm f})$. By the inequality $P \geq \mathbb{1} - N - Q$ in the sense, we have

$$(\Psi_\nu, P\Psi_\nu) \geq 1 - (\Psi_\nu, N\Psi_\nu) - (\Psi_\nu, Q\Psi_\nu). \tag{4.13}$$

Adopting the arguments in the proof of Lemma 3.3 of Ref. 14, we can estimate (4.13) and conclude

$$(\Psi, P\Psi) \geq 1 - \left(\frac{\alpha}{m_{\rm eff}(\alpha)} \frac{C\|\hat\varphi/\omega^{5/2}\|}{\kappa^3}\right)^2 - \frac{D/\kappa}{|\Sigma| - a - D/\kappa}, \tag{4.14}$$

where $D = C' \frac{\alpha}{m_{\rm eff}(\alpha)}(2\|\hat\varphi/\omega^2\|/\kappa + \|\hat\varphi/\omega^{3/2}\|)$, C, and C' are positive constants independent of κ and α. Since $m_{\rm eff}(\alpha) > m_\epsilon > m_{\epsilon/2}$,

$$\Sigma \leq \inf\sigma(H_{\rm p}(m_\epsilon)) \leq -\frac{\epsilon}{2m_\epsilon} \tag{4.15}$$

by Corollary 3.5. Since

$$\gamma = \sup_\alpha \frac{\alpha}{m_{\rm eff}(\alpha)} = (2\sqrt{m}\|\hat\varphi/\omega\|)^{-1}, \tag{4.16}$$

there exists κ_ϵ such that for all $\kappa > \kappa_\epsilon$ and all $\alpha \in \mathbb{R}$,

$$(\Psi, P\Psi) \geq 1 - \left(\frac{A}{\kappa^3}\right)^2 - \frac{D'}{\kappa B - D'} > 0, \tag{4.17}$$

where $A = \gamma C\|\hat\varphi/\omega^{5/2}\|$, $B = \frac{\epsilon}{2m_\epsilon} - a > 0$, and $D' = C'\gamma(2\|\hat\varphi/\omega^2\|/\kappa + \|\hat\varphi/\omega^{3/2}\|)$. Then $\Psi \neq 0$ for all $\kappa > \kappa_\epsilon$. Thus, the ground state exists for all $\alpha > \alpha_\epsilon$ and all $\kappa > \kappa_\epsilon$ and (1) is complete.

We next show (2). Notice that

$$U_\kappa^{-1} H_\kappa U_\kappa = -\frac{1}{2m_{\rm eff}(\alpha)}\Delta + H_{\rm f} + T^{-1}V_\kappa T + g.$$

Define the unitary operator u_κ by $(u_\kappa f)(x) = k^{d/2} f(x/\kappa)$. Then, we infer $V_\kappa = \kappa^{-2} u_\kappa V u_\kappa^{-1}$, $-\Delta = \kappa^{-2} u_\kappa(-\Delta)u_\kappa^{-1}$, and

$$\||V_\kappa|^{1/2}(-\Delta)^{-1}|V_\kappa|^{1/2}\| = \kappa^{-2}\|u_\kappa|V|^{1/2}u_\kappa^{-1}(-\Delta)^{-1}u_\kappa|V|^{1/2}u_\kappa^{-1}\| = \|K_0\|.$$

(2) follows from Theorem 3.7. □

Corollary 4.3: Let arbitrary $\delta > 0$ be given. Then there exists an external potential $\tilde V$ and constants $\alpha_+ > \alpha_-$ such that

(1) $0 < \alpha_+ - \alpha_- < \delta$;
(2) *$H(\alpha)$ has a ground state for $\alpha > \alpha_+$ but no ground state for $\alpha < \alpha_-$.*

Proof: Suppose that V satisfies Assumption 3.1. For $\delta > 0$, we take $\epsilon > 0$ such that $\alpha_\epsilon - \alpha_0 < \delta$. Take a sufficiently large κ such that (4.17) is fulfilled, and set $\tilde V(x) = V(x/\kappa)/\kappa^2$. Define $H(\alpha)$ by the Pauli-Fierz Hamiltonian with potential $\tilde V$. Then $H(\alpha)$ satisfies (1) and (2) with $\alpha_+ = \alpha_\epsilon$ and $\alpha_- = \alpha_0$. □

Remark 4.4 (Upper and lower bound of $m_{\rm c}(\alpha)$)*:* Corollary 4.3 implies the upper and lower bounds

$$\begin{aligned} m_-(\alpha) &\leq m_{\rm c}(\alpha) \leq m_+(\alpha), \\ m_{\rm c}(0) &= m_{\rm c}, \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} m_-(\alpha) &= m_0 - \alpha^2 \tfrac{d-1}{d}\|\hat\varphi/\omega\|^2, \\ m_+(\alpha) &= m_\epsilon - \alpha^2 \tfrac{d-1}{d}\|\hat\varphi/\omega\|^2. \end{aligned}$$

Fix the coupling constant α. If $m < m_-(\alpha)$, then there is no ground state and if $m > m_+(\alpha)$, then the ground state exists, compare with Fig. 1.

Remark 4.5 ($m_c(\alpha)$ for sufficiently large α)*:* Let $(\frac{d-1}{d}\|\hat{\varphi}/\omega\|^2)^{-1}m_\epsilon < \alpha^2$. Then by Remark 4.4, $H(\alpha)$ has a ground state for arbitrary $m > 0$. It is an open problem to establish whether this is an artifact of the dipole approximation or in fact holds also for the Pauli-Fierz operator.

V. EXAMPLES OF EXTERNAL POTENTIALS

In this section we give examples of potentials V satisfying Assumption 3.1. The self-adjoint operator h_0^{-1} has the integral kernel

$$h_0^{-1}(x, y) = \frac{b_d}{|x-y|^{d-2}}, \quad d \geq 3,$$

with $b_d = 2\Gamma((d/2)-1)/\pi^{(d/2)-2}$. It holds that

$$(f, K_0 f) = \int dx \int dy \overline{f(x)} K_0(x, y) f(y), \tag{5.1}$$

where

$$K_0(x, y) = b_d \frac{|V(x)|^{1/2}|V(y)|^{1/2}}{|x-y|^{d-2}}, \quad d \geq 3, \tag{5.2}$$

is the integral kernel of operator K_0. We recall the Rollnik class $\mathscr{R}$ of potentials is defined by

$$\mathscr{R} = \left\{ V \Big| \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \frac{|V(x)V(y)|}{|x-y|^2} < \infty \right\}.$$

By the Hardy-Littlewood-Sobolev inequality, $\mathscr{R} \supset L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ with $1/p + 1/r = 4/3$. In particular, $L^{3/2}(\mathbb{R}^3) \subset \mathscr{R}$.

Example 5.1 ($d = 3$ and Rollnik class)*:* Let $d = 3$. Suppose that V is negative and $V \in \mathscr{R}$. Then $K_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Hence K_0 is Hilbert-Schmidt and Assumption 3.1 is satisfied.

The example can be extended to dimensions $d \geq 3$.

Example 5.2 ($d \geq 3$ and $V \in L^{d/2}(\mathbb{R}^d)$)*:* Let $L_w^p(\mathbb{R}^d)$ be the set of Lebesgue measurable function u such that $\sup_{\beta>0} \beta \left|\{x \in \mathbb{R}^d | |u(x) > \beta\}\right|_L^{1/p} < \infty$, where $|E|_L$ denotes the Lebesgue measure of $E \subset \mathbb{R}^d$. Let $g \in L^p(\mathbb{R}^d)$ and $u \in L_w^p(\mathbb{R}^d)$ for $2 < p < \infty$. Define the operator $B_{u,g}$ by

$$B_{u,g} h = (2\pi)^{-d/2} \int e^{ikx} u(k) g(x) h(x) dx.$$

It is shown in Theorem (p. 97) of Ref. 8, that $B_{u,g}$ is a compact operator on $L^2(\mathbb{R}^d)$. It is known that $u(k) = 2|k|^{-1} \in L_w^d(\mathbb{R}^d)$ for $d \geq 3$. Let F denote Fourier transform on $L^2(\mathbb{R}^d)$, and suppose that $V \in L^{d/2}(\mathbb{R}^d)$. Then $B_{u,|V|^{1/2}}$ is compact on $L^2(\mathbb{R}^d)$ and then $R_0^* = F B_{u,V^{1/2}} F^{-1}$ is compact. Thus, R_0 is also compact. (See 8, 15, and 18).

Assume that $V \in L^{d/2}(\mathbb{R}^d)$. Let us now see the critical mass of zero coupling $m_c = m_0$. By the Hardy-Littlewood-Sobolev inequality, we have

$$|(f, K_0 f)| \leq D_V \|f\|_2^2, \tag{5.3}$$

where

$$D_V = \sqrt{2}\pi \frac{\Gamma((d/2)-1)}{\Gamma((d/2)+1)} \left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{2/d} \|V\|_{d/2}^2, \tag{5.4}$$

a constant in (5.4) is proved by Lieb.[16] Then

$$\|K_0\| \leq D_V. \tag{5.5}$$

By (5.5) we have $m_c \geq D_V^{-1}$. In particular in the case of $d = 3$,

$$m_c \geq \frac{3}{\sqrt{2}\pi^{2/3} 4^{5/3}} \|V\|_{3/2}^{-2}. \tag{5.6}$$

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