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バージョン：

ON THE $SO(N)$ AND $Sp(N)$ FREE ENERGY OF A RATIONAL HOMOLOGY 3-SPHERE

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Dedicated to Professor Akio Kawauchi on the occasion of his 60th birthday

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Abstract. We give an explicit presentation of the $SO(N)$ and $Sp(N)$ free energy of lens spaces and show that the genus g terms of it are analytic in a neighborhood at zero, where we can choose the neighborhood independently of g. Moreover, we prove that for any rational homology 3-sphere M and any g, the genus g terms of $SO(N)$ and $Sp(N)$ free energy of M agree up to sign. We also observe new weight systems related to the free energy.

1. introduction

Let G_N be a compact Lie group parameterized by N such as $SU(N)$, $SO(N)$ or $Sp(N)$, and let $\mathfrak{g}_N$ be the Lie algebra of G_N. The LMO invariant $Z_M \in \mathcal{A}(\emptyset)$ [6] of a closed 3-manifold M is presented by a linear sum of (a kind of) trivalent graphs, where $\mathcal{A}(\emptyset)$ denotes the $\mathbb{Q}$ vector space spanned by such trivalent graphs (subject to some relations). The $\mathfrak{g}_N$ weight system $W_{\mathfrak{g}_N}$ is a map $\mathcal{A}(\emptyset) \to \mathbb{Q}[[h]]$ which "substitutes" $\mathfrak{g}_N$ to trivalent graphs, such that $W_{\mathfrak{g}_N}(D)$ of a trivalent graph D of degree d is defined to be h^d times some polynomial in N of degree $\leq d+2$. When we fix a value of N, $W_{\mathfrak{g}_N}(\log Z_M)$ is a power series in h with $\mathbb{Q}$ coefficients, which presents the perturbative expansion of the path integral of the Chern-Simons theory on the trivial G_N bundle over M. When we regard N as a variable, the weight system can be regarded as a map $W_{\mathfrak{g}_\star} : \mathcal{A}(\emptyset) \to \mathbb{Q}[N][[h]]$, and $W_{\mathfrak{g}_\star}(\log Z_M)$ is a power series in h whose coefficients are polynomials in N. Putting τ to be Nh if $G_N = SU(N)$, $(N-1)h$ if $G_N = SO(N)$, and $(N+1)h$ if $G_N = Sp(N)$, $W_{\mathfrak{g}_\star}(\log Z_M)$ is a power series in τ and h. We denote it by $F_M^{G_N}(\tau, h) \in h^{-2}\mathbb{Q}[[\tau, h]]$, and call it the G_N *free energy* of M [5]. Further, we put the coefficient of h^{g-2} in $F_M^{G_N}(\tau, h)$ to be $F_{M,g}^{G_N}(\tau) \in \mathbb{Q}[[\tau]]$, *i.e.*,

$$F_M^{G_N}(\tau, h) = \sum_{g=0}^{\infty} h^{g-2} F_{M,g}^{G_N}(\tau),$$

where the value of g implies the genus of some surface appearing in the definition of the weight system.

Recently, in [4], S. Garoufalidis, T.T.Q. Le and M. Mariño proved that the power series $F_{M,g}^{SU(N)}(\tau)$ of a closed oriented 3-manifold M for any g is analytic in a neighborhood of zero, where the neighborhood is independent of g, and gave an explicit presentation of the $SU(N)$ free energy for lens spaces to illustrate the analyticity. Further, S. Sinha and C. Vafa [8] gave a formula of the $SO(N)$ and $Sp(N)$ free energy of S^3 from Chern-Simons gauge theory.

In this paper, when $G_N = SO(N)$ and $Sp(N)$, we give an explicit presentation of the G_N free energy for lens spaces, and show that $F^{G_N}_{L(d,b),g}(\tau)$ of the lens space $L(d,b)$ is analytic in a neighborhood of zero, where we can choose the neighborhood independently of g (Theorem 2). This analyticity has been conjectured by Le, Garoufalidis and Mariño [4]. Moreover, we show that for any g, the genus g terms of $SO(N)$ and $Sp(N)$ free energy agree up to sign (Corollary 2) and observe new weight systems related to the G_N free energy (Section 4).

An idea of the proof of Theorem 2 is to use a presentation of $F^{G_N}_{L(d,b),g}(\tau)$ given in [4], which is a presentation in terms of the sum of some function of h over positive roots of $\mathfrak{g}_N$. We calculate this sum concretely when $G_N = SO(N)$ and $Sp(N)$, to present $F^{G_N}_{L(d,b),g}(\tau)$ by a function of τ and h.

The paper is organized as follows. In Section 2, we review the definition of the G_N free energy and results on the $SU(N)$ free energy for lens spaces obtained by Garoufalidis, Le and Mariño. In Section 3, we present an explicit presentation of the $SO(N)$ and $Sp(N)$ free energy for lens spaces and study these analyticity. We also show that the genus g terms of $SO(N)$ and $Sp(N)$ free energy for a rational homology 3-sphere agree up to sign. In Section 4, we recall properties of the $\mathfrak{sl}_N$ and $\mathfrak{so}_N$ weight systems and observe new weight systems related to the free energy. In Section 5, we prove a relation between the $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ weight systems.

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2. Preliminaries

In this section, we review the definition of the free energy and some results about the $SU(N)$ free energy of lens spaces in [4].

We briefly review the LMO invariant Z_M of a closed oriented 3-manifold M, constructed by T.T.Q. Le, J. Murakami and T. Ohtsuki in [6]. We denote by $\mathcal{A}(\emptyset)$ the vector space over $\mathbb{Q}$ spanned by trivalent graphs whose vertices are oriented, modulo the AS, IHX and STU relations and denote by $\mathcal{A}(\emptyset)_{\mathrm{conn}}$ the subspace of $\mathcal{A}(\emptyset)$ spanned by connected trivalent graphs. The degree of a trivalent graph is half the number of vertices. The LMO invariant Z_M takes values in $\mathcal{A}(\emptyset)$. It is known that $\log Z_M$ takes values in $\mathcal{A}(\emptyset)_{\mathrm{conn}}$.

Let us recall the weight system associated with a semi-simple Lie algebra $\mathfrak{g}$. It is known that for a semi-simple Lie algebra $\mathfrak{g}$, one obtains a $\mathbb{Q}$ linear map $W_{\mathfrak{g}} : \mathcal{A}(\emptyset) \to \mathbb{Q}[[h]]$, called the weight system associated with $\mathfrak{g}$ (for general references, see [2, 7]). From a trivalent graph D of degree d in $\mathcal{A}(\emptyset)$, $W_{\mathfrak{g}}(D)$ is obtained by substituting $\mathfrak{g}$ into D, contracting a tensor at vertices and multiplying by h^d. When $\mathfrak{g} = \mathfrak{g}_N = \mathfrak{sl}_N, \mathfrak{so}_N$ or $\mathfrak{sp}_N$, regarding N as a variable, $W_{\mathfrak{g}_N}(D)$ of a connected trivalent graph D of degree d is h^d times some polynomial in N of degree $\leq d+2$ by Lemma 1 below, and we regard the weight system $W_{\mathfrak{g}_N}$ as a map $W_{\mathfrak{g}_\star} : \mathcal{A}(\emptyset) \to \mathbb{Q}[N][[h]]$.

Lemma 1. *For $\mathfrak{g}_N = \mathfrak{sl}_N, \mathfrak{so}_N, \mathfrak{sp}_N$ and a connected trivalent graph D of degree d, $W_{\mathfrak{g}_N}(D)$ can be presented in the following form,*

$$W_{\mathfrak{g}_N}(D) = \sum_{0 \leq g \leq d+1} a_{\mathfrak{g}_N,g}(D) N^{d+2-g} h^d, \tag{1}$$

for some $a_{\mathfrak{g}_N,g}(D) \in \mathbb{Z}$.

We show a proof of the lemma in Section 4.

Let G_N be a simple compact Lie group $SU(N)$, $SO(N)$ or $Sp(N)$ and let $\mathfrak{g}_N$ be the Lie algebra of G_N. Putting τ to be Nh for $\mathfrak{g} = \mathfrak{sl}$, $(N-1)h$ for $\mathfrak{g} = \mathfrak{so}$, and $(N+1)h$ for $\mathfrak{g} = \mathfrak{sp}$,

$W_{\mathfrak{g}_\star}(D)$ has the following form,

$$W_{\mathfrak{g}_\star}(D) = \sum_{0\le g\le d+1} c_{\mathfrak{g},g}(D)\tau^{d+2-g}h^{g-2}, \tag{2}$$

for some $c_{\mathfrak{g},g}(D) \in \mathbb{Z}$. Since $\log Z_M \in A(\emptyset)_{conn}$, $W_{\mathfrak{g}_\star}(\log Z_M)$ can be presented in the following form,

$$W_{\mathfrak{g}_\star}(\log Z_M) = \sum_{d>0}\sum_{0\le g\le d+1} c_{\mathfrak{g},d,g}(M)\tau^{d+2-g}h^{g-2} \in h^{-2}\mathbb{Q}[[\tau,h]], \tag{3}$$

for some $c_{\mathfrak{g},d,g}(M) \in \mathbb{Q}$. As in [4], we define the G_N *free energy* of a rational homology 3-sphere M by

$$F_M^{G_N}(\tau,h) := W_{\mathfrak{g}_\star}(\log Z_M) \in h^{-2}\mathbb{Q}[[\tau,h]],$$

and put the coefficient of h^{g-2} in $F_M^{G_N}(\tau,h)$ to be $F_{M,g}^{G_N}(\tau) \in \mathbb{Q}[[\tau]]$, *i.e.*,

$$F_M^{G_N}(\tau,h) = \sum_{g=0}^{\infty} F_{M,g}^{G_N}(\tau)h^{g-2}.$$

Let us review a presentation of the G_N free energy of a lens space given in [4]. Let $L(d,b)$ be the lens space of type (d,b). It is shown in [4] that, for any semi-simple Lie algebra $\mathfrak{g}$,

$$W_{\mathfrak{g}_\star}(Z_{L(d,b)}) = \exp\left(\frac{\lambda_{L(d,b)}}{4}C_{\mathfrak{g}}\cdot\dim\mathfrak{g}\cdot h\right) d^{|\Psi_+|}\prod_{\alpha\in\Psi_+}\frac{\sinh((\alpha,\rho)h/(2d))}{\sinh((\alpha,\rho)h/2)}, \tag{4}$$

where λ_M denotes Casson-Walker invariant for a rational homology 3-sphere M, Ψ_+ denotes the set of positive roots of $\mathfrak{g}$, $|\Psi_+|$ denotes the number of positive roots, and $C_{\mathfrak{g}}$ is the quadratic Casimir of $\mathfrak{g}$. Since $F_{L(d,b)}^{G_N}(\tau,h) = W_{\mathfrak{g}_\star}(\log Z_{L(d,b)}) = \log W_{\mathfrak{g}_\star}(Z_{L(d,b)})$ by definition, we obtain the following proposition from (4).

Proposition 2 ([4, Proposition 6.1])**.**

$$F_{L(d,b)}^{G_N}(\tau,h) = \frac{\lambda_{L(d,b)}}{4}C_{\mathfrak{g}_N}\cdot\dim\mathfrak{g}_N\cdot h + \sum_{\alpha\in\Psi_+}\left(f((\alpha,\rho)h/d) - f((\alpha,\rho)h)\right), \tag{5}$$

where we define the function f by

$$f(x) := \log\left(\frac{\sinh(x/2)}{x/2}\right).$$

By a concrete computation of (5) in the case that $G_N = SU(N)$, Garoufalidis, Le, and Mariño gave an explicit presentation of the $SU(N)$ free energy of the lens space $L(d,b)$:

Theorem 3 ([4, Theorem 1.4])**.** *The $SU(N)$ free energy of the lens space $L(d,b)$ is presented by*

$$F_{L(d,b),g}^{SU(N)}(\tau) = \begin{cases} (g-1)\dfrac{B_g}{g!}(d^{2-g}\mathrm{Li}_{3-g}(e^{\tau/d}) - \mathrm{Li}_{3-g}(e^{\tau})) + a_g(\tau), & \text{if } g \text{ is even,}\\ 0 & \text{if } g \text{ is odd,}\end{cases}$$

where

$$a_g(\tau) = \begin{cases} -\dfrac{\tau^3}{12}(d^{-1}-1) - \dfrac{\pi^2\tau}{6}(d-1) + \dfrac{\tau^2}{2}\log d + (d^2-1)\zeta(3) + \lambda_{L(d,b)}\dfrac{\tau^3}{2} & \text{if } g=0,\\ -\dfrac{\tau}{24}(d^{-1}-1) + \dfrac{1}{12}\log d - \lambda_{L(d,b)}\dfrac{\tau}{2} & \text{if } g=2,\\ 0 & \text{if } g\ge 4.\end{cases}$$

Here the kth Bernoulli number B_k is defined by the generating series

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

and the polylogarithm function Li_p *is defined by*

$$\mathrm{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$$

for any integer p and $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$.

One sees that the power series $F^{SU(N)}_{L(d,b),g}(\tau)$ with even g are analytic in a common neighborhood at zero, independently of g. Moreover, it is proved in [4] that for any closed 3-manifold M, the power series $F^{SU(N)}_{M,g}(\tau)$ with even g is analytic in a neighborhood at zero, where the neighborhood can be chosen independently of g. They conjectured such analyticity of the $SO(N)$ and $Sp(N)$ free energy of a closed oriented 3-manifold M, which was discussed in [5].

In the next section, when $G_N = SO(N)$ or $Sp(N)$, by a concrete computation of the second term in the formula (5), we show their conjecture for lens spaces.

3. Results

In this section, we give an explicit presentation of the $SO(N)$ and $Sp(N)$ free energy for lens spaces and show that the genus g terms of $SO(N)$ and $Sp(N)$ free energy for a rational homology 3-sphere agree up to sign.

We have

Theorem 4. *The $SO(N)$ and $Sp(N)$ free energy of the lens space $L(d,b)$ is presented by*

$$F^{G_N}_{L(d,b),g}(\tau) = \begin{cases} \frac{1}{2}\{(g-1)\frac{B_g}{g!}(d^{2-g}\mathrm{Li}_{3-g}(e^{\tau/d}) - \mathrm{Li}_{3-g}(e^{\tau})) + a_g(\tau)\} & \text{if } g \text{ is even,} \\ \varepsilon_{G_N}\left[\frac{(2^{g-2}-1)B_{g-1}}{(g-1)!}\left\{d^{2-g}(2^{2-g}\mathrm{Li}_{3-g}(e^{\tau/2d}) - \frac{1}{2}\mathrm{Li}_{3-g}(e^{\tau/d})) \right.\right. & \\ \qquad \left.\left. -2^{2-g}\mathrm{Li}_{3-g}(e^{\tau/2}) + \frac{1}{2}\mathrm{Li}_{3-g}(e^{\tau})\right\} + a'_g(\tau)\right] & \text{if } g \text{ is odd,} \end{cases}$$

where ε_{G_N} is 1 for $G_N = SO(N)$ and -1 for $G_N = Sp(N)$,

$$a_g(\tau) = \begin{cases} -\frac{\tau^3}{12}(d^{-1}-1) - \frac{\pi^2\tau}{6}(d-1) + \frac{\tau^2}{2}\log d + (d^2-1)\zeta(3) + \lambda_{L(d,b)}\frac{\tau^3}{2} & \text{if } g = 0, \\ -\frac{\tau}{24}(d^{-1}-1) + \frac{1}{12}\log d - \lambda_{L(d,b)}\frac{\tau}{2} & \text{if } g = 2, \\ 0 & \text{if } g \geq 4, \end{cases}$$

$$a'_g(\tau) = \begin{cases} \frac{\tau}{2}\log d - \frac{\pi^2}{4}(d-1) & \text{if } g = 1, \\ 0 & \text{if } g \geq 3. \end{cases}$$

In particular, $F^{SO(N)}_{L(d,b),g}(\tau)$ and $F^{Sp(N)}_{L(d,b),g}(\tau)$ are analytic in a neighborhood at zero, where we can choose the neighborhood independently of g.

Proof. In the case that $G_N = SO(N)$ with even $N = 2n$, we show the required formula by calculating the right-hand side of (5) as follows.

The first term of the right-hand side of (5) is given by

$$\frac{\lambda_{L(d,b)}}{4} C_{\mathfrak{so}_N} \cdot \dim \mathfrak{so}_N \cdot h = \frac{\lambda_{L(d,b)}}{4} N(N-1)(N-2)h = \frac{\lambda_{L(d,b)}}{4} \left(\frac{\tau^3}{h^2} - \tau \right),$$

where $\tau = (N-1)h$.

We calculate the second term of the right-hand side of (5). For $j \in \mathbb{N}$, let $m(j)$ be the number of positive roots α such that $(\alpha, \rho) = j$. By definition,

$$\sum_{\alpha \in \Psi_+} f((\alpha, \rho)h) = \sum_{j \in \mathbb{N}} m(j) f(jh).$$

Further, by Lemma 5 below,

$$\begin{aligned} \sum_{\alpha \in \Psi_+} f((\alpha,\rho)h) &= \sum_{\substack{j:\text{odd} \\ 1 \le j \le n-1}} \frac{2n-j+1}{2} f(jh) + \sum_{\substack{j:\text{even} \\ 1 \le j \le n-1}} \frac{2n-j}{2} f(jh) \\ &\quad + \sum_{\substack{j:\text{odd} \\ n \le j \le 2n-3}} \frac{2n-j-1}{2} f(jh) + \sum_{\substack{j:\text{even} \\ n \le j \le 2n-3}} \frac{2n-j-2}{2} f(jh). \end{aligned}$$

From the definition of sinh, we have the following presentation of $f(x)$,

$$f(x) = \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} x^{2k}, \tag{6}$$

where B_k is the kth Bernoulli number. So, it follows that

$$\begin{aligned} &\sum_{\alpha \in \Psi_+} f((\alpha,\rho)h) \\ &= \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k} \left\{ \sum_{1 \le j \le 2n-2} \frac{2n-j-1}{2} j^{2k} + \sum_{1 \le j \le n-1} j^{2k} - \frac{1}{2} \sum_{\substack{j:\text{even} \\ 1 \le j \le 2n-2}} j^{2k} \right\} \\ &= \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k} \sum_{1 \le j \le 2n-2} \frac{2n-j-1}{2} j^{2k} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k} (1 - 2^{2k-1}) \sum_{1 \le j \le n-1} j^{2k}. \end{aligned} \tag{7}$$

By using $2n-1 = N-1 = \tau/h$, from the formulas (6.7), (6.8), and (6.10) in [4], the first term of (7) is presented by

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k} \sum_{1 \le j \le 2n-2} \frac{2n-j-1}{2} j^{2k} &= \frac{1}{2} \sum_{s=0}^{\infty} \frac{(1-2s)B_{2s} h^{2s-2}}{(2s)!} \sum_{l=0}^{\infty} \frac{B_{2l+2s}}{(2l+2)!(2l+2s)} \tau^{2l+2} \\ &= \frac{1}{2} \sum_{s=0}^{\infty} \frac{(1-2s)B_{2s} h^{2s-2}}{(2s)!} F_s^{even}(\tau), \end{aligned}$$

where we put

$$F_s^{even}(\tau) := \sum_{l=0}^{\infty} \frac{B_{2l+2s}}{(2l+2)!(2l+2s)} \tau^{2l+2}.$$

Using the formula

$$\sum_{j=1}^{n} j^{2k} = \frac{(n+\frac{1}{2})^{2k+1}}{2k+1} + \sum_{s=1}^{k} \frac{2^{1-2s}-1}{2k+1} \left(\begin{array}{c} 2k+1 \\ 2g \end{array} \right) B_{2s}(n+\frac{1}{2})^{2k+1-2s},$$

the second term of (7) is presented by

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k}(1-2^{2k-1}) \sum_{1\le j\le n-1} j^{2k}$$

$$= \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k}(1-2^{2k-1}) \left\{ \frac{(n-\frac{1}{2})^{2k+1}}{2k+1} + \sum_{s=1}^{k} \frac{2^{1-2s}-1}{2k+1} \left(\begin{array}{c} 2k+1 \\ 2s \end{array} \right) B_{2s}(n-\frac{1}{2})^{2k+1-2s} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k}(1-2^{2k-1})$$

$$\times \left\{ \frac{(2n-1)^{2k+1}}{2k+1} (\frac{1}{2})^{2k+1} + \sum_{s=1}^{k} \frac{2^{1-2s}-1}{2k+1} \left(\begin{array}{c} 2k+1 \\ 2s \end{array} \right) B_{2s}(2n-1)^{2k+1-2s} (\frac{1}{2})^{2k+1-2s} \right\}$$

$$= h^{-1} \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} (1-2^{2k-1}) \frac{\tau^{2k+1}}{2k+1} (\frac{1}{2})^{2k+1}$$

$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2g-1}(1-2^{2k-1}) \sum_{s=1}^{k} \frac{2^{1-2s}-1}{2k+1} \left(\begin{array}{c} 2k+1 \\ 2s \end{array} \right) B_{2s} \tau^{2k+1-2s} (\frac{1}{2})^{2k+1-2s}$$

$$= h^{-1} \left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} (\frac{\tau}{2})^{2k+1} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} (\frac{1}{2})^{2} \tau^{2k+1} \right)$$

$$+ \sum_{s=1}^{\infty} \frac{(2^{1-2s}-1)B_{2s}}{(2s)!} h^{2s-1}$$

$$\times \left(\sum_{k=s}^{\infty} \frac{B_{2k}}{2k(2k+1-2s)!} (\frac{\tau}{2})^{2k+1-2s} - (\frac{1}{2})^{2-2s} \sum_{k=s}^{\infty} \frac{B_{2k}}{2k(2k+1-2s)!} \tau^{2k+1-2s} \right)$$

$$= h^{-1} \left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} (\frac{\tau}{2})^{2k+1} - (\frac{1}{2})^{2} \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} \tau^{2k+1} \right)$$

$$+ \sum_{s=1}^{\infty} \frac{(2^{1-2s}-1)B_{2s}}{(2s)!} h^{2s-1}$$

$$\times \left(\sum_{l=0}^{\infty} \frac{B_{2l+2s}}{(2l+1)!(2l+2s)} (\frac{\tau}{2})^{2l+1} - (\frac{1}{2})^{2-2s} \sum_{l=0}^{\infty} \frac{B_{2l+2s}}{(2l+1)!(2l+2s)} \tau^{2l+1} \right)$$

$$= \sum_{s=0}^{\infty} \frac{(1-2^{2s-1})B_{2s}}{(2s)!} h^{2s-1} (2^{1-2s} F_s^{odd}(\tau/2) - \frac{1}{2} F_s^{odd}(\tau)),$$

where we put

$$F_s^{odd}(\tau) := \sum_{l=0}^{\infty} \frac{B_{2l+2s}}{(2l+2s)(2l+1)!} \tau^{2l+1}.$$

Thus, it turns out that

$$
\begin{aligned}
&\sum_{\alpha\in\Psi_+} f((\alpha,\rho)h) \\
&= \frac{1}{2}\sum_{s=0}^{\infty}\frac{(1-2s)B_{2s}h^{2s-2}}{(2s)!}F_s^{even}(\tau)+\sum_{s=0}^{\infty}\frac{(1-2^{2s-1})B_{2s}}{(2s)!}h^{2s-1}(2^{1-2s}F_s^{odd}(\tau/2)-\frac{1}{2}F_s^{odd}(\tau)).
\end{aligned}
$$

Hence, from the formula (5), with the replacement of (τ,h) to $(\tau/d,h/d)$, we obtain

$$
F_{M,g}^{SO(N)}(\tau)=\begin{cases}
\frac{1}{2}\left\{\frac{(1-2s)B_{2s}}{(2s)!}\left(d^{2-2s}F_s^{even}(\tau/d)-F_s^{even}(\tau)\right)+\frac{\lambda_{L(d,b)}}{2}(\tau^3\delta_{s,0}-\tau\delta_{s,1})\right\} & \text{if} \quad g=2s,\\
\frac{(1-2^{2s-1})B_{2s}}{(2s)!}\left\{d^{1-2s}\left(2^{1-2s}F_s^{odd}(\tau/2d)-\frac{1}{2}F_s^{odd}(\tau/d)\right)\right. \\
\qquad \left.-\left(2^{1-2s}F_s^{odd}(\tau/2)-\frac{1}{2}F_s^{odd}(\tau)\right)\right\} & \text{if} \quad g=2s+1.
\end{cases}
$$

From Lemma 7 below, we obtain the required formula for $G_N=SO(N)$ with even N.

In the case that $G_N=SO(N)$ with odd $N=2n+1$, from Lemma 6 below, it follows that

$$
\sum_{\alpha\in\Psi_+} f((\alpha,\rho)h)=\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k)!}h^{2k}\left\{\sum_{1\le j\le 2n-1}\frac{2n-j}{2}j^{2k}+\frac{1-2^{2k-1}}{2^{2k}}\sum_{\substack{j:\mathrm{odd}\\1\le j\le 2n-1}}j^{2k}\right\}.
$$

By a similar calculation, we obtain the required formula for $G_N=SO(N)$ with odd $N=2n+1$.

We obtain the required formula for $G_N=Sp(N)$, since

$$
F_{L(d,b),g}^{Sp(N)}(\tau)=(-1)^g F_{L(d,b),g}^{SO(N)}(\tau),
$$

by Proposition 8 below.

In particular, we see that for any g, $F_{L(d,b),g}^{SO(N)}(\tau)$ and $F_{L(d,b),g}^{Sp(N)}(\tau)$ are analytic in the unit disk, which is not trivial, since the function $\mathrm{Li}_{3-g}(e^\tau)$ for $g\ge 4$ has poles at $2\pi\sqrt{-1}\mathbb{Z}$. Hence, $F_{L(d,b),g}^{SO(N)}(\tau)$ and $F_{L(d,b),g}^{Sp(N)}(\tau)$ are analytic in a neighborhood of zero, where we can choose the neighborhood independently of g. □

Lemma 5. *For $j\in\mathbb{N}$, let $m(j)$ be the number of positive roots α of $\mathfrak{so}_{2n}$ such that $(\alpha,\rho)=j$. We have that*

$$
m(j)=\begin{cases}
\frac{2n-j+1}{2} & \text{if } j:\text{odd}, 1\le j\le n-1,\\
\frac{2n-j}{2} & \text{if } j:\text{even}, 1\le j\le n-1,\\
\frac{2n-j-1}{2} & \text{if } j:\text{odd}, n\le j\le 2n-3,\\
\frac{2n-j-2}{2} & \text{if } j:\text{even}, n\le j\le 2n-3,\\
0 & \textit{otherwise.}
\end{cases}
$$

Proof. The set of positive roots of $\mathfrak{so}_{2n}$ is

$$
\Psi_+=\{\varepsilon_k\pm\varepsilon_l \mid 1\le k<l\le n\},
$$

$(\varepsilon_k,\varepsilon_l)=\delta_{kl}$, and $\rho=\sum_{k=1}^{n-1}(n-k)\varepsilon_k$. Since $(\varepsilon_k-\varepsilon_l,\rho)=l-k$ for $1\le k<l\le n$, it holds that for $j\in\mathbb{N}$, the number of $\varepsilon_k-\varepsilon_l$ with $(\varepsilon_k-\varepsilon_l,\rho)=j$ is $n-j$ if $1\le j\le n-1$ and 0 otherwise.

Since $(\varepsilon_k+\varepsilon_l,\rho)=2n-k-l$ for $1\le k<l\le n$, it holds that for $j\in\mathbb{N}$, the number of $\varepsilon_k+\varepsilon_l$ with $(\varepsilon_k+\varepsilon_l,\rho)=j$ is

$$\begin{array}{ll} \frac{j+1}{2} & \text{if } j:\text{odd}, 1\le j\le n-1,\\ \frac{j}{2} & \text{if } j:\text{even}, 1\le j\le n-1,\\ \frac{2n-j-1}{2} & \text{if } j:\text{odd}, n\le j\le 2n-3,\\ \frac{2n-j-2}{2} & \text{if } j:\text{even}, n\le j\le 2n-3,\\ 0 & \textit{otherwise}. \end{array}$$

Then, we obtain the required formula. □

Lemma 6. *For $j\in\mathbb{N}$, let $m(j)$ be the number of positive roots α of $\mathfrak{so}_{2n+1}$ such that $(\alpha,\rho)=j$. We have that*

$$m(j)=\begin{cases} 1 & \text{if } j=\frac{2l-1}{2}, 1\le l\le n,\\ \frac{2n-j-1}{2} & \text{if } j:\text{odd}, n+1\le j\le 2n-1,\\ \frac{2n-j}{2} & \text{if } j:\text{even}, n+1\le j\le 2n-2,\\ 0 & \text{otherwise}. \end{cases}$$

Lemma 7. *We have*

$$F_s^{even}(\tau)=-\mathrm{Li}_{3-2s}(e^\tau)+\begin{cases} -\dfrac{\tau^2}{2}\log(-\tau)-\dfrac{\tau^3}{12}+\dfrac{3\tau^2}{4}-\dfrac{\pi^2\tau}{6}+\zeta(3) & \text{if } s=0,\\ -\log(-\tau)-\dfrac{\tau}{2} & \text{if } s=1,\\ (2s-3)!\tau^{2-2s}-\dfrac{B_{2s-2}}{2s-2} & \text{if } s\ge 2, \end{cases}$$

$$F_s^{odd}(\tau)=-\mathrm{Li}_{2-2s}(e^\tau)+\begin{cases} -\tau\log(-\tau)-\dfrac{1}{4}\tau^2-\dfrac{\pi^2}{6}+\tau & \text{if } s=0,\\ -\dfrac{1}{\tau}-\dfrac{1}{2} & \text{if } s=1,\\ -(2s-2)!\tau^{1-2s} & \text{if } s\ge 2. \end{cases}$$

Proof. The first formula follows from [4], by noting that $F_s^{even}(\tau)$ equals (6.8) in [4]. As $F_s^{odd}(\tau)=\partial_\tau F_s^{even}(\tau)$ and $\partial_\tau \mathrm{Li}_p(e^\tau)=\mathrm{Li}_{p-1}(e^\tau)$ for any integer p, the second formula follows from the first formula. □

We show a relation between the genus g terms of $SO(N)$ and $Sp(N)$ free energy for a rational homology 3-sphere, which we used in the proof of Theorem 4.

Proposition 8. *For any rational homology 3-sphere M and any g,*

$$F_{M,g}^{Sp(N)}(\tau)=(-1)^g F_{M,g}^{SO(N)}(\tau).$$

Proof. Noting that $\tau=N-1$ for $\mathfrak{g}=\mathfrak{so}$ and that $\tau=N+1$ for $\mathfrak{g}=\mathfrak{sp}$, it follows from (2) that

$$W_{\mathfrak{sp}_\star}(D)=\sum_{0\le g\le d+1} c_{\mathfrak{sp},g}(D)(N+1)^{d+2-g}h^{g-2},$$

$$W_{\mathfrak{so}_\star}(D)=\sum_{0\le g\le d+1} c_{\mathfrak{so},g}(D)(N-1)^{d+2-g}h^{g-2}$$

for a connected trivalent graph D of degree d. Hence,

$$(-1)^d W_{\mathfrak{so}_\star}(D)|_{N\to -N} = (-1)^d\sum_{0\le g\le d+1} c_{\mathfrak{so},g}(D)(-N-1)^{d+2-g}h^{g-2}$$

$$= \sum_{0 \le g \le d+1} (-1)^g c_{\mathfrak{so},g}(D)(N+1)^{d+2-g} h^{g-2}.$$

Comparing $W_{\mathfrak{sp}_\star}(D)$ and $(-1)^d W_{\mathfrak{so}_\star}(D)|_{N \to -N}$ by Proposition 9 below, we have

$$c_{\mathfrak{sp},g}(D) = (-1)^g c_{\mathfrak{so},g}(D)$$

for any g. Since $\log Z_M$ is a linear sum of such D, it follows from (3) that

$$c_{\mathfrak{sp},d,g}(M) = (-1)^g c_{\mathfrak{so},d,g}(M)$$

for any rational homology 3-sphere M, any d, and any g. Further, since

$$F^{G_N}_{M,g}(\tau) = \sum_{d>0, d \ge g-1} c_{\mathfrak{g},d,g}(M) \tau^{d+2-g}$$

by definition, we obtain the required formula. □

Proposition 9. *For a connected trivalent graph D of degree d, $W_{\mathfrak{sp}_N}(D)$ is obtained from $(-1)^d W_{\mathfrak{so}_N}(D)$ by replacing N with $-N$, i.e., $W_{\mathfrak{sp}_N}(D) = (-1)^d W_{\mathfrak{so}_N}(D)|_{N \to -N}$.*

This proposition was proved up to sign in [3, Chapter 13], while we give a complete proof in another way in Section 5. As a corollary of Theorems 3 and 4, we obtain

Corollary 10. *For the lens space $L(d,b)$ and any even g,*

$$\frac{1}{2} F^{SU(N)}_{L(d,b),g}(\tau) = F^{SO(N)}_{L(d,b),g}(\tau) = F^{Sp(N)}_{L(d,b),g}(\tau).$$

Proof. The first equality follows from Theorems 3 and 4 and the second equality follows from Proposition 8. □

4. Observation

In this section, we review the descriptions of $W_{\mathfrak{sl}_N}$ and $W_{\mathfrak{so}_N}$ given by Bar-Natan in [1, 2] and observe new weight systems related to the free energy.

We consider the weight system $W_{\mathfrak{sl}_N}$. We double any edge and replace any trivalent vertex of D in the following:

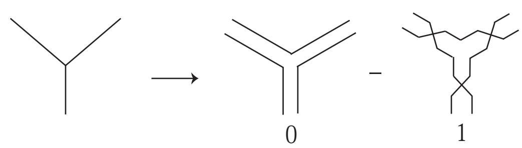


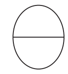
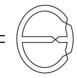
Figure 1

This diagrammatic interpretation comes from the fact that $\mathfrak{gl}_N = V \otimes V^*$ for the defining representation V of $\mathfrak{gl}_N$ and the $\mathfrak{gl}_N$ weight system at a trivalent vertex is defined by the Lie bracket. We note that the $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$ weight systems agree on a trivalent graph, since an abelian ideal of $\mathfrak{gl}_N$ does not contribute on any trivalent vertex applied with the $\mathfrak{gl}_N$ weight system. Let D be a connected trivalent graph and $v(D)$ the set of trivalent vertices. Given a map $m_v : v(D) \to \{0,1\}$, called a vertex marking of D, choosing one of the two possibilities for the replacement of a trivalent vertex depending on m_v, connecting up, we obtain an orientable surface S_{D,m_v} of the genus $g(S_{D,m_v})$ with b_{D,m_v} boundary components. It is showed that for a connected trivalent graph D of degree d,

$$W_{\mathfrak{sl}_N}(D) = \sum_{m_v} (-1)^{s_{m_v}} N^{b_{D,m_v}} h^d, \tag{8}$$

where $s_{m_v} = \sum_{x \in v(D)} m_v(x)$ and the sum is over all possible vertex marking m_v of D. On the other hand, It holds that $2 - 2g(S_{D,m_v}) = \chi(D) + b_{D,m_v}$, where $\chi(D)$ denotes the Euler characteristic of D. As the degree of D is a half of the number of trivalent vertices and $\chi(D) = -d$, we get

$$W_{\mathfrak{sl}_N}(D) = \sum_{m_v} (-1)^{s_{m_v}} N^{d+2-2g(S_{D,m_v})} h^d. \tag{9}$$

For example, if $D = x_1$  x_2 and $m_v(x_1) = 0$, $m_v(x_2) = 1$, then $s_{m_v} = 1$ and $S_{D,m_v} =$  is a torus with one boundary component, i.e., $g(S_{D,m_v}) = 1$, $b_{D,m_v} = 1$. This contributes $-Nh$ to $W_{\mathfrak{sl}_N}(D)$. We get that $W_{\mathfrak{gl}_N}(D) = 2N^3h - 2Nh = 2N(N^2 - 1)h$.

Moreover, we have the following description of the weight system $W_{\mathfrak{so}_N}$. We replace any trivalent vertex and any edge in the following:

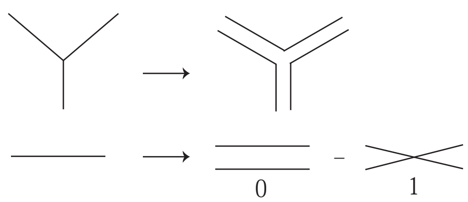
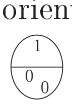
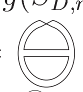
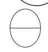


FIGURE 2

We denote by $e(D)$ the set of edges of a connected trivalent graph D. Given a map $m_e : e(D) \to \{0, 1\}$, called an edge marking of D, choosing one of the two possibilities for the replacement of an edge depending on m_e, connecting up, we obtain an orientable or a nonorientable surface S_{D,m_e} of the genus $g(S_{D,m_e})$ with b_{D,m_e} boundary components. Then, we have

$$W_{\mathfrak{so}_N}(D) = \sum_{m_e} (-1)^{s_{m_e}} N^{b_{D,m_e}} h^d = \sum_{m_v} (-1)^{s_{m_v}} N^{d+2-g'_{D,m_v}} h^d, \tag{10}$$

where $s_{m_e} = \sum_{y \in e(D)} m_e(y)$, the sum is over all possible edge marking m_e of D, and $g'_{D,m_e} = 2g(S_{D,m_e})$ if the surface S_{D,m_e} is orientable and $g'_{D,m_e} = g(S_{D,m_e})$ if the surface S_{D,m_e} is nonorientable. For example, from , we obtain $S_{D,m_e} =$  is a projective plane with two boundary components. This contributes $-N^2h$ to $W_{\mathfrak{so}_N}($$)$. We get that $W_{\mathfrak{so}_N}($$) = N^3h - 3N^2h + 3Nh - Nh = N(N-1)(N-2)h$. We remark that the inner product for $\mathfrak{so}_N$ here is the one in [2] multiplied by $\frac{1}{2}$.

Using the above descriptions of $W_{\mathfrak{sl}_N}$ and $W_{\mathfrak{so}_N}$, we show Lemma 1.

Proof of Lemma 1 By noting that $b_{D,m_v} > 0$ in (8) and that $b_{D,m_e} > 0$ in (10), Lemma 1 follows from the above descriptions (9) and (10) and Proposition 9. □

Let us observe new weight systems related to the G_N free energy. We recall the presentation (2) of $W_{\mathfrak{g}_\star}(D)$ for $\mathfrak{g} = \mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}$ and a connected trivalent graph D of degree d,

$$W_{\mathfrak{g}_\star}(D) = \sum_{0 \le g \le d+1} c_{\mathfrak{g},g}(D) \tau^{d+2-g} h^{g-2}, \tag{11}$$

for some $c_{\mathfrak{g},g}(D) \in \mathbb{Z}$. For $\mathfrak{g} = \mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}$ and any g, we get the weight system $w_{\mathfrak{g}_\star,g} : \mathcal{A}(\emptyset)_{\mathrm{conn}} \to \mathbb{Q}[[\tau]]$ defined by

$$w_{\mathfrak{g}_\star,g}(D) := \begin{cases} c_{\mathfrak{g},g}(D)\tau^{d+2-g} & \text{if } d \ge g-1, \\ 0 & \text{otherwise,} \end{cases}$$

for a connected trivalent graph D of degree d.

We study relations among the weight systems $w_{\mathfrak{sl}_\star,g}$, $w_{\mathfrak{so}_\star,g}$ and $w_{\mathfrak{sl}_\star,g}$. Since only orientable surface appears in the above description of the weight system $W_{\mathfrak{sl}_N}$, $w_{\mathfrak{sl}_\star,g} \equiv 0$ for any odd g, and Proposition 9 implies

Proposition 11. *For any connected trivalent graph D and any g,*

$$w_{\mathfrak{so}_\star,g}(D) = (-1)^g w_{\mathfrak{sp}_\star,g}(D).$$

We consider the weight systems $w_{\mathfrak{sl}_\star,g}$ for even g and $w_{\mathfrak{so}_\star,g}$ for any g. In the case that $g = 0$, we have

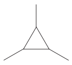
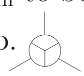
Proposition 12. *For any connected trivalent graph D,*

$$w_{\mathfrak{so}_\star,0}(D) = \frac{1}{2} w_{\mathfrak{sl}_\star,0}(D).$$

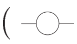
Proof. One sees that two different vertex markings m_v and m'_v of D induce the same edge marking of D if and only if $m'_v(x) - m_v(x) = 1 \pmod 2$ for any vertex x of D. Conversely, if an edge marking m_e of D gives an orientable surface, then there exists a vertex marking of D which induces the edge marking m_e. Noting that only edge marking of D such that gives orientable surface contributes to $w_{\mathfrak{so}_N,0}(D)$, we obtain the required formula. □

Moreover, we obtain

Proposition 13. *The family* $\{w_{\mathfrak{sl}_\star,g} \,|\, g \text{ is even}, g > 0\} \cup \{w_{\mathfrak{so}_\star,g} \,|\, g \ge 0\}$ *of the weight systems are linearly independent in the space spanned over $\mathbb{Q}$ by these weight systems.*

To show Proposition 13, we need some lemmas. We define tD (resp. uD) for a connected trivalent graph D in $\mathcal{A}(\emptyset)_{\mathrm{conn}}$ to be a connected trivalent graph obtained by replacing a trivalent vertex in D with  (resp. ), which was introduced by Vogel in [9]. By the AS and IHX relations, these replacements are independent of a choice of a trivalent vertex. We have the following lemma (for example, see [9]).

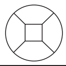
Lemma 14. *For any simple Lie algebra $\mathfrak{g}$ and any connected trivalent graph D in $\mathcal{A}(\emptyset)_{\mathrm{conn}}$,*

$W_{\mathfrak{g}}($  $) = C_{\mathfrak{g}} W_{\mathfrak{g}}($ $)$, $\; W_{\mathfrak{g}}(\, tD \,) = \frac{1}{2} C_{\mathfrak{g}} W_{\mathfrak{g}}(\, D \,),$

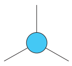
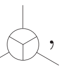
where $C_{\mathfrak{g}}$ is the quadratic Casimir of $\mathfrak{g}$.

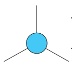
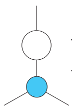
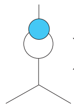
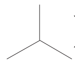
From Lemma 14, we also have

Lemma 15. *For any simple Lie algebra* $\mathfrak{g}$ *and any connected trivalent graph* D *in* $\mathcal{A}(\emptyset)_{\mathrm{conn}}$*,*

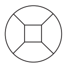
$W_{\mathfrak{g}}(uD) = W_{\mathfrak{g}}($  $) / W_{\mathfrak{g}}(\ominus) \, W_{\mathfrak{g}}(D) = \frac{\sum d_i \alpha_i^4}{C_{\mathfrak{g}} \dim \mathfrak{g}} W_{\mathfrak{g}}(D).$

Here $\{\alpha_i\}$ *are the eigenvalues of the* $\mathfrak{g}$*-homomorphism from* $\mathfrak{g} \otimes \mathfrak{g}$ *to itself defined by* $x \otimes y \mapsto \sum_\alpha [x, \mathfrak{g}_\alpha] \otimes [\mathfrak{g}'_\alpha, y]$ *with the Casimir element* $\sum_\alpha \mathfrak{g}_\alpha \mathfrak{g}'_\alpha$*, where* $\{\mathfrak{g}_\alpha\}$ *is a basis of* $\mathfrak{g}$*,* $\{\mathfrak{g}'_\alpha\}$ *is the dual basis on the Killing form, and* d_i *is the dimension of the eigenspace of* α_i*.*

Proof. From Lemma 14, for  = , there exists a scalar $\lambda_{\mathfrak{g}}$ such that

$W_{\mathfrak{g}}($  $) = \frac{1}{C_{\mathfrak{g}}} W_{\mathfrak{g}}($  $) = \frac{1}{C_{\mathfrak{g}}} W_{\mathfrak{g}}($  $) = \lambda_{\mathfrak{g}} W_{\mathfrak{g}}($  $),$

and so $W_{\mathfrak{g}}(uD) = \lambda_{\mathfrak{g}} W_{\mathfrak{g}}(D)$. Applying to this $D = \ominus$, we get that

$\lambda_{\mathfrak{g}} = W_{\mathfrak{g}}($  $)/W_{\mathfrak{g}}(\ominus).$

The second equality can be obtained from [9, Proposition 6.2]. □

Using Lemmas 14 and 15, we get the following lemma.

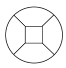
Lemma 16. *For the connected trivalent graph* $T_{m,n} := t^m u^n \ominus$ *of degree* $m + 3n + 1$*,*

$$\begin{aligned}
W_{\mathfrak{sl}_\star}(T_{m,n}) &= 2N^{m+n+1}(N^2+12)^n(N^2-1)h^{m+3n+1} \\
&= 2\tau^{m+n+1}((\frac{\tau}{h})^2+12)^n((\frac{\tau}{h})^2-1)h^{2n}, \\
W_{\mathfrak{so}_\star}(T_{m,n}) &= (N-2)^{m+1}(N^3-9N^2+54N-104)^n N(N-1)h^{m+1} \\
&= \tau(\frac{\tau}{h}-1)^m((\frac{\tau}{h})^3-6(\frac{\tau}{h})^2+39(\frac{\tau}{h})-58)^n((\frac{\tau}{h})^2-1)h^m,
\end{aligned}$$

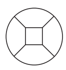
where $\tau = Nh$ *for* $W_{\mathfrak{sl}_\star}$ *and* $\tau = (N-1)h$ *for* $W_{\mathfrak{so}_\star}$*.*

Proof. We have that $C_{\mathfrak{sl}_N} = 2N$ and $C_{\mathfrak{so}_N} = 2(N-2)$ and calculate

$W_{\mathfrak{sl}_N}(\ominus) = 2N(N^2-1),$

$W_{\mathfrak{sl}_N}($  $) = 2N^2(N^2-1)(N^2+12),$

$W_{\mathfrak{so}_N}(\ominus) = N(N-1)(N-2),$

$W_{\mathfrak{so}_N}($  $) = N(N-1)(N-2)(N^3-9N^2+54N-104).$

From Lemmas 14 and 15, we obtain the required formulas. □

Now let us show Proposition 13.

Proof of Proposition 13 From Lemma 16, we calculate that for $g \geq 3$,

$$\begin{aligned}
&w_{\mathfrak{sl}_\star,2}(T_{g-2,0}) = -2\tau^{g-1}, \ w_{\mathfrak{sl}_\star,m}(T_{g-2,0}) = 0 \text{ if } m \geq 4, \ m \text{ is even}, \\
&w_{\mathfrak{so}_\star,0}(T_{g-2,0}) = \tau^{g+1}, w_{\mathfrak{so}_\star,1}(T_{g-2,0}) = -(g-2)\tau^g, \\
&w_{\mathfrak{so}_\star,2}(T_{g-2,0}) = \frac{(g-1)(g-4)}{2}\tau^{g-1}, \ w_{\mathfrak{so}_\star,3}(T_{g-2,0}) = -\frac{(g-1)(g-2)(g-6)}{6}\tau^{g-2},
\end{aligned}$$

$$w_{\mathfrak{so}_\star,g}(T_{g-2,0}) = (-1)^{g-1}\tau,\ w_{\mathfrak{so}_\star,m}(T_{g-2,0}) = 0 \text{ if } m > g,$$

and that for any even g with $g \geq 4$,

$$w_{\mathfrak{sl}_\star,g}(T_{0,\frac{g-2}{2}}) = -2 \cdot 12^{\frac{g-2}{2}}\tau^{\frac{g}{2}},\ w_{\mathfrak{sl}_\star,m}(T_{0,\frac{g-2}{2}}) = 0 \text{ if } m \geq g+2,\ m \text{ is even.}$$

Then, we get the proposition. □

5. Proof of Proposition 9

Let us state some results about the $\mathfrak{sp}_N$ weight system. From [1], we get the following diagrammatic description of the $\mathfrak{sp}_N$ weight system with $N = 2n$, which comes from that $\mathfrak{sp}_N$ has a basis $E_{ij} - E_{n+j\,n+i}$ $(1 \leq i, j \leq n)$, $E_{i\,n+j} + E_{j\,n+i}$ $(1 \leq i \leq j \leq n)$, and $E_{n+i\,j} + E_{n+j\,i}$ $(1 \leq i \leq j \leq n)$. and that the inner product is given by $(E_{ij}, E_{kl}) = \frac{1}{2}\mathrm{tr}(E_{ij}E_{kl})$ $(1 \leq i, j, k, l \leq 2n)$. Let D be a connected trivalent graph, $v(D)$ the set of vertices of D, and $Y_0(\circ, \bullet)$ the set of the diagrams and the diagrams obtained by the $\frac{2\pi}{3}$-rotation or $\frac{4\pi}{3}$-rotation of the above

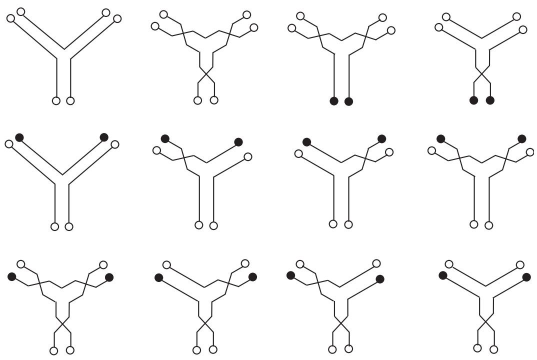


Figure 3

diagrams except the first and second diagrams. We double any edge in D and replace each vertex with one diagram in $Y := Y_0(\circ, \bullet) \cup Y_0(\bullet, \circ)$, in such a way that connecting up, the two ends of each edge in any double edge have the same symbol. Such a replacement defines a map $m : v(D) \to Y$, called an admissible vertex marking of D, and we obtain an orientable or a nonorientable surface $S_{D,m}$ with $b_{D,m}$ boundary components with even symbols $\circ$ and even symbols $\bullet$. We comment that the symbol $\circ$ (resp. $\bullet$) corresponds to index i (resp. $n+i$) with $1 \leq i \leq n$ in the above basis of $\mathfrak{sp}_N$. Then, we have

$$W_{\mathfrak{sp}_N}(D) = 2^{-3d}\sum_m (-1)^{s_m} n^{b_{D,m}} h^d,$$

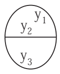
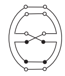
where s_m is the number of  and  in $S_{D,m}$, and the sum is over all possible admissible vertex marking m of D. We note that the symbols $\circ$ and $\bullet$ correspond to the symbols P and Q respectively in [1].

We have a simpler description of the weight system $W_{\mathfrak{sp}_N}$. We denote by $e(D)$ the set of edges of a connected trivalent graph D and Y' the set of the diagrams We replace any trivalent

Figure 4

vertex in the same way as the weight system $W_{\mathfrak{so}_N}$ and replace each edge with one diagram in Y', in such a way that connecting up, the two ends of each arc in any doubled vertex have the same symbol. Such a replacement defines a map $m' : e(D) \to Y'$, called an admissible edge marking of D, and we obtain an orientable or a nonorientable surface $S_{D,m'}$ with $b_{D,m'}$ boundary components with even symbols $\circ$ and even symbols $\bullet$. Then, we have

$$W_{\mathfrak{sp}_N}(D) = \sum_{m'} (-1)^{s_{m'}} n^{b_{D,m'}} h^d,$$

where $s_{m'}$ is the number of and in $S_{D,m'}$ and the sum is over all possible admissible edge marking m' of D. For example, when $D =$ , $m'(y_1) =$, $m'(y_2) =$, and $m'(y_3) =$, the surface $S_{D,m'}$  is a nonorientable surface of the genus 1 with 2 boundary components and so contributes n^2h to $W_{\mathfrak{sp}_N}(\ )$. We compute that $W_{\mathfrak{sp}_N}(\ ) = 8n^3h + 12n^2h + 4nh = 2n(2n+1)(2n+2)h = N(N+1)(N+2)h$.

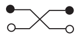
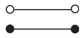
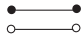
Now let us prove Proposition 9.

Proof of Proposition 9 Let D be a connected trivalent graph. One sees that an admissible edge marking $m' : v(D) \to Y$ in the above description of $W_{\mathfrak{sp}_N}$ induces an edge marking $m_e : e(D) \to \{0,1\}$ in the description of $W_{\mathfrak{so}_N}$ in Section 4, by ignoring the symbols $\circ$ and $\bullet$. Let $m_e : e(D) \to \{0,1\}$ be an edge marking. We construct an admissible marking $m' : v(D) \to Y'$ which induces m_e as follows. Let B be a boundary component of the surface S_{D,m_e}. We decompose B into a sequence $\alpha_1\beta_1 \dots \alpha_k\beta_k$ of arcs, where α_i is one of two arcs in the diagram or and β_i is one of three arcs in the diagram . Let p_i be the intersection point of β_{i-1} and α_i for $1 \le i \le k$, where $\beta_0 := \beta_k$, and q_i be the intersection point of α_i and β_i for $1 \le i \le k$. Next, we assign p_i and q_i with $\circ$ or $\bullet$ in such a way that q_{i-1} and p_i for $1 \le i \le k$ are assigned with the same symbol, where $q_0 := q_k$, and that if α_i is an arc in the diagram (resp.), then p_i and q_i are assigned with the same symbol (resp. the different symbol). As the number of α_i which is an arc in the diagram is even and an assignment of p_1 determines such an assignment, such two assignments exist. A surface S_{D,m_e} with any boundary component given one of two possible assignments is said to be decorated. It follows from the definition of Y' that a decorated surface S_{D,m_e} determines $S_{D,m'}$ for an admissible edge marking $m' : e(D) \to Y'$ inducing m_e. For any edge marking m_e, there exist $2^{b_{D,m_e}}$ admissible edge markings m' that induces m_e. Moreover, it holds that for any admissible edge marking m', there exists an edge marking m_e such that a decorated surface S_{D,m_e} coincides with $S_{D,m'}$. Noting that the number of , on each boundary component of $S_{D,m'}$ is even, one also sees that if admissible edge markings m'_1 and m'_2 induce the same edge marking m_e, then $s_{m'_1} \equiv s_{m'_2} \pmod 2$. Consequently, we obtain that

$$W_{\mathfrak{sp}_N}(D) = \sum_{m_e} (-1)^{s_{m'}} 2^{b_{D,m_e}} n^{b_{D,m_e}} h^d, \tag{12}$$

where the sum is over all possible edge marking $m_e : e(D) \to \{0,1\}$, m' is an admissible edge marking inducing m_e, and d is the degree of D. Moreover, by the definition of s_{m_e} and $s_{m'}$, we have that $s_{m_e} = s_{m'} + j_{m'}$, where $j_{m'}$ is the number of , in $S_{D,m'}$. Hence, we obtain that

$$W_{\mathfrak{sp}_N}(D) = \sum_{m_e} (-1)^{s_{m_e} - j_{m'}} (2n)^{b_{D,m_e}} h^d = \sum_{m_e} (-1)^{s_{m_e} - j_{m'}} N^{b_{D,m_e}} h^d. \tag{13}$$

From the formula (10), to prove Proposition 9, it is enough to show that $d + b_{D,m_e} \equiv j_{m'}$ (mod 2). We remark that $2 - g'_{D,m'} = -d + b_{D,m_e}$. In the case that $j_{m'} = 0$, one sees that $S_{D,m'}$ is an orientable surface and that $-d + b_{D,m_e} = 2 - 2g(S_{D,m'}) \equiv 0 = j_{m'}$ (mod 2). Suppose that $j_{m'} \neq 0$. From the definition of an admissible edge marking, we see that the surface $S_{D,m'}$ is nonorientable. Replacing all  (resp. ) with  (resp. ), we get an orientable surface $S^o_{D,m'}$. Then, it follows that $g'_{D,m'} \equiv 2g(S^o_{D,m'}) + j_{m'} \equiv j_{m'}$ (mod 2) and so we get that $-d + b_{D,m_e} = 2 - g'_{D,m'} \equiv j_{m'}$ (mod 2). This completes the proof of Proposition 9. □

Remark. Proposition 9 is noted as Exercise 6.37 in [2]. It can also be obtained from a result on the weight system associated with the super Lie algebra $\mathfrak{osp}(m, n)$ in [9], noting that $\mathfrak{osp}(m, 1) = \mathfrak{so}_m$ and that $\mathfrak{osp}(1, n) = \mathfrak{sp}_n$.

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Dedicated to Professor Akio Kawauchi on the occasion of his 60th birthday

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