

Fundamental group of simple C^* -algebras with unique trace II

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FUNDAMENTAL GROUP OF SIMPLE C^* -ALGEBRAS WITH UNIQUE TRACE II

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ABSTRACT. We show that any countable subgroup of the multiplicative group \mathbb{R}_+^\times of positive real numbers can be realized as the fundamental group $\mathcal{F}(A)$ of a separable simple unital C^* -algebra A with unique trace. Furthermore for any fixed countable subgroup G of \mathbb{R}_+^\times , there exist uncountably many mutually nonisomorphic such algebras A with $G = \mathcal{F}(A)$.

1. INTRODUCTION

Let M be a factor of type II_1 with a normalized trace τ . Murray and von Neumann introduced the fundamental group $\mathcal{F}(M)$ of M in [13]. The fundamental group $\mathcal{F}(M)$ of M is a subgroup of the multiplicative group \mathbb{R}_+^\times of positive real numbers. They showed that if M is hyperfinite, then $\mathcal{F}(M) = \mathbb{R}_+^\times$. In our previous paper [14], we introduced the fundamental group $\mathcal{F}(A)$ of a simple unital C^* -algebra A with a unique normalized trace τ based on the computation of Picard groups by Kodaka [8], [9], [10]. We compute the fundamental groups of several nuclear or nonnuclear C^* -algebras. K -theoretical obstruction enable us to compute the fundamental group easily.

There has been many works on the computation of fundamental groups of the factors of type II_1 . Voiculescu [22] showed that the fundamental group $\mathcal{F}(L(\mathbb{F}_\infty))$ of the group factor $L(\mathbb{F}_\infty)$ of the free group \mathbb{F}_∞ contains the positive rationals and Radulescu proved that $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+^\times$ in [19]. Connes [3] showed that if G is an countable ICC group with property (T), then $\mathcal{F}(L(G))$ is a countable group. Recently, Popa showed that any countable subgroup of \mathbb{R}_+^\times can be realized as the fundamental group of some factor of type II_1 with separable predual in [17]. Furthermore Popa and Vaes [18] exhibited a large family \mathcal{S} of subgroups of \mathbb{R}_+^\times , containing \mathbb{R}_+^\times itself, all of its countable subgroups, as well as uncountable subgroups with any Hausdorff dimension in $(0, 1)$, such that for each $G \in \mathcal{S}$ there exist many free ergodic measure preserving actions of \mathbb{F}_∞ for which the associated II_1 factor M has fundamental group equal to G .

In this paper we show that any countable subgroup of \mathbb{R}_+^\times can be realized as the fundamental group of a separable simple unital C^* -algebra with unique trace. Furthermore for any fixed countable subgroup G of \mathbb{R}_+^\times , there exist uncountably many mutually nonisomorphic such algebras A with $G = \mathcal{F}(A)$. We apply a method of Blackadar [1] and Phillips [16] to the type II_1 factors of Popa [17]. Our new examples are nonnuclear.

On the other hand, for an additive subgroup E of \mathbb{R} containing 1, we define the positive inner multiplier group $IM_+(E)$ of E by

$$IM_+(E) = \{t \in \mathbb{R}_+^\times \mid t \in E, t^{-1} \in E, \text{ and } tE = E\}.$$

Then we have $\mathcal{F}(A) \subset IM_+(\tau_*(K_0(A)))$. Almost all examples provided in [14] satisfy $\mathcal{F}(A) = IM_+(\tau_*(K_0(A)))$. We should note that not all countable subgroups of \mathbb{R}_+^\times arise as $IM_+(E)$. For example, $\{9^n \in \mathbb{R}_+^\times \mid n \in \mathbb{Z}\}$ does not arise as $IM_+(E)$ for any additive subgroup E of \mathbb{R} containing 1. Therefore if the fundamental group of a C^* -algebra A is equal to $\{9^n \in \mathbb{R}_+^\times \mid n \in \mathbb{Z}\}$ and A is in a classifiable class by the Elliott invariant, then $\tau_* : K_0(A) \rightarrow \tau_*(K_0(A))$ cannot be an order isomorphism. Matui informed us that there exists such an AF-algebra.

2. HILBERT C^* -MODULES AND PICARD GROUPS

We recall some definitions and notations in [14]. Let A be a simple unital C^* -algebra with a unique normalized trace τ and \mathcal{X} a right Hilbert A -module. (See [11], [12] for the basic facts on Hilbert modules.) We denote by $L_A(\mathcal{X})$ the algebra of the adjointable operators on \mathcal{X} . For $\xi, \eta \in \mathcal{X}$, a "rank one operator" $\Theta_{\xi, \eta}$ is defined by $\Theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle_A$ for $\zeta \in \mathcal{X}$. We denote by $K_A(\mathcal{X})$ the closure of the linear span of "rank one operators" $\Theta_{\xi, \eta}$. We call a finite set $\{\xi_i\}_{i=1}^n \subseteq \mathcal{X}$ a *finite basis* of \mathcal{X} if $\eta = \sum_{i=1}^n \xi_i \langle \xi_i, \eta \rangle_A$ for any $\eta \in \mathcal{X}$, see [7], [23]. It is also called a frame as in [6]. If A is unital and there exists a finite basis for \mathcal{X} , then $L_A(\mathcal{X}) = K_A(\mathcal{X})$. Let $\mathcal{H}(A)$ denote the set of isomorphic classes $[\mathcal{X}]$ of right Hilbert A -modules \mathcal{X} with finite basis.

Let B be a C^* -algebra. An A - B -equivalence bimodule is an A - B -bimodule \mathcal{F} which is simultaneously a full left Hilbert A -module under a left A -valued inner product ${}_A\langle \cdot, \cdot \rangle$ and a full right Hilbert B -module under a right B -valued inner product $\langle \cdot, \cdot \rangle_B$, satisfying ${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$ for any $\xi, \eta, \zeta \in \mathcal{F}$. We say that A is *Morita equivalent* to B if there exists an A - B -equivalence bimodule. The dual module \mathcal{F}^* of an A - B -equivalence bimodule \mathcal{F} is a set $\{\xi^*; \xi \in \mathcal{F}\}$ with the operations such that $\xi^* + \eta^* = (\xi + \eta)^*$, $\lambda \xi^* = (\bar{\lambda} \xi)^*$, $b \xi^* a = (a^* \xi b^*)^*$, ${}_B\langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle_B$ and $\langle \xi^*, \eta^* \rangle_A = {}_A\langle \eta, \xi \rangle$. Then \mathcal{F}^* is a B - A -equivalence bimodule. We refer the reader to [20], [21] for the basic facts on equivalence bimodules and Morita equivalence.

We review elementary facts on the Picard groups of C^* -algebras introduced by Brown, Green and Rieffel in [2]. For A - A -equivalence bimodules \mathcal{E}_1 and \mathcal{E}_2 , we say that \mathcal{E}_1 is isomorphic to \mathcal{E}_2 as an equivalence bimodule if there exists a \mathbb{C} -linear one-to-one map Φ of \mathcal{E}_1 onto \mathcal{E}_2 with the properties such that $\Phi(a\xi b) = a\Phi(\xi)b$, ${}_A\langle \Phi(\xi), \Phi(\eta) \rangle = {}_A\langle \xi, \eta \rangle$ and $\langle \Phi(\xi), \Phi(\eta) \rangle_A = \langle \xi, \eta \rangle_A$ for $a, b \in A$, $\xi, \eta \in \mathcal{E}_1$. The set of isomorphic classes $[\mathcal{E}]$ of the A - A -equivalence bimodules \mathcal{E} forms a group under the product defined by $[\mathcal{E}_1][\mathcal{E}_2] = [\mathcal{E}_1 \otimes_A \mathcal{E}_2]$. We call it the *Picard group* of A and denote it by $\text{Pic}(A)$. The identity of $\text{Pic}(A)$ is given by the A - A -bimodule $\mathcal{E} := A$ with ${}_A\langle a_1, a_2 \rangle = a_1 a_2^*$ and $\langle a_1, a_2 \rangle_A = a_1^* a_2$ for $a_1, a_2 \in A$. The inverse element of $[\mathcal{E}]$ in the Picard group of A is the dual module $[\mathcal{E}^*]$. Let α be an automorphism of A , and let $\mathcal{E}_\alpha^A = A$ with the obvious left A -action and the obvious A -valued inner product. We define the right A -action on \mathcal{E}_α^A by $\xi \cdot a = \xi \alpha(a)$ for any $\xi \in \mathcal{E}_\alpha^A$ and $a \in A$, and the right A -valued inner product by $\langle \xi, \eta \rangle_A = \alpha^{-1}(\xi^* \eta)$ for

any $\xi, \eta \in \mathcal{E}_\alpha^A$. Then \mathcal{E}_α^A is an A - A -equivalence bimodule. For $\alpha, \beta \in \text{Aut}(A)$, \mathcal{E}_α^A is isomorphic to \mathcal{E}_β^A if and only if there exists a unitary $u \in A$ such that $\alpha = \text{ad } u \circ \beta$. Moreover, $\mathcal{E}_\alpha^A \otimes \mathcal{E}_\beta^A$ is isomorphic to $\mathcal{E}_{\alpha \circ \beta}^A$. Hence we obtain an homomorphism ρ_A of $\text{Out}(A)$ to $\text{Pic}(A)$. An A - B -equivalence bimodule \mathcal{F} induces an isomorphism Ψ of $\text{Pic}(A)$ to $\text{Pic}(B)$ by $\Psi([\mathcal{E}]) = [\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]$ for $[\mathcal{E}] \in \text{Pic}(A)$. Therefore if A is Morita equivalent to B , then $\text{Pic}(A)$ is isomorphic to $\text{Pic}(B)$. Since A is unital, any A - A -equivalence bimodule is a finitely generated projective A -module as a right module with a finite basis $\{\xi_i\}_{i=1}^n$. Put $p = (\langle \xi_i, \xi_j \rangle_A)_{ij} \in M_n(A)$. Then p is a projection and \mathcal{E} is isomorphic to pA^n as a right Hilbert A -module with an isomorphism of A to $pM_n(A)p$ as a C^* -algebra.

Define a map $\hat{T}_A : \mathcal{H}(A) \rightarrow \mathbb{R}_+$ by $\hat{T}_A([\mathcal{X}]) = \sum_{i=1}^n \tau(\langle \xi_i, \xi_i \rangle_A)$, where $\{\xi_i\}_{i=1}^n$ is a finite basis of \mathcal{X} . Then $\hat{T}_A([\mathcal{X}])$ does not depend on the choice of basis and \hat{T}_A is well-defined. We can define a map T_A of $\text{Pic}(A)$ to \mathbb{R}_+ by the same way of \hat{T}_A . We showed that T_A is a multiplicative map and $T_A(\mathcal{E}_{id}^A) = 1$ in [14]. Moreover, we can show the following proposition by a similar argument in the proof of Proposition 2.1 in [14].

Proposition 2.1. Let A and B be simple unital C^* -algebras with unique trace. Assume that \mathcal{F} is an A - B -equivalence bimodule and \mathcal{X} is a right Hilbert A -module. Then

$$\hat{T}_B([\mathcal{X} \otimes \mathcal{F}]) = \hat{T}_A([\mathcal{X}])\hat{T}_B([\mathcal{F}]).$$

We denote by Tr the usual unnormalized trace on $M_n(\mathbb{C})$. Put

$$\mathcal{F}(A) := \{\tau \otimes Tr(p) \in \mathbb{R}_+^\times \mid p \text{ is a projection in } M_n(A) \text{ such that } pM_n(A)p \cong A\}.$$

Then $\mathcal{F}(A)$ is equal to the image of T_A and a multiplicative subgroup of \mathbb{R}_+^\times by Theorem 3.1 in [14]. We call $\mathcal{F}(A)$ the *fundamental group* of A . If A is separable, then $\mathcal{F}(A)$ is countable. We shall show that the fundamental group is a Morita equivalence invariant for simple unital C^* -algebras with unique trace.

Proposition 2.2. Let A and B be simple unital C^* -algebras with unique trace. If A is Morita equivalent to B , then $\mathcal{F}(A) = \mathcal{F}(B)$.

Proof. By assumption, there exists an A - B -equivalence bimodule \mathcal{F} , and \mathcal{F} induces an isomorphism Ψ of $\text{Pic}(A)$ to $\text{Pic}(B)$ such that $\Psi([\mathcal{E}]) = [\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]$ for $[\mathcal{E}] \in \text{Pic}(A)$. Since $\mathcal{F}^* \otimes \mathcal{F}$ is isomorphic to \mathcal{E}_{id}^B , Proposition 2.1 implies

$$\hat{T}_A([\mathcal{F}^*])\hat{T}_B([\mathcal{F}]) = T_B([\mathcal{F}^* \otimes \mathcal{F}]) = 1.$$

For $[\mathcal{E}] \in \text{Pic}(A)$,

$$T_B([\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]) = \hat{T}_A([\mathcal{F}^*])\hat{T}_B([\mathcal{E} \otimes \mathcal{F}]) = \hat{T}_A([\mathcal{F}^*])T_A([\mathcal{E}])\hat{T}_B([\mathcal{F}])$$

by Proposition 2.1. Therefore $T_B([\Psi(\mathcal{E})]) = T_A([\mathcal{E}])$ and $\mathcal{F}(A) = \mathcal{F}(B)$. \square

3. NEW EXAMPLES

An idea of our construction comes from the following results of Blackadar, Proposition 2.2 of [1] and Phillips, Lemma 2.2 of [16].

Lemma 3.1 ([1](Blackadar)). Let M be a simple C^* -algebra, and let $A \subset M$ be a separable C^* -subalgebra. Then there exists a simple separable C^* -subalgebra B with $A \subset B \subset M$.

Lemma 3.2 ([16](Phillips)). Let M be a unital C^* -algebra, and let $A \subset M$ be a separable C^* -subalgebra. Then there exists a separable C^* -subalgebra B with $A \subset B \subset M$ such that every tracial state on B is the restriction of a tracial state on M .

The following lemma is just a combination of the two results above.

Lemma 3.3. Let M be a simple C^* -algebra with unique trace $\hat{\tau}$, and let $A \subset M$ be a separable C^* -subalgebra. Then there exists a simple separable C^* -subalgebra B with $A \subset B \subset M$ such that B has a unique trace τ that is a restriction of $\hat{\tau}$.

Theorem 3.4. Let G be a countable subgroup of \mathbb{R}_+^\times . Then there exist uncountably many mutually nonisomorphic separable simple nonnuclear unital C^* -algebras A with unique trace such that the fundamental group $\mathcal{F}(A) = G$.

Proof. First we shall show that there exists a separable simple unital C^* -algebra A with unique trace such that $\mathcal{F}(A) = G$. There exists a type II_1 factor M with separable predual such that $\mathcal{F}(M) = G$, which is constructed by Popa [17]. Let $S_1 \subset M$ be a countable subset that is weak operator dense in M . We denote by $\hat{\tau}$ the unique trace of M . We enumerate the countable semigroup $G \cap (0, 1]$ by $\{t_m : m \in \mathbb{N}\}$. Since $\mathcal{F}(M) = G$ and M is a factor of type II_1 , for any $m \in \mathbb{N}$ there exist a projection p_m in M such that $\hat{\tau}(p_m) = t_m$ and an isomorphism ϕ_m of M onto $p_m M p_m$. Define $B_0 \subset M$ be the unital C^* -subalgebra of M generated by S_1 and $\{p_m : m \in \mathbb{N}\}$. By Lemma 3.3, there exists a separable simple unital C^* -algebra A_0 with a unique trace τ_0 such that $B_0 \subset A_0 \subset M$. Let $B_1 \subset M$ be the C^* -subalgebra of M generated by A_0 , $\cup_{m \in \mathbb{N}} \phi_m(A_0)$ and $\cup_{m \in \mathbb{N}} \phi_m^{-1}(p_m A_0 p_m)$. By the same way, there exists a separable simple unital C^* -algebra A_1 with a unique trace τ_1 such that $B_1 \subset A_1 \subset M$. We construct inductively C^* -algebras $B_n \subset A_n \subset M$ as follows: Let $B_n \subset M$ be the C^* -subalgebra of M generated by A_{n-1} , $\cup_{m \in \mathbb{N}} \phi_m(A_{n-1})$ and $\cup_{m \in \mathbb{N}} \phi_m^{-1}(p_m A_{n-1} p_m)$. By Lemma 3.3, there exists a separable simple unital C^* -algebra A_n with a unique trace τ_n such that $B_n \subset A_n \subset M$. Then we have

$$B_0 \subset A_0 \subset B_1 \subset A_1 \subset \dots B_n \subset A_n \dots \subset M,$$

and $\phi_m(A_{n-1}) \subset p_m A_n p_m$ and $\phi_m^{-1}(p_m A_{n-1} p_m) \subset A_n$ for any $m \in \mathbb{N}$. Set $A = \overline{\cup_{n=0}^\infty A_n}$. Then A is a separable simple unital C^* -algebra A with a unique trace τ . By construction, $\phi_m(A) = p_m A p_m$ for any $m \in \mathbb{N}$. Hence $G \subset \mathcal{F}(A)$. Since $\pi_\tau(A)''$ is isomorphic to M ,

$$\mathcal{F}(A) \subset \mathcal{F}(\pi_\tau(A)'') = \mathcal{F}(M) = G$$

by Proposition 3.29 of [14]. Thus $\mathcal{F}(A) = G$. Moreover A is not nuclear, because A is weak operator dense in a factor M that is not hyperfinite.

Next we shall show that there exist uncountably many mutually nonisomorphic such examples. Let E be a countable additive subgroup of \mathbb{R} . We enumerate by $\{r_m : m \in \mathbb{N}\}$ the positive elements of E . Since M is a factor

of type II_1 , there exist a natural number k and a projection $q_m \in M_k(M)$ such that $\hat{\tau}(q_m) = r_m$ for any $m \in \mathbb{N}$. Define $S_2 \subset M$ to be the union of the matrix elements of q_m for running $m \in \mathbb{N}$. Let C_0 the C^* -subalgebra of M generated by S_2 and A . By a similar argument as the first paragraph, we can construct a separable simple unital C^* -algebra C with unique trace such that $\mathcal{F}(C) = G$ and $C_0 \subset C \subset M$. Then it is clear that E is contained in $\tau_*(K_0(C))$. Since no countable union of countable subgroups of \mathbb{R} can contain all countable subgroups of \mathbb{R} , we can construct uncountably many mutually nonisomorphic examples by the choice of E . \square

Remark 3.5. In fact, we show that there exist uncountably many Morita inequivalent separable simple nonnuclear unital C^* -algebras A with unique trace such that the fundamental group $\mathcal{F}(A) = G$ in the proof above.

Remark 3.6. We can choose a C^* -algebra A in the theorem above so that A has stable rank one and real rank zero and $\tau_* : K_0(A) \rightarrow \tau_*(K_0(A))$ is an order isomorphism by using Lemma 2.3, Lemma 2.4 and Lemma 2.5 of [16]. Then we have the following exact sequence by Proposition 3.26 of [14]:

$$1 \longrightarrow \text{Out}(A) \xrightarrow{\rho_A} \text{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1.$$

Remark 3.7. We do not know whether any countable subgroup of \mathbb{R}_+^\times can be realized as the fundamental group of a separable unital simple *nuclear* C^* -algebra with unique trace.

We denote by $\text{Ell}(A)$ the Elliott invariant $(K_0(A), K_0(A)_+, [1]_0, K_1(A))$.

Corollary 3.8. For any countable subgroups G_1 and G_2 of \mathbb{R}_+^\times , there exist separable simple nonnuclear unital C^* -algebras A and B with unique trace such that $\text{Ell}(A) \cong \text{Ell}(B)$, $\mathcal{F}(A) = G_1$ and $\mathcal{F}(B) = G_2$.

Proof. Let M_1 and M_2 be a type II_1 factors with separable preduals such that $\mathcal{F}(M_1) = G_1$ and $\mathcal{F}(M_2) = G_2$ constructed by Popa [17]. By Theorem 3.4, there exist separable simple nonnuclear unital C^* -algebras $A_0 \subset M_1$ and $B_0 \subset M_2$ with unique trace such that $\mathcal{F}(A_0) = G_1$ and $\mathcal{F}(B_0) = G_2$. Moreover, we can assume that the traces τ_{A_0} and τ_{B_0} of A_0 and B_0 induces order isomorphisms of K_0 -groups and $K_1(A_0) = K_1(B_0) = 0$ by Remark 3.6. By a similar argument in the proof of Theorem 3.4, we can construct separable simple nonnuclear unital C^* -algebras $A_0 \subset A_1 \subset M_1$ and $B_0 \subset B_1 \subset M_2$ with unique trace τ_{A_1} and τ_{B_1} such that $\mathcal{F}(A_1) = G_1$, $\mathcal{F}(B_1) = G_2$, $(\tau_{A_0})_*(K_0(A_0)) \subset (\tau_{B_1})_*(K_0(B_1))$ and $(\tau_{B_0})_*(K_0(B_0)) \subset (\tau_{A_1})_*(K_0(A_1))$. We may assume that τ_{A_1} and τ_{B_1} induce order isomorphisms and $K_1(A_1) = K_1(B_1) = 0$. Moreover, we may assume that for any projection $p \in A_0$ (resp. B_0) with $\tau(p) \in G_1$ (resp. G_2), there exists an isomorphism ϕ of A_1 to pA_1p (resp. B_1 to pB_1p) such that the restriction $\phi|_{A_0}$ (resp. B_0) is an isomorphism of A_0 to pA_0p (resp. B_0 to pB_0p) by the proof of Theorem 3.4. We construct inductively C^* -algebras $A_n \subset A_{n+1} \subset M_1$ and $B_n \subset B_{n+1} \subset M_2$. Define $A = \overline{\bigcup_{n=0}^\infty A_n}$ and $B = \overline{\bigcup_{n=0}^\infty B_n}$. Then A and B are separable simple nonnuclear unital C^* -algebras with unique trace such that the traces τ_A and τ_B induce order isomorphisms of K_0 -groups, $\mathcal{F}(A) = G_1$, $\mathcal{F}(B) = G_2$ and $K_1(A) = K_1(B) = 0$. A similar argument in the proof of Theorem 3.4 shows $(\tau_A)_*(K_0(A)) = (\tau_B)_*(K_0(B))$. Since τ_A and τ_B induce order isomorphisms of K_0 -groups, $\text{Ell}(A) \cong \text{Ell}(B)$. \square

For a positive number λ , let $G_\lambda = \{\lambda^n \in \mathbb{R}_+^\times \mid n \in \mathbb{Z}\}$ be the multiplicative subgroup of \mathbb{R}_+^\times generated by λ . In the below we shall consider whether G_λ can be realized as the fundamental group of a nuclear C^* -algebra.

Proposition 3.9. Let λ be a prime number or a positive transcendental number. Then there exists a simple AF -algebra A with unique trace such that $\mathcal{F}(A) = G_\lambda$.

Proof. Let λ be a prime number. Consider a UHF-algebra $A = M_{\lambda^\infty}$. Then $\mathcal{F}(A) = G_\lambda$ as in Example 3.11 of [14]. Next we assume that λ is a positive transcendental number. Let R_λ be the unital subring of \mathbb{R} generated by λ . Then the set $(R_\lambda)_+^\times$ of positive invertible elements in R_λ is equal to G_λ . The proof of Theorem 3.14 of [14] shows that there exists a simple unital AF -algebra A with unique trace such that $\mathcal{F}(A) = G_\lambda$. \square

Let \mathcal{O} be an order of a real quadratic field or a real cubic field with one real embedding. Then $\mathcal{O}_+^\times = G_\lambda$ is singly generated and the generator $\lambda > 1$ is called the fundamental unit of \mathcal{O} by Dirichlet's unit theorem. We refer the reader to [15] for details. The proof of Theorem 3.14 of [14] implies the following proposition.

Proposition 3.10. Let λ be a fundamental unit of an order of a real quadratic field or a cubic field with one real embedding. Then there exists a simple AF -algebra A with unique trace such that $\mathcal{F}(A) = G_\lambda$.

Note that if p is a prime number and $n \geq 2$, then the subgroup G_λ of R_+^\times generated by $\lambda = p^n$ can not be the positive inner multiplier group $IM_+(E)$ for any additive subgroup E of \mathbb{R} containing 1. In fact, on the contrary, suppose that $G_\lambda = IM_+(E)$ for some E . Then there exists a unital subring R of \mathbb{R} such that $G_\lambda = R_+^\times$ by Lemma 3.6 of [14]. Then $\frac{1}{p} = \frac{1}{\lambda} + \dots + \frac{1}{\lambda} \in R_+^\times$. This contradicts that $\frac{1}{p} \notin G_\lambda$. However, we have another construction.

Example 3.11. For $\lambda = 3^2 = 9$, Matui shows us the following example: Let A be an AF -algebra such that

$$K_0(A) = \{(\frac{b}{9^a}, c) \in \mathbb{R} \times \mathbb{Z} \mid a, b, c \in \mathbb{Z}, b \equiv c \pmod{8}\},$$

$$K_0(A)_+ = \{(\frac{b}{9^a}, c) \in K_0(A) : \frac{b}{9^a} > 0\} \cup \{(0, 0)\} \quad \text{and} \quad [1_A]_0 = (1, 1).$$

Then

$$\mathcal{F}(A) = G_9 := \{9^n \in \mathbb{R}_+^\times \mid n \in \mathbb{Z}\}$$

Moreover $\tau_* : K_0(A) \rightarrow \tau_*(K_0(A))$ is not an order isomorphism and $\mathcal{F}(A) \neq IM_+(\tau_*(K_0(A)))$.

Furthermore Katsura suggests us the following examples: Let $\lambda = p^n$ for a prime number p and a natural number $n \geq 2$. Then there exists a simple AF -algebra A with unique trace such that $\mathcal{F}(A) = G_\lambda$.

First consider the case that $\lambda \geq 8$. Define

$$E = \{(\frac{b}{p^{na}}, c) \in \mathbb{R} \times \mathbb{Z} \mid a, b, c \in \mathbb{Z}, b \equiv c \pmod{p^n - 1}\}$$

$$E_+ = \{(\frac{b}{p^{na}}, c) \in E : \frac{b}{p^{na}} > 0\} \cup \{(0, 0)\} \quad \text{and} \quad [u]_0 = (1, 1).$$

Then there exists a simple AF -algebra A such that $(K_0(A), K_0(A)_+, [1_A]_0) = (E, E_+, u)$ by [4]. The classification theorem of [5] and some computation yield that $\mathcal{F}(A) = G_\lambda$.

Next consider the case that $\lambda = 2^2 = 4$. Let

$$E = \{(\frac{b}{16^a}, c) \in \mathbb{R} \times \mathbb{Z} \mid a, b, c \in \mathbb{Z}, b \equiv c \pmod{5}\}$$

$$E_+ = \{(\frac{b}{16^a}, c) \in E : \frac{b}{16^a} > 0\} \cup \{(0, 0)\} \quad \text{and} \quad [u]_0 = (1, 1).$$

Consider a simple AF -algebra A such that $(K_0(A), K_0(A)_+, [1_A]_0) = (E, E_+, u)$. Then $\mathcal{F}(A) = G_4$.

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