Certain rigidity theorem for compact manifolds with almost nonpositive Ricci curvature

中村, 拓也

https://doi.org/10.15017/2534379

Doctoral Dissertation

Certain rigidity theorem for compact manifolds with almost nonpositive Ricci curvature

Takuya Nakamura

Graduate School of Mathmatics, Kyushu University

Superviser: Professor Atsushi Katsuda

July 17, 2019

Contents

1 Introduction **1**

2 Preliminaries **3**
2.1 Vector bundle and linear connection 3
2.2 Riemannian geometry 6
2.3 Lie group, Lie algebra, and homogeneous space 21

3 Isometry group and Killing vector fields **24**
3.1 Definition and some properties 24
3.2 Correspondence between Killing vector fields and left invariant vector fields 26
3.3 Dimension of isometry group and Bochner's theorem 28

4 Riemannian curvature tensor on Lie group with left invariant metric **31**

5 Estimates of isoperimetric constants **33**
5.1 Some isoperimetric inequalities 33
5.2 Isoperimetric constant and isoperimetric function 33
5.3 Almgren's theorem and mean curvature 37
5.4 Estimate of isoperimetric constant $I_1(M)$ by Gallot 42
5.5 Etimate of isoperimetric constant $I_{(n-1)/n}(M)$ by Gallot 46

6 Gallot's two results **49**
6.1 Gallot's Sobolev inequality 49
6.2 Gallot's estimate of L^∞-norm from above by L^2-norm 53

7 Proof of Proposition 1.1 **56**

8 Proof of Theorem 1.4 **59**

1 Introduction

Let M be a compact connected n-dimensional Riemannian manifold. Bochner's celebrated theorem asserts that if M has nonpositive Ricci curvature, then the dimension of the space of Killing vector fields, i.e. that of the isometry group $\mathrm{Isom}(M)$, of M is smaller than or equal to n. Moreover, if it is equal to n, then M is isometric to a flat torus. The purpose of this paper is to give a perturbative result of this theorem. For a Riemannian manifold M, we denote by g_M the Riemannian metric, by Ric_M the Ricci tensor, and by $\mathrm{diam}(M)$ the diameter of M.

First we give the following proposition.

Proposition 1.1. *For constants $k, D > 0$, there exists a constant $\varepsilon = \varepsilon(n, k, D) > 0$ such that if a compact connected n-dimensional Riemannian manifold M satisfies*

$$-kg_M \leq \mathrm{Ric}_M \leq \varepsilon g_M,$$
$$\mathrm{diam}(M) \leq D,$$

then we have

$$\dim \mathrm{Isom}(M) \leq n$$

The proof of this proposition can be obtained by an easy modification of the proof due to Gallot [7] for the following theorem, which is a positive counterpart of Proposition 1.1.

Theorem 1.2 (Gromov [10], Gallot [7]). *For a constant $D > 0$, there exists a constant $\varepsilon = \varepsilon(n, D) > 0$ such that if a compact connected orientable n-dimensional Riemannian manifold M satisfies*

$$\mathrm{Ric}_M \geq -\varepsilon g_M,$$
$$\mathrm{diam}(M) \leq D,$$

then the first Betti number $b_1(M)$ of M satisfies

$$b_1(M) \leq n.$$

Our main concern is the case when $\dim \mathrm{Isom}(M) = n$. A positive counterpart is already obtained by Colding [6], Cheeger-Colding [4].

Theorem 1.3. *For a constant $D > 0$, there exists a constant $\varepsilon = \varepsilon(n, D) > 0$ such that if a compact connected n-dimensional Riemannian manifold M satisfies*

$$\mathrm{Ric}_M \geq -\varepsilon g_M,$$
$$\mathrm{diam}(M) \leq D,$$
$$b_1(M) = n,$$

then M is diffeomorphic to an n-torus $\mathbb{T}^n$.

Note that their proof does not give an explicit estimate of the constant $\varepsilon = \varepsilon(n, D)$ since it utilizes some compactness arguments. Our main result in this note is the followings.

THEOREM 1.4. *For constants $k, D > 0$, there exists a constant $\varepsilon = \varepsilon(n, k, D) > 0$ such that if a compact connected n-dimensional Riemannian manifold M satisfies*

$$-kg_M \leq \mathrm{Ric}_M \leq \varepsilon g_M,$$
$$\mathrm{diam}(M) \leq D,$$
$$\dim \mathrm{Isom}(M) = n,$$

then M is isometric to a flat n-torus $\mathbb{T}^n$.

The proof of this theorem is different from that of Theorem 1.3. Moreover, since we do not use any kind of compactness or convergence arguments, we can estimate $\varepsilon = \varepsilon(n,\ k,\ D)$ explicitly.

This paper ia organized as follows. In section 2, we prepare basic notions and terminologies of Riemannian geometry and Lie group theory. In section 3, we recall basic properties of isometry groups and Killing vector fields. In section 4, we recall the Riemannian curvature tensor of Lie group with left invariant metric. To prove Theorem 1.4, we consider the curvature of isometry group, which is a Lie group, with left ivariant metric. In section 5, we introduce the notion of the isoperimetric constant and recall Gallot's estimate of isoperimetric constant. In section 6, we recall Gallot's two results. One is a Sobolev inequality, and the other is an estimate of L^∞-norm by L^2-norm, which is used the isoperimetric constant. In section 7, we give a proof of Proposition 1.1, which is used Gallot's results in section 5 and section 6. In section 8, we give a proof of Theorem 1.4. To prove this, we shall show that a given Riemannian manifold is homogeneous and almost flat, and apply the structure theorem of compact Lie group to the identity component of the isometry group.

Acknowledgements

I am deeply grateful to my supervisor Professor Atsushi Katsuda who provided enormous support and insightful suggestions. I would also like to thank Professor Shouhei Honda for useful comments and information on [16] and the referee of [11] for pointing out a short-cut of our proof.

2 Preliminaries

In this section, we prepare basic notion and terminologiy with respect to Riemannian geometry, and Lie group theory. We refer to [15] and [3]. Throughout this paper, we assume that manifold is Hausdorff and second countable.

2.1 Vector bundle and linear connection

DEFINITION 2.1 (Vector bundle). Let M and E be smooth manifolds, let $\pi : E \to M$ be a smooth map, and let k be a nonnegative integer. If the following two conditions are satisfied, then the triple (E, M, π) is said to be a *vector bundle* of rank k over M.
(i) For every $p \in M$, $E_p := \pi^{-1}(p)$ has the structure of a k-dimensional real vector space.
(ii) For every $p \in M$, there exists a coordinate neighborhood U of p and a diffeomorphism $\tilde{\varphi} : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that the following hold:

(a) The equality $\mathrm{pr}_1 \circ \tilde{\varphi} = \pi|\pi^{-1}(U)$ holds, where pr_1 is the canonical projection from $U \times \mathbb{R}^k$ to U.

(b) For all $q \in U$, the map $\mathrm{pr}_2 \circ \tilde{\varphi}|\pi^{-1}(q) : \pi^{-1}(q) \to \mathbb{R}^k$ is a linear isomorphism, where pr_2 is the canonical projection from $U \times \mathbb{R}^k$ to $\mathbb{R}^k$.

For a vector bundle (E, M, π), the manifold E is called a *total space*, the manifold M is called a *base space*, the map π is called a *projection*, and the vector space $E_p = \pi^{-1}(p)$ is called a *fiber* at $p \in M$. A pair $(U, \tilde{\varphi})$ in Definition 2.1 is called a *local trivialization*. If $(U, \tilde{\varphi}), (V, \tilde{\psi})$ is local trivializations, then the map $\tilde{\psi} \circ \tilde{\varphi}^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ is diffeomorphic by the definition of the local trivialization. We also say just vctor bundle E for short, omitting M and π.

DEFINITION 2.2 (Bundle map and bundle isomorphism). Let (E, M, π) and (F, N, ρ) is vector bundles, let $\Phi : E \to F$, $f : M \to N$ be smooth maps. The map Φ is called a *bundle map* from (E, M, π) to (F, N, ρ) covering f provided the follwing properties hold.

(i) $\rho \circ \Phi = f \circ \pi$,

(ii) for every $p \in M$ the image $\Phi(E_p)$ is a vetor subspace of $F_{f(p)}$,

(iii) for every $p \in M$ the map $\Phi|E_p : E_p \to F_{f(p)}$ is a linear map.

In Particular, if $M = N$, $f = \mathrm{id}_M$, and the bundle map Φ is bijective, then Φ is called a *bundle isomorhism* and the vector bundle (E, M, π) is called to be *isomorphic* to (F, N, ρ).

For a vector space V we denote by V^* the space of dual space of V, defined by $V^* := \{f : V \to \mathbb{R} \,|\, \text{linear map}\}$.

Definition 2.3 (Dual bundle). Let $E = (E, M, \pi)$ be a vector bundle and take the dual space $(E_p)^*$ of each fibre E_p. Put

$$E^* := \bigsqcup_{p \in M} (E_p)^*$$

and define the map $\pi^* : E^* \to M$ as

$$\pi^*(v_p^*) := p.$$

for any $v_p^* \in (E_p)^*$. Then, we can natulally define the structure of a vector bundle with respect to the triple (E^*, M, π^*) which is called a *dual bundle* of E.

Definition 2.4 (Tangent bundle and cotangent bundle). Let M be an n-dimensional manifold, let TM be the disjoint union of the family of tanget spaces $\{T_pM\}_{p \in M}$, and define the map $\pi_M : TM \to M$ as $\pi_M(v_p) := p$ for $v_p \in T_pM$. Then, we can natulally define the structure of a vector bundle with respect to the triple (TM, M, π_M). The vector bundle TM is called a *tangent bundle* and the dual bundle $T^*M := (TM)^*$ of TM is called a *cotangent bundle*.

Definition 2.5 (Tensor and tensor space). Let V be an n-dimensionl real vector space and V^* be a dual space of V. Then, for nonnegative integers r, s, the (r, s)-*tensor space* $T_s^r(V)$ is defined by

$$T_s^r(V) := \{t : \overbrace{V^* \times \cdots \times V^*}^{r} \times \overbrace{V \times \cdots \times V}^{s} \to \mathbb{R} \mid f \text{ is a multilinear map}\}$$
$$\text{(In the case } (r, s) = (0, 0) \text{, we identify } T_s^r(V) = \mathbb{R}).$$

A element of the (r, s)-tensor space $T_s^r(V)$ is called a (r, s)-*tensor* on V

Let $\{v_i\}_i$ be a basis of V and let $\{v^i\}_i$ be the dual basis of $\{v_i\}_i$. We define $v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s} \in T_s^r(V)$ $(i_k = 1, \ldots, n\,(k = 1, \ldots, r), j_l = 1, \ldots, n\,(l = 1, \ldots, s))$ as

$$v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s}(u_1^*, \ldots, u_r^*, w_1, \ldots, w_s) := \prod_{k=1}^{r} u_k^*(v_{i_k}) \prod_{l=1}^{s} v^{j_l}(v_l).$$

Then, $\{v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s}\}_{i_1, \ldots, i_r, j_1, \ldots, j_s}$ becomes a basis of the vector space $T_s^r(V)$, and $\dim T_s^r(V) = n^{r+s}$.

Definition 2.6 (Tensor bundle). Let (E, M, π) be a vector bundle of rank k. Put

$$T_s^r(E) := \bigsqcup_{p \in M} T_s^r(E_p),$$

and define the map $\pi^r_s : T^r_s(E) \to M$ as

$$\pi^r_s(t_p) := p$$

for any $t_p \in T^r_s(E_p)$. Then, we can natulally define the structure of a vector bundle of rank k^{r+s} with respect to the triple $(T^r_s(E), M, \pi^r_s)$. This vector bundle $T^r_s(E)$ is called a (r,s)-*tensor bundle* of E. In particular, if the vector bundle E is the tangent bandle TM, then $T^r_s(M) := T^r_s(TM)$ is called a (r,s)-tensor bundle over M. Note that $T^1_0(E) = E^{**}$ and $T^0_1(E) = E^*$.

Remark 2.7. A $(1,0)$-tensor bundle $T^1_0(E) = E^{**}$ is identified with the tangent bundle E by the natural bundle isomorphism $T : E \to E^{**}$, defined by $(T(u_p))(v^*_p) := v^*_p(u_p)$ for $u_p \in E_p$ and $v^*_p \in E^*_p$.

Definition 2.8 (Section of vector bundle). Let $E = (E, M, \pi)$ be a vector bundle. A smooth map $\xi : M \to E$ is called a *section* of E if $\pi \circ \xi = \mathrm{id}_M$. We denote by $\Gamma(E)$ the space of all sections, which has a structure of a $C^\infty(M)$-module.

Remark 2.9. $T^0_0(E)$ is identified with $C^\infty(M)$. $\Gamma(T^1_0(E)) = \Gamma(E^{**})$ is also identified with $\Gamma(E)$ (see Remark 2.7). If $E = TM$, then $\Gamma(TM)$ is the space $\mathcal{X}(M)$ of all vector fields and $\Gamma(T^*M)$ is the space $\Omega(M)$ of all differential 1-forms on M. In particular, $(1,0)$-tensor fields on M are identified with vector fields on M.

Definition 2.10 (Tensor field). Let $E = (E, M, \pi)$ be a vector bundle. Then, a section of $T^r_s(E)$ is called a (r,s)-*tensor field*.

Let $T \in T^r_s(E)$, $\omega_i \in \Gamma(E^*)$ $(i = 1, \ldots, r)$, and $X_j \in \Gamma(E)$ $(j = 1, \ldots, s)$. The C^∞-function $T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \in C^\infty(M)$ on M is defined by

$$T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s)(p) := T_p(\omega_1(p), \ldots, \omega_r(p), X_1(p), \ldots, X_s(p)).$$

Then, the map $T : (\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \mapsto T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s)$ becomes a $C^\infty(M)$-multilinear map from $\overbrace{\Gamma(E^*) \times \cdots \times \Gamma(E^*)}^{r} \times \overbrace{\Gamma(E) \times \cdots \times \Gamma(E)}^{s}$ to $C^\infty(M)$. Conversely, let $T : \overbrace{\Gamma(E^*) \times \cdots \times \Gamma(E^*)}^{r} \times \overbrace{\Gamma(E) \times \cdots \times \Gamma(E)}^{s} \to C^\infty(M)$ be a C^∞-multilinear map. For each $p \in M$ we define the (r,s)-tensor $T_p \in T^r_s(E_p)$ as for $\alpha_i \in E^*_p$ $(i = 1, \ldots r)$ and for $v_i \in E_p$ $(j = 1, \ldots, s)$

$$T_p(\alpha_1, \ldots, \alpha_r, v_1, \ldots, v_s) := T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s)(p)$$

where $\omega_i \in \Gamma(E^*)$ $(i = 1, \ldots r)$ with $\omega_i(p) = \alpha_i$ and $X_j \in \Gamma(E)$ $(j = 1, \ldots, s)$ with $X_i(p) = v_i$, which is well-difined. These give a correspondence between a (r,s)-tensor field and a $C^\infty(M)$-multiliniear map from $\overbrace{\Gamma(E^*) \times \cdots \times \Gamma(E^*)}^{r} \times \overbrace{\Gamma(E) \times \cdots \times \Gamma(E)}^{s}$ to $C^\infty(M)$.

Example 2.11. The identification between a $(1,0)$-tensor fields on M and a vector fields on M in Remark 2.9 is also given as follows: Let $X \in \mathcal{X}(M)$. Then, the corresponding element $T_X \in \Gamma(T_0^1(M))$, which is is given by $T_X(\omega) = \omega(X)$ for $\omega \in \Omega(M)$. Conversely, let $T \in \Gamma(T_0^1(M))$. Then, the corresponding element $X_T \in \mathcal{X}(M)$ is given by $X_T(f) = T(df)$ for $f \in C^\infty(M)$.

Example 2.12. By Example 2.11, a $C^\infty(M)$-multilinaer map

$$T : \overbrace{\Omega(M) \times \cdots \times \Omega(M)}^{r} \times \overbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}^{s} \to \mathcal{X}(M)$$

is regarded as a C^∞-multilnilear map

$$T : \overbrace{\Omega(M) \times \cdots \times \Omega(M)}^{r+1} \times \overbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}^{s} \to C^\infty(M).$$

Thus, we can regard T as $(r+1, s)$-tensor field on M.

Definition 2.13 (Linear connection and covariant derivative). Let (E, M, π) be a vector bundle. Then, a real bilinear map $\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), (X, \xi) \mapsto \nabla_X \xi$ is called a *linear connection* on E provided ∇ satisfies that

$$\begin{cases} \nabla_{fX}\xi = f\nabla_X\xi, \\ \nabla_X(f\xi) = X(f)\xi + f\nabla_X\xi. \end{cases}$$

for any $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, and $\xi \in \Gamma(E)$. A section $\nabla_X\xi \in \Gamma(E)$ is also called a *covariant derivative* of ξ with resoect to X.

Definition 2.14 (Positive definite and symmetric $(0,2)$-tensor field). Let E be a vector bundle and g be a $(0,2)$-tensor field on E.

(i) The $(0,2)$-tensor field g is called to be *positive definite* provided for any $X \in \Gamma(E)$ the inequality $g(X,X) \geq 0$ holds, and equality holds if and only if $X \equiv 0$.

(ii) The $(0,2)$-tensor field g is called to be *symmetric* provided for any $X, Y \in \Gamma(E)$ the inequality $g(X,Y) = g(Y,X)$ holds.

Definition 2.15 (Bundle metric and Riemannian vector bundle). Let E be a vector bundle. Then, a positive definite symmetric $(0,2)$-tensor field is called a *bundle metric* on E and the pair (E, g) is called a *Riemannian vector bundle*.

2.2 Riemannian geometry

Definition 2.16 (Riemannian metric and Riemannian manifold). Let M be a smooth manifold. Then, a positive definite symmetric $(0,2)$-tensor field g on M is called a *Riemannian metric* on M and the pair (M, g) is called a *Riemannian Manifold*. Note that for each $p \in M$ the $(0,2)$-tensor g_p become an inner product on T_pM.

For simplicity, for tangent vectors $u, v \in T_pM$ and vector fields $X, Y \in \mathcal{X}(M)$, we denote the inner product by $\langle u, v\rangle = g_p(u, v)$ and $\langle X, Y\rangle = g(X, Y)$, and denote the norm by $|u| = g_p(u, u)^{1/2}$ and $|X| = g(X, X)^{1/2}$.

EXAMPLE 2.17. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real inner product space, $u \in V$. We define the linear isomorphism $\iota_u : V \to T_uV$ as $\iota_u(v) := \dot{c_v}(0) \in T_uV$ for any $v \in V$, where c_v is the curve on V defined by $c_v(t) := u + tv$ for any $t \in \mathbb{R}$. Then, the canonical Riemannian metric g_V on V is defined by

$$(g_V)_u(\iota_u(v), \iota_u(w)) := \langle v, w\rangle$$

for any $u, v, w \in V$.

The canonical Riemannian metirc on the Euclidean space $\mathbb{R}^n$ is defined as in Example 2.17. For any Riemannian manifold M and $p \in M$ we shall also consider the Riemannian metric on T_pM defined as in Example 2.17 (consider $(g_M)_p$ as the inner product on $V = T_pM$) and identify $\iota_u(v) \in T_u(T_pM)$ with $v \in T_pM$.

DEFINITION 2.18 (Induced metric). Let (M, g) be a Riemannian manifold and let N be a manifold and let $\varphi : N \to M$ be a immersion map. Then, the map φ induces Riemannian metric h on N as follows: for $u, v \in T_pN$

$$h_p(u, v) := g_{\varphi(p)}(d\varphi(u), d\varphi(v)).$$

The metric h is called a *induced metric* (*pullback metric*) of g by φ, and denoted by φ^*g.

DEFINITION 2.19 (Riemannian submanifold). Let N be a submanifold of a Riemannian manifold (M, g). Then, for the inclusion map $\iota : N$, the Riemannian manifold (N, ι^*g) is called a *Riemannian submanifold* of (M, g).

DEFINITION 2.20 (Local isometry and isometry). Let (M, g) and (N, h) be Riemannian manifolds and let $\varphi : M \to N$ is a smooth map. If the map φ satisfies $\varphi^*h = g$, then φ is called a *local isometry*. Moreover, if the map φ is diffeomorphism, then φ is called a an *isometry*.

PROPOSITION 2.21. *Let M, N be Riemannian manifolds and $f : M \to N$ be a smooth map. Then, the following are equivalent.*

(i) f is an isometry.

(ii) f is a bijective local isometry.

DEFINITION 2.22 (Riemannian covering). Let M, N be Riemannian manifolds and $\pi : M \to N$ be a covering map. If π is a local isometry, then π is called a *Riemannian covering*. In particular, if M is simply connected, then π is called a universal Riemannian covering.

Proposition 2.23. *Let M be a Riemannian manifold and N be a smooth manifold. Let $\pi : M \to N$ be a covering map. If for all deck transformations of π are isometries, then there exists a unique Riemannian metric on N such that π is a Riemannian covering.*

Definition 2.24 (Length of curves). Let M be a Riemannian manifold and let $c : [a, b] \to M$ be a smooth curve on M. Then, the *length* $L(c)$ of the curve c, is defined by

$$L(c) := \int_a^b |\dot{c}(t)|\, dt$$

where $\dot{c}(t) \in T_{c(t)}M$ is the velocity vector of c at $t \in [a, b]$.

Definition 2.25 (Riemannian distance function). Let M be a connected Riemannian manifold. Then, the *Riemannian distance function* $d : M \times M$ is defined by

$$d(p, q) := \inf\{L(c) \mid c \text{ is a smooth curve from } p \text{ to } q\}.$$

Proposition 2.26. *The Riemannian distance function is a distance. Moreover, the topology induced by the Riemannian distance function coincides with the topology as a manifold.*

Definition 2.27 (Levi-Civita connection). Let M be a Riemannian manifold. Then, the *Levi-Civita connection* ∇ is a connection on the tangent bundle TM satisfying that for vector fields $X, Y, Z \in \mathcal{X}(M)$,

$$\nabla_X Y - \nabla_Y X = [X, Y],$$
$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

where $[X, Y]$ is the Lie bracket of vector fields X and Y.

Proposition 2.28. *The Levi-Civita connection exists and is unique, and for smooth vector fields X, Y, Z the folloing equality holds:*

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\{&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &+ \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle\}.\end{aligned}$$

Remark 2.29. If vector fields $X, X' \in \mathcal{X}(M)$ satisfy that $X_p = X'_p$ for some $p \in M$, then we have $(\nabla_X Y)_p = (\nabla_{X'} Y)_p$ for any $Y \in \mathcal{X}(M)$. Thus, for $v \in T_pM$ and $Y \in \mathcal{X}(M)$ we can define the vector $\nabla_v Y \in T_pM$

$$\nabla_v Y := (\nabla_X Y)_p$$

for some $X \in \mathcal{X}(M)$ with $X_p = v$, which is well-defined. Moreover, if $Z \in \mathcal{X}(M)$ satisfies that $Y = Z$ on some smooth curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$ and $\dot{c}(0) = v$, then we have $\nabla_v X = \nabla_v Y$.

Let M, N be Riemannian manifolds and φ be a diffeomorphism from M to N. For a vector field $X \in \mathcal{X}(M)$, we define the vector field $d\varphi(X)$ as

$$d\varphi(X)_p := d\varphi(X_{\varphi^{-1}(p)})$$

for any $p \in M$.

Proposition 2.30. *Let M, N be Riemannian manifolds and $\nabla, \bar{\nabla}$ be Levi-Civita connectons on M and N, respectively. Suppose that a smooth map $\varphi : M \to N$ is isometric. Then, the follwing holds: for any vector fields $X, Y \in \mathcal{X}(M)$,*

$$d\varphi(\nabla_X Y) = \bar{\nabla}_{d\varphi(X)} d\varphi(Y).$$

Definition 2.31 (Gradient, Hessian, divergence, and Laplacian). Let M be a Riemannian manifold.
(1) For $f \in C^\infty(M)$, the *gradient* vector field of f, denoted by ∇f, is the smooth vector field on M defined by

$$Xf := \langle \nabla f, X \rangle, \qquad X \in \mathcal{X}(M).$$

(2) For $f \in C^\infty(M)$, the *Hessian* of f, denoted by $\mathrm{Hess}(f)$, is the symmetric $(0,2)$-tensor field on M defined by

$$\mathrm{Hess}(f)(X, Y) := \langle \nabla_X \nabla f, Y \rangle, \qquad X, Y \in \mathcal{X}(M).$$

(3) For $X \in \mathcal{X}(M)$, the *divergence* of X, denoted by $\mathrm{div}(X)$, is the smooth function on M defined by

$$\operatorname{div} X(p) := \mathrm{trace}(T_pM \to T_pM, u \mapsto \nabla_u X).$$

(4) For $f \in C^\infty(M)$, the *Laplacian* of f, denoted by Δf, is the smooth function on M defined by

$$\Delta f := -\,\mathrm{trace}\,\mathrm{Hess}(f) = -\,\mathrm{div}(\nabla f).$$

Levi-Civita connection is extended on the (r,s)-tensor bundle $T^r_s(M)$ as follows: for any $T \in T^r_s(M)$ and $X \in \mathcal{X}(M)$, we define $\nabla_X T$ as

- $(r,s) = (0,0)$ $(T \in C^\infty(M))$

$$\nabla_X T := X(T),$$

- $(r,s) = (0,1)$

$$\nabla_X T(Y) := X(T(Y)) - T(\nabla_X Y)$$

for any $X \in \mathcal{X}(M)$,

- $(r,s) \neq (0,0),(0,1)$

$$\begin{aligned}\nabla_X T(\omega_1,\dots,\omega_r,X_1,\dots,X_s) := {} & X(T(\omega_1,\dots,\omega_r,X_1,\dots,X_s)) \\ & - \sum_{i=1}^{r} T(\omega_1,\dots,\nabla_X\omega_i,\dots,\omega_r,X_1,\dots,X_s) \\ & - \sum_{j=1}^{s} T(\omega_1,\dots,\omega_r,X_1,\dots,\nabla_X X_j,\dots,X_s),\end{aligned}$$

for any $X_i \in \mathcal{X}(M)$ $(i=1,\dots,r)$ and $\omega_j \in \Omega(M)$ $(j=1,\dots,s)$.

For a vector field X on M and a $(0,1)$-tensor field T on M which corresponds to X in the sense of the identification as in Remark 2.9, $\nabla_Y X$ also corresponds to $\nabla_Y T$ for every $Y \in \mathcal{X}(M)$ by Example 2.11.

Definition 2.32 (∇T and ∇X $(T \in \Gamma(T^r_s(M)),\ X \in \mathcal{X}(M))$). Let T be a (r,s)-tensor field on a Riemannian manifold M. Then, the $(r,s+1)$-tensor ∇X is defined by

$$\nabla T(\omega_1,\dots,\omega_r,X,X_1,\dots,X_s) := \nabla_X T(\omega_1,\dots,\omega_r,X_1,\dots,X_s)$$

for any $X, X_i \in \mathcal{X}(M)$ $(i=1,\dots,r)$ and $\omega_j \in \Omega(M)$ $(j=1,\dots,s)$. Similarly, for a vector field X, we define the $C^\infty(M)$-linear map $\nabla X : \mathcal{X}(M) \to \mathcal{X}(M)$ as

$$\nabla X(Y) := \nabla_Y X$$

for $Y \in \mathcal{X}(M)$.

Definition 2.33 (Parallel tensor field and parallel vector field). We say that the tensor field T on a Riemannian manifold is *parallel* provided $\nabla T \equiv 0$. Similarly, we say that the vector field X on a Riemannian manifold is parallel provided $\nabla X = 0$.

For a smooth manifolds M, N and a smooth map $\varphi : N \to M$, we denote by $\mathcal{X}(\varphi, M)$ the space of smooth functions from N to TM such that for any $p \in N$ the image of p is an element of $T_{\varphi(p)}M$. $\mathcal{X}(\varphi, M)$ become a $C^\infty(N)$-module.

Definition 2.34 (Vector field along curve). Let M be a smooth manifold and $c : (a,b) \to M$ be a smooth cureve on M. Then, an element of $\mathcal{X}(c, M)$ is called a vector field along the curve c.

Definition 2.35 (Covariant derivative of $Y \in \mathcal{X}(\varphi, M)$). Let $\varphi : N \to M$ be a smooth map from a smooth manifold N to an n-dimensional Riemannian manifold N and ∇ be a Levi-Civita connection on M. Then, we define the covariant derivative $\nabla_X Y \in \mathcal{X}(\varphi, M)$ of $Y \in \mathcal{X}(\varphi, M)$ with respect to $X \in \mathcal{X}(N)$ as for any $p \in N$

$$(\nabla_X Y)(p) := \sum_{i=1}^{n} \left(X_p(Y^i) \frac{\partial}{\partial x^i}(\varphi(p)) + \nabla_{d\varphi(X_p)} \frac{\partial}{\partial x^i} \right) \qquad \left(\in T_{\varphi(p)}M\right),$$

where $(x^1, \dots, x^n)$ is a coordinate neighbourhood of $\varphi(p) \in M$ and Y^i $(i = 1, \dots, n)$ is the smooth function on the coordinate neighbourhood such that $Y(q) = \sum_{i=1}^n Y^i(q)\partial/\partial x^i(\varphi(q))$. This definition is well-defined. Moreover, since for any $f \in C^\infty(N)$, $X \in \mathcal{X}(N)$, and $Y \in \mathcal{X}(\varphi, M)$ the equalities

$$\begin{cases} \nabla_{fX} Y = f\nabla_X Y, \\ \nabla_X(fY) = X(f)Y + f\nabla_X Y \end{cases}$$

hold, the real bilinear map $\nabla : \mathcal{X}(N) \times \mathcal{X}(\varphi, M) \to \mathcal{X}(\varphi, M), (X, Y) \mapsto \nabla_X Y$ regard as a linear connection. In the same way as Remark 2.29, for $v \in T_pN$ and $Y \in \mathcal{X}(\varphi, M)$ we can define the vector $\nabla_v Y \in T_{\varphi(p)}M$.

For a vector field $X(t)$ along a curve $c(t)$ on a Riemannian manifold, we sometimes denote by ∇X the covariant derivative $\nabla_{\frac{d}{dt}} X$ for simplicity.

Definition 2.36 (Parallel vector field along curve). Let $c : (a, b) \to M$ be a curve on a Riemannian manifold M and X be a vector field along c. Then, X is called parallel if $\nabla X(t) = 0$ for any $t \in (a, b)$.

Proposition 2.37. *Let $c : (a, b) \to M$ be a smooth curve on a Riemannian manifold M. Then, the following hold.*

(i) For any $t_0 \in (a, b)$ and $u \in T_c(t)M$, there exists a unique parallel vector field X along c such that $X(t_0) = u$.

(ii) For any parallel vector fields X, Y along c, the following holds: for any $s, t \in (a, b)$,

$$\langle X(s), Y(s)\rangle = \langle X(t), Y(t)\rangle.$$

In particular, $|X(t)|$ is a constant function.

Now we define the curvatures.

Definition 2.38 (Riemannian curvature tensor). Let M be a Riemannian manifold. Then, the *Riemannian curvature tensor* of M, denoted by R, is $(1, 3)$-tensor field on M defined by

$$R(X, Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in \mathcal{X}(M).$$

Proposition 2.39. *The Riemannian curvature tensor R on a Riemannian manifold M satisfies the following inequalities: for any $X, Y, Z, W \in \mathcal{X}(M)$,*

$$\begin{aligned}
&R(X, Y)Z = -R(Y, X)Z, \\
&R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \\
&\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle, \\
&\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle, \\
&(\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W + (\nabla_Z R)(X, Y)W = 0.
\end{aligned}$$

Proposition 2.40. *Let M, N be Riemannian manifolds and $R, \bar{R}$ be Riemannian curvature tensor on M, N. Suppose that a smooth map $\varphi : M \to N$ is locally isometric. Then, the following holds: for any $u, v, w \in T_pM$ $(p \in M)$,*

$$d\varphi(R(u,v)w) = \bar{R}(d\varphi(u), d\varphi(v))d\varphi(w).$$

Definition 2.41 (Sectional curvature). Let M be a Riemannian manifold. Let σ be a two-dimensional subspace of T_pM and $\{u, v\}$ be a basis of σ. We define the *sectional curvature* K_σ of σ as

$$K_\sigma := \frac{\langle R(u,v)v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2},$$

which dose not depend on the choice of the basis $\{u, v\}$ of σ. We sometimes denote by $K(u, v)$ instead of K_σ.

If there exists a constant $c \in \mathbb{R}$ such that for any $p \in M$ and two-dimensional subspace σ of T_pM the sectional curvature K_σ is equal to c, then we say that M has constant curvature c. A Riemannian manifold with constant curvature 0 is asid to be flat. The Euclidean space with canonical Riemannian metric has constant curvature 0.

Definition 2.42 (Flat torus). Let Γ be a descrete subgroup of $\mathbb{R}^n$ such that Γ is isomorphic to $\mathbb{Z}^n$. By Proposition 2.23, there exists a unique metric $g_{\mathbb{T}^n}$ on $\mathbb{T}^n := \mathbb{R}^n/\Gamma$ such that the projection map $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is a universal Riemannian covering. The Riemannian manifold $(\mathbb{T}^n, g_{\mathbb{T}^n})$ is flat and called a *flat torus*.

Definition 2.43 (Ricci curvature tensor). Let M be a Riemannian manifold. We define the *Ricci curvature tensor*, which is a symmetric $(0, 2)$-tensor field, as follows: for any $u, v \in T_pM$ $(p \in M)$,

$$\mathrm{Ric}(u, v) := \mathrm{trace}(w \mapsto R(w, u)v).$$

Definition 2.44 (Normal bundle and normal vector field). Let M be a Riemannian manifold and N be a submanifold of M. Put

$$TN^\perp := \bigsqcup_{p \in M} T_pN^\perp,$$

where $T_pN^\perp$ is the orthogonal complement of $T_pN \subset T_pM$. Then, $TN^\perp$ becomes a submanifold of TM. Moreover, for the smooth map $\pi_N^\perp := \pi_M|TN^\perp : TN^\perp \to N$, the triple $(TN^\perp, \pi_N^\perp, N)$ become a vector bundle. This vector bundle is called a *normal bundle*, and a section of $TN^\perp$ is called a normal vector field on N.

Definition 2.45 (Shape operator). Let M be a Riemannian manifold and N be a submanifold of M. Let ∇ be the Levi-Civita connecton on M. For $\xi \in T_pN^{\perp}$ $(p \in N)$, we define the *shape operator* $A_\xi : T_pN \to T_pN$ as

$$A_\xi(u) := (\nabla_u X)^{\top},$$

where X is a normal vector field on N such that $X_p = \xi$ and $(\nabla_u X)^{\top}$ is the horizontal component of $\nabla_u X$ with respect to T_pN. This definition is not depend on the choice of X, and A_ξ become a symmetric linear operator.

Definition 2.46 (Mean curvature). Let H be a hypersurface of an n-dimensional Riemannian manifold M and ν be a unit normal vector to H. Then, the *mean curvature* η of H with respect to ν is defined as

$$\eta := \frac{1}{n-1} \operatorname{trace} A_\nu = \frac{1}{n-1} \sum_{i=1}^{n-1} \langle A_\nu(e_i), e_i \rangle$$

where $\{e_i\}_{i=1}^{n-1}$ is an orthonormal basis of T_pN.

Next, we define geodesics, exponetinal maps, and Jacobi fields.

Definition 2.47 (Geodesic). Let γ be a smooth curve on a Riemannian manifold. We say that γ is a *geodesic* provided γ satisfies

$$\nabla_{\frac{d}{dt}} \dot{\gamma}(t) \equiv 0.$$

Remark 2.48. Geodesics on a Riemannian manifold exist and are unique in the following sense:

(i) For any $t_0 \in \mathbb{R}$ and $u \in TM$, there exist an open interbal (a, b) and a geodesic $\gamma : (a, b) \to M$ such that $t_0 \in (a, b)$, $\gamma(t_0) = \pi_M(u)$, and $\dot{\gamma}(t_0) = u$.

(ii) For geodesics $\gamma : (a, b) \to M$ and $\delta : (c, d) \to M$, if $\gamma(t_0) = \delta(s_0)$ $(t_0 \in (a, b)$, $s_0 \in (c, d)$) and $d\gamma/dt(t_0) = d\delta/ds$, then the equality

$$\gamma(t) = \delta(t - t_0 + s_0)$$

holds for any $t \in (a, b) \cap (t_0 + c - s_0, t_0 + d - s_0)$. Moreover, γ is extend on the open interval (a', b'), where $a' := \min\{a, t_0 + c - s_0\}$ and $b' := \max\{b, t_0 + d - s_0\}$).

For $u \in TM$, we denote by γ_u the geodesic on a Riemannian manifold M with $\gamma_u(0) = \pi_M(u)$ and $\dot{\gamma}(0) = u$. For $a \in \mathbb{R}$ and $u \in TM$, if $\gamma_{au}(t)$ is defiend, then the inequality

$$\gamma_{au}(t) = \gamma_u(at)$$

holds. In particular, $\gamma_u(t) = \gamma_{tu}(1)$.

Proposition 2.49. *Let M be a Riemannian manifold and $u_0 \in TM$. Then, there exist $\varepsilon > 0$ and a neighbourhood $U \subset TM$ of u_0 such that for any $u \in U$ the geodesic γ_u is defined on the open interval $(-\varepsilon, \varepsilon)$ and the map $U \times (-\varepsilon, \varepsilon) \to M, (u, t) \mapsto \gamma_u(t)$ is smooth.*

Definition 2.50 (Exponential map). Let M be aRiemannian manifold and $p \in M$. Put

$$\tilde{U} := \{u \in T_pM \mid \text{the geodesic } \gamma_u(t) \text{ is defined at } t = 1\},$$

which is an open set in T_pM. Then, the *exponential map* $\exp_p : \tilde{U} \to M$ at p is defined as

$$\exp_p u := \gamma_u(1),$$

which is smooth.

Proposition 2.51. *The exponetial map $\exp_p$ is a diffeomorphism from a neighbourhood of the origin of T_pM to a neighbourhood of p.*

Definition 2.52 (Normal coordinate system). Let M be a Riemannian manifold, $p \in M$, and $\{e_i\}_{i=1}^n$ be an orthonormal basis of T_pM. By Proposition 2.51, a differomorphism f from a neibourhood $\tilde{U}$ of $0 \in \mathbb{R}^n$ to a neibourhood U of p is defined as

$$f(x^1, \ldots, x^n) := \exp_p(x^1 e_1 + \ldots + x^n e_n).$$

Then, $(U, \varphi := f^{-1})$ become a coordinate system on M. The coordinate system (U, φ) is called a *normal coordinate system* at p.

Proposition 2.53. *Let M be a Riemannian manifold and $p \in M$. For the normal coordinate sysytem $(U, (x^1, \ldots, x^n))$ at p, we put $\partial_i := \partial/\partial x^i$ $(i = 1, \ldots, n)$. Then, the following hold:*

(i)

$$\langle \partial_i, \partial_j \rangle (p) = \delta_{ij}.$$

(ii) For any $X \in \mathcal{X}(M)$,

$$\nabla_X \partial_i(p) = 0.$$

Definition 2.54 (Injective radius). Let M be a Riemannian manifold. Then, the *injectivity radius* at $p \in M$ is defined as

$$i_p(M) := \sup\{r > 0 \mid \exp_p |B(o_p, r) \text{ is a diffeomorphism}\},$$

where o_p is the origin of the tangent space T_pM. The injectivity radius of M is defined as

$$i(M) := \inf\{i_p(M) \mid p \in M\}.$$

PROPOSITION 2.55. *For a compact Riemannian manifold, the injectivity radius $i(M)$ is positive.*

DEFINITION 2.56 (Normal exponetial map). Let M be a Riemannian manifold and N be a submanifold of M. Put

$$\tilde{V} := \{u \in TN^{\perp} \mid \text{the geodesic } \gamma_u(t) \text{ is defined at } t = 1\},$$

which is an open set in $TN^{\perp}$. Then, the *normal exponential map* $\exp_N : \tilde{V} \to M$ of N is defined as

$$\exp_N u := \gamma_u(1),$$

which is smooth.

DEFINITION 2.57 (Jacobi field). The vector field Y along a geodesic γ on a Riemannian manifold is called a *Jacobi field* provided the equality

$$\nabla\nabla Y(t) = R(Y(t), \dot{\gamma}(t))\dot{\gamma}(t)$$

holds.

PROPOSITION 2.58. *Let $\gamma : (a, b) \to M$ be a geodesic on a Riemannian manifold and $t_0 \in (a, b)$. Then, for any $u, v \in T_{c(t_0)}M$, there exists a unique Jacobi field along γ with $Y(t_0) = u$, $\nabla Y(t_0) = v$.*

EXAMPLE 2.59. Let M be a Riemannian manifold and $u, v \in T_pM$. Then, the Jacobi field Y along γ_u with $Y(0) = 0, \nabla Y(0) = v$ can be written as

$$Y(t) = t d\exp_p(tu)v.$$

EXAMPLE 2.60. Let M be a complete Riemanian manifold with constant curvature k and $\gamma : (-\infty, \infty) \to M$ be a normal geodesic on M. Let Y be a normal Jacobi field along γ and take the parallel vector fields E_1, E_2 along γ with $E_1(0) = Y(0), E_2(0) = \nabla Y(0)$ respectively. Put

$$s_k(t) = \begin{cases} \sin(\sqrt{k}t)/\sqrt{k} & (k > 0) \\ t & (k = 0) \\ \sinh(\sqrt{|k|}t)/\sqrt{|k|} & (k < 0) \end{cases},$$

$$c_k(t) = \begin{cases} \cos(\sqrt{k}t) & (k > 0) \\ 1 & (k = 0) \\ \cosh(\sqrt{|k|}t) & (k < 0) \end{cases}.$$

Then, $Y(t)$ can be written as

$$Y(t) = c_k(t)E_1(t) + s_k(t)E_2(t).$$

Proposition 2.61 (Gauss's Lemma). *Let M be a Riemmanian manifold, $p \in M$, and $u, v \in T_pM$. Then, for any $u, v \in T_pM$ the following inequality holds.*

$$\langle d\exp_p(u)v, d\exp_p(u)u\rangle = \langle u, v\rangle.$$

Definition 2.62 (Conjugate point). Let M be a Riemannian manifold and $\gamma : [a, b] \to M$ be a geodesic. Put $p := \gamma(a), q := \gamma(b)$. We say that q is *conjugate* to p along γ provided there exists a nonzero Jacobi field Y along γ satisfying that $Y(a) = 0, Y(b) = 0$.

Definition 2.63 (N-Jacobi field). Let N be a submanifold of a Riemannian manifold M and $u \in T_pN^{\perp}$ ($p \in N$). Then, a Jacobi field Y along γ_u is called a N-Jacobi field provided Y satisfies

$$Y(0) \in T_pN, \qquad \nabla Y(0) - A_uY(0) \in T_pN^{\perp},$$

where A_u is the shape operator of N with respect to u.

Example 2.64. Let H be a hypersurface in a Riemannian manifold M, ν be a normal vector field on N, and $u \in T_pN$. Define the function $\Psi : N \times \mathbb{R}$ as

$$\Psi(p, t) := t\nu_p.$$

Then, the H-Jacobi field Y along γ_{ν_p} ($p \in N$) with $Y(0) = u, \nabla Y(0) = A_{\nu_p}$ can be written as

$$Y(t) = d(\exp \circ \Psi)(p, t)(u, 0),$$

where $(u, 0) \in T_pN \oplus T_t\mathbb{R} = T_{(p,t)}(N \times \mathbb{R})$.

Next we recall the Riemannian measure.

Definition 2.65 (Inner product on exterior power). Let V be an n-dimennsional real inner product space and $\{e_i\}_{i=1}^n$ be an orthonormal basis on V. We denote by $\bigwedge^r(V)$ the rth exterior power of V^*. Then, the inner product on $\bigwedge^r(V)$ is defined as

$$\left\langle \sum_{i_1<\cdots<i_r} a_{i_1,\ldots,i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}, \sum_{j_1<\cdots<j_r} b_{j_1,\ldots,j_r} e_{j_1} \wedge \cdots \wedge e_{j_r} \right\rangle := \sum_{i_1<\cdots<i_r} a_{i_1,\ldots,i_r} b_{i_1,\ldots,i_r},$$

which does not depend on the choice of the orthonormal basis $\{e_i\}_{i=1}^n$.

Remark 2.66. Let V be an n-dimennsional real inner product space and $\{e_i\}_{i=1}^n$ be an orthonormal basis on V. Then, for $v_1, \ldots, v_n \in V$, the follwing equalities hold:

$$v_1 \wedge \cdots \wedge v_n = \det(\langle v_i, e_j\rangle)_{i,j} e_1 \wedge \cdots \wedge e_n.$$

$$|v_1 \wedge \cdots \wedge v_n| = |\det(\langle v_i, e_j\rangle)_{i,j}| = \sqrt{\det(\langle v_i, v_j\rangle)_{i,j}}.$$

Definition 2.67 (Determinant of linear map between inner product spaces). Let V, W be n-dimensional inner product spaces and $\{e_1, \ldots, e_n\}, \{f_1, \ldots, f_n\}$ be orthonormal bases of V and W, respectively. Let $T : V \to W$ be a linear map. We define the *determinant* of T as

$$\det T := \det(\langle T(e_i), f_j\rangle_W)_{ij}.$$

By the definition, the following holds.

$$|\det T| = \sqrt{\det(\langle T(e_i), T(e_j)\rangle_W)_{ij}} = |T(e_1) \wedge \cdots \wedge T(e_n)|\,.$$

For $A \subset M$, we denote by χ_A the indicator function of A.

Definition 2.68 (Riemannian measure). Let M be an n-dimensional Riemannian manifold M and $\mathcal{B}$ be the Borel algebra on M (or the σ-algebra generated by the family of inverse images of Lebesgue measurable sets by coordinate systems on M). Take an atlas $\{(U_\alpha, \varphi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n))\}_{\alpha \in A}$ and a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$. Define the function $J_\alpha : U_\alpha \to \mathbb{R}$ as

$$J_\alpha(p) := |\det d\varphi_\alpha^{-1}(\varphi_\alpha(p))| = \sqrt{\det\left(\left\langle \frac{\partial}{\partial x_\alpha^i}(p), \frac{\partial}{\partial x_\alpha^j}(p)\right\rangle\right)_{ij}} = \left|\frac{\partial}{\partial x_\alpha^1}(p) \wedge \cdots \wedge \frac{\partial}{\partial x_\alpha^n}(p)\right|$$

Then, the *Riemannian measure* $v_M : \mathcal{B} \to [0, +\infty]$ is defined as

$$v_M(B) := \sum_{\alpha \in A} \int_{\varphi_\alpha(U_\alpha)} (\rho_\alpha \chi_B J_\alpha) \circ \varphi_\alpha^{-1}(x_\alpha^1, \ldots, x_\alpha^n)\, dx_\alpha^1 \cdots dx_\alpha^n,$$

which does not depend on the choice of the atlas $\{(U_\alpha, \varphi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n))\}_{\alpha \in A}$ and the partition of unity $\{\rho_\alpha\}_{\alpha \in A}$. Then, a triple $(M, \mathcal{B}, v_M)$ become a measure space. $v_M(B)$ is called a volume of B, denoted by $\mathrm{vol}(B)$.

Proposition 2.69. *Let M, N be Riemannian manifolds with $\dim M = \dim N$ and $\varphi : N \to M$ be a diffeomorphism. Then, for any integrable function f on M, the following equality holds:*

$$\int_M f\, dv_M = \int_N f \circ \varphi |\det d\varphi|\, dv_N.$$

In particular, for an isometry $\varphi : N \to M$,

$$\int_M f\, dv_M = \int_N f \circ \varphi\, dv_N.$$

For a Riemannian manifold M and $p \in M$, we define the diffeomorphism $\Theta_p : (0, \infty) \times S_pM \to T_pM \setminus \{o_p\}$,

$$\Theta_p(t, u) := tu$$

where $S_pM := \{u \in T_pM \mid |u| = 1\} \subset T_pM$ the unit sphere in T_pM and o_p is the origin of T_pM. For any $r > 0$ define the map $\Theta_{p,r} : S_pM \to T_pM \setminus \{o_p\}$ as $\Theta_{p,r}(u) := \Theta_p(r, u)$. We shall estimate the $(n-1)$-dimesional volume $\mathrm{vol}_{n-1}(\partial B(p, r))$ applying Proposition 2.69 for the diffeomorphism $\exp_p \circ \Theta_{p,r} : S_pM \to \partial B(p, r)$ ($r > 0$ is sufficiently small).

PROPOSITION 2.70. *We put*

$$\theta_p(t, u) := |\det d(\exp_p \circ \Theta_p)(t, u)|$$

$$\theta_{p,r}(u) := |\det d(\exp_p \circ \Theta_{p,r})(u)|$$

For $(t, u) \in (0, +\infty) \times S_pM$ *and an orthonormal basis* $\{e_1, \ldots, e_n\}$ *of* T_pM *with* $e_n = u$, *we take the Jacobi fields* Y_i $(i = 1, \ldots, n-1)$ *along the normal geodesic* γ_u *with* $Y_i(0) = 0, \nabla Y_i(0) = e_i$. *Then, the following equalities hold.*

$$\begin{aligned}\theta_p(t, u) &= t^{n-1}\sqrt{\det\langle d\exp_p(u)e_i, d\exp_p(u)e_j\rangle_{1\le i,j\le n-1}} \\ &= \sqrt{\det\langle Y_i(t), Y_j(t)\rangle_{1\le i,j\le n-1}},\\ \theta_{p,r}(u) &= \theta_p(r, u).\end{aligned}$$

EXAMPLE 2.71. Let r_0 be a positive constant such that $\exp_p |B(o_p, r_0)$ is a diffeomorphism. Then, by Proposition 2.70 we have, for $0 < r < r_0$,

$$\mathrm{vol}(B(p, r)) = \int_{S_pM}\int_0^r \theta_p(t, u)\, dt dv_{S_pM}(u)$$

$$\mathrm{vol}_{n-1}(\partial B(p, r)) = \int_{S_pM} \theta_{p,r}(u)\, dv_{S_pM}(u)$$

COROLLARY 2.72. *Let* M *be a* n*-dimensional Riemannian manifold and* $p \in M$. *Then, the following hold:*

$$\lim_{\varepsilon\to 0}\frac{\mathrm{vol}(B(p,\varepsilon))}{\varepsilon^n} = \mathrm{vol}(B_0^n(1)),$$

$$\lim_{\varepsilon\to 0}\frac{\mathrm{vol}_{n-1}(\partial B(p,\varepsilon))}{\varepsilon^{n-1}} = \mathrm{vol}_{n-1}(\partial B_0^n(1)),$$

$$\lim_{\varepsilon\to 0}\frac{\mathrm{vol}_{n-1}(\partial B(p,\varepsilon))}{\mathrm{vol}(B(p,\varepsilon))^{\frac{n-1}{n}}} = \frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}},$$

where $B_0^n(1)$ *is a unit ball in* n*-dimensional Euclidean space* $\mathbb{R}^n$.

If M has a constant curvature k, then a Jacobi field Y along the normal geodesic γ_u $(u \in S_pM)$ with $Y(0) = 0$ can be written as

$$Y(t) = s_k(t)E(t),$$

where E is the parallel vector field along γ_u with $E(0) = \nabla Y(0)$. In this case, the following corollary holds.

Corollary 2.73. *Let M be a n-dimensional Riemannian manifold with constant curvature k. Then, we have*

$$\theta_p(t,u) = s_k^{n-1}(t),$$
$$\theta_{p,r}(u) = s_k^{n-1}(r),$$

and for a constant r_0 such that $\exp_p|B(o_p,r_0)$ is a diffeomorphism, the following hold: for any $0 < r < r_0$,

$$\mathrm{vol}(B(p,r)) = \mathrm{vol}_{n-1}(\partial B_0^n(1)) \int_0^r s_k^{n-1}(t)\,dt,$$
$$\mathrm{vol}_{n-1}(\partial B(p,r)) = \mathrm{vol}_{n-1}(\partial B_0^n(1)) s_k^{n-1}(r).$$

In particular,

$$\mathrm{vol}(B_0^n(1)) = \frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{n}.$$

Proposition 2.74 (Coarea formula). *Let M be a Riemannian manifold and f be a proper smooth function on M. By Sard's theorem, the set of critical values is a null set in $\mathbb{R}$, and for almost everywhere regular point $t \in \mathbb{R}$, $f^{-1}(t)$ is a compact hypersurface in M. Then, for any integrable function u, the following equality holds.*

$$\int_m u|\nabla f| dv_M = \int_{-\infty}^{+\infty} \left[\int_{f^{-1}(t)} u\, dv_{f^{-1}(t)} \right] dt,$$

where $v_{f^{-1}(t)}$ is the Riemannian measure of the Riemannian submanifold $f^{-1}(t)$ of M.

Theorem 2.75 (Divergence theorem). *Let M be a Riemannian manifold. Then, for any C^1-vector field X on M with compact support, the following equality holds.*

$$\int_M \mathrm{div}\, X dv_M = 0.$$

Next, we define the completeness and recall some comparison theorems.

Definition 2.76 (Geodesically complete). A Riemannian manifold M is called to be *geodesically complete* at $p \in M$ provided for any $u \in T_pM$ the geodesic γ_u is defined on $\mathbb{R}$. M is called to be geodesically complete provided for any $p \in M$, M is geodesically complete at p.

Theorem 2.77 (Hopf-Rinow theorem). *Let M be a connected Riemannian manifold. Then, the following are equivalent.*

(i) M is a complete metric space for the Riemannian distance d of M.

(ii) There exists $p \in M$ shuch that M is geodesically complete at p.

(iii) M is geodesically complete.

(iv) There exists $p \in M$ such that for any $r > 0$, $\overline{B(p,r)} := \{q \in M \mid d(p,q) \le r\}$ is compact.

(v) For any $p \in M$ and $r > 0$, $\overline{B(p,r)}$ is compact.

We simply say that a connected Riemannian manifold M is complete provided M satisfies the condition in Theorem 2.77. We see that compact connected Riemannian manifolds are complete.

PROPOSITION 2.78. *Let M, N be connected Riemannian manifolds and $f : M \to N$ be a local isometry. If M is complete, then f is a Riemannian covering.*

THEOREM 2.79 (Rauch comparison theorem). *Let M be a complete Riemannian manifold and K_M be a sectional curvature of M. Let $\gamma : [0, \infty) \to M$ be a normal geodesic and Y be a normal Jacobi field along γ with $Y(0) = 0$.*

(i) Assume $K_M \leq \Delta$ and put $t_0 := \sup\{t > 0 \mid 0 < \forall t' < t, s\Delta(t') > 0\}$. Then, for $0 < t < t_0$,

$$|Y(t)| \geq |\nabla Y(0)| s_\Delta(t).$$

(ii) Assune $K_M \geq \delta$ and let t_0 be the minimum positive value of t such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ (if for all $t > 0$ the point $\gamma(t)$ is not conjugate to $\gamma(0)$ along γ, then put $t_0 := +\infty$). Then, for $0 < t < t_0$,

$$|Y(t)| \leq |\nabla Y(0)| s_\delta(t).$$

COROLLARY 2.80. *Let M be a complete Riemannian manifold satisfying $\delta \leq K_M \leq \Delta$.*

(i) Let $p \in M$ and $u \in T_pM$ be a non-zero tangent vector such that $0 < |u| < \pi/\sqrt{\Delta}$ (when $\Delta \geq 0$, we interpret $\pi/\sqrt{\Delta} = +\infty$). Then, for all non-zero tangent vector $v \in T_pM$ with $u \perp v$ the following inequalities hold.

$$\frac{s_\Delta(|u|)}{|u|} \leq \frac{|d\exp_p(u)v|}{|v|} \leq \frac{s_\delta(|u|)}{|u|}.$$

(ii) Assume M is compact. Then, for $\varepsilon > 0$ there exists a positive constant $r = r(M, g_M, \varepsilon) < i(M)$ such that for any $p \in M$ and domain $\Omega \subset B(r, p)$ with smooth boudary the following inequlities hold.

$$(1 - \varepsilon)\operatorname{vol}(\tilde{\Omega}) < \operatorname{vol}(\Omega) < (1 + \varepsilon)\operatorname{vol}(\tilde{\Omega}),$$

$$(1 - \varepsilon)\operatorname{vol}(\partial\tilde{\Omega}) < \operatorname{vol}(\partial\Omega) < (1 + \varepsilon)\operatorname{vol}(\partial\tilde{\Omega}).$$

where $\tilde{\Omega} := \exp_p^{-1}(\Omega)$, $\operatorname{vol}(\tilde{\Omega})$ is the Euclidean volume on T_pM induced by the its innner product g_p, and $\operatorname{vol}(\partial\tilde{\Omega})$ is the $(n-1)$-dimensional volume as the Riemanian submanifold $\partial\tilde{\Omega} \subset T_pM$.

PROPOSITION 2.81 (Bishop's inequality). *Let M be a complete n-dimensiona Riemannian manifold with $\operatorname{Ric}_M \geq k$ ($k \in \mathbb{R}$). Then, for any $p \in M$ and $r > 0$ the following inequality holds.*

$$\operatorname{vol}(B(p, r)) \leq \operatorname{vol}(B_k^n(r)),$$

where $B_k^n(r)$ is the ball of radius r on the n-dimensional simply connected space form with constant curvature k.

THEOREM 2.82 (Heintze-Karcher). *Let M be a complete Riemannian manifold with $\mathrm{Ric}_M \geq kg_M$ for some $k \in \mathbb{R}$ and H be a hypersurface in M. Let ν be a unit normal vetor field on H and η be the mean curvature function of H with respect to ν. Then, the following inequality holds.*

$$|\det d\Psi|(p,t) \leq (c_k(t) + \eta(p)s_k(t))^{n-1}, \qquad 0 \leq t \leq t_0(H, \nu_p),$$

where $t_0(H,\nu_p) := \sup\{t > 0 \mid \textit{for any } t' \in (0,t),\ \mathrm{rank}\, d\Psi(p,t') = n\}$.

Finally, we recall the Gromov's almost flat theorem [9].

PROPOSITION 2.83 (Gromov's almost flat theorem). *Let M be a compact connected n-dimensional Riemannian manifold. Then, there exists an explicit postitive constant $\varepsilon = \varepsilon(n)$ such that if $|K_M \operatorname{diam}(M)^2| < \varepsilon$, then the universal covering of M is diffeomorphic to $\mathbb{R}^n$.*

2.3 Lie group, Lie algebra, and homogeneous space

DEFINITION 2.84 (Lie Group). Let G be a group with structure of a C^∞-manifold. Then, G is called a *Lie group* provided the map $G \times G \to G$ defined by $(a,b) \mapsto ab^{-1}$ is smooth.

PROPOSITION 2.85. *The product of two Lie groups is also a Lie group.*

DEFINITION 2.86 (Lie subgroup). Let H be a subgroup of a Lie group G. Then, H is called a *Lie subgroup* provided H is a Lie group and the inclusion map $\iota : H \hookrightarrow G$ is a smooth immersion.

PROPOSITION 2.87. *Let G be a subgroup and N be a normal Lie subgroup of G. Then, the quatient G/N is also a Lie group.*

DEFINITION 2.88 (Lie group homomorphism and Lie group isomorphism). Let G and H be Lie groups. Then, a smooth homomorhism $F : G \to H$ is called a *Lie group homomorphism*. Moreover, if $F : G \to H$ is bijective and $F^{-1} : H \to G$ is a Lie group homomorphism, then F is called a *Lie group isomorphism* and we say that G is isomorphic to H.

For a group G, we denote by L_a (resp. R_a) the left (resp. right) translation of $a \in G$ defined by $L_a : G \to G, x \to ax$ (resp. $R_a : G \to G, x \to xa$). Note that if G is a Lie group, then for any $a \in G$ the left translation L_a and the right translation R_a are diffeomorphisms on M

Definition 2.89 (Left invariant vector field and Right invariant vector field). Let $\tilde{X}$ be a smooth vector field on a Lie group G. Then, $\tilde{X}$ is called a *left* (resp. *right*) *invariant vector field* on G provided for any $a \in G$ the equality

$$dL_a(\tilde{X}) = \tilde{X} \qquad \text{(resp. } dR_a(\tilde{X}) = \tilde{X}\text{)}$$

holds.

Definition 2.90 (Lie algebra). Let $\mathfrak{g}$ be a real vector field with binary operation $[\cdot, \cdot]$ on $\mathfrak{g}$. The binary operation $[\cdot, \cdot]$ is called a Lie bracket on $\mathfrak{g}$ provided the following hold:

(i) For any $a, b \in \mathbb{R}$ and $x, y, z \in \mathfrak{g}$,

$$\begin{aligned}[ax + by, z] =& a[x, z] + b[y, z],\\ [x, ay + bz] =& a[x, y] + b[x, z].\end{aligned}$$

(ii) For any $x, y \in \mathfrak{g}$,

$$[x, y] = -[y, x].$$

(iii) For any $x, y, z \in \mathfrak{g}$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

If $[\cdot, \cdot]$ is a Lie bracket, then we say that $\mathfrak{g}$ is a Lie algebra.

For instance, the space $\mathcal{X}(M)$ of vector fields on a smooth manifold M with Lie bracket $[\cdot, \cdot]$, defined by $[X, Y](f) := X(Y(f)) - Y(X(f))$ $(f \in C^\infty(M))$, is a Lie algebra.

Definition 2.91 (Adjoint endmorphism). Let $\mathfrak{g}$ be a Lie algebra. For any $a \in \mathfrak{g}$, we define the *adjoint endmorphism* $\mathrm{ad}_a : \mathfrak{g} \to \mathfrak{g}$ as

$$\mathrm{ad}_a(x) := [a, x].$$

Definition 2.92 (Lie algebra of Lie group). Let G be a Lie group. Since for any left invariant vector fields $\tilde{X}$ and $\tilde{Y}$ on G the Lie bracket $[\tilde{X}, \tilde{Y}]$ is also a left invariant vector field, the space $\mathrm{Lie}(G)$ of left invariant vector fields on G become a Lie subalgebra of $\mathcal{X}(G)$. We say that $\mathrm{Lie}(G)$ is the *Lie algebra* of G.

For a Lie group G, we see that the dimension of G as a manifold is equal to the dimension of $\mathrm{Lie}(G)$ as a vector field.

Definition 2.93 (Universal covering group). Let G be a connected Lie group and take a universal cover $\pi : \tilde{G} \to G$. Then, there exists a structure of Lie group on $\tilde{G}$ such that π become a group homomorphism and a smooth map. We say that $\tilde{G}$ is a *universal covering group*. The universal covering group $\tilde{G}$ is unique up to group isomorphism.

Theorem 2.94 (Structure theorem of compact Lie group [3]). *Let G be a compact connected Lie group. Then, there exist a nonnegative integer k ($\leq n$), a simply connected compact Lie group G_0, and a finite central subgroup Z of $\mathbb{T}^n \times G_0$ such that G is ismorphic to $(\mathbb{T}^n \times G_0)/Z$.*

Remark 2.95. Note that the simply connected Lie group $\mathbb{R}^n \times G_0$ is a universal covering group of $G = (\mathbb{T}^n \times G_0)/Z$.

Next, we define homogeneous spaces.

Definition 2.96 (Lie transformation group). Let G be a Lie group and M be a smooth manifold. If G acts on M (on the left) and the group action $(g, p) \mapsto g \cdot p$ is smooth, then G is called a *Lie transformation group* acting on M.

Definition 2.97 (Homogeneous space). Let G be a Lie transformation group acting a smooth manifold M. If G acts transitively, then M is called a *homogeneous space.*

Definition 2.98 (Isotoropy group). Let G be a Lie transformation group acting a smooth manifold M and $p \in M$. Then, the closed Lie subgroup H of G defined as

$$H := \{g \in G \mid g \cdot p = p\}$$

is called an *isotoropy group.*

Proposition 2.99. *Let G be a Lie transformation group acting a smooth manifold M, $p \in M$. Assume that M is a homogeneous space. Then, for the isotoropy group $H := \{g \in G \mid g \cdot p = p\}$, G/H becomes a smooth manifold. Moreover, the map $G/H \to M, gH \mapsto g \cdot p$ is a diffeomorphism.*

3 Isometry group and Killing vector fields

3.1 Definition and some properties

DEFINITION 3.1. Let M be a Riemannian manifold and $\mathrm{Isom}(M)$ be the set of isometries on M. Then, considering the composition of isometries as the group law, $\mathrm{Isom}(M)$ become a group. The group $\mathrm{Isom}(M)$ is called an *isometry group*.

PROPOSITION 3.2. *For a Riemannian manifold M, there exists a unique way to make* $\mathrm{Isom}(M)$ *a Lie group as follws:*

(i) The action $\mathrm{Isom}(M) \times M \to M, (\varphi, p) \mapsto \varphi(p)$ *is smooth.*

(ii) A one-parameter group $\alpha : \mathbb{R} \to \mathrm{Isom}(M)$ *is smooth as a map if the map* $\mathbb{R} \times M \to M, (t, p) \mapsto \alpha(t)(p)$ *is smooth.*

DEFINITION 3.3 (Riemannian homogeneous space). Let M be a Riemannian manifold. Then, if the isometry group $\mathrm{Isom}(M)$ acts on M transitively, then M is called a *Riemannian homogeneous space*

DEFINITION 3.4 (Killing vector field). A smooth vector field X on a Riemannian manifold on M is called a *Killing vector field* provided

$$\mathcal{L}_X g_M = 0,$$

where $\mathcal{L}_X g_M$ be the Lie derivative of g_M with respect to X.

By the definition of Lie derivative, X is a Killing vector field on a Riemannian manifold M if and only if for any vector fields $Y, Z \in \mathcal{X}(M)$ the following eaualiy holds:

$$\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0.$$

PROPOSITION 3.5. *Let X be a Killing vector field on a Riemannian maniofld M. Then, the following hold.*

(i) For any isometry $\varphi \in \mathrm{Isom}(M)$, the vector field $d\varphi(X)$ is also a Killing vector filed.

(ii) For any (local) flow $(\varphi_t)_t$ of X the map φ_t is (locally) isometric if and only if X is a Killing vector field. In particular, if X is a complete vector field, then $\varphi_t \in \mathrm{Isom}(M)$ for the flow $(\varphi_t)_t$ of X.

(iii) For any geodesic γ on M, $X(t) := X_{\gamma(t)}$ is a Jacobi field along γ.

(iv) For any integral cunrve c of X, c have a constant speed.

(v) If the manifold M is complete, then X is a complete vector field.

Proposition 3.6 (Bochner formula [14]). *Let X be a Killing vector field on a n-dimensional Riemannian maniofld M. Then, the following equality holds:*

$$\frac{1}{2}\Delta|X|^2 = -|\nabla X|^2 + \mathrm{Ric}(X, X),$$

where $|\nabla X|(p) := \left(\sum_{i=1}^n |\nabla_{e_i} X|^2\right)^{1/2}$ *($\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM).*

Proof. By the definition of Killing vector fields, for any vector field $Y \in \mathcal{X}(M)$ we have

$$\langle \nabla|X|^2, Y\rangle = Y(|X|^2) = 2\langle \nabla_Y X, X\rangle = -2\langle \nabla_X X, Y\rangle,$$

and thus

$$\nabla|X|^2 = -2\nabla_X X.$$

From this equality, we have

$$\frac{1}{2}\Delta|X|^2 = \operatorname{div} \nabla_X X.$$

Let $(U, (x^1, \ldots, x^n))$ be a normal coordinate system at $p \in M$ and put $\partial_i := \partial/\partial x^i$ $(i = 1, \ldots, n)$. By the definition of divergene and curvature tensor, we have

$$\begin{aligned}\frac{1}{2}\Delta|X|^2 &= \sum_{i=1}^n \langle \nabla_{\partial_i}\nabla_X X, \partial_i\rangle \\ &= \sum_{i=1}^n \left[\langle R(\partial_i, X) X, \partial_i\rangle + \langle \nabla_X \nabla_{\partial_i} X, \partial_i\rangle + \left\langle \nabla_{[\partial_i, X]} X, \partial_i\right\rangle\right] \\ &= \mathrm{Ric}(X, X) + \sum_{i=1}^n \left[\langle \nabla_X \nabla_{\partial_i} X, \partial_i\rangle + \left\langle \nabla_{[\partial_i, X]} X, \partial_i\right\rangle\right].\end{aligned}$$

Since $\langle \nabla_{\partial_i} X, \partial_i\rangle \equiv 0$, we have

$$\begin{aligned}\langle \nabla_X \nabla_{\partial_i} X, \partial_i\rangle + \left\langle \nabla_{[\partial_i, X]} X, \partial_i\right\rangle &= \langle \nabla_X \nabla_{\partial_i} X, \partial_i\rangle - \langle \nabla_{\partial_i} X, [\partial_i, X]\rangle \\ &= \langle \nabla_X \nabla_{\partial_i} X, \partial_i\rangle + \langle \nabla_{\partial_i} X, \nabla_X \partial_i\rangle - |\nabla_{\partial_i} X|^2 \\ &= X\langle \nabla_{\partial_i} X, \partial_i\rangle - |\nabla_{\partial_i} X|^2 \\ &= -|\nabla_{\partial_i} X|^2.\end{aligned}$$

Thus, we get the conclusion. □

Proposition 3.7 (Kato's inequality [1]). *Let X be a Killing vector field on a Riemannian maniofld M. Then, the inequality*

$$|\nabla|X|| \leq |\nabla X|$$

holds on $\{p \in M \mid |X_p| > 0\}$.

PROOF. Let $p \in M$ such that $|X_p| > 0$ and $\{e_i\}_{i=1}^n$ be an orthonormal basis of T_pM. Then,

$$|\nabla|X||^2(p) = \sum_{i=1}^n \langle \nabla|X|, e_i\rangle^2 = \sum_{i=1}^n e_i(|X|)^2 = \sum_{i=1}^n \frac{\langle \nabla_{e_i}X, X_p\rangle^2}{|X_p|^2} \leq \sum_{i=1}^n |\nabla_{e_i}X|^2 = |\nabla X|^2.$$

□

3.2 Correspondence between Killing vector fields and left invariant vector fields

LEMMA 3.8. *Let $\tilde{X}$ be a left invariant vctor field on a Lie group G. Then, for the flow $(\tilde{\varphi}_t)_t$ of $\tilde{X}$, the follwing inequlity holds:*

$$\tilde{\varphi}_t = R_{g_t},$$

where $g_t := \tilde{\varphi}_t(e) \in G$ (e is the identity element of G).

PROOF. Let $h \in G$ and define the curve c on G as $c(t) := R_{g_t}(h)$. Then, we have

$$\frac{d}{dt}c(t) = \frac{d}{dt}R_{g_t}(h) = \frac{d}{dt}hg_t = \frac{d}{dt}L_h(g_t) = \frac{d}{dt}(\tilde{\varphi}_t(e)) = dL_h(\tilde{X}_{\tilde{\varphi}_t(e)}) = \tilde{X}_{L_h(g_t)} = \tilde{X}_{c(t)},$$

and thus $c(t)$ is a integral curve of $\tilde{X}$ with $c(0) = h$. By the uniqueness of integral curve and the definition of the flow $\tilde{\varphi}$ of vector field $\tilde{X}$, we get $\tilde{\varphi}_t(h) = R_{g_t}(h)$, which is the conclusion. □

LEMMA 3.9. *Let M be a Riemannian manifold and $\tilde{X}$ be a left invariant vector field on the isometry group* $\mathrm{Isom}(M)$. *For the flow $(\tilde{\varphi}_t)_t$ of $\tilde{X}$, we put $\varphi_t := \tilde{\varphi}_t(\mathrm{id})$, where id is the identity map on M which is the identity element of* $\mathrm{Isom}(M)$. *Then, $(\varphi_t)_t$ become a one-parameter transformation group on M and induce a Killing vector field X on M.*

PROOF. By Proposition 3.2 (i), the map $\mathbb{R} \times M \to M, (t,p) \mapsto \varphi_t(p)$ is smooth. By Lemma 3.8, we have

$$\varphi_s \circ \varphi_t = R_{\varphi_t}(\varphi_s) = \tilde{\varphi}_t(\varphi_s) = \tilde{\varphi}_t(\tilde{\varphi}_s(\mathrm{id})) = \tilde{\varphi}_{s+t}(\mathrm{id}) = \varphi_{s+t}.$$

Thus, $(\varphi_t)_t$ is a one-parameter transformation group on M. Moreover, by Proposition 3.5 (ii), X is a complete Killing vector field. □

LEMMA 3.10. *Let M be a Riemannian manifold and X be a complete Killing vector field on the isometry group* $\mathrm{Isom}(M)$. *For the flow $(\varphi_t)_t$ of X, we put $\tilde{\varphi}_t := R_{\varphi_t}$. Then, $(\tilde{\varphi}_t)_t$ become a one-parameter transformation group on G and induce a left invariant vector field $\tilde{X}$ on G.*

PROOF. By Proposition 3.2 (ii) and the definition of Lie group, the map $\mathbb{R}\times \mathrm{Isom}(M) \to \mathrm{Isom}(M), (t, \varphi) \mapsto \tilde{\varphi}_t(\psi) = \psi \circ \varphi_t$ is smooth. By the definition of $\tilde{\varphi}_t$, for any ψ we have

$$\tilde{\varphi}_s \circ \tilde{\varphi}_t(\psi) = \tilde{\varphi}_s(\psi \circ \varphi_t) = \psi \circ \varphi_t \circ \varphi_s = \psi \circ \varphi_{s+t} = \tilde{\varphi}_{s+t}(\psi),$$

and thus we get $\tilde{\varphi}_s \circ \tilde{\varphi}_t = \tilde{\varphi}_{s+t}(\psi)$. Hence, $(\varphi_t)_t$ is a one-parameter transformation group on $\mathrm{Isom}(M)$. Moreover, we have

$$\begin{aligned} (dL_\psi(\tilde{X}))_\phi = dL_\psi(\tilde{X}_{\psi^{-1}\circ\phi}) &= \left.\frac{d}{dt}\right|_{t=0} L_\psi(\tilde{\varphi}_t(\psi^{-1} \circ \phi)) \\ = \left.\frac{d}{dt}\right|_{t=0} L_\psi(\psi^{-1} \circ \phi \circ \varphi_t^X) &= \left.\frac{d}{dt}\right|_{t=0} \phi \circ \varphi_t^X = \left.\frac{d}{dt}\right|_{t=0} \tilde{\varphi}_t(\phi) = \tilde{X}_\phi. \end{aligned}$$

Thus, $\tilde{X}$ is a left invariant vector field on $\mathrm{Isom}(M)$ □

We denote by $\mathrm{Lie}(\mathrm{Isom}(M))$ the space of left invariatn vector fields on $\mathrm{Isom}(M)$ and by $\mathcal{K}(M)$ the space of complete Killing vector fields on M. Then, by Lemma 3.9, we can define the map $T : \mathrm{Lie}(\mathrm{Isom}(M)) \to \mathcal{K}(M)$. By the Lemma 3.10, we see that T is surjective. Morerover, for $\mathcal{K}(M)$ and T the following folds:

PROPOSITION 3.11. *$\mathcal{K}(M)$ is a Lie subalgebra of the algebra $\mathcal{X}(M)$ of smooth vector fields on M and the map $T : \mathrm{Lie}(\mathrm{Isom}(M)) \to \mathcal{K}(M)$ is a linear isomorphism satisfying the relation $T([\tilde{X}, \tilde{Y}]) = -[X, Y]$, where $X := T(\tilde{X})$, $Y := T(\tilde{Y})$.*

PROOF. Let $\tilde{X}$ be a left invariant vector field and $(\tilde{\varphi}_t)_t$ be the flow of $\tilde{X}$. Put $\varphi_t := \tilde{\varphi}_t(\mathrm{id})$ and $X := T(\tilde{X})$. For each $p \in M$, define the map $\pi_p : \mathrm{Isom}(M) \to M$ as

$$\pi_p(\varphi) := \varphi(p),$$

which is smooth by Proposition 3.2 (i). Then, we have

$$T(\tilde{X})_p = X_p = \left.\frac{d}{dt}\right|_{t=0} \varphi_t(p) = \left.\frac{d}{dt}\right|_{t=0} \pi_p(\varphi_t) = \left.\frac{d}{dt}\right|_{t=0} \pi_p(\tilde{\varphi}_t(\mathrm{id})) = d\pi_p(\tilde{X}_{\mathrm{id}}).$$

It follows that T is a linear isomorphism.

Next let $\tilde{X}, \tilde{Y}$ be a left invariant vector fields and $(\tilde{\varphi}_t)_t$ be the flow of $\tilde{X}$. Put $\varphi_t := \tilde{\varphi}_t(\mathrm{id})$ and $X := T(\tilde{X}), Y := T(\tilde{Y})$. Then, for any $\psi \in \mathrm{Isom}(M)$ we have

$$\pi_p \circ R_{\varphi_{-t}} \circ L_{\varphi_t}(\psi) = \varphi_t \circ \psi \circ \varphi_{-t}(p) = \varphi_t \circ \pi_{\varphi_{-t}(p)}(\psi),$$

and thus $\pi_p \circ R_{\varphi_{-t}} \circ L_{\varphi_t} = \varphi_t \circ \pi_{\varphi_{-t}(p)}$. From this equality, we get the relation

$$\begin{aligned}
T([\tilde{X},\tilde{Y}])_p =& d\pi_p([\tilde{X},\tilde{Y}]_{\mathrm{id}}) \\
=& d\pi_p\left(\frac{d}{dt}\bigg|_{t=0} \frac{dR_{\varphi_{-t}}dL_{\varphi_t}(\tilde{Y}_{\mathrm{id}}) - \tilde{Y}_{\mathrm{id}}}{t}\right) \\
=& \frac{d}{dt}\bigg|_{t=0} \frac{d(\pi_p \circ R_{\varphi_{-t}} \circ L_{\varphi_t})(\tilde{Y}_{\mathrm{id}}) - d\pi_p(\tilde{Y}_{\mathrm{id}})}{t} \\
=& \frac{d}{dt}\bigg|_{t=0} \frac{d(\varphi_t \circ \pi_{\varphi_{-t}(p)})(\tilde{Y}_{\mathrm{id}}) - d\pi_p(\tilde{Y}_{\mathrm{id}})}{t} \\
=& \frac{d}{dt}\bigg|_{t=0} \frac{d\varphi_t(Y_{\varphi_{-t}(p)}) - Y_p}{t} \\
=& -[X,Y]_p.
\end{aligned}$$

Moreove, It follows that Lie bracket $[X,Y]$ of Killing vector fields X and Y is also a Killing vector filed, and hence the conclusion follows. □

3.3 Dimension of isometry group and Bochner's theorem

PROPOSITION 3.12. *Let M be a complete n-domensional Riemannian manifold. Then, for the dimension of isometry group* $\mathrm{Isom}(M)$*, the follwing inequality holds*

$$\dim \mathrm{Isom}(M) \leq \frac{1}{2}n(n+1).$$

THEOREM 3.13 (Bochner). *Let M be a compact connected n-dimensional Riemannian manifold with* $\mathrm{Ric} \geq 0$*. Then, for the dimension of isometry group* $\mathrm{Isom}(M)$*, the inequality*

$$\dim \mathrm{Isom}(M) \leq n$$

holds, and equality holds if and only if M is isometric to an n-dimensional flat torus.

PROOF. (a) Let X be a Killing vector field on M. Integrating the Bochner formula (see Proposition 3.6), we get

$$\int_M |\nabla X|^2 \, dv_M = \int_M \mathrm{Ric}(X,X) \, dv_M \leq 0.$$

If follws that Killing vector fields on M are parallel. Take $p \in M$ and put

$$V_p := \{X_p \in T_pM \mid X \in \mathcal{K}(M)\}.$$

Then, the linear map $\Phi_p : \mathcal{K}(M) \to V_p, X \mapsto X_p$ is injective, and thus we have

$$\dim \mathrm{Isom}(M) = \dim \mathcal{K}(M) = \dim \mathrm{Im}\, \Phi \leq n.$$

(b) Assume that $\dim \mathrm{Isom}(M) = n$. Then, it holds that $V_p = T_pM$. For a Killing vector field X, we denote by $(\varphi_t^X)_t$ the flow of X. Since Killing vector fields X, Y are prallel, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X = 0$$

and

$$\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X.$$

Moreover, the equality

$$\varphi_1^{sX+tY} = \varphi_s^X \circ \varphi_t^Y$$

holds. In fact, $c(r) := \varphi_{rs}^X \circ \varphi_{rt}^Y(q)$ is the integral curve of the Killing vector field $sX + tY$ since

$$\begin{aligned}
\frac{dc}{dr}(r) &= \left.\frac{d}{dr'}\right|_{r'=r} \varphi_{r's}^X \circ \varphi_{rt}^Y(q) + \left.\frac{d}{dr'}\right|_{r'=r} \varphi_{rs}^X \circ \varphi_{r't}^Y(q) \\
&= \left.\frac{d}{dr'}\right|_{r'=r} \varphi_{r's}^X \circ \varphi_{rt}^Y(q) + \left.\frac{d}{dr'}\right|_{r'=r} \varphi_{r't}^Y \circ \varphi_{rs}^X(q) \\
&= sX_{\varphi_{rs}^X \circ \varphi_{rt}^Y(q)} + tY_{\varphi_{rt}^Y \circ \varphi_{rs}^X(q)} \\
&= (sX + tY)_{c(r)}.
\end{aligned}$$

Since a Killing vector field X is parallel, the curve $t \mapsto \varphi_t^X(p)$ is a geodesic and the equality

$$\exp_p tX_p = \varphi_t^X(p)$$

holds. From the above, we have

$$d\exp_p(X_p)Y_p = \left.\frac{d}{dt}\right|_{t=0} \exp_p(X_p + tY_p) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^Y \circ \varphi_1^X(p) = Y_{\exp_p X_p}.$$

Considering that the Riemannian metric on T_pM is defined in Example 2.17 and that vector field Y is parallel, it follows that the exponetial map $\exp_p : T_pM \to M$ is a local isometry. In particular, by Proposition 2.78, the exponetial map $\exp_p : T_pM \to M$ is a universal Riemannian covering. Put

$$\Gamma := (\exp_p)^{-1}(p).$$

Then, Γ is a descrete subgroup of T_pM. In fact, for any $X, Y \in \mathcal{K}(M)$ such that $X_p, Y_p \in \Gamma$, the following hold:

$$\exp_p(X_p + Y_p) = \varphi_1^X \circ \varphi_1^Y(p) = \varphi_1^X(\exp_p Y_p) = \varphi_1^X(p) = \exp_p X_p = p,$$

$$\exp_p(-X_p) = \varphi_{-1}^X(p) = \varphi_{-1}^X(\exp_p X_p) = \varphi_{-1}^X \circ \varphi_1^X(p) = p.$$

Moreover, for any $X, Y \in \mathcal{K}(M)$ such that $\exp_p X_p = \exp_p Y_p$, we have

$$\exp_p(X_p - Y_p) = \varphi_{-1}^Y \circ \varphi_1^X(p) = \varphi_{-1}^Y(\exp_p X_p) = \varphi_{-1}^Y(\exp_p Y_p) = \varphi_{-1}^Y \circ \varphi_1^Y(p) = p,$$

$$X_p - Y_p \in \Gamma,$$

and thus the bijective map $F : T_pM/\Gamma \to M, X_p + \Gamma \mapsto \exp_p X_p$ is well-defined. Note that the projection map $\pi : T_pM \to T_pM/\Gamma$ is a covering map. Let f be a deck transformation of π such that $f(o_p) = X_p$ $(X \in \mathcal{K}(M), X_p \in \Gamma)$. For each $u \in T_pM$, put

$$\varphi_t^{f(u)} := \varphi_t^Z, \qquad (t \in \mathbb{R})$$

where $Z \in \mathcal{K}(M)$ such that $Z_p = f(u)$. Then, we have

$$\exp_p(f(u) - X_p) = \varphi_1^{f(u)} \circ \varphi_{-1}^X(p) = \varphi_1^{f(u)}(\exp_p(-X_p)) = \varphi_1^{f(u)}(p) = \exp \circ f(p) = \exp_p u,$$

and thus $\tilde{f} := f - X_p$ is also a deck transformation of π. However, from $\tilde{f}(o_p) = o_p$, $\tilde{f}$ is the identitiy map, and we see that

$$f(u) = u + X_p$$

is an isometry on T_pM. By Proposition 2.23, there exists a unique Riemannian metric on T_pM/Γ such that the projection map $\pi : T_pM \to T_pM/\Gamma$ is a local isometry. Since the projection $T_pM \to T_pM/\Gamma$ and the exponential map $\exp_p : T_pM \to M$ are universal Riemannian coverings, we see that the map F is a local isometry from T_pM/Γ to M. Moreover, by Proposition 2.21, F is an isomety. Thus, T_pM/Γ is compact and Γ is ismorphic to $\mathbb{Z}^n$. Hence the conclusion follows. $\square$

4 Riemannian curvature tensor on Lie group with left invariant metric

Definition 4.1 (Left invariant metric). Let G be a Lie group with Riemannian metric g_G. Then, g_G is called to be *left invariant* provided for any $h \in G$ the equality

$$L_h^* g_G = g_G,$$

namely for any $h \in G$ the left translation L_h is an isometry on G.

Proposition 4.2. *Let G be a Lie group with left invariant metric. Then, for any left invariant vector fields $\tilde{X}, \tilde{Y}$ on G, the following hold.*
(i) The function $\langle \tilde{X}, \tilde{Y} \rangle$ is constant.
(ii) The covariant derivative $\nabla_{\tilde{X}} \tilde{Y}$ is also a left invariant vector field.

Proposition 4.3. *Let G be a Lie group with left invariant metric. Then, for any left invariant vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ on G, the following hold.*
(i)

$$\nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2}\{[\tilde{X}, \tilde{Y}] - (\mathrm{ad}_{\tilde{X}})^*(\tilde{Y}) - (\mathrm{ad}_{\tilde{Y}})^*(\tilde{X})\}.$$

(ii)

$$\langle R(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W} \rangle = \langle \nabla_{\tilde{X}} \tilde{Z}, \nabla_{\tilde{Y}} \tilde{W} \rangle - \langle \nabla_{\tilde{Y}} \tilde{Z}, \nabla_{\tilde{X}} \tilde{W} \rangle - \langle \nabla_{[\tilde{X}, \tilde{Y}]} \tilde{Z}, \tilde{W} \rangle.$$

(iii)

$$\begin{aligned} \langle R(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X} \rangle =& |(\mathrm{ad}_{\tilde{X}})^*(\tilde{Y}) + (\mathrm{ad}_{\tilde{Y}})^*(\tilde{X})|^2 - \langle (\mathrm{ad}_{\tilde{X}})^*(\tilde{X}), (\mathrm{ad}_{\tilde{Y}})^*(\tilde{Y}) \rangle \\ & - \frac{3}{4}|[\tilde{X}, \tilde{Y}]|^2 - \frac{1}{2}\langle [[X, Y], Y], X \rangle - \frac{1}{2}\langle [[Y, X], X], Y \rangle. \end{aligned}$$

Proof.
(i) By Proposition 4.2 (i), we have

$$\begin{aligned} 0 &= \tilde{X}\langle \tilde{Y}, \tilde{Z} \rangle = \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \langle \tilde{Y}, \nabla_{\tilde{X}} \tilde{Z} \rangle, \\ 0 &= \tilde{Y}\langle \tilde{X}, \tilde{Z} \rangle = \langle \nabla_{\tilde{Y}} \tilde{X}, \tilde{Z} \rangle + \langle \tilde{X}, \nabla_{\tilde{Y}} \tilde{Z} \rangle, \\ 0 &= \tilde{Z}\langle \tilde{X}, \tilde{Y} \rangle = \langle \nabla_{\tilde{Z}} \tilde{X}, \tilde{Y} \rangle + \langle \tilde{X}, \nabla_{\tilde{Z}} \tilde{Y} \rangle. \end{aligned}$$

By combining these equalities, we have

$$\tilde{X}\langle \tilde{Y}, \tilde{Z} \rangle = \frac{1}{2}\{\langle [\tilde{X}, \tilde{Y}], \tilde{Z} \rangle - \langle \tilde{Y}, [\tilde{X}, \tilde{Z}] \rangle - \langle \tilde{X}, [\tilde{Y}, \tilde{Z}] \rangle\},$$

and thus (i) follows.

(ii) By Proposition 4.2, we have

$$
\begin{aligned}
0 &= \tilde{X}\langle \nabla_{\tilde{Y}}\tilde{Z}, \tilde{W}\rangle = \langle \nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z}, \tilde{W}\rangle + \langle \nabla_{\tilde{Y}}\tilde{Z}, \nabla_{\tilde{X}}\tilde{W}\rangle, \\
0 &= \tilde{Y}\langle \nabla_{\tilde{X}}\tilde{Z}, \tilde{W}\rangle = \langle \nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{Z}, \tilde{W}\rangle + \langle \nabla_{\tilde{X}}\tilde{Z}, \nabla_{\tilde{Y}}\tilde{W}\rangle, \\
0 &= \widetilde{[X,Y]}\langle \tilde{Z}, \tilde{W}\rangle = \langle \nabla_{\widetilde{[X,Y]}}\tilde{Z}, \tilde{W}\rangle + \langle \tilde{Z}, \nabla_{\widetilde{[X,Y]}}\tilde{W}\rangle.
\end{aligned}
$$

Thus, by the definition of Riemannian curvature tensore, we get (ii).

(iii) From (i) and (ii), we obtain (iii). □

5 Estimates of isoperimetric constants

In this section, a domain in a manifold means an open set which is not necessarily connected. Let M_k^n be the n-dimensional simply connected space form with constant curvature k. We denote by $B_k^n(r)$ the open ball of radius r in M_k^n.

5.1 Some isoperimetric inequalities

PROPOSITION 5.1. *Let Ω be a domain in Euclidean space $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then, the following ineqauality holds*

$$\frac{\mathrm{vol}_{n-1}(\partial\Omega)}{\mathrm{vol}(\Omega)^{\frac{n-1}{n}}} \geq \frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}} \tag{1}$$

and equality holds only if Ω is a metric ball in $\mathbb{R}^n$.

The inequality (1) is called a *isoperimetric inequality.* Since a Riemanian manifold is locally apploximated by the Euclidean space locally, we have the followings:

COROLLARY 5.2. *Let M be a compact connected Riemannian n-dimensional manifold. Then, for any $\varepsilon > 0$ there exists a positive constant $r = r(M, g_M, \varepsilon) < i(M)$ such that for any $p \in M$ and domain $\Omega \subset B(p, r)$ with smooth boundary the following inequality holds*

$$\frac{\mathrm{vol}_{n-1}(\partial\Omega)}{\mathrm{vol}(\Omega)^{\frac{n-1}{n}}} \geq (1-\varepsilon)\frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}}. \tag{2}$$

PROOF. By the Rauch comparison theorem (Corollary 2.80 (ii)), there exist a constant $0 < r < i(M)$ such that for any $p \in M$ and domain $\Omega \subset B(p, r)$ with smooth boundary,

$$\frac{\mathrm{vol}_{n-1}(\partial\Omega)}{\mathrm{vol}(\Omega)^{\frac{n-1}{n}}} \geq (1-\varepsilon)\frac{\mathrm{vol}_{n-1}(\partial\tilde{\Omega})}{\mathrm{vol}(\tilde{\Omega})^{\frac{n-1}{n}}}$$

where $\tilde{\Omega} := \exp_p^{-1}(\Omega)$, $\mathrm{vol}(\tilde{\Omega})$ is the Euclidean volume on T_pM induced by the its innner product g_p, and $\mathrm{vol}(\partial\tilde{\Omega})$ is the $(n-1)$-dimensional volume as the Riemanian submanifold $\partial\tilde{\Omega} \subset T_pM$. Since T_pM is isometric to the Euclidean space $\mathbb{R}^n$, by Proposition 5.1 the inequality (2) follows. $\square$

5.2 Isoperimetric constant and isoperimetric function

DEFINITION 5.3 (Isoperimetric constant). Let M be a compact connected n-dimensional Riemannian manifold. Then, we define a *isoperimetric constant* $I_a(M)$ as follows: for $a > 0$,

$$I_a(M) := \inf\left\{ \frac{\mathrm{vol}_{n-1}(\partial\Omega)}{\mathrm{vol}(\Omega)^a} \,\middle|\, \Omega \subset M \text{ is a domain with smooth boundary}, \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(M)} \leq \frac{1}{2} \right\}.$$

DEFINITION 5.4 (Isoperimetric function). Let M be a compact connected Riemannian manifold. Then, the *isoperimetric function* $h_M = h : (0,1) \to \mathbb{R}_{\geq 0}$ is defined as for $\beta \in (0,1)$

$$h(\beta) := \inf\{v_{n-1}(\partial\Omega) \,|\, \Omega \in \mathcal{O}_\beta\}$$

where $\mathcal{O}_\beta := \{\Omega \subset M \,|\, \Omega \text{ is a domain with smooth boundary}, \operatorname{vol}(\Omega)/\operatorname{vol}(M) = \beta\}$.

The isoperimetric function $h(\beta)$ has the following properties.

PROPOSITION 5.5.

(i) $h(\beta) = h(1-\beta)$.

(ii) $\inf_{\beta\in(0,1/2]} h(\beta)/(\beta \operatorname{vol}(M))^a = I_a(M)$.

(iii) $h(\beta)$ *is continuous.*

(iv) $\lim_{\beta\to 0} h(\beta)/(\beta\operatorname{vol}(M))^{(n-1)/n} = \operatorname{vol}_{n-1}(\partial B_0^n(1))/\operatorname{vol}(B_0^n(1))^{(n-1)/n}$. *In particular,* $I_a(M) = 0$ *if* $0 < a < (n-1)/n$.

PROOF.

(i) It is obvious, because for any $\Omega \in \mathcal{O}_\beta$ we have $M \setminus \overline{\Omega} \in \mathcal{O}_{1-\beta}$ and $\partial\Omega = \partial(M \setminus \overline{\Omega})$.

(ii)

$$\begin{aligned}
I_a(M) &= \inf \bigsqcup_{\beta\in(0,1/2]} \left\{ \frac{\operatorname{vol}_{n-1}(\partial\Omega)}{\operatorname{vol}(\Omega)^a} \,\middle|\, \Omega \in \mathcal{O}_\beta \right\} \\
&= \inf_{\beta\in(0,1/2]} \inf \left\{ \frac{\operatorname{vol}_{n-1}(\partial\Omega)}{\operatorname{vol}(\Omega)^a} \,\middle|\, \Omega \in \mathcal{O}_\beta \right\} \\
&= \inf_{\beta\in(0,1/2]} \inf \left\{ \frac{\operatorname{vol}_{n-1}(\partial\Omega)}{(\beta\operatorname{vol}(M))^a} \,\middle|\, \Omega \in \mathcal{O}_\beta \right\} \\
&= \inf_{\beta\in(0,1/2]} \frac{h(\beta)}{(\beta\operatorname{vol}(M))^a}.
\end{aligned}$$

(iii) By the Rauch comparison theorem (Corollary 2.80 (ii)), there exists a constant $0 < r_0 < i(M)$ such that for any $p \in M$ and $0 < r \leq r_0$

$$\operatorname{vol}(B(p,r)) > \frac{1}{2}\operatorname{vol}(B_0^n(1))r^n$$

$$\operatorname{vol}(\partial B(p,r)) < \frac{3}{2}\operatorname{vol}(\partial B_0^n(1))r^{n-1}.$$

Let $\beta \in (0,1)$ and $\Omega \in \mathcal{O}_\beta$. By the Fubini's theorem we get

$$\begin{aligned}
\int_M \operatorname{vol}(\Omega \cap B(p,r))\, dv_M(p) &= \int_M \int_\Omega \chi_{B(p,r)}(q)\, dv_M(q) dv_M(p) \\
&= \int_\Omega \int_M \chi_{B(q,r)}(p)\, dv_M(p) dv_M(q) \\
&= \int_\Omega \operatorname{vol}(B(q,r))\, dv_M(q) \\
&> \frac{1}{2}\operatorname{vol}(B_0^n(1))r^n \operatorname{vol}(\Omega),
\end{aligned}$$

and thus there exists $p = p_{r,\Omega} \in M$ such that

$$\mathrm{vol}(\Omega \cap B(p,r)) > \frac{1}{2}\,\mathrm{vol}(B_0^n(1))r^n\beta.$$

Then,

$$\begin{aligned}\frac{\mathrm{vol}(\Omega \setminus \overline{B(p,r)})}{\mathrm{vol}(M)} &= \frac{\mathrm{vol}(\Omega) - \mathrm{vol}(\Omega \cap B(p,r))}{\mathrm{vol}(M)}\\ &< \beta - \frac{\mathrm{vol}(B_0^n(1))}{2\,\mathrm{vol}(M)}r^n\beta =: t(r,\beta).\end{aligned}$$

Put $t_0(\beta) := \max\{0, t(r_0,\beta)\}$. For $\beta' \in (t_0(\beta), \beta)$, set

$$r := \left[\frac{2(\beta-\beta')\,\mathrm{vol}(M)}{\beta\,\mathrm{vol}(B_0^n(1))}\right]^{1/n} \left(\leq \left[\frac{2(\beta - t(r_0,\beta))\,\mathrm{vol}(M)}{\beta\,\mathrm{vol}(B_0^n(1))}\right]^{1/n} = r_0\right).$$

Then, we have

$$\frac{\mathrm{vol}(\Omega \setminus \overline{B(p,r)})}{\mathrm{vol}(M)} < t(r,\beta) = \beta'$$

and thus there exists $0 < r' < r$ such that

$$\frac{\mathrm{vol}(\Omega \setminus \overline{B(p,r')})}{\mathrm{vol}(M)} = \beta'.$$

Since we can show that $\Omega \setminus \overline{B(p,r')}$ is a limit of a sequence of elements of $\mathcal{O}_{\beta'}$, we get

$$\begin{aligned}h(\beta') &\leq \mathrm{vol}_{n-1}(\partial(\Omega \setminus \overline{B(p,r')}\,)) \leq \mathrm{vol}(\partial\Omega) + \mathrm{vol}_{n-1}(\partial B(p,r'))\\ &< \mathrm{vol}(\partial\Omega) + \frac{3}{2}\,\mathrm{vol}(\partial B_0^n(1))(r')^{n-1} < \mathrm{vol}(\partial\Omega) + \frac{3}{2}\,\mathrm{vol}_{n-1}(\partial B_0^n(1))r^{n-1}\\ &= \mathrm{vol}(\partial\Omega) + C\left(\frac{\beta-\beta'}{\beta}\right)^{(n-1)/n}\end{aligned}$$

where $C := \frac{3}{2}\,\mathrm{vol}_{n-1}(\partial B_0^n(1))\left[\frac{2\,\mathrm{vol}(M)}{\mathrm{vol}(B_0^n(1))}\right]^{(n-1)/n}$. Since for any $\Omega \in \mathcal{O}_\beta$, the inquality

$$h(\beta') < \mathrm{vol}(\partial\Omega) + C\left(\frac{\beta-\beta'}{\beta}\right)^{(n-1)/n}$$

holds, we have

$$h(\beta') \leq h(\beta) + C\left(\frac{\beta-\beta'}{\beta}\right)^{(n-1)/n}.$$

On the other hand, put $t_1(\beta) := 1 - t_0(1-\beta)$ and let $\beta' \in (\beta, t_1(\beta))$. Then, since $1-\beta' \in (t_0(1-\beta), 1-\beta)$, we have

$$h(1-\beta') \leq h(1-\beta) + C\left(\frac{\beta'-\beta}{1-\beta}\right)^{(n-1)/n},$$

and thus by Proposition 5.5 (i),

$$h(\beta') \leq h(\beta) + C\left(\frac{\beta' - \beta}{1-\beta}\right)^{(n-1)/n}.$$

Therefore, for any $\beta' \in (t_0(\beta), t_1(\beta))$ the following inequality holds

$$|h(\beta') - h(\beta)| \leq C\left[\frac{|\beta' - \beta|}{\min\{\beta, 1-\beta\}}\right]^{(n-1)/n}$$

which show that $h(\beta)$ is continuous at β.

(iv) Let $\varepsilon > 0$ and take a constant $0 < r < i(M)$ as in Corollary 5.2. By the compactness of M, there exisits a finite family of balls $(B(p_i, r/2))_{i=1}^N$ which covers M. Let Ω be a domain with smooth boudary. By the coarea formula, we get

$$\begin{aligned}
\mathrm{vol}(\Omega) &\geq \mathrm{vol}(\Omega \cap B(r, p_i) \setminus B(r/2, p_i)) \\
&= \int_{B(r,p_i)\setminus B(r/2,p_i)} \chi_\Omega |\nabla d_{p_i}|\, dv_M \\
&= \int_{r/2}^{r} \int_{\partial B(p_i,t)} \chi_\Omega \, dv_{\partial B(p_i,t)} dt \\
&= \int_{r/2}^{r} \mathrm{vol}_{n-1}(\Omega \cap \partial B(p_i, t))\, dt
\end{aligned}$$

for every i, and thus there exists $r/2 \leq r_i \leq r$ such that

$$\mathrm{vol}_{n-1}(\Omega \cap \partial B(p_i, r_i)) \leq \frac{2}{r}\,\mathrm{vol}(\Omega).$$

We denote by $(\tilde{\Omega}_j)_j$ the family of connected componets of $\Omega \setminus \bigcup_{i=1}^N B(p_i, r_i)$. Then, we get

$$\begin{aligned}
\mathrm{vol}_{n-1}(\partial\Omega) &= \sum_j \mathrm{vol}_{n-1}(\partial\tilde{\Omega}_j) - 2\sum_{i=1}^N \mathrm{vol}_{n-1}(\Omega \cap \partial B(p_i, r_i)) \\
&\geq \sum_j \mathrm{vol}_{n-1}(\partial\tilde{\Omega}_j) - \frac{4N}{r}\,\mathrm{vol}(\Omega).
\end{aligned}$$

Since we have taken the constant r as in Corollary 5.2, we can get the following estimates.

$$\begin{aligned}
\sum_j \mathrm{vol}_{n-1}(\partial\tilde{\Omega}_j) &\geq (1-\varepsilon)\frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}} \sum_j \mathrm{vol}(\tilde{\Omega}_j)^{\frac{n-1}{n}} \\
&\geq (1-\varepsilon)\frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}} \left[\sum_j \mathrm{vol}(\tilde{\Omega}_j)\right]^{\frac{n-1}{n}} \\
&= (1-\varepsilon)\frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}}\, \mathrm{vol}(\Omega)^{\frac{n-1}{n}},
\end{aligned}$$

and thus

$$\frac{\operatorname{vol}(\partial\Omega)}{\operatorname{vol}(\Omega)^{\frac{n-1}{n}}} \geq (1-\varepsilon)\frac{\operatorname{vol}_{n-1}(\partial B_0^n(1))}{\operatorname{vol}(B_0^n(1))^{\frac{n-1}{n}}} - \frac{4N}{r}\operatorname{vol}(\Omega)^{\frac{1}{n}}.$$

Note that, for any $\Omega \in \mathcal{O}_\beta$, the above inequalities hold. Therefore, we get

$$\frac{h(\beta)}{(\beta\operatorname{vol}(M))^{\frac{n-1}{n}}} \geq (1-\varepsilon)\frac{\operatorname{vol}_{n-1}(\partial B_0^n(1))}{\operatorname{vol}(B_0^n(1))^{\frac{n-1}{n}}} - \frac{4N}{r}(\beta\operatorname{vol}(M))^{\frac{1}{n}}.$$

Let β tends to 0. Then, we have

$$\lim_{\beta\to 0}\frac{h(\beta)}{(\beta\operatorname{vol}(M))^{\frac{n-1}{n}}} \geq (1-\varepsilon)\frac{\operatorname{vol}_{n-1}(\partial B_0^n(1))}{\operatorname{vol}(B_0^n(1))^{\frac{n-1}{n}}}.$$

Since $\varepsilon > 0$ is arbitary, we get

$$\lim_{\beta\to 0}\frac{h(\beta)}{(\beta\operatorname{vol}(M))^{\frac{n-1}{n}}} \geq \frac{\operatorname{vol}_{n-1}(\partial B_0^n(1))}{\operatorname{vol}(B_0^n(1))^{\frac{n-1}{n}}}.$$

The inverse inequality is obtained by Corollary 2.72 □

5.3 Almgren's theorem and mean curvature

Note that a domain which attain

Proposition 5.6. *Let M be a compact connected n-dimensional Riemannian manifold and $\beta \in (0,1)$. Then, there exists a domain Ω in M such that*

(i) $\operatorname{vol}(\Omega)/\operatorname{vol}(M) = \beta$.

(ii) $\partial\Omega$ is a submanifold of M with codimension 1 which is not necessarily smooth.

(iii) Let H be the set of all smooth points in $\partial\Omega$. Then, H is an open dense subset of $\partial\Omega$ and $h(\beta) = \operatorname{vol}_{n-1}(H)$.

(iv) For any $p \in M \setminus \partial\Omega$, it follows that if $q \in \partial\Omega$ satisfies $d(p,\partial\Omega) = d(p,q)$, then $q \in H$.

Take a domain Ω and a hypersurface $H \subset \partial\Omega$ as in Proposition 5.6. Let ν be the unit outward normal vector field on H with respect to Ω and $f \in C_0^\infty(H)$ where $C_0^\infty(H)$ is the set of all smooth functions on H with compact support. Define the functions $\Psi_f : H \times \mathbb{R} \to M$ and $\Psi_{f,\tau} : H \to M$ as

$$\Psi_f(p,\tau) := \exp_H \tau f(p)\nu_p,$$

$$\Psi_{f,\tau}(p) := \Psi_f(p,\tau).$$

Put

$$\Omega_f := \Omega \cup \Psi_f(f^{-1}(0,+\infty)\times[0,1)) \setminus \Psi_f(f^{-1}(-\infty,0)\times(0,1])$$

$$H_f := \Psi_{f,1}(H).$$

Note that H_f is the set of all smooth points in $\partial\Omega_f$ if $\|f\|_{L^\infty(\underline{v})}$ is sufficiently small. Also, it is know that

$$h(\mathrm{vol}(\Omega_f)/\,\mathrm{vol}(M)) \leq \mathrm{vol}_{n-1}(H_f)$$

if $\|f\|_{L^\infty(\underline{v})}$ is sufficiently small.

LEMMA 5.7. *Let $f \in C_0^\infty(H)$. For each $p \in H$, take an orthonormal basis $\{e_{p,1}, \dots, e_{p,n-1}\}$ of T_pH and H-Jacobi fields $Y_{p,i}$ $(i = 1, \dots, n-1)$ along the normal geodesic γ_{ν_p} with $Y_{p,i}(0) = e_{p,i}$, $\nabla Y_{p,i}(0) = A_{\nu_p} e_{p,i}$, where A_{ν_p} is the shape oparator of H. Then, if $\|f\|_{L^\infty(\underline{v})}$ is sufficiently small, following hold.*

$$\mathrm{vol}(\Omega_f) = \mathrm{vol}(\Omega) + \int_H f(p) \int_0^1 |Y_{p,1}(\tau f(p))) \wedge \cdots \wedge Y_{p,n-1}(\tau f(p))|\, d\tau dv_H(p),$$

$$\begin{aligned}\mathrm{vol}_{n-1}(H_f) = \int_H \big| &\left[Y_{p,1}(f(p)) + df(e_{p,1})\dot\gamma_{\nu_p}(f(p))\right] \wedge \\ &\cdots \wedge \left[Y_{p,n-1}(f(p)) + df(e_{p,n-1})\dot\gamma_{\nu_p}(f(p))\right]\big|\, dv_H(p).\end{aligned}$$

PROOF. Define the smooth maps $\Psi : H \times \mathbb{R} \to M$ and $T_f : H \times \mathbb{R} \to N \times \mathbb{R}$ as

$$\Psi(p,\tau) := \exp_H \tau\nu_p,$$

$$T_f(p,\tau) := (p, \tau f(p))$$

, which satisfy $\Psi_f = \Psi \circ T_f$. Then, the following holds.

$$d\Psi(p,\tau)(e_{p,i}, 0) = Y_{p,i}(\tau),$$

$$d\Psi(p,\tau)\left(o_p, \frac{d}{d\tau}\right) = \dot\gamma_{\nu_p}(\tau),$$

$$dT_f(p,\tau)(e_{p,i}, 0) = \left(e_{p,i}, \tau df(e_{p,i})\frac{d}{d\tau}\right),$$

$$dT_f(p,\tau)\left(o_p, \frac{d}{d\tau}\right) = \left(o_p, f(p)\frac{d}{d\tau}\right).$$

From this, we have

$$|\det d\Psi|(p,\tau) = |Y_{p,1}(t) \wedge \cdots \wedge Y_{p,n-1}(t) \wedge \dot\gamma_{\nu_p}(t)| = |Y_{p,1}(t) \wedge \cdots \wedge Y_{p,n-1}(t)|,$$

$$\begin{aligned}|\det dT_f|(p,\tau) &= \left|\left(e_1, \tau df(e_{p,1})\frac{d}{d\tau}\right) \wedge \cdots \wedge \left(e_{p,n-1}, \tau df(e_{p,n-1})\frac{d}{d\tau}\right) \wedge \left(o_p, f(p)\frac{d}{d\tau}\right)\right| \\ &= |f(p)| \left|(e_{p,1}, 0) \wedge \cdots \wedge (e_{p,n-1}, 0) \wedge \left(o_p, \frac{d}{d\tau}\right)\right| = |f(p)|,\end{aligned}$$

$$
\begin{aligned}
|\det d\Psi_f|(p,\tau) &= (|\det d\Psi| \circ T_f(p,\tau))|\det dT_f|(p,\tau) \\
&= |f(p)||Y_{p,1}(\tau f(p)) \wedge \cdots \wedge Y_{p,n-1}(\tau f(p))|.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\mathrm{vol}(\Omega_f) = \mathrm{vol}(\Omega) &+ \int_{f^{-1}(0,+\infty)} \int_0^1 |\det d\Psi_f|(p,\tau)\, d\tau dv_H(p) \\
&- \int_{f^{-1}(-\infty,0)} \int_0^1 |\det d\Psi_f|(p,\tau)\, d\tau dv_H(p) \\
&= \mathrm{vol}(\Omega) + \int_H f(p) \int_0^1 |Y_{p,1}(\tau f(p))) \wedge \cdots \wedge Y_{p,n-1}(\tau f(p))|\, d\tau dv_H(p).
\end{aligned}
$$

On the other hand, since for $i = 1, \ldots, n-1$

$$
\begin{aligned}
d\Psi_{f,1}(e_{p,i}) &= d\Psi(p, f(p)) dT_f(p,t)(e_{p,i}, 0) \\
&= d\Psi(p, f(p)) \left(e_i, df(e_{p,i}) \frac{d}{d\tau} \right) \\
&= d\Psi(p, f(p))(e_{i,p}, 0) + df(e_{p,i}) \Psi(p, f(p)) \left(o_p, \frac{d}{d\tau} \right) \\
&= Y_{p,i}(f(p)) + df(e_{p,i}) \gamma_{\nu_p}(f(p)),
\end{aligned}
$$

we get

$$
\begin{aligned}
|\det d\Psi_{f,1}|(p) = \big| \big[Y_{p,1}(f(p)) + df(e_{p,1}) \dot\gamma_{\nu_p}(f(p)) \big] \wedge \\
\cdots \wedge \big[Y_{p,n-1}(f(p)) + df(e_{p,n-1}) \dot\gamma_{\nu_p}(f(p)) \big] \big|,
\end{aligned}
$$

so that the conclusion follows. □

Lemma 5.8. *For any $f \in C_0^\infty(H)$ the following hold.*

$$
\left. \frac{d}{dt} \right|_{t=0} \mathrm{vol}(\Omega_{tf}) = \int_H f\, dv_H,
$$

$$
\left. \frac{d}{dt} \right|_{t=0} \mathrm{vol}_{n-1}(H_{tf}) = (n-1) \int_H \eta f\, dv_H,
$$

where η is the mean curvature function of H.

Proof. From Lemma 5.7,

$$
\frac{\mathrm{vol}(\Omega_{tf}) - \mathrm{vol}(\Omega)}{t} = \int_H f(p) \int_0^1 |Y_{p,1}(\tau t f(p))) \wedge \cdots \wedge Y_{p,n-1}(\tau t f(p))|\, d\tau dv_H(p)
$$

if $|t|$ is sufficiently small. Thus, letting $t \to 0$, we get

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\Omega_{tf}) &= \int_H f(p) \int_0^1 |Y_{p,1}(0) \wedge \cdots \wedge Y_{p,n-1}(0)| \; d\tau dv_H(p) \\ &= \int_H f(p) \int_0^1 |e_{p,1} \wedge \cdots \wedge e_{p,n-1}| \; d\tau dv_H(p) \\ &= \int_H f(p) dv_H(p).\end{aligned}$$

Also, from Lemma 5.7, putting $\tilde{Y}_{p,i}(t) := Y_{p,i}(t) + tdf(e_{p,i})\dot{\gamma}_{\nu_p}(t)$ $(i = 1, \ldots, n-1)$,

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_{n-1}(H_{tf}) \\ &= \int_H \left.\frac{d}{dt}\right|_{t=0} |\tilde{Y}_{p,1}(tf(p)) \wedge \cdots \wedge \tilde{Y}_{p,n-1}(tf(p))| \, dv_H \\ &= \int_H \sum_{i=1}^{n-1} \frac{\langle \tilde{Y}_1(0) \wedge \cdots \wedge f(p)\nabla \tilde{Y}_i(0) \wedge \cdots \wedge \tilde{Y}_{n-1}(0), \tilde{Y}_1(0) \wedge \cdots \wedge \tilde{Y}_{n-1}(0)\rangle}{|\tilde{Y}_1(0) \wedge \cdots \wedge \tilde{Y}_{n-1}(0)|} \, dv_H(p) \\ &= \int_H \sum_{i=1}^{n-1} \frac{\langle e_{p,1} \wedge \cdots \wedge f(p)[A_{\nu_p} e_{p,i} + df(e_{p,i})\nu_p] \wedge \cdots \wedge e_{p,n-1}, e_{p,1} \wedge \cdots \wedge e_{p,n-1}\rangle}{|e_{p,1} \wedge \cdots \wedge e_{p,n-1}|} \, dv_H(p) \\ &= \int_H f(p) \sum_{i=1}^{n-1} \langle A_{\nu_p} e_{p,i}, e_{p,i}\rangle \, dv_H(p) \\ &= (n-1) \int_H \eta(p) f(p) \, dv_H(p).\end{aligned}$$

□

LEMMA 5.9. *$\int_H \eta f \, dv_H = 0$ for any $f \in C_0^\infty(H)$ with $\int_H f \, dv_H = 0$.*

PROOF. Let $f, g \in C_0^\infty(H)$ and assume $\int_H f \, dv_H = 0$, $\int_H g \, dv_H = 1$. Define the function $V : \mathbb{R}^2 \to \mathbb{R}$ as

$$V(s,t) := \mathrm{vol}(\Omega_{sf+tg})$$

for any $(s,t) \in \mathbb{R}^2$, which is smooth at sufficiently small neighbourhood of $(0,0) \in \mathbb{R}^2$. From Lemma 5.8, we obtain

$$\frac{\partial V}{\partial s}(0,0) = \int_H f \, dv_H = 0$$

$$\frac{\partial V}{\partial t}(0,0) = \int_H g \, dv_H = 1 (\neq 0).$$

Applying the implicit function theorem for V, there exsits a smooth function $t : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$t(0) = 0,$$

$$V(s,t(s)) = V(0,0)$$
$$(\Leftrightarrow \mathrm{vol}(\Omega_{sf+t(s)g}) = \mathrm{vol}(\Omega)),$$
$$\frac{dt}{ds}(s) = \frac{\frac{\partial V}{\partial s}(s,f(s))}{\frac{\partial V}{\partial t}(s,t(s))}.$$

Note that

$$\mathrm{vol}_{n-1}(H) = h\left(\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(M)}\right) = h\left(\frac{\mathrm{vol}(\Omega_{sf+t(s)g})}{\mathrm{vol}(M)}\right) \leq \mathrm{vol}_{n-1}(H_{sf+t(s)g}),$$

so that

$$\left.\frac{d}{dt}\right|_{s=0} \mathrm{vol}_{n-1}(H_{sf+t(s)g}) = 0.$$

On the other hand, applying Lemma 5.7 for $sf + t(s)g$ and in a similar way to the proof of Lemma 5.8, we get

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{vol}_{n-1}(H_{sf+t(s)g}) = (n-1)\int_H \eta f \, dv_H,$$

and thus the conclusion follows. □

By Lemma 5.9 we obtain the following proposition.

PROPOSITION 5.10. *H has the constant mean curvature.*

PROOF. Assume that there exist $p_0, p_1 \in H$ such that $\eta(p_0) < \eta(p_1)$. Take $\alpha \in (\eta(p_0), \eta(p_1))$ and cutoff functions φ_0, φ_1 on H such that

$$\int_H \varphi_0 \, dv_H = \int_H \varphi_1 \, dv_H = 1,$$
$$\eta(p) < \alpha, \qquad \forall p \in \mathrm{supp}(\varphi_0),$$
$$\eta(p) > \alpha, \qquad \forall p \in \mathrm{supp}(\varphi_1).$$

Then, the function $\varphi := \varphi_0 - \varphi_1$ satisfies that $\varphi \in C_0^\infty(H)$ and $\int_H \varphi \, dv_H = 0$. Applying Lemma 5.9 for φ, we obtain

$$\int_H \eta\varphi \, dv_H = 0$$
$$\Leftrightarrow \int_H (\eta - \alpha)\varphi \, dv_H = 0$$
$$\Leftrightarrow \int_H (\eta - \alpha)\varphi_0 \, dv_H = \int_H (\eta - \alpha)\varphi_1 \, dv_H.$$

However, by the choice of α and φ_0, φ_1,

$$\int_H (\eta - \alpha)\varphi_0 \, dv_H < 0 < \int_H (\eta - \alpha)\varphi_1 \, dv_H,$$

which is contradiction. □

5.4 Estimate of isoperimetric constant $I_1(M)$ by Gallot

In this subsection, we give a lower bound of the isoperimetric constant $I_1(M)$ for Riemanian manifolds with $\mathrm{Ric}_M \geq k g_m$ $(k \in \mathbb{R})$.

PROPOSITION 5.11. *Let M be a compact connected n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq k g_m$. Then, the following inequality holds.*

$$I_1(M) \geq \left[\int_0^{\mathrm{diam}(M)/2} c_k(t)^{n-1}\, dt\right]^{-1}.$$

PROOF. By Proposition 5.5, there exists $\beta \in (0, 1/2]$ such that $I_1(M) = h(\beta)/(\beta \operatorname{vol}(M))$. For the constant β, take a domain Ω and a hypersurface $H \subset \partial\Omega$ as in Proposition 5.6. Let ν be the unit outward normal vetor field on H with respect to Ω, whose mean curvature function of H with respect to ν is constant η by Proposition 5.10. Put

$$d_0 := \sup\{d(p, H) \mid p \in \Omega\},$$

$$d_1 := \sup\{d(p, H) \mid p \in M \setminus \overline{\Omega}\},$$

which satisfy $d_0 + d_1 \leq \mathrm{diam}(M)$. Applying the Heintze-Karcher theorem (see Theorem 2.82), we get

$$\operatorname{vol}(\Omega) \leq \operatorname{vol}(H) \int_0^{d_0} (c_k(t) - \eta s_k(t))^{n-1}\, dt,$$

$$\operatorname{vol}(M \setminus \overline{\Omega}) \leq \operatorname{vol}(H) \int_0^{d_1} (c_k(t) + \eta s_k(t))^{n-1}\, dt.$$

Since $\operatorname{vol}(\Omega) \leq \operatorname{vol}(M)/2 \leq \operatorname{vol}(M \setminus \Omega)$, we obtain

$$\begin{aligned} I_1(M) = \frac{\operatorname{vol}_{n-1}(H)}{\operatorname{vol}(\Omega)} &\geq \frac{\operatorname{vol}(H)}{\min\{\operatorname{vol}(\Omega), \operatorname{vol}(M \setminus \overline{\Omega})\}} \\ &\geq \min\left\{\int_0^{d_0} (c_k(t) - \eta s_k(t))^{n-1}\, dt, \int_0^{diam(M)-d_0} (c_k(t) + \eta s_k(t))^{n-1}\, dt\right\}^{-1}. \end{aligned}$$

Next we shall show the following claim. Let $k \in \mathbb{R}$ and $d > 0$ ($d \leq \sqrt{k/(n-1)}\pi$ if $k > 0$). Define the function $J_k : (0, d) \times \mathbb{R} \to \mathbb{R}$ as

$$J_k(t, \zeta) := \max\{c_k(t) + \zeta s_k(t), 0\}^{n-1}.$$

Claim: for any $\zeta \in \mathbb{R}$ and $a \in (0, d)$ the following inequality holds.

$$m(a, \zeta) := \min\left\{\int_0^a J_k(t, -\zeta)\, dt, \int_0^{d-a} J_k(t, \zeta)\, dt\right\} \leq \int_0^{2/d} c_k(t)^{n-1}\, dt.$$

To prove this claim, we define the C^1-function $F : (0,d) \times \mathbb{R} \to \mathbb{R}$ as

$$F(a,\zeta) := \int_0^a J_k(t,-\zeta)\,dt - \int_0^{d-a} J_k(t,\zeta)\,dt$$

for $(a,\zeta) \in (0,d) \times \mathbb{R}$. For every $a \in (0,d)$ $\lim_{\zeta\to-\infty} F(a,\zeta) = +\infty$, $\lim_{\zeta\to-\infty} F(d,\zeta) = -\infty$, and the function $a \mapsto F(a,\zeta)$ is strictly decreasing, so that there exsits a unique $\zeta(a) \in \mathbb{R}$ such that $F(a,\zeta(a)) = 0$. Then, for any $\zeta \in \mathbb{R}$ the inequality $m(a,\zeta) \le m(a,\zeta(a))$ holds. Since the equalities

$$0 = -F(a,\zeta(a)) = \int_0^{d-a} J_k(t,\zeta(a))\,dt - \int_0^a J_k(t,-\zeta(a))\,dt = F(d-a,-\zeta(a)) \quad (\forall a \in (0,d))$$

holds, by the uniqueness of $\zeta(d-a)$, we get

$$\zeta(d-a) = -\zeta(a). \tag{3}$$

In particular, we obtain

$$\zeta\left(\frac{d}{2}\right) = 0.$$

For any $a, a' \in (0,d)$ $(a < a')$

$$F(a,\zeta(a)) = 0 = F(a',\zeta(a')) > F(a,\zeta(a'))$$

holds. By the monotonicity of the function $\zeta \mapsto F(a,\zeta)$ we know the function $a \mapsto \zeta(a)$ is strictly increasing. Moreover, we see that the function $a \mapsto \zeta(a)$ is class C^1. In fact, for any $t, a \in (0,d)$ and $\zeta \in \mathbb{R}$,

$$\frac{\partial J_k}{\partial \zeta}(t,\zeta) = \begin{cases} (n-1)s_k(t)(c_k(t) + \zeta s_k(t))^{n-2} \ (>0) & \text{if } J_k(t,\zeta) > 0 \\ 0 & \text{if } J_k(t,\zeta) = 0, \end{cases}$$

$$\frac{\partial F}{\partial \zeta}(a,\zeta) = -\int_0^a \frac{\partial J_k}{\partial \zeta}(t,-\zeta)\,dt - \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t,\zeta)\,dt < 0,$$

and thus the function $a \mapsto \zeta(a)$ coincides with the function obtained by the implicit function theorem. In particular, the following holds.

$$\begin{aligned} \frac{d\zeta}{da}(a) &= -\frac{\partial F}{\partial a}(a,\zeta(a)) \left[\frac{\partial F}{\partial \zeta}(a,\zeta(a))\right]^{-1} \\ &= [J_k(a,-\zeta(a)) + J_k(d-a,\zeta(a))] \left[\int_0^a \frac{\partial J_k}{\partial \zeta}(t,-\zeta(a))\,dt + \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t,\zeta(a))\,dt\right]^{-1}. \end{aligned}$$

Next we define the function $G : (0,d) \to \mathbb{R}$ as $G(a) := m(a,\zeta(a))$. For any $a \in (0,d)$, we express $G(a)$ as follows.

$$G(a) = \frac{1}{2}\left[\int_0^a J_k(t,-\zeta(a))\,dt + \int_0^{d-a} J_k(t,\zeta(a))\,dt\right].$$

By the equality (5.4), we have the equality $G(d-a) = G(a)$. Thus, to prove the claim, it is sufficient to prove that

$$G(a) \le G\left(\frac{d}{2}\right) \left(= \int_0^{d/2} c_k(t)^{n-1} dt\right)$$

for any $a \in (d/2, d)$ and $t \in (0, a)$. To prove this we show that the function G is strictly decreasing on $(d/2, d)$. Note that

$$\zeta(a) > \zeta(d/2) = 0,$$

$$\begin{aligned} s_k(a) &\ge s_k(d-a), \\ c_k(a) &< c_k(d-a), \\ J_k(a, -\zeta) \le J_k(d-a, -\zeta) &\le J_k(d-a, \zeta), \end{aligned}$$

for any $a \in (d/2, d)$ and $\zeta > 0$. Put

$$a_0 := \sup\{a \in (d/2, d) \mid c_k(a) - \zeta(a) s_k(a) > 0\}.$$

Then, since the function $t \mapsto c_k(t)/s_k(t)$ on $(0, d)$ is strictly decreasing, for any $a \in (d/2, a_0)$ and $t \in (0, a)$,

$$\begin{aligned} c_k(t) - \zeta(a) s_k(t) &= s_k(t) \left(\frac{c_k(t)}{s_k(t)} - \zeta(a)\right) > s_k(t) \left(\frac{c_k(a)}{s_k(a)} - \zeta(a)\right) \\ &= \frac{s_k(t)}{s_k(a)} (c_k(a) - \zeta(a) s_k(a)) > 0, \end{aligned}$$

and thus

$$J(t, \zeta(a)) > J(t, -\zeta(a)) > 0.$$

Define the function $\varphi : (0, d) \to \mathbb{R}$ as

$$\varphi(t, \zeta) := \begin{cases} (n-1) s_k(t) \left[c_k(t) + \zeta s_k(t)\right]^{-1} & (\text{if } J_k(t, \zeta) > 0) \\ 0 & (\text{if } J_k(t, \zeta) = 0). \end{cases}$$

Then, we have

$$\frac{\partial J_k}{\partial \zeta}(t, \zeta) = \varphi(t, \zeta) J_k(t, \zeta), \qquad (\forall t \in (0, d), \forall \zeta \in \mathbb{R}),$$

$$\varphi(t, -\zeta(a)) > \varphi(t, \zeta(a)) > 0, \qquad (\forall a \in (d/2, a_0), \forall t \in (0, a)).$$

Also, since

$$\varphi(t, -\zeta(a)) = (n-1) \left[\frac{c_k(t)}{s_k(t)} - \zeta(a)\right]^{-1}, \qquad (\forall a \in (d/2, a_0), \forall t \in (0, a))$$

for any $a \in (2/d, a_0)$ the function $t \mapsto \varphi(t, -\zeta(a))$ on $(0, a)$ is strictly incleasing. From the above, for any $a \in (d/2, a_0)$

$$
\begin{aligned}
2\frac{dG}{da}(a) = {} & J_k(a, -\zeta(a)) - J_k(d-a, \zeta(a)) + \\
& + \frac{d\zeta}{da}(a)\left[-\int_0^a \frac{\partial J_k}{\partial \zeta}(t, -\zeta(a))\, dt + \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t, \zeta(a))\, dt \right],
\end{aligned}
$$

and

$$
\begin{aligned}
& 2\left[\int_0^a \frac{\partial J_k}{\partial \zeta}(t, -\zeta(a))\, dt + \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t, \zeta(a))\, dt \right] \frac{dG}{da}(a) \\
&= [J_k(a, -\zeta(a)) - J_k(d-a, \zeta(a))] \left[\int_0^a \frac{\partial J_k}{\partial \zeta}(t, -\zeta(a))\, dt + \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t, \zeta(a))\, dt \right] \\
&\quad + [J_k(a, -\zeta(a)) + J_k(d-a, \zeta(a))] \left[-\int_0^a \frac{\partial J_k}{\partial \zeta}(t, -\zeta(a))\, dt + \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t, \zeta(a))\, dt \right] \\
&= 2\left[J_k(a, -\zeta(a)) \int_0^{d-a} \frac{\partial J_k}{\partial \zeta}(t, \zeta(a))\, dt - J_k(d-a, \zeta(a)) \int_0^a \frac{\partial J_k}{\partial \zeta}(t, -\zeta(a))\, dt \right] \\
&= 2\left[J_k(a, -\zeta(a)) \int_0^{d-a} \varphi(t, \zeta(a)) J_k(t, \zeta(a))\, dt \right. \\
&\qquad\qquad \left. - J_k(d-a, \zeta(a)) \int_0^a \varphi(t, -\zeta(a)) J_k(t, -\zeta(a))\, dt \right] \\
&< 2J_k(d-a, \zeta(a)) \left[\int_0^{d-a} \varphi(t, \zeta(a)) J_k(t, \zeta(a))\, dt - \int_0^a \varphi(t, -\zeta(a)) J_k(t, -\zeta(a))\, dt \right] \\
&< 2J_k(d-a, \zeta(a)) \left[\int_0^{d-a} \varphi(t, -\zeta(a)) J_k(t, \zeta(a))\, dt - \int_0^a \varphi(t, -\zeta(a)) J_k(t, -\zeta(a))\, dt \right] \\
&= 2J_k(d-a, \zeta(a)) \left[\int_0^{d-a} \varphi(t, -\zeta(a)) \left[J_k(t, \zeta(a)) - J_k(t, -\zeta(a)) \right] dt \right. \\
&\qquad\qquad \left. - \int_{d-a}^a \varphi(t, -\zeta(a)) J_k(t, -\zeta(a)) \right] \\
& 2J_k(d-a, \zeta(a)) \left[\varphi(d-a, -\zeta(a)) \int_0^{d-a} \left[J_k(t, \zeta(a)) - J_k(t, -\zeta(a)) \right] dt \right. \\
&\qquad\qquad \left. - \varphi(d-a, -\zeta(a)) \int_{d-a}^a J_k(t, -\zeta(a)) \right] \\
&= 2J_k(d-a, \zeta(a)) \varphi(d-a, -\zeta(a)) \left[\int_0^{d-a} J_k(t, \zeta(a))\, dt - \int_0^a J_k(t, -\zeta(a))\, dt \right] \\
&= -2J_k(d-a, \zeta(a)) \varphi(d-a, -\zeta(a)) F(a, \zeta(a)) = 0.
\end{aligned}
$$

Thus, the function G is strictly decreasing on $(2/d, a_0)$. If $a_0 = d$, then the proof is completed. Now we consider the case when $a_0 < d$. Since

$$c(a_0) - \zeta(a_0) s(a_0) = 0$$

by the definition of a_0, for any $a \in (a_0, d)$ and $t \in (a_0, a)$, we have

$$c_k(t) - \zeta(a)s_k(t) = s_k(t)\left(\frac{c_k(t)}{s_k(t)} - \zeta(a)\right) < s_k(t)\left(\frac{c_k(a_0)}{s_k(a_0)} - \zeta(a_0)\right) = 0,$$

and

$$J_k(t, -\zeta(a)) = 0.$$

Thus, for any $a \in (a_0, d)$, $G(a)$ is written as

$$G(a) = \int_0^a J(t, -\zeta(a))\, dt = \int_0^{a_0} J(t, -\zeta(a))\, dt,$$

so that G is also stlictly decreasing on (a_0, d), which is the conclusion. $\square$

5.5 Etimate of isoperimetric constant $I_{(n-1)/n}(M)$ by Gallot

In this subsection, we give a lower bound of the isoperimetric constant $I_{(n-1)/n}1(M)$ for Riemanian manifolds with $\mathrm{Ric}_M \geq k g_m$ $(k < 0)$.

If $\beta \in (0, 1/2)$ satisfies that $I_a(M) = h(\beta)/(\beta \operatorname{vol}(M))^a$ for some $a \geq (n-1)/n$ (see Proposition 5.5 (ii)), then the mean curvature η of H is determined following.

Lemma 5.12. *Let $\beta \in (0, 1/2)$, $a \geq (n-1)/n$, and assume $I_a(M) = h(\beta)/(\beta \operatorname{vol}(M))^a$. For the constant β, take a domain Ω and a hypersurface $H \subset \partial\Omega$ as in Proposition 5.6. Let ν be the unit outward normal vetor field on H with respect to Ω. Then, for the mean curvaure function η of H with respect to ν the following holds.*

$$\eta = \frac{a}{n-1}\frac{\operatorname{vol}_{n-1}(H)}{\operatorname{vol}(\Omega)} = \frac{a}{n-1}\frac{h(\beta)}{\beta \operatorname{vol}(M)} \left(\geq \frac{a}{n-1} I_1(M)\right)$$

Proof. Take $f \in C_0^\infty(H)$ satisfying $\int_H f\, dv_H = 1$, and put $\beta_t := \operatorname{vol}(\Omega_{tf})/\operatorname{vol}(M)$. Since $\beta_t \leq 1/2$ if $|t|$ is sufficiently small, by Proposition 5.5 (ii),

$$\frac{\operatorname{vol}_{n-1}(H)}{\operatorname{vol}(\Omega)^a} = \frac{h(\beta)}{(\beta \operatorname{vol}(M))^a} = I_a(M) \leq \frac{h(\beta_t)}{(\beta_t \operatorname{vol}(M))^a} \leq \frac{\operatorname{vol}_{n-1}(H_{tf})}{\operatorname{vol}(\Omega_{tf})^a}.$$

Thus, the function $t \to \operatorname{vol}_{n-1}(H_t f)/\operatorname{vol}(\Omega_{tf})$ has a minimal value at $t = 0$, so that

$$\left.\frac{d}{dt}\right|_{t=0} \frac{\operatorname{vol}_{n-1}(H_t f)}{\operatorname{vol}(\Omega_{tf})^a} = 0.$$

On the other hand, by Lemma 5.8

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \frac{\operatorname{vol}_{n-1}(H_t f)}{\operatorname{vol}(\Omega_{tf})^a} &= \frac{\left[\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}_{n-1}(H_{tf})\right] \operatorname{vol}(\Omega) - a \operatorname{vol}_{n-1}(H) \left[\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol}(\Omega_{tf})\right]}{\operatorname{vol}(\Omega)^{a+1}} \\
&= \frac{(n-1)\int_H \eta f\, dv_H \operatorname{vol}(\Omega) - a \operatorname{vol}_{n-1}(H) \int_H f\, dv_H}{\operatorname{vol}(\Omega)^{a+1}} \\
&= \frac{(n-1)\eta \operatorname{vol}(\Omega) - a \operatorname{vol}_{n-1}(H)}{\operatorname{vol}(\Omega)^{a+1}}.
\end{aligned}$$

Thus,

$$\eta = \frac{a}{n-1}\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)} = \frac{a}{n-1}\frac{h(\beta)}{\beta\,\mathrm{vol}(M)}.$$

$\square$

PROPOSITION 5.13. *Let M be a compact connected n-dimensional Riemanian manifold with $\mathrm{Ric}_M \geq k g_M$ ($k<0$). Then, the following estimate holds.*

$$I_{(n-1)/n}(M) \geq \mathrm{vol}(M)^{\frac{1}{n}}\left[\int_0^{\mathrm{diam}(M)}\left(\frac{c_k(t)}{I_1(M)}+\frac{s_k(t)}{n}\right)^{n-1}dt\right]^{-\frac{1}{n}}.$$

PROOF.

(a) The case $I_{(n-1)/n}(M) = h(\beta)/(\beta\,\mathrm{vol}(M))^{(n-1)/n}$ for some $\beta \in (0,1/2]$
For the constant β, take a domain Ω and a hypersurface $H \subset \partial\Omega$ as in Proposition 5.6. Let ν be the unit outward normal vetor field on H with respect to Ω, whose mean curvature function of H with respect to ν is constant η by Proposition 5.10.

(a-1) The case $\beta \in (0,1/2)$
Put

$$d_0 := \sup\{d(p,H) \mid p \in \Omega\},$$
$$d_1 := \sup\{d(p,H) \mid p \in M \setminus \overline{\Omega}\},$$

which satisfy $d_0 + d_1 \leq \mathrm{diam}(M)$. Applying the Heintze-Karcher theorem, we get

$$\begin{aligned}\mathrm{vol}(M) &\leq \mathrm{vol}(H)\int_{-d_0}^{d_1}(c_k(t)+\eta s_k(t))^{n-1}\,dt\\ &\leq \mathrm{vol}(H)\int_0^{\mathrm{diam}(M)}(c_k(t)+\eta s_k(t))^{n-1}\,dt,\end{aligned}$$

and thus by Lemma 5.12,

$$\begin{aligned}I_{(n-1)/n}(M) = \frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)^{\frac{n-1}{n}}} &\geq \left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{n-1}{n}}\left[\frac{\mathrm{vol}(M)}{\int_0^{\mathrm{diam}(M)}(c_k(t)+\eta s_k(t))^{n-1}\,dt}\right]^{\frac{1}{n}}\\ &= \mathrm{vol}(M)^{\frac{1}{n}}\left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{n-1}{n}}\left[\int_0^{\mathrm{diam}(M)}\left(c_k(t)+\frac{1}{n}\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}s_k(t)\right)^{n-1}dt\right]^{-\frac{1}{n}}\\ &= \mathrm{vol}(M)^{\frac{1}{n}}\left[\int_0^{\mathrm{diam}(M)}\left(\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}_{n-1}(H)}c_k(t)+\frac{s_k(t)}{n}\right)^{n-1}dt\right]^{-\frac{1}{n}}\\ &\geq \mathrm{vol}(M)^{\frac{1}{n}}\left[\int_0^{\mathrm{diam}(M)}\left(\frac{c_k(t)}{I_1(M)}+\frac{s_k(t)}{n}\right)^{n-1}dt\right]^{-\frac{1}{n}}\end{aligned}$$

(a-2) The case $\beta = 1/2$
By Proposition ,

$$\begin{aligned}
I_{(n-1)/n}(M) &= \left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{n-1}{n}} \left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{1}{n}} \mathrm{vol}(\Omega)^{\frac{1}{n}} \\
&\geq \left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{n-1}{n}} I_1(M)^{\frac{1}{n}} \left[\frac{\mathrm{vol}(M)}{2}\right]^{\frac{1}{n}} \\
&\geq \left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{n-1}{n}} \left[\frac{\mathrm{vol}(M)}{2\int_0^{\mathrm{diam}(M)/2} c_k(t)^{n-1} dt}\right]^{\frac{1}{n}} \\
&\geq \left[\frac{\mathrm{vol}_{n-1}(H)}{\mathrm{vol}(\Omega)}\right]^{\frac{n-1}{n}} \left[\frac{\mathrm{vol}(M)}{\int_0^{\mathrm{diam}(M)} (c_k(t) + \eta s_k(t))^{n-1}\, dt}\right]^{\frac{1}{n}} \\
&\geq \mathrm{vol}(M)^{\frac{1}{n}} \left[\int_0^{\mathrm{diam}(M)} \left(\frac{c_k(t)}{I_1(M)} + \frac{s_k(t)}{n}\right)^{n-1} dt\right]^{-\frac{1}{n}}.
\end{aligned}$$

(b) The case $I_{(n-1)/n}(M) = \lim_{\beta\to 0} h(\beta)/(\beta\, \mathrm{vol}(M))^{(n-1)/n}$
By Proposition 5.5, Corollarly 2.73, and Bishop's inequality

$$\begin{aligned}
I_{(n-1)/n}(M) &= \frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{\mathrm{vol}(B_0^n(1))^{\frac{n-1}{n}}} \\
&= \mathrm{vol}_{n-1}(\partial B_0^n(1)) \left[\frac{\mathrm{vol}_{n-1}(\partial B_0^n(1))}{n}\right]^{-\frac{n-1}{n}} \\
&= n^{\frac{n-1}{n}} \left[\mathrm{vol}_{n-1}(\partial B_0^n(1))\right]^{\frac{1}{n}} \\
&= n^{\frac{n-1}{n}} \left[\mathrm{vol}_{n-1}(\partial B_0^n(1)) \int_0^{\mathrm{diam}(M)} s_k(t)^{n-1}\, dt\right]^{\frac{1}{n}} \left[\int_0^{\mathrm{diam}(M)} s_k(t)^{n-1}\, dt\right]^{-\frac{1}{n}} \\
&= \mathrm{vol}(B_k^n(\mathrm{diam}(M)))^{\frac{1}{n}} \left[\int_0^{\mathrm{diam}(M)} \left(\frac{s_k(t)}{n}\right)^{n-1} dt\right]^{-\frac{1}{n}} \\
&\geq \mathrm{vol}(M)^{\frac{1}{n}} \left[\int_0^{\mathrm{diam}(M)} \left(\frac{c_k(t)}{I_1(M)} + \frac{s_k(t)}{n}\right)^{n-1} dt\right]^{-\frac{1}{n}}.
\end{aligned}$$

□

Remark 5.14. If $k \geq 0$, then the following estimate holds:

$$I_{(n-1)/n}(M) \geq 2^{\frac{n-1}{n}} \mathrm{vol}(M)^{\frac{1}{n}} \mathrm{diam}(M)^{-1}.$$

6 Gallot's two results

In this section, a domain in a manifold means an open set which is not necessarily connected. Let M be a compact connected Riemannian manifold. For a domain Ω with smooth boundary $\partial\Omega$, we put

$$\underline{\mathrm{vol}}(\Omega) := \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(M)},$$

$$\underline{\mathrm{vol}}_{n-1}(\partial\Omega) := \frac{\mathrm{vol}(\partial\Omega)}{\mathrm{vol}(M)}.$$

Similarly, define the measure $\underline{v}$ as

$$\underline{v} := \frac{v_M}{\mathrm{vol}(M)}.$$

Moreover, we define the isoperimetric constant $\underline{I}_a$ as

$$\underline{I}_a(M) := \inf\left\{ \frac{\underline{\mathrm{vol}}_{n-1}(\partial\Omega)}{\underline{v}(\Omega)^a} \,\middle|\, \Omega \subset M \text{ is a domain with smooth boundary}, \underline{\mathrm{vol}}(\Omega) \le \frac{1}{2} \right\}.$$

Note that the equality

$$\underline{I}_a(M) = \frac{I_a(M)}{\mathrm{vol}(M)^{1-a}}$$

holds.

6.1 Gallot's Sobolev inequality

Proposition 6.1 (Gallot). *Let M be a compact connected n-dimensional Riemannian manifold, $n/(n-1) \ge p \ge 1$, $q \ge 1$, and assume $2(q-1) \le pq$. Then, for any C^1-function $f : M \to \mathbb{R}$ the following inequality holds:*

$$\|f\|_{L^{pq}(\underline{v})} \le \frac{2q}{\underline{I}_{1/p}(M)} \|\nabla f\|_{L^2(\underline{v})} + \|f\|_{L^2(\underline{v})}, \tag{4}$$

where $\|f\|_{L^r(\underline{v})} := (\int_M f^r \, d\underline{v})^{1/r}$ $(r > 0)$ is the $L^r(\underline{v})$-norm of f.

Lemma 6.2. *Let M be a compact connected Riemannian manifold and $p \ge 1$. If a bounded measurable function satisfies*

$$\int_M \mathrm{sgn}(f)|f|^{p-1} d\underline{v} = 0, \tag{5}$$

then for all $t \in \mathbb{R}$ the following inequality holds.

$$\int_M |f - t|^p d\underline{v} \ge \int_M |f|^p d\underline{v} \tag{6}$$

PROOF.

• ($p = 1$) By the equality (5)

$$\underline{\mathrm{vol}}(f^{-1}(-\infty, 0)) = \underline{\mathrm{vol}}(f^{-1}(0, \infty)).$$

If $t > 0$, then

$$\begin{aligned}
\int_M |f - t| \, d\underline{v} - \int_M |f| \, d\underline{v} &= \int_M (|f - t| - |f|) \, d\underline{v} \\
&= \int_{f^{-1}(-\infty,0)} t \, d\underline{v} + \int_{f^{-1}[0,t)} (t - 2f) \, d\underline{v} - \int_{f^{-1}[t,\infty)} t \, d\underline{v} \\
&= t \, \underline{\mathrm{vol}}(f^{-1}(-\infty, 0)) + \int_{f^{-1}[0,t)} (t - 2f) \, d\underline{v} - t \, \underline{\mathrm{vol}}(f^{-1}[t, \infty)) \\
&= t \, \underline{\mathrm{vol}}(f^{-1}(0, \infty)) + \int_{f^{-1}[0,t)} (t - 2f) \, d\underline{v} - t \, \underline{\mathrm{vol}}(f^{-1}[t, \infty)) \\
&= \int_{f^{-1}[0,t)} (t - 2f) \, d\underline{v} + t \, \underline{\mathrm{vol}}(f^{-1}(0, t)) \\
&\geq 2 \int_{f^{-1}(0,t)} (t - f) \, d\underline{v} \\
&\geq 0.
\end{aligned}$$

Similarly, if $t < 0$, then we can show that $\int_M |f - t| \, d\underline{v} - \int_M |f| \, d\underline{v} \geq 0$.

• ($p > 1$) Let

$$\varphi(t) = \int_M |f - t|^p d\underline{v}.$$

Then, we have

$$\frac{d\varphi}{dt}(t) = -p \int_M \mathrm{sgn}(f - t)|f - t|^{p-1} d\underline{v}.$$

If $t > 0$, then by the equality (5)

$$\begin{aligned}
&\int_M \mathrm{sgn}(f - t)|f - t|^{p-1} d\underline{v} \\
&= -\int_{f^{-1}(-\infty,0)} |f - t|^{p-1} d\underline{v} - \int_{f^{-1}[0,t)} |f - t|^{p-1} d\underline{v} + \int_{f^{-1}[t,\infty)} |f - t|^{p-1} d\underline{v} \\
&\leq -\int_{f^{-1}(-\infty,0)} |f|^{p-1} d\underline{v} + 0 + \int_{f^{-1}[t,\infty)} |f|^{p-1} d\underline{v} \\
&= \int_{f^{-1}(-\infty,0) \cap f^{-1}[t,\infty)} \mathrm{sgn}(f)|f|^{p-1} d\underline{v} \\
&= -\int_{f^{-1}[0,t)} \mathrm{sgn}(f)|f|^{p-1} d\underline{v} \\
&\leq 0
\end{aligned}$$

and therefore $d\varphi/dt(t) \geq 0$. Similarly, we can prove $d\varphi/dt(t) \leq 0$ if $t < 0$. Thus, $\varphi(t) \geq \varphi(0)$ for every $t \in \mathbb{R}$. Hence the conclusion follows. □

Proposition 6.3 (Bombieri). *Let M be a compact connected n-dimensional Riemannian manifold and $1 \le p \le n/(n-1)$. Then, for any C^1-function $f : M \to \mathbb{R}$ with $\int_M \operatorname{sgn}(f)|f|^{p-1} d\underline{v} = 0$ the following inequality holds:*

$$\|f\|_{L^p(\underline{v})} \le \frac{1}{\underline{I}_{1/p}(M)} \|\nabla f\|_{L^1(\underline{v})}. \tag{7}$$

Proof. Since the set of Morse functions on M is dense in $C^1(M)$, it is sufficient to show the inequality (7) for any Morse function f satisfying that $\int_M \operatorname{sgn}(f)|f|^{p-1} d\underline{v} = 0$. Take $\alpha \in \mathbb{R}$ satisfying that

$$\underline{\mathrm{vol}}(f^{-1}(-\infty, \alpha)) \le \frac{1}{2}, \qquad \underline{\mathrm{vol}}(f^{-1}(\alpha, \infty)) \le \frac{1}{2}. \tag{8}$$

(a) the case $\alpha = 0$.
Put $\Omega_t := f^{-1}(t, \infty)$. Since f is a Morse function, $\underline{v}(f^{-1}(t)) = 0$ for any $t \in \mathbb{R}$ and thus the map $t \mapsto \underline{v}(\Omega_t)$ is continuous. By the Fubini's theorem

$$\begin{aligned} \|f_+\|^p_{L^p(\underline{v})} &= \int_M |f_+|^p d\underline{v} = \int_M \int_0^{f_+^p} dt\, d\underline{v} = \int_0^\infty \int_{\Omega_{t^{1/p}}} d\underline{v}\, dt \\ &= \int_0^\infty \underline{\mathrm{vol}}(\Omega_{t^{1/p}})\, dt = p \int_0^\infty \underline{\mathrm{vol}}(\Omega_t) t^{p-1}\, dt. \end{aligned}$$

where $f_+ := \max\{f, 0\}$. On the other hand, by the co-area fomula, the definition of $\underline{I}_{1/p}(M)$, and the condition (8), we have

$$\|\nabla f_+\|_{L^1(\underline{v})} = \int_M |\nabla f_+|\, d\underline{v} = \int_0^\infty \underline{\mathrm{vol}}_{n-1}(\partial\Omega_t)\, dt \ge \underline{I}_{1/p}(M) \int_0^\infty \underline{\mathrm{vol}}(\Omega_t)^{1/p}\, dt.$$

Combining the above, we see that if the inequality

$$p \int_0^\infty \underline{\mathrm{vol}}(\Omega_t) t^{p-1}\, dt \le \left(\int_0^\infty \underline{\mathrm{vol}}(\Omega_t)^{1/p}\, dt \right)^p, \tag{9}$$

holds, then we get the inequality (7) for the function f_+.

$$\frac{d}{ds}\left(p \int_0^s \underline{\mathrm{vol}}(\Omega_t) t^{p-1}\, dt \right) = p s^{p-1} \underline{\mathrm{vol}}(\Omega_s),$$

$$\frac{d}{ds}\left(\int_0^s \underline{\mathrm{vol}}(\Omega_t)^{1/p}\, dt \right)^p = p \left(\int_0^s \underline{\mathrm{vol}}(\Omega_t)^{1/p}\, dt \right)^{p-1} \underline{v}(\Omega_s^{1/p}).$$

Since the function $t \mapsto \underline{\mathrm{vol}}(\Omega_t)$ is monotone decreasing, we have

$$\int_0^s \underline{\mathrm{vol}}(\Omega_t)^{1/p}\, dt \ge s \underline{\mathrm{vol}}(\Omega_s)^{1/p}$$

and

$$\frac{d}{ds}\left(p\int_0^s \underline{\mathrm{vol}}(\Omega_t)t^{p-1}\,dt\right) \le \frac{d}{ds}\left(\int_0^s \underline{\mathrm{vol}}(\Omega_t)^{1/p}\,dt\right)^p.$$

Therefore, we get the inequality (9). Hence, for the function f_+ the inequality (7) holds. Similarly, for the function $f_- := \max\{-f, 0\}$ the inequality (7) also holds. Thus, we obtain

$$\begin{aligned}
\|f\|_{L^p(\underline{v})} &\le \|f_+\|_{L^p(\underline{v})} + \|f_-\|_{L^p(\underline{v})} \\
&\le \frac{1}{\underline{I}_{1/p}(M)}\|\nabla f_+\|_{L^1(\underline{v})} + \frac{1}{\underline{I}_{1/p}(M)}\|\nabla f_-\|_{L^1(\underline{v})} \\
&= \frac{1}{\underline{I}_{1/p}(M)}\|\nabla f\|_{L^1(\underline{v})}.
\end{aligned}$$

(b) the case $\alpha \neq 0$.

Put $\tilde{f} := f - \alpha$. Then, the inequalities

$$\underline{\mathrm{vol}}(f^{-1}(-\infty, 0)) \le \frac{1}{2}, \qquad \underline{\mathrm{vol}}(\tilde{f}^{-1}(0, \infty)) \le \frac{1}{2}$$

hold. Note that we did not use the condition $\int_M \mathrm{sgn}(f)|f|^{p-1} d\underline{v} = 0$ in the proof of the case $\alpha = 0$. It follws that the inequality (7) also holds for $\tilde{f}$. Thus, by Lemma 6.2, we have

$$\|f\|_{L^p(\underline{v})} \le \|\tilde{f}\|_{L^p(\underline{v})} = \frac{1}{\underline{I}_{1/p}(M)}\|\nabla\tilde{f}\|_{L^1(\underline{v})} = \frac{1}{\underline{I}_{1/p}(M)}\|\nabla f\|_{L^1(\underline{v})}.$$

□

Proof of Theorem 6.1. By the density of Morse functions in $C^1(M)$, we may assume that f is a Morse function f. By the dominated convergence theorem it is sufficient to show that for $q > 1$. By the similar consideration of the proof of Proposition 6.3, we can take $\alpha \in \mathbb{R}$ satisfying that

$$\int_M \mathrm{sgn}(f-\alpha)|f-\alpha|^{q(p-1)}\,d\underline{v} = 0.$$

Applying the Proposition 6.3 for the function $\mathrm{sgn}(f-\alpha)|f-\alpha|^q$, we get

$$\||f-\alpha|^q\|_{L^p(\underline{v})} \le \frac{1}{\underline{I}_{1/p}(M)}\|\nabla|f-\alpha|^q\|_{L^1(\underline{v})}.$$

Since

$$\||f-\alpha|^q\|_{L^p(\underline{v})} = \|f-\alpha\|^q_{L^{pq}(\underline{v})}$$

and, by Hölder's inequality,

$$\begin{aligned}
\|\nabla|f-\alpha|^q\|_{L^1(\underline{v})} &= q\||f-\alpha|^{q-1}\nabla f\|_{L^1(\underline{v})} \le q\||f-\alpha|^{q-1}\|_{L^2(\underline{v})}\|\nabla f\|_{L^2(\underline{v})} \\
&= q\|f-\alpha\|^{q-1}_{L^{2(q-1)}(\underline{v})}\|\nabla f\|_{L^2(\underline{v})} \le q\|f-\alpha\|^{q-1}_{L^{pq}(\underline{v})}\|\nabla f\|_{L^2(\underline{v})},
\end{aligned}$$

we have

$$\|f-\alpha\|_{L^{pq}(\underline{v})} \le \frac{q}{\underline{I}_{1/p}(M)}\|\nabla f\|_{L^2(\underline{v})}.$$

On the other hand, we have

$$\|\alpha\|_{L^{pq}(\underline{v})} = |\alpha| = \|\alpha\|_{L^1(\underline{v})} \le \|f-\alpha\|_{L^1(\underline{v})} + \|f\|_{L^1(\underline{v})} \le \|f-\alpha\|_{L^{pq}(\underline{v})} + \|f\|_{L^2(\underline{v})}.$$

Thus, we obtain

$$\begin{aligned}\|f\|_{L^{pq}(\underline{v})} &\le \|f-\alpha\|_{L^{pq}(\underline{v})} + \|\alpha\|_{L^{pq}(\underline{v})} \le 2\|f-\alpha\|_{L^{pq}(\underline{v})} + \|f\|_{L^2(\underline{v})} \\ &\le \frac{2q}{\underline{I}_{1/p}(M)}\|\nabla f\|_{L^2(\underline{v})} + \|f\|_{L^2(\underline{v})}.\end{aligned}$$

□

6.2 Gallot's estimate of L^∞-norm from above by L^2-norm

PROPOSITION 6.4. *Let M be a compact connected Riemannian manifold and $\lambda \ge 0$. If a nonnegative continuous function f on M satisfies that*
(i) f is C^2 on $M_+ := f^{-1}(0,\infty)$,
(ii) ∇f, $f\Delta f$ is bounded on M_+,
(iii) $\Delta f \le \lambda^2 f$ on M_+,
then the following inequality holds:

$$\|f\|_{L^\infty(\underline{v})} \le L_n\left(\frac{\lambda}{c_M}\right)\|f\|_{L^2(\underline{v})}. \tag{10}$$

where $c_M := \underline{I}_{(n-1)/n}(M)$ and L_n is a strictly increasing contiuous function from $[0,\infty)$ to $\mathbb{R}$ defined as

$$L_n(t) := \prod_{i=0}^{\infty}\left(1+\frac{4p^i}{\sqrt{2p^i-1}}t\right)^{p^{-i}} \qquad \left(p := \frac{n}{n-1}\right)$$

REMARK 6.5. Note that $L_n(0) = 1$ and that $L_n(t)$ is finite for all $t > 0$. In fact,

$$\begin{aligned}L_n(t) &= \prod_{i=0}^{\infty}\left(1+\frac{4p^i}{\sqrt{2p^i-1}}t\right)^{p^{-i}} \le \prod_{i=0}^{\infty}\left(1+4p^{i/2}t\right)^{p^{-i}} \le \prod_{i=0}^{\infty}(\exp(4p^{i/2}t))^{p^{-i}} \\ &= \prod_{i=0}^{\infty}\exp(4p^{-i/2}t) = \exp\left(4t\sum_{i=0}^{\infty}p^{-i/2}\right) = \exp\left(\frac{4t}{1-p^{-1/2}}\right) < \infty.\end{aligned}$$

LEMMA 6.6. *Assume the same notations and assumptions as in Propsition 6.4. Then, for any $a > 1$, the following inequality holds:*

$$\|\nabla f^a\|_{L^2(\underline{v})} \le \frac{a\lambda}{\sqrt{2a-1}}\|f\|^a_{L^{2a}(\underline{v})}. \tag{11}$$

PROOF. Define the vector field $\overline{\nabla f}$ and function $\overline{f\Delta f}$ on M as

$$\overline{\nabla f}(p) := \begin{cases} \nabla f & (p \in M_+) \\ 0 & (p \in f^{-1}(0)) \end{cases}, \overline{f\Delta f}(p) := \begin{cases} f\Delta f & (p \in M_+) \\ 0 & (p \in f^{-1}(0)). \end{cases}$$

By the condition (iii), we have $\overline{f\Delta f} \leq \lambda^2 f^2$. We also see that $f^{2a-1}\overline{\nabla f}$ is a C^1-vector field on M and the equality

$$\mathrm{div}(f^{2a-1}\overline{\nabla f}) = -f^{2a-2}\overline{f\Delta f} + \langle \nabla f^{2a-1}, \overline{\nabla f} \rangle$$

holds. Thus, by the divergence theorem

$$\begin{aligned} \|\nabla f^a\|_{L^2(\underline{v})} = a\|f^{a-1}\overline{\nabla f}\|_{L^2(\underline{v})} &= \frac{a}{\sqrt{2a-1}} \left[\int_M \langle \nabla f^{2a-1}, \overline{\nabla f} \rangle \, d\underline{v} \right]^{1/2} \\ &= \frac{a}{\sqrt{2a-1}} \left[\int_M f^{2a-2}\overline{f\Delta f} \, d\underline{v} \right]^{1/2} \leq \frac{a\lambda}{\sqrt{2a-1}} \left[\int_M f^{2a} \, d\underline{v} \right]^{1/2} \\ &= \frac{a\lambda}{\sqrt{2a-1}} \|f\|^a_{L^{2a}(\underline{v})}. \end{aligned}$$

$\square$

Proof of Theorem 6.4. For all $a > 1$ the function f^a is class C^2. Applying the Proposition 6.1 for the function f^a, for $p = n/(n-1)$, $q = 2$, we have

$$\|f^a\|_{L^{2p}(\underline{v})} \leq \frac{4}{c_M}\|\nabla f^a\|_{L^2(\underline{v})} + \|f^a\|_{L^2(\underline{v})}.$$

From Lemma 6.6, we have

$$\begin{aligned} \|f\|^a_{L^{2ap}(\underline{v})} = \|f^a\|_{L^{2p}(\underline{v})} &\leq \frac{4}{c_M}\|\nabla f^a\|_{L^2(\underline{v})} + \|f^a\|_{L^2(\underline{v})} \\ &\leq \frac{4}{c_M}\frac{a\lambda}{\sqrt{2a-1}}\|f\|^a_{L^{2a}(\underline{v})} + \|f^a\|_{L^2(\underline{v})} \\ &= \left(1 + \frac{4a}{\sqrt{2a-1}}\frac{\lambda}{c_M}\right)\|f\|^a_{L^{2a}(\underline{v})}. \end{aligned}$$

Note that this inequality also holds for $a = 1$ by the dominated convergence theorem. It follows that for all $i \in \{0\} \cup \mathbb{N}$

$$\|f\|_{L^{2p^{i+1}}(\underline{v})} \leq \left(1 + \frac{4p^i}{\sqrt{2p^i - 1}}\frac{\lambda}{c_M}\right)^{p^{-i}} \|f\|_{L^{2p^i}(\underline{v})}$$

and

$$\|f\|_{L^{2p^{i+1}}(\underline{v})} \leq \prod_{j=0}^{i} \left(1 + \frac{4p^j}{\sqrt{2p^j - 1}}\frac{\lambda}{c_M}\right)^{p^{-j}} \|f\|_{L^2(\underline{v})}.$$

Thus,

$$\|f\|_{L^\infty(\underline{v})} = \lim_{i\to\infty} \|f\|_{L^{2p^{i+1}}(\underline{v})} \leq \prod_{i=0}^{\infty} \left(1 + \frac{4p^i}{\sqrt{2p^i - 1}} \frac{\lambda}{c_M}\right)^{p^{-i}} \|f\|_{L^2(\underline{v})} = L_n\left(\frac{\lambda}{c_M}\right) \|f\|_{L^2(\underline{v})}.$$

□

7 Proof of Proposition 1.1

First we note that by Proposition 5.11, Proposition 5.13, and Proposition 6.4, we have the following corollary:

Corollary 7.1. *Let M be a compact connected n-dimensional Riemannian manifold such that*

$$\mathrm{Ric}_M \geq -kg_M \ (k > 0),$$

$$\mathrm{diam}(M) \leq D.$$

Put

$$\tilde{G}_{n,k,D} := \left[\int_0^{\mathrm{diam}(M)/2} c_{-k}(t)^{n-1}\, dt \right]^{-1},$$

$$G_{n,k,D} := \left[\int_0^{\mathrm{diam}(M)} \left(\frac{c_{-k}(t)}{\tilde{G}_{n,k,D}} + \frac{s_{-k}(t)}{n} \right)^{n-1} dt \right]^{-\frac{1}{n}},$$

and define the the continuous function $L_{n,k,D} : [0,\infty) \to \mathbb{R}$ as

$$L_{n,k,D}(t) := L_n \left(\frac{t^2}{G_{n,k,D}} \right),$$

where L_n is the function defined in Proposition 6.4. The function $L_{n,k,d}$ is strictly increasing and continuous, and satisfies that $L_{n,k,d}(0) = 1$. Then, for any nonnegative continuous function f on M satisfying the conditions (i), (ii), (iii) in Proposition 6.4, the following inequality holds:

$$\|f\|_{L^\infty(\underline{v})} \leq L_{n,k,D}(\lambda^2) \|f\|_{L^2(\underline{v})}.$$

By Corollary 7.1, we have the following proposition:

Proposition 7.2. *If a compact connected n-dimensional Riemannian manifold M satisfies*

$$-kg_M \leq \mathrm{Ric}_M \leq \varepsilon g_M,$$

$$\mathrm{diam}(M) \leq D,$$

then, for a Killing vector field X on M, the following inequality holds.

$$\|X\|_{L^\infty(\underline{v})} \leq L_{n,k,D}(\varepsilon) \|X\|_{L^2(\underline{v})}$$

Proof. We apply Proposition 6.4 for $f = |X|$. By the Kato's inequality (Proposition 3.7)

$$|\nabla |X|| \leq |\nabla X|$$

and the equality

$$\frac{1}{2}\Delta|X|^2 = |X|\Delta|X| - |\nabla|X||^2,$$

we see that $\nabla|X|$ and $|X|\Delta|X|$ are bounded on $M_+ = \{p \in M \mid |X| > 0\}$. Combining with the Bochner formula (Proposition 3.6)

$$\frac{1}{2}\Delta|X|^2 = -|\nabla X|^2 + \mathrm{Ric}(X,\, X),$$

we have

$$|X|\Delta|X| \leq \varepsilon|X|^2$$

and

$$\Delta|X| \leq \varepsilon|X|.$$

Thus, by Corollary 7.1, the conclusion follows. □

The following proposition is essentially given by Li [12]. Since followings are slightly different to the corresponding statements in [12], we give a proof for the sake of completeness.

Proposition 7.3. *Let E be any Riemannian vector bundle over M of the rank n. Let $\Gamma(E)$ be a subspace of the space $\Gamma(E)$ of all sections of E. Assume that there esists a constant $a > 0$ such that for any $\omega \in \Gamma$ the inequality*

$$\|\omega\|_{L^\infty(\underline{v})} \leq a\|\omega\|_{L^2(\underline{v})}$$

holds. Then, the following hold:

$$\begin{aligned}\dim\Gamma &\leq a^2 \max_{p\in M} \dim\{\omega(p) \in E_p \,|\, \omega \in \Gamma\} \\ &\leq a^2 n,\end{aligned}$$

where E_p is the fibre of E at $p \in M$.

Proof. Let Γ' be a finite dimensional subspace of Γ. Take an $L^2(\underline{v})$-orthonormal basis $\{\omega_i\}_{i=1}^m$ of Γ' and put

$$F(p) = \sum_{i=1}^m |\omega_i(p)|^2.$$

Note that F can be independent to the choice of an $L^2(\underline{v})$-orthonormal basis. Then we have

$$\dim\Gamma' = \sum_{i=1}^m \|\omega_i\|^2_{L^2(\underline{v})} = \int_M \sum_{i=1}^m |\omega_i(p)|^2\, d\underline{v}(p) = \int_M F(p)\, d\underline{v}(p).$$

For each $p \in M$, take the evaluation map $\Phi_p : \Gamma' \to E_p$ with $\Phi_p(\omega) = \omega(p)$ and take an $L^2(\underline{v})$-orthonormal basis $\{\omega_i\}_{i=1}^m$ such that the vectors $\Phi(\omega_i) = \omega_i(p)$, $i = 1, \ldots, k$ form the basis of the orthogonal complement $(\ker \Phi_p)^\perp$ of the kernel of Φ_p. Then, we have

$$F(p) = \sum_{i=1}^{k} |\omega_i(p)|^2 \leq a^2\, k \leq a^2\, \beta$$

where $\beta = \max_{p \in M} \dim\{\omega(p) \in E_p \,|\, w \in \Gamma\}$. Thus, we have

$$\dim \Gamma' = \int_M F(p)\, d\underline{v}(p) \leq a^2\, \beta \leq a^2 n.$$

By the choice of Γ', we get the conclusion.

$\square$

We apply this proposition for the case that $E = TM$, the tangent bundle of M, and Γ is the space of Killing vector fields on M. Then, combining with Proposition 7.2, it implies Proposition 1.1.

8 Proof of Theorem 1.4

We start from the following lemma which follows from Proposition 7.3 here and Theorem 2.2 in [16]. Since the proof given in [16] is somewhat sketchy, we give a proof here.

Lemma 8.1. *For constants k, $D > 0$, there exists a constant $\varepsilon = \varepsilon(n, k, D) > 0$ such that if M satisfies the assumption in Theorem 1.4, then M is a Riemannian homogeneous space.*

Proof. We also apply Proposition 7.3 for the case mentioned in the sentence right after the proof of Proposition 7.3. Take $p \in M$ satisfying

$$\dim\{X_p \in T_pM \mid X \text{ is a Killing vector field.}\} = n$$

Let $B \subset M$ be a set whose element q is an image of p by an isometry of M. We shall prove that B is open. Take Killing vector fields $X_i, i = 1, 2, \dots n$ such whose vectors $X_{i,p}$ at p form a basis of T_pM. Let φ_t^X denote the flow generated by a vector field X. We define a map $F : \mathbb{R}^n \to M$ by

$$F(t_1, \dots, t_n) = \varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_n}^{X_n}(p).$$

Then, the rank of the differential dF at the origin of F is n. By the inverse function theorem, we see that F is a local diffeomorphism near the origin ,and thus p is an interior point of B. For $q \in B$, take an isometry φ such that $\varphi(p) = q$. Since φ is homeomorphism, we see that q is also an interior point of B. Thus B is open in M.

To prove the closedness of B, take a point q in the closure of B. For a sequence $q_i \in B$ converging q, take isometries φ_i with $\varphi_i(p) = q_i$, Since the isometry group $\mathrm{Isom}(M)$ is a compact Lie group, there is a subsequence of $\{\varphi_i\}$ converge to some isometry φ. Note that $\varphi(p) = q$, and thus $q \in B$, hence the conclusion follows. □

Lemma 8.2. *If M satisfies*

$$-kg_M \leq \mathrm{Ric}_M \leq \varepsilon g_M,$$
$$\mathrm{diam}(M) \leq D,$$
$$\dim \mathrm{Isom}(M) = n,$$

then for any Killing vector field X on M, we have

$$|X|^2 \geq (n - (n-1)L_{n,k,D}^2(\varepsilon))\|X\|_{L^2(\underline{v})}^2.$$

Proof. Take an $L^2(\underline{v})$-orthonomal basis $\{X_i\}$ of the space of Killing vector fields on M. Since the function $F(p) = \sum_i |X_i|^2$ does not depend on the choice of an $L^2(\underline{v})$-orthonomal basis and for any isometry φ, $\{d\varphi(X_i)\}$ is also $L^2(\underline{v})$-orthonomal basis, we have

$$F(\varphi(p)) = \sum_i |d\varphi(X_i)(\varphi(p))|^2 = \sum_i |d\varphi(X_i(p))|^2 = \sum_i |X_i(p)|^2 = F(p).$$

Thus, F is a constant function. Since $\int_M F\, d\underline{v} = n$, we know that $F \equiv n$. Thus,

$$|X_1|^2 = n - \sum_{i \neq 1} |X_i|^2 \geq n - (n-1)L^2_{n,k,D}(\varepsilon).$$

We get the conclusion by putting $X_1 = X/\|X\|_{L^2(\underline{v})}$. □

Lemma 8.3. *There exists a finite covering $\pi : \hat{M} \to M$ such that $\hat{M}$ is isometric to the identity component G of the isometry group* $\mathrm{Isom}(M)$ *of M, which is equipped with a certain left invariant Riemannian metric.*

Proof. From Lemma 8.1, M can be written as

$$M = G/K,$$

where K is the isotropy subgroup of G at p.

We shall prove that K is a finite group. Note that we can identify the Lie algebra $\mathfrak{g}$ of G and the space of the Killing vector fields on M. Since $\mathrm{Isom}(M)$ is a compact Lie group, it suffices to show that the Lie algebra $\mathfrak{k}$ of K, which corresponds to the space of Killing vector field X with $X_p = 0$, is trivial. By Lemma 8.2, we see that the evaluation map $\Phi_p : \mathfrak{g} \to T_pM$ defined by $\Phi_p(X) = X(p)$ is a linear isomorphism for sufficiently small ε, and thus $\dim \mathfrak{k} = \dim \ker(\Phi_p) = 0$.

A left invariant metric g_G on G is given as follows; Take a point $p \in M$. We define a map $\pi_p : G \to M$ by $\pi_p(\psi) = \psi(p)$ and g_G on G by induced metric $g_G = {\pi_p}^* g_M$ from the Riemannian metric g_M on M. We shall show that g_G is left invariant. Let L_φ denote the left translation of φ on G. Then we have

$$\pi_p \circ L_\varphi(\psi) = \pi_p(\varphi \circ \psi) = \varphi \circ \psi(p) = \varphi \circ \pi_p(\psi),$$

namely, $\pi_p \circ L_\varphi = \varphi \circ \pi_p$. Then, we get the conclusion by

$${L_\varphi}^* g_G = {L_\varphi}^* {\pi_p}^* g_M = (\pi_p \circ L_\varphi)^* g_M = (\varphi \circ \pi_p)^* g_M = {\pi_p}^* \varphi^* g_M = {\pi_p}^* g_M = g_G$$

Here we have used $\varphi^* g_M = g_M$ which is implied by the fact that φ is isometry. □

LEMMA 8.4. *M is an almost flat manifold. Namely, for any $\delta > 0$, there exists $\varepsilon = \varepsilon(k, D, \delta) > 0$ such that if*

$$-kg_M \leq \mathrm{Ric} \leq \varepsilon g_M,$$
$$\mathrm{diam}(M) \leq D,$$
$$\dim \mathrm{Isom}(M) = n,$$

then

$$|K_M D^2| \leq \delta$$

where K_M is the sectional curvaure of M.

PROOF. First we give a pointwise estimate of the Lie bracket $[X, Y]$ of Killing vector fields X, Y. Integrating the Bochner formula

$$\frac{1}{2}\Delta|X|^2 = \mathrm{Ric}(X, X) - |\nabla X|^2,$$

we have

$$\|\nabla X\|^2_{L^2(\underline{v})} = \int_M \mathrm{Ric}(X, X)\, d\underline{v} \leq \varepsilon\|X\|^2_{L^2(\underline{v})}.$$

Since $[X, Y]$ is also Killing, we can apply Proposition 7.2. Then, we have, for $p \in M$, by Lemma 8.2,

$$\begin{aligned}
|[X,Y]_p| &\leq \|[X,Y]\|_{L^\infty(\underline{v})} \\
&\leq L_{n,k,D}(\varepsilon)\|[X,Y]\|_{L^2(\underline{v})} \\
&= L_{n,k,D}(\varepsilon)\,\|\nabla_X Y - \nabla_Y X\|_{L^2(\underline{v})} \\
&\leq L_{n,k,D}(\varepsilon)\left(\|\nabla Y\|_{L^2(\underline{v})}\|X\|_{L^\infty(\underline{v})} + \|\nabla X\|_{L^2(\underline{v})}\|Y\|_{L^\infty(\underline{v})}\right) \\
&\leq L^2_{n,k,D}(\varepsilon)\left(\|\nabla Y\|_{L^2(\underline{v})}\|X\|_{L^2(\underline{v})} + \|\nabla X\|_{L^2(\underline{v})}\|Y\|_{L^2(\underline{v})}\right) \\
&= 2\sqrt{\varepsilon}L^2_{n,k,D}(\varepsilon)\|X\|_{L^2(\underline{v})}\|Y\|_{L^2(\underline{v})} \\
&\leq \frac{2\sqrt{\varepsilon}L^2_{n,k,D}(\varepsilon)}{n-(n-1)L^2_{n,k,D}(\varepsilon)}|X_p||Y_p| \\
&\leq 4\sqrt{\varepsilon}|X_p||Y_p|.
\end{aligned} \tag{12}$$

for sufficiently small $\varepsilon > 0$.

Next we estimate K_M. For this purpose, it suffices to estimate the sectional curvature K_G of $G = \hat{M}$ by Lemma 8.3. By Lemma 3.11, Killing vector fields on M correspond to left invariant vector fields on G. By Proposition 4.3, for left invariant vector fileds $\tilde{X}, \tilde{Y}$ on G

$$\begin{aligned}
\langle R(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X}\rangle &= |(\mathrm{ad}_{\tilde{X}})^*(\tilde{Y}) + (\mathrm{ad}_{\tilde{Y}})^*(\tilde{X})|^2 - \langle(\mathrm{ad}_{\tilde{X}})^*(\tilde{X}), (\mathrm{ad}_{\tilde{Y}})^*(\tilde{Y})\rangle \\
&\quad - \frac{3}{4}|[\tilde{X},\tilde{Y}]|^2 - \frac{1}{2}\langle[[\tilde{X},\tilde{Y}],\tilde{Y}],\tilde{X}\rangle - \frac{1}{2}\langle[[\tilde{Y},\tilde{X}],\tilde{X}],\tilde{Y}\rangle,
\end{aligned} \tag{13}$$

where R is the Riemannian curvature tensor and $(\mathrm{ad}_{\tilde{X}})^*$ is the (formal) adjoint of the linear transformation $\mathrm{ad}_{\tilde{X}}$ defined by $\mathrm{ad}_{\tilde{X}}(\tilde{Y}) = [\tilde{X}, \tilde{Y}]$ with respect to the Riemannian inner product $\langle \cdot, \cdot \rangle$ on G.

For $p \in M$, we induce left invariant metric on G by π_p. Take Killing vector field $X_i, i = 1, 2, \dots n$ such that the vectors $X_{i,p}$ of X_i at p forms an orthonornal basis in T_pM and put $\tilde{X}_i = T^{-1}(X_i)$, then $\tilde{X}_i$ forms an orthonomal basis on each tangent space of G. Then, we have an estimate of the numerator of the sectional curvature

$$K_G(\tilde{X}_i, \tilde{X}_j) = \frac{\langle R(\tilde{X}_i, \tilde{X}_j)\tilde{X}_j, \tilde{X}_i \rangle}{|\tilde{X}_i|^2 |\tilde{X}_j|^2 - \langle \tilde{X}_i, \tilde{X}_j \rangle^2}.$$

by (12) and (13). The denominator is equal to 1. We have an estimate of K_G which is independent to the choice of orthonormal basis, and thus K_M at p. Since M is homogeneous, we have uniform estimate of K_M. Hence the conclusion follows follows. $\square$

We finally give a proof of Theorem 1.4. By the structure theorem of compact Lie group (Theorem 2.94), the universal covering of $G = \hat{M}$ can be split as a product $\mathbb{R}^k \times G_0$ of abelian group $\mathbb{R}^k$ and a simply connected semi-simple compact Lie group G_0. By Lemma 8.4 and Gromov's almost flat theorem (Theorem 2.83), $\mathbb{R}^k \times G_0$ is diffeomorphic to $\mathbb{R}^n$. Thus we see that G_0 is trivial and thus G is an abelian group by the structure theorem. Then by the formula (13), we see that M and $G = \hat{M}$ are flat manifolds. Bochner's classical theorem mentioned in the introduction implies the conclusion.

Remark 8.5. The referee of [11] pointed out that the following short-cut of the proof is possible. By Proposition 4.3 (i), Proposition 3.11 and the estimate (12), we have

$$|\nabla_U V| \le 12\sqrt{\varepsilon}|U||V|$$

for any left invariant vector fields U, V on G. This implies the following estimate of the Maurer-Cartan form ω of G:

$$|d\omega| \le \delta(n, k, D, \varepsilon),$$

where $\delta(n, k, D, \varepsilon) \to 0$ if $\varepsilon \to 0$. Then the Zassenhaus and Kazhdan-Margulis lemma (Theorem 1.4. in [8]) implies G is nilpotent. Since G is compact Lie group, we see that G is abelian. Then, combining with (2), we conclude that M is a flat torus by the Bochner's classical theorem.

References

[1] P. Bérard, From vanishing theorems and estimating theorems: The Bochner technique revisited, Bull. Amer. Math. Soc. 19(1988), 371–406.

[2] M. Berger, Lecture notes at Osaka University "Ricci curvature and Topology", noted by T. Tsujishita, (in Japanese), 1982.

[3] T. Bröcker and T. tom Dieck, Representations of compact Lie groups. Graduate Texts in Mathematics, 98. Springer, New York, 1985.

[4] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I, J. Differential Geom. 46, no. 3, 1997, 406–480.

[5] J. Cheeger and D. G. Ebin, Comparison Theorems in Riemannian Geometry, AMS Chelsea Publishing, 1975.

[6] T. H. Colding, Ricci curvature and volume convergence, Ann. of Math. (2) 145 (1997), no. 3, 477–501.

[7] S. Gallot, A Sobolev inequality and some geometric applications, Spectra of Riemannian Manifolds, Kaigai, Tokyo, 1983, 45–55.

[8] P. Ghanaat, Almost Lie groups of type $\mathbb{R}^n$, J. Reine Angew. Math.401 (1989), 6081.

[9] M. Gromov, Almost flat manifolds, J. Differential Geom. 13 (1978), no. 2, 231–241.

[10] M. Gromov, Structures métriques pour les variétés riemanniennes, Cedric, Paris 1982.

[11] A. Katsuda and T. Nakamura, A rigidity theorem for Killing vector fields on compact manifolds with almost nonpositive Ricci curvature, to apper in Proceeding of the AMS.

[12] P. Li, On the Sobolev constant and the p-spectrum of a compact riemannian manifold, Ann. scient. Éc. Norm. Sup. 13, 1980, 451–469.

[13] B. O'Neill, Semi-Riemannian Geometry, with Application to Relativity, Academic Press, New York, 1983.

[14] P. Petersen, Riemannian Geometry, Graduate Texts in Mathematics, 171. Springer, New York, second edition 2006.

[15] T. Sakai, Riemannian geometry, Translations of Mathematical Monographs, 149, American Mathematical Society, 1996.

[16] A. K. Stehney and R. S. Millman, Riemannian manifolds with many Killing vector fields. Fund. Math. 105 (1979/80), no. 3, 241–247.