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Abstract

We investigate the massive Schwinger model quantized on the light cone with great care on the bosonic zero modes by putting the system in a finite (light-cone) spatial box. After Dirac quantization for the constrained system, the zero mode of A^+ survives as a dynamical degree of freedom. We show that the physical state condition relates the fermion Fock states to the zero mode of the gauge field. We construct the physical vacuum by imposing the physical state condition carefully. The periodicity of physics in θ can be understood as that of the Bloch spectrum for the periodic "effective" potential. We calculate the θ-dependence of the vacuum energy density quantitatively and find the signal of the phase transition at $\theta = \pi$.

1 Introduction

Recently there has been growing interest in the light-front Tamm-Dancoff (LFTD) approximation [1] as a new promising numerical approach to nonperturbative problems [1-10]. It has been successfully applied to two-dimensional models [4-10], although the application to QCD_{1+3} is still beyond our control despite of much recent effort [2]. It is important to note that this new approach not only reproduces known results correctly, but also brings us new results [6-9] which have never been obtained by other methods. (See also Refs. [11] for some of such "new" results in the discretized light-cone quantization (DLCQ) approach [12].)

The LFTD approximation is the Tamm-Dancoff approximation (a truncation of the infinite dimensional Fock space by limiting the number of constituents [13]) applied to field theory quantized on the light cone. The light-cone quantization is essential for the validity of the Tamm-Dancoff approximation: Tamm-Dancoff approximation is a kind of the valence quark approximation and on the light cone, pair creations/annihilations are kinematically suppressed [3], then it is plausible that the sea quark/gluon contributions are small in this framework. Typically, the lightest particles are expected to be in the valence states. It is generally true in the models so far investigated.

It has several attractive points compared to the lattice formulation: (i) It is based on the diagonalization of the (light-cone) Hamiltonian. It is therefore intuitively appealing, and gives eigenvalues and eigenvectors simultaneously. (ii) One does not need to make "Wick rotations" at all. Thus one may include topological terms (such as a θ-term) which usually cause trouble in the lattice formulation.

There are however some problems in light-front field theory [3]: (i) The vacuum problem. (It is also known as the "zero-mode problem.") In the light-cone quantization, it is not clear how the complex structure of the vacuum emerges, which is supposed to be responsible for spontaneous symmetry breaking, the vacuum angle, etc. It is widely

believed that the zero modes of the field variables play an important role. (ii) The renormalization problem. Even the power-counting rules are different from the usual covariant ones and there are infinitely many relevant and marginal operators [14]. Although there is an attempt to overcome this difficulty, we still do not know how to renormalize such theories.

A nice way to attack these difficult problems is to separate them each other. In this paper we study the massive Schwinger model (QED_{1+1}) in the LFTD approximation, and investigate the vacuum angle, θ. Since it is a two-dimensional model, we can avoid the renormalization problem and concentrate on the zero-mode problem.

The massive Schwinger model has been studied by many authors because it shares several important features with QCD_{1+3} such as quark confinement, anomalous $U(1)_A$ breaking as well as θ-vacuum [4, 6, 20, 21]. In his seminal paper [21], Coleman showed that the vacuum angle θ can be regarded as an external constant electric field. One of his important results is that the periodicity of physics in θ is a consequence of dynamical structure of the vacuum. Namely, it comes out from the fact that a pair creation of a fermion and an anti-fermion from the vacuum is energetically favorable in a background electric field stronger than a certain critical value. He also obtains that at $\theta = \pi$ the theory undergoes a phase transition as it passes from strong to weak coupling.

How can this dynamical feature of θ be understood in the light-cone quantization with a simple vacuum? The main purpose of this thesis is to explain how the dynamical zero modes of the gauge fields are related to the vacuum structure (the vacuum angle). In order to explicitly extract the zero modes, we first put the system into a finite light-cone spatial box ($x^- \in [-L, L]$) and impose the periodic boundary condition [15], keeping in mind that we should eventually take the limit $L \to \infty$. Even after fixing a gauge, the zero mode of the longitudinal component of the gauge field ($\overset{o}{A}{}^+$) remains to be dynamical while the other gauge components ($\tilde{A}^+, \overset{\bullet}{A}{}^-, \tilde{A}^-$) do not. We did not include the fermionic zero modes tentatively, although they might be important in other respects.

The vacuum angle, θ has not been discussed much in the light-cone context. Although there are several papers on the θ-term (vacuum) in the Schwinger model, the *massive* Schwinger model has been rarely studied. Our approach may appear similar to that of Heinzl, Krusche and Werner [17], who discussed the zero modes of the gauge field in the massless Schwinger model and how the θ-vacuum arises. There are, however, critical differences; (i) Their treatment of the regularized current is not adequate. In their paper the chiral anomaly is *not* derived through the point-splitting regularization but as a consequence of the *classical* equation of motion. (ii) They impose a regularized charge density as an additional constraint, which leads to a *second-class* Gauss law (the chargeless condition). Of course these are different form the usual treatment of the regularized currents and the Gauss law, and they lost the dynamical zero mode A^+ because of this constraint. In this thesis, we take a great care of the relationship between the regularization of the currents and the Gauss law. We find that the regularization does not affect the structure of constraints. We end up with the first-class Gauss law and the usual chiral anomaly which arises from a gauge-invariant regularization procedure.

There are several papers on the massive Schwinger model on the light cone. Bergknoff [4] first applied the LFTD approximation and Mo and Perry [5] refined his calculations by using the method of basis functions. We have achieved six-body LFTD calculations in order to investigate two-meson and three-meson bound states [7, 8]. Eller, Pauli and Brodsky [16] considered the discretized light-cone quantization (DLCQ) of the massive Schwinger model. This thesis is based on these. These papers include the essence of the LFTD approximation although they did not treat the zero mode at all. We refer the readers to them.

In Sec. 2, we review the essence of the light-front field theory and the LFTD approximation, and show that the quantization on the light cone is essential for the validity of the Tamm-Dancoff approximation. We summarize our results of the 6-body LFTD approximation applied to the massive Schwinger model.

In Sec. 3, we first examine the constraints and eliminate dependent degrees of freedom, paying attention to bosonic zero modes. We find that the zero mode of A_- and its canonically conjugate momentum remain to be dynamical. To quantize the theory, we need to regularize operators to make them well-defined. We show that the regularization does not affect the structure of the constraints.

In Sec. 4, we investigate the ground state, namely the vacuum structure of the massive Schwinger model. The ground states are found to be infinitely degenerate. We define the vacuum state by making coherent superpositions of these infinite number of states. We find out the physical vacuum (θ-vacuum) by imposing the physical state condition. Interestingly, the physical state condition relates the fermion Fock states to the zero mode of the gauge field.

Sec. 5 is devoted to the conclusion and discussions. In Appendix A, we collect the notations in this thesis and the explicit equations of the 6-body LFTD approximation for the massive Schwinger model. Appendix B contains a proof of the "iterative property" of Dirac brackets [30], which may be of some use when treating a system with many constraints. We give a detailed discussion on current regularization and chiral anomaly in Appendix C.

2 Light-front Tamm-Dancoff approximation

In this section we discuss the elementary idea of the light front field theory and the merit of the Tamm-Dancoff approximation on the light cone. For simplicity we work on the two dimensional case. To higher dimensional cases also this idea is basically applicable with some trivial extensions.

The light-cone space-time is defined as a linear combination of the usual space-time:

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \tag{2.1}$$

where x^0 and x^1 are the (usual) time and space, respectively. We call x^+ the light-cone time and x^- the light-cone space. The element of the metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\mu, \nu = +, -). \tag{2.2}$$

Usually in the equal-time quantization the momentum operator, P^1, is the generator of the translation in the spatial direction and the dynamics is described by the time-development which is generated by the Hamiltonian, P^0. In the light-cone flame, the translation in the light-cone space is generated by the light-cone momentum, P^+, and the dynamics is defined as the light-cone time development which is generated by the light-cone Hamiltonian, P^-.

The most important feature on the light-front field theory is that the momenta of particles are positive. It is easy to derive this feature for the on-shell particles: $p^2 = 2p^+p^- = m^2$. The energy p^- is always positive, and so is the momentum p^+. For the off-shell particles it becomes almost the same situation as the on-shell case when one first integrates the p^- in the propagator [31]. Because every momentum of the particles is positive, the pair creation(annihilation) is forbidden due to the momentum conservation. In the same manner there exist no such interactions as four fermions emerging(vanishing) from(into) the vacuum in the four point interactions such as the current-current Coulomb interaction or the four-Fermi interaction. In the light-cone quantization scheme therefore

the vacuum decouples from the particle states and it is usually the simple Fock vacuum. For example, see Fig. 1.

In the following we solve the Einstein-Schrödinger equation:

$$2P^- P^+ |\psi(P)\rangle = M^2 |\psi(P)\rangle, \tag{2.3}$$

where M^2 is the mass-squared and $|\psi(P)\rangle$ is the particle state with the total momentum P. We consider the "mesonic" states in the following for simplicity. This state $|\psi(P)\rangle$ is decomposed into the states with j fermions and j antifermions, $|2j(P)\rangle$,

$$|\psi(P)\rangle = |2(P)\rangle + |4(P)\rangle + |6(P)\rangle + \cdots \tag{2.4}$$

where the fermion number must be same as the antifermion number for the "mesonic" state. These states are constructed by operating the fermion and antifermion creation operator on the Fock vacuum. (See Appendix A for the explicit expressions for these states of the massive Schwinger model.) If we investigate a model which contains the dynamical gauge fields in such a case as 4 dimensional QCD, we must include the number of the gauge field in this decomposition.

This Einstein-Schrödinger equation is rewritten in the matrix form:

$$\begin{pmatrix} H_{22} & H_{24} & H_{26} & \cdots \\ H_{42} & H_{44} & H_{46} & \cdots \\ H_{62} & H_{64} & H_{66} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_4 \\ \psi_6 \\ \vdots \end{pmatrix} = M^2 \begin{pmatrix} \psi_2 \\ \psi_4 \\ \psi_6 \\ \vdots \end{pmatrix}, \tag{2.5}$$

where H_{ii} consists of the kinetic energy and interactions which do not change the particle number, and H_{ij} is an interaction which changes i particles into j particles. ψ_2, ψ_4, ψ_6, $\cdots$ are the wave functions of the 2-body, 4-body, 6-body, $\cdots$ fermionic states, respectively. On the light cone, there exist no such interactions as to change the particle number by four (or more) in the four point interactions. Therefore we have the matrix element $H_{ij} = 0$ for $j \geq i+4$ and $j \leq i-4$ while we would have $H_{ij} \neq 0$ for $j = i+4$ and $j = i-4$ in the equal-time quantization. The matrix of the Hamiltonian on the light cone is slimmer than that

of the equal-time quantization. To solve this equation we truncate particle numbers, which is called the Tamm-Dancoff approximation. (In Appendix A we truncate up to including 6-body states.) After diagonalizing the Hamiltonian we simultaneously get masses and wave functions as eigenvalues and eigenfunctions, respectively. We emphasize that this approximation is not based on the perturbation. Namely this approach is nonperturbative one. By direct diagonalization of the matrix for any value of the coupling constants, we can get the non-perturbative results.

After solving Eqn. (2.5) we must make sure that this approximation is plausible or not for the states which we look at. In the perturbative sense it is plausible that one "mesonic" state does not have 4 fermionic component or higher because the off-diagonal elements of the matrix is proportional to the coupling constant. If the coupling constant is weak enough that the perturbation makes sense, the "mesonic" state is not affected by the higher Fock states. For this state it is justified to truncate the Fock space only up to including 2-body state. For the strong coupling constant, it is nontrivial whether this approximation is good or not. We can make sure it by looking at obtained wave functions, ψ_2, ψ_4, ψ_6, $\cdots$. If ψ_4 or higher are negligibly small compared with ψ_2 for the "mesonic" state, then the approximation is plausible for this state.

In the previous works on the massive Schwinger model [7], we obtained the following results by using the LFTD approximation up to including six-body states.

(1) The masses of the lowest states do not change if we include four- and six-body states.

(2) In particular, the state which can be regarded as a bound state of two mesons has a negligible six-body component.

(3) We find a candidate for the bound state of three mesons.

(4) The wave function of the relative motion of the two-meson bound state describes the bound state well. We can have a picture that in the strong coupling region it is loosely bounded, while in the weak coupling region it is tightly bounded, compared with the size of the meson.

In Fig. 2 we show the mass spectrum in the strong coupling region where the fermion mass is given in the unit of the coupling constant $e/\sqrt{\pi}$.

We emphasize that the information about states (wave functions) is very useful. It was used for identifying the three-meson bound state. The three-meson bound state, if it exists, must be in the continuum spectrum unless it is lighter than two mesons. It is therefore apparently difficult to identify the state among the continuum. We could however find a candidate among several states by looking at the wave functions. The point is that below the three-meson threshold, six-body components should be very small except for three-meson bound states. As another example, we introduce the wave function of the relative motion in the two-meson bound state and try to describe the bound state in terms of the wave function. Although the concept of "relative motion" of a relativistic bound state is somewhat awkward, we however find that the two-meson bound state is well described in terms of the wave function of the relative motion, in the sense that a smaller set of basis functions motivated by the concept of the relative motion gives a good approximation. It gives us a qualitative picture of the bound state. The readers who are interested in these matters, see Ref. [7].

3 Light-cone quantization in a box

3.1 Constraints of QED_2 on the light cone

In this section we analyze the structure of constraints of the massive Schwinger model, including zero modes, and derive the Hamiltonian in a canonical way. In order to explicitly separate the zero modes from the non-zero modes of the bosonic variables, we put the system into a finite light-cone spatial box ($x^- \in [-L, L]$) with the periodic boundary condition. For the fermionic variables we impose the anti-periodic boundary condition and disregard their zero-modes completely. We will discuss possible consequences of the inclusion of the fermionic zero modes in Section 5.

The Lagrangian density of the massive Schwinger model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi \tag{3.1}$$

$$= \frac{1}{2}(\partial_+ \overset{o}{A}_-)^2 + \frac{e\theta}{2\pi}\partial_+ \overset{o}{A}_- + \frac{1}{2}(\partial_+\tilde{A}_- - \partial_-\tilde{A}_+)^2 + \sqrt{2}(\psi_R^\dagger i\partial_+\psi_R + \psi_L^\dagger i\partial_-\psi_L)$$
$$-m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R) - e(\sqrt{2}\psi_R^\dagger\psi_R A_+ + \sqrt{2}\psi_L^\dagger\psi_L A_-), \tag{3.2}$$

where $\overset{o}{A}_\pm$ stands for the zero mode of $A_\pm$ and $\tilde{A}_\pm$ the non-zero mode. We will use similar notations hereafter. we refer the readers to Appendix A for other notations. The conjugate momenta are obtained as follows:

$$\overset{o}{\pi}^+ \equiv 2L\,\overset{o}{E}{}^+ \approx 0, \quad \tilde{\pi}^+ \equiv \tilde{E}^+ \approx 0, \tag{3.3}$$

$$\overset{o}{\pi}^- \equiv 2L\,\overset{o}{E}{}^- = 2L\,(\partial_+ \overset{o}{A}_-), \quad \tilde{\pi}^- \equiv \tilde{E}^- = \partial_+\tilde{A}_- - \partial_-\tilde{A}_+ \tag{3.4}$$

$$\pi_R^\dagger = i\sqrt{2}\psi_R^\dagger, \quad \pi_R \approx 0, \quad \pi_L^\dagger \approx 0, \quad \pi_L \approx 0. \tag{3.5}$$

Note that because the zero modes do not depend on x^-, it is useful to extract the factor L from the conjugate momenta, and that, for the fermionic variables, the dagger ed/undaggered momenta are conjugate to undaggered/daggered variables, respectively, e.g., $\pi_R^\dagger \equiv \delta L/\delta(\partial_+\psi_R)$.

From these we see that the primary constraints are as follows:

$$\theta_1 \equiv \overset{o}{E}{}^+, \quad \theta_2 \equiv \tilde{E}^+, \quad \theta_3 \equiv \pi_R^\dagger - i\sqrt{2}\psi_R^\dagger, \quad \theta_4 \equiv \pi_R, \quad \theta_5 \equiv \pi_L^\dagger, \quad \theta_6 \equiv \pi_L. \tag{3.6}$$

The total Hamiltonian[30] becomes

$$\begin{aligned} H = \int_{-L}^{L} dx^- [&\frac{1}{2}(\overset{o}{E}{}^-)^2 + \frac{1}{2}(\tilde{E}^-)^2 + \tilde{E}^- \partial_- \tilde{A}_+ - \sqrt{2}\psi_L^\dagger i\partial_- \psi_L \\ &+ m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R) + e(\sqrt{2}\psi_R^\dagger \psi_R A_+ + \sqrt{2}\psi_L^\dagger \psi_L A_-) + \sum_{i=1}^{6} \theta_i \lambda^i], \end{aligned} \tag{3.7}$$

where λ^i $(i = 1, \ldots, 6)$ are Lagrange multipliers. The consistency conditions for θ_3 and θ_4 only determine the Lagrange multipliers λ^4 and λ^3 respectively. The rest leads to further (secondary) constraints.

$$\varphi_1 \equiv \frac{1}{2L} \int_{-L}^{L} dx^- \sqrt{2} e \psi_R^\dagger \psi_R(x), \tag{3.8}$$

$$\varphi_2 \equiv \partial_- \tilde{E}^- - \sqrt{2} e (\psi_R^\dagger \psi_R(x))_\sim, \tag{3.9}$$

$$\varphi_5 \equiv i\partial_- \psi_L^\dagger + \frac{m}{\sqrt{2}} \psi_R^\dagger + e\psi_L^\dagger A_-, \tag{3.10}$$

$$\varphi_6 \equiv i\partial_- \psi_L - \frac{m}{\sqrt{2}} \psi_R - eA_- \psi_L. \tag{3.11}$$

The consistency conditions for these constraints do not lead to any further constraints. (The consistency conditions of φ_5 and φ_6 determine the multipliers λ_6 and λ_5 respectively, while those of φ_1 and φ_2 are satisfied automatically.) As usual we can arrange these constraints into first- or second-class ones. We find the following first-class constraints,

$$\theta_1 = \overset{o}{E}{}^+, \quad \theta_2 = \tilde{E}^+ \tag{3.12}$$

$$\varphi_1 = \frac{-ie}{2L} \int_{-L}^{L} dx^- (\pi_R^\dagger \psi_R(x) + \psi_R^\dagger \pi_R(x) + \pi_L^\dagger \psi_L(x) + \psi_L^\dagger \pi_L(x)), \tag{3.13}$$

$$\varphi_2 = \partial_- \tilde{E}^- + ie(\pi_R^\dagger \psi_R(x) + \psi_R^\dagger \pi_R(x) + \pi_L^\dagger \psi_L(x) + \psi_L^\dagger \pi_L(x))_\sim. \tag{3.14}$$

We choose the following gauge-fixing conditions,

$$\chi_1 \equiv \overset{o}{A}_+, \quad \chi_2 \equiv \tilde{A}_-, \quad \chi_3 \equiv \tilde{E}^- + \partial_- \tilde{A}_+. \tag{3.15}$$

Note that the consistency of χ_2 gives the third constraint χ_3. The consistency of χ_1 and χ_3 determine the multiplier λ_1 and λ_2 respectively. Interestingly one cannot choose $\overset{o}{A}_- \approx 0$ because it does not have non-vanishing Poisson brackets with any of the first-class constraints. We end up with a single first-class constraint φ_1, the charge. We will impose it as a physical state condition after quantization.

$$\varphi_1 |phys\rangle = 0, \tag{3.16}$$

which eliminates charged states from the physical space.

We use second-class constraints to eliminate dependent degrees of freedom. It is easy to see that the independent variables are $\overset{o}{A}_-$, $\overset{o}{E}{}^-$, ψ_R and $\psi_R^\dagger$. Non-vanishing Dirac brackets[30] for these variables are calculated as

$$\{\psi_R(x^-), \psi_R^\dagger(y^-)\}_D = \frac{-i}{\sqrt{2}}\delta(x^- - y^-), \quad \{\overset{o}{A}_-, \overset{o}{E}{}^-\}_D = \frac{1}{2L}. \tag{3.17}$$

(In order to get the Dirac brackets, one usually needs to obtain the inverse matrix of a big matrix. This routine burden reduces considerably if we use the "iterative property". For the readers' convenience the proof is included in Appendix B.) In terms of independent degrees of freedom, the Hamiltonian can be written as

$$P^- = P^-_{zero} + P^-_{fmass} + P^-_{inter}, \tag{3.18}$$

$$P^-_{zero} = L(\overset{o}{E}{}^-)^2, \tag{3.19}$$

$$P^-_{fmass} = \frac{m^2}{\sqrt{2}} \int_{-L}^{L} dx^- [\psi_R^\dagger(x^-) e^{-ie\overset{o}{A}_- x^-} \frac{1}{i\partial_-} e^{ie\overset{o}{A}_- x^-} \psi_R(x^-)], \tag{3.20}$$

$$P^-_{inter} = \frac{e^2}{2} \int_{-L}^{L} dx^- \tilde{j}^+(x^-) \left(\frac{1}{i\partial_-}\right)^2 \tilde{j}^+(x^-), \tag{3.21}$$

where the inverse of the derivative operator is understood as the principal value in the Fourier transforms[25]. Note that the dynamical zero modes ($\overset{o}{A}_-$, $\overset{o}{E}{}^-$) come into the expression in a nontrivial way. The first term P^-_{zero} is the energy of the constant electric field. The second term P^-_{fmass} contains the interaction of the zero mode of the gauge field with the fermion, and requires a special care. It is interesting to note that only the non-zero mode of the current appears in the third term P^-_{inter}.

3.2 Current and charge regularization, and subsidiary condition

In order to quantize the theory, we replace Dirac brackets with the corresponding ($(-i)$ times) equal-x^+ commutators. In addition, we need to regularize composite operators to make them well-defined. In two dimensions, one can eliminate all divergences by normal-ordering. In the following, we carefully define the current operators, Hamiltonian, and charge in a well-defined way so that the structure of constraints analyzed in the previous subsection is not altered by the regularization.

First of all, we have to define the "normal-ordering." For this purpose, we treat the gauge field $\overset{o}{A}_-$ (or, $q \equiv (L/\pi)e\overset{o}{A}_-$, which is nothing but the Chern-Simons term in one dimension) as an external field and quantize the fermionic variables in this external field.

We make the Fourier expansion of the fermionic variable ψ_R,

$$\psi_R(x) = \frac{1}{2^{1/4}\sqrt{2L}} \sum_{n=-\infty}^{\infty} a_{n+\frac{1}{2}} e^{-i\frac{\pi}{L}(n+\frac{1}{2})x^-}. \tag{3.22}$$

From the corresponding Dirac brackets, $a_{n+\frac{1}{2}}$ is assumed to satisfy the following anti-commutation relations, $\{a_{n+\frac{1}{2}}, a^\dagger_{m+\frac{1}{2}}\} = \delta_{n,m}$, $\{a_{n+\frac{1}{2}}, a_{m+\frac{1}{2}}\} = \{a^\dagger_{n+\frac{1}{2}}, a^\dagger_{m+\frac{1}{2}}\} = 0$. Using these operators, we define a set of reference states, so-called "N-vacua," in analogy of Dirac sea,

$$|0\rangle_N \equiv \prod_{n=-\infty}^{N-1} a^\dagger_{n+\frac{1}{2}} |\tilde{0}\rangle, \tag{3.23}$$

where $|\tilde{0}\rangle$ is the 'empty' state, i.e., $a_{n+\frac{1}{2}}|0\rangle = 0$ for any n. At this moment, N is an arbitrary integer. (The use of the "N-vacua" is rather standard in the Schwinger model in the equal-time quantization. See Refs. [22, 23].) This state is satisfied with the normalization condition: ${}_N\langle 0|0\rangle_{N'} = \delta_{N,N'}$. It is easy to derive the vacuum expectation value of the fermionic part of the Hamiltonian, ${}_N\langle 0|P^-_{fermion}|0\rangle_{N'} = 0$ if $N \neq N'$, where $P^-_{fermion} \equiv P^-_{fmass} + P^-_{current}$. (The use of the

We regularize the current by point-splitting. We define the current operator $j^\mu(x)$ in

a gauge invariant way,

$$j^\mu = \lim_{\epsilon\to 0}\frac{1}{2}[\bar{\psi}(x+\epsilon)\gamma^\mu\psi(x)\exp\{-ie\int_x^{x+\epsilon} dx^\mu A_\mu\}$$
$$-\psi(x)\bar{\psi}(x-\epsilon)\gamma^\mu\exp\{+ie\int_{x-\epsilon}^{x} dx^\mu A_\mu\}], \tag{3.24}$$

where only $\overset{o}{A}_-$ and $\tilde{A}_+$ are non-zero. A straightforward calculation shows

$$j^+(x) = \sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N + \frac{1}{2L}(N-q), \tag{3.25}$$

where

$$\sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N = \frac{1}{2L}\{(\sum_{n\geq N}\sum_{m\geq N} + \sum_{n<N}\sum_{m\geq N} + \sum_{n\geq N}\sum_{m<N})a^\dagger_{n+\frac{1}{2}}a_{m+\frac{1}{2}}$$
$$-\sum_{n<N}\sum_{m<N} a_{m+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}}\}e^{i\pi(n-m)x^-/L}. \tag{3.26}$$

In Appendix C, we discuss how to obtain the Schwinger term and the anomalous conservation law of the axial vector current.

A problem arises when we treat zero-modes with care. Because of the relation $j_5^\mu = -\epsilon^{\mu\nu}j_\nu$, the +-components of these two currents coincide. Naively, therefore, the charges should be the same. On the other hand, because the vector current is conserved and the axial-vector current is not conserved anomalously as well as explicitly, one would expect that the vector charge is conserved while the axial-vector charge is not. This apparent contradiction is resolved formally by thinking that the zero modes of the charge have no direct connection with the non-zero modes. Probably an elaborate work on zero-modes may explain the precise relation between the zero modes and non-zero modes of the currents. At this moment, however, we take a pragmatic way and simply "adjust" the zero modes (charges) so that they satisfy desired properties. (See Appendix C for the axial-vector charge.)

Because the Hamiltonian does not contain the zero modes of the currents, it is free from this ambiguity. What we should do is to regularize P^-_{fmass}, which is essentially the mass term $m\int dx^-\bar{\psi}\psi$ written in terms of the independent fields. But there is a rather

surprising fact; the mass term $\bar{\psi}\psi$ is *not* invariant under charge conjugation on the light-cone. In equal-time quantization, in order to prove the charge conjugation invariance of the mass term, we use the fact that ψ_R anti-commutes with $\psi_L^\dagger$. In light-cone quantization, on the other hand, they do not anti-commute,

$$\{\psi_R(x), \psi_L^\dagger(y)\} = \frac{m}{4\pi}\sum_n \frac{1}{n+\frac{1}{2}-q} e^{i\frac{\pi}{L}(n+\frac{1}{2})(x^- - y^-)}. \tag{3.27}$$

Therefore, if we wish to preserve charge conjugation invariance of P^- at the quantum level, we have to define it in an invariant way. The simplest way is to replace $\bar{\psi}\psi$ with $(\bar{\psi}\psi - \psi^T(\bar{\psi})^T)/2$, where the superscript T stands for transpose. By using this definition in the quantum theory, we get [26]

$$\begin{aligned}
&\frac{m^2}{2\sqrt{2}}\int_{-L}^{L} dx^-[\psi_R^\dagger e^{-ie\mathring{A}_- x^-}\frac{1}{i\partial_-}e^{ie\mathring{A}_- x^-}\psi_R - e^{-ie\mathring{A}_- x^-}\frac{1}{i\partial_-}e^{ie\mathring{A}_- x^-}\psi_R\psi_R^\dagger] \\
&= \frac{m^2 L}{2\pi}\{\sum_{n\geq N}\frac{1}{n+\frac{1}{2}-q}a^\dagger_{n+\frac{1}{2}}a_{n+\frac{1}{2}} - \sum_{n<N}\frac{1}{n+\frac{1}{2}-q}a_{n+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}}\} \\
&+ \frac{m^2 L}{4\pi}[\sum_{n<N}\frac{1}{n+\frac{1}{2}-q} - \sum_{n\geq N}\frac{1}{n+\frac{1}{2}-q}],
\end{aligned} \tag{3.28}$$

where the last term may be regularized by using ζ-function. It is rewritten as

$$\frac{m^2 L}{4\pi}[\psi(\frac{1}{2}+q-N) + \psi(\frac{1}{2}-q+N)] \tag{3.29}$$

after dropping q-independent infinity, where ψ is a digamma function.

We are now going to discuss a very interesting symmetry. Even after fixing a gauge, there is a residual symmetry, called "large" gauge symmetry. The theory is invariant under a large gauge transformation U,

$$U\psi_R(x)U^\dagger = e^{i\frac{\pi}{L}x^-}\psi_R(x), \tag{3.30}$$

$$U\,\mathring{A}_-\,U^\dagger = \mathring{A}_- - \frac{1}{e}\frac{\pi}{L}. \tag{3.31}$$

In terms of $a_{n+\frac{1}{2}}$ and q, we have

$$U a_{n+\frac{1}{2}} U^\dagger = a_{n+\frac{3}{2}}, \tag{3.32}$$

$$U\hat{q}U^\dagger = \hat{q} - 1. \tag{3.33}$$

(In order to avoid possible confusions, we have denoted $\hat{q}$ for the operator.) Note that this transformation change neither the gauge conditions, nor the boundary conditions for ψ_R and A_-. This transformation generates an additive group Z and decreases q by one. It is easy to prove the following transformation properties,

$$U|0\rangle_N = |0\rangle_{N+1} \tag{3.34}$$

$$U|q\rangle = |q+1\rangle \tag{3.35}$$

$$UP^-U^\dagger = P^- \tag{3.36}$$

$$\begin{aligned} U(\sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N U^\dagger &= \sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_{N+1} \\ &= \sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N + \frac{q}{2L}. \end{aligned} \tag{3.37}$$

At this point it is useful to introduce $M(q)$, the integer closest to q, i.e.,

$$-\frac{1}{2} < q - M(q) \leq \frac{1}{2}, \tag{3.38}$$

which transforms in the following way,

$$UM(q)U^\dagger = M(q-1) = M(q) - 1. \tag{3.39}$$

Let us define the charge operator. As we have explained, we do not require that the charge must be just the (light-cone) spatial integral of the current. In fact, it is easy to show that such a "naive" definition of the charge

$$Q_{\text{naive}} \equiv 2L\, \overset{o}{j}{}^+ = \sum_{n \geq N} a^\dagger_{n+\frac{1}{2}} a_{n+\frac{1}{2}} - \sum_{n<N} a_{n+\frac{1}{2}} a^\dagger_{n+\frac{1}{2}} + N - q$$

does not commute with the Hamiltonian,

$$[Q_{\text{naive}}, P^-] = -\frac{ie}{2\pi} L\, \overset{o}{E}{}^-$$

though it is invariant under a large gauge transformation.

We define the charge in the following way,

$$Q = \sum_{n \geq N} a^\dagger_{n+\frac{1}{2}} a_{n+\frac{1}{2}} - \sum_{n<N} a_{n+\frac{1}{2}} a^\dagger_{n+\frac{1}{2}} + N - M(q). \tag{3.40}$$

In defining this charge operator, we have used the arbitrariness of the constant vector in the gauge invariant point-splitting of the current (3.24). Note that it is invariant under a large gauge transformation and commutes with the Hamiltonian. (The momentum operator $\overset{o}{E}^-$ generates an infinitesimal translation of the coordinate q. The integer part $M(q)$ is invariant under such a translation.)

We can now impose the physical state condition,

$$Q|\text{phys}\rangle = 0. \tag{3.41}$$

Because the charge is conserved and is invariant under a large gauge transformation, this definition of physical states is gauge invariant and is consistent with (light-cone) "time" evolution.

4 Physical vacuum

In this section we impose the physical state condition (3.41) and find out the physical states. In this thesis we especially look at the vacuum state, but this scheme is applicable to other particle states.

We consider a generic state for the total system,

$$|\phi\rangle = \int_{-\infty}^{\infty} dq|q\rangle\langle q|\phi\rangle \tag{4.1}$$

$$= \int_{-\infty}^{\infty} dq|q\rangle \sum_{\alpha} \phi_\alpha(q)|\alpha(q)\rangle \tag{4.2}$$

where $\phi_\alpha(q)$ is the wave function for the zero mode in the q-representation, and α parameterizes fermion Fock states. The ket $|\alpha(q)\rangle$ is a fermion Fock state in the presence of the external field q.

Let us now consider the ground state. We first notice that the state $|0\rangle_N$ has a smaller energy than that of any of the states made on the N-vacua, $|0\rangle_N$, by acting the fermionic operators, $a^\dagger_{n+\frac{1}{2}}$ for $n \geq N$ and $a_{n+\frac{1}{2}}$ for $n \leq N-1$. The problem is that the state $|0\rangle_N$

is not a physical state nor U-invariant. We therefore consider a linear combination of N-vacua and seek for the conditions under which it satisfies all the desired properties. Consider the state $|vac\rangle$,

$$|vac\rangle = \int dq|q\rangle \sum_{n=-\infty}^{\infty} \phi_n(q)|0\rangle_n, \tag{4.3}$$

which is constructed as the linear combinations of the general N-vacua. We require the physical state condition, $Q|vac\rangle = 0$,

$$Q|vac\rangle = \int dq|q\rangle \sum_{n=-\infty}^{\infty} \phi_n(q)(n - M(q))|0\rangle_n = 0. \tag{4.4}$$

This is satisfied when $\phi_n(q) = \tilde{\phi}_n(q)\delta_{n,M(q)}$ for all integer n. In this case,

$$|vac\rangle = \int dq\tilde{\phi}_{M(q)}(q)|q\rangle|0\rangle_{M(q)}. \tag{4.5}$$

From now on we write $\phi(q)$ instead of $\tilde{\phi}_{M(q)}(q)$ for simplicity. It is important to note that this physical state condition connects the fermionic Dirac sea with the configuration of the zero mode of the gauge field.

This state is an eigenstate of the Hamiltonian, P^-, because the charge operator, Q, commutes with the Hamiltonian. We solve the eigenvalue equation, $P^-|vac\rangle = 2L\epsilon|vac\rangle$, where ϵ is the energy density. In the unit of $e/\sqrt{\pi} = 1$, this equation becomes

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial q^2} + V(q)\right]\phi(q) = \tilde{\epsilon}\phi(q) \tag{4.6}$$

where $V(q)$ is a potential (density) and $\tilde{\epsilon} \equiv 4\pi\epsilon$. We call this equation (4.6) the vacuum equation.

The potential, $V(q)$, arises only from the fermion mass term, P^-_{fmass}, because the current term, $P^-_{current}$, is made only of the non-zero mode of the current, $\tilde{j^+}$, and annihilates the vacuum. The potential is defined as follows:

$$\begin{aligned} V(q)\delta(q'-q) &= \frac{2\pi}{L}{}_{M(q')}\langle 0|\langle q'|P^-_{fmass}(\hat{q})|q\rangle|0\rangle_{M(q)} \\ &= \frac{m^2}{2}[\psi(\frac{1}{2} + q' - M(q')) + \psi(\frac{1}{2} - q' + M(q'))]\delta(q'-q). \end{aligned} \tag{4.7}$$

It is easy to derive that this potential is a periodic potential with the period one in the q space, i.e. $V(q+1) = V(q)$. This feature comes from the fact that our theory is invariant under the large gauge transformation. We emphasize again that we have regularized the charge and imposed the physical state condition correctly. As the consequence, we obtained the periodicity of the potential.

The vacuum equation (4.6) is just the usual Schrödinger equation in a periodic potential. Now it is clear that we have stated Bloch theorem in the other way around; usually, it states that the eigenfunction of a Schrödinger equation in a periodic potential with period a has the form

$$\phi(q) = e^{-ipq/a}\varphi(q), \tag{4.8}$$

where p is a Bloch momentum and $\varphi(q)$ is a periodic function [28]. We can consider in the reduced Brillouin zone, $-\pi/a \leq p \leq \pi/a$.

In our case the period of the potential is one and we can write the similar equation as (4.8) as follows:

$$\phi(q) = e^{-i\theta q}\varphi(q), \tag{4.9}$$

where θ is the Bloch momentum reduced in the first Brillouin zone, $-\pi \leq \theta \leq \pi$. This fact decides the transformation property of the vacuum state (4.5) under the large gauge transformation:

$$\begin{aligned} U|vac\rangle &= \int dq\phi(q)|q+1\rangle|0\rangle_{M(q)+1} \\ &= \int dq\phi(q-1)|q\rangle|0\rangle_{M(q)} \\ &= e^{i\theta}|vac\rangle, \end{aligned} \tag{4.10}$$

where we have used $\phi(q-1) = e^{i\theta}\phi(q)$ which is derived easily by (4.9). It indicates that our vacuum state is an eigenstate of the large gauge transformation. This is consistent with the fact that the Hamiltonian P^- commutes with U.

The vacuum equation (4.6) is rewritten in terms of the wave function φ:

$$\left[-\frac{1}{2}(\frac{\partial}{\partial q} - i\theta)^2 + V(q)\right]\varphi(q) = \tilde{\epsilon}\varphi(q). \tag{4.11}$$

Of course it gives the same eigenvalue as Eqn. (4.6). It is interesting that it coincides with the equation which is obtained if we start from the Lagrangian density with the θ-term, $-\frac{e\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu$ in addition to Eqn. (3.1). In this case the transformation property of the vacuum state is $U|vac\rangle = |vac\rangle$. It is now evident that the θ-parameter is identified with the Bloch momentum. The periodicity of θ is now easily understood. Note that, though it is known that θ *is analogous* to a Bloch momentum [29], what we have shown is that θ *is nothing but* a Bloch momentum in the massive Schwinger model, with the explicit periodic potential.

We solve the vacuum equation. It is easier to treat Eqn. (4.11) than Eqn. (4.6), and we solve Eqn. (4.11). To solve it, it is convenient to make the Fourier decomposition. The potential and wave function is periodic functions with period one, then the potential and the wave function are decomposed as follows:

$$\varphi(q) = \sum_n a_n e^{2n\pi iq},$$
$$V(q) = \sum_n U_n e^{2n\pi iq},$$

where the Fourier coefficients U_{-n} are the same as U_n, because the potential is symmetric to the origin. Therefore the equation (4.11) becomes

$$\lambda_n a_n + \sum_m U_{n-m} a_m = \tilde{\epsilon} a_n, \tag{4.12}$$

where $\lambda_n = (\theta - 2n\pi)^2/2$, or in the matrix form:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \lambda_{-1} + U_0 & U_1 & U_2 & \cdots \\ \cdots & U_1 & \lambda_0 + U_0 & U_1 & \cdots \\ \cdots & U_2 & U_1 & \lambda_1 + U_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} = \tilde{\epsilon} \begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix}, \tag{4.13}$$

From this equation we get a lot of interesting properties. At first the eigenvalue of this equation is invariant under the transformation, $\theta \to -\theta$ because the matrix element is invariant if we change $n \to -n$ simultaneously. This property is the same as the statement that the physics is invariant under changing the sign of the back ground constant electric field θ, which is plausible since this transformation is nothing but the redefinition of the spatial coordinate, $x^- \to -x^-$. Second interesting feature which is derived from this equation is the periodicity of physics in θ. Even if we transform $\theta \to \theta + 2\pi$, the eigenvalue of this equation is invariant because this transformation is absorbed by the redefinition of the integer n. Coleman derived this feature by using the dynamical property of the vacuum in the equal-time quantization scheme [21]. In our scheme it is understood in the context of the zero mode of the gauge field and its property under the large gauge transformation.

We have solved this eigenvalue equation numerically. The potential we are investigating is the singular potential, and its behavior in the vicinity of $q = \pm 1/2$ is known as $-1/2x$, then we take the principal value prescription for it: $\frac{1}{x} \to \frac{1}{2}(\frac{1}{x+i\epsilon} + \frac{1}{x-i\epsilon})$. We get $V(\pm\frac{1}{2}) = -\gamma$, where γ is the Euler constant ($\gamma = 0.57721\cdots$). The size of the Hamiltonian of Eqn. (4.13) is infinite, and then we must truncate the matrix up to some large enough value where the eigenvalue is considered to be convergent. Note that the size of the matrix must be odd $(2M+1)$, otherwise it is not invariant under the transformation, $\theta \to -\theta$. We use these two features for the consistency check whether the value we got is plausible or not.

We summarize the results for the vacuum equation below. We show the three lowest energy densities in Fig. 3. The excited states are interpreted as the unstable vacua where the constant electric field θ is larger than π. We can see it by thinking the massless case. In this case, the potential is zero and the vacuum equation is the free equation. The eigenfunction is a plane wave and the eigenvalue is given as $\theta^2/2$. We are now reducing the range of θ, and the spectrum out of this region is turned up in the first Brillouin zone.

This is the original form of this "excited"states. For massive case, this also happens in the same way.

We show the lowest (stable) energy densities of the vacuum for the various value θ in the Fig's. 4-9. This is symmetric in θ and periodic in the period 2π. The most interesting points in these figures are the behaviors at $\theta = \pi$. For the strong coupling constant (or the small mass limit, where the fermion mass is now in the unit of the coupling constant $e/\sqrt{\pi}$), the point at $\theta = \pi$ is a cusp and not differentiable, but as the fermion mass is increasing, the cusp at $\theta = \pi$ becomes milder. In the weak coupling the derivative of the vacuum density by θ gets equal to zero. This is a kind of the signal of the phase transition at this point. The critical value is $m_c \approx 0.5$ from these figures. The potential term originally comes from the vacuum expectation values of the fermionic part of Hamiltonian, therefore the potential is the energy of the Dirac sea. This potential plays such a role as to reduce the energy peak at $\theta = \pi$ as the fermion mass increases. It is considered as the polarization of the vacuum becomes larger along increasing the fermion mass.

5 Conclusions and Discussion

We investigate the massive Schwinger model with great care of the zero mode of the gauge fields by putting the system into the finite (light-cone) spatial box. After Dirac procedure of quantization for the constrained system, we regularize the composite operators, the currents and the charge by the point-splitting in gauge invariant way. The chiral anomaly of the axial $U(1)$ current is derived from the anomalous commutation relations of the currents, j^+'s. Especially the chargeless condition, which is a zero mode of the Gauss law, is derived to be still the first class constraint by using the arbitrariness of the regularization of the current. Therefore the zero mode of A^+ is still dynamical degree of freedom and the chargeless condition is consistently imposed and relates the

fermion Fock states to the zero mode of the gauge field.

We discuss the residual gauge transformation, the large gauge transformation, and by the invariance of the physics under this transformation it is derived that the physics is periodic in θ-parameter and symmetric under changing the signs of the background constant electric field θ. This is exactly same as the Bloch Theorem in the solid state physics in which the system has the symmetry under the translation of a lattice of the crystal.

We get the signal of the phase transition at $\theta = \pi$ by looking at the energy density of the vacuum. In the strong coupling the vacuum energy density is not differentiable because it has a cusp. However the cusp becomes gradually milder from the strong coupling to the weak coupling and become differentiable at some critical point.

The physical meaning of this phase transition is a little unclear in our scheme. Coleman says that this is the phase transition of the charge conjugation symmetry [21]. Our result is basically consistent with the lattice calculation [34]. They looked at the slope of the string tension at $\theta = \pi$. The string tension is in essence the difference of the energy density for $\theta \neq 0$ and for $\theta = 0$. They did not include the energy density $\theta^2/2$ in the strong coupling limit which is corresponded to the massless Schwinger model, then the string tension becomes $T(\theta) = \theta^2/2 - (\epsilon(\theta) - \epsilon(0))$ in our language, where ϵ is our vacuum density. This value is the same behavior as our results. Therefore it is justified for us to have found the phase transition at $\theta = \pi$. We succeeded to look at the phase transition dynamically in the light-cone quantization scheme.

In this thesis we ignored the zero mode of the fermions. Even if we take the anti-periodic boundary condition for the fermions, the dynamical zero modes of the fermion come arise from the zero point of the Dirac operator of the Eqn. (3.11). However it is dependent on the configuration of the gauge field q. Let us define the Fourier component of the fermions, $\tilde{\psi}_L$ and $\tilde{\psi}_R$, respectively. The Eqn. (3.11) becomes as $\frac{\pi}{L}[(n+1/2)-q]\tilde{\psi}_L(n) = \frac{m}{\sqrt{2}}\tilde{\psi}_R(n)$ and there is a zero point of the Dirac operator when q is a half-integer. In this

case, this constraint becomes $\tilde{\psi}_R(n) \approx 0$ and $\tilde{\psi}_L(n)$ remains as the dynamical degree of the freedom only in *one* value of n. Happily, the Hamiltonian does not involve such a term because its terms always take the forms which are the summation of the combination $\tilde{\psi}_R^\dagger(n)\tilde{\psi}_L(n)$. Therefore the vacuum equation does not change basically in this case. It changes the value of the potential just at the singular point where the gauge field q is a half-integer. We have solved the singular potential problem with the principal value prescription and expect that the singularity disappears if we investigate the zero mode of the left-handed fermion seriously.

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APPENDIX

A 6-body LFTD approximation of QED_2

At first we summarize notation and conventions used in this paper. They are essentially the same as those by Perry and Harindranath [10]. The metric is given by

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\mu, \nu = 0, 1),$$
$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\mu, \nu = +, -), \tag{A.1}$$

where

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1). \tag{A.2}$$

We treat x^+ as our "time". Accordingly, gamma matrices are defined as follows;

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\gamma^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix},$$
$$\gamma^+\gamma^- = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \gamma^-\gamma^+ = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}; \tag{A.3}$$

thus $\psi = (\psi_R, \psi_L)^T$. The totally antisymmetric tensor $\epsilon^{\mu\nu}$ is defined by

$$\epsilon^{01} = \epsilon^{-+} = +1 \tag{A.4}$$

The massive Schwinger model [20, 21] is two-dimensional QED with a massive fermion. It is not exactly solvable in contrast to the massless one [19, 24]. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right) - m\right]\psi \ , \tag{A.5}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. In two dimensions, the coupling constant e has mass dimension. It is therefore useful to measure all dimensionful quantities in units of $e/\sqrt{\pi}$. We hereafter set $e/\sqrt{\pi} = 1$. In this unit, strong couplings correspond to small masses.

In the light-cone gauge ($A^{+} = 0$), only the independent variable is ψ_R in the light-cone quantization. A^{-} and ψ_L are expressed in terms of ψ_R as follows:

$$A^{-} = \sqrt{\pi}\frac{1}{(i\partial_{-})^{2}}j^{+}, \tag{A.6}$$

$$\psi_L = \frac{m}{\sqrt{2}}\frac{1}{i\partial_{-}}\psi_{R}., \tag{A.7}$$

with $j^{+}(x^{-}) = \sqrt{2} : \psi_R^{\dagger}(x^{-})\psi_R(x^{-}) :$. Eliminating A^{-} and ψ_L by using (A.6) and (A.7), one obtains the light-cone Hamiltonian P^{-}.

$$\begin{aligned} P^{-} &= P^{-}_{free} + P^{-}_{int} \ , \\ P^{-}_{free} &= \frac{m^2}{\sqrt{2}}\int_{-\infty}^{\infty} dx^{-}\psi_R^{\dagger}(x^{-})\frac{1}{i\partial_{-}}\psi_R(x^{-}) \ , \\ P^{-}_{int} &= \frac{\pi}{2}\int_{-\infty}^{\infty} dx^{-} j^{+}(x^{-})\frac{1}{(i\partial_{-})^{2}}j^{+}(x^{-}) \ . \end{aligned} \tag{A.8}$$

We expand ψ_R in terms of the creation and annihilation operators,

$$\psi_R(x^{-}) = \frac{1}{2^{1/4}}\int_{0}^{\infty}\frac{dk^{+}}{(2\pi)\sqrt{k^{+}}}\left[b(k^{+})e^{-ik^{+}x^{-}} + d^{\dagger}(k^{+})e^{ik^{+}x^{-}}\right] \ , \tag{A.9}$$

where $b(k^{+})$ and $d(k^{+})$ satisfy the following anti-commutation relations,

$$\left\{b(k^{+}), b^{\dagger}(l^{+})\right\} = \left\{d(k^{+}), d^{\dagger}(l^{+})\right\} = (2\pi)k^{+}\delta(k^{+} - l^{+}) \ , \tag{A.10}$$

derived from $\left\{\psi_R(x^{-}), \psi_R^{\dagger}(y^{-})\right\} = (1/\sqrt{2})\delta(x^{-} - y^{-})$. One may express P^{-} entirely in terms of $b(k^{+})$ and $d(k^{+})$ (and their Hermitian conjugates). We refer the reader to Ref. [5] for the explicit form.

We work in a truncated Fock space in which a state with total light-cone momentum $P^+ = \mathcal{P}$ is expressed as

$$
\begin{aligned}
|\psi(P)\rangle &= |2(P)\rangle + |4(P)\rangle + |6(P)\rangle \ , \qquad \text{(A.11)}\\
|2(P)\rangle &= \int_0^{\mathcal{P}} \frac{dk_1 dk_2}{\sqrt{(2\pi)^2 k_1 k_2}} \delta(k_1 + k_2 - \mathcal{P}) \psi_2(k_1, k_2) b_1^\dagger d_2^\dagger |0\rangle \ ,\\
|4(P)\rangle &= \frac{1}{2} \int_0^{\mathcal{P}} \prod_{i=1}^{4} \frac{dk_i}{\sqrt{(2\pi)k_i}} \delta(\sum_{i=1}^{4} k_i - \mathcal{P}) \psi_4(k_1, k_2; k_3, k_4) b_1^\dagger b_2^\dagger d_3^\dagger d_4^\dagger |0\rangle \ ,\\
|6(P)\rangle &= \frac{1}{3!} \int_0^{\mathcal{P}} \prod_{i=1}^{6} \frac{dk_i}{\sqrt{(2\pi)k_i}} \delta(\sum_{i=1}^{6} k_i - \mathcal{P}) \psi_6(k_1, k_2, k_3; k_4, k_5, k_6) b_1^\dagger b_2^\dagger b_3^\dagger d_4^\dagger d_5^\dagger d_6^\dagger |0\rangle \ ,
\end{aligned}
$$

where we use the abbreviated notations, $b_i^\dagger = b^\dagger(k_i)$, $d_i^\dagger = d^\dagger(k_i)$. In these equations, we rescaled momenta, $k_i \to x_i = k_i/\mathcal{P}$ and the wave functions, $\psi_2(k_1, k_2)$, $\psi_4(k_1, k_2; k_3, k_4)$, and $\psi_6(k_1, k_2, k_3; k_4, k_5, k_6)$ are replaced by $\psi_2(x_1, x_2)$, $\mathcal{P}^{-1}\psi_4(x_1, x_2; x_3, x_4)$, and $\mathcal{P}^{-2}\psi_6(x_1, x_2, x_3; x_4, x_5, x_6)$ with $\sum_i x_i = 1$, respectively.

The wave functions ψ_2, ψ_4 and ψ_6 must satisfy the following symmetry properties due to Fermi statistics,

$$
\begin{aligned}
&\psi_4(x_1, x_2; x_3, x_4) = -\psi_4(x_2, x_1; x_3, x_4) = -\psi_4(x_1, x_2; x_4, x_3) = \psi_4(x_2, x_1; x_4, x_3) \ ,\\
&\psi_6(x_1, x_2, x_3; x_4, x_5, x_6) = -\psi_6(x_2, x_1, x_3; x_4, x_5, x_6) = -\psi_6(x_1, x_3, x_2; x_4, x_5, x_6)\\
&= -\psi_6(x_1, x_2, x_3; x_5, x_4, x_6) = -\psi_6(x_1, x_2, x_3; x_4, x_6, x_5) \ \text{etc.} \ .
\end{aligned}
$$

If we require that this state has a definite property under charge conjugation transformation, we have further conditions on these wave functions,

$$
\begin{aligned}
&\psi_2(x_1, x_2) = \mp\psi_2(x_2, x_1) \ ,\\
&\psi_4(x_1, x_2; x_3, x_4) = \pm\psi_4(x_3, x_4; x_1, x_2) \ ,\\
&\psi_6(x_1, x_2, x_3; x_4, x_5, x_6) = \mp\psi_6(x_4, x_5, x_6; x_1, x_2, x_3) \ .
\end{aligned}
$$

The upper/lower sign in (A.12) corresponds to charge conjugation even/odd. This property is derived by the fact that if $\psi_2(x, 1-x)$, $\psi_4(x_1, x_2; x_3, x_4)$ and $\psi_6(x_1, x_2, x_3; x_4, x_5, x_6)$

are a solution of the Einstein-Schrödinger equations then $-\psi_2(1-x,x)$, $\psi_4(x_3,x_4;x_1,x_2)$ and $-\psi_6(x_4,x_5,x_6;x_1,x_2,x_3)$ satisfy them too.

We solve the Einstein-Schrödinger equation:

$$M^2|\psi\rangle_{\mathcal{P}} = 2P^-P^+|\psi\rangle_{\mathcal{P}}. \tag{A.12}$$

With (A.11), we obtain the following coupled integral equations:

$$\begin{aligned} M^2\psi_2(x,1-x) &= (m^2-1)\psi_2(x,1-x)\frac{1}{x(1-x)} \\ &+ \int_0^1 dy\psi_2(y,1-y)\left(1-\frac{1}{(x-y)^2}\right) \\ &+ 2\int_0^1 dy_1dy_2\psi_4(y_1,y_2;x-y_1-y_2,1-x)\frac{1}{(x-y_1)^2} \\ &- 2\int_0^1 dy_1dy_2\psi_4(1-x-y_1-y_2,x;y_1,y_2)\frac{1}{(1-x-y_1)^2}\ , \end{aligned} \tag{A.13}$$

$$\begin{aligned} M^2\psi_4(x_1,x_2;x_3,x_4) &= (m^2-1)\psi_4(x_1,x_2;x_3,x_4)\sum_{i=1}^4\frac{1}{x_i} \\ &+\int_0^1 dy_1dy_2\psi_4(y_1,y_2;x_3,x_4)\delta(x_1+x_2-y_1-y_2)\frac{1}{(x_1-y_1)^2} \\ &+\int_0^1 dy_1dy_2\psi_4(x_1,x_2;y_1,y_2)\delta(x_3+x_4-y_1-y_2)\frac{1}{(x_3-y_1)^2} \\ &+4\int_0^1 dy_1dy_2\psi_4(x_1,y_1;x_3,y_2)\delta(x_2+x_4-y_1-y_2)\left[\frac{1}{(x_2+x_4)^2}-\frac{1}{(x_2-y_1)^2}\right] \\ &+2\int_0^1 dy\psi_2(y,x_4)\delta(x_1+x_2+x_3-y)\frac{1}{(x_1-y)^2} \\ &-2\int_0^1 dy\psi_2(x_2,y)\delta(x_1+x_3+x_4-y)\frac{1}{(x_3-y)^2} \\ &-6\int_0^1 dy_1dy_2dy_3\psi_6(y_1,y_2,x_2;y_3,x_3,x_4)\delta(y_1+y_2+y_3-x_1)\frac{1}{(x_1-y_1)^2} \\ &+6\int_0^1 dy_1dy_2dy_3\psi_6(y_1,x_1,x_2;y_2,y_3,x_4)\delta(y_1+y_2+y_3-x_3)\frac{1}{(x_3-y_2)^2}\ , \end{aligned} \tag{A.14}$$

where $\sum_{i=1}^4 x_i = 1$, and

$$
\begin{aligned}
M^2\psi_6(x_1,x_2,x_3;x_4,x_5,x_6) &= (m^2-1)\psi_6(x_1,x_2,x_3;x_4,x_5,x_6)\sum_{i=1}^{6}\frac{1}{x_i} \qquad \text{(A.15)}\\
&+3\int_0^1 dy_1dy_2\psi_6(y_1,y_2,x_3;x_4,x_5,x_6)\delta(x_1+x_2-y_1-y_2)\frac{1}{(x_1-y_1)^2}\\
&+3\int_0^1 dy_1dy_2\psi_6(x_1,x_2,x_3;y_1,y_2,x_6)\delta(x_4+x_5-y_1-y_2)\frac{1}{(x_4-y_1)^2}\\
&+9\int_0^1 dy_1dy_2\psi_6(y_1,x_2,x_3;y_2,x_5,x_6)\delta(x_1+x_4-y_1-y_2)\left[\frac{1}{(x_1+x_4)^2}-\frac{1}{(x_1-y_1)^2}\right]\\
&-6\int_0^1 dy\psi_4(y,x_3;x_5,x_6)\delta(x_1+x_2+x_4-y)\frac{1}{(x_1-y)^2}\\
&+6\int_0^1 dy\psi_4(x_2,x_3;y,x_6)\delta(x_1+x_4+x_5-y)\frac{1}{(x_4-y)^2}\,.
\end{aligned}
$$

where $\sum_{i=1}^{6} x_i = 1$.

This is very complicated and one may think too difficult to solve it, but it can be converted to a single matrix eigenvalue equation by expanding the wave functions in terms of basis functions. The basis functions are decided according to its symmetries above and its behavior at $x = 0, 1$. It is well known that the wave function $\psi_2(x, 1-x)$ behaves as x^β in the vicinity of $x = 0$ [4], with β being the solution of the equation $m^2 - 1 + \pi\beta\cot(\pi\beta) = 0$. By taking it into account, Mo and Perry concluded that a useful choice of the basis functions for the wave functions is given in terms of Jacobi polynomials, $P_n^{(\beta,\beta)}$. In a previous paper [6, 7], we propose a simpler set of basis functions, essentially equivalent to that of Mo and Perry. For example $\psi_2(x, 1-x)$ is decomposed by polynomials, $[x(1-x)]^{\beta+k}$ and $[x(1-x)]^{\beta+k}(2x-1)$, where k is integer. In the 2-body, the charge conjugation is defined by $x \leftrightarrow 1-x$, they correspond to the charge conjugation odd and even, respectively due to Fermi statistics. For the explicit form of the basis function for the 4-body and 6-body wave functions, see Refs. [7]. It is important that we can obtain the matrix element analytically by using these basis functions. We now get a simple matrix equation, then we diagonalize it and obtain the eigenvalues and eigenfunctions with the computer.

B "Iterative property" of Dirac brackets

In the light-cone quantization, one usually face with a lot of constraints. It is sometimes very tedious to calculate Dirac brackets directly. In this appendix we explain the "iterative method"[30] of calculating Dirac brackets for the readers who are not familiar with it.

The most tedious part of the calculation is to get the inverse of the constraint matrix $C_{IJ} = \{\theta_I, \theta_J\}$, where θ_I, $(I = 1, \cdots, 2N)$ are the second-class constraints of the theory. (For simplicity, we assume that there is no first-class constraints. The generalization to such cases with first-class constraints is trivial.) We divide these constraints into two groups, for example: $\theta_I = (\phi_i, \varphi_j)$, where $1 \leq i \leq 2m$, $1 \leq j \leq 2N - 2m$ for arbitrary m $(1 \leq m < N)$. We denote this constraint matrix as follows:

$$C = \begin{pmatrix} \{\phi, \phi\} & \{\phi, \varphi\} \\ \{\varphi, \phi\} & \{\varphi, \varphi\} \end{pmatrix} \equiv \begin{pmatrix} X & Z \\ -Z^T & Y \end{pmatrix},$$

where we introduce the abbreviations X, Y and Z. What we want to do is to calculate the inverse of C in (more than) two steps. First we calculate the 'Dirac brackets' constrained on the surface $\phi_i \approx 0$

$$\{F, G\}_{D1} \equiv \{F, G\} - \{F, \phi_i\}(X^{-1})^{ij}\{\phi_j, G\}.$$

Let us call these brackets "D1 brackets." We have to calculate "D1 brackets" for all the variables and set ϕ_i strongly equal to zero. Note that the constraint matrix for the rest φ_j changes because of this procedure, namely,

$$\begin{aligned} M_{ij} &\equiv \{\varphi_i, \varphi_j\}_{D1} = \{\varphi_i, \varphi_j\} - \{\varphi_i, \phi_k\}(X^{-1})^{kl}\{\phi_l, \varphi_j\}, \\ M &= Y + Z^T X^{-1} Z. \end{aligned} \tag{B.1}$$

The iterative property is the property that the correct Dirac bracket for physical variables F and G is obtained in the following way:

$$\{F, G\}_D = \{F, G\}_{D1} - \{F, \varphi_i\}_{D1}(M^{-1})^{ij}\{\varphi_j, G\}_{D1}, \tag{B.2}$$

namely, we can calculate the Dirac brackets in two steps. Obviously we can generalize the above procedure to many step calculations.

The proof is quite simple. In terms of Poisson brackets, the Dirac bracket (B.2) is given as follows:

$$
\begin{aligned}
\{F,G\}_D = \{F,G\} &- \{F,\phi_i\}(X^{-1})^{ij}\{\phi_j,G\} \\
&+ \{F,\phi_k\}(X^{-1})^{kl}Z_{li}(M^{-1})^{ij}Z_{jm}(X^{-1})^{mn}\{\phi_n,G\} \\
&+ \{F,\phi_k\}(X^{-1})^{kl}Z_{li}(M^{-1})^{ij}\{\varphi_j,G\} \qquad \text{(B.3)} \\
&- \{F,\varphi_i\}(M^{-1})^{ij}Z_{jk}(X^{-1})^{kl}\{\phi_l,G\} \\
&- \{F,\varphi_i\}(M^{-1})^{ij}\{\varphi_j,G\} \\
&= \{F,G\} - \{F,\theta_I\}\Omega^{IJ}\{\theta_J,G\}, \qquad \text{(B.4)}
\end{aligned}
$$

where Ω is given by

$$
\Omega = \begin{pmatrix} X^{-1} - X^{-1}ZM^{-1}Z^TX^{-1} & -X^{-1}ZM^{-1} \\ M^{-1}Z^TX^{-1} & M^{-1} \end{pmatrix}. \qquad \text{(B.5)}
$$

One can easily see that Ω is the inverse of C.

C Currents and anomaly

In this appendix, we discuss the regularization of the current, the Schwiger term, and chiral anomaly. In order to have a well-defined quantum theory, one must regularize the current properly so that it reproduces the well-known chiral anomaly.

The massive Schwinger model is a gauge invariant theory. One should preserve gauge invariance in any regularization. Actually it is possible. On the other hand, axial symmetry is broken anomalously at the quantum level. There is no consistent way to preserve both symmetries.

Let us begin with our Fourier expansion of the fermion field (3.22). By substituting

it into the gauge invariant definition of the current (3.24), we get

$$j^+(x) = \sqrt{2} : \bar{\psi}_R^\dagger \psi_R(x) :_N + \frac{1}{2L}(N - q), \tag{C.1}$$

$$j^-(x) = \sqrt{2} : \bar{\psi}_L^\dagger \psi_L(x) :_N - \frac{e}{2\pi}\tilde{A}^-, \tag{C.2}$$

in our gauge condition. The normal-ordering is with respect to the N-vacua. See Eqn. (3.26). In deriving these, we used the following properties[4],

$${}_N\langle 0|\psi_R^\dagger(x+\epsilon)\psi_R(x)|0\rangle_N = \frac{-i}{2\sqrt{2}\pi}\frac{e^{i\frac{\pi}{L}N\epsilon^-}}{\epsilon^- - i0}, \tag{C.3}$$

$${}_N\langle 0|\psi_L^\dagger(x+\epsilon)\psi_L(x)|0\rangle_N = \frac{-i}{2\sqrt{2}\pi}\frac{e^{i\frac{\pi}{L}N\epsilon^-}}{\epsilon^+ - i0}. \tag{C.4}$$

As explained in the text, we think that the zero mode (the charge) has nothing to do with the non-zero modes and "adjust" the zero mode so that it satisfies desired properties. In Sec. 3.2, we have constructed such a charge. We only require that the nonzero modes of the vector and axial vector currents satisfy the conservation and anomalous conservation laws respectively.

In order to calculate the divergences of the currents, we need the commutator of the current, $[\tilde{j}^+(x), \tilde{j}^+(y)]$. By a straightforward calculation, we get

$$\begin{aligned}[\tilde{j}^+(x), \tilde{j}^+(y)] = \frac{1}{(2L)^2}[(\sum_{n=0}^{\infty} \exp\{-i\frac{\pi}{L}(n+\frac{1}{2})(x-y)\})^2 \\ - (\sum_{n=0}^{\infty} \exp\{i\frac{\pi}{L}(n+\frac{1}{2})(x-y)\})^2],\end{aligned} \tag{C.5}$$

where the sums do not converge. We make them convergent by adding (or subtracting) a small imaginary part in the exponents. We get

$$\begin{aligned}[\tilde{j}^+(x), \tilde{j}^+(y)] &= \frac{1}{4\pi}\lim_{\epsilon\to 0}[\frac{1}{(x-y+i\epsilon)^2} - \frac{1}{(x-y-i\epsilon)^2}] \\ &= \frac{i}{2\pi}\delta'(x-y) + \mathcal{O}(L^{-2}).\end{aligned} \tag{C.6}$$

In this way, we can reproduce the correct Schwinger term in the "continuum" limit.

It is now easy to calculate the current divergences. By using the anomalous commutation relation (C.6), one get

$$\partial_+ \tilde{j}^+(x) = -i[\tilde{j}^+(x), P^-] = im(:\psi_L^\dagger \psi_R(x) - \psi_R^\dagger \psi_L(x):)_\sim + \frac{e}{2\pi}\partial_- \tilde{A}^-. \tag{C.7}$$

The spatial derivative of $\tilde{j}^-(x)$ is

$$\partial_- \tilde{j}^-(x) = -im(:\psi_L^\dagger \psi_R(x) - \psi_R^\dagger \psi_L(x):)_\sim - \frac{e}{2\pi}\partial_- \tilde{A}^-. \tag{C.8}$$

From these we get the divergences of the vector current and the axial vector current:

$$\partial_\mu \tilde{j}^\mu(x) = \partial_+ \tilde{j}^+(x) + \partial_- \tilde{j}^-(x) = 0, \tag{C.9}$$

$$\begin{aligned} \partial_\mu \tilde{j}_5^{\ \mu}(x) &= \partial_+ \tilde{j}^+(x) - \partial_- \tilde{j}^-(x) \\ &= 2im(:\bar{\psi}\gamma_5\psi:)_\sim + \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu \tilde{A}_\nu, \end{aligned} \tag{C.10}$$

where we use the relation $\gamma^\mu\gamma_5 = -\epsilon^{\mu\nu}\gamma_\nu, (\epsilon^{+-} = -1)$.

How about the axial charge? As explained in the text, it is formally equal to the (vector) charge. But because the axial vector current is not conserved, we expect that the axial charge is *not* conserved. In conclusion, there is no such a charge on the light-cone. Remember that the left-handed field ψ_L is not an independent field. The independent fields are ψ_R and $\overset{o}{A}_-$. It is well-known that axial-vector transformations are inconsistent on the light cone[35], i.e., they are inconsistent with the constraint equation (3.11). What if one wants to define the axial-vector transformations only for the independent field ψ_R? Because of $\gamma_5\psi_R = \psi_R$ it is equivalent to the usual (vector) phase transformations. One cannot define an axial-vector transformation, different from the usual (vector) phase transformation, in a self-consistent way. It means that the axial charge, which is supposed to be the generator of the transformation does not exit.

Mustaki proposed another definition of the axial-vector current which is conserved even for massive fermions[35]. Does it lead us to another definition of axial charge? Unfortunately it does not. Mustaki's conserved current is nothing but the vector current in the massive Schwinger model.

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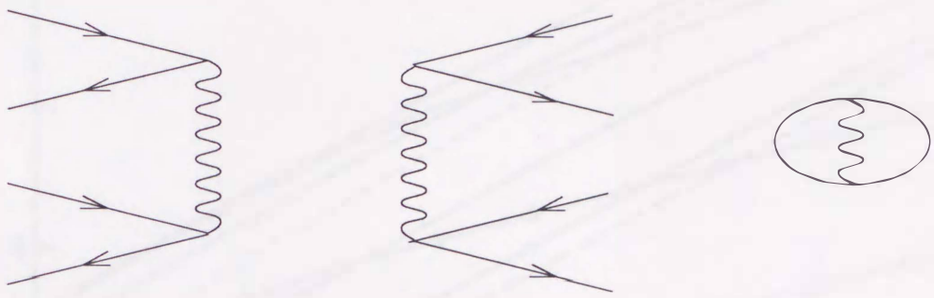


Fig. 1: These are the examples for the interactions forbidden in the light-cone formulation, which exist in the (usual) equal time quantization. The first one is the annihilation process of the four body fermionic state to the vacuum. The second is its reverse process. The third is vacuum bubble diagram in which the fermion-antifermion pair emerge from the vacuum and vanish into the vacuum after changing one photon. From this feature we can make sense the Tamm-Dancoff approximation in the light-front quantization scheme.

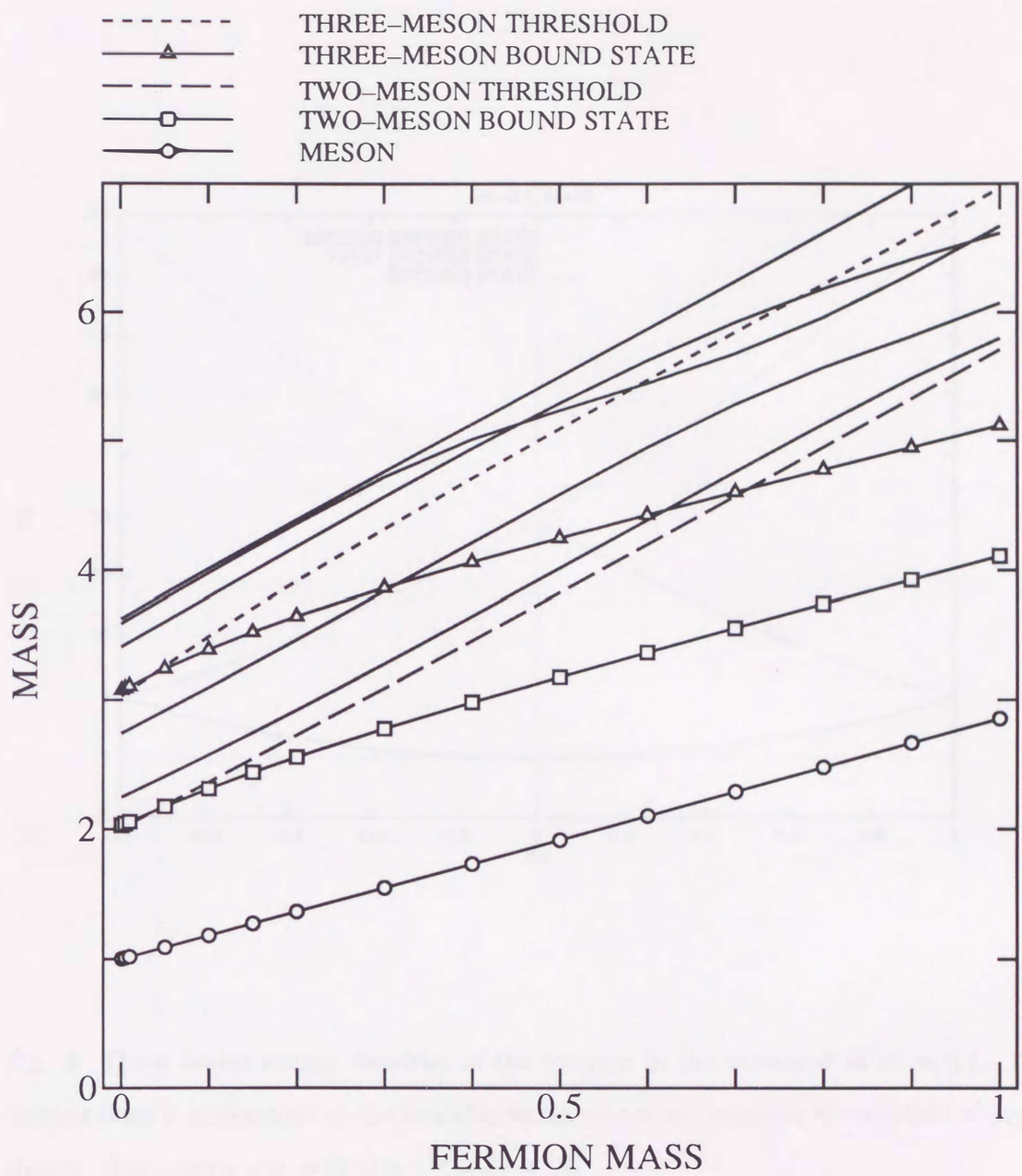


Fig. 2: Fermion mass dependence of the mass eigenvalues. Fermion mass m is in the unit of $e/\sqrt{\pi}$. The dashed and dotted lines stand for the two-meson and three-meson thresholds, respectively.

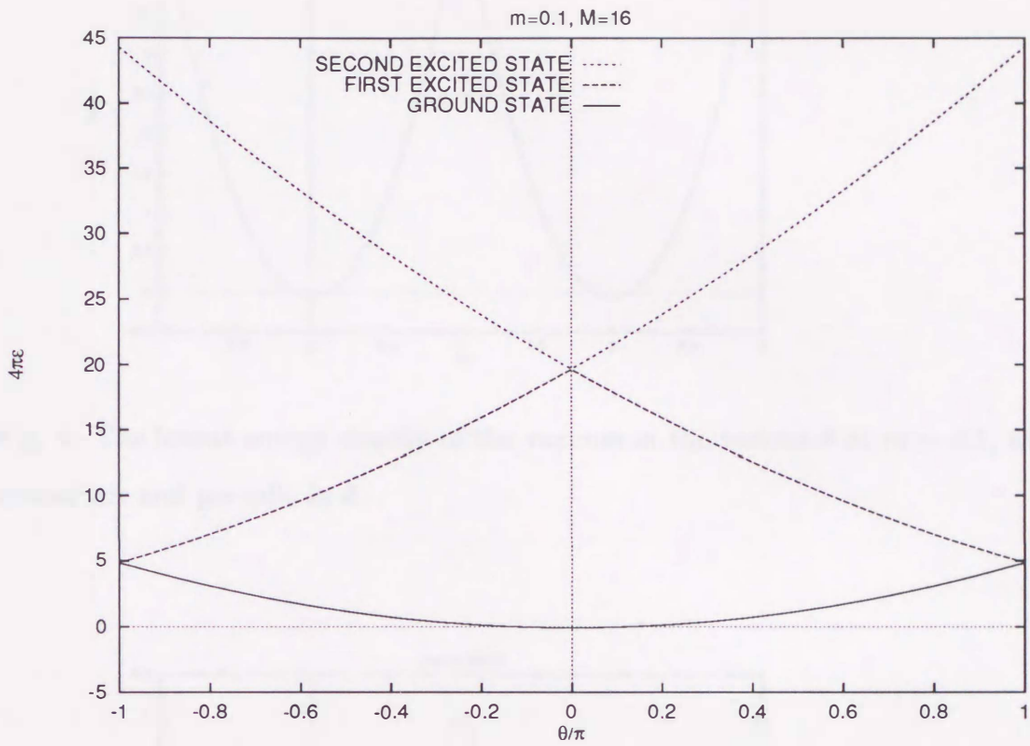


Fig. 3: Three lowest energy densities of the vacuum in the various θ at $m = 0.1$. The excited state is understood as the unstable vacua where the constant electric field is larger than π. The matrix size is $(2M+1)^2 = 33 \times 33$.

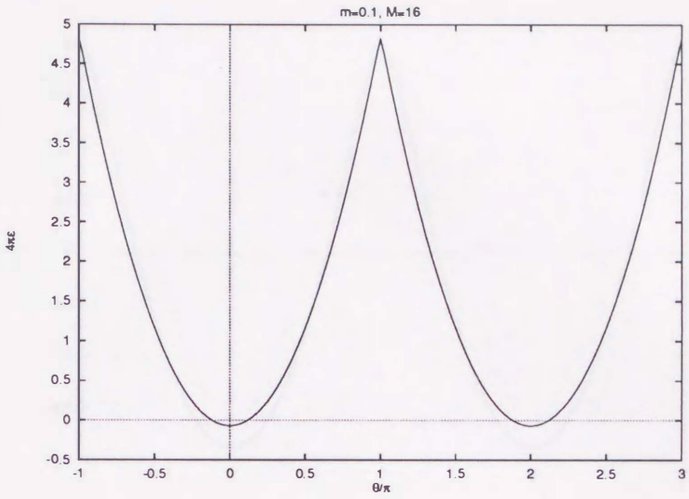


Fig. 4: The lowest energy density of the vacuum in the various θ at $m = 0.1$, which are symmetric and periodic in θ.

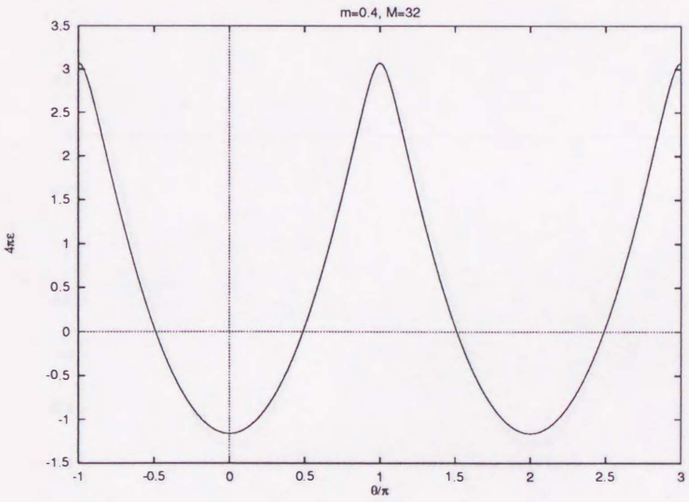


Fig. 5: The lowest energy density of the vacuum in the various θ at $m = 0.4$, which is symmetric and periodic in θ.

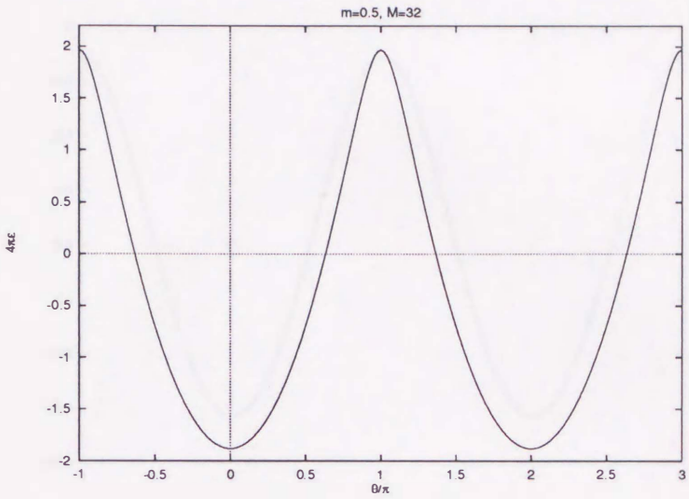


Fig. 6: The lowest energy density of the vacuum in the various θ at $m = 0.5$, which is symmetric and periodic in θ.

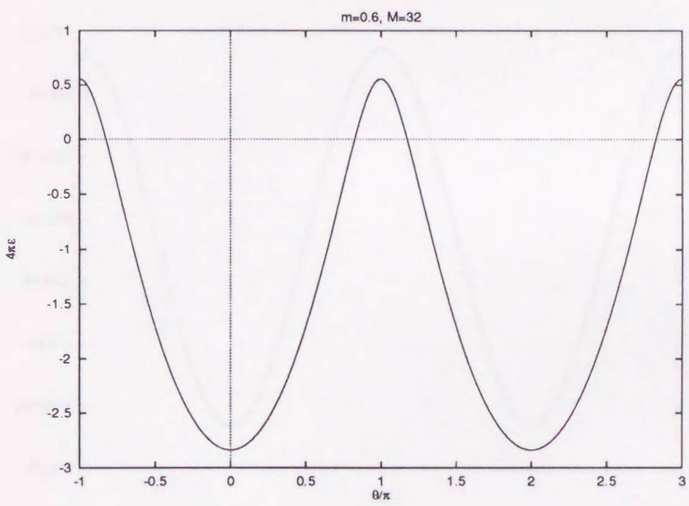


Fig. 7: The lowest energy density of the vacuum in the various θ at $m = 0.6$, which is symmetric and periodic in θ.

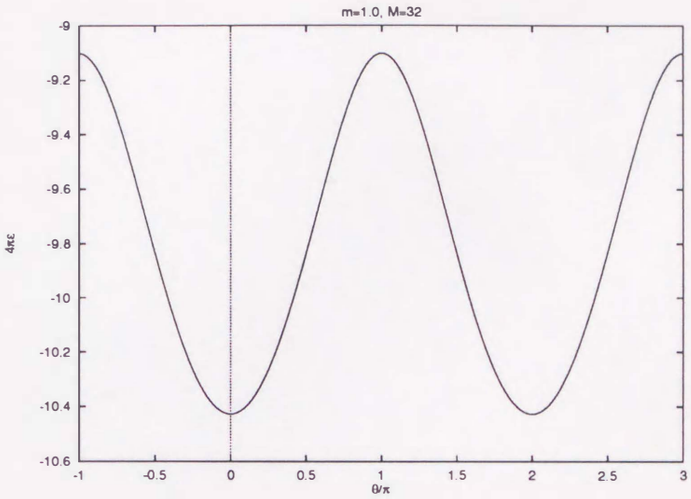


Fig. 8: The lowest energy density of the vacuum in the various θ at $m = 1.0$, which is symmetric and periodic in θ.

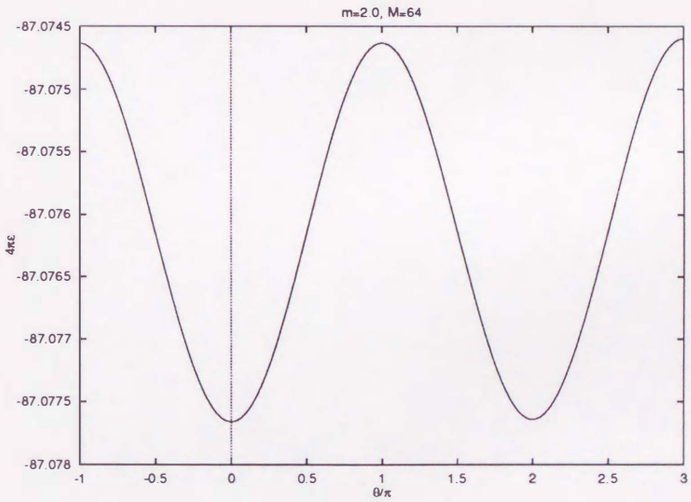
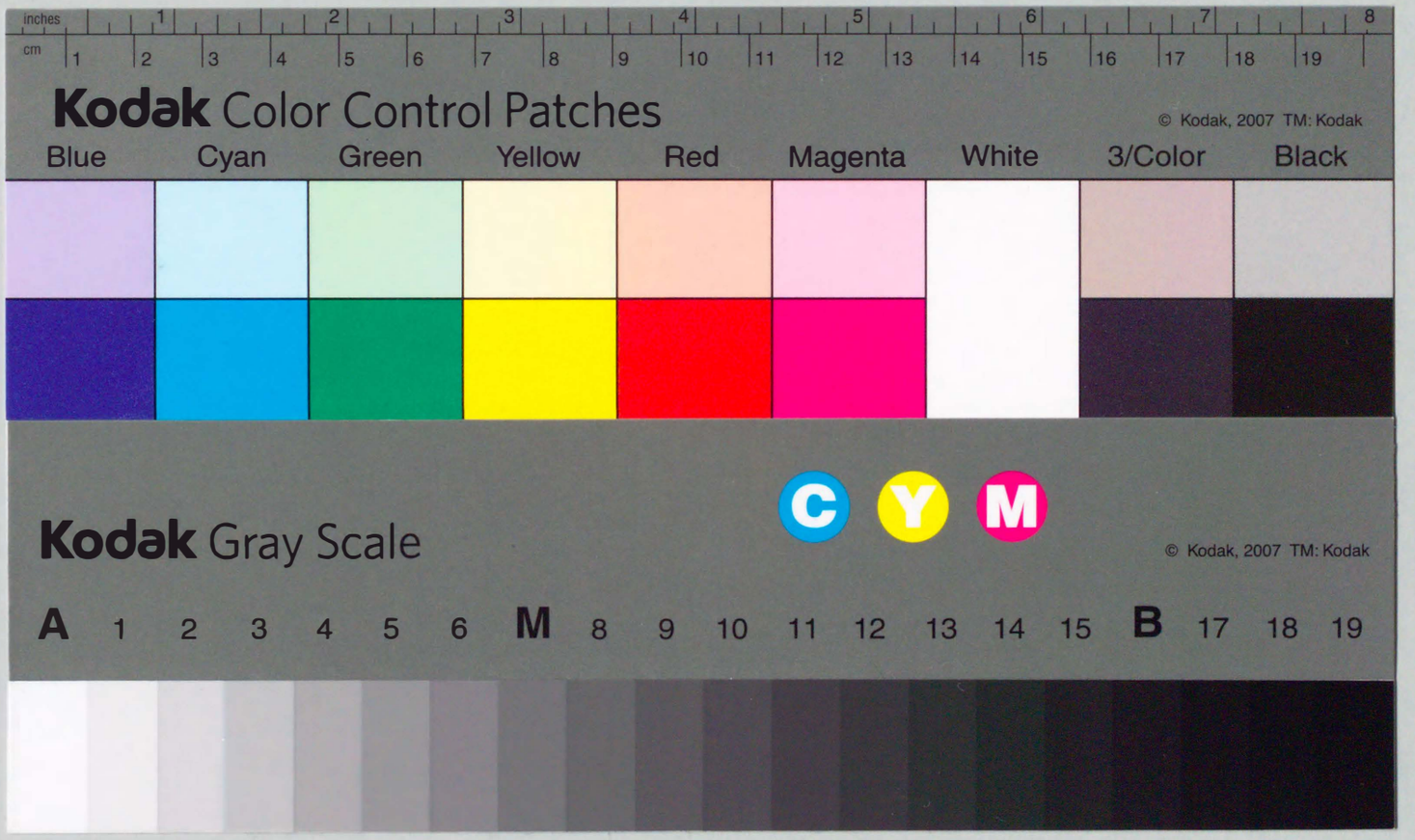


Fig. 9: The lowest energy density of the vacuum in the various θ at $m = 2.0$, which is symmetric and periodic in θ.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