ハーディー空間に関する研究

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STUDIES ON HARDY SPACES
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Let $n \geq 1$ be an integer. Let $H(B_n)$ denote the space of all holomorphic functions in the open unit ball $B_n$ of the $n$-dimensional Euclidean space $\mathbb{C}^n$. Every function $f$ in $H(B_n)$ has its nonnegative order $\text{Ord}_f$ of the Nevanlinna characteristic $T_r(f)$, $0 \leq r < 1$. All the functions in $H(B_n)$ are ranged into uncountable classes according to their order. The set $\Omega_0(B_n)$ of those functions in $H(B_n)$ which are of zero order is thus a very 'thin' subclass of the space $H(B_n)$, nevertheless it is a very significant object of study in the analytic function theory in $B_n$. The Hardy spaces $H^p(B_n)$, $0 < p \leq \infty$, and the Nevanlinna space $N(B_n)$ are extremely important subspaces of $\Omega_0(B_n)$. The inclusion relations between these subspaces are as follows:

$$H^\infty(B_n) \subset H^q(B_n) \subset H^p(B_n) \subset N(B_n) \quad \text{if} \quad 0 < p < q \leq \infty.$$  


The present thesis is devoted to mainly on the zero sets of the Hardy spaces $H^p(B_n)$, $n \geq 1$, $0 < p \leq \infty$, and the inclusion relation between them. We call a nonnegative real-valued function $\varphi$ defined on $[0, \infty)$ a modulus function if it is a nondecreasing and nonconstant function
such that \( \phi(t) = \phi(e^t) \) is a convex function on \((-\infty, \infty)\). For each modulus function \( \phi \), we can define a Hardy-Orlicz space \( H_\phi(B_n) \). The Hardy spaces \( H^p(B_n) \), \( 0 < p \leq \infty \), and the Nevanlinna space \( N(B_n) \) are special Hardy-Orlicz spaces. In this thesis, we also treat general Hardy-Orlicz spaces \( H_\phi(B_n) \).

It is well known that (in the case of the dimension \( n=1 \)) all the spaces \( H^p(B_1) \), \( 0 < p \leq \infty \), and the Nevanlinna space \( N(B_1) \) admit the same zero sets which are completely characterized by the Blaschke condition. When \( n \geq 2 \), the situation is considerably more complicated.

In [14], R.O.Kujala proved the following:

1. The zero sets of \( N(B_n) \) satisfy a generalized Blaschke condition (P).
2. The zero sets of \( H_\infty(B_n) \) have a property (Q) of the Blaschke condition type.
3. There exists a non-constant function \( f \in N(B_n) \) whose zero set \( Z(f) \) is a determining set for \( H_\infty(B_n) \), that is, if \( g \in H_\infty(B_n) \) vanishes on \( Z(f) \), then \( g \equiv 0 \) in \( B_n \). Hence the zero sets of \( N(B_n) \) are different from those of \( H_\infty(B_n) \).

To each of these three results, Kujala raised the following problems:

(a) Is the necessary condition (P) in (1) also sufficient?
(b) Is the necessary condition (Q) in (2) also sufficient?
(c) Can determining sets for \( H_\infty(B_n) \) or \( N(B_n) \) be reasonably characterized?

The problem (a) was solved affirmatively by G.M.Henkin [8] and H.Skoda [25], independently. The zero sets of the Nevanlinna space \( N(B_n) \) are thus characterized completely by the (generalized) Blaschke condition (P). The second problem (b) was solved negatively by the
author in [15]. One of our purposes here is to discuss the last problem (c). This thesis consists of six parts. Two parts (Part 3 and Part 5) of them are concerned with the problem (c). In Part 3, we characterize completely the determining sets for \( H(B^\infty_n) \), by using the Henkin-Skoda theorem. (See Theorem 1 in §3.3.) The characterization of the determining sets for \( H^p(B^n) \), \( 0 < p \leq \infty \), is much more complicated. We shall only show the existence of various determining sets and non-determining sets for \( H^p(B^n) \), \( 0 < p \leq \infty \). (See Theorem 2 ~ Theorem 5 in §3.4 and §3.5.) One of these results is as follows:

**Theorem.** Let \( n \geq 3 \) be an integer. Then there exists a function \( f \in \bigcap_{0 < p < \infty} H^p(B^n) \) satisfying the following two conditions:

1. The zero set of \( f \) is not a determining set for \( H^\infty(B^n) \).
2. The zero-divisor \( v_f \) does not equal \( v_g \) for any \( g \in H^\infty(B^n) \).

In Part 5, we prove that the above theorem is still valid when \( n = 2 \).

In [20], W. Rudin proved that, if \( n \geq 2 \) and \( 0 < p < q < \infty \), then the zero sets of \( H^p(B^n) \) are different from those of \( H^q(B^n) \). C. Horowitz [10] and J. H. Shapiro [23] proved analogous results regarding the zero sets of the Bergman spaces \( A^p(B^n) \), \( n \geq 1 \), \( 0 < p < \infty \). In Part 1, we shall establish the following \( 1 \)-dimensional version of the Rudin’s theorem:

**Theorem.** (Theorem 1 in §1.1) Assume \( \varphi \) and \( \psi \) are two modulus functions. If \( \lim_{t \to \infty} \frac{\psi(t)}{\varphi(t)} = \infty \), then there exists an \( f \in H^\varphi(B^1) \) such that \( 2f \notin H^\psi(B^1) \).

By applying this theorem and the Rudin’s theorem, we shall describe the strict inclusion relation between the Hardy spaces \( H^p(B^n) \), \( n \geq 1 \), \( 0 < p \leq \infty \). For example, Theorem 3 in §1.3 is as follows:
\[ \bigcup_{p<q<\infty} H^q(B_n) \subseteq H^p(B_n) \subseteq \bigcap_{0<q<p} H^q(B_n) \quad (0<p<\infty). \]

For a given modulus function \( \varphi \), the (generalized) Bergman space \( A_\varphi(B_n) \) is defined as
\[
A_\varphi(B_n) = \{ f \in H(B_n) : \int_{B_n} \varphi(|f|) \lambda < \infty \},
\]
where \( \lambda \) is the Lebesgue measure on \( \mathbb{C}^n \). If \( \varphi(t) = t^P \), \( 0<p<\infty \), then \( A_\varphi(B_n) \) are the ordinary Bergman spaces \( A^P(B_n) \). In Part 2, we prove the following generalization of the Shapiro's theorem:

**Theorem.** (Theorem 1 in §2.1) Let \( \varphi \) and \( \psi \) be modulus functions. Assume that \( \lim_{t \to \infty} \varphi(t)/\varphi(t+1) = \infty \) and that there exists a real number \( T \) such that \( \varphi(T) > 0 \) and \( \sup_{t \geq T} \varphi(t+1)/\varphi(t) < \infty \). Then there exists an \( f \in A_\varphi(B_1) \) with the following property:

If \( m \) is a positive integer, \( b \in H^\infty(B_1) \), \( g \in H(B_1) \), \( g \not= 0 \), and \( h = (f^m + b)g \), then some constant multiple of \( h \) fails to be in \( A_\psi(B_1) \), where \( \psi_m(t) = \psi(t/m) \).

By making use of this theorem, we shall study the inclusion relation between \( A^P(B_1) \), \( 0<p<\infty \), \( N(B_1) \) and \( H^P(B_1) \), \( 0<p<\infty \). (See §2.4 and §2.5.)

In Part 4, we first prove two other generalizations of the Shapiro's theorem, to "weighted" Bergman spaces on the one hand, and to the case of the dimension \( n \geq 2 \) on the other hand. (See Theorem 1 and Theorem 2 in §4.3.) By using them, we shall amplify the Rudin's theorem and the Horowitz-Shapiro's theorem. For example, Theorem 3 in §4.4 is as follows:

**Theorem.**

(a) For any \( p \in (0,\infty) \) and any integer \( n \geq 1 \),
\[
\nu(\bigcup_{p<q<\infty} A^q(B_n)) \subseteq \nu(A^P(B_n)) \subseteq \nu(\bigcap_{0<q<p} A^q(B_n)).
\]
(b) For any $p \in (0, \infty)$ and any integer $n \geq 2$,
\[
\nu\left( \bigcup_{p < q < \infty} H^q(B_n) \right) \subseteq \nu(H^p(B_n)) \subseteq \nu\left( \bigcap_{0 < q < p} H^q(B_n) \right).
\]
Here $\nu$ denotes the divisor map from $H(B_n)$ into $\mathbb{D}^*(B_n)$ (see §5.2).

Each Hardy-Orlicz space $H_{\varphi}(B_n)$ contains a subclass $H^+_{\varphi}(B_n)$ of the Smirnov type. In 1990, Z. Jianzhong conjectured that, for given two modulus functions $\varphi$ and $\psi$, $H^+_{\varphi}(B_1) = H^+_{\psi}(B_1)$ if and only if $H_{\varphi}(B_1) = H_{\psi}(B_1)$. In Part 6, we prove that this conjecture is true not only on the unit disc $B_1$ of $\mathbb{C}$ but also on the unit ball $B_n$ of $\mathbb{C}^n$ for any dimension $n \geq 1$.

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Part 1

The Strict Inclusion Relation between the Spaces $H_\phi (U)$ on the Open Unit Disc $U$ in $\mathbb{C}$
1.1. Introduction.

Let $n \geq 1$ be an integer. Let $H(B_n)$ denote the space of all holomorphic functions in the open unit ball $B_n$ of the complex $n$-dimensional Euclidean space $\mathbb{C}^n$. Let $\varphi:(-\infty, \infty) \to [0, \infty)$ be a non-decreasing convex function, not identically 0, and let $H_{\varphi}(B_n)$ be the class of all $f \in H(B_n)$ whose growth is restricted by the requirement

$$\sup_{0<r<1} \int_{\partial B_n} \varphi(\log |f(r\omega)|) d\sigma(\omega) < \infty,$$

where $\partial B_n$ is the boundary of $B_n$ and $\sigma$ is the Euclidean volume element on the unit sphere $\partial B_n$ in $\mathbb{C}^n$ normalized so that the volume of the sphere is 1. If $\varphi(x) = \max\{0, x\}$, then $H(B_n)$ is called to be the Nevanlinna space $N(B_n)$. If $\varphi(x) = e^{px}$, $0 < p < \infty$, then $H_{\varphi}(B_n)$ are called to be the Hardy spaces $H^p(B_n)$. By $N(B_n)$ we shall denote the space of all bounded holomorphic functions in $B_n$.

In [20], W. Rudin proved the following theorem:

Theorem A (Rudin [20], p.58). Fix $n \geq 2$. Assume that $\varphi$ and $\psi$ are nonconstant, nondecreasing, nonnegative convex functions defined on $(-\infty, \infty)$, and that

$$\lim_{t \to \infty} \psi(t)/\varphi(t) = \infty.$$

Then there exists an $f \in H_{\varphi}(B_n)$ with the following property:

If $b \in H_{\psi}(B_n)$, $g \in H(B_n)$, $g \neq 0$, and

$$h = (f+b)g,$$

then some constant multiple of $h$ fails to be in $H_{\psi}(B_n)$.

In the case $n=1$, this theorem is not valid. Indeed, if $\varphi = e^{px}$, $0 < p < \infty$, and $\psi = (2 + p^2x^2)e^{px}$, then Theorem A implies that the zero sets of functions in $H^p(B_1)$ differ from the zero sets of functions in $H^q(B_1)$, for any $q > p$. But this is false when $n=1$. 

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The main purpose of Part 1 is to prove the following analogue of Theorem A in the case of \( n=1 \):

(The open unit disc in \( \mathbb{C} \) and the unit circle will be denoted by \( U \) and \( T \), in place of \( B_1 \) and \( \partial B_1 \), respectively.)

**Theorem 1.** Assume that \( \varphi \) and \( \psi \) are as in Theorem A. Then there exists an \( f \in H_\varphi(U) \) such that \( 2f \notin H_\psi(U) \).

Applying Theorem A and Theorem 1, in §1.3 we shall describe the strict inclusion relation between the Hardy spaces \( H^p(B_1^n) \), \( 0<p<\infty \), and the Nevanlinna space \( N(B_1^n) \).

### 1.2. Proof of Theorem 1 in §1.1.

We need the lemma which was used to prove the Rudin’s theorem (Theorem A in §1.1) in [20].

**Lemma** (Rudin [20], pp.59-60). Suppose

(a) \( \mu \) is a finite positive measure on a set \( \Omega \);

(b) \( \nu \) is a real measurable function on \( \Omega \), with \( 0<\nu<1 \) a.e., whose essential supremum is \( 1 \);

(c) \( \Lambda \) is a continuous nondecreasing real function on \( (0,\infty) \), with \( \Lambda(0)=0 \) and \( \Lambda(t)\to \infty \) as \( t\to \infty \);

(d) \( 0<\delta<\infty \).

Then there exist constants \( c_k \in (0,\infty) \), for \( k=1,2,3,\ldots \), such that

\[
\int_{\Omega} \Lambda(c_k \nu^k) d\mu = \delta.
\]

These \( c_k \) also satisfy

\[
\lim_{k\to \infty} c_k \alpha^k = 0
\]

whenever \( |\alpha|<1 \).

If \( 0<t<\infty \) and if \( Y_k(t) \) is the set of all \( x \in \Omega \) at which
Proof of Theorem 1. Without loss of generality, we can assume that
\[
\varphi(t) = 0 \quad \text{if} \quad t \leq 0.
\]
Choose a sequence \( \{X_j\}_{j=1,2,3,\ldots} \) of nonempty connected open subsets of \( T \) so that
\[
X_j \subset \{ e^{i\theta} \in T: 0 < \theta < \frac{\pi}{2} \}
\]
and \( X_j \cap X_k = \emptyset \) if \( j \neq k \). For each \( j = 1,2,3,\ldots \), pick \( u_j \in X_j \).

Define
\[
s_j(z) = \frac{1}{2}(u_j^{-1}z + 1) \quad (z \in \mathbb{C})
\]
for each \( j \) and
\[
D = \{ z \in \mathbb{C}: |z - \frac{1}{2}| = \frac{1}{2} \}.
\]
Then
\[
s_j(U) = D, \quad s_j(u_j) = 1,
\]
\[
\max_{z \in T} |s_j(z)| = \max_{z \in U} |s_j(z)| = 1,
\]
and
\[
|s_j(z)| < 1 \quad \text{for} \quad z \in \mathbb{U}, \quad z \neq u_j.
\]
Moreover, the following inequalities hold:
\[
\frac{1}{2}(1+r)|s_j(z)| < |s_j(rz)| \leq |s_j(z)|
\]
for \( 0 < r < 1, \ z \in \bigcup_{i=1}^{t} X_i \). In fact, fix \( r \in (0,1) \). Put
\[
V(z) = \frac{s_j(rz)}{s_j(z)} = \frac{u_j^{-1}rz+1}{u_j^{-1}z+1}.
\]
Then \( V(T) = \{ w \in \mathbb{C}: \text{Re } w = \frac{1+r}{2} \} \). Hence
\[
\min_{z \in T} |V(z)| = \frac{1+r}{2}.
\]
Let $z \in \bigcup_{i=1}^{\infty} X_i$. Then, by (2), we can write $z = u_j e^{i\theta_j}$ for some $\theta_j$ with $-\frac{\pi}{2} < \theta_j < \frac{\pi}{2}$. A simple calculation shows that

$$|V(z)|^2 = \frac{1 + r^2 + 2r \cos \theta_j}{2 + 2 \cos \theta_j} < 1,$$

since $-\frac{\pi}{2} < \theta_j < \frac{\pi}{2}$ and $0 < r < 1$. (5), (6) and (7) give (4).

Since $\lim_{t \to \infty} \psi(t)/\varphi(t) = \infty$, there are numbers $t_j > 3j$ such that

$$\psi(\log(1+t)) > j^3 \psi(\log(1+t)) \quad \text{if} \quad t > t_j.$$

We now apply the lemma, for each positive integer $j$, with $(T, \sigma)$ in place of $(\Omega, \mu)$, and with

$$v_j(z) = |s_j(z)|,$$
$$\Lambda(t) = \varphi(\log(1+t)),$$
$$\delta_j = 2j^{-2},$$
$$\alpha_j = \max_{z \in T \setminus X_j} |s_j(z)|.$$

Then $0 < \alpha_j < 1$.

The lemma shows that there exist positive numbers $a_j = a_{k_j}$ (where $k_j$ is a sufficiently large positive integer) such that, setting

$$F_j(z) = a_j |s_j(z)|^{k_j} \quad (z \in C),$$

we have

$$\int_T \psi(\log(1+|F_j|)) d\sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\log(1+|F_j(e^{i\theta})|)) d\theta = 2j^{-2},$$

$$|F_j(z)| < 2^{-j} \text{ on } T \setminus X_j \text{ and for } |z| < 1 - j^{-1},$$

$$\int_{Y_j} \psi(\log(1+|F_j|)) d\sigma > j^{-2},$$

where $Y_j = \{z \in T: |F_j(z)| > t_j\}.$

By (11), $Y_j \subset X_j$. By (8) and (12),

$$\int_{Y_j} \psi(\log(1+|F_j|)) d\sigma > j.$$
We now define

\[(14) \quad f(z) = \sum_{j=1}^{\infty} F_j(z) \quad (z \in U).\]

The series converges uniformly on compact subsets of \(U\), by (11). Hence \(f \in \mathcal{H}(U)\).

To prove that \(f \in B_0(U)\), for \(N=1,2,3,\ldots\), define

\[(15) \quad H_N(z) = |F_1(z) + \cdots + F_N(z)| \quad (z \in \mathbb{C}),\]

\[(16) \quad H(z) = \sum_{j=1}^{\infty} |F_j(z)| \quad (z \in T).\]

Since the sets \(X_j\) are disjoint, (11) implies that

\[
H(z) \leq \begin{cases} 
1 & \text{in } \bigcap_{j=1}^{\infty} X_j \\
1 + |F_j(z)| & \text{in } X_j.
\end{cases}
\]

It follows from (1) that

\[
|\varphi(\log H(z))| \leq \begin{cases} 
0 & \text{in } \bigcap_{j=1}^{\infty} X_j \\
|\varphi(\log(1 + |F_j(z)|))| & \text{in } X_j.
\end{cases}
\]

Hence (10) implies

\[(17) \quad \int_T \varphi(\log H) d\sigma \leq \sum_{j=1}^{\infty} 2j^{-2} = 3^{-1} \pi^2 < 4.
\]

Since \(F_1 + \cdots + F_N\) is a holomorphic function in \(\mathbb{C}\), \(\log H_N\) is subharmonic in \(\mathbb{C}\), for each \(N\), and so is \(\varphi(\log H_N)\), because \(\varphi\) is convex and nondecreasing. Moreover, \(H_N(z) \leq H(z)\) for \(z \in T\), by (15) and (16).

It follows from (17) that

\[(18) \quad \int_T \varphi(\log H_N(rz)) d\sigma(z) \leq \int_T \varphi(\log H) d\sigma < 4
\]

for \(0 < r < 1\). If we fix \(r\) and let \(N \to \infty\), \(H_N(rz) \to |f(rz)|\) uniformly on \(T\). Hence (18) gives

\[
\int_T \varphi(\log |f(rz)|) d\sigma(z) \leq 4 \quad (0 < r < 1).
\]

Thus \(f \in B_0(U)\).
We turn to proving that $2f \mathcal{H}_\psi(U)$. Fix $j \in \{1, 2, 3, \ldots\}$ and choose $r_j$ so that

$$0 < r_j < 1 \quad \text{and} \quad \left(1 - \frac{1 + r_j}{2}\right) \|F_j\|_\infty < 2^{-j},$$

where $\|F_j\|_\infty = \max_{z \in \Lambda_j} |F_j(z)| = a_j$. For $z \in \Lambda_j$, by (14), (9), (4), (11) and (19),

$$|f(r_j z)| = \left| \sum_{i=1}^\infty F_i(r_j z) \right|$$

$$\geq |F_j(r_j z)| - \sum_{i \neq j} |F_i(r_j z)|$$

$$\geq \frac{1 + r_j}{2} k_j \|F_j(z)\| \left(1 - 2^{-j}\right)$$

$$= |F_j(z)| \left(1 - \frac{1 + r_j}{2} k_j\right) \|F_j(z)\| - 1 + 2^{-j}$$

$$\geq |F_j(z)| \left(1 - \frac{1 + r_j}{2} k_j\right) \|F_j\|_\infty - 1 + 2^{-j}$$

$$> |F_j(z)| - 1.$$

Since $|F_j(z)| > t_j > 3j > 3$ for $z \in \Lambda_j$,

$$|2f(r_j z)| > |F_j(z)| + 1$$

for $z \in \Lambda_j$. It follows from (13) that

$$\int_{\Lambda_j} \psi(\log |2f(r_j z)|) d\sigma(z) > j.$$

Thus

$$\int_{\Lambda_j} \psi(\log |2f(r_j z)|) d\sigma(z) > j \quad (j = 1, 2, 3, \ldots),$$

so that,

$$\sup_{0 < r < 1} \int_{\Lambda_j} \psi(\log |2f(r z)|) d\sigma(z) = \infty.$$

This means $2f \mathcal{H}_\psi(U)$. The proof is complete.

1.3. The strict inclusion relation between the Hardy spaces $\mathcal{H}_n(B^n)$ and the Nevanlinna space $N(B^n)$.

By Theorem A and Theorem 1, we obtain the following

Theorem 2. Let $n \geq 1$ be an integer. Assume that $\psi$ and $\psi$ are
nonconstant, nondecreasing, nonnegative convex functions defined on 
\((-\infty, \infty)\), and that

\[
\lim_{t \to \infty} \frac{\psi(t)}{\varphi(t)} = \infty.
\]

Then there exists an \(f \in \mathcal{H}_\varphi(B_n)\) such that some constant multiple of \(f\) fails to be in \(\mathcal{H}_\psi(B_n)\).

Remark 1. If \(\psi\) satisfies the growth condition

\[
\limsup_{t \to \infty} \frac{\psi(t+1)}{\psi(t)} < \infty,
\]

then \(\mathcal{H}_\psi(B_n)\) is closed under scalar multiplication. (See Rudin [20], p.58.) In that case, the conclusion of Theorem 2 is simply

\[
\mathcal{H}_\varphi(B_n) \subset \mathcal{H}_\psi(B_n).
\]

Now we apply Theorem 2 to the description of the strict inclusion relation between the spaces \(\mathcal{H}_p(B_n), 0 < p < \infty,\) and the space \(N(B_n)\).

First we note that

\[
\mathcal{H}_p(B_n) \subset \mathcal{H}_q(B_n) \subset \mathcal{H}_P(B_n) \subset N(B_n)
\]

if \(0 < p < q < \infty\). For each \(p \in (0, \infty)\), we define

\[
\mathcal{H}_P^-(B_n) = \bigcup_{0 < q < p} \mathcal{H}_q(B_n), \quad \mathcal{H}_P^+(B_n) = \bigcap_{0 < q < p} \mathcal{H}_q(B_n).
\]

Then

\[
\mathcal{H}_P^-(B_n) \subset \mathcal{H}_P(B_n) \subset \mathcal{H}_P^+(B_n) \quad (0 < p < \infty).
\]

**Theorem 3.**

\[
\mathcal{H}_P^-(B_n) \not\subset \mathcal{H}_P(B_n) \not\subset \mathcal{H}_P^+(B_n) \quad (0 < p < \infty).
\]

**Proof.** (cf. Rudin [20], p.59, (c).)

(1) Put

\[
\varphi(t) = e^{pt} \quad (-\infty < t < \infty),
\]

\[
\psi(t) = \begin{cases} 
te^{pt} & (t \geq 0) \\ 0 & (t < 0). \end{cases}
\]

Then \(\varphi\) and \(\psi\) satisfy the assumptions in Theorem 2. Moreover, \(\psi\)
satisfies the condition
\[ \lim_{t \to +\infty} \frac{\psi(t+1)}{\psi(t)} = e^p < \infty. \]

It follows from Remark 1 that
\[ H^p_\psi(B_n) \subset H^p(B_n) = H^p(B_n). \]
Since \( H^p_\psi(B_n) \subset H^p_\psi(B_n) \), this implies
\[ H^p_\psi(B_n) \subset H^p(B_n). \]

(ii) Put
\[ \varphi(t) = \begin{cases} 
  t^{-1}e^{pt} & (t \geq p^{-1}) \\
  1 & (t < p^{-1}). 
\end{cases} \]

\[ \psi(t) = e^{pt} \quad (-\infty < t < \infty). \]

Then Theorem 2 and Remark 1 imply that
\[ H^p(B_n) = H^p_\psi(B_n) \subset H^p(B_n) \subset H^p(\bar{B}_n). \]

Remark 2. In the case \( n=1 \), some outer functions give another proof of Theorem 3. Let \( f \) be a positive measurable function on \( T \) such that \( \log f \in L^1(T) \). Define
\[ Q_f(z) = \exp \left\{ (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log f(e^{it}) \, dt \right\} \quad (z \in \mathbb{U}). \]

Then \( Q_f(z) \) is called the outer function with respect to the function \( f \). We note the following theorem:

**Theorem B** (See e.g. Rudin [19], Theorem 7.16.). Fix \( p \in (0, \infty) \).
Let \( f \) be a positive measurable function on \( T \) such that \( \log f \in L^1(T) \).
Then \( Q_f \in H^p(\mathbb{U}) \) if and only if \( f \in L^p(T) \).

Now fix \( p \in (0, \infty) \). Put
\[ f(e^{it}) = \begin{cases} 
  t^{-1/p}(-\log t)^{-2/p} & (0 < t < e^{-1}) \\
  1 & (t \in [-\pi, \pi] \setminus (0, e^{-1})), \\
  t^{-1/p} & (t \in [-\pi, \pi]). 
\end{cases} \]

Then
\[ f \in L^p(T), \ g \in L^q(T), \ p < q < \infty \]
\[ g \notin L^p(T), \ g \notin \bigcap_{0 < q < p} L^q(T) \]

and

\[ \log f \notin L^1(T), \ \log g \notin L^1(T). \]

(See Hardy-Littlewood-Pólya [6], §6.1.) It follows from Theorem B that

\[ Q_f \in H^p(U), \ Q_f \notin \bigcup_{p < q < \infty} H^q(U) = H^p_-(U), \]
\[ Q_g \in H^p(U), \ Q_g \notin \bigcap_{0 < q < p} H^q(U) = H^p_+(U). \]

**Theorem 4.**

\[ H^\omega(B_n) \subset \bigcap_{0 < p < \infty} H^p(B_n). \]

**Proof.**

With

\[ \varphi(t) = \left\{ \begin{array}{ll} \exp(t^2) & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{array} \right. \]
\[ \psi(t) = \left\{ \begin{array}{ll} \exp(t^3) & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{array} \right. \]

Theorem 2 establishes the existence of an \( f \in H_\varphi(B_n) \) such that \( c\phi \psi(B_n) \) for some constant \( c \). Note that

\[ H^\omega(B_n) \subset H_\psi(B_n) \subset H_\varphi(B_n) \subset \bigcap_{0 < p < \infty} H^p(B_n). \]

Hence \( f \notin \bigcap_{0 < p < \infty} H^p(B_n) \) but \( f \in H_\psi(B_n) \).

**Remark 3.** In the case \( n=1 \), as well as in Theorem 3, some outer functions give another proof of Theorem 4. Put

\[ f(\phi t) = \left\{ \begin{array}{ll} \log t & (0 < t \leq e^{-1}) \\ 1 & (t \in [-\pi, \pi] \setminus (0, e^{-1})) \end{array} \right. \]

Then \( f \notin \bigcap_{0 < p < \infty} L^p(T) \setminus L^\omega(T) \) and \( \log f \notin L^1(T) \). (See [6], §6.1.) It follows from Theorem B that

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Theorem 5.

Proof. Put

\[ \varphi(t) = \max\{0, t\} \quad (-\infty < t < \infty), \]

\[ \psi(t) = \begin{cases} \exp(\sqrt{t}) & (t \geq 1) \\ e & (t < 1) \end{cases}. \]

Then \( \varphi \) and \( \psi \) satisfy the assumptions in Theorem 2. In addition, \( \psi \) satisfies the growth condition

\[ \lim_{t \to \infty} \frac{\psi(t+1)}{\psi(t)} = 1 < \infty. \]

Hence

\[ \bigcup_{0<p<\infty} H^p(B_n) \subseteq H^\varphi(B_n) \subseteq H^\psi(B_n) = N(B_n). \]

Remark 4. In the case \( n=1 \), a simple function gives another proof of Theorem 5. Put

\[ f(z) = \exp(\frac{1+z}{1-z}) \quad (z \in U). \]

Then \( f \in N(U) \), \( |f^*| = 1 \) a.e., and

\[ \log |f(0)| = 1 > 0 = (2\pi)^{-1} \int_{-\pi}^{\pi} \log |f^*(e^{it})| \, dt. \]

Here \( f^* \) denotes the radial limits of \( f \). (See Rudin [19], §17.19.)

If \( f \in \bigcup_{0<p<\infty} H^p(U) \), then

\[ \log |f(0)| \leq (2\pi)^{-1} \int_{-\pi}^{\pi} \log |f^*(e^{it})| \, dt. \]

(See [19], Theorem 17.17.) Hence we conclude that

\[ f \in \bigcup_{0<p<\infty} H^p(U). \]
Part 2

The Inclusion Relation between
the Bergman Spaces $A^p(U)$,
the Hardy Spaces $H^p(U)$
and the Nevanlinna Space $N(U)$
on the Open Unit Disc in $\mathbb{C}$
2.1. Introduction.

By $U$ and $T$ we shall denote the open unit disc in the complex plane $\mathbb{C}$ and the unit circle, respectively. The space of all holomorphic functions in $U$ will be denoted by $H(U)$. Let $\varphi:(-\infty,\infty) \rightarrow [0,\infty)$ be a nondecreasing convex function, not identically 0, and let $H_\varphi(U)$ (resp. $A_\varphi(U)$) be the class of all $f \in H(U)$ whose growth are restricted by the requirement

$$\sup_{0<r<1} (2\pi)^{-1} \int_0^\pi \varphi(\log |f(re^{i\theta})|)d\theta < \infty$$

(resp. $\pi^{-1} \int_0^{2\pi} \varphi(\log |f(re^{i\theta})|)rdrd\theta < \infty$).

If $\varphi(x) = \max\{0, x\}$, $H_\varphi(U)$ is said to be the Nevanlinna space $N(U)$ and $A_\varphi(U)$ will be denoted by $BN(U)$. If $\varphi(x) = e^{\alpha x}$, $0 < \alpha < \infty$, then $H_\varphi(U)$ (resp. $A_\varphi(U)$) are said to be the Hardy spaces $H^p(U)$ (resp. the Bergman spaces $A^p(U)$). By $H^\infty(U)$ we shall denote the space of all bounded holomorphic functions in $U$.

In [23], J.H. Shapiro proved the following theorem:

**Theorem C ([23], Theorem 2.1).** Assume $\varphi$ and $\psi$ are strictly positive, convex, increasing, unbounded functions defined on $(-\infty,\infty)$, and that

$$\sup_{-\infty < t < \infty} \varphi(t+1)/\varphi(t) < \infty, \quad \sup_{-\infty < t < \infty} \psi(t+1)/\psi(t) < \infty,$$

$$\lim_{t \to -\infty} \varphi(t) = 0, \quad \lim_{t \to -\infty} \psi(t) = 0,$$

$$\lim_{t \to \infty} \psi(t)/\varphi(t) = \infty.$$

Then there exists an $f \in A_\varphi(U)$ with the following property:

If $n$ is a positive integer, $b \in H^\infty(U)$, $g \in H(U)$, $g \not\equiv 0$, and

$$h = (f^n + b)g,$$

then $h \in A_{\psi_n}(U)$, where $\psi_n(t) = \psi(t/n)$. 

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(In fact, Shapiro proved this theorem more generally on weighted Bergman spaces.)

Applying Theorem 1 in Part 1, we studied there the inclusion relation between the Hardy spaces $H^p(U)$, $0 < p < \infty$. The main purpose of Part 2 is to study the inclusion relation between the Bergman spaces $A^p(U)$, $0 < p < \infty$, the Nevanlinna space $N(U)$ and the Hardy spaces $H^p(U)$, $0 < p < \infty$. To do so, we need the following generalization of Theorem C:

**Theorem 1.** Let $\varphi$ and $\psi$ be nonconstant, nondecreasing, non-negative, convex functions on $(-\infty, \infty)$. Assume that

$$\lim_{t \to \infty} \frac{\psi(t)}{\varphi(t)} = \infty,$$

and that there exists a number $t_0 \in (-\infty, \infty)$ such that $\varphi(t_0) > 0$ and

$$\sup_{t \geq t_0} \frac{\psi(t+1)}{\varphi(t)} < \infty.$$

Then there exists an $f \in A^\varphi(U)$ with the following property:

If $n$ is a positive integer, $b \in H^\varphi(U)$, $g \in H(U)$, $g \not\equiv 0$, and

$$h = (f^n + b)g,$$

then some constant multiple of $h$ fails to be in $A^\varphi(U)$.

**2.2. Preliminaries.**

It is easily shown that

$$H^\varphi(U) \subset H^\psi(U) \subset H^p(U) \subset N(U),$$

$$A^\varphi(U) \subset A^\psi(U) \subset A^p(U) \subset BN(U)$$

if $0 < p < q < \infty$. Here we write $A^\infty(U) = A^\infty(U)$. For each $p \in (0, \infty)$, we define

$$H^{p_1}_-(U) = \bigcup_{p < q < \infty} H^q(U), \quad H^{p_1}_+(U) = \bigcap_{0 < q < p} H^q(U),$$

$$A^{p_1}_-(U) = \bigcup_{p < q < \infty} A^q(U), \quad A^{p_1}_+(U) = \bigcap_{0 < q < p} A^q(U).$$

Then
\[ H^P(U) \subset A^P(U) \subset H^P(U), \]
\[ A^P(U) \subset A^P(U) \subset A^P(U). \]

Let \( f \in H(U) \). Take a point \( a \in U \). Assume \( f \neq 0 \) in \( U \). Then a power series

\[ f(z) = \sum_{k=m}^{\infty} c_k (z-a)^k \]

converges in some neighborhood of \( a \) and represents \( f \) in this neighborhood. Here \( c_\infty \neq 0 \). The integer

\[ v_f(a) = m \geq 0 \]

is called the zero multiplicity of \( f \) at \( a \). The integer-valued function \( v_f \) defined in \( U \) is called the zero-divisor of \( f \).

Let \( \mu \) be a nonnegative integer-valued function defined in \( U \). Then \( \mu \) is called a positive divisor on \( U \) if and only if it is locally the zero-divisor of some holomorphic function, that is, for each point \( a \in U \) there exist a connected neighborhood \( V \) of \( a \) and a holomorphic function \( f \) in \( V \) such that \( f \neq 0 \) and \( \mu = v_f \) in \( V \).

We denote by \( D^+(U) \) the set of all positive divisors on \( U \). Then we have the divisor map \( v \) from \( H(U) \) into \( D^+(U) \) defined by letting \( v(f) \) for \( f \) in \( H(U) \) be \( v_f \). Here, for any subspace \( X \) of \( H(U) \) we write

\[ X^* = \{ f \in X : f \neq 0 \text{ in } U \}. \]

We recall that \( \mu \in D^+(U) \) satisfies the Blaschke condition if and only if

\[ \sum_{z \in U} \mu(z)(1-|z|) < \infty. \]

The set of positive divisors on \( U \) which satisfy the Blaschke condition will be denoted by \( D_0 \). The following classical theorem will be used in §2.5:

**Theorem D** (See e.g. Duren [2], §2.2.). For any \( p \in (0, \infty) \),
The following is an immediate consequence of Theorem 1:

**Theorem 2.** Assume that \( \varphi \) and \( \psi \) are as in Theorem 1. In addition, assume that \( \psi \) satisfies the condition

\[
\limsup_{t \to \infty} \frac{\psi(t+1)}{\psi(t)} < \infty.
\]

Then

\[
\nu(A_{\psi}(U)^*) \subseteq \nu(A_{\varphi}(U)^*)
\]

Hence

\[
A_{\psi}(U) \subseteq A_{\varphi}(U).
\]

2.2. **Proof of Theorem 1 in §2.1.**

Our proof is a modification of the Shapiro's one of Theorem C. (cf. Shapiro [23], pp.248-251.)

**Step 1.** Without loss of generality, we can assume that

(1) \( \varphi(t) = 0 \) if \( t \leq 0 \).

In fact, when \( \varphi(0) > 0 \), we put

(2) \( \varphi_0(t) = \begin{cases} \varphi(t) - \varphi(0) & (t > 0) \\ 0 & (t \leq 0) \end{cases} \).

Then \( \varphi_0 \) has the same properties as \( \varphi \) does. In addition, \( \varphi_0 \) satisfies (1). Because of (2), \( \varphi - \varphi_0 \) is bounded, hence \( A_{\varphi}(U) = A_{\varphi_0}(U) \).

For \( t \geq 0 \), define

(3) \( \Phi(t) = \varphi(\log t), \ \Phi_0(t) = \varphi(\log t+1), \)
\( \Psi(t) = \psi(\log t) \).

Then \( \Phi_0 \) is a continuous nondecreasing nonnegative function on \( (0, \infty) \) and \( \Phi_0(t) \to \infty \) as \( t \to \infty \). By (1), \( \Phi_0(0) = 0 \). Since \( \lim_{t \to \infty} \psi(t)/\varphi(t+1) = \infty \),

(4) \( \lim_{t \to \infty} \frac{\psi(t)}{\Phi_0(t)} = \infty \).
Put
\[ H = \sup_{t \geq t_0} \varphi(t+1)/\varphi(t). \]
Since \( \varphi \) is nondecreasing, it follows from (3) that
\[ \Phi(s+t) \leq H(\Phi(s)+\Phi(t)) \quad (s \geq s_0, t \geq s_0), \]
where \( s_0 = \exp(t_0) \). And \( 1 < H < \infty \), by the hypothesis.

**Step 2.** By \( \lambda \) we shall denote the Lebesgue measure on \( \mathbb{C} = \mathbb{R}^2 \), so normalized that \( \lambda(\mathbb{U}) = 1 \). We now apply Rudin's lemma (in §1.2), for each positive integer \( k \), with \( (U, \lambda) \) in place of \( (\Omega, \mu) \), and with
\[ v(z) = |z|, \]
\[ \Lambda(t) = \Phi_0(t), \]
\[ \delta = (k^2 H)^{-1}. \]
Then the following holds:

**Lemma 1.** There exist sequences \{c_kn\} \( n = 1, 2, 3, \ldots \) of real numbers such that
(a) \( \int_U \Phi_0(1|c_kn z^n|) d\lambda = (k^2 H)^{-1}; \)
(b) \( 0 < c_{k1} \leq c_{k2} \leq c_{k3} \leq \ldots, \lim_{n \to \infty} c_{kn} = \infty; \)
(c) \( \lim_{n \to \infty} c_{kn} \gamma^n = 0 \) whenever \( |\gamma| < 1; \)
(d) \( \lim_{n \to \infty} \int_{\{|c_kn z^n| > t\}} \Phi_0(1|c_kn z^n|) d\lambda = (k^2 H)^{-1} \) for each \( t > 0. \)

**Lemma 2.** There exist four sequences \( \{t_k\}, \{a_k\}, \{r_k\} \) and \( \{\rho_k\} \) of real numbers, and one sequence \( \{n_k\} \) of integers with
\[ 0 < t_1 < t_2 < t_3 < \ldots, \lim_{k \to \infty} t_k = \infty, \]
\[ 0 < a_1 < a_2 < a_3 < \ldots, \lim_{k \to \infty} a_k = \infty, \]
\[ 0 < n_1 < n_2 < n_3 < \ldots, \lim_{k \to \infty} n_k = \infty, \]
\[ 0 < r_1 < \rho_1 < r_2 < \rho_2 < \ldots, \lim_{k \to \infty} r_k = \lim_{k \to \infty} \rho_k = 1. \]
such that if \( u_k(z) = a_k z^{n_k} \) and \( R_k = \{ z \in \mathbb{C} : r_k < |z| \leq \rho_k \} \), then for \( k \geq 2 \) the following conditions hold:

(a) \( t_k \geq 4 \sum_{j=1}^{k-1} a_j \) and \( \Psi(t)/\Phi_0(t) > kH \) if \( t \geq t_k \);

(b) \( \int U \Phi_0(|u_k'|)d\lambda = (k^{-2H})^{-1} \);

(c) \( \int R_k \Phi_0(|u_k'|)d\lambda > (2k^{-2H})^{-1} \);

(d) \( |u_k(z)| \geq t_k \) if \( |z| \geq r_k \);

(e) \( |u_k(z)| \leq |u_{k-1}(z)|/5 \) if \( r_1 \leq |z| \leq \rho_{k-1} \).

Proof. We prove the lemma by induction. Choose any positive integer \( n_1 \) and any positive numbers \( t_1, a_1, r_1, \rho_1 \), with \( s_0 < t_1 < a_1 \), and \( 0 < r_1 < \rho_1 < 1 \). Suppose \( k \geq 2 \), and suppose the five sequences have been successfully chosen for all indices less than or equal to \( k-1 \). By (4), there exists a positive number \( t_k \) such that

\[
  t_k > t_{k-1}, \quad t_k > a_{k-1}, \quad t_k > 4 \sum_{j=1}^{k-1} a_j, \quad \Psi(t)/\Phi_0(t) > kH \quad \text{for} \quad t \geq t_k.
\]

By Lemma 1, there exists a positive integer \( n_k \) with \( n_{k-1} < n_k \), such that, letting \( a_k = c_k n_k \), we have

\[
  a_k > t_k, \quad a_k > a_{k-1}, \quad a_k \rho_{k-1} < a_{k-1} r_1, \quad a_k \rho_{k-1} < a_{k-1} \rho_{k-1}/5, \quad \int_U \Phi_0(|a_k z^{n_k}|)d\lambda = (k^{-2H})^{-1},
\]

and

\[
  \int \{ |a_k z^{n_k}| > t_k \} \Phi_0(|a_k z^{n_k}|)d\lambda > (2k^{-2H})^{-1}.
\]

Put

\[
  r_k = (t_k/a_k)^{1/n_k}.
\]

Then \( r_{k-1} < r_k < 1 \), by (7). Because of (8) and (9), there exists a

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positive number $\rho_k$ with $r_k < \rho_k < 1$ such that

$$\int \{r_k < |z| \leq \rho_k \} \Phi_0(\{a_k z^\frac{n}{2k}\}) d\lambda > (2k^2 H)^{-1}.$$ 

This completes the proof of the lemma.

**Step 3.** We now define

$$f(z) = \sum_{k=1}^{\infty} u_k(z) \quad (z \in U).$$ 

The series converges uniformly on compact subsets of $U$, by (6-e). Hence $f \in H(U)$.

**Lemma 3.**

(10) $|f| \leq 5|u_k|/4 + 5|u_{k+1}|/4$ on $\{r_k \leq |z| \leq \rho_{k+1}\}$.

(11) $|f| \geq |u_k|/2$ on $R_k$.

**Proof.** By (6-a) and (6-d),

$$\sum_{j=1}^{k-1} |u_j| \leq |u_k|/4 \quad \text{on} \quad \{z \leq r_k\}.$$ 

By (6-e),

$$\sum_{j=k+1}^{\infty} |u_j| \leq 5|u_{k+1}|/4 \quad \text{on} \quad \{r_1 \leq |z| \leq \rho_{k+1}\},$$

$$\sum_{j=k+1}^{\infty} |u_j| \leq |u_k|/4 \quad \text{on} \quad \{r_1 \leq |z| \leq \rho_k\}.$$ 

(10) and (11) follow from (12), (13) and (14).

**Lemma 4.** $f \in A_\phi(U)$.

**Proof.**

$$\int_{U} \Phi(\log |f|) d\lambda = \int_{U} \Phi(|f|) d\lambda$$

$$= \int \{z \leq r_1\} \Phi(|f|) d\lambda + \sum_{k=1}^{\infty} \int \{r_k \leq |z| \leq r_{k+1}\} \Phi(|f|) d\lambda.$$ 

Fix $k \in \{1, 2, 3, \ldots\}$. By (10),

$$\int \{r_k \leq |z| \leq r_{k+1}\} \Phi(|f|) d\lambda \leq \int \{r_k \leq |z| \leq r_{k+1}\} \Phi(5|u_k|/4 + 5|u_{k+1}|/4) d\lambda.$$ 

Put
\[ E_1 = \{ z \in \mathbb{C} : r_k < |z| \leq r_{k+1}, \ 5|u_{k+1}(z)|/4 \geq s_0 \}, \]
\[ E_2 = \{ z \in \mathbb{C} : r_k < |z| \leq r_{k+1}, \ 5|u_{k+1}(z)|/4 < s_0 \}. \]

By (6-d),
\[ 5|u_k(z)|/4 \geq |u_k(z)| \geq t_k > s_0 \quad \text{if} \quad |z| \geq r_k. \]

It follows from (5) that
\[
\int_{E_1} \Phi(5|u_k|/4+5|u_{k+1}|/4)d\lambda \leq M\int_{E_1} \Phi(5|u_k|/4)d\lambda + M\int_{E_1} \Phi(5|u_{k+1}|/4)d\lambda
\]
\[ = M\int_{E_1} \Phi(\log|u_k|+\log(5/4))d\lambda + M\int_{E_1} \Phi(\log|u_{k+1}|+\log(5/4))d\lambda
\]
\[ \leq M\int_{E_1} \Phi(\log|u_k|+1)d\lambda + M\int_{E_1} \Phi(\log|u_{k+1}|+1)d\lambda
\]
\[ = M\int_{E_1} \Phi_0(|u_k|)d\lambda + M\int_{E_1} \Phi_0(|u_{k+1}|)d\lambda.
\]

On the other hand, by (6-d),
\[
\int_{E_2} \Phi(5|u_k|/4+5|u_{k+1}|/4)d\lambda \leq \int_{E_2} \Phi(5|u_k|/4+s_0)d\lambda
\]
\[ \leq \int_{E_2} \Phi(5|u_k|/4+t_k)d\lambda \leq \int_{E_2} \Phi(9|u_k|/4)d\lambda
\]
\[ = \int_{E_2} \Phi(\log|u_k|+\log(9/4))d\lambda \leq \int_{E_2} \Phi(\log|u_k|+1)d\lambda
\]
\[ = \int_{E_2} \Phi_0(|u_k|)d\lambda.
\]

Thus
\[
\int_{\{r_k < |z| \leq r_{k+1}\}} \Phi(|f|)d\lambda
\]
\[ \leq M\int_{E_1} \Phi_0(|u_k|)d\lambda + M\int_{E_1} \Phi_0(|u_{k+1}|)d\lambda + \int_{E_2} \Phi_0(|u_k|)d\lambda
\]
\[ \leq M\int_{U} \Phi_0(|u_k|)d\lambda + M\int_{U} \Phi_0(|u_{k+1}|)d\lambda.
\]

It follows from (6-b) that
\[
\int_{\{r_k < |z| \leq r_{k+1}\}} \Phi(|f|)d\lambda \leq k^{-2} + (k+1)^{-2}.
\]

Hence
\[
\int_{U} \Phi(\log|f|)d\lambda \leq \int_{\{|z| \leq r_1\}} \Phi(|f|)d\lambda + \sum_{k=1}^{\infty} \{k^{-2}+(k+1)^{-2}\} < \infty.
\]

This means \( f \in A_{\Phi}(U) \).
Step 5. Let \( n \) be a positive integer, \( b \in \mathcal{H}(U) \), \( g \in \mathcal{H}(U)^* \), and 
\[ h = (f^n + b)g. \]

Put 
\[ \alpha = (2\pi)^{-1} \int_{-\pi}^{\pi} \log |g(r_1 e^{i\theta})| d\theta, \]
\[ \beta = \sup_{z \in U} |b(z)|. \]

Then \( 0 \leq \beta < \infty \). Since \( \log |g| \) is subharmonic in \( U \), 
\[ (15) \quad -\infty < \alpha \leq (2\pi)^{-1} \int_{-\pi}^{\pi} \log |g(re^{i\theta})| d\theta < \infty \quad (r_1 \leq r < 1). \]

Choose a positive number \( c \) so that 
\[ (16) \quad \log c + \alpha - n \log 4 > 0. \]

Lemma 5. \( c h \not\in \mathcal{A}_n(U) \).

Proof. Define 
\[ \Psi_n(t) = \psi_n(\log t) \quad (t > 0). \]

Then 
\[ \int_U \psi_n(\log |ch|) d\lambda = \int_U \psi_n(|ch|) d\lambda \]
\[ \geq \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} \psi_n(|ch|) d\lambda \]
\[ = \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} 2r dr (2\pi)^{-1} \int_{-\pi}^{\pi} \psi_n(|ch(re^{i\theta})|) d\theta. \]

Fix \( r \in (r_k, r_{k+1}) \). By Jensen's convexity theorem and (15), 
\[ (2\pi)^{-1} \int_{-\pi}^{\pi} \psi_n(|ch(re^{i\theta})|) d\theta = (2\pi)^{-1} \int_{-\pi}^{\pi} \psi_n(\log |ch(re^{i\theta})|) d\theta \]
\[ \geq \psi_n((2\pi)^{-1} \int_{-\pi}^{\pi} \log |ch(re^{i\theta})| d\theta) \]
\[ = \psi_n(\log c + (2\pi)^{-1} \int_{-\pi}^{\pi} \log |f^n + b(re^{i\theta})| d\theta + (2\pi)^{-1} \int_{-\pi}^{\pi} \log |g(re^{i\theta})| d\theta) \]
\[ \geq \psi_n(\log c + \alpha + (2\pi)^{-1} \int_{-\pi}^{\pi} \log |f^n + b(re^{i\theta})| d\theta). \]

Since \( \lim_{k \to \infty} t_k = \infty \), there exists a positive integer \( K \) such that 
\[ -21 - \]
(19) 
\[(t_{k/4})^n > 1 \text{ if } k \geq K.\]

By (11), (19) and (6-d),
\[|f_n+b| \geq |f_n^\beta > (|u_k|/2)^n - (t_{k/4})^n \geq (|u_k|/2)^n \geq (|u_k|/4)^n\]
for \(k \geq K\) and \(z \in \mathbb{R}_k\). Hence, for \(k \geq K\) and \(r \in (r_k, \rho_k]\),
\[(20) \quad (2\pi)^{-1} \int_{-\pi}^{\pi} \log |f_n+b(re^{i\theta})| d\theta \geq (2\pi)^{-1} \int_{-\pi}^{\pi} n \log |u_k(re^{i\theta})| d\theta - n \log 4.
\]

By (18), (20) and (16), for \(k \geq K\) and \(r \in (r_k, \rho_k]\),
\[(2\pi)^{-1} \int_{-\pi}^{\pi} \psi_n(|\log (re^{i\theta})|) d\theta \geq \psi_n(\log \sigma + \alpha - n \log 4 + (2\pi)^{-1} \int_{-\pi}^{\pi} n \log |u_k(re^{i\theta})| d\theta)
\]
\[\geq \psi_n((2\pi)^{-1} \int_{-\pi}^{\pi} \log |u_k(re^{i\theta})| d\theta)
\]
\[= \psi((2\pi)^{-1} \int_{-\pi}^{\pi} \log |u_k(re^{i\theta})| d\theta)
\]
\[= \psi(\log(a_k r^{k})) = \psi(\log u_k(r)) = \psi(u_k(r))
\]
\[= \psi((2\pi)^{-1} \int_{-\pi}^{\pi} \psi(|u_k(re^{i\theta})|) d\theta).
\]

It follows from (17) that
\[\int_u \psi_n(|\log| d\lambda \geq \sum_{k=K}^{\infty} \int_{R_k} \Psi_{\rho_k} 2\pi r (2\pi)^{-1} \int_{-\pi}^{\pi} \psi(|u_k(re^{i\theta})|) d\theta
\]
\[= \sum_{k=K}^{\infty} \int_{R_k} \Psi(|u_k|) d\lambda.
\]

By (6-a), (6-c) and (6-d),
\[\int_{R_k} \psi(|u_k|) d\lambda \geq k \delta \int_{R_k} \Phi_0(|u_k|) d\lambda > (2k)^{-1} \quad (k=1, 2, 3, \ldots).
\]

Thus
\[\int_u \psi_n(|\log| d\lambda \geq \sum_{k=K}^{\infty} (2k)^{-1} = \infty.
\]

This means \(\chi A_{\psi_n}(U)\). The proof of Theorem 1 is now complete.
2.4. The inclusion relation between the spaces $A^p(U)$.

Theorem 3. For any $p \in (0, \infty)$

$$
u(A^{p-}(U)^*) \subseteq \nu(A^p(U)^*) \subseteq \nu(A^{p+}(U)^*).$$

Consequently,

$$A^{p-}(U) \subseteq A^p(U) \subseteq A^{p+}(U) \quad (0 < p < \infty).$$

Proof (cf. Shapiro [23], Corollary 2.2; Horowitz [10], Theorem 4.6).

(1) Theorem 2 with

$$\varphi(t) = e^{pt} \quad (-\infty < t < \infty),$$

$$\psi(t) = \begin{cases} t e^{pt} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

implies that

$$\nu(A^{p-}(U)^*) \subseteq \nu(A_{\varphi}(U)^*) \subseteq \nu(A_{\psi}(U)^*) = \nu(A^p(U)^*).$$

(11) Theorem 2 with

$$\varphi(t) = \begin{cases} t^{-1} e^{pt} & (t > 0) \\ p e^{pt} & (t < 0) \end{cases},$$

$$\psi(t) = e^{pt} \quad (-\infty < t < \infty),$$

implies that

$$\nu(A^p(U)^*) = \nu(A_{\varphi}(U)^*) \subseteq \nu(A_{\psi}(U)^*) \subseteq \nu(A^{p+}(U)^*).$$

Theorem 4.

$$H^\infty(U) \subseteq \bigcap_{0 < p < \infty} A^p(U).$$

Proof. This is an immediate consequence of the following two facts:

(1) $H^p(U) \subseteq A^p(U) \quad (0 < p < \infty)$,

(11) $H^\infty(U) \subseteq \bigcap_{0 < p < \infty} H^p(U)$.

The latter is just the case of the dimension $n=1$ of Part 1, Theorem 4.

Theorem 5.
\[ \nu(\cup_{0<p<\infty} A^p(U)^*) \subsetneq \nu(BN(U)^*). \]

Consequently,
\[ \cup_{0<p<\infty} A^p(U) \subsetneq BN(U). \]

Proof. Put
\[
\begin{align*}
\varphi(t) &= \max(0,t) \quad (-\infty<t<\infty), \\
\psi(t) &= \exp(\sqrt{t}) \quad (t \geq 1), \\
\phi(t) &= \begin{cases} 
1 & (t<0), \\
0 & (t=0), \\
0 & (t>0).
\end{cases}
\end{align*}
\]
Then \( \varphi \) and \( \psi \) satisfy the assumptions in Theorem 2. Hence
\[ \nu(\cup_{0<p<\infty} A^p(U)^*) \subset \nu(A_\varphi(U)^*) \subsetneq \nu(A_\psi(U)^*) = \nu(BN(U)^*). \]

2.5. The inclusion relation between the spaces \( A^p(U), H^p(U) \) and \( \Lambda(U) \).

Theorem 6. Suppose \( 0<p<\infty \). Then we have
\[
(1) \quad H^p(U) \subsetneq A^p(U), \\
(1i1) \quad H^{p-}(U) \subsetneq A^{p-}(U), \\
(1ii1) \quad H^{p+}(U) \subsetneq A^{p+}(U).
\]

Proof. Choose \( q \) with \( p<q<\infty \). Then
\[ H^\infty(U) \subset A^q(U) \subset A^p(U), \]
so that,
\[ D_0 = \nu(H^\infty(U)^*) \subset \nu(A^q(U)^*) \subset \nu(A^p(U)^*). \]

On the other hand, by Theorem 3,
\[ \nu(A^q(U)^*) \subset \nu(A^{p-}(U)^*) \subset \nu(A^p(U)^*). \]

Hence
\[ D_0 \subset \nu(A^p(U)^*). \]

It follows from Theorem C that
\[ \nu(H^p(U)^*) \subset \nu(A^p(U)^*). \]
Since \( h^p(U) \subset A^p(U) \), this implies (i). The same arguments prove (ii) and (iii).

**Theorem 7.** For any \( p \in (0, \infty) \) and any \( q \in (0, \infty) \),
\[ A^p(U) \subseteq H^q(U). \]

**Proof.** If \( A^p(U) \subset H^q(U) \), then
\[ \nu(A^p(U)^\ast) \subseteq \nu(H^q(U)^\ast) = \mathcal{D}_0. \]
But this is impossible.

**Theorem 8.** Suppose \( 0 < 2p < q < \infty \). Then
\[ h^p(U) \not\subseteq A^q(U). \]

**Proof.** For \( z \in U \), \( c \in (-\infty, \infty) \), define
\[ I_c(z) = (2\pi)^{-1} \int _{-\pi} ^{\pi} |1-ze^{it}|^{1-c} dt. \]
If \( c < 0 \), then \( I_c \) is bounded in \( U \). If \( c > 0 \), then there exists a positive constant \( M_c \) such that
\[ I_c(z) \geq M_c (1-|z|)^{-c} \quad (z \in U). \]
(See Rudin [21], Proposition 1.4.10.)

Choose \( a \) with \( 0 < a < p^{-1} - 2q^{-1} \). Put \( b = p^{-1} - a \). Then
\[ 0 < 2q^{-1} < b < p^{-1}. \]

Define
\[ f(z) = (1-z)^{-b} \quad (z \in U). \]
Then, for \( r \in (0, 1) \),
\[ (2\pi)^{-1} \int _{-\pi} ^{\pi} |f(re^{it})|^p dt = (2\pi)^{-1} \int _{-\pi} ^{\pi} |1-re^{it}|^{-bp} dt = (2\pi)^{-1} \int _{-\pi} ^{\pi} |1-re^{it}|^{-1+ap} dt = I_{-ap}(r). \]
Since \(-ap < 0\),
\[ \sup _{0<r<1} I_{-ap}(r) < \infty. \]
Hence \( f \in h^p(U) \).
We turn to proving that $f \notin A^q(U)$. Put $\gamma = bq-1$. Then $\gamma > 1$, since $2q^{-1} < b$. For $r \in (0,1)$

$$(2\pi)^{-1} \int_{-\pi}^{\pi} |f(re^{it})|^q \, dt = (2\pi)^{-1} \int_{-\pi}^{\pi} |1-re^{it}|^{-bq} \, dt = I_\gamma(r).$$

Since $\gamma > 1$,

$$\int_0^1 I_\gamma(r) \, dr > \int_0^1 N_\gamma(1-r)^{-\gamma} \, dr = \infty.$$

Hence $f \notin A^q(U)$.

**Theorem 9.**

$$U \ 
\cup \ H^p(U) \subseteq \ U \ 
\cup \ A^p(U). \quad 0 < p < \infty \quad 0 < p < \infty$$

**Proof.** This follows from the fact

$$\nu(\cup \ H^p(U)^*) = D_0 \subseteq \nu(\cup \ A^p(U)^*).$$

**Theorem 10.**

$$A^p(U) \not\subseteq N(U) \quad (0 < p < \infty).$$

**Proof.** This follows from the fact

$$\nu(N(U)^*) = D_0 \subseteq \nu(A^p(U)^*).$$

**Corollary.**

$$\cup \ A^p(U) \not\subseteq N(U). \quad 0 < p < \infty$$

**Theorem 11.**

$$N(U) \not\subseteq \cup \ A^p(U). \quad 0 < p < \infty$$

**Proof.** Define

$$f(z) = \exp\left(\frac{1+z}{1-z}\right) \quad (z \in U).$$

Then $f \in N(U)$. (See Rudin [19], §17.19.) A simple computation shows

$$\lim_{|z| \to 1} (1-|z|)^{2/p} |f(z)| = \infty \quad (0 < p < \infty).$$

If $f \in A^p(U)$, $0 < p < \infty$, then

$$\lim_{|z| \to 1} (1-|z|)^{2/p} |f(z)| = 0.$$
(See Rudin [21], Theorem 7.2.5.) Hence \( f \notin \bigcup_{0<p<\infty} A^p(U) \).

**Corollary.**

\[ \mathcal{H}(U) \notin A^p(U) \quad (0<p<\infty). \]
Part 3

Determining Sets for $N(U_n)$ and $H^P(U_n)$
3.1. Introduction.

Let $n \geq 2$ be an integer. Let $N(B_n)$ denote the Nevanlinna space on the open unit ball $B_n$ of the complex $n$-dimensional Euclidean space $\mathbb{C}^n$. Let $H^p(B_n)$, $0 < p \leq \infty$, denote the Hardy spaces on $B_n$. $H^\infty(B_n)$ is the space of all bounded holomorphic functions in $B_n$.

In [14], R.O. Kujala proposed three problems. One of them is on the complete characterization of the zero sets of functions in $N(B_n)$, by the Blaschke condition. This problem was solved affirmatively by G.M. Henkin [8] and H. Skoda [25], independently. (See below Theorem E in §3.2.) Secondly Kujala asked whether a certain necessary condition for the zero sets in $H^\infty(B_n)$ (which is easily obtained through the Jensen Formula) is also sufficient. This problem was solved negatively in [15]. In this Part 3 we shall study the Kujala's last problem. He asked ([14], p.260):

Can determining sets (or divisors) for $H^\infty(B_n)$ or $N(B_n)$ be reasonably characterized?

We shall consider when zero sets (of holomorphic functions) in $B_n$ are determining sets for $H^p(B_n)$, $0 < p \leq \infty$, or $N(B_n)$. By using the Henkin-Skoda theorem (Theorem E in §3.2), we can characterize completely the determining sets for $N(B_n)$ (Theorem 1 in §3.1). The characterization of the determining sets for $H^p(B_n)$, $0 < p \leq \infty$, is much more complicated. We shall only show the existence of various determining sets and non-determining sets for $H^p(B_n)$, $0 < p \leq \infty$. (See Theorem 2 ~ Theorem 5 in §3.4. and §3.5.)

3.2. Preliminaries.

Let $H(B_n)$ denote the space of all holomorphic functions in $B_n$. 
Put

\[ H(B_n)^* = \{ f \in H(B_n) : f \neq 0 \text{ in } B_n \}, \]
\[ N(B_n)^* = N(B_n) \cap H(B_n)^*, \]
\[ H^p(B_n)^* = H^p(B_n) \cap H(B_n)^* \quad (0 < p < \infty). \]

We note that

\[ H^\infty(B_n) \subset H^q(B_n) \subset H^p(B_n) \subset N(B_n) \subset H(B_n) \quad \text{if} \quad 0 < p < q < \infty. \]

Let \( f \in H(B_n)^* \). In a neighborhood of each point \( a \in B_n \), \( f \) can be expanded in a series of homogeneous polynomials:

\[ f(z) = \sum_{k=0}^{\infty} P_k(z-a). \]

The integer

\[ \nu_f(a) = \min \{ k \geq 0 : P_k \neq 0 \} \]

is called the zero multiplicity of \( f \) at \( a \). The integer-valued function \( \nu_f \) defined in \( B_n \) is called the zero-divisor of \( f \).

Let \( \mu \) be a nonnegative integer-valued function defined in \( B_n \). Then \( \mu \) is called a positive divisor on \( B_n \) if and only if it is locally the zero-divisor of some holomorphic function, that is, for each point \( a \in B_n \) there exist a connected neighborhood \( V \) of \( a \) and a holomorphic function \( f \) in \( V \) such that \( f \neq 0 \) and \( \mu = \nu_f \) in \( V \).

We denote by \( D^+(B_n) \) the set of all positive divisors on \( B_n \). Then we have the divisor map \( \nu \) from \( H(B_n)^* \) into \( D^+(B_n) \) defined by letting \( \nu(f) \) for \( f \) in \( H(B_n)^* \) be \( \nu_f \).

Let \( \mu \) be a positive divisor on the open unit disc \( B_1 \) in the complex plane. Define

\[ n_\mu(r) = \sum_{\lambda \in rB_1} \mu(\lambda) \quad \text{for } 0 < r \leq 1, \]

and
\[ N_\mu(r,s) = \int_s^r \frac{\mu(t)}{t} \, dt \]
for \( 0 < s < r \leq 1 \), where \( rB_1 = \{ \lambda \in \mathbb{C} : |\lambda| < r \} \).

Let \( n \geq 2 \) be an integer. Let \( \mu \in \mathcal{D}^+(B_n) \). Take a point \( \xi \in \partial B_n \), where \( \partial B_n \) is the boundary of \( B_n \). Define
\[ \mu[\xi](\lambda) = \mu(\lambda \xi) \quad \text{for} \quad \lambda \in B_1. \]
Put \( E = \{ \xi \in \partial B_n : \mu[\xi] \in \mathcal{D}^+(B_1) \} \). Then \( \sigma(\partial B_n \setminus E) = 0 \), where \( \sigma \) is the rotation invariant positive Borel measure on \( \partial B_n \) for which \( \sigma(\partial B_n) = 1 \). (See e.g. Stoll [28], p.13.) We write
\[ N_\mu(r,s;\xi) = N_{\mu[\xi]}(r,s) \]
if \( \xi \in E \), and we define
\[ N_\mu(r,s) = \int_{\partial B_n} N_\mu(r,s;\xi) \, d\sigma(\xi) \]
for \( 0 < s < r \leq 1 \).

Let \( f \in \mathcal{H}(B_n)^* \). We denote by \( Z(f) \) the zero set of \( f \):
\[ Z(f) = \{ z \in B_n : f(z) = 0 \}. \]
Then \( Z(f) = \{ z \in B_n : \nu_f(z) > 0 \} \).

We shall say that a positive divisor \( \mu \in \mathcal{D}^+(B_n) \) (resp. a zero set \( Z(f) \) of some \( f \in \mathcal{H}(B_n)^* \)) satisfies the Blaschke condition if and only if
\[ N_\mu(1,s) < \infty \quad \text{(resp.} \quad N_{\nu_f}(1,s) < \infty) \]
for some \( s \in (0,1) \). (See Kujala [14], p.252 and Stoll [28], p.41.)

**Theorem F** (Henkin [8]; Skoda [25]). For \( \mu \in \mathcal{D}^+(B_n) \), the following two conditions are equivalent:

(a) \( \mu \in \mathcal{V}(N(B_n)^*) \).

(b) \( \mu \) satisfies the Blaschke condition.

Let \( X \) be a subspace of \( \mathcal{H}(B_n) \). A zero set \( M \) in \( B_n \) (i.e. \( M = Z(g) \) for some \( g \in \mathcal{H}(B_n)^* \)) is said to be a determining set for \( X \) if the
assumptions $f \in \mathcal{X}$, $\mathcal{M} \subseteq \mathcal{Z}(f)$ force $f \equiv 0$. Here the symbol $\mathcal{M} \subseteq \mathcal{Z}(f)$ means the inclusion relation with multiplicity; i.e. for two zero sets $\mathcal{M}(=\mathcal{Z}(g))$ and $\mathcal{Z}(f)$, we write $\mathcal{M} \subseteq \mathcal{Z}(f)$ if and only if $\nu_g \leq \nu_f$ in $B_n$.

We recall some results about the Hardy spaces $H^p(B_n)$ and the Bergman spaces $A^p(B_n)$. Assume $0 < p < \infty$. For $f \in H(B_n)$, we define the $H^p$-norm and the $A^p$-norm as follows:

\[
\|f\|_{H^p} = \sup_{0 < r < 1} \frac{1}{r^n} \left\{ \int_{\partial B_r^n} |f(z)|^p \, d\sigma(z) \right\}^{1/p},
\]

\[
\|f\|_{A^p} = \left\{ \int_{B_1^n} |f(z)|^p \, d\lambda(z) \right\}^{1/p}.
\]

Here $\lambda$ is the Lebesgue measure on $\mathbb{C}^n$ normalized so that $\lambda(B_1^n) = 1$.

Then

\[
H^p(B_n) = \{ f \in H(B_n) : \|f\|_{H^p} < \infty \},
\]

\[
A^p(B_n) = \{ f \in H(B_n) : \|f\|_{A^p} < \infty \}.
\]

We note that $H^p(B_n) \subseteq A^p(B_n)$ ($0 < p < \infty$).

Suppose $n \geq 2$. Let $f$ and $g$ be functions defined in $B_n$ and $B_{n-1}$, respectively, and define a restriction operator $\rho$ and an extension operator $E$ by

\[
(\rho f)(z') = f(z',0) \quad (z' \in B_{n-1}),
\]

\[
(Eg)(z',z_n) = g(z') \quad ((z',z_n) \in B_n).
\]

We note that $\rho E$ is the identity operator on $B_{n-1}$.

The following two theorems will be used in §3.5:

**Theorem F** (Rudin [21], p.127). Assume $n \geq 2$, $0 < p < \infty$.

(a) The extension $E$ is a linear isometry of $A^p(B_{n-1})$ into $H^p(B_n)$.

(b) The restriction $\rho$ is a linear norm-decreasing map of $H^p(B_n)$ onto $A^p(B_{n-1})$.

**Theorem G** (Rudin [21], p.128). Assume $n \geq 1$, $0 < p < \infty$. If $f \in H^p(B_n)$, then $|f(z)| \leq 2^{n/p} \|f\|_{H^p}(1-|z|)^{-n/p}$ ($z \in B_n$).
3.3. Determining sets for $N(B^n)$.

Theorem 1. Suppose $n \geq 1$. Let $M$ be a zero set in $B^n$. Then $M$ is a determining set for $N(B^n)$ if and only if $M$ does not satisfy the Blaschke condition.

Proof. Suppose that $M$ satisfies the Blaschke condition. Then, by Theorem F, there exists a $g \in N(B^n)^*$ such that $Z(g) = M$. Hence $M$ cannot be a determining set for $N(B^n)$.

Conversely, suppose that $M$ is not a determining set for $N(B^n)$. Then there exists an $h \in N(B^n)^*$ such that $Z(h) \supset M$. By Theorem F, $Z(h)$ satisfies the Blaschke condition, so does $M$.

Since $N(B^n) \supset H^P(B^n)$ ($0 < p < \infty$), we obtain

Corollary. Suppose $n \geq 1$. Let $M$ be a zero set in $B^n$. If $M$ does not satisfy the Blaschke condition, then $M$ is a determining set for $H^P(B^n)$ ($0 < p < \infty$).

3.4. Determining sets for $H^P(B^n)$, $0 < p < \infty$.

In this section we need a result of Rudin [20]. In [20], Rudin remarked that the following theorem is obtained as a special case of Theorem A ([20], p.58) in §1.1.

Theorem H ([20], p.59). Let $n \geq 2$ be an integer and $0 < p < \infty$. Then there exists an $f \in H^P(B^n)$ such that $Z(f)$ is a determining set for $\bigcup_{q > p} H^Q(B^n) \supset H^\infty(B^n)$.

Now we show that Theorem A furnishes two similar results to Theorem H.

Theorem 2 (cf. Rudin [18], pp.60-62, Theorem 4.1.1.). Let $n \geq 2$
be an integer. Then there exists an \( f \in \bigcap_{0 < p < \infty} H^p(B_n) \) such that \( Z(f) \) is determining set for \( H^\infty(B_n) \).

**Proof.** Put

\[
\varphi(t) = \begin{cases} 
\exp(t^2) & \text{if } t > 0 \\
1 & \text{if } t < 0,
\end{cases} \quad \psi(t) = \begin{cases} 
\exp(t^3) & \text{if } t > 0 \\
1 & \text{if } t < 0.
\end{cases}
\]

Then \( \varphi \) and \( \psi \) satisfy the assumptions in Theorem A. Hence there is an \( f \in H^\varphi(B_n) \) such that \( f \) has the property described in the conclusion of Theorem A. We note that

\[
H^\varphi(B_n) \subset H^\psi(B_n) \subset H^\varphi(B_n) \subset \bigcap_{0 < p < \infty} H^p(B_n).
\]

Since \( f \in H^\varphi(B_n) \), we have \( f \in \bigcap_{0 < p < \infty} H^p(B_n) \).

Suppose \( h \in H^\varphi(B_n) \) and \( Z(h) \supseteq Z(f) \) i.e. \( \nu_h \geq \nu_f \) in \( B_n \). Then \( h = gf \) for some \( g \in H(B_n) \). If \( g \not= 0 \) in \( B_n \), then there is a constant \( c \in \mathbb{C} \) such that \( ch \notin H^\varphi(B_n) \). Since \( H^\varphi(B_n) \subset H^\psi(B_n) \), we have \( ch \notin H^\psi(B_n) \). This contradicts the assumption \( h \in H^\varphi(B_n) \). Therefore \( g \equiv 0 \) in \( B_n \), and so, \( h \equiv 0 \) in \( B_n \). Thus \( Z(f) \) is a determining set for \( H^\varphi(B_n) \).

**Theorem 3.** Let \( n \geq 2 \) be an integer. Then there exists an \( f \in H(B_n) \) such that \( Z(f) \) is a determining set for \( \bigcup_{0 < p < \infty} H^p(B_n) \).

**Proof.** Put

\[
\varphi(t) = \max(0, t), \quad (\varphi) (-\infty < t < \infty),
\psi(t) = \begin{cases} 
\exp(\sqrt{t}) & \text{if } t \geq 1 \\
e & \text{if } t < 1.
\end{cases}
\]

Then \( \varphi \) and \( \psi \) satisfy the assumptions in Theorem A. Hence we can find an \( f \in H^\varphi(B_n) \) with the property described in the conclusion of Theorem A. We note that
The same reasoning as in the proof of Theorem 2 shows that $Z(f)$ is a determining set for $\bigcup_{0<p<\infty} H^p(B_n)$.

3.5. Non-determining sets for $H^p(B_n)$, $0<p<\infty$.

In this section, in addition to the theorems (Theorem F and Theorem G) described in §3.2, we shall use the following J.H. Shapiro’s result:

Theorem I (cf. Shapiro [23], p.245, Corollary 2.2 and p.246, Corollary 2.5). Let $n \geq 1$ be an integer and $0<p<\infty$. Then there exists an $f \in A^p(B_n)$ such that $\nu_f \not\equiv \nu(\bigcup_{q>p} H^q(B_n))$.

Theorem 4 (cf. Rudin [21], §7.3.4.). Let $n \geq 2$ be an integer and $0<p<\infty$. Then there exists an $f \in H^p(B_n)$ which satisfies the following two conditions:

(a) $Z(f)$ is not a determining set for $H^\infty(B_n)$.

(b) $\nu_f \not\equiv \nu(\bigcup_{q>p} H^q(B_n))$.

Proof. By Theorem I, there is a $g \in A^p(B_{n-1})$ such that $\nu_g \not\equiv \nu(\bigcup_{q>p} A^p(B_{n-1}))$. Define $f = Eg$ in $B_n$, where $E$ is the extension operator in §3.2. Then it follows from Theorem F that $f \in H^p(B_n)$.

If $\nu_f \not\equiv \nu(\bigcup_{q>p} H^q(B_n))$, then there is an $h \in \bigcup_{q>p} H^q(B_n)$ with $\nu_h = \nu_f$.

Therefore $h = fh$ for some $k \in H(B_n)$ with $Z(k) = \phi$. Put $h' = \rho h$ and $k' = \rho k$, where $\rho$ is the restriction operator defined in §3.2. Then $h'(z') = g(z')k'(z')$ for $z' \in B_{n-1}$.

Since $Z(k') = \phi$, we have $\nu_g = \nu_{h'}$. By Theorem F we have $h' \in \bigcup_{q>p} A^q(B_{n-1})$. 

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since $\mathcal{H} \subseteq \mathcal{H}^q(B_n)$. Thus
\[ v_g \in v(\bigcup_{q>p} A^q(B_{n-1})). \]
This contradicts the choice of the function $g$. Hence
\[ v_f \notin v(\bigcup_{q>p} \mathcal{H}^q(B_n)). \]

We turn to the proof of (a). Since $f \in \mathcal{H}^p(B_n)$, Theorem G gives
\[ |f(z)| \leq 2^{n/p} \|f\|_{L^p} (1-|z|)^{-n/p} \quad (z \in B_n). \]
Hence
\[ |g(z')| \leq 2^{n/p} \|f\|_{L^p} (1-|z'|)^{-n/p} \quad (z' \in B_{n-1}). \]

Choose a positive integer $m$ with $n/p < m$. Define
\[ F(z) = f(z) z_n^{2m} = g(z') z_n^{2m} \quad (z = (z', z_n) \in B_n). \]
Then $F \in \mathcal{H}(B_n)$. By (1) and (2),
\[ |F(z)| \leq 2^{n/p} \|f\|_{L^p} (1-|z'|)^{-n/p} |z_n|^{2m} \]
for $z = (z', z_n) \in B_n$. Since
\[ |z_n| < 1 - |z'|^2 < 2(1 - |z'|), \]
we have
\[ (1 - |z'|)^{-n/p} < 2^{n/p} |z_n|^{-2n/p}. \]
It follows from (3) and (4) that
\[ |F(z)| \leq 2^{2n/p} \|f\|_{L^p} |z_n|^{2m-2n/p} \leq 2^{2n/p} \|f\|_{L^p} < \infty \]
for $z = (z', z_n) \in B_n$. Thus $F \in \mathcal{H}^m(B_n)$.

On the other hand, (2) gives
\[ Z(F) \supset Z(f) \quad \text{and} \quad F \neq 0 \quad \text{in} \quad B_n. \]
Hence $Z(f)$ is not a determining set for $\mathcal{H}^m(B_n)$.

**Theorem 5.** Let $n \geq 3$ be an integer. Then there exists an $f \in \bigcap_{0<p<\infty} \mathcal{H}^p(B_n)$ which satisfies the following two conditions:

(a) $Z(f)$ is not a determining set for $\mathcal{H}^m(B_n)$. 
(b) $Z(f)$ is not a determining set for $\mathcal{H}^m(B_n)$. 

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(b) $\forall f \notin \nu(H^\infty(B_n)^*)$.

Proof. Since $n-1 \geq 2$, Theorem 2 establishes the existence of a function $g \in \bigcap_{0 < p < \infty} L^p(B_n)$ such that $Z(g)$ is a determining set for $H^\infty(B_n)$. Since $H^p(B_n) \subset L^p(B_n)$ $(0 < p < \infty)$, we have $g \in \bigcap_{0 < p < \infty} L^p(B_n)$.

Put $f = Eg$. Then Theorem F implies

(5) $f \in \bigcap_{0 < p < \infty} H^p(B_n)$.

Choose a positive number $p$ with $n < p$. Define

(6) $F(z) = f(z)z_n^2 = g(z')z_n^2$

for $z = (z', z_n) \in B_n$. By (4), (5), (6) and Theorem G, we have

$$|F(z)| \leq 2^{2n/p} \|f\|_{L^p} |z_n|^{2-2n/p} \leq 2^n \|f\|_{L^p} < \infty$$

for $z = (z', z_n) \in B_n$. Hence $F \in H^\infty(B_n)$. It follows from (6) that $Z(F) \supset Z(f)$ and $F \neq 0$ in $B_n$.

This proves (a).

The repetition of the argument used to prove that $\forall f \notin \nu(\bigcup_{q > p} H^q(B_n)^*)$ in Theorem 4 gives now (b).
Part 4

The Zero Sets of Functions
in the Bergman Spaces and the Hardy Spaces
4.1. Introduction.

Let $n \geq 1$ be an integer. Let $H(B_n)$ denote the space of all holomorphic functions in the open unit ball $B_n$ of the complex $n$-dimensional Euclidean space $\mathbb{C}^n$. Let $\Gamma$ denote the class of all functions defined on $(-\infty, \infty)$ which are nonconstant, nonnegative, nondecreasing and convex. For each $\varphi \in \Gamma$, we define

$$A_\varphi(B_n) = \{f \in H(B_n): \int_{B_n} \varphi(\log |f|) d\lambda < \infty\},$$

$$H_\varphi(B_n) = \{f \in H(B_n): \sup_{0<\tau<1} \int_{\partial B_n} \varphi(\log |f(\tau \xi)|) d\sigma(\xi) < \infty\}.$$ 

Here $\lambda$ is the usual Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$, $\partial B_n$ is the boundary of $B_n$ and $\sigma$ is the rotation invariant positive Borel measure on $\partial B_n$ for which $\sigma(\partial B_n) = 1$. If $\varphi(t) = e^{pt}$, $0 < p < \infty$, then $A_\varphi(B_n)$ are the Bergman spaces $A^p(B_n)$ and $H_\varphi(B_n)$ are the Hardy spaces $H^p(B_n)$. If $\varphi(t) = \max\{0, t\}$, then $H_\varphi(B_n)$ is the Nevanlinna space $N(B_n)$ and $A_\varphi(B_n)$ will be denoted by $A^0(B_n)$ throughout this Part 4. $H^\infty(B_n)$ stands for the space of all bounded holomorphic functions in $B_n$.

The open unit disc in the complex plane $\mathbb{C}$ will be denoted by $U$ in place of $B_1$. It is well known that all of the spaces $H^p(U)$ ($0 < p \leq \infty$) and the space $N(U)$ admit the same zero sets which are completely characterized by the Blaschke condition. (See Part 2, Theorem D.) When $n \geq 2$, the situation is much more complicated. It was proved by W. Rudin [20] that for two different values of $p > 0$ the zero sets of functions in the corresponding $H^p(B_n)$ differ.

Regarding the Bergman spaces $A^p(U)$, an analogous result was proved by C. Horowitz [10]: If $0 < p < q < \infty$, then the zero sets of functions in $A^p(U)$ and those of functions in $A^q(U)$ are different. J. H. Shapiro [23] extended this theorem to the weighted Bergman
spaces and to the case of several variables.

The main purpose of the present Part 4 is to amplify the above results of Rudin, Horowitz and Shapiro. The summary of our results will be stated at the end of §4.2.

4.2. Preliminaries.

First we note that

\[ H^p(B_n) \subset \Lambda^p(B_n) \]

for any \( p \in (0, \infty) \), and that

\[ H^\infty(B_n) \subset H^q(B_n) \subset H^p(B_n) \subset N(B_n), \]
\[ H^\infty(B_n) \subset \Lambda^q(B_n) \subset \Lambda^p(B_n) \subset A^0(B_n) \]

if \( 0 < p < q < \infty \). For each \( p \in (0, \infty) \), we define

\[ H^p(B_n) = \bigcup_{p < q < \infty} H^q(B_n), \quad H^p(B_n) = \bigcap_{0 < q < p} H^q(B_n), \]
\[ \Lambda^p(B_n) = \bigcup_{p < q < \infty} \Lambda^q(B_n), \quad \Lambda^p(B_n) = \bigcap_{0 < q < p} \Lambda^q(B_n). \]

Then

\[ H^p(B_n) \subset H^p(B_n) \subset H^p(B_n), \]
\[ \Lambda^p(B_n) \subset \Lambda^p(B_n) \subset \Lambda^p(B_n). \]

As in §3.2, we denote by \( D^+(B_n) \) the set of all positive divisors on \( B_n \). Then we have the divisor map \( \nu \) from \( H(B_n)^* \) into \( D^+(B_n) \) defined by letting \( \nu(f) \) for \( f \) in \( H(B_n) \) be \( \nu_f \). Here, for any subspace \( X \) of \( H(B_n) \) we write

\[ X^* = \{ f \in X : f \neq 0 \text{ in } B_n \}. \]

We recall that \( \mu \in D^+(U) \) satisfies the Blaschke condition if and only if

\[ \sum_{z \in U} \mu(z) (1 - |z|) < \infty. \]

\( D_0 \) stands for the set of all positive divisors which satisfy the
Blaschke condition.

Applying Theorem A in §1.1 to the case
\[ \psi(t) = e^{pt}, \ \varphi(t) = (2 + p^2 t^2) e^{pt}, \ 0 < p < \infty, \]
Rudin showed the following:

**Theorem J** (Rudin [20], p. 59; cf. Theorem H in §3.4.). For any positive integer \( n \geq 2 \) and any \( p \in (0, \infty) \),
\[ \nu(h^n(B_n)^*) \not\subset \nu(h^p(B_n)^*). \]

To describe the results C. Horowitz [10] and J. H. Shapiro [23], we define the "weighted" Bergman spaces \( A^w_\varphi [\mu] \). From now on, \( \mu \) will denote a finite, positive, rotation invariant Borel measure on \( U \) which gives positive mass to each annulus
\[ \{ z \in U : r < |z| < 1 \}, \]
where \( 0 < r < 1 \). For each \( \varphi \in \Gamma \), we define
\[ A^w_\varphi [\mu] = \{ f \in H(U) : \int U \varphi(\log|f|)d\mu < \infty \}. \]

The main result of J. H. Shapiro [23] is the following:

**Theorem K** ([23], Theorem 2.1; cf. Theorem C in §2.1.). Assume that \( \varphi \) and \( \psi \) are strictly positive, convex, increasing, unbounded functions defined on \( (-\infty, \infty) \), and that
\[ \sup_{-\infty < t < \infty} \varphi(t+1)/\varphi(t) < \infty, \sup_{-\infty < t < \infty} \psi(t+1)/\psi(t) < \infty, \]
\[ \lim_{t \to -\infty} \varphi(t) = 0, \ \lim_{t \to -\infty} \psi(t) = 0, \ \lim_{t \to \infty} \psi(t)/\varphi(t) = \infty. \]

Then there is an \( f \in A^w_\varphi [\mu] \) such that for any positive integer \( m \), any \( b \in H^\infty(U) \) and any \( g \in H(U)^* \),
\[ (f^m + b)g \in A^w_\varphi [\mu], \]
where \( \psi_m(t) = \psi(t/m) \).

C. Horowitz [10] considered the case
\[ d\mu(z) = (1-|z|)^\alpha dx dy, \ \alpha > -1. \]
Shapiro noticed that with 
\[ \phi(t) = e^{pt}, \quad \psi(t) = (2 + p^2 t^2)e^{pt}, \quad 0 < p < \infty. \]

Theorem K gives the Horowitz's result:

**Theorem L** ([10], Theorem 4.6 and Theorem 6.11; [23], Corollary 2.2 and Corollary 2.5). For any integer \( n \geq 1 \) and any \( p \in (0, \infty) \),

\[ \nu(A^p(B_n)^*) \subseteq \nu(A^p(B_n)^*). \]

In §4.3, we shall prove some generalizations of Theorem K. In §4.4, making use of them and Theorem A (in §1.1), we shall describe the zero sets of functions in the spaces \( A^p(B_n) \) and \( H^p(B_n) \). The summary of our results is the following:

**Theorem.**

(a) \( \nu(\bigcap_{0 < q < \infty} A^q(U)^*) \not\subseteq D_0 \), so that, \( \bigcap_{0 < q < \infty} A^q(U) \not\subseteq \bigcap_{0 < q < \infty} H^q(U) \).

(b) For any integer \( n \geq 2 \) and any \( p \in (0, \infty) \),

\[ \nu(H^\infty(B_n)^*) \not\subseteq \nu(\bigcap_{0 < q < \infty} H^q(B_n)^*) \subseteq \nu(H^p(B_n)^*) \subseteq \nu(H^p(B_n)^*) \]

\[ \not\subseteq \nu(H^p(B_n)^*) \subseteq \nu(\bigcup_{0 < q < \infty} H^q(B_n)^*) \subseteq \nu(H(B_n)^*). \]

(c) For any integer \( n \geq 1 \) and any \( p \in (0, \infty) \),

\[ \nu(H^\infty(B_n)^*) \not\subseteq \nu(\bigcap_{0 < q < \infty} A^q(B_n)^*) \subseteq \nu(A^p(B_n)^*) \not\subseteq \nu(A^p(B_n)^*) \]

\[ \not\subseteq \nu(A^p(B_n)^*) \subseteq \nu(\bigcup_{0 < q < \infty} A^q(B_n)^*) \not\subseteq \nu(A(B_n)^*). \]

**4.3. Generalizations of the Shapiro's theorem.**

**Theorem 1.** Suppose \( \phi, \psi \in \Gamma \) and

\[ \lim_{t \to \infty} \frac{\psi(t)}{\phi(t+1)} = \infty. \]

Then there exists an \( f \in A_\psi[\mu] \) such that for any positive integer \( m \), any \( b \in H^\infty(U) \) and any \( g \in H(U)^* \),

\[ c(f^m + b)g \not\in A_\psi[\mu]. \]
for some constant $c$, where $\psi_m(t) = \psi(t/m)$.

Proof. Our proof follows the same lines as [23], §3, pp.248-251
Without loss of generality, we may assume that

$$\varphi(t) = 0 \quad \text{for } t \leq 0.$$ 

For $t \geq 0$, we define

$$\Phi(t) = \varphi(\log t), \quad \Psi(t) = \varphi(\log t + 1), \quad \psi_m(t) = \psi_m(\log t).$$

Then $\Phi_0$ is a continuous nondecreasing nonnegative function on $[0, \infty)$, and

$$\Phi_0(0) = 0, \quad \lim_{t \to \infty} \Phi_0(t) = \infty, \quad \lim_{t \to \infty} \psi(t)/\Phi_0(t) = 0.$$ 

Using Rudin's lemma (Lemma in §1.2), we can show the following lemma (cf. Lemma 2 in §2.2; Shapiro [23], p.248, Lemma):

Lemma. There exist sequences $\{t_k\}$ and $\{a_k\}$ of positive numbers increasing to $\infty$, and $\{n_k\}$ of positive integers increasing to $\infty$, and $\{r_k\}$ and $\{\rho_k\}$ with

$$0 < r_1 < \rho_1 < r_2 < \rho_2 < \ldots, \quad \lim_{k \to \infty} r_k = \lim_{k \to \infty} \rho_k = 1,$$

such that if $u_k(z) = a_k z^{n_k}$ and $R_k = \{z \in U: r_k < |z| \leq \rho_k\}$, then for $k \geq 2$ the following conditions hold:

(a) $t_k \geq 4 \sum_{j=1}^{k-1} a_j$ and $\psi(t)/\Phi_0(t) > k$ for $t \geq t_k$;

(b) $\int_U \Phi_0(|u_k|) d\mu = k^{-2}$;

(c) $\int_{R_k} \Phi_0(|u_k|) d\mu > (2k^2)^{-1}$;

(d) $|u_k(z)| \geq t_k$ if $|z| \geq r_k$;

(e) $|u_k(z)| \leq |u_{k-1}(z)|/5$ if $r_{k-1} \leq |z| \leq \rho_{k-1}$.

We now define

$$f(z) = \sum_{k=1}^{\infty} u_k(z) \quad (z \in U).$$

The series converges uniformly on compact subsets of $U$, by (1-e).
Hence $f \in \mathcal{H}(U)$. By (1-a), (1-d) and (1-e), we have

\begin{align*}
(2) 
|f| & \leq 5|u_k|/4 + 5|u_{k+1}|/4 
\quad \text{on } \{z \in U: r_k \leq |z| \leq r_{k+1}\}, \\
(3) 
|f| & \geq |u_k|/2 
\quad \text{on } R_k.
\end{align*}

Using (2), we have

\begin{align*}
\int_{\{r_k < |z| \leq r_{k+1}\}} \Phi(|f|) \, d\mu & \leq \int_{\{r_k < |z| \leq r_{k+1}\}} \Phi(5|u_k|/4 + 5|u_{k+1}|/4) \, d\mu \\
& \leq \int_{\{r_k < |z| \leq r_{k+1}\}} (\Phi(5|u_k|/2) + \Phi(5|u_{k+1}|/2)) \, d\mu \\
& \leq \int_U \Phi_0(|u_k|) \, d\mu + \int_U \Phi_0(|u_{k+1}|) \, d\mu.
\end{align*}

It follows from (1-b) that

\begin{align*}
\int_U \Phi(|f|) \, d\mu & = \left( \int_{\{|z| \leq r_1\}} + \sum_{k=1}^{\infty} \int_{\{r_k < |z| \leq r_{k+1}\}} \right) \Phi(|f|) \, d\mu \\
& \leq \int_{\{|z| \leq r_1\}} \Phi(|f|) \, d\mu + \sum_{k=1}^{\infty} \left\{ k^{-2} + (k+1)^{-2} \right\} < \infty.
\end{align*}

Thus $f \in A_\Psi[\mu]$.

Fix a positive integer $m$. Suppose that $b \in \mathcal{H}_m(U)$, $g \in \mathcal{H}(U)^*$ and 

\[ h = (f^m, b) g. \]

Put

\[ \beta = \sup_{z \in U} |b(z)|, \quad \delta = (2\pi)^{-1} \int_{-\pi}^{\pi} \log |g(r_1 e^{i\theta})| \, d\theta. \]

Since $\log |g|$ is subharmonic in $U$,

\begin{align*}
(4) 
-\infty < \delta \leq (2\pi)^{-1} \int_{-\pi}^{\pi} \log |g(r_1 e^{i\theta})| \, d\theta < \infty \quad (r_1 \leq r < 1).
\end{align*}

Choose a positive number $c$ so that

\begin{align*}
(5) 
\log c + \delta - m \log 4 > 0.
\end{align*}

We shall see that $ch$ is not in $A_{\Psi_\mu}[\mu]$.

Since $t_k \rightarrow \infty$, there exists a positive integer $K \geq 2$ such that 

\[ (t_k/4)^m > \beta \quad \text{if } k \geq K.\]

It follows from (1-d) and (3) that

\begin{align*}
(6) 
|f^m, b| \geq (|u_k|/4)^m 
\quad \text{on } R_k.
\end{align*}
for \( k \geq K \). Fix \( k \geq K \) and \( r \in (r_k, \rho_k) \). By Jensen's convexity theorem, (4), (6) and (5), we have

\[
(2\pi)^{-1} \int_{-\pi}^{\pi} \Psi_{\infty}(\frac{1}{\log u_k(re^{i\theta})}) \, d\theta \geq \Psi((2\pi)^{-1} \int_{-\pi}^{\pi} \log u_k(re^{i\theta}) \, d\theta) = \Psi(\log u_k(r))
\]

Hence

\[
\int_{R_k} \Psi_{\infty}(\frac{1}{\log u_k}) \, d\mu \geq \int_{R_k} \Psi(\frac{1}{u_k}) \, d\mu \quad \text{for } k \geq K.
\]

By (1-d), (1-a), and (1-c),

\[
\int_{R_k} \Psi(\frac{1}{u_k}) \, d\mu > (2k)^{-1} \quad \text{for } k \geq 2.
\]

It follows from (7) that

\[
\int_U \Psi_{\infty}(\frac{1}{\log u_k}) \, d\mu \geq \sum_{k=K}^{\infty} \int_{R_k} \Psi_{\infty}(\frac{1}{\log u_k}) \, d\mu \geq \sum_{k=K}^{\infty} (2k)^{-1} = \infty
\]

Hence \( ch \Psi_{\infty} \mu \). This completes the proof.

Remark. If \( \psi \) satisfies the growth condition

\[
\lim \sup_{t \to \infty} \psi(t+1)/\psi(t) < \infty,
\]

then \( A_{\psi, \infty} \mu \) is closed under scalar multiplication. (cf. [20], p.58.)

In that case, the conclusion \( ch \Psi_{\infty} \mu \) is simply that \( h \Psi_{\infty} \mu \).

Moreover, if \( \phi \) also satisfies the growth condition

\[
\lim \sup_{t \to \infty} \phi(t+1)/\phi(t) < \infty,
\]

then the condition \( \lim_{t \to \infty} \psi(t)/\phi(t) = \infty \) implies that \( \lim_{t \to \infty} \psi(t)/\phi(t+1) = \infty \).

Theorem \( K \) is therefore a special case of Theorem 1.

Using Theorem 1, we obtain its analogue in the case of several complex variables:

**Theorem 2.** Let \( n \geq 2 \) be an integer. Assume that \( \phi \) and \( \psi \) are as in Theorem 1. Then there exists an \( f \in A_{\phi, \psi} \mu \) with the following property:
If \( m \) is a positive integer, \( b \in \mathcal{H}(B_n), \ g \in \mathcal{H}(B_n)^* \) with 
\[ g(z_1,0,...,0) \neq 0 \text{ in } U, \]
and
\[ h = (f^m + b)g, \]
then some constant multiple of \( h \) fails in \( A_{\phi}(B_n) \).

Proof. (cf. [23], pp.246-247, Proof of Corollary 2.5.) By Theorem 1, there exists an \( f_0 \in \mathcal{H}(U) \) which satisfies the following two conditions:

(a) \( \int_\partial \varphi(\log |f(z)|)(1-|z|^2)^{n-1}d\lambda(z) < \infty; \)
(b) If \( m \) is a positive integer, \( b_0 \in \mathcal{H}(U), \ g_0 \in \mathcal{H}(U)^* \), and 
\[ h_0 = (f_0^m + b_0)g_0, \]
then there exists a constant \( c \) such that 
\[ \int_\partial \psi_m(\log |ch_0(z)|)(1-|z|^2)^{n-1}d\lambda(z) = \infty. \]

Define
\[ f(z_1,...,z_n) = f_0(z_1) \text{ for } (z_1,...,z_n) \in B_n. \]

By Fubini’s theorem and (a),
\[ \int_\partial \varphi(\log |f|)d\lambda \]
\[ = \pi^{n-1}((n-1)!)^{-1}\int_\partial \varphi(\log |f_0(z)|)(1-|z|^2)^{n-1}d\lambda(z) < \infty, \]
so that \( f \in A_{\phi}(B_n) \).

Suppose that \( m \) is a positive integer, \( b \in \mathcal{H}(B_n), \ g \in \mathcal{H}(B_n)^* \) with 
\[ g(z_1,0,...,0) \neq 0 \text{ in } U, \]
and
\[ h = (f^m + b)g. \]

Define 
\[ h_0(z_1) = h(z_1,0,...,0), \]
\[ b_0(z_1) = b(z_1,0,...,0), \]
\[ g_0(z_1) = g(z_1, 0, \ldots, 0), \]
for \( z_1 \in U \). Then we have
\[ b_0 \in H^\infty(U), \quad \theta_0 \in H(B_n)^*, \]
and
\[ h_0 = (f_{0*}^m, b_0)g_0. \]
It follows from Fubini's theorem and (b) that for some constant \( c \)
\begin{equation}
\int_{B_n} \psi_m(1/(z_1, \ldots, 0))d\lambda(z_1, z_2, \ldots, z_n)
= \pi^{n-1}((n-1))^{-1} \int_U \psi_m(1/(z_1)) \cdot (1 - |z_1|^2)^{n-1}d\lambda(z_1) = \infty.
\end{equation}

Fix \( z_1 \in U \). Put
\[ \rho(z_1) = (1 - |z_1|^2)^{1/2} \]
and
\[ D(r) = \{(z_2, \ldots, z_n) \in C^{n-1} : |z_2|^2 + \ldots + |z_n|^2 < r^2\} \]
for \( r \in (0, \rho(z_1)) \).

Define
\[ G(z_1)[z_2, \ldots, z_n] = \psi_m(1/(z_1, z_2, \ldots, z_n)) \]
for \( (z_2, \ldots, z_n) \in D(\rho(z_1)) \). Since \( ch \in H(B_n)^* \) and \( \psi_m \) is a
nondecreasing convex function on \((-\infty, \infty)\), \( G(z_1) \) is plurisubharmonic
in \( D(\rho(z_1)) \). Hence
\[ \pi^{n-1}((n-1))^{-1}(1 - |z_1|^2)^{n-1}\psi_m(1/(z_1, 0, \ldots, 0)) \]
\[ = \lim_{r \to \rho(z_1)} \pi^{n-1}((n-1))^{-1} r^{2n-2} G(z_1)(0, \ldots, 0) \]
\[ \leq \lim_{r \to \rho(z_1)} \int_{D(r)} G(z_1)[z_2, \ldots, z_n]d\lambda(z_2, \ldots, z_n) \]
\[ = \int_{D(\rho(z_1))} G(z_1)[z_2, \ldots, z_n]d\lambda(z_2, \ldots, z_n) \]
It follows from Fubini's theorem that
\begin{equation}
\int_{B_n} \psi_m(1/(z_1, 0, \ldots, 0))d\lambda(z_1, z_2, \ldots, z_n)
= \int_U \pi^{n-1}((n-1))^{-1}(1 - |z_1|^2)^{n-1}\psi_m(1/(z_1, 0, \ldots, 0))d\lambda(z_1)
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\end{equation}
By (8) and (9), we have
\[ \int_{B^n} \psi_m(\log lchl) d\lambda = \infty. \]
This completes the proof.

4.4. Zero sets of functions in the spaces \( A^P(B_n) \) and \( H^P(B_n) \).

Theorem 3. (cf. Theorem J and Theorem L in §4.2.)
(a) For any \( p \in (0,\infty) \) and any integer \( n \geq 1 \),
\[ v(A^P(B_n)^*) \subseteq v(A^P(B_n)^*) \subsetneq v(A^P(B_n)^*). \]
(b) For any \( p \in (0,\infty) \) and any integer \( n \geq 2 \),
\[ v(H^P(B_n)^*) \subseteq v(H^P(B_n)^*) \subsetneq v(H^P(B_n)^*). \]

Proof. Put
\[ \varphi(t) = \begin{cases} \frac{1}{t} e^{pt} & (t \geq p^{-1}) \\ e^{pt} & (t < p^{-1}) \end{cases} \]
\[ \psi(t) = e^{pt} \quad (-\infty < t < \infty). \]
Then \( \varphi \) and \( \psi \) satisfy the assumptions in Theorem 1. Hence Theorem 1 and theorem 2 give
\[ v(A^P(B_n)^*) \subseteq v(A^P(B_n)^*) \quad (n \geq 1, \ 0 < p < \infty). \]

Likewise, Theorem A (in §1.1) gives
\[ v(H^P(B_n)^*) \subseteq v(H^P(B_n)^*) \quad (n \geq 2, \ 0 < p < \infty). \]

Theorem 4. (cf. §2.4, Theorem 5; §3.4, Theorem 3.)
(a) For any integer \( n \geq 1 \),
\[ v(\bigcup_{0 < p < \infty} A^P(B_n)^*) \subseteq v(A^0(B_n)^*). \]
(b) For any integer \( n \geq 2 \),
\[ v(\bigcup_{0 < p < \infty} H^P(B_n)^*) \subseteq v(H^0(B_n)^*). \]
Proof. Put
\[
\begin{align*}
\varphi(t) &= \max\{0, t\} \quad (-\infty < t < \infty), \\
\psi(t) &= \begin{cases} 
\exp(\sqrt{t}) & (t \geq 1) \\
1 & (t < 1).
\end{cases}
\end{align*}
\]
By applying Theorem 1, Theorem 2 and Theorem A to these \(\varphi\) and \(\psi\), we have
\[
\nu(\bigcup_{0<p<\infty} A^p(B_n)^*) \subseteq \nu(A^0(B_n)^*) = \nu(A^p(B_n)^*)
\]
for any \(n \geq 1\) and
\[
\nu(\bigcup_{0<p<\infty} H^p(B_n)^*) \subseteq \nu(H^0(B_n)^*) = \nu(H^p(B_n)^*)
\]
for any \(n \geq 2\).

Theorem 5. (cf. §3.4, Theorem 2.)
(a) For any integer \(n \geq 1\),
\[
\nu\left(\bigcap_{0<p<\infty} A^p(B_n)^*\right) \not\subseteq \nu(H^0(B_n)^*).
\]
(b) For any integer \(n \geq 2\),
\[
\nu\left(\bigcap_{0<p<\infty} H^p(B_n)^*\right) \not\subseteq \nu(H^0(B_n)^*).
\]
Proof. Fix \(n \geq 1\). Put
\[
\begin{align*}
\varphi(t) &= \begin{cases} 
\exp(t^2) & (t \geq 0) \\
1 & (t < 0),
\end{cases} \\
\psi(t) &= \begin{cases} 
\exp(t^3) & (t \geq 0) \\
1 & (t < 0).
\end{cases}
\end{align*}
\]
Then Theorem 2 (or Theorem 1) then establishes the existence of an \(f \in A^\varphi(B_n)\) with the following property:

If \(g \in H(B_n)^*\) and
\[
g(z_1, 0, \ldots, 0) \neq 0
\]
in \(U\), then
\[
\sigma f g \notin A^\psi(B_n)^*.
\]
for some constant $c$.

Suppose that $v_f \in \nu(H^\infty(B_n)^*)$. Then there exists an $h \in H^\infty(B_n)^*$ and a $g \in H(B_n)^*$ such that the zero set of $g$ is empty and $h = fg$. Hence

$$ch \notin A(B_n)$$

for some constant $c$. Since $H^\infty(B_n) \subseteq A(B_n)$, it follows that $ch \notin H^\infty(B_n)$.

This contradicts the fact $h \in H^\infty(B_n)$. Thus $v_f \notin \nu(H^\infty(B_n)^*)$.

On the other hand,

$$v_f \in \nu(A(B_n)^* \subseteq \nu(\bigcap_{0<p<\infty} A^p(B_n)^*)$$

Hence we obtain (a).

Apply Theorem A instead of Theorem 2 (or Theorem 1). The same reasoning as above now gives (b).

**Corollary.**

(a) There exists an $f \in \bigcap_{0<p<\infty} A^p(U)$ whose zero set does not satisfy the Blaschke condition.

(b) $\bigcap_{0<p<\infty} H^p(U) \subseteq \bigcap_{0<p<\infty} A^p(U)$.

**Proof.** These are immediate consequences of Theorem 5 and Theorem D (in §2.1).
Part 5

Determining Sets for $H^p(B_0^*)$ II.
5.1. Introduction.

In Part 3, we described the characterization of the determining sets for the Nevanlinna space $N(B_n)$ on the open unit ball $B_n$ of $\mathbb{C}^n$, and showed the existence of various determining sets and non-determining sets for the Hardy spaces $H^p(B_n)$, $0<p<\infty$. One of the results in Part 3 is the following:

Theorem M (§3.5, Theorem 5). Let $n \geq 3$ be an integer. Then there exists an $f \in \bigcap_{0<p<\infty} H^p(B_n)$ satisfying the following conditions:

(a) The zero set of $f$ is not a determining set for $H^\infty(B_n)$.

(b) The zero-divisor $\nu_f$ does not equal $\nu_g$ for any $g \in H^\infty(B_n)$.

The purpose of this Part 5 is to prove that the above theorem is still valid when $n=2$. For the proof we shall make use of a result in Part 4. (See §4.4, Theorem 5.)

5.2. The main result.

Theorem. Let $n \geq 2$ be an integer. Then there exists an

$$f \in \bigcap_{0<p<\infty} H^p(B_n)$$

that has the following two conditions:

(a) The zero set $Z(f)$ is not a determining set for $H^\infty(B_n)$.

(b) $\nu_f \neq \nu(H^\infty(B_n)^*)$.

Proof. By Theorem 5 in §4.4, there is a $g \in \bigcap_{0<p<\infty} H^p(B_{n-1})^*$ such that $\nu_g \neq \nu(H^\infty(B_{n-1})^*)$. Define

$$f = Eg,$$

where $E$ is the extension operator defined in §3.2. By Theorem F in §3.2, we have

$$f \in \bigcap_{0<p<\infty} H^p(B_n)^*.$$
If $v_f \in \nu(\mathcal{H}^\omega(B_n)^*)$, then there is an $h \in \mathcal{H}^\omega(B_n)^*$ with $\nu_h = v_f$. It follows that $h = fk$ for some $k \in \mathcal{H}(B_n)$ with $Z(k) = \phi$. Put

$$h' = \rho h \quad \text{and} \quad k' = \rho k,$$

where $\rho$ is the restriction operator defined in §3.2. Then

$$h' = gk' \quad \text{in} \quad B_{n-1}.$$

Since $Z(k') = \phi$, we have $\nu_g = \nu_h$. In addition, $h' \in \mathcal{H}^\omega(B_{n-1})$, because $h \in \mathcal{H}^\omega(B_n)$. Thus $\nu_g \in \nu(\mathcal{H}^\omega(B_{n-1})^*)$. This contradicts the choice of the function $g$. Hence $\nu_f \notin \nu(\mathcal{H}^\omega(B_n)^*)$.

We turn to show that the condition (a) holds. Since

$$f \in \bigcap_{0 < p < \infty} \mathcal{H}^p(B_n),$$

it follows from Theorem G (in §3.2) that

$$|f(z)| \leq \frac{2^n/p}{p} \| f \|_{\mathcal{H}^p} (1 - |z|)^{-n/p} \quad (z \in B_n, \ 0 < p < \infty).$$

Hence

$$|g(z')| \leq \frac{2^n/p}{p} \| f \|_{\mathcal{H}^p} (1 - |z'|)^{-n/p} \quad (z' \in B_{n-1}, \ 0 < p < \infty).$$

Choose a positive number $p$ with $n < p$. Define

$$F(z) = f(z)z_n^2 = g(z')z_n^2$$

for $z = (z', z_n) \in B_n$. Then

$$|F(z)| \leq \frac{2^n/p}{p} \| f \|_{\mathcal{H}^p} (1 - |z'|)^{-n/p} |z_n|^2$$

for $z = (z', z_n) \in B_n$. Since

$$|z_n|^2 < 1 - |z'|^2 < 2(1 - |z'|),$$

we have

$$(1 - |z'|)^{-n/p} < \frac{2^n/p}{p} |z_n|^{-2n/p}.$$

It follows that

$$|F(z)| \leq \frac{2^{2n/p}}{p} \| f \|_{\mathcal{H}^p} |z_n|^{2-2n/p} < 2^{2n/p} \| f \|_{\mathcal{H}^p} < \infty$$

for $z = (z', z_n) \in B_n$. Hence $F \in \mathcal{H}^\omega(B_n)$. By the definition of $F$,
\[ v_F \geq v_f \text{ in } B_n. \]

Therefore, we conclude that \( Z(f) \) is not a determining set for \( H^\infty(B_n) \).
Part 6

On a Conjecture of Z. Jianzhong
6.1. Introduction.

We call a nonnegative real-valued function \( \phi \) defined on \([0, \infty)\) a modulus function if it is a nondecreasing and nonconstant function such that \( \Phi(t) = \phi(e^t) \) is a convex function on \((-\infty, \infty)\). According to Deeb and Marzuq [1], for a given modulus function \( \phi \), the Hardy-Orlicz space \( H_{\phi}(B_n) \) is defined as
\[
H_{\phi}(B_n) = \{ f \in H(B_n) : \sup_{0 < r < 1} \int_S \phi(|f(r\xi)|) d\sigma(\xi) < \infty \},
\]
where \( S = \partial B_n \) is the unit sphere of \( \mathbb{C}^n \) and \( \sigma \) is the rotation invariant positive Borel measure on \( S \) for which \( \sigma(S) = 1 \). Let
\[
H^+_{\phi}(B_n) = \{ f \in H(B_n) : \lim_{r \to 1} f(r\xi) = f^*(\xi) \ a.e. \ [\sigma] \ on \ S \}.
\]
The space \( H^+_{\phi}(B_n) \) is defined to be the class of all those functions \( f \in H^+(B_n) \cap H_{\phi}(B_n) \) satisfying the condition
\[
\sup_{0 < r < 1} \int_S \phi(|f(r\xi)|) d\sigma(\xi) = \int_S \phi(|f^*|) d\sigma.
\]
Let \( N(B_n) \) and \( N^+(B_n) \) denote the Nevanlinna space and the Smirnov space, respectively; that is,
\[
N(B_n) = \{ f \in H(B_n) : \sup_{0 < r < 1} \int_S \log^+|f(r\xi)| d\sigma(\xi) < \infty \},
\]
\[
N^+(B_n) = \{ f \in N(B_n) : \lim_{r \to 1} \int_S \log^+|f(r\xi)| d\sigma(\xi) = \int_S \log^+|f^*| d\sigma \}.
\]
In [11, p.32, Remark 4], Jianzhong conjectured (for dimension \( n = 1 \)) that for two modulus functions \( \phi \) and \( \psi \), \( H^+_{\phi}(B_n) = H^+_{\psi}(B_n) \) if and only if \( H_{\phi}(B_n) = H_{\psi}(B_n) \). The main purpose of this Part 6 is to prove that Jianzhong's conjecture is true for any dimension \( n \geq 1 \).

6.2. Inclusion relation between the spaces \( H_{\phi}(B_n) \).

To prove the Proposition 1 described below, we recall some notations used in Rudin [22]. For \( 0 < p < \infty \), the Lebesgue spaces \( L^p(\sigma) \)
have their customary meaning. $L^0(\sigma)$ stands for the set of all measurable functions $u$ for which
\[ \int_S \log^+ |u| d\sigma < \infty. \]
$LSC$ denotes the set of all lower semicontinuous functions on $S$. The following theorem is proved in Rudin [22, pp.19-20].

Theorem N. Suppose $u \in LSC \cap L^0(\sigma)$, $u > 0$ on $S$. Then there is an $f \in H^+(B_n)$ whose boundary values $f^*$ satisfy
\[ |f^*(\xi)| = u(\xi) \]
almost everywhere $[\sigma]$ on $S$.

The following Proposition 1 is proved in Hasumi and Kataoka [7, Theorem 1.3] for the case $n = 1$.

Proposition 1. Let $\phi$ and $\psi$ be modulus functions. If
\[ \lim_{t \to \infty} \frac{\phi(t)}{\psi(t)} = \infty, \]
then there exists an $f \in H_\phi(B_n) \cap H^+(B_n)$ which does not belong to $H_\psi(B_n)$.

Proof. The proof for arbitrary dimension $n$ closely follows that of Hasumi and Kataoka for $n = 1$ [7, Proof of Theorem 1.3]. Put $\Phi(t) = \phi(e^t)$, $\Psi(t) = \psi(e^t)$ for $-\infty < t < \infty$. Then $\Phi$ and $\Psi$ are nondecreasing nonconstant convex functions on $[-\infty, \infty)$, and
\[ \lim_{t \to \infty} \frac{\Phi(t)}{\Psi(t)} = \infty. \]
Hence we can choose a sequence $\{t_j\}$ such that
\[ 0 < t_1 < t_2 < t_3 < \ldots, \lim_{j \to \infty} t_j = \infty, \]
\[ \Psi(t_j) > 2^j j^{-2} \quad \text{and} \quad \Phi(t_j)/\Psi(t_j) > j, \quad j=1,2,3,\ldots. \]
Set $\varepsilon_j = (j^2 \Psi(t_j))^{-1}$, for each $j$. Then we see that $\varepsilon_j < 2^{-j}$. 

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\[ j = 1, 2, 3, \ldots, \text{ and so } \sum \varepsilon_j < 1. \] Consequently, there is a sequence \( (E_j) \) of disjoint open subsets of the unit sphere \( S \) of \( \mathbb{C}^n \) such that \( \sigma(E_j) = \varepsilon_j, j = 1, 2, 3, \ldots. \) We define a function \( u \) on \( S \) by

\[ u = \sum_{j=1}^{\infty} t_j \chi_j, \]

where \( \chi_j \) is the characteristic function of the set \( E_j. \) Since \( E_j \) is an open subset of \( S, \) \( \chi_j \) is lower semicontinuous on \( S, \) that is, \( \chi_j \in \text{LSC}. \) Since each number \( t_j \) is positive, it follows that \( u \in \text{LSC}. \) The function \( \Psi * u \) is Borel measurable on \( S, \) and it holds that

\[
\int_S (\Psi * u) \, d\sigma = \sum_j \Psi(t_j) \sigma(E_j) + \Psi(0)(1 - \sum_j \sigma(E_j)) \\
\leq \sum_j \Psi(t_j) \varepsilon_j + \Psi(0) = \sum_j j^{-2} + \Psi(0) < \infty.
\]

so we have \( \Psi * u \in L^1(\sigma). \) Since \( \Psi \) is convex, nondecreasing and nonconstant, \( \Psi(t) \geq Ct \) for some constant \( C > 0 \) and for all sufficiently large \( t. \) Thus we see that \( u \in L^1(\sigma). \) On the other hand, the same way as in the case of \( \Psi * u \) gives that

\[
\int_S (\Phi * u) \, d\sigma = \sum_j \Phi(t_j) \varepsilon_j + \Phi(0)(1 - \sum_j \varepsilon_j) \\
\geq \sum_j j^2 j^{-2} = \sum_j j^{-1} = \infty.
\]

This means that \( \Phi * u \) does not belong to \( L^1(\sigma). \)

Now we put \( v = e^u \) on \( S. \) Since \( u \in \text{LSC} \cap L^1(\sigma) \) and \( 0 \leq u < \infty, \) it follows that \( v \in \text{LSC} \cap L^0(\sigma) \) and \( 1 \leq v < \infty. \) By Theorem N, there exists an \( f \in N^*(B_n) \) whose boundary values \( f^* \) satisfy \( |f^*| = v \) almost everywhere \( \{\sigma\}. \) Since \( f \in N^*(B_n), \) we have

\[ \log |f| \leq P(\log |f^*|) \]

in \( B_n, \) where \( P \) is the Poisson kernel in \( B_n. \) (See for example, M.Stoll [27, Lemma 3.1].) It follows from Jensen's inequality that

\[ \Psi(\log |f|) \leq P(\Psi \log |f^*|) = P(\Psi \log v) = P(\Psi * u) \]

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in $B_n$, because $\Psi$ is convex and nondecreasing on $(-\infty, \infty)$. Since $\psi \ast u \in L^1(\sigma), P[\psi \ast u]$ is harmonic in $B_n$. Noting $\psi(|f|) = \psi(|f|)$, we see that $f \in H^\psi(B_n)$. Finally, we shall show that $f$ does not belong to $H^\psi(B_n)$. By Fatou's lemma, we have

$$\lim_{r \to 1} \int_{S} \phi(|f(r\xi)|)d\sigma(\xi) \geq \int_{S} \phi(|f^*|)d\sigma$$

$$= \int_{S} \Phi(\log|f^*|)d\sigma$$

$$= \int_{S} (\psi \ast u)d\sigma = \infty.$$ 

Thus $f$ is not in $H^\psi(B_n)$. This completes the proof.

Now we consider the converse of Proposition 1. In the case of $n = 1$, the following Proposition 2 is proved in Jianzhong [11, Proposition 5]. (See also [7, Theorem 1.3] and [12, Theorem 2.1]. The proof is the same for any dimension $n \geq 1$.

**Proposition 2.** Let $\varphi$ and $\psi$ be modulus functions. If

$$\lim_{t \to \infty} \frac{\varphi(t)}{\psi(t)} < \infty,$$

then $H^\psi(B_n) \subset H^\varphi(B_n)$.

Proposition 1 and Proposition 2 give the following

**Theorem 1.** Suppose $\varphi$ and $\psi$ are two modulus functions. Then the following hold:

1. $H^\psi(B_n) \subset H^\varphi(B_n)$ if and only if

$$\lim_{t \to \infty} \frac{\varphi(t)}{\psi(t)} < \infty.$$ 

2. $H^\psi(B_n) = H^\varphi(B_n)$ if and only if

$$\lim_{t \to \infty} \frac{\varphi(t)}{\psi(t)} < \infty$$

and

$$\lim_{t \to \infty} \frac{\varphi(t)}{\psi(t)} > 0.$$ 

3. $H^\psi(B_n) \subsetneq H^\varphi(B_n)$ if and only if
\[
\lim_{t \to 0} \frac{\phi(t)}{\psi(t)} < \infty
\]

and
\[
\lim_{t \to 0} \frac{\phi(t)}{\psi(t)} = 0.
\]

We remark that this is a generalization of a result of Hasumi and Kataoka. They proved this in the case of the dimension \( n = 1 \).
See [7, Theorem 1.3 and Corollary 4.1].

6.3. Proof that \( H_\phi(B_n^+) \cap N^+(B_n^+) = H^+_\phi(B_n^+) \).

This equality is conjectured in Jianzhong [11, Remark 4]. In the case of \( n = 1 \), Hasumi and Kataoka [7, Theorem 2.1] proved that the equality is valid. To prove the general case we need the following lemmas:

Lemma 1. Suppose \( \{u_j : j = 1, 2, 3, \ldots \} \) is a sequence of nonnegative \( L^1(\sigma) \) functions such that \( \lim u_j(\xi) = u(\xi) \) almost everywhere \( \sigma \) on \( S \). Then \( \{u_j\} \) is uniformly integrable if and only if
\[
\lim_{j \to \infty} \int_S u_j \, d\sigma = \int_S u \, d\sigma < \infty.
\]

Proof. See Privalov [16, Satz 3.2]. He proved the lemma for a compact interval \([a, b]\) in place of the unit sphere \( S \), but the proof is the same for \( S \).

Lemma 2. Let \( f \in \mathcal{H}(B_n^+) \). Suppose that there is a real function \( u \in L^1(\sigma) \) such that \( \log |f| \leq P(u) \) in \( B_n^+ \). Then we have \( f \in N^+(B_n^+) \).

Proof. (See Hahn [4, Theorem 4]; Rudin [18, Theorem 3.3.5].)

Put \( u^* = \max(u, 0) \). Then \( u^* \geq 0, u \leq u^* \) on \( S \), and \( u^* \in L^1(\sigma) \).

Since \( \log |f| \leq P(u) \) in \( B_n^+ \), it follows that \( \log |f| \leq P(u^*) \) in \( B_n^+ \).

This shows \( f \in \mathcal{H}(B_n^+) \). Put \( v = P(u^*) \) in \( B_n^+ \). For \( 0 < r < 1 \) and \( \xi \in S \), we define \( v_r(\xi) = v(r\xi) \). Then \( v \) is a positive harmonic function in \( B_n^+ \) and \( \{v_r : 0 < r < 1\} \subset L^1(\sigma) \).

Hence we have
\[ \lim_{r \to 1} \int_S v_r \, d\sigma = v(0) = \int_S u^+ \, d\sigma. \]

By Fatou's theorem (see for example, Rudin [21, Theorem 5.4.8]),
\[ v^*(\xi) = \lim_{r \to 1} v_r(\xi) = u^*(\xi) \]
almost everywhere on \( S \). Since \( v_r \geq 0 \) on \( S \) (\( 0 < r < 1 \)), it follows from Lemma 1 that \( \{v_r\} \) is uniformly integrable. Note that
\[ \log^+ |f_r| \leq v_r \quad \text{on} \quad S \quad (0 < r < 1). \]
We therefore see that \( \{\log^+ |f_r| : 0 < r < 1\} \) is uniformly integrable.

Consequently, Lemma 1 gives
\[ \lim_{r \to 1} \int_S \log^+ |f_r| d\sigma = \int_S \log^+ |f^*| d\sigma. \]
This completes the proof.

Lemma 3. For every modulus function \( \varphi \), \( H_\varphi(B_n) \subset H(B_n) \).

Proof. Put \( \Phi(t) = \varphi(e^t) \). Then \( \Phi \) is a nonnegative nonconstant nondecreasing convex function on \( [-\infty, \infty) \), and so \( \Phi(t) \geq Ct \) for some positive constant \( C \) and for all sufficiently large \( t \). Now we define \( \psi(t) = \log^+ t \) for \( 0 \leq t < \infty \) and \( \psi(t) = \psi(e^t) \) for \( -\infty < t < 0 \). Then \( \psi \) is a modulus function and \( H_\psi(B_n) = H(B_n) \). Moreover, it holds that \( \Phi(t) \geq C\psi(t) \) for all sufficiently large \( t \). Hence we have
\[
\lim_{t \to \infty} \frac{\psi(t)}{\Phi(t)} = \lim_{t \to \infty} \frac{\psi(t)}{\phi(t)} \leq C^{-1} < \infty.
\]
It follows from Proposition 2 that \( H_\varphi(B_n) \subset H_\psi(B_n) = H(B_n) \).

Now we prove the following

Theorem 2. For every modulus function \( \varphi \), it holds that
\[ H_\varphi(B_n) \cap H^+(B_n) = H^+_\varphi(B_n). \]

Proof. (See Hasumi and Kataoka [7, Theorem 2.1]; Rudin [18, Theorem 3.4.2].) Suppose that \( f \in H^+_\varphi(B_n) \). Put \( \Phi(t) = \varphi(e^t) \) for \( -\infty < t < \infty \). Then we have
\[
\sup_{0<r<1} \int_{S} \Phi(log|f(r\xi)|)d\sigma(\xi) = \int_{S} \Phi(log|f^*|)d\sigma < \infty.
\]
Since \( \Phi \) is nonnegative, nonconstant, nondecreasing and convex on \([-\infty, \infty) \), there exists a positive finite Borel measure \( \mu \) on \( S \) such that \( \varphi(|f|) = \Phi(log|f|) \leq P[\mu] \) in \( B_n \) and \( \|\mu\| = \int_{S} \Phi(log|f^*|)d\sigma. \) (See for example, Rudin [21, Theorem 5.6.2].) We set \( u = P[\mu] \) in \( B_n \). By Fatou's theorem, \( u \) has radial limits
\[
u^*(\xi) = \lim_{r \to 1^-} u(r\xi)
\]
for almost all \( \xi \in S \{ \sigma \} \) and \( d\mu = u^* d\sigma + dv \), where \( u^* \in L^1(\sigma) \) and \( v \) is a finite positive singular Borel measure on \( S \). Since \( f \in H(\phi_n') \), Lemma 3 gives \( f \in N(B_n') \). Since \( \varphi(|f|) \leq u \) in \( B_n \), we have \( \varphi(|f^*|) \leq u^* \) almost everywhere \( \{ \sigma \} \) on \( S \). Consequently,
\[
\|\mu\| = \int_{S} \Phi(log|f^*|)d\sigma = \int_{S} \varphi(|f^*|)d\sigma \leq \int_{S} u^* d\sigma \leq \int_{S} u^* d\sigma + \int_{S} dv = \|\mu\|.
\]
This shows \( v = 0 \), and so \( \Phi(log|f|) \leq u = P[u^*] \) in \( B_n \). Since \( \Phi \) is nonnegative, nonconstant, nondecreasing and convex on \([-\infty, \infty) \), there are two positive constants \( C_1 \) and \( C_2 \) such that \( t \leq C_1 \Phi(t) + C_2 \) for all real \( t \). Thus we have
\[
log|f| \leq C_1 \Phi(log|f|) + C_2 \leq C_1 P[u^*] + C_2 = P[C_1 u^* + C_2]
\]
in \( B_n \). Since \( C_1 u^* + C_2 \in L^1(\sigma) \), it follows from Lemma 2 that \( f \in N^*(B_n) \).

Conversely, we suppose \( f \in H(\phi_n) \cap N^*(B_n) \). Then
\[
log|f| \leq P[log|f^*|] \in B_n. \text{ By Jensen's inequality, we have}
\]
\[
\varphi(|f|) = \Phi(log|f|) \leq P[\Phi(log|f^*|)] = P[\varphi(|f^*|)]
\]
in \( B_n \). Since \( \varphi(|f|) \) is subharmonic in \( B_n \), Fatou's lemma gives
\[
\int_{S} \varphi(|f^*|)d\sigma \leq \lim_{r \to 1} \int_{S} \varphi(|f(r\xi)|)d\sigma(\xi) \leq P[\varphi(|f^*|)](0) = \int_{S} \varphi(|f^*|)d\sigma.
\]
Hence it follows that

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\[
\sup_{0 < r < 1} \int_S \varphi(|f(r\xi)|) d\sigma(\xi) = \lim_{r \to 1} \int_S \varphi(|f(r\xi)|) d\sigma(\xi) = \int_S \varphi(|f^*|) d\sigma.
\]

This completes the proof.

6.4. Proof of the main result.

Now we can prove the main result of this Part 6:

**Theorem 3.** Let \( \varphi \) and \( \psi \) be modulus functions. Then

\[
H_\varphi(B_n) = H_\psi(B_n) \text{ if and only if } H_\varphi(B_n) = H_\psi(B_n).
\]

**Proof.** If \( H_\varphi(B_n) = H_\psi(B_n) \), we have \( H_\varphi(B_n) = H_\psi(B_n) \) as an immediate consequence of Theorem 2. Conversely, suppose that \( H_\varphi(B_n) = H_\psi(B_n) \). If \( \lim_{t \to \infty} \varphi(t)/\psi(t) = \infty \), then it follows from Proposition 1 that there is an \( f \in H_\psi(B_n) \cap N_\psi(B_n) \) such that \( f \notin H_\varphi(B_n) \). Theorem 2 gives \( f \in H_\psi(B_n) \), but \( f \notin H_\varphi(B_n) \). This contradicts the assumption \( H_\varphi(B_n) = H_\psi(B_n) \). So we have \( \lim_{t \to \infty} \varphi(t)/\psi(t) < \infty \). Similarly, we have \( \lim_{t \to \infty} \psi(t)/\varphi(t) < \infty \).

By Theorem 1, we can thus conclude that \( H_\varphi(B_n) = H_\psi(B_n) \). The proof is complete.
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