A p-ADIC INTERPOLATING FUNCTION OF THE GENERALIZED EULER NUMBERS-AND ITS INVARIANTS

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CONTENTS

INTRODUCTION

Let $u \neq 1$ be an algebraic number. The n-th Euler number $H^n(u)$ belonging to u is defined by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} \frac{H^n(u)}{n!} t^n.$$

Let p be a prime number and χ a primitive Dirichlet character. Shiratani-Yamamoto ([15]) constructed a p-adic interpolating function $G_p(s,u)$ of the Euler numbers $H^n(u)$, and as its applications to the p-adic L-functions $L_p(s,\chi)$, derived an explicit formula for $L_p'(0,\chi)$ including the Ferrero-Greenberg formula ([2]), and gave an explanation of Diamond's formula ([1]).

Let f_χ be the conductor of χ. As analogue to the generalized Bernoulli numbers, Tsumura ([19]) defined the n-th generalized Euler number $H_\chi^n(u)$ for χ belonging to u by

$$\sum_{a=0}^{f_\chi-1} \frac{(1-u^{f_\chi})\chi(a)e^{at}u^{f_\chi-a-1}}{e^{f_\chi t}-u^{f_\chi}} = \sum_{n=0}^{\infty} \frac{H_\chi^n(u)}{n!} t^n, \tag{0.1}$$

and he constructed a p-adic interpolating function $l_p(u,s,\chi)$, which is an extension of $G_p(s,u)$. Further, by considering the expansion of $l_p(u,s,\chi)$ at $s=1$, he obtained some congruences for the generalized Euler numbers.

As for the p-adic L-functions $L_p(s,\chi)$, Ferrero-Washington ([3]) showed that when χ is even, the μ-invariant of the interpolating

power series of $L_p(s,\chi)$ is zero. This implies that Iwasawa's μ-invariant is zero for the basic $\mathbf{Z}_p$-extension of any finite abelian number field ([3]). Friedman ([4]) generalized the Ferrero-Washington theorem to the basic $\mathbf{Z}_{p_1}\times\cdots\times\mathbf{Z}_{p_t}$-extension of any finite abelian number field.

Sinnott ([16]) showed how to calculate the μ-invariants of the Γ-transforms of rational functions and gave a new proof of the Ferrero-Washington theorem. By similar technique, an analytic property of the interpolating power series of $L_p(s,\chi)$ was investigated in [17] and a new proof of Friedman's theorem was given in [18].

In the present paper, by using the methods in the above references, we shall investigate p-adic properties of the generalized Euler numbers.

In Chapter 1, we first summarize some of the basic properties of the p-adic measures, and then, we reconstruct the function $l_p(u,s,\chi)$ by constructing an interpolating power series $F_{\chi,u}(T)\in O_{\chi,u}[[T-1]]$. Here, $O_{\chi,u}$ is the ring of integers of the field generated by u and the values of χ over the p-adic rational number field $\mathbf{Q}_p$. Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$, $|\ |$ denote the multiplicative valuation of $\mathbf{C}_p$ normalized by $|p|=1/p$ and $\mathrm{ord}_p(\ )$ denote the additive valuation of $\mathbf{C}_p$ normalized by $\mathrm{ord}_p(p)=1$. As usual, we put $\mathrm{ord}_p(0)=\infty$ and understand $\infty>c$ for all rational numbers c. Let $F_{\chi,u}(T)=\sum_{n=0}^{\infty} a_{\chi,n,u}(T-1)^n$ and we define its μ-invariant $\mu_{\chi,u}$ by $\mu_{\chi,u}=\min\{\mathrm{ord}_p(a_{\chi,n,u})\,|\,n\geq 0\}$. The purpose of Chapter 1 is to calculate the value of $\mu_{\chi,u}$. Denote by $\mathbf{N}$ the set of positive integers as usual and

we put $\bar{N}=N\cup\{0\}$. We shall obtain the following

THEOREM 1. *Suppose that we have*

$$|1-u^{f_\chi p^n}|\geq 1 \quad \textit{for all } n\in\bar{N}. \tag{0.2}$$

Then,

$$\mu_{\chi,u}=\begin{cases} -\mathrm{ord}_p(u) & \textit{if } |u|>1 \\ \mathrm{ord}_p(u) & \textit{if } |u|<1 \\ \mathrm{ord}_p(1+u) & \textit{if } |u|=1 \textit{ and if } \chi \textit{ is even} \\ 0 & \textit{otherwise.} \end{cases}$$

We shall give two proofs, firstly by the methods of Sinnott ([16]) and secondly by the methods of Ferrero-Washington ([3]) and Gillard ([5]). Further, we shall note that Theorem 1 includes the Ferrero-Washington theorem under the appropriate condition on the conductor of χ.

In Chapter 2, we shall investigate an analytic property of the function $F_{\chi,u}(z)$ defined on the set $D=\{z\in\mathbf{C}_p \mid |z-1|<1\}$. (Here and in the sequel, "analytic" means "Krasner analytic" ([10]).) We shall prove the following

THEOREM 2. *Assume that the condition* (0.2) *holds. Then,* $F_{\chi,u}(z)$ *is an analytic function on* D. *If* $u\neq 0$ *and if either* $\chi(-1)=-1$ *or* $u\neq -1$ *holds, then* $F_{\chi,u}(z)$ *has no analytic continuation to any quasi-connected subset of* $\mathbf{C}_p$ *properly containing* D.

In Chapter 3, fixing a finite set $S=\{p_1,\cdots,p_t\}$ of prime numbers distinct from p, and denoting by Ψ (resp. Φ) the set of Dirichlet characters of the second kind for S (resp. p), we shall investigate the p-adic valuation of $H^n_{\chi\omega^{-n}\psi\varphi}(u)$ for $\psi\in\Psi$, $\varphi\in\Phi$ and $n\in\mathbb{N}$, where ω is the Teichmüller character for p. The main result is as follows.

THEOREM 3. *Suppose that we have*

$$(0.3)\qquad |1-u^{f_{\chi p_1^{n_1}\cdots p_t^{n_t}p^n}}|\geq 1 \quad \textit{for all } n_1,\cdots,n_t,n\in\mathbb{N}.$$

For each $n\in\mathbb{N}$, *put* $\chi_n=\chi\omega^{-n}$. *Then, we have*

$$(0.4)\qquad \mathrm{ord}_p\left(\frac{u}{1-u^{f_{\chi_n\psi\varphi}}}H^n_{\chi_n\psi\varphi}(u)\right)=\mu_{\chi,u}$$

for almost all $\psi\in\Psi$ *and for all* $\varphi\in\Phi$ *and all* $n\in\mathbb{N}$. *In particular, if* $|u|\neq 1$, *then* (0.4) *holds for all* $\psi\in\Psi$, *all* $\varphi\in\Phi$ *and all* $n\in\mathbb{N}$, *except for the case that* $n=0$, $\chi\psi\varphi=1$ *and* $|u|>1$, *in which case, we have* $\mathrm{ord}_p(\frac{u}{1-u}H^0(u))=0$. (*Here and in the sequel, "almost all" means "all but finitely many".*)

Now, let $B_{n,\chi}$ denote the n-th generalized Bernoulli number for χ as usual. Friedman ([4]) showed that if χ is even, then we have

$$(0.5)\qquad \left|\frac{B_{n,\chi\psi\varphi}}{2n}\right|=1$$

for almost all $\psi\in\Psi$ and for all $\varphi\in\Phi$ and $n\in\mathbf{N}$. We can view Theorem 3 as an analogous result on Euler numbers to Friedman's theorem described above. Further, we shall note that Theorem 3 includes Friedman's theorem under the appropriate condition on χ.

Throughout this paper, denoting by $\bar{\mathbf{Q}}$ the algebraic closure of the rational number field $\mathbf{Q}$ in the complex number field $\mathbf{C}$, we fix an embeddeng of $\bar{\mathbf{Q}}$ into $\mathbf{C}_p$ and regard $\bar{\mathbf{Q}}$ also as a field contained in $\mathbf{C}_p$. A Dirichlet character always means a primitive one. In general, if R is a ring, we write $R^\times$ for the multiplicative group of units of R, $R[T]$ for the ring of polynomials in an indeterminate T with coefficients in R, and $R[[T-1]]$ for the ring of formal power series in an indeterminate $T-1$ with coefficients in R. If F is a field, we write $F(T)$ for the field of rational functions with coefficients in F. Denoting by $\mathbf{Z}_p$ the ring of integers of $\mathbf{Q}_p$ as usual, we put $\langle x\rangle = x/\omega(x)$ for any $x\in\mathbf{Z}_p^\times$.

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CHAPTER 1. An interpolating power series and the μ-invariant

1. The p-adic measures

In this section, we summarize some of the basic properties of the p-adic measures which will be used later ([12],[13],[16]).

If O is the ring of integers of a finite extension k of $\mathbf{Q}_p$, we denote by Λ_O the ring of O-valued measures on $\mathbf{Z}_p$. If $\alpha\in\Lambda_O$, we put

$$\hat{\alpha}(T)=\sum_{n=0}^{\infty}\left(\int_{\mathbf{Z}_p}\binom{x}{n}d\alpha(x)\right)(T-1)^n\in O[[T-1]].$$

Then,

$$(1.1.1)\qquad \hat{\alpha}(T)\equiv\sum_{a=0}^{p^n-1}\alpha(a+p^n\mathbf{Z}_p)T^a\ \left(\mathrm{mod}(T^{p^n}-1)O[[T-1]]\right)$$

for all $n\in\mathbf{N}$, and the map $\alpha\to\hat{\alpha}(T)$ gives an isomorphism of Λ_O with $O[[T-1]]$. If $a\in\mathbf{Z}_p^{\times}$, we write $\alpha\circ a$ for the measure defined by $\alpha\circ a(X)=\alpha(aX)$ for any compact and open subset X of $\mathbf{Z}_p$. Then $\widehat{(\alpha\circ a)}(T)=\hat{\alpha}(T^{a^{-1}})$, where we put $T^x=\sum_{n=0}^{\infty}\binom{x}{n}(T-1)^n$ for $x\in\mathbf{Z}_p$.

We define the Γ-transform $\Gamma_\alpha:\mathbf{Z}_p\to O$ of $\alpha\in\Lambda_O$ by

$$\Gamma_\alpha(s)=\int_{\mathbf{Z}_p^{\times}}\langle x\rangle^s d\alpha(x).$$

We put $q=4$ or p according as $p=2$ or $p\geq 3$ and fix a topological generator u_0 of the multiplicative group $1+q\mathbf{Z}_p$. We define a continuous homomorphism $\ell:\mathbf{Z}_p^{\times}\to\mathbf{Z}_p$ by $\langle x\rangle=u_0^{\ell(x)}$ for each $x\in\mathbf{Z}_p^{\times}$. Put

$$F_\alpha(T)=\sum_{n=0}^{\infty}\left(\int_{\mathbf{Z}_p^{\times}}\binom{\ell(x)}{n}d\alpha(x)\right)(T-1)^n\in O[[T-1]].$$

Then,

$$\Gamma_\alpha(s)=F_\alpha(u_0^{s}).$$

If α^* denote the measure defined by

$$\alpha^*(a+p^n\mathbf{Z}_p)=\alpha(\ell^{-1}(a+p^n\mathbf{Z}_p))$$

for all $a\in\mathbf{Z}_p$ and $n\in\mathbb{N}$, then we have

$$(1.1.2)\qquad F_\alpha(T)\equiv\sum_{a=1}^{p^n-1}\alpha^*(a+p^n\mathbf{Z}_p)T^a\ (\mathrm{mod}(T^{p^n}-1)O[[T-1]]).$$

For any $\alpha\in\Lambda_O$, we denote by $\tilde{\alpha}$ the measure on $\mathbf{Z}_p$ defined by restricting α to $\mathbf{Z}_p^{\times}$ and extending by 0. For any $g(T)\in O[[T-1]]$, we put $\tilde{g}(T)=g(T)-1/p\sum_{\zeta^p=1}g(\zeta T)\in O[[T-1]]$. Then, we have

$$(\hat{\tilde{\alpha}})(T)=(\tilde{\hat{\alpha}})(T).$$

We also note that $F_\alpha(T)=F_{\tilde{\alpha}}(T)$.

For any $g(T)\in O[[T-1]]$ and any Dirichlet character ν with conductor p^n, we put

$$g_\nu(T)=1/\tau(\nu^{-1},\zeta_{p^n})\sum_{a=1}^{p^n}\nu^{-1}(a)g(\zeta_{p^n}{}^aT),$$

where ζ_{p^n} is a primitive p^n-th root of unity and $\tau(\nu^{-1},\zeta_{p^n})=\sum_{a=1}^{p^n}\nu^{-1}(a)\zeta_{p^n}{}^a$. It is easy to see that $g_\nu(T)$ is independent of the choice of ζ_{p^n}. Let O_ν' denote the ring of integers of the field $k(\zeta_{p^n})$, which contains the values of ν. Then $g_\nu(T)\in O_\nu'[[T-1]]$. For any $\alpha\in\Lambda_O$ and any $n\in\mathbb{N}$, we have

$$(1.1.3)\qquad F_\alpha(u_0{}^n)=(T\cdot d/dT)^n((\hat{\tilde{\alpha}}))_{\omega^{-n}}(T)|_{T=1}$$
$$=(d/dz)^n((\hat{\tilde{\alpha}}))_{\omega^{-n}}(e^z)|_{z=0}.$$

For any $\alpha\in\Lambda_O$ and any Dirichlet character φ of the second kind for p, we denote by α_φ the measure in Λ_{O_φ}, satisfying $(\widehat{\alpha_\varphi})(T)=(\hat{\alpha})_\varphi(T)\in O_\varphi'[[T-1]]$. Then, we have

$$(1.1.4)\qquad F_{\alpha_\varphi}(T)=F_\alpha(\varphi(u_0)T).$$

2. Construction of an interpolating power series $F_{\chi,u}(T)$

Let χ be a Dirichlet character with conductor f_χ and $u\neq 1$ an element of $\bar{\mathbb{Q}}$. As in Introduction, we define the n-th generalized Euler number $H_\chi^n(u)$ by (0.1). Let $\mathbb{Q}_p(\chi,u)$ be the field generated by u and the values of χ over $\mathbb{Q}_p$ and, as described in Introduction, we denote by $O_{\chi,u}$ the ring of integers of $\mathbb{Q}_p(\chi,u)$. In this section, we assume the condition (0.2) and construct a power series $F_{\chi,u}(T)$ $\in O_{\chi,u}[[T-1]]$ which interpolates the generalized Euler numbers.

If $g\in\mathbb{N}$ is any multiple of f_χ, we put

$$R_{\chi,u}(T)=\sum_{a=0}^{g-1}\frac{\chi(a)u^{g-a}T^a}{T^g-u^g}\in\mathbb{Q}_p(\chi,u)(T), \tag{1.2.1}$$

which is independent of the choice of g.

PROPOSITION 1.2.1. *$R_{\chi,u}(T)$ lies in $O_{\chi,u}[[T-1]]$.*

Proof. Put $g=f_\chi$. If $|u|\leq 1$, then the condition (0.2) implies $|1-u^g|=1$, and so, $T^g-u^g=(1-u^g)+\sum_{a=0}^{g-1}\binom{g}{a}(T-1)^a\in(O_{\chi,u}[[T-1]])^\times$. Hence, $R_{\chi,u}(T)$ lies in $O_{\chi,u}[[T-1]]$. If $|u|>1$, then $|(1/u)^g-1|=1$, and so, $(T/u)^g-1\in(O_{\chi,u}[[T-1]])^\times$. Hence, $R_{\chi,u}(T)=\sum_{a=0}^{g-1}\chi(a)u^{-a}T^a/((T/u)^g-1)\in$ $O_{\chi,u}[[T-1]]$.

Now, we put $\Lambda_{\chi,u}=\Lambda_{O_{\chi,u}}$ and let $\alpha_{\chi,u}\in\Lambda_{\chi,u}$ be the measure satisfying $\widehat{\alpha_{\chi,u}}(T)=R_{\chi,u}(T)$. Put

$$F_{\chi,u}(T)=F_{\alpha_{\chi,u}}(T)\,(=F_{\widetilde{\alpha_{\chi,u}}}(T)).$$

LEMMA 1.2.2. (1) *If* $g\in\mathbb{N}$ *is a common multiple of* f_χ *and* p, *then*,

$$\widetilde{R_{\chi,u}}(T)=\sum_{\substack{a=0\\(a,p)=1}}^{g-1}\frac{\chi(a)u^{g-a}T^a}{T^g-u^g}=R_{\chi,u}(T)-\chi(p)R_{\chi,u^p}(T^p).$$

(2) *If* ν *is a Dirichlet character with conductor a power of* p, *then*

$$(\widetilde{R_{\chi,u}})_\nu(T)=\widetilde{R_{\chi\nu,u}}(T).$$

Proof. If $g\in\mathbb{N}$ is divisible by f_χ and p, then

$$\sum_{\zeta^p=1}R_{\chi,u}(\zeta T)=\sum_{\zeta^p=1}\sum_{a=0}^{g-1}\frac{\chi(a)u^{g-a}\zeta^aT^a}{T^g-u^g}=p\sum_{\substack{a=0\\p\mid a}}^{g-1}\frac{\chi(a)u^{g-a}T^a}{T^g-u^g}.$$

Considering the special case $g=f_\chi p$, we see that $\sum_{\zeta^p=1}R_{\chi,u}(\zeta T)=p\chi(p)R_{\chi,u^p}(T^p)$. Hence, we obtain the assertion of (1).

As for (2), the assertion is obvious if $\nu=1$. Suppose that $\nu\neq1$ and $f_\nu=p^n$, and put $g=f_\chi p^n$. Then, a direct calculation shows

$$\widetilde{(R_{\chi,u})}_{\nu}(T)=1/\tau(\nu^{-1},\zeta_{p^n})\sum_{a=1}^{p^n}\sum_{\substack{b=0\\(b,p)=1}}^{g-1}\frac{\nu^{-1}(a)\chi(b)u^{g-b}\zeta_{p^n}^{ab}T^b}{T^g-u^g}$$

$$=\sum_{\substack{b=0\\(b,p)=1}}^{g-1}\frac{\chi(b)\nu(b)u^{g-b}T^b}{T^g-u^g}$$

$$=R_{\chi\nu,u}(T)-1/p\sum_{\zeta^p=1}R_{\chi\nu,u}(\zeta T).$$

Hence, we deduce our assertion.

In the sequel, we put $\chi_n=\chi\omega^{-n}$ for each $n\in\mathbb{N}$.

PROPOSITION 1.2.3. *For each $n\in\mathbb{N}$, let f_n denote the conductor of χ_n. Then, we have*

$$F_{\chi,u}(u_0{}^n)=\frac{u}{1-u^{f_n}}H^n_{\chi_n}(u)-\frac{\chi_n(p)p^nu^p}{1-u^{pf_n}}H^n_{\chi_n}(u^p).$$

Proof. From (1.1.3) and the definition of $F_{\chi,u}(T)$, we have

$$F_{\chi,u}(u_0{}^n)=(d/dz)^n\widehat{(\widetilde{\alpha_{\chi,u}})}_{\omega^{-n}}(e^z)|_{z=0}.$$

Since $\widehat{(\widetilde{\alpha_{\chi,u}})}(T)=\widetilde{(\widehat{\alpha_{\chi,u}})}(T)=\widetilde{R_{\chi,u}}(T)$, Lemma 1.2.2 shows that

$$\widehat{(\widetilde{\alpha_{\chi,u}})}_{\omega^{-n}}(T)=R_{\chi_n,u}(T)-\chi_n(p)R_{\chi_n,u^p}(T^p).$$

Hence, the assertion follows from the definition of the generalized

Euler numbers (0.1).

PROPOSITION 1.2.4. *Let φ be a Dirichlet character of the second kind for p. Then,*

$$F_{\chi\varphi,u}(T)=F_{\chi,u}(\varphi(u_0)T).$$

Proof. Lemma 1.2.2 shows that $\widetilde{\alpha_{\chi\varphi,u}}=(\widetilde{\alpha_{\chi,u}})_{\varphi}$. Hence, the assertion follows from (1.1.4).

Remark. From Proposition 1.2.3 and Theorem 1 of [19], we see that the function $l_p(u,s,\chi)$ in [19] is equal to $F_{\chi,u}(u_0^{-s})$.

3. The μ-invariant of $F_{\chi,u}(T)$

In this section, we calculate the μ-invariant of $F_{\chi,u}(T)$ by the methods in [16] and prove Theorem 1.

For any power series $f(T)=\sum_{n=0}^{\infty} a_n(T-1)^n \in \mathbf{C}_p[[T-1]]$ with $|a_n|\le 1$ for all $n\in\mathbb{N}$, we define its μ-invariant by $\mu(f(T))=\inf\{\mathrm{ord}_p(a_n)\,|\,n\in\mathbb{N}\}$. Note that if $f(T)$ is a polynomial such that $f(T)=\sum_{n=0}^{m} b_n T^n$, then $\mu(f(T))=\min\{\mathrm{ord}_p(b_n)\,|\,0\le n\le m\}$. As described in Introduction, we put $\mu_{\chi,u}=\mu(F_{\chi,u}(T))$.

Proof of Theorem 1. Theorem 1 of [16] states that $\mu_{\chi,u}=\mu(\widetilde{R_{\chi,u}}(T)+\widetilde{R_{\chi,u}}(T^{-1}))$. Put $g=f_{\chi}p$. Then, a direct calculation shows

$$(1.3.1)\quad \widetilde{R_{\chi,u}}(T)+\widetilde{R_{\chi,u}}(T^{-1})=-\sum_{\substack{a=0\\(a,g)=1}}^{g-1}\frac{\chi(a)u^a(u^{2(g-a)}-\chi(-1))(T^a+T^{-a})}{(T^g-u^g)(T^{-g}-u^g)}$$

$$=\sum_{\substack{a=0\\(a,g)=1}}^{g-1}\frac{\chi(a)u^a(u^{2(g-a)}-\chi(-1))(T^{g+a}+T^{g-a})}{(T^g-u^g)(T^gu^g-1)}.$$

If $|u|>1$, then

$$\mu_{\chi,u}=\mu\Big(\sum_{\substack{a=0\\(a,g)=1}}^{g-1}\frac{\chi(a)u^{a-2g}(u^{2(g-a)}-\chi(-1))(T^{g+a}+T^{g-a})}{((T/u)^g-1)(T^g-(1/u)^g)}\Big).$$

Since $\mu(((T/u)^g-1)(T^g-(1/u)^g))=0$, we obtain

$$\mu_{\chi,u}=\min\{\mathrm{ord}_p(u^{a-2g}(u^{2(g-a)}-\chi(-1)))\,|\,1\le a<g,(a,g)=1\}$$
$$=-\mathrm{ord}_p(u).$$

Next, suppose that $|u|\le 1$. Then $\mu((T^g-u^g)(T^gu^g-1))=0$, hence we have

$$\mu_{\chi,u}=\min\{\mathrm{ord}_p(u^a(u^{2(g-a)}-\chi(-1)))\,|\,1\le a<g,(a,g)=1\}.$$

If $|u|<1$, then $\mu_{\chi,u}=\mathrm{ord}_p(u)$. If $|u|=1$ and if χ is even, then

$$\mu_{\chi,u}=\min\{\mathrm{ord}_p(u^{2a}-1)\,|\,1\le a<g,(a,g)=1\}=\mathrm{ord}_p(u^2-1).$$

Moreover (0.2) implies $|u-1|=1$ if $|u|=1$. Hence, $\mu_{\chi,u}=\mathrm{ord}_p(1+u)$. If $|u|=1$ and if χ is odd, then we have

$$\mu_{\chi,u}=\min\{\mathrm{ord}_p(u^{2a}+1)\mid 1\le a<g,(a,g)=1\}.$$

If $2|g$, then $4|f_\chi$ or $p=2$, and so, $4|f_\chi p^2$. Then, (0.2) implies $|1-u^4|=1$ and we deduce $|1+u^2|=1$. Thus, we obtain $\mu_{\chi,u}=0$. If $2\nmid g$, then both u^2+1 and u^4+1 belong to the set $\{u^{2a}+1\mid 1\le a<g,(a,g)=1\}$. If $|u^2+1|=1$, then we immediately have $\mu_{\chi,u}=0$. If $|u^2+1|<1$, then $|u+1|=|(u^2+1)+(u-1)|=1$, hence, $|u^4+1|=|u^2(u+1)(u-1)+(u^2+1)|=1$. Thus, we obtain $\mu_{\chi,u}=0$.

Remark. Suppose that χ is odd. If c is an integer with $c>1$ and $(c,f_\chi p)=1$, we have

$$\sum_{\substack{\zeta^c=1\\ \zeta\ne 1}} l_p(\zeta,s,\chi)=(1-\chi\omega(c)\langle c\rangle^{1-s})L_p(s,\chi\omega)$$

([19]). In particular, if $(2,f_\chi p)=1$, then

$$l_p(-1,s,\chi)=(1-\chi\omega(2)\langle 2\rangle^{1-s})L_p(s,\chi\omega).$$

Consequently

$$L_p(s,\chi\omega)=g_{\chi\omega}(u_0{}^{1-s}),$$

where $g_{\chi\omega}(T)=F_{\chi,-1}(u_0{}^{-1}T)/(1-\chi\omega(2)T^{\ell(2)})$. Hence, by Theorem 1, we deduce the Ferrero-Washington theorem under the restricted condition that $(2,f_\chi p)=1$.

4. Another proof of Theorem 1

In this section, using the methods in [3] and [5], we give another proof of Theorem 1. We first introduce notations.

We write $f_\chi = dp^r$ with $(d,p)=1$ and $r\in\mathbb{N}$. For each $n\in\mathbb{N}$, the canonical ring homomorphism $\rho_n : \mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}_p \to \mathbb{Z}/dqp^n\mathbb{Z}$ is defined, and we can view χ also as a function on $\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}_p$.

For any $x\in\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}_p$, we write $s_{n,d}(x)$ for the unique integer satisfying $0\le s_{n,d}(x) < dqp^n$ and $s_{n,d}(x) \bmod dqp^n = \rho_n(x)$. Similarly, for any $y\in\mathbb{Z}_p$, we write $s_n(y)$ for the unique integer satisfying $0\le s_n(y) < qp^n$ and $s_n(y)\equiv y \pmod{qp^n}$.

We identify an element $x\in\mathbb{Z}_p^\times$ with $(1 \bmod d, x)\in\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}_p^\times$. We decompose the character χ as

$$\chi = \theta\varphi,$$

where θ is a character of the first kind for p and φ is a character of the second kind for p. We denote by V the torsion subgroup of $\mathbb{Z}_p^\times$ and put $W=\mathbb{Z}/d\mathbb{Z}\times V$.

LEMMA 1.4.1. *For all $n\in\mathbb{N}$ and $a\in\mathbb{Z}$ with $0\le a < p^{n+r}$, we have*

$$\alpha_{\chi,u}(a+p^{n+r}\mathbb{Z}_p) = \sum_{i=0}^{d-1} \chi(a+ip^{n+r}) \frac{u^{f_\chi p^n - (a+ip^{n+r})}}{1-u^{f_\chi p^n}}.$$

Proof. Put $g=f_\chi p^n=dp^{n+r}$ and $\mathfrak{a}=(T^{p^{n+r}}-1)O_{\chi,u}[[T-1]]$.

If $|u|\le 1$, then $T^g-u^g\in(O_{\chi,u}[[T-1]])^\times$ and $(T^g-u^g)^{-1}\equiv(1-u^g)^{-1}$ (mod $\mathfrak{a}$). Then,

$$\begin{aligned}\widehat{\alpha_{\chi,u}}(T)&=R_{\chi,u}(T)\\&=\sum_{a=0}^{g-1}\frac{\chi(a)u^{g-a}T^a}{T^g-u^g}\\&\equiv\sum_{a=0}^{p^{n+r}-1}\sum_{i=0}^{d-1}\chi(a+ip^{n+r})u^{g-(a+ip^{n+r})}T^{a+ip^{n+r}}(1-u^g)^{-1}\pmod{\mathfrak{a}}\\&\equiv\sum_{a=0}^{p^{n+r}-1}\sum_{i=0}^{d-1}\chi(a+ip^{n+r})\frac{u^{f_\chi p^n-(a+ip^{n+r})}}{1-u^{f_\chi p^n}}T^a\pmod{\mathfrak{a}}.\end{aligned}$$

Hence, by (1.1.1), we obtain our assertion.

If $|u|>1$, then $(T/u)^g-1\in(O_{\chi,u}[[T-1]])^\times$ and $((T/u)^g-1)^{-1}\equiv((1/u)^g-1)^{-1}$ (mod $\mathfrak{a}$). Then,

$$\begin{aligned}\widehat{\alpha_{\chi,u}}(T)&=\frac{\sum_{a=0}^{g-1}\chi(a)u^{-a}T^a}{(T/u)^g-1}\\&\equiv\sum_{a=0}^{p^{n+r}-1}\sum_{i=0}^{d-1}\frac{\chi(a+ip^{n+r})u^{-(a+ip^{n+r})}}{(1/u)^g-1}T^{a+ip^{n+r}}\pmod{\mathfrak{a}}\\&\equiv\sum_{a=0}^{p^{n+r}-1}\sum_{i=0}^{d-1}\chi(a+ip^{n+r})\frac{u^{f_\chi p^n-(a+ip^{n+r})}}{1-u^{f_\chi p^n}}T^a\pmod{\mathfrak{a}}.\end{aligned}$$

Hence our assertion holds also in the case $|u|>1$.

PROPOSITION 1.4.2. *If $\mu_{\chi,u}\neq\infty$, then $\mu_{\chi,u}$ is the largest c in* $\mathbb{Q}$

such that

$$(1.4.1)\qquad \left|\sum_{\eta\in W}\theta(\eta)\frac{u^{dqp^n-s_{n,d}(x\eta)}}{1-u^{dqp^n}}\right|\le|p|^c$$

holds for all $x\in\mathbb{Z}_p^\times$ *and all* $n\in\mathbb{N}$. *Further,* $\mu_{\chi,u}=\infty$ *holds if and only if* (1.4.1) *holds for all* $c\in\mathbb{Q}$, $x\in\mathbb{Z}_p^\times$ *and* $n\in\mathbb{N}$.

Proof. For each $n\in\mathbb{N}$, put $\mathfrak{a}_n=(T^{p^n}-1)\mathcal{O}_{\chi,u}[[T-1]]$. From (1.1.2) and the definition of $F_{\chi,u}(T)$, we have

$$(1.4.2)\qquad F_{\chi,u}(T)\equiv\sum_{a=0}^{p^n-1}\alpha_{\chi,u}^*(a+p^n\mathbb{Z}_p)T^a \pmod{\mathfrak{a}_n}.$$

Note that, for any $f(T)=\sum_{m=0}^{\infty}c_m(T-1)^m\in\mathcal{O}_{\chi,u}[[T-1]]$, there are uniquely determined elements $b_{0,n},\cdots,b_{p^n-1,n}\in\mathcal{O}_{\chi,u}$ satisfying $f(T)\equiv\sum_{j=0}^{p^n-1}b_{j,n}T^j$ (mod $\mathfrak{a}_n$), and that, for a given $c\in\mathbb{Q}$, $\mathrm{ord}_p(c_m)>c$ holds for all $m\in\mathbb{N}$ if and only if $\mathrm{ord}_p(b_{j,n})>c$ holds for all $n\in\mathbb{N}$ and $j\in\mathbb{Z}$ with $0\le j\le p^n-1$. Hence, for a given $c\in\mathbb{Q}$, $\mu_{\chi,u}>c$ holds if and only if

$$|\alpha_{\chi,u}^*(v+p^n\mathbb{Z}_p)|<|p|^c$$

holds for all $v\in\mathbb{Z}_p$ and $n\in\mathbb{N}$.

For each $x\in\mathbb{Z}_p^\times$, we have

$$\alpha_{\chi,u}{}^{*}(\ell(x)+p^{n}\mathbf{Z}_{p})=\sum_{\eta\in V}\alpha_{\chi,u}(s_{n}(x\eta)+qp^{n}\mathbf{Z}_{p}).$$

If $p^{r}|qp^{n}$, then Lemma 1.4.1 shows that

$$\alpha_{\chi,u}{}^{*}(\ell(x)+p^{n}\mathbf{Z}_{p})=\sum_{\eta\in V}\sum_{i=0}^{d-1}\chi(s_{n}(x\eta)+iqp^{n})\frac{u^{f_{\chi}qp^{n-r}-(s_{n}(x\eta)+iqp^{n})}}{1-u^{f_{\chi}qp^{n-r}}}.$$

Recall that $\chi=\theta\varphi$ and that $f_{\chi}=dp^{r}$ with $(d,p)=1$. Then, the conductor of φ divides p^{r}, and so, $\chi(s_{n}(x\eta)+iqp^{n})=\theta(s_{n}(x\eta)+iqp^{n})\varphi(x)$. Further the integers $s_{n}(x\eta)+iqp^{n}$ with $\eta\in V$ and $0\le i\le d-1$ are precisely $s_{n,d}(x\eta)$ with $\eta\in W$. Since the conductor of θ is d or dq, we obtain

$$(1.4.3)\quad \alpha_{\chi,u}{}^{*}(\ell(x)+p^{n}\mathbf{Z}_{p})=\varphi(u_{0})^{\ell(x)}\theta(x)\sum_{\eta\in W}\theta(\eta)\frac{u^{dqp^{n}-s_{n,d}(x\eta)}}{1-u^{dqp^{n}}}$$

for all $n\in\mathbb{N}$ with $p^{r}|qp^{n}$. In particular, we have

$$(1.4.3)'\quad \alpha_{\theta,u}{}^{*}(\ell(x)+p^{n}\mathbf{Z}_{p})=\theta(x)\sum_{\eta\in W}\theta(\eta)\frac{u^{dqp^{n}-s_{n,d}(x\eta)}}{1-u^{dqp^{n}}}$$

for all $x\in\mathbf{Z}_{p}{}^{\times}$ and $n\in\mathbb{N}$.

From (1.4.2), (1.4.3), (1.4.3)' and the fact that $\ell(\mathbf{Z}_{p}{}^{\times})=\mathbf{Z}_{p}$, we obtain $F_{\chi,u}(T)=F_{\theta,u}(\varphi(u_{0})T)$. (Note that this can be obtained also from Proposition 1.2.4.) Hence, we see $\mu_{\chi,u}=\mu_{\theta,u}$ and it follows that, for a given $c\in\mathbf{Q}$, $\mu_{\chi,u}>c$ holds if and only if $|\alpha_{\theta,u}{}^{*}(v+p^{n}\mathbf{Z}_{p})|<|p|^{c}$

holds for all $v\in\mathbb{Z}_p$ and $n\in\bar{\mathbb{N}}$. Taking account of the fact that $\ell(\mathbb{Z}_p^{\times})=\mathbb{Z}_p$ again, we obtain from (1.4.3)' the required assertion.

Now, we prove Theorem 1.

If $|u|>1$, then $\left|\sum_{\eta\in W}\theta(\eta)\frac{u^{dqp^n-s_{n,d}(x\eta)}}{1-u^{dqp^n}}\right|\le|u|^{-1}$ for all $x\in\mathbb{Z}_p^{\times}$ and $n\in\bar{\mathbb{N}}$. In particular, we have

$$\left|\sum_{\eta\in W}\theta(\eta)\frac{u^{dq-s_{0,d}(\eta)}}{1-u^{dq}}\right|=\left|\sum_{j=0}^{dq-1}\theta(j)\frac{u^{dq-j}}{1-u^{dq}}\right|=|u|^{-1}.$$

Hence, we see from Proposition 1.4.2 that $\mu_{\chi,u}=-\mathrm{ord}_p(u)$. In a similar way, we obtain $\mu_{\chi,u}=\mathrm{ord}_p(u)$ if $|u|<1$.

Now, we assume in the sequel that $|u|=1$. Then, by (0.2), we have $|1-u^{dqp^n}|=1$ for all $n\in\bar{\mathbb{N}}$. In order to prove Theorem 1, we use the following lemma (Proposition 4 of [5], Chapter 11 of [13]).

LEMMA 1.4.3. *Let $g,m\in\mathbb{N}$ with $(g,p)=1$. Then, we can choose a complete set R_0 of representatives of V modulo $\{1,-1\}$ such that, for any sufficiently large integer l, there exist $x_1,x_2\in 1+p^m\mathbb{Z}_p$ and $\eta_0\in R_0$ satisfying the following properties.*

(1) $s_{l+m}(x_1\eta)=s_l(x_1\eta)\equiv 0 \pmod g$ *for all* $\eta\in R_0$,

(2) $s_{l+m}(x_2\eta)=s_l(x_2\eta)\equiv 0 \pmod g$ *for all* $\eta\in R_0-\{\eta_0\}$,

(3) $s_{l+m}(x_2\eta_0)=s_l(x_2\eta_0)+qp^l\equiv 0 \pmod g$.

Proof. If $p\neq 2$, the proof is already given in [5]. If $p=2$, we can easily obtain our assertion as in the following way. Put $R_0=\{1\}$. Choose $n_1\in\mathbb{N}$ such that $g^{n_1}\equiv 1 \pmod{2^m}$, and let the 2-adic expansion of g^{n_1} be $g^{n_1}=1+2^m a_m+2^{m+1}a_{m+1}+\cdots+2^{m_1}$, where $m_1\geq m$ and $a_i=0$ or 1 for each i with $m\leq i\leq m_1-1$. Suppose that $l\in\mathbb{N}$ is an integer such that $l>2m_1-2$. Then, putting $x_1=g^{n_1}$ and $x_2=(1+2^{l+2-m_1})g^{n_1}$, we have $x_1\equiv x_2\equiv 1 \pmod{2^m}$, and x_1 and x_2 satisfy the conditions (1),(2) and (3) for $\eta_0=1$.

Now, we continue to prove Theorem 1. For any $x\in\mathbb{Z}_p^{\times}$ and any $n\in\bar{\mathbb{N}}$, we put

$$H(x,n)=H(\theta,u,x,n)=\theta(x)\sum_{\eta\in W}\theta(\eta)u^{-s_{n,d}(x\eta)}.$$

Then, for any complete set R of representatives of V modulo$\{1,-1\}$,

$$H(x,n)=\sum_{\eta\in R}\sum_{i=0}^{d-1}\theta(s_n(x\eta)+iqp^n)\{u^{-s_n(x\eta)-iqp^n}+\theta(-1)u^{-dqp^n+s_n(x\eta)+iqp^n}\}.$$

We put further

$$G(x,\eta,n)=G(\theta,u,x,\eta,n)$$
$$=\sum_{i=0}^{d-1}\theta(s_n(x\eta)+iqp^n)\{u^{-s_n(x\eta)-iqp^n}+\theta(-1)u^{-dqp^n+s_n(x\eta)+iqp^n}\}$$

for any $\eta\in V$. Then,

(1.4.4) $$H(x,n)=\sum_{\eta\in R} G(x,\eta,n).$$

Let $\mathfrak{p}_{\chi,u}$ denote the maximal ideal of $O_{\chi,u}$ and $g_1\in\mathbb{N}$ the order of u mod $\mathfrak{p}_{\chi,u}$ in the multiplicative group $(O_{\chi,u}/\mathfrak{p}_{\chi,u})^{\times}$. Write $g_1=d_0p^{r_0}$ with $(d_0,p)=1$ and $r_0\geq 0$, and put $g=d_0d$. Let m be an integer with $qp^{r_0}|p^m$ and suppose that $l>m$ is a sufficiently large integer. We can choose a complete set R_0 of representatives of V modulo $\{1,-1\}$, $x_1,x_2\in 1+p^m\mathbb{Z}_p$ and $\eta_0\in R_0$ satisfying the conditions (1),(2) and (3) of Lemma 1.4.3. Then, we have

$$s_l(x_1\eta)\equiv s_l(x_2\eta) \pmod{g_1dq} \text{ for all } \eta\in R_0-\{\eta_0\},$$

and so,

(1.4.5) $$\theta(s_l(x_1\eta)+iqp^l)=\theta(s_l(x_2\eta)+iqp^l)$$
$$\text{and } u^{s_l(x_1\eta)}\equiv u^{s_l(x_2\eta)} \pmod{\mathfrak{p}_{\chi,u}}$$

hold for all $\eta\in R_0-\{\eta_0\}$. On the other hand, we have

(1.4.6) $$s_l(x_1\eta_0)\equiv s_l(x_2\eta_0)+qp^l \pmod{g_1dq},$$

and we obtain

(1.4.7) $$\theta(s_l(x_1\eta_0)+iqp^l)=\theta(s_l(x_2\eta_0)+(i+1)qp^l) \text{ and}$$
$$u^{s_l(x_1\eta_0)}\equiv u^{s_l(x_2\eta_0)+qp^l} \pmod{\mathfrak{p}_{\chi,u}}.$$

In the first place, we prove Theorem 1 in the case $g_1=2$, that is, $u\equiv -1 \pmod{\mathfrak{p}_{\chi,u}}$. Then, since g_1 divides the order of the group $(\mathcal{O}_{\chi,u}/\mathfrak{p}_{\chi,u})^{\times}$, we have $(p,2)=1$, and so, $d_0=2$, $r_0=0$ and by (0.2) we must have $(d,2)=1$. It is sufficient to show that $\mu_{\chi,u}=\mathrm{ord}_p(u+\theta(-1))$. We first have

$$
\begin{aligned}
&G(x,\eta,n)\\
&=\sum_{i=0}^{d-1}\theta(s_n(x\eta)+iqp^n)\{(u^{-s_n(x\eta)-iqp^n}-u^{s_n(x\eta)+iqp^n})+(u^{s_n(x\eta)+iqp^n}\\
&\qquad\qquad +\theta(-1)u^{-dqp^n+s_n(x\eta)+iqp^n})\}\\
&=(u+\theta(-1))\sum_{i=0}^{d-1}\theta(s_n(x\eta)+iqp^n)\{u^{-s_n(x\eta)-iqp^n}(\theta(-1)-u)\sum_{j=0}^{s_n(x\eta)+iqp^n-1}u^{2j}\\
&\qquad\qquad +u^{-dqp^n+s_n(x\eta)+iqp^n}\sum_{j=0}^{dqp^n-1}(-\theta(-1)u)^j\}
\end{aligned}
$$

for all $x\in\mathbb{Z}_p^{\times}$, all $\eta\in V$ and all $n\in\mathbb{N}$. The case $u+\theta(-1)=0$ occurs if and only if both $\theta(-1)=1$ and $u=-1$ hold, and in this case we have $G(x,\eta,n)=0$. Then by (1.4.1) we have $H(x,n)=0$ and it follows from Proposition 1.4.2 and the definition of $H(x,n)$ that $\mu_{\chi,u}=\infty$. Consequently we obtain $\mu_{\chi,u}=\mathrm{ord}_p(u+\theta(-1))=\infty$. Now suppose that $u+\theta(-1)\neq 0$. Then, $\frac{1}{u+\theta(-1)}G(x,\eta,n)\in\mathcal{O}_{\chi,u}$ and we have

$$
\begin{aligned}
\frac{1}{u+\theta(-1)}G(x,\eta,n)\equiv\sum_{i=0}^{d-1}\theta(s_n(x\eta)+iqp^n)(-1)^{s_n(x\eta)+i}&\{(\theta(-1)+1)x\eta\\
&-\sum_{j=0}^{dqp^n-1}\theta(-1)^j\}\pmod{\mathfrak{p}_{\chi,u}}.
\end{aligned}
$$

Since $g=d_0d=2d$, we have $s_l(x_1\eta_0)\equiv 0 \pmod 2$. Hence by (1.4.5) and (1.4.7)

$$\frac{1}{u+\theta(-1)}G(x_1,\eta,l)\equiv\frac{1}{u+\theta(-1)}G(x_2,\eta,l) \pmod{\mathfrak{p}_{\chi,u}} \text{ for all } \eta\in R_0-\{\eta_0\}$$

and

$$\frac{1}{u+\theta(-1)}\{G(x_1,\eta_0,l)-G(x_2,\eta_0,l)\}\equiv -2\theta(s_l(x_2\eta_0))((\theta(-1)+1)\eta_0 - \sum_{j=0}^{dqp^n-1}\theta(-1)^j) \pmod{\mathfrak{p}_{\chi,u}}.$$

It follows from (1.4.3) that

$$\begin{aligned}\frac{1}{u+\theta(-1)}\{H(x_1,l)-H(x_2,l)\} &\equiv \frac{1}{u+\theta(-1)}\{G(x_1,\eta_0,l)-G(x_2,\eta_0,l)\} \pmod{\mathfrak{p}_{\chi,u}}\\ &\equiv \begin{cases} -4\theta(s_l(x_2\eta_0))\eta_0 \pmod{\mathfrak{p}_{\chi,u}} & \text{if } \theta(-1)=1\\ 2\theta(s_l(x_2\eta_0)) \pmod{\mathfrak{p}_{\chi,u}} & \text{if } \theta(-1)=-1\end{cases}\\ &\not\equiv 0 \pmod{\mathfrak{p}_{\chi,u}}.\end{aligned}$$

Consequently we must have

$$|H(x_1,l)|=|u+\theta(-1)| \text{ or } |H(x_2,l)|=|u+\theta(-1)|.$$

Therefore, we see from Proposition 1.4.2 and the definition of $H(x,n)$ that $\mu_{\chi,u}=\mathrm{ord}_p(u+\theta(-1))$. Thus, Theorem 1 is proved in the case $g_1=2$. In particular, we have

(1.4.8) $$\mu_{\chi,-1}=0 \text{ if } \theta(-1)=-1.$$

We shall use this equation later.

Next, suppose that $g_1\neq 2$, namely $u^2\not\equiv 1 \pmod{p_{\chi,u}}$. We must show that $\mu_{\chi,u}=0$. By (1.4.5), we have

$$G(x_1,\eta,l)\equiv G(x_2,\eta,l) \pmod{p_{\chi,u}} \text{ for all } \eta\in R_0-\{\eta_0\}.$$

Hence by (1.4.4) and (1.4.7),

$$\begin{aligned} H(x_1,l)-H(x_2,l) &\equiv G(x_1,\eta_0,l)-G(x_2,\eta_0,l) \pmod{p_{\chi,u}} \\ &\equiv \theta(s_l(x_2\eta_0))\{(u^{-s_l(x_2\eta_0)-dpq^l}+\theta(-1)u^{s_l(x_2\eta_0)}) \\ &\qquad -(u^{-s_l(x_2\eta_0)}+\theta(-1)u^{-dqp^l+s_l(x_2\eta_0)})\} \pmod{p_{\chi,u}} \\ &\equiv \theta(-s_l(x_2\eta_0))u^{-s_l(x_2\eta_0)}(u^{2s_l(x_2\eta_0)}-\theta(-1))(1-u^{-dqp^l}) \\ &\qquad \pmod{p_{\chi,u}}. \end{aligned}$$

If $u^{2s_l(x_2\eta_0)}\not\equiv\theta(-1) \pmod{p_{\chi,u}}$, then $|H(x_1,l)-H(x_2,l)|=1$, and so, $|H(x_1,l)|=1$ or $|H(x_2,l)|=1$. Hence, from Proposition 1.4.2 and the definition of $H(x,n)$, we obtain $\mu_{\chi,u}=0$. Now suppose that $u^{2s_l(x_2\eta_0)}\equiv\theta(-1) \pmod{p_{\chi,u}}$. From (1.4.6) and the congruence $s_l(x_1\eta_0)\equiv 0 \pmod{d_0}$, we see $(s_l(x_2\eta_0),g_1)=1$. If $\theta(-1)=1$, then $u^2\equiv 1 \pmod{p_{\chi,u}}$, which contradicts the assumption that $g_1\neq 2$. If $\theta(-1)=-1$, then we obtain $u^2\equiv -1 \pmod{p_{\chi,u}}$.

In order to complete the proof of Theorem 1, we must deduce $\mu_{\chi,u}=0$ under the condition that both $\theta(-1)=-1$ and $u^2\equiv -1 \pmod{p_{\chi,u}}$

hold. In this case we have $(p,2)=1$. We assume $\mu_{\chi,u}>0$ under the above condition and derive a contradiction.

Proposition 1.4.2 shows that $\mu_{\chi,u}>0$ holds if and only if $|H(\theta,u,x,n)|<1$ holds for all $x\in\mathbb{Z}_p^{\times}$ and $n\in\mathbb{N}$. Now we have

$$\begin{aligned}
&H(\theta,u^{-1},x,n)\\
&=\theta(x)\sum_{\eta\in W}\theta(\eta)u^{s_{n,d}(x\eta)}=\theta(x)\sum_{\eta\in W}\theta(-\eta)u^{s_{n,d}(-x\eta)}\\
&=\theta(-x)\sum_{\eta\in W}\theta(\eta)u^{dqp^n-s_{n,d}(x\eta)}=u^{dqp^n}\theta(-1)H(\theta,u,x,n).
\end{aligned}$$

Hence, we see from Proposition 1.4.2 that $\mu_{\chi,u}=\mu_{\chi,u^{-1}}$. The condition $u^2\equiv-1 \pmod{p_{\chi,u}}$ implies $|u-\sqrt{-1}|<1$ or $|u^{-1}-\sqrt{-1}|<1$. In either case, by the assumption that $\mu_{\chi,u}>0$, we obtain $|H(\theta,\sqrt{-1},x,n)|=|H(\theta,-\sqrt{-1},x,n)|<1$. Consequently, $\mu_{\chi,\sqrt{-1}}=\mu_{\chi,-\sqrt{-1}}>0$. Now, from $u^2\equiv-1 \pmod{p_{\chi,u}}$ and (0.2), we must have $(f_\chi,2)=1$. Then, a direct calculation shows

$$\sum_{a=0}^{f_\chi-1}\frac{\chi(a)e^{at\sqrt{-1}^{f_\chi}-a}}{e^{f_\chi t}-\sqrt{-1}^{f_\chi}}+\sum_{a=0}^{f_\chi-1}\frac{\chi(a)e^{at(-\sqrt{-1})^{f_\chi}-a}}{e^{f_\chi t}-(-\sqrt{-1})^{f_\chi}}=2\chi(2)\sum_{a=0}^{f_\chi-1}\frac{\chi(a)e^{2at(-1)^{f_\chi}-a}}{e^{2f_\chi t}-(-1)^{f_\chi}}.$$

It follows from the definition of the generalized Euler numbers that

$$\frac{\sqrt{-1}}{1-\sqrt{-1}^{f_\chi}}H_\chi^n(\sqrt{-1})+\frac{-\sqrt{-1}}{1-(-\sqrt{-1})^{f_\chi}}H_\chi^n(-\sqrt{-1})=-\chi(2)2^nH_\chi^n(-1).$$

Since $(p,2)=1$, we see from Proposition 1.2.3 and the above equation that

$$F_{\chi,\sqrt{-1}}(T)+F_{\chi,-\sqrt{-1}}(T)=2\chi(2)T^{\ell(2)}F_{\chi,-1}(T)$$
$$=2\chi(2)\sum_{n=0}^{\infty}\binom{\ell(2)}{n}(T-1)^{n}F_{\chi,-1}(T).$$

Therefore, if $\mu_{\chi,\sqrt{-1}}=\mu_{\chi,-\sqrt{-1}}>0$, we must have $\mu_{\chi,-1}>0$, which contradicts the equation (1.4.8) in the case $\theta(-1)=-1$. Thus, we obtain the assertion of Theorem 1 in the final case.

CHAPTER 2. Analytic property of the function $F_{\chi,u}(z)$

In this chapter, we view $F_{\chi,u}(z)$ as an analytic function on the set $D=\{z\in\mathbf{C}_p \mid |z-1|<1\}$ and prove Theorem 2.

We put $\widetilde{R_{\chi,u}}^{+}(T)=\widetilde{R_{\chi,u}}(T)+\widetilde{R_{\chi,u}}(T^{-1})$ and $\widetilde{\alpha_{\chi,u}}^{+}=\widetilde{\alpha_{\chi,u}}+\widetilde{\alpha_{\chi,u}}\circ(-1)$. Then, $\widetilde{\alpha_{\chi,u}}^{+}$ is an even measure supported on $\mathbf{Z}_p^{\times}$ and we have $(\widehat{\widetilde{\alpha_{\chi,u}}^{+}})(T)=\widetilde{R_{\chi,u}}^{+}(T)$ and $\frac{1}{2}\Gamma_{\widetilde{\alpha_{\chi,u}}^{+}}(s)=F_{\chi,u}(u_0^{s})$. Now, we apply Theorem 1 or 2 of [17].

Case 1. $|u|=1$. Put $F=O_{\chi,u}/\mathfrak{p}_{\chi,u}$. We first consider the case $\mu_{\chi,u}=0$. Suppose that the assertion of Theorem 2 does not hold. Then, it follows from the first paragraph of Section 3 of [17] that $F_{\chi,u}(T)\bmod \mathfrak{p}_{\chi,u}\in F(T)$, and Theorem 1 of [17] shows that $T^{n}\widetilde{R_{\chi,u}}^{+}(T)\bmod \mathfrak{p}_{\chi,u}\in F[T]$ for a sufficiently large $n\in\mathbf{N}$. Putting $g=f_{\chi}p$, it follows from (1.3.1) that $(T^{g}-u^{g})(T^{g}u^{g}-1)\bmod \mathfrak{p}_{\chi,u}$ divides $\sum_{\substack{a=0\\(a,g)=1}}^{g-1}\chi(a)u^{a}(u^{2(g-a)}-\chi(-1))(T^{g+a}-T^{g-a})\bmod \mathfrak{p}_{\chi,u}$ in $F[T]$. Considering the degrees of these polynomials, we deduce $u^{2a}-\chi(-1)\equiv 0 \pmod{\mathfrak{p}_{\chi,u}}$ for any a with $1\le a<g$ and $(a,g)=1$. This contradicts the fact that $|u^{2a}-\chi(-1)|=1$ for some a with $1\le a<g$ and $(a,g)=1$, as is known from the proof of Theorem 1 in Section 3 of the preceding Chapter. Hence, Theorem 2 must hold in the case $\mu_{\chi,u}=0$.

Next, if $\mu_{\chi,u}\neq 0$, then Theorem 1 shows that $\chi(-1)=1$ and that $\mu_{\chi,u}=\mathrm{ord}_p(1+u)$. Applying the above argument to $(1+u)^{-1}F_{\chi,u}(T)$ and

$(1+u)^{-1}\widetilde{R_{\chi,u}}^{+}(T)$ instead of $F_{\chi,u}(T)$ and $\widetilde{R_{\chi,u}}^{+}(T)$, we obtain the assertion of Theorem 2.

Case 2. $|u|>1$. In this case, putting $g=f_\chi p$, we have

$$\widetilde{R_{\chi,u}}(T)=\sum_{\substack{a=0\\(a,g)=1}}^{g-1}\frac{\chi(a)u^{-a}T^{a}}{(T/u)^{g}-1}=-\sum_{\substack{a=0\\(a,g)=1}}^{g-1}\chi(a)u^{-a}T^{a}\sum_{m=0}^{\infty}(u^{-1}T)^{gm}$$
$$=-\sum_{\substack{m=0\\(m,g)=1}}^{\infty}\chi(m)u^{-m}T^{m},$$

and so

$$\widetilde{R_{\chi,u}}^{+}(T)=-\sum_{\substack{m=0\\(m,g)=1}}^{\infty}\chi(m)u^{-m}(T^{m}+T^{-m}). \tag{2.1}$$

In order to apply Theorem 2 of [17], we first prove the following

LEMMA 2.1. *Let O be the ring of integers of a finite extension of $\mathbb{Q}_p$, and let $\{a_m\}$ and $\{b_m\}$ be sequences of O such that $\lim_{m\to\infty}a_m=\lim_{m\to\infty}b_m=0$. If both of the elements $\sum_{m=0}^{\infty}a_m(T^m+T^{-m})$ and $\sum_{m=0}^{\infty}b_m(T^m+T^{-m})$ in $O[[T-1]]$ are equal, then we have $a_m=b_m$ for all $m\in\mathbb{N}$.*

Proof. Put $f(T)=\sum_{m=0}^{\infty}a_m(T^m+T^{-m})$. It is sufficient to show that if $f(T)=0$, then we have $a_m=0$ for all $m\in\mathbb{N}$. For each $c\in\mathbb{Z}_p$, let δ_c denote the Dirac measure of mass 1 supported at c. Then, $\widehat{\delta_c}(T)=T^c$. Let $\alpha_f\in\Lambda_O$

be the measure satisfying $\widehat{\alpha_f}(T)=f(T)$. Then, for any $n\in\mathbb{N}$ and any integer l, we have $\alpha_f(l+p^n\mathbb{Z}_p)=\sum_{\substack{m=0\\ m\equiv\pm l\,(\mathrm{mod}\ p^n)}}^{\infty} a_m$. Now, assume that $a_{m_0}\neq 0$ for some $m_0\in\mathbb{N}$. Since $\lim_{m\to\infty} a_m=0$, there is an integer $m_1\in\mathbb{N}$ such that $|a_m|<|a_{m_0}|$ for all $m>m_1$. Choose an integer $n\in\mathbb{N}$ with $p^n-m_0>m_1$. Then, we have $\alpha_f(m_0+p^n\mathbb{Z}_p)\neq 0$. In fact, if $m_0\neq 0$, then $|\alpha_f(m_0+p^n\mathbb{Z}_p)|$ $=|a_{m_0}+\sum_{m=1}^{\infty}(a_{m_0+mp^n}+a_{-m_0+mp^n})|=|a_{m_0}|\neq 0$, and if $m_0=0$, then $|\alpha_f(m_0+p^n\mathbb{Z}_p)|$ $=|2(a_0+\sum_{m=1}^{\infty}a_{mp^n})|=|2a_0|\neq 0$. Thus, we see that if $\alpha_f=0$, we have $a_m=0$ for all $m\in\mathbb{N}$. Now, $\alpha_f=0$ holds if and only if $f(T)=0$. Hence, if $f(T)=0$, we have $a_m=0$ for all $m\in\mathbb{N}$.

Let V be the torsion subgroup of $\mathbb{Z}_p^{\times}$. Lemma 1 of [17] states that there is a rational number $r\geq 1$ such that

$$Vu_0^{\mathbb{Z}}\cap\mathbb{Q}^{\times}=\{1,-1\}r^{\mathbb{Z}}.$$

Now, we continue to prove Theorem 1 in the case $|u|>1$. From (2.1), Lemma 2.1 and Theorem 2 of [17], it is sufficient to show that there exists an integer $n\in\mathbb{N}$ prime to g which does not belong to $r^{\mathbb{Z}}$. Indeed, for any $k\in\mathbb{N}$, the integer $kg+1$ is prime to g, but not all of the integers of this form belong to $r^{\mathbb{Z}}$.

Case 3. $|u|<1$. From Lemma 1.2.2, a direct calculation shows that $\widetilde{R_{\chi,u^{-1}}}(T)=-\chi(-1)\widetilde{R_{\chi,u}}(T^{-1})$, namely $\widetilde{\alpha_{\chi,u^{-1}}}=-\chi(-1)(\widetilde{\alpha_{\chi,u}}\circ(-1))$. Hence, the case $|u|<1$ is reduced to Case 2.

CHAPTER 3. The p-adic valuation of the generalized Euler numbers

1. The p-adic valuation of the generalized Euler numbers

Let $S=\{p_1,\cdots,p_t\}$ be a finite set of prime numbers distinct from p and, as described in Introduction, we denote by Ψ (resp. Φ) the set of Dirichlet characters of the second kind for S (resp. p). Let χ be a Dirichlet character with conductor f_χ. In this section, we assume the condition (0.3) and, by using the methods in [18], we prove Theorem 3.

Fix $\psi\in\Psi$, $\varphi\in\Phi$ and $n\in\mathbb{N}$ arbitrarily, and denote the conductor of $\chi_n\psi\varphi$ simply by f. We denote by $\mathbb{Q}_p(\chi,\psi,\varphi,u)$ the field generated by u and the values of χ,ψ and φ over $\mathbb{Q}_p$, by $O_{\chi,\psi,\varphi,u}$ the ring of integers of $\mathbb{Q}_p(\chi,\psi,\varphi,u)$ and by $\mathfrak{p}_{\chi,\psi,\varphi,u}$ the maximal ideal of $O_{\chi,\psi,\varphi,u}$. Let e denote the ramification index of the extension $\mathbb{Q}_p(\chi,\psi,\varphi,u)/\mathbb{Q}_p$.

In the case $|u|\neq 1$ and $n\geq 1$, we see from Lemma 1 of [19] and Theorem 1 that for any sufficiently large $N\in\mathbb{N}$,

$$\frac{u}{1-u^f}H^n_{\chi_n\psi\varphi}(u)\equiv\sum_{a=1}^{fp^N-1}\chi_n\psi\varphi(a)a^n\frac{u^{fp^N-a}}{1-u^{fp^N}}\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}^{e\mu_{\chi,u}+1}}$$

$$\equiv\sum_{\substack{a=1\\(a,p)=1}}^{fp^N-1}\chi_n\psi\varphi(a)a^n\frac{u^{fp^N-a}}{1-u^{fp^N}}\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}^{e\mu_{\chi,u}+1}}.$$

On account of Theorem 1 again, we see

$$\mathrm{ord}_p\Big(\sum_{\substack{a=1\\(a,p)=1}}^{fp^N-1} \chi_n\psi\varphi(a)a^n\frac{u^{fp^N-a}}{1-u^{fp^N}}\Big)=\mu_{\chi,u}.$$

Hence, we obtain (0.4).

In the case $|u|\neq 1$ and $n=0$, since we have $H^0_{\chi\psi\varphi}(u)=\sum_{a=0}^{f-1}\chi\psi\varphi(a)u^{f-a-1}$, it is easy to deduce the assertion of Theorem 3.

In the case $|u|=1$, we use the methods in [18]. For that purpose, we first introduce notations and propositions without assuming $|u|=1$.

We put $\mathbf{Z}_S=\mathbf{Z}_{p_1}\times\cdots\times\mathbf{Z}_{p_t}$ and $P_S=p_1\cdots p_t$. For each $m\in\mathbf{N}$, let μ_m denote the group of m-th root of unity in $\bar{\mathbf{Q}}$ and put $\mu_S=\bigcup_{n=0}^{\infty}\mu_{P_S{}^n}$. Put $k=\mathbf{Q}_p(\chi,u)$ and $k_S=k(\mu_S)$.

For any k_S-valued measure ν on $\mathbf{Z}_S$, its Fourier transform $\hat{\nu}:\mu_S\to k_S$ is defined by $\hat{\nu}(\zeta)=\int_{\mathbf{Z}_S}\zeta^x d\nu(x)$. If there exists $R(T)\in k_S(T)$ such that $\hat{\nu}(\zeta)=R(\zeta)$ holds for almost all $\zeta\in\mu_S$, we call ν a rational function measure and $R(T)$ the associated rational function of ν. Any rational function in $k_S(T)$ can occur as the associated rational function of a certain k_S-valued rational function measure on $\mathbf{Z}_S$ ([18]§2).

Let $\nu_{\chi,u}$ be a rational function measure on $\mathbf{Z}_S$ whose associated rational function is $R_{\chi,u}(T)$. We assume that in the case $|u|\neq 1$, $R_{\chi,u}(\zeta)=\widehat{\nu_{\chi,u}}(\zeta)$ holds for all $\zeta\in\mu_S$. For any $\psi\in\Psi$, regarding ψ as a character of $\mathbf{Z}_S{}^\times$, we put

$$\Gamma_{\nu_{\chi,u}}(\psi)=\int_{\mathbf{Z}_S{}^\times}\psi(x)d\nu_{\chi,u}(x).$$

For any $S'\subset S$, we put

$$\Psi_{S'}=\{\psi\in\Psi \mid l\mid f_{\chi\psi} \text{ if } l\in S' \text{ and } l\nmid f_{\chi\psi} \text{ if } l\in S-S'\}.$$

PROPOSITION 3.1.1. *For almost all* $\psi\in\Psi_S$, *we have*

$$\Gamma_{\nu_{\chi,u}}(\psi)=\frac{u}{1-u^{f_{\chi\psi}}}H^0_{\chi\psi}(u). \tag{3.1.1}$$

If $|u|\neq 1$, *then* (3.1.1) *holds for all* $\psi\in\Psi_S$.

Proof. For any $\psi\in\Psi$, choose $n_{1,\psi},\cdots,n_{t,\psi}\in\mathbb{N}$ such that the conductor f_ψ of ψ divides $F_\psi=\prod_{i=1}^{t}p_i^{n_{i,\psi}}$. As in the proof of Proposition 2.2 of [18], we have

$$\Gamma_{\nu_{\chi,u}}(\psi)=\sum_{\zeta\in\mu_{F_\psi}}a_\psi(\zeta)\widehat{\nu_{\chi,u}}(\zeta),$$

where $a_\psi(\zeta)=1/F_\psi\sum_{\substack{a=0\\(a,P_S)=1}}^{F_\psi-1}\psi(a)\zeta^{-a}$. If ζ is a F_ψ-th root of unity whose order is not divisible by f_ψ, then $a_\psi(\zeta)=0$. Since $\nu_{\chi,u}$ is a rational function measure with the associated rational function $R_{\chi,u}(T)$, there is an integer $L\in\mathbb{N}$ divisible only by the primes in S such that $\widehat{\nu_{\chi,u}}(\zeta)=R_{\chi,u}(\zeta)$ for all $\zeta\in\mu_S-\mu_L$. Now, there are only finitely many ψ with $f_\psi\mid L$. If $f_\psi\nmid L$, then we have $\zeta^L\neq 1$ for any F_ψ-th root of unity ζ

whose order is divisible by f_ψ. Hence,

$$\Gamma_{\nu_{\chi,u}}(\psi)=\sum_{\zeta\in\mu_{F_\psi}} a_\psi(\zeta)R_{\chi,u}(\zeta T)|_{T=1}. \tag{3.1.2}$$

Put $g=f_\chi F_\psi$. Then, a direct calculation shows

$$\sum_{\zeta\in\mu_{F_\psi}} a_\psi(\zeta)R_{\chi,u}(\zeta T)=\sum_{\substack{a=1\\(a,P_S)=1}}^{g-1}\frac{\chi\psi(a)u^{g-a}T^a}{T^g-u^g}. \tag{3.1.3}$$

In the case $\psi\in\Psi_S$, we have

$$\sum_{\zeta\in\mu_{F_\psi}} a_\psi(\zeta)R_{\chi,u}(\zeta T)=R_{\chi\psi,u}(T). \tag{3.1.4}$$

Since $R_{\chi\psi,u}(1)=\frac{u}{1-u}\frac{}{f_{\chi\psi}}H^0_{\chi\psi}(u)$, we see that (3.1.1) holds for any $\psi\in\Psi_S$ with $f_\psi\nmid L$.

If $|u|\neq 1$, (3.1.2) holds for all $\psi\in\Psi$, and so, if $\psi\in\Psi_S$, we see from (3.1.4) that (3.1.1) holds. This completes the proof.

PROPOSITION 3.1.2. *For almost all* $\psi\in\Psi$, *we have*

$$\mathrm{ord}_p(\Gamma_{\nu_{\chi,u}}(\psi))=\mu_{\chi,u}. \tag{3.1.5}$$

If $|u|\neq 1$, *then* (3.1.5) *holds for all* $\psi\in\Psi$.

Proof. Let $\nu_{\chi,u}^{*}$ be the measure on $\mathbf{Z}_S$ obtained by restricting $\nu_{\chi,u}$ to $\mathbf{Z}_S^{\times}$ and extending by 0. Let $\phi(x)$ be the characteristic function on $\mathbf{Z}_S^{\times}$. Then,

$$\phi(x)=\sum_{\zeta\in\mu_{P_S}} a_\zeta \zeta^x,$$

where $a_\zeta = P_S^{-1} \sum_{\substack{a=1 \\ (a,P_S)=1}}^{P_S} \zeta^{-a}$. Put

$$R_{\chi,u}^{*}(T)=\sum_{\zeta\in\mu_{P_S}} a_\zeta R_{\chi,u}(\zeta T).$$

Then, $\nu_{\chi,u}^{*}$ is a rational function measure whose associated rational function is $R_{\chi,u}^{*}(T)$. Let $\nu_{\chi,u}^{*}\circ(-1)$ be the measure on $\mathbf{Z}_S$ defined by $\nu_{\chi,u}^{*}\circ(-1)(X)=\nu_{\chi,u}^{*}(-X)$ for any compact and open subset X of $\mathbf{Z}_S$. Then, $\nu_{\chi,u}^{*}\circ(-1)$ is also a rational function measure whose associated rational function is $R_{\chi,u}^{*}(T^{-1})$. Put $g=f_\chi P_S$. Then, a direct calculation shows

$$R_{\chi,u}^{*}(T)=\sum_{\substack{a=0 \\ (a,P_S)=1}}^{g-1} \frac{\chi(a)u^{g-a}T^a}{T^g-u^g}$$

and in the same way as we have shown Proposition 1.2.1, we see that $R_{\chi,u}^{*}(T)\in O_{\chi,u}[[T-1]]$. From the remark after Theorem 3.1 of [18], we have, for almost all $\psi\in\Psi$,

$$\mathrm{ord}_p(\Gamma_{\nu_{\chi,u}}(\psi))=\mu(R_{\chi,u}{}^*(T)+R_{\chi,u}{}^*(T^{-1})).$$

Further, in the same way as in the proof of Theorem 1 in the second section of Chapter 1, we see

$$\mu(R_{\chi,u}{}^*(T)+R_{\chi,u}{}^*(T^{-1}))=\mu_{\chi,u}.$$

Hence (3.1.5) holds for almost all $\psi\in\Psi$.

In the case $|u|\neq 1$, (3.1.2) holds for all $\psi\in\Psi$. Hence, we see from (3.1.3) and Theorem 1 that (3.1.5) holds for all $\psi\in\Psi$.

PROPOSITION 3.1.3. *Suppose that* $\mu_{\chi,u}\neq\infty$. *Let* π *be a prime of* $O_{\chi,\psi,\varphi,u}$. *Then, for almost all* $\psi\in\Psi_S$,

$$F_{\chi\psi,u}(T)/\pi^{e\mu_{\chi,u}}\in(O_{\chi\psi,u}[[T-1]])^{\times}. \tag{3.1.6}$$

If $|u|\neq 1$, *then* (3.1.6) *holds for all* $\psi\in\Psi_S$.

Proof. If p divides f_χ, then p divides $f_{\chi\psi}$ and Proposition 1.2.3 shows $F_{\chi\psi,u}(1)=\dfrac{u}{1-u^{f_{\chi\psi}}}H^0_{\chi\psi}(u)$. Hence by Propositions 3.1.1, 3.1.2, we have, for almost all $\psi\in\Psi_S$,

$$\mathrm{ord}_p(F_{\chi\psi,u}(1))=\mu_{\chi,u}. \tag{3.1.7}$$

Hence (3.1.6) holds for all $\psi\in\Psi_S$ satisfying (3.1.7).

Unless p divides f_χ, then p divides $f_{\chi\varphi}$ for all $\varphi\in\Phi-\{1\}$. For any $\psi\in\Psi$, we have $\psi\in\Psi_S$ if and only if $P_S|f_{\chi\psi\varphi}$. Hence, for almost all $\psi\in\Psi_S$, we have

$$\text{(3.1.8)} \qquad \mathrm{ord}_p(F_{\chi\psi\varphi,u}(1))=\mu_{\chi\psi\varphi,u}.$$

From Proposition 1.2.4 and Theorem 1, we see that (3.1.8) is equivalent to (3.1.6). Therefore, (3.1.6) holds for almost all $\psi\in\Psi_S$.

Finally, if $|u|\neq 1$, we see from Propositions 3.1.1, 3.1.2 that, for all $\psi\in\Psi_S$, (3.1.7) or (3.1.8) holds. Hence, (3.1.6) holds for all $\psi\in\Psi_S$. This completes the proof.

Proof of Theorem 3 in the case $|u|=1$. Again, we fix $\psi\in\Psi$, $\varphi\in\Phi$ and $n\in\bar{\mathbb{N}}$, and write f for the conductor of $\chi_n\psi\varphi$.

We first assume that $\psi\in\Psi_S$. Suppose that $n\geq 1$ in the first place.

Case 1. $\mu_{\chi,u}=0$. From Lemma 1 of [19], we have, for any sufficiently large $N\in\bar{\mathbb{N}}$,

$$\begin{aligned}\frac{u}{1-u^f}H^n_{\chi_n\psi\varphi}(u) &\equiv \sum_{a=0}^{fp^N-1}\chi_n\psi\varphi(a)a^n\frac{u^{fp^N-a}}{1-u^{fp^N}} \pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}\\ &\equiv \sum_{\substack{a=0\\(a,p)=1}}^{fp^N-1}\chi\psi(a)\frac{u^{fp^N-a}}{1-u^{fp^N}} \pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}.\end{aligned}$$

Hence, we have

$$\frac{u}{1-u^f}H^n_{\chi_n\psi\varphi}(u)\equiv\widetilde{R_{\chi\psi,u}}(1)\equiv F_{\chi\psi,u}(1)\ (\mathrm{mod}\ \mathfrak{p}_{\chi,\psi,\varphi,u}).$$

Therefore, if ψ satisfies (3.1.6), we have $\left|\frac{u}{1-u^f}H^n_{\chi_n\psi\varphi}(u)\right|=|F_{\chi\psi,u}(1)|=1$ and (0.4) holds.

Case 2. $\mu_{\chi,u}>0$. In this case, Theorem 1 states that χ is even and that $\mu_{\chi,u}=\mathrm{ord}_p(1+u)$, and (0.3) implies $2\nmid fp$. Choose $N_0\in\mathbb{N}$ arbitrarily. Then, Lemma 1 of [19] shows that, for any sufficiently large $N\in\mathbb{N}$, we have

$$\begin{aligned}\frac{u}{1-u^f}H^n_{\chi_n\psi\varphi}(u)&\equiv\sum_{a=1}^{fp^N-1}\chi_n\psi\varphi(a)a^n\frac{u^{fp^N-a}}{1-u^{fp^N}}\ (\mathrm{mod}\ \mathfrak{p}_{\chi,\psi,\varphi,u}^{N_0})\\&\equiv\sum_{a=1}^{\frac{fp^N-1}{2}}\chi_n\psi\varphi(a)\frac{a^nu^{fp^N-a}+\omega^n(-1)(fp^N-a)^nu^a}{1-u^{fp^N}}(\mathrm{mod}\ \mathfrak{p}_{\chi,\psi,\varphi,u}^{N_0})\\&\equiv\sum_{a=1}^{\frac{fp^N-1}{2}}\chi_n\psi\varphi(a)a^n\frac{u^a(u^{fp^N-2a}+1)}{1-u^{fp^N}}\ (\mathrm{mod}\ \mathfrak{p}_{\chi,\psi,\varphi,u}^{N_0}).\end{aligned}$$

Since $2\nmid fp^N$, we have $|u^{fp^N-2a}+1|\le|u+1|$. If $u=-1$, then $H^n_{\chi_n\psi\varphi}(u)=0$, and (0.4) holds. If $u\neq-1$, choose the above N_0 such that $\mathfrak{p}_{\chi,\psi,\varphi,u}^{N_0}\subset(1+u)\mathfrak{p}_{\chi,\psi,\varphi,u}$. Then,

$$\frac{u}{(1-u^f)(1+u)}H^n_{\chi_n\psi\varphi}(u)\equiv\sum_{a=1}^{\frac{fp^N-1}{2}}\chi_n\psi\varphi(a)a^n\frac{u^a(u^{fp^N-2a}+1)}{(1-u^{fp^N})(1+u)}\ (\mathrm{mod}\ \mathfrak{p}_{\chi,\psi,\varphi,u})$$

$$\equiv \sum_{\substack{a=1\\(a,p)=1}}^{\frac{fp^N-1}{2}} \chi\psi(a)\frac{u^a(u^{fp^N-2a}+1)}{(1-u^{fp^N})(1+u)} \pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}$$

$$\equiv \frac{1}{1+u}\sum_{\substack{a=1\\(a,p)=1}}^{fp^N-1} \chi\psi(a)\frac{u^{fp^N-a}}{1-u^{fp^N}} \pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}.$$

Thus, we have shown

$$\frac{u}{(1-u^f)(1+u)}H^n_{\chi_n\psi\varphi}(u)\equiv\frac{1}{1+u}F_{\chi\psi,u}(1)\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}.$$

Therefore, if ψ satisfies (3.1.6), then (0.4) holds.

Next, we consider the case $n=0$. If $p|f_{\chi\varphi}$, then Propositions 1.2.3, 1.2.4 show that $F_{\chi\psi,u}(\varphi(u_0))=\frac{u}{1-u^f}H^0_{\chi\psi\varphi}(u)$. In the case $\mu_{\chi,u}=\infty$, we have $H^0_{\chi\psi\varphi}(u)=0$, and (0.4) holds. In the case $\mu_{\chi,u}\neq\infty$, if ψ satisfies (3.1.6), then (0.4) holds. If $p\nmid f_{\chi\varphi}$, then Theorem 1 and Propositions 3.1.1, 3.1.2 show that $\mathrm{ord}_p(\frac{u}{1-u^f}H^0_{\chi\psi'\varphi}(u))=\mu_{\chi\varphi,u}=\mu_{\chi,u}$ holds for almost all $\psi'\in\Psi_S$. Further, we note that such φ is unique in Φ if it exists.

Thus, we conclude that (0.4) holds for almost all $\psi\in\Psi_S$ and for all $\varphi\in\Phi$ and $n\in\bar{\mathbb{N}}$.

Finally, let S' be a proper subset of S. If S' is empty, then $\Psi_{S'}$ is empty or a set consisting of only one element. If $\Psi_{S'}$ is an infinite set, then each element ψ of $\Psi_{S'}$ is expressed as $\psi=\psi_{S'}\psi'_{S'}$, where $\psi_{S'}$ (resp. $\psi'_{S'}$) is a Dirichlet character of the second kind for S' (resp. $S-S'$), and it is easy to see that $\psi'_{S'}$ depends only on the set S'. Hence, putting $\Psi'_{S'}=\{\psi\psi'^{-1}_{S'}|\psi\in\Psi_{S'}\}$ and applying the above argument to S', $\Psi'_{S'}$ and $\chi\psi'_{S'}$ instead of S, Ψ_S and χ, we deduce that

(0.4) holds for almost all $\psi\in\Psi_S$, and for all $\varphi\in\Phi$ and $n\in\mathbb{N}$. Since $\Psi=\bigcup_{S'\subset S}\Psi_{S'}$, we complete the proof of Theorem 3.

Remark. If $\chi(-1)=1$ and if $(2,f_\chi pp_1\cdots p_t)=1$, then

$$(1-2^n\chi_n\psi\varphi(2))\frac{B_{n,\chi_n\psi\varphi}}{n} = \frac{1}{2}H^{n-1}_{\chi_n\psi\varphi}(-1)$$

for all $n\in\mathbb{N}$, $\psi\in\Psi$ and $\varphi\in\Phi$ ([19]). Since $\chi_n=(\chi\omega^{-1})_{n-1}$ and $\chi\omega^{-1}(-1)=-1$, we obtain from Theorems 1, 3 the result (0.5) of Friedman under the restricted condition that $(2,f_\chi pp_1\cdots p_t)=1$.

2. Another proof of Theorem 3

In this section, using the methods in [4] and [20], we give another proof of Theorem 3 in the case $|u|=1$. Note that if $|u|\neq 1$, the assertion of Theorem 3 was obtained easily from Lemma 1 of [19].

As in Section 1, let $S=\{p_1,\cdots,p_t\}$ be a finite set of prime numbers distinct from p and put $\mathbf{Z}_S=\mathbf{Z}_{p_1}\times\cdots\times\mathbf{Z}_{p_t}$. Let $q_i=p_i$ or $q_i=4$ according as $p_i>2$ or $p_i=2$ and put $Q=\prod_{i=1}^{t} q_i$. For $N=(n_1,\cdots,n_t)\in\mathbf{Z}^t$, put $\pi^N=\prod_{i=1}^{t} p_i^{n_i}$. For $N=(n_1,\cdots,n_t),M=(m_1,\cdots,m_t)\in\mathbf{Z}^t$, we write $N\geq M$ to mean $n_i\geq m_i$ for each $1\leq i\leq t$. We write simply $N\geq 0$ (resp. $N>0$) for $N\geq(0,\cdots,0)$ (resp. $N\geq(1,\cdots,1)$). We write $N>>0$ to mean $N\geq N_0$ for some fixed $N_0\geq 0$. For each $x=(x_1,\cdots,x_t)\in\mathbf{Z}_S$ and $N=(n_1,\cdots,n_t)\geq 0$, we denote by $S_N(x)$ the

unique integer satisfying $0 \le S_N(x) < Q\pi^N$ and $S_N(x) \equiv x_i \pmod{q_i p_i^{n_i} \mathbb{Z}_{p_i}}$ for $1 \le i \le t$. We regard each $\psi \in \Psi$ also as a character of $\mathbb{Z}_S^\times$. Let H denote the torsion subgroup of $\mathbb{Z}_S^\times$. For each $m \in \mathbb{N}$, let ζ_m denote an arbitrarily chosen primitive m-th root of unity. In the case $|u|=1$, Theorem 3 can be expressed as in the following

THEOREM 3'. *Suppose that* $|u|=1$ *and that the condition* (0.3) *holds. Then, we have*

$$(3.2.1) \qquad |H^n_{\chi_n \psi \varphi}(u)| = \begin{cases} |1+u| & \text{if } \chi \text{ is even} \\ 1 & \text{if } \chi \text{ is odd} \end{cases}$$

for almost all $\psi \in \Psi$ *and for all* $\varphi \in \Phi$ *and* $n \in \mathbb{N}$.

In order to prove this, we use the following

LEMMA 3.2.1 (*Friedman* [4]). *Let* $M \in \mathbb{Z}^t$ *with* $M>0$ *and* $g \in \mathbb{N}$ *with* $(g,Q)=1$. *Then, we can choose a complete set* E *of representatives of* H *modulo its subgroup* $\{1,-1\}=\{(1,\cdots,1),(-1,\cdots,-1)\}$, $\eta_0 \in E$ *and* $N_0 \in \mathbb{Z}^t$ *with* $N_0 \ge M$, *such that for all* $N \ge N_0$, *there exist* x_1 *and* x_2 *in* $\mathbb{Z}_S^\times$ *satisfying*

(1) $S_N(x_1\eta)=S_{N-M}(x_1\eta) \equiv 0 \pmod g$ *for all* $\eta \in E$,

(2) $S_N(x_2\eta)=S_{N-M}(x_2\eta) \equiv 0 \pmod g$ *for all* $\eta \in E-\{\eta_0\}$,

(3) $S_N(x_2\eta_0)=S_{N-M}(x_2\eta_0)+Q\pi^{N-M} \equiv 0 \pmod g$,

(4) $S_M(x_1)=S_M(x_2)=1$.

Now, we prove Theorem 3'. Without loss of generality, we assume that χ is of the first kind for each $p_1,\cdots,p_t$ and p. For each $\psi\in\Psi$, let $N_\psi\in\mathbb{Z}^t$ be such that $f_\psi=Q\pi^{N_\psi}$. Applying induction on t, it is sufficient to prove that (3.2.1) holds for all $\psi\in\Psi$ with $N_\psi>>0$, all $\varphi\in\Phi$ and all $n\in\mathbb{N}$.

In the case where χ is even and $u=-1$, we can deduce $H^n_{\chi_n\psi\varphi}(u)=0$ for all $\psi\in\Psi$, $\varphi\in\Phi$ and $n\in\mathbb{N}$ directly from Lemma 1 of [19]. We exclude this case in the rest of this section.

We deduce from Lemma 1 of [19] that

$$\frac{u}{1-u^{f_{\chi_n\psi}}}H^n_{\chi_n\psi}(u)\equiv\frac{u}{1-u^{f_{\chi_n\psi\varphi}}}H^n_{\chi_n\psi\varphi}(u)\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}$$

holds for all $\varphi\in\Phi$ and $n\in\mathbb{N}$ and that

$$\frac{u}{1-u^{f_{\chi_n\psi\varphi}}}H^n_{\chi_n\psi\varphi}(u)\equiv\frac{u}{1-u^{f_{\chi\psi\varphi}}}H^0_{\chi\psi\varphi}(u)\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}$$

holds for all $\varphi\in\Phi-\{1\}$ and $n\in\mathbb{N}$. If χ is even, we have

$$\frac{1}{1+u}\cdot\frac{u}{1-u^{f_{\chi_n\psi}}}H^n_{\chi_n\psi}(u)\equiv\frac{1}{1+u}\cdot\frac{u}{1-u^{f_{\chi_n\psi\varphi}}}H^n_{\chi_n\psi\varphi}(u)\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}$$

for all $\varphi\in\Phi$ and $n\in\mathbb{N}$ and

$$\frac{1}{1+u}\cdot\frac{u}{1-u^{f_{\chi_n\psi\varphi}}}H^n_{\chi_n\psi\varphi}(u)\equiv\frac{1}{1+u}\cdot\frac{u}{1-u^{f_{\chi\psi\varphi}}}H^0_{\chi\psi\varphi}(u)\pmod{\mathfrak{p}_{\chi,\psi,\varphi,u}}$$

for all $\varphi\in\Phi-\{1\}$ and $n\in\mathbb{N}$. Hence, it is sufficient to prove that, for an arbitrarily fixed $\varphi\in\Phi$, (3.2.1) holds for any $\psi\in\Psi$ with $N_\psi>>0$ and $n=0$.

We write $f_{\chi\varphi}=d\prod_{i=1}^{t}q_i^{a_i}$ with $(d,Q)=1$ and $a_i=0$ or 1. We fix $\psi\in\Psi$ and denote the conductor of $\chi\psi\varphi$ simply by f. Then, $f=dQ\pi^{N_\psi}$.

Let k be the field obtained by adjoining u and the values of χ and φ to $\mathbf{Q}$, and put $k_N=k(\xi_{\pi^N})$ for each $N\in\mathbf{Z}^t$ with $N\geq 0$. We can choose $M\in\mathbf{Z}^t$ with $M>0$ such that, for any $N\geq M$, $[k_N:k_M]=\pi^{N-M}$ and that the prime ideal of k_M defined by the valuation $|\ |$ does not split in k_N. Then, denoting by $T_{N,M}$ the trace map from k_N to k_M, we have $|T_{N,M}(x)|<1$ for any $x\in k_N$ with $|x|<1$.

Let f' be the least positive integer such that $|u^{f'}-1|<1$ and write $f'=d_1\pi^{M_1}$ with $(d_1,Q)=1$ and $M_1\in\mathbf{Z}^t$. In what follows, we fix $M\in\mathbf{Z}^t$ satisfying $M\geq M_1$ and the property described above. Further, we always suppose that $N\geq M$ and write simply T for $T_{N,M}$. We denote the prime ideal of the completion of k_N in $\mathbf{C}_p$ simply by p. For each $x\in\mathbf{Z}_S^\times$, $j\in\mathbf{Z}$ and $\eta\in H$, we put $\tau_{x,j,\eta}=S_{N-M}(x\eta)+jQ\pi^{N-M}$, $A(x,j,\eta)=\psi(x)^{-1}\chi\psi\varphi(\tau_{x,j,\eta})$ and $B(x,j,\eta)=u^{-\tau_{x,j,\eta}}+\chi(-1)u^{-f+\tau_{x,j,\eta}}$. We first prove the following

LEMMA 3.2.2. *Let E be a complete set of representatives of H modulo $\{1,-1\}$, and suppose that $N=N_\psi$. Then, for any $x\in\mathbf{Z}_S^\times$, we have*

$$T(\psi(x)^{-1}H^0_{\chi\psi\varphi}(u))=u^{f-1}\pi^{N-M}\sum_{\eta\in E}\sum_{j=0}^{d\pi^{M}-1}A(x,j,\eta)B(x,j,\eta).$$

Proof. We have $H^0_{\chi\psi\varphi}(u)=\sum_{a=0}^{f-1}\chi\psi\varphi(a)u^{f-a-1}$. Hence,

$$T(\psi(x)^{-1}H^0_{\chi\psi\varphi}(u))=u^{f-1}\sum_{a=0}^{f-1}\chi\varphi(a)u^{-a}T(\psi(x^{-1}a)).$$

As in the proof of Lemma 1 of [4], we see that $T(\psi(x^{-1}a))\neq 0$ holds if and only if $\psi(x^{-1}a)^{\pi^M}=1$ and in that case, we have $T(\psi(x^{-1}a))=\pi^{N-M}\psi(x^{-1}a)$. Further, we have $\psi(x^{-1}a)^{\pi^M}=1$ if and only if $a\equiv\pm\eta x$ (mod $Q\pi^{N-M}$) for some $\eta\in E$. Since the integers a with $0\leq a<f$ satisfying the above property are exactly the integers $\tau_{x,j,\eta}$ and $f-\tau_{x,j,\eta}$ with $0\leq j<d\pi^M$, we obtain the assertion of this lemma.

We continue to prove Theorem 3'. In the first place, we consider the case $|1+\chi(-1)u|=1$, that is, χ is odd or both $\chi(-1)=1$ and $|1+u|=1$ hold. We put $g=dd_1$ and apply Lemma 3.2.1. Suppose that $N=N_\psi\geq M$ and that E, x_1 and x_2 in $\mathbf{Z}_S^\times$ and $\eta_0\in E$ satisfy the conditions (1)~(4) of Lemma 3.2.1. We prove $|H^0_{\chi\psi\varphi}(u)|=1$.

If $\eta\neq\eta_0$, we have $x_1^{-1}\tau_{x_1,j,\eta}\equiv x_2^{-1}\tau_{x_2,j,\eta}$ (mod f_ψ) and $\tau_{x_1,j,\eta}$ is congruent to $\tau_{x_2,j,\eta}$ modulo both $f_{\chi\varphi}$ and f'. Hence,

$$A(x_1,j,\eta)B(x_1,j,\eta)\equiv A(x_2,j,\eta)B(x_2,j,\eta)\ (\mathrm{mod}\ \mathfrak{p}).$$

On the other hand, if $\eta=\eta_0$, we have $x_1^{-1}\tau_{x_1,j-1,\eta_0}\equiv x_2^{-1}\tau_{x_2,j,\eta_0}$ (mod f_ψ) and $\tau_{x_1,j-1,\eta_0}$ is congruent to τ_{x_2,j,η_0} modulo both $f_{\chi\varphi}$ and f'. Hence,

$$A(x_1,j-1,\eta_0)B(x_1,j-1,\eta_0)\equiv A(x_2,j,\eta_0)B(x_2,j,\eta_0)\pmod{p}.$$

Therefore, we see from Lemma 3.2.2 that

$$\begin{aligned}(3.2.2)\quad & u^{-f+1}\pi^{M-N}T((\psi(x_1)^{-1}-\psi(x_2)^{-1})H^0_{\chi\psi\varphi}(u))\\ &\equiv A(x_1,d\pi^M-1,\eta_0)B(x_1,d\pi^M-1,\eta_0)-A(x_1,-1,\eta_0)B(x_1,-1,\eta_0)\pmod{p}\\ &\equiv A(x_1,-1,\eta_0)u^{\tau_{x_1,-1,\eta_0}}(u^{-2\tau_{x_1,-1,\eta_0}}-\chi(-1))(u^{-dQ\pi^M}-1)\pmod{p}.\end{aligned}$$

Now, it is easy to see that $(\tau_{x_1,-1,\eta_0},f_{\chi\psi\varphi})=1$, and so, $|A(x_1,-1,\eta_0)|=1$. From (0.3), we have $|u^{-dQ\pi^M}-1|=1$.

Suppose that $|H^0_{\chi\psi\varphi}(u)|<1$. Then, we have $|u^{-2\tau_{x_1,-1,\eta_0}}-\chi(-1)|<1$. If χ is even, then we have $2\tau_{x_1,-1,\eta_0}\equiv 0\pmod{f'}$ and we deduce $f'=2$, namely, $|u+1|<1$, which contradicts the assumption that $|u+1|=1$. If χ is odd, then $4\tau_{x_1,-1,\eta_0}\equiv 0\pmod{f'}$ and we deduce $f'=4$. Further, we see easily that $\sum_{a=0}^{2f-1}\chi\psi\varphi(a)u^a\equiv\sum_{a=0}^{2f-1}\chi\psi\varphi(a)u^{-a}\equiv 0\pmod{p}$, and so $\sum_{a=0}^{2f-1}\chi\psi\varphi(a)(u^a+u^{-a})\equiv 0\pmod{p}$. Since $f'=4$, we have $u^2\equiv -1\pmod{p}$. Hence, we obtain $2\chi\psi\varphi(2)\sum_{a=0}^{f-1}\chi\psi\varphi(a)(-1)^a\equiv 0\pmod{p}$. From (0.3) and $f'=4$, we deduce $(2,fpQ)=1$. Hence, $H^0_{\chi\psi\varphi}(-1)\equiv 0\pmod{p}$. Then, in the same way as we have deduced (3.2.2), we obtain

$$((-1)^{-2\tau_{x_1,-1,\eta_0}}-\chi(-1))((-1)^{dQ\pi^M}-1)\equiv 0\pmod{p},$$

namely $2(-2)\equiv 0\pmod{p}$, which is a contradiction. Thus, we conclude

that $|H^0_{\chi\psi\varphi}(u)|=1$.

Finally, suppose that χ is even and that $|1+u|<1$. Then, (0.3) implies $(fpQ,2)=1$, and we have $|(1+u)^{-1}H^0_{\chi\psi\varphi}(u)|\leq 1$. We must show that $|(1+u)^{-1}H^0_{\chi\psi\varphi}(u)|=1$. Put $g=2dp$ and apply Lemma 3.2.1. Suppose that $N=N_\psi\geq M$ and that E, x_1 and x_2 in $\mathbb{Z}_S^\times$ and $\eta_0\in E$ satisfy the conditions (1)~(4) of Lemma 3.2.1. We prove $|(1+u)^{-1}H^0_{\chi\psi\varphi}(u)|=1$. We have, for $\eta\neq\eta_0$,

$$(3.2.3)\qquad (u^{S_{N-M}(x_1\eta)}-u^{S_{N-M}(x_2\eta)})/(u+1)\equiv 0 \pmod{p}.$$

In fact, putting $a=\max\{S_{N-M}(x_1\eta)-S_{N-M}(x_2\eta),\ S_{N-M}(x_2\eta)-S_{N-M}(x_1\eta)\}$, we have $(u^a-1)/(u+1)=\sum_{j=0}^{a-1}(-u)^j\equiv a \pmod{p}$. Since $p|g$, we have $a\equiv 0 \pmod{p}$. Hence (3.2.3) holds. In the case $\eta=\eta_0$, we have

$$(3.2.4)\qquad (u^{S_{N-M}(x_1\eta_0)-Q\pi^{N-M}}-u^{S_{N-M}(x_2\eta_0)})/(u+1)\equiv 0 \pmod{p}.$$

Using a similar method that we have deduced (3.2.2) and taking account of (3.2.3) and (3.2.4), we deduce

$$u^{-f+1}\pi^{N-M}T((\psi(x_1)^{-1}-\psi(x_2)^{-1})H^0_{\chi\psi\varphi}(u))/(u+1)$$
$$\equiv A(x_1,-1,\eta_0)u^{\tau_{x_1,-1,\eta_0}}(u^{-dQ\pi^M}-1)(u^{-2\tau_{x_1,-1,\eta_0}}-1)/(u+1) \pmod{p}.$$

Now,

$$(u^{2\tau_{x_1,-1,\eta_0}}-1)/(u+1)=\sum_{j=0}^{2\tau_{x_1,-1,\eta_0}-1}(-u)^j\equiv 2\tau_{x_1,-1,\eta_0}\pmod{p},$$

and since $(p,2)=1$ and $\tau_{x_1,-1,\eta_0}\equiv -Q\pi^{N-M}\not\equiv 0\pmod{p}$, we see

$$T((\psi(x_1)^{-1}-\psi(x_2)^{-1})H^0_{\chi\psi\varphi}(u))/(u+1)\not\equiv 0\pmod{p}.$$

Thus, we obtain $|(1+u)^{-1}H^0_{\chi\psi\varphi}(u)|=1$.

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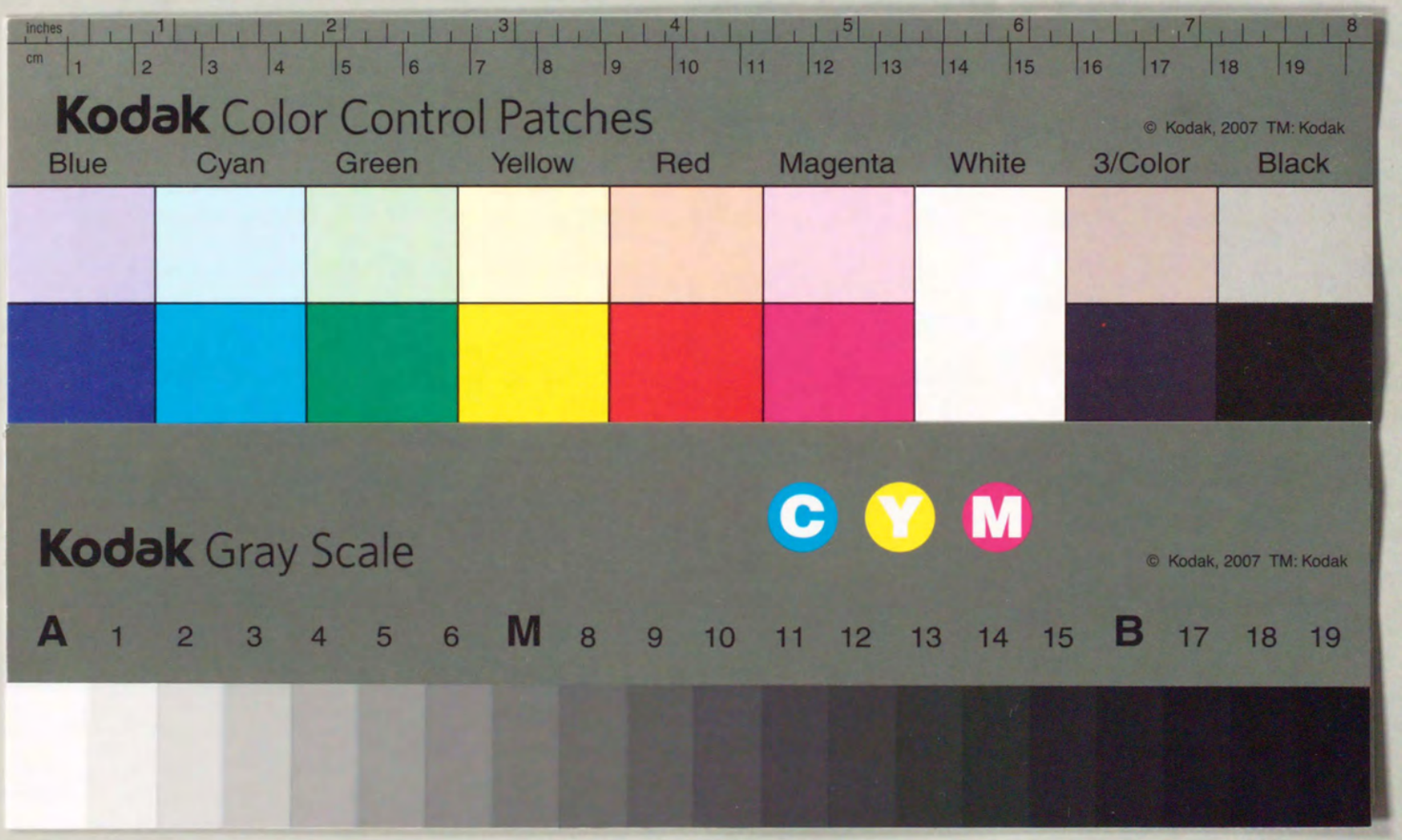
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