A p-ADIC INTERPOLATING FUNCTION OF THE GENERALIZED EULER NUMBERS--AND ITS INVARIANTS

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https://doi.org/10.11501/3054108

出版情報：九州大学，1990，理学博士，課程博士
バージョン：
権利関係：
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Let \( u \neq 1 \) be an algebraic number. The \( n \)-th Euler number \( H^n(u) \) belonging to \( u \) is defined by

\[
\frac{1-u}{e^{t-u}} = \sum_{n=0}^{\infty} \frac{H^n(u)}{n!} t^n.
\]

Let \( p \) be a prime number and \( x \) a primitive Dirichlet character. Shiratani-Yamamoto ([15]) constructed a \( p \)-adic interpolating function \( G_p(s,u) \) of the Euler numbers \( H^n(u) \), and as its applications to the \( p \)-adic \( L \)-functions \( L_p(s,x) \), derived an explicit formula for \( L_p'(0,x) \) including the Ferrero-Greenberg formula ([2]), and gave an explanation of Diamond's formula ([1]).

Let \( f \) be the conductor of \( x \). As analogue to the generalized Bernoulli numbers, Tsumura ([19]) defined the \( n \)-th generalized Euler number \( H^n_{\chi}(u) \) for \( \chi \) belonging to \( u \) by

\[
(0.1) \quad \frac{f_{\chi}^{-1}}{2} \sum_{a=0}^{f_{\chi}} \frac{f_{\chi}(a)}{f_{\chi} t} \frac{a t u \chi^{-a-1}}{e^{\chi t - u \chi}} = \sum_{n=0}^{\infty} \frac{H^n_{\chi}(u)}{n!} t^n,
\]

and he constructed a \( p \)-adic interpolating function \( l_p(u,s,\chi) \), which is an extension of \( G_p(s,u) \). Further, by considering the expansion of \( l_p(u,s,\chi) \) at \( s=1 \), he obtained some congruences for the generalized Euler numbers.

As for the \( p \)-adic \( L \)-functions \( L_p(s,x) \), Ferrero-Washington ([3]) showed that when \( \chi \) is even, the \( \mu \)-invariant of the interpolating

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The power series of $L_p(s,\chi)$ is zero. This implies that Iwasawa's $\mu$-invariant is zero for the basic $\mathbb{Z}_p$-extension of any finite abelian number field ([3]). Friedman ([4]) generalized the Ferrero-Washington theorem to the basic $\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$-extension of any finite abelian number field.

Sinnott ([16]) showed how to calculate the $\mu$-invariants of the $\Gamma$-transforms of rational functions and gave a new proof of the Ferrero-Washington theorem. By similar technique, an analytic property of the interpolating power series of $L_p(s,\chi)$ was investigated in [17] and a new proof of Friedman's theorem was given in [18].

In the present paper, by using the methods in the above references, we shall investigate $p$-adic properties of the generalized Euler numbers.

In Chapter 1, we first summarize some of the basic properties of the $p$-adic measures, and then, we reconstruct the function $l_p(u,s,\chi)$ by constructing an interpolating power series $F_{x,u}(T) \in O_{x,u}[\mathbb{Z}^\mathbb{Z}]$. Here, $O_{x,u}$ is the ring of integers of the field generated by $u$ and the values of $x$ over the $p$-adic rational number field $\mathbb{Q}_p$. Let $\mathbb{C}_p$ denote the completion of the algebraic closure of $\mathbb{Q}_p$, $\| \|$ denote the multiplicative valuation of $\mathbb{C}_p$ normalized by $\|p\|=1/p$ and ord$_p(\ )$ denote the additive valuation of $\mathbb{C}_p$ normalized by ord$_p(p)=1$. As usual, we put ord$_p(0)=\infty$ and understand $\infty > c$ for all rational numbers $c$. Let $F_{x,u}(T) = \sum_{n=0}^{\infty} a_{x,n,u}(T-1)^n$ and we define its $\mu$-invariant $\mu_{x,u}$ by $\mu_{x,u} = \min(\text{ord}_p(a_{x,n,u})|n \geq 0)$. The purpose of Chapter 1 is to calculate the value of $\mu_{x,u}$. Denote by $N$ the set of positive integers as usual and
we put \( \mathbb{N} = \mathbb{N} \cup \{0\} \). We shall obtain the following

**THEOREM 1.** Suppose that we have

\[
|1-uX^p|^n \geq 1 \quad \text{for all } n \in \mathbb{N}.
\]

Then,

\[
\mu_{x,u} = \begin{cases} 
-\text{ord}_p(u) & \text{if } |u| > 1 \\
\text{ord}_p(u) & \text{if } |u| < 1 \\
\text{ord}_p(1+u) & \text{if } |u| = 1 \text{ and } x \text{ is even} \\
0 & \text{otherwise.}
\end{cases}
\]

We shall give two proofs, firstly by the methods of Sinnott ([16]) and secondly by the methods of Ferrero-Washington ([3]) and Gillard ([5]). Further, we shall note that Theorem 1 includes the Ferrero-Washington theorem under the appropriate condition on the conductor of \( x \).

In Chapter 2, we shall investigate an analytic property of the function \( F_{x,u}(z) \) defined on the set \( D = \{ z \in \mathbb{C}_p | |z-1| < 1 \} \). (Here and in the sequel, "analytic" means "Krasner analytic" ([10]).) We shall prove the following

**THEOREM 2.** Assume that the condition (0.2) holds. Then, \( F_{x,u}(z) \) is an analytic function on \( D \). If \( u \neq 0 \) and if either \( x(-1) = -1 \) or \( u \neq -1 \) holds, then \( F_{x,u}(z) \) has no analytic continuation to any quasi-connected subset of \( \mathbb{C}_p \) properly containing \( D \).
In Chapter 3, fixing a finite set $S=(p_1, \cdots, p_t)$ of prime numbers distinct from $p$, and denoting by $\Psi$ (resp. $\Phi$) the set of Dirichlet characters of the second kind for $S$ (resp. $p$), we shall investigate the $p$-adic valuation of $H_{\chi_\omega^{-n}\psi \varphi}^{n}(u)$ for $\psi \in \Psi$, $\varphi \in \Phi$ and $n \in \mathbb{N}$, where $\omega$ is the Teichmüller character for $p$. The main result is as follows.

**THEOREM 3.** Suppose that we have

$$1 -\frac{u}{1-u} \prod_{i=1}^{n} p_{i}^{n_{i}} \geq 1 \quad \text{for all } n_1, \cdots, n_t, n \in \mathbb{N}. \tag{0.3}$$

For each $n \in \mathbb{N}$, put $\chi_n = \chi_\omega^{-n}$. Then, we have

$$\ord_p \left( \frac{u}{1-u} H_{\chi_\omega^{-n}\psi \varphi}^{n}(u) \right) = \mu \chi, u \tag{0.4}$$

for almost all $\psi \in \Psi$ and for all $\varphi \in \Phi$ and all $n \in \mathbb{N}$. In particular, if $|u| > 1$, then (0.4) holds for all $\psi \in \Psi$, all $\varphi \in \Phi$ and all $n \in \mathbb{N}$, except for the case that $n=0$, $\chi \varphi = 1$ and $|u| > 1$, in which case, we have

$$\ord_p \left( \frac{u}{1-u} H_{\chi_0}^{0}(u) \right) = 0. \tag{Here and in the sequel, "almost all" means "all but finitely many".}$$

Now, let $B_{n, \chi}$ denote the $n$-th generalized Bernoulli number for $\chi$ as usual. Friedman (14) showed that if $\chi$ is even, then we have

$$\left| \frac{B_{n, \chi \varphi \chi}}{2n} \right| = 1 \tag{0.5}$$
for almost all \( \psi \in \Phi \) and for all \( \varphi \in \Phi \) and \( n \in \mathbb{N} \). We can view Theorem 3 as an analogous result on Euler numbers to Friedman's theorem described above. Further, we shall note that Theorem 3 includes Friedman's theorem under the appropriate condition on \( \chi \).

Throughout this paper, denoting by \( \bar{Q} \) the algebraic closure of the rational number field \( Q \) in the complex number field \( C \), we fix an embedding of \( \bar{Q} \) into \( C_p \) and regard \( \bar{Q} \) also as a field contained in \( C_p \). A Dirichlet character always means a primitive one. In general, if \( R \) is a ring, we write \( R^\times \) for the multiplicative group of units of \( R \), \( R[T] \) for the ring of polynomials in an indeterminate \( T \) with coefficients in \( R \), and \( R[[T^{-1}]] \) for the ring of formal power series in an indeterminate \( T^{-1} \) with coefficients in \( R \). If \( F \) is a field, we write \( F(T) \) for the field of rational functions with coefficients in \( F \).

Denoting by \( \mathbb{Z}_p \) the ring of integers of \( Q_p \) as usual, we put \( \langle x \rangle = x/\omega(x) \) for any \( x \in \mathbb{Z}_p^\times \).

The author wishes here to express his deep gratitude to Professor Shiratani for his constant encouragement.
1. The $p$-adic measures

In this section, we summarize some of the basic properties of the $p$-adic measures which will be used later ([12],[13],[16]).

If $O$ is the ring of integers of a finite extension $k$ of $\mathbb{Q}_p$, we denote by $\Lambda_O$ the ring of $O$-valued measures on $\mathbb{Z}_p$. If $\alpha \in \Lambda_O$, we put

$$a(T) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} \frac{x}{n!} \alpha(x) \right) (T-1)^n \in O[[T-1]].$$

Then,

$$(1.1.1) \quad a(T) = \sum_{a=0}^{p-1} \alpha(a+p \mathbb{Z}_p) T^a \mod (T^{p^n}-1) O[[T-1]].$$

for all $n \in \mathbb{N}$, and the map $\alpha \mapsto a(T)$ gives an isomorphism of $\Lambda_O$ with $O[[T-1]]$. If $\alpha \in \mathbb{Z}_p^\times$, we write $\alpha \cdot a$ for the measure defined by $\alpha \cdot a(X) = \alpha(aX)$ for any compact and open subset $X$ of $\mathbb{Z}_p$. Then $a(T) = a(T^{-1})$.

where we put $T^x = \sum_{n=0}^{\infty} (x)(T-1)^n$ for $x \in \mathbb{Z}_p$.

We define the $\Gamma$-transform $\Gamma_{\alpha} : \mathbb{Z}_p \to O$ of $\alpha \in \Lambda_O$ by

$$\Gamma_{\alpha}(s) = \int_{\mathbb{Z}_p} \frac{x}{s} \alpha(x) \cdot \delta(x).$$
We put \( q = 4 \) or \( p \) according as \( p = 2 \) or \( p \geq 3 \) and fix a topological generator \( u_0 \) of the multiplicative group \( 1 + q \mathbb{Z}_p \). We define a continuous homomorphism \( t: \mathbb{Z}_p^\times \to \mathbb{Z}_p \) by \( \langle x \rangle = u_0^{t(x)} \) for each \( x \in \mathbb{Z}_p^\times \). Put

\[
\mathcal{F}_\alpha(T) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^n d\alpha(x) \right) (T-1)^n \in \mathbb{O}[[T-1]].
\]

Then,

\[
\Gamma_\alpha(s) = \mathcal{F}_\alpha(u_0^s).
\]

If \( \alpha^* \) denote the measure defined by

\[
\alpha^*(a + p^n \mathbb{Z}_p) = \alpha(t^{-1}(a + p^n \mathbb{Z}_p))
\]

for all \( a \in \mathbb{Z}_p \) and \( n \in \mathbb{N} \), then we have

\[
(1.1.2) \quad \mathcal{F}_\alpha(T) = \sum_{a=1}^{p^n-1} \alpha^*(a + p^n \mathbb{Z}_p) T^a \pmod{(T^{p^n}-1) \mathbb{O}[[T-1]]}).
\]

For any \( \alpha \in \mathbb{A}_0 \), we denote by \( \tilde{\alpha} \) the measure on \( \mathbb{Z}_p \) defined by restricting \( \alpha \) to \( \mathbb{Z}_p^\times \) and extending by 0. For any \( g(T) \in \mathbb{O}[[T-1]] \), we put \( \tilde{g}(T) = g(T) - 1/p \sum_{\xi = 1}^{p-1} g(\xi T) \in \mathbb{O}[[T-1]] \). Then, we have

\[
(\tilde{\alpha})(T) = (\tilde{\alpha})(1).
\]
We also note that $F_\alpha(T)=F_{-\alpha}(T)$.

For any $g(T)\in\mathcal{O}[[T-1]]$ and any Dirichlet character $\nu$ with conductor $p^n$, we put

$$g_\nu(T)=\frac{1}{\tau(\nu^{-1},\zeta_p^n)} \sum_{a=1}^{p^n} \nu^{-1}(a)g(\zeta_p^n aT),$$

where $\zeta_p^n$ is a primitive $p^n$-th root of unity and $\tau(\nu^{-1},\zeta_p^n)=$

$$\sum_{a=1}^{p^n} \nu^{-1}(a)\zeta_p^n a.$$

It is easy to see that $g_\nu(T)$ is independent of the choice of $\zeta_p^n$. Let $\mathcal{O}_\nu'$ denote the ring of integers of the field $k(\zeta_p^n)$, which contains the values of $\nu$. Then $g_\nu(T)\in\mathcal{O}_\nu'[[T-1]]$. For any $\alpha\in\Lambda_0$ and any $n\in\mathbb{N}$, we have

$$F_\alpha(u_0^n)=(T\cdot d/dT)^n((\hat{\alpha}))^\omega_{-n}(T)|_{T=1}$$

$$=(d/dz)^n((\hat{\alpha}))^\omega_{-n}(e^z)|_{z=0}.$$  

For any $\alpha\in\Lambda_0$ and any Dirichlet character $\phi$ of the second kind for $\rho$, we denote by $\alpha_\phi$ the measure in $\Lambda_{\mathcal{O}_\phi}$, satisfying $(\hat{\alpha}_\phi)(T)=(\hat{\alpha})_\phi(T)$ $\in\mathcal{O}_\phi'[[T-1]]$. Then, we have

$$F_{\alpha_\phi}(T)=F_{\alpha}(\phi(u_0)T).$$
2. Construction of an interpolating power series $F_{\chi, u}(T)$

Let $\chi$ be a Dirichlet character with conductor $f_{\chi}$ and $u \neq 1$ an element of $\bar{Q}$. As in Introduction, we define the $n$-th generalized Euler number $H_{\chi}^n(u)$ by (0.1). Let $Q_p(\chi, u)$ be the field generated by $u$ and the values of $\chi$ over $Q_p$ and, as described in Introduction, we denote by $O_{\chi, u}$ the ring of integers of $Q_p(\chi, u)$. In this section, we assume the condition (0.2) and construct a power series $F_{\chi, u}(T) \in O_{\chi, u}[[T^{-1}]]$ which interpolates the generalized Euler numbers.

If $g \in \mathbb{N}$ is any multiple of $f_{\chi}$, we put

$$R_{\chi, u}(T) = \sum_{a=0}^{g-1} \frac{\chi(a) u^{g-a} T^a}{T^g - u^g} \in O_{\chi, u}[[T^{-1}]],$$

which is independent of the choice of $g$.

**PROPOSITION 1.2.1.** $R_{\chi, u}(T)$ lies in $O_{\chi, u}[[T^{-1}]]$.

**Proof.** Put $g = f_{\chi}$. If $|u| \leq 1$, then the condition (0.2) implies $|1-u^g|=1$, and so, $T^g - u^g = (1-u^g) + \sum_{a=0}^{g-1} \binom{g}{a} (T-1)^a \in O_{\chi, u}[[T^{-1}]]$. Hence, $R_{\chi, u}(T)$ lies in $O_{\chi, u}[[T^{-1}]]$. If $|u| > 1$, then $|(1/u)^{g-1}| = 1$, and so, $(T/u)^{g-1} \in O_{\chi, u}[[T^{-1}]]$. Hence, $R_{\chi, u}(T) = \sum_{a=0}^{g-1} \frac{\chi(a) u^{-a} T^a}{(T/u)^g - 1} \in O_{\chi, u}[[T^{-1}]]$.

Now, we put $\Lambda_{\chi, u} = \Lambda_{\chi, u}$ and let $\alpha_{\chi, u} \in \Lambda_{\chi, u}$ be the measure satisfying $\sum_{a=0}^{g-1} \frac{\chi(a) u^{-a} T^a}{(T/u)^g - 1} \in O_{\chi, u}[[T^{-1}]]$. Put

$-g-$
\[ F_{\chi, u}(T) = F_{\alpha_{\chi}, u}(T) = F_{\alpha_{\chi}, u}(T). \]

**Lemma 1.2.2.** (1) If \( g \in \mathbb{N} \) is a common multiple of \( f_{\chi} \) and \( p \), then,

\[
R_{\chi, u}(T) = \sum_{a=0}^{g-1} \frac{\mu(a)u^{g-a_{\gamma}a}}{\tau^{g-u}g} = R_{\chi, u}(T) - x(p)R_{\chi, u}(T^p).
\]

(2) If \( \nu \) is a Dirichlet character with conductor a power of \( p \), then

\[
(R_{\chi, u\nu}(T)) = R_{\chi, u\nu}(T).
\]

**Proof.** If \( g \in \mathbb{N} \) is divisible by \( f_{\chi} \) and \( p \), then

\[
\sum_{\xi^p=1} R_{\chi, u}(\xi T) = \sum_{\xi^p=1} \frac{\mu(a)u^{g-a_{\gamma}a}}{\tau^{g-u}g} = \frac{\mu(a)u^{g-a_{\gamma}a}}{\tau^{g-u}g}.
\]

Considering the special case \( g = f_{\chi}p \), we see that

\[
R_{\chi, u}(T) = \sum_{\xi^p=1} R_{\chi, u}(\xi T) = \sum_{\xi^p=1} \frac{\mu(a)u^{g-a_{\gamma}a}}{\tau^{g-u}g}.
\]

Hence, we obtain the assertion of (1).

As for (2), the assertion is obvious if \( \nu = 1 \). Suppose that \( \nu \neq 1 \) and \( f_{\nu} = p^n \), and put \( g = f_{\chi}p^n \). Then, a direct calculation shows...
\[
\langle R_{\chi, u} \rangle_{u(T)} = \frac{1}{\tau(v^{-1}, \xi)} \sum_{\substack{a=1 \ b=0 \ (b,p)=1 \ g-1 \ v^{-1}(b(b)v(b))u^g-b+b \ g-1 \ b=0 \ b=0 \ b=0 \ b=0 \ g-1 \ (b,p)=1 \ p_n \ a=1}} \sum_{b=0}^{p_n} \frac{\chi(b)v(b)u^{g-b+b}}{T^g - u^g} \]

\[
= R_{\chi, u(T)} - \frac{1}{p} \sum_{\psi^p = 1} R_{\chi, u}(\xi T).
\]

Hence, we deduce our assertion.

In the sequel, we put \( \chi_n = \chi^{\omega^{-n}} \) for each \( n \in \mathbb{N} \).

**PROPOSITION 1.2.3.** For each \( n \in \mathbb{N} \), let \( f_n \) denote the conductor of \( \chi_n \). Then, we have

\[
F_{\chi, u}(u^0, n) = \frac{u}{f_n} \frac{u^{-n}}{1-u} \frac{\chi_n(p)^n u^p}{1-u} - \frac{\chi_n(p)^n u^p}{1-u} \frac{h_n(u^p)}{1-u}.
\]

**Proof.** From (1.1.3) and the definition of \( F_{\chi, u}(T) \), we have

\[
\frac{\chi_n(\omega^{-n})}{(d/dz) \big|_{z=0}} \bigg( \frac{\chi_n(\omega^{-n})}{\omega^{-n}(e^z)} \bigg) = 0.
\]

Since \( \langle \alpha(\chi, u) \rangle(T) = \langle \alpha(\chi, u) \rangle(T) = \hat{R}_{\chi, u}(T) \), Lemma 1.2.2 shows that

\[
\langle \alpha(\chi, u) \rangle_{\omega^{-n}}(T) = R_{\chi_n, u(T)} - \chi_n(p) R_{\chi_n, u(T)}.
\]

Hence, the assertion follows from the definition of the generalized...
Euler numbers (0.1).

**PROPOSITION 1.2.4.** Let \( \phi \) be a Dirichlet character of the second kind for \( p \). Then,

\[
F_{\chi \phi, u}(T) = F_{\chi, u}(\phi(u_0)T).
\]

**Proof.** Lemma 1.2.2 shows that \( \alpha_{\chi \phi, u} = (\alpha_{\chi, u}) \phi \). Hence, the assertion follows from (1.1.4).

**Remark.** From Proposition 1.2.3 and Theorem 1 of [19], we see that the function \( L_p(u, s, x) \) in [19] is equal to \( F_{\chi, u}(u_0^{-S}) \).

3. The \( \mu \)-invariant of \( F_{\chi, u}(T) \)

In this section, we calculate the \( \mu \)-invariant of \( F_{\chi, u}(T) \) by the methods in [16] and prove Theorem 1.

For any power series \( f(T) = \sum_{n=0}^{\infty} a_n(T-1)^n \in \mathbb{C}_p[[T-1]] \) with \( |a_n| \leq 1 \) for all \( n \in \mathbb{N} \), we define its \( \mu \)-invariant by \( \mu(f(T)) = \inf_{n \in \mathbb{N}} \{ \text{ord}_p(a_n) \} \). Note that if \( f(T) \) is a polynomial such that \( f(T) = \sum_{n=0}^{m} b_n T^n \), then \( \mu(f(T)) = \min_{0 \leq n \leq m} \{ \text{ord}_p(b_n) \} \). As described in Introduction, we put \( \mu_{\chi, u} = \mu(F_{\chi, u}(T)) \).

**Proof of Theorem 1.** Theorem 1 of [16] states that \( \mu_{\chi, u} = \mu(R_{\chi, u}(T)) \). Put \( g = f \chi \rho \). Then, a direct calculation shows
(1.3.1) \( R_{x,u}(T) + R_{x,u}(T^{-1}) = \sum_{a=0}^{g-1} \frac{\chi(a)u^a(u^2(g-a)-\chi(-1)) \left( T^a + T^{-a} \right)}{(T^a - u^a)(T^{-a} - u^a)} \)

\[
= \sum_{a=0}^{g-1} \frac{\chi(a)u^a(u^2(g-a)-\chi(-1)) \left( T^a + T^{-a} \right)}{(T^a - u^a)(T^{-a} - u^a)}. 
\]

If \(|u| > 1\), then

\[
\mu_{x,u} = \mu \left( \sum_{a=0}^{g-1} \frac{\chi(a)u^a(u^2(g-a)-\chi(-1)) \left( T^a + T^{-a} \right)}{(T^a - u^a)(T^{-a} - u^a)} \right). 
\]

Since \(\mu((T/u)^g - 1)(T^{-g} - (1/u)^g) = 0\), we obtain

\[
\mu_{x,u} = \min \{ \text{ord}_p (u^a(u^2(g-a)-\chi(-1))) \mid 1 \leq a < g, (a,g) = 1 \} = -\text{ord}_p(u).
\]

Next, suppose that \(|u| \leq 1\). Then \(\mu((T^g - u^g)(T^{-g} - u^g)) = 0\), hence we have

\[
\mu_{x,u} = \min \{ \text{ord}_p (u^a(u^2(g-a)-\chi(-1))) \mid 1 \leq a < g, (a,g) = 1 \}.
\]

If \(|u| < 1\), then \(\mu_{x,u} = \text{ord}_p(u)\). If \(|u| = 1\) and if \(x\) is even, then

\[
\mu_{x,u} = \min \{ \text{ord}_p (u^2a - 1) \mid 1 \leq a < g, (a,g) = 1 \} = \text{ord}_p(u^2 - 1).
\]

Moreover (0.2) implies \(|u-1| = 1\) if \(|u| = 1\). Hence, \(\mu_{x,u} = \text{ord}_p(1+u)\). If \(|u| = 1\) and if \(x\) is odd, then we have

\[
-13-
\]
\[ \mu_{\chi, u} = \min \{ \text{ord}_p (u^{2a} + 1) \mid 1 \leq a < g, (a, g) = 1 \}. \]

If \( 2 \mid g \), then \( 4 \mid f_{\chi} \) or \( p = 2 \), and so, \( 4 \mid f_{\chi} p^2 \). Then, \( (0.2) \) implies \( |1-u^4| = 1 \) and we deduce \( |1+u^2| = 1 \). Thus, we obtain \( \mu_{\chi, u} = 0 \). If \( 2 \nmid g \), then both \( u^2 + 1 \) and \( u^4 + 1 \) belong to the set \( \{ u^{2a} + 1 \mid 1 \leq a < g, (a, g) = 1 \} \). If \( |u^2 + 1| = 1 \), then we immediately have \( \mu_{\chi, u} = 0 \). If \( |u^2 + 1| < 1 \), then \( |u + 1| = |(u^2 + 1)(u - 1)| = 1 \), hence, \( |u^4 + 1| = |u^2(u + 1)(u - 1)(u^2 + 1)| = 1 \). Thus, we obtain \( \mu_{\chi, u} = 0 \).

Remark. Suppose that \( \chi \) is odd. If \( c \) is an integer with \( c > 1 \) and \( (c, f_{\chi} p) = 1 \), we have

\[ \sum_{\xi \in \mathbb{Z}} l_p (\xi, \chi, \sigma) = (1 - \chi \omega (c)^{<c\sigma>^{1-g}}) L_p (\sigma, \chi \omega) \]

(119). In particular, if \( (2, f_{\chi} p) = 1 \), then

\[ l_p (-1, \sigma, \chi) = (1 - \chi \omega (2)^{<2\sigma>^{1-g}}) L_p (\sigma, \chi \omega). \]

Consequently

\[ L_p (\sigma, \chi \omega) = g_{\chi \omega} (u_0^{1-g}), \]

where \( g_{\chi \omega} (T) = F_{\chi, -1} (u_0^{1-T}) / (1 - \chi \omega (2)^T (2)^{<2\sigma>}). \) Hence, by Theorem 1, we deduce the Ferrero-Washington theorem under the restricted condition that \( (2, f_{\chi} p) = 1 \).
4. Another proof of Theorem 1

In this section, using the methods in [3] and [5], we give another proof of Theorem 1. We first introduce notations.

We write $f_\chi = dp^n$ with $(d, p) = 1$ and $r \in \mathbb{N}$. For each $n \in \mathbb{N}$, the canonical ring homomorphism $\rho_n : \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/dp^n\mathbb{Z}$ is defined, and we can view $\chi$ also as a function on $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$.

For any $x \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$, we write $s_{n,d}(x)$ for the unique integer satisfying $0 \leq s_{n,d}(x) < dp^n$ and $s_{n,d}(x) \mod dp^n = \rho_n(x)$. Similarly, for any $y \in \mathbb{Z}/p^n$, we write $s_n(y)$ for the unique integer satisfying $0 \leq s_n(y) < qp^n$ and $s_n(y) \equiv y \mod qp^n$.

We identify an element $x \in \mathbb{Z}/p^n$ with $(1 \mod d, x) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$. We decompose the character $\chi$ as

$$\chi = \Theta \phi,$$

where $\Theta$ is a character of the first kind for $p$ and $\phi$ is a character of the second kind for $p$. We denote by $V$ the torsion subgroup of $\mathbb{Z}/p^n$ and put $W = \mathbb{Z}/d\mathbb{Z} \times V$.

LEMMAN 1.4.1. For all $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $0 \leq a < p^{n+r}$, we have

$$\alpha_{\chi, u}(a+p^n+r \mathbb{Z}/p^n) = \sum_{i=0}^{d-1} x(a+ip^n+r) \frac{f_x p^n - (a+ip^n+r)}{1-u x p^n}.$$

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Proof. Put $g = f p^{-1} d p^{n+r}$ and $a = (T p^{n+r} - 1) O_{x, u[[T^{-1}]]}$.

If $|u| \leq 1$, then $T^g - u^g \in (O_{x, u[[T^{-1}]]})^\times$ and $(T^g - u^g)^{-1} \equiv (1 - u^g)^{-1} \pmod{a}$. Then,

$$
\alpha_{x, u}(T) = R_{x, u}(T) \sum_{a=0}^{g-1} \chi(a) u^{-aT} \frac{p^{n+r} - 1}{T^g - u^g} \equiv \sum_{a=0}^{p^{n+r} - 1} \sum_{i=0}^{d-1} \chi(a+i p^{n+r}) u^{g-(a+i p^{n+r})} (1-i p^{n+r})(1-u^g)^{-1} \pmod{a}.
$$

Hence, by (1.1.1), we obtain our assertion.

If $|u| > 1$, then $(T/u)^{g-1} \in (O_{x, u[[T^{-1}]]})^\times$ and $((T/u)^{g-1})^{-1} \equiv ((1/u)^{g-1})^{-1} \pmod{a}$. Then,

$$
\alpha_{x, u}(T) = \sum_{a=0}^{g-1} \chi(a) u^{-aT} \frac{T^{g-1}}{(T/u)^{g-1}} \equiv \sum_{a=0}^{p^{n+r} - 1} \sum_{i=0}^{d-1} \chi(a+i p^{n+r}) u^{g-(a+i p^{n+r})} (1-i p^{n+r})(1/u)^{g-1} \pmod{a}.
$$

Hence our assertion holds also in the case $|u| > 1$.

**Proposition 1.4.2.** If $\nu_{x, u}$, then $\nu_{x, u}$ is the largest $c$ in $\mathbb{Q}$
such that

\begin{equation}
\left| \sum_{\eta \in \mathbb{W}} \frac{d\eta_{\Phi}^{m} g_{\eta, d_{\Phi}(x\eta)}}{i - u d\eta_{\Phi}^{m}} \right| \leq |\alpha|_{p}^{c}
\end{equation}

holds for all \( x \in \mathbb{Z}_{p}^{\times} \) and all \( n \in \mathbb{N} \). Further, \( \mu_{x, u} = \infty \) holds if and only if (1.4.1) holds for all \( c \in \mathbb{Q} \), \( x \in \mathbb{Z}_{p}^{\times} \) and \( n \in \mathbb{N} \).

Proof. For each \( n \in \mathbb{N} \), put \( a_{n} = (T_{p}^{n} - 1)\mathcal{O}_{x, u}[[T_{-1}]] \). From (1.1.2) and the definition of \( F_{x, u}(T) \), we have

\begin{equation}
F_{x, u}(T) = \sum_{a=0}^{p_{n-1}} a_{x, u}(a + p_{n}^{\mathbb{Z}_{p}}) T^{a} \pmod{a_{n}}.
\end{equation}

Note that, for any \( f(T) = \sum_{m=0}^{\infty} c_{m} (T-1)^{m} \in \mathcal{O}_{x, u}[[T_{-1}]] \), there are uniquely determined elements \( b_{0, n}, \ldots, b_{p_{n-1}, n} \in \mathcal{O}_{x, u} \) satisfying

\begin{equation}
f(T) \equiv \sum_{j=0}^{p_{n-1}} b_{j, n} T^{j} \pmod{a_{n}}.
\end{equation}

and that, for a given \( c \in \mathbb{Q} \), \( \operatorname{ord}_{p}(c_{m}) > c \) holds for all \( m \in \mathbb{N} \) if and only if \( \operatorname{ord}_{p}(b_{j, n}) > c \) holds for all \( n \in \mathbb{N} \) and \( j \in \mathbb{Z} \) with \( 0 \leq j < p_{n-1} \).

Hence, for a given \( c \in \mathbb{Q} \), \( \mu_{x, u} > c \) holds if and only if

\begin{equation}
|c_{x, u}(v + p_{n}^{\mathbb{Z}_{p}})| < |\rho|^{c}
\end{equation}

holds for all \( v \in \mathbb{Z}_{p} \) and \( n \in \mathbb{N} \).

For each \( x \in \mathbb{Z}_{p}^{\times} \), we have

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If \( p^n | q^n \), then Lemma 1.4.1 shows that

\[
\alpha_{x, u}^* (t(x) + p^n\mathbb{Z}_p^n) = \sum_{\eta \in \mathcal{V}} \alpha_{x, u} (s_n(x\eta) + iq^n_p)p^n\mathbb{Z}_p^n)
\]

Recall that \( \chi=0\varphi \) and that \( f_x=dp^n \) with \( (d, p)=1 \). Then, the conductor of \( \varphi \) divides \( p^n \), and so, \( \chi(s_n(x\eta) + iq^n) = \theta(s_n(x\eta) + iq^n)\varphi(x) \). Further the integers \( s_n(x\eta) + iq^n \) with \( \eta \in \mathcal{V} \) and \( 0 \leq i \leq d-1 \) are precisely \( g_n(x\eta) \) with \( \eta \in \mathcal{W} \). Since the conductor of \( \theta \) is \( d \) or \( dq \), we obtain

\[
(1.4.3) \quad \alpha_{x, u}^* (t(x) + p^n\mathbb{Z}_p^n) = \varphi(u_0) t(x) \sum_{\eta \in \mathcal{W}} \frac{d\varphi^n - s_n(x\eta)}{1 - u d\varphi^n}
\]

for all \( \eta \in \mathcal{W} \) with \( p^n | q^n \). In particular, we have

\[
(1.4.3)' \quad \alpha_{\theta, u}^* (t(x) + p^n\mathbb{Z}_p^n) = \theta(x) \sum_{\eta \in \mathcal{W}} \frac{d\varphi^n - s_n(x\eta)}{1 - u d\varphi^n}
\]

for all \( x \in \mathbb{Z}_p^\times \) and \( \eta \in \mathcal{W} \).

From (1.4.2), (1.4.3), (1.4.3)', and the fact that \( t(\mathbb{Z}_p^\times) = \mathbb{Z}_p^\times \), we obtain \( F_{x, u}(T) = F_{\theta, u}(\varphi(u_0)T) \). (Note that this can be obtained also from Proposition 1.2.4.) Hence, we see \( \mu_{x, u} = \mu_{\theta, u} \) and it follows that, for a given \( \alpha \in \mathcal{Q} \), \( \mu_{x, u} \succ \alpha \) holds if and only if \( |\alpha_{\theta, u}^*(v + p^n\mathbb{Z}_p^n)| < |p|^c \)
holds for all $v \in \mathbb{Z}_p$ and $\pi \in \mathbb{N}$. Taking account of the fact that $t(\mathbb{Z}_p^\times) = \mathbb{Z}_p$ again, we obtain from (1.4.3)' the required assertion.

Now, we prove Theorem 1.

If $|u| > 1$, then $|\sum_{\pi \in \mathbb{Z}} \theta(\pi) \frac{d_{\pi}^{\pi - \pi, d_n(x)}}{1 - u d_{\pi}^{\pi - \pi, d_n(x)}}| \leq |u|^{-1}$ for all $x \in \mathbb{Z}_p^\times$ and $\pi \in \mathbb{N}$. In particular, we have

$$|\sum_{\pi \in \mathbb{Z}} \theta(\pi) \frac{d_{\pi}^{\pi - \pi, d_n(x)}}{1 - u d_{\pi}^{\pi - \pi, d_n(x)}}| = |\sum_{\pi = 0}^{\pi - j} \theta(\pi) \frac{d_{\pi}^{\pi - \pi, d_n(x)}}{1 - u d_{\pi}^{\pi - \pi, d_n(x)}}| = |u|^{-1}.$$ 

Hence, we see from Proposition 1.4.2 that $\mu_x, u = -\text{ord}_p(u)$. In a similar way, we obtain $\mu_x, u = \text{ord}_p(u)$ if $|u| < 1$.

Now, we assume in the sequel that $|u| = 1$. Then, by (0.2), we have $|1 - u d_{\pi}^{\pi - \pi, d_n(x)}| = 1$ for all $\pi \in \mathbb{N}$. In order to prove Theorem 1, we use the following lemma (Proposition 4 of [5], Chapter 11 of [13]).

**Lemma 1.4.3.** Let $g, \pi \in \mathbb{N}$ with $(g, \pi) = 1$. Then, we can choose a complete set $R_0$ of representatives of $V$ modulo $(1, -1)$ such that, for any sufficiently large integer $l$, there exist $x_1, x_2 \in \mathbb{Z}_p^\times$ and $\eta_0 \in R_0$ satisfying the following properties.

1. $s_{l + m}(x_1, \eta) = s_{l}(x_1, \eta) \equiv 0 \text{ (mod g)}$ for all $\eta \in R_0$,
2. $s_{l + m}(x_2, \eta) = s_{l}(x_2, \eta) \equiv 0 \text{ (mod g)}$ for all $\eta \in R_0 - \{\eta_0\}$,
3. $s_{l + m}(x_2, \eta_0) = s_{l}(x_2, \eta_0) + \pi p^l \equiv 0 \text{ (mod g)}$. 

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Proof. If $p \neq 2$, the proof is already given in [5]. If $p = 2$, we can easily obtain our assertion as in the following way. Put $R_0 = \{1\}$.

Choose $n_1 \in \mathbb{N}$ such that $g^{n_1} \equiv 1 \pmod{2^m}$, and let the 2-adic expansion of $g^{n_1}$ be $g^{n_1} = 1 + 2^m a_m + 2^{m+1} a_{m+1} + \cdots + 2^{l-1} a_l$, where $m \geq m$ and $a_i = 0$ or 1 for each $i$ with $m \leq i \leq m_1 - 1$. Suppose that $l \in \mathbb{N}$ is an integer such that $l > 2m_1 - 2$.

Then, putting $x_1 = g^{n_1}$ and $x_2 = (1 + 2^{l-1}) g^{n_1}$, we have $x_1 \equiv x_2 \equiv 1 \pmod{2^m}$, and $x_1$ and $x_2$ satisfy the conditions (1), (2) and (3) for $n_0 = 1$.

Now, we continue to prove Theorem 1. For any $x \in \mathbb{Z}_p^\times$ and any $n \in \mathbb{N}$, we put

$$H(x, n) = H(0, u, x, n) = \theta(x) \sum_{\eta \in \mathbb{W}} (-s_n, d(x\eta)) - s_n(x\eta) - iq^{p^n}.$$ 

Then, for any complete set $R$ of representatives of $V$ modulo $(1, -1)$,

$$H(x, n) = \sum_{\eta \in R} \sum_{i=0}^{d-1} \theta(s_n(x\eta) + iq^{p^n}) (u^{-s_n(x\eta)} - iq^{p^n}) - dqp^n s_n(x\eta) + iq^{p^n} + \theta(-1)u.$$ 

We put further

$$G(x, n, n) = G(0, u, x, n, n)$$

$$= \sum_{i=0}^{d-1} \theta(s_n(x\eta) + iq^{p^n}) (u^{-s_n(x\eta)} - iq^{p^n}) - dqp^n s_n(x\eta) + iq^{p^n} + \theta(-1)u$$

for any $\eta \in V$. Then,
(1.4.4) \[ H(x,n) = \sum_{\eta \in R} G(x, \eta, n). \]

Let \( p_{x,u} \) denote the maximal ideal of \( \mathcal{O}_{x,u} \) and \( g_1 \in \mathbb{N} \) the order of \( u \mod p_{x,u} \) in the multiplicative group \( (\mathcal{O}_{x,u}/p_{x,u})^\times \). Write \( g_1 = d_0 p_{x,u}^{r_0} \) with \( (d_0, p) = 1 \) and \( r_0 > 0 \), and put \( g = d_0 d \). Let \( m \) be an integer with \( qp^m | p^m \) and suppose that \( l \) is a sufficiently large integer. We can choose a complete set \( R_0 \) of representatives of \( V \) modulo \( (1, -1) \), \( x_1, x_2 \in 1 + p^m \mathbb{Z}_p \) and \( \eta_0 \in R_0 \) satisfying the conditions (1), (2) and (3) of Lemma 1.4.3. Then, we have

\[ s_l(x_1 \eta) \equiv s_l(x_2 \eta) \pmod{g_1 dq} \]

for all \( \eta \in R_0 - \{\eta_0\} \),

and so,

\[ (1.4.5) \theta(s_l(x_1 \eta) + iqp^l) = \theta(s_l(x_2 \eta) + iqp^l) \]

and

\[ u s_l(x_1 \eta) \equiv u s_l(x_2 \eta) \pmod{p_{x,u}} \]

hold for all \( \eta \in R_0 - \{\eta_0\} \). On the other hand, we have

\[ (1.4.6) s_l(x_1 \eta_0) \equiv s_l(x_2 \eta_0) + qp^l \pmod{g_1 dq}, \]

and we obtain

\[ (1.4.7) \theta(s_l(x_1 \eta_0) + iqp^l) = \theta(s_l(x_2 \eta_0) + (i+1)qp^l) \]

and

\[ u s_l(x_1 \eta_0) \equiv u s_l(x_2 \eta_0) + qp^l \pmod{p_{x,u}}. \]
In the first place, we prove Theorem 1 in the case $g_1 = 2$, that is, $u \equiv -1 \pmod{p_{\chi,u}}$. Then, since $g_1$ divides the order of the group $(O_{\chi,u}, p_{\chi,u})^X$, we have $(p, 2) = 1$, and so, $d_0 = 2$, $r_0 = 0$ and by (0.2) we must have $(d, 2) = 1$. It is sufficient to show that $\mu_{\chi,u} = \text{ord}_p(u + \theta(-1))$.

We first have

$$G(x, n, n)$$

$$d-1 \sum_{i=0}^{d-1} \theta(s_n(xn) + iqpn^n) + \sum_{j=0}^{d-1} u^j$$

for all $x \in \mathbb{Z}_{p_{\chi,u}}$, all $n \in \mathbb{V}$ and all $n \in \mathbb{N}$. The case $u + \theta(-1) = 0$ occurs if and only if both $\theta(-1) = 1$ and $u = -1$ hold, and in this case we have $G(x, n, n) = 0$. Then by (1.4.1) we have $H(x, n) = 0$ and it follows from Proposition 1.4.2 and the definition of $H(x, n)$ that $\mu_{\chi,u} = \infty$.

Consequently, we obtain $\mu_{\chi,u} = \text{ord}_p(u + \theta(-1)) = \infty$. Now suppose that $u + \theta(-1) \neq 0$. Then, $\frac{1}{u + \theta(-1)}G(x, n, n) \in O_{\chi,u}$ and we have

$$\frac{1}{u + \theta(-1)}G(x, n, n) \equiv \sum_{i=0}^{d-1} \theta(s_n(xn) + iqpn^n)((\theta(-1) + 1)xn$$

$$- \sum_{j=0}^{d-1} \theta(-1)^j (\text{mod } p_{\chi,u}).$$

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Since \( g = d_0 d = 2d \), we have \( g_1 (x_1 \eta_0) \equiv 0 \pmod{2} \). Hence by (1.4.5) and (1.4.7)

\[
\frac{1}{u+\theta(-1)} G(x_1, \eta, l) \equiv \frac{1}{u+\theta(-1)} G(x_2, \eta, l) \pmod{p_{x,u}} \text{ for all } \eta \in R_0 - (\eta_0)
\]

and

\[
\frac{1}{u+\theta(-1)} (G(x_1, \eta_0, l) - G(x_2, \eta_0, l)) \equiv -2\theta(s_L(x_2 \eta_0)) \pmod{p_{x,u}}.
\]

It follows from (1.4.3) that

\[
\frac{1}{u+\theta(-1)} (H(x_1, l) - H(x_2, l)) \equiv \frac{1}{u+\theta(-1)} (G(x_1, \eta_0, l) - G(x_2, \eta_0, l)) \pmod{p_{x,u}}
\]

\[
\equiv \begin{cases} 
-4\theta(s_L(x_2 \eta_0)) \eta_0 & \text{if } \theta(-1) = 1 \\
2\theta(s_L(x_2 \eta_0)) & \text{if } \theta(-1) = -1
\end{cases} \pmod{p_{x,u}}.
\]

Consequently we must have

\[
|H(x_1, l)| = |u+\theta(-1)| \text{ or } |H(x_2, l)| = |u+\theta(-1)|.
\]

Therefore, we see from Proposition 1.4.2 and the definition of \( H(x, \eta) \) that \( \mu_{x,u} = \text{ord}_{p} (u+\theta(-1)) \). Thus, Theorem 1 is proved in the case \( g_1 = 2 \). In particular, we have

\[
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\]
\( \mu_{x,-1} = 0 \) if \( \theta(-1) = -1 \).

We shall use this equation later.

Next, suppose that \( g_1 \neq 2 \), namely \( u^2 \not\equiv 1 \pmod{p_x,u} \). We must show that \( \mu_{x,u} = 0 \). By (1.4.5), we have

\[
G(x_1, n, l) \equiv G(x_2, n, l) \pmod{p_x,u} \quad \text{for all } n \in \mathbb{Z}.
\]

Hence by (1.4.4) and (1.4.7),

\[
H(x_1, l) - H(x_2, l) \equiv G(x_1, n_0, l) - G(x_2, n_0, l) \pmod{p_x,u}
\]

\[
\equiv \theta(s_l(x_2 n_0))((u^{-1} - dp^l) + \theta(-1)u)
\]

\[
+ \theta(s_l(x_2 n_0))u^{-1} - dp^l + s_l(x_2 n_0)(u^{-1} - \theta(-1)u^{-1})
\]

\[
\equiv \theta(s_l(x_2 n_0))u^{-1} - dp^l + s_l(x_2 n_0)(u^{-1} - \theta(-1)u^{-1})
\]

(1.4.8)

If \( u \not\equiv 0 \pmod{p_x,u} \), then \( |H(x_1, l) - H(x_2, l)| = 1 \), and so, \( |H(x_1, l)| = 1 \) or \( |H(x_2, l)| = 1 \). Hence, from Proposition 1.4.2 and the definition of \( H(x,n) \), we obtain \( \mu_{x,u} = 0 \). Now suppose that \( u^2 \not\equiv s_l(x_2 n_0) \equiv 0 \pmod{p_x,u} \). From (1.4.6) and the congruence \( s_l(x_1 n_0) \equiv 0 \pmod{d_0} \), we see \( s_l(x_2 n_0) \equiv 0 \). If \( \theta(-1) = 1 \), then \( u^2 \equiv 1 \pmod{p_x,u} \), which contradicts the assumption that \( g_1 \neq 2 \). If \( \theta(-1) = -1 \), then we obtain \( u^2 \equiv -1 \pmod{p_x,u} \).

In order to complete the proof of Theorem 1, we must deduce \( \mu_{x,u} = 0 \) under the condition that both \( \theta(-1) = -1 \) and \( u^2 \equiv -1 \pmod{p_x,u} \).
hold. In this case we have $(p, 2) = 1$. We assume $\mu_x, u > 0$ under the above condition and derive a contradiction.

Proposition 1.4.2 shows that $\mu_x, u > 0$ holds if and only if $|H(\theta, u, x, n)| < 1$ holds for all $x \in \mathbb{Z}_p^x$ and $n \in \mathbb{Q}$. Now we have

\[
H(\theta, u^{-1}, x, n) = \theta(x) \sum_{\eta \in \mathbb{W}} \theta(\eta) u^s_n, d(\eta x) = \theta(x) \sum_{\eta \in \mathbb{W}} \theta(-\eta) u^s_n, d(-\eta x) \]
\[
= \theta(-x) \sum_{\eta \in \mathbb{W}} \theta(\eta) u^s_n, d(\eta x) = u^d q^n \theta(-1) H(\theta, u, x, n). 
\]

Hence, we see from Proposition 1.4.2 that $\mu_x, u = u_{x, u}^{\mu}$. The condition $u^2 \equiv -1 (\text{mod } p_x, u)$ implies $|u - \sqrt{-1}| < 1$ or $|u^{-1} - \sqrt{-1}| < 1$. In either case, by the assumption that $\mu_x, u > 0$, we obtain $|H(\theta, \sqrt{-1}, x, n)| = |H(\theta, -\sqrt{-1}, x, n)| < 1$. Consequently, $\mu_{x, \sqrt{-1}} = \mu_{x, -\sqrt{-1}} > 0$. Now, from $u^2 \equiv -1 (\text{mod } p_x, u)$ and (0.2), we must have $(x, 2) = 1$. Then, a direct calculation shows

\[
f_{\chi}^{-1} \sum_{a=0}^{f_{\chi}} x(a) e^{\frac{at}{x}} \frac{f_{\chi}^{-1}}{f_{\chi}^{-1}} x^{-a} + \sum_{a=0}^{f_{\chi}} x(a) e^{\frac{a}{x} \left(-\frac{1}{x}ight)} \frac{f_{\chi}^{-1}}{f_{\chi}^{-1}} x^{-a} = 2 \sum_{a=0}^{f_{\chi}} x(a) e^{\frac{2at}{x} \left(-\frac{1}{x}ight)} x^{-a}.
\]

It follows from the definition of the generalized Euler numbers that

\[
\frac{f_{\chi}^{-1}}{f_{\chi}^{-1}} H_x^{\mu}(\sqrt{-1}) + \frac{f_{\chi}^{-1}}{f_{\chi}^{-1}} H_x^{\mu}(-\sqrt{-1}) = x(2) 2^n H_x^{\mu}(-1).
\]

Since $(p, 2) = 1$, we see from Proposition 1.2.3 and the above equation that
\[
\frac{F_{\chi,\sqrt{-1}}(T)+F_{\chi,-\sqrt{-1}}(T)}{2X(2)T^{\ell(2)}F_{\chi,-1}(T)} = 2X(2) \sum_{n=0}^{\infty} \frac{t(2)}{(T-1)^n} F_{\chi,-1}(T).
\]

Therefore, if \( \mu_{\chi,\sqrt{-1}} > 0 \), we must have \( \mu_{\chi,-1} > 0 \), which contradicts the equation (1.4.8) in the case \( \Theta(-1) = -1 \). Thus, we obtain the assertion of Theorem 1 in the final case.
CHAPTER 2. Analytic property of the function $F_{x,u}(z)$

In this chapter, we view $F_{x,u}(z)$ as an analytic function on the set $D=\{z \in \mathbb{C} \mid |z-1|<1\}$ and prove Theorem 2.

We put $R_{x,u}(t) = R_{x,u}(t) + R_{x,u}(t^{-1})$ and $\alpha_{x,u}(t) = \alpha_{x,u}(t) + \alpha_{x,u}(t^{-1})(-1)$.

Then, $\alpha_{x,u}$ is an even measure supported on $\mathbb{Z}_{p}^\times$ and we have

$(\alpha_{x,u}(t) + \alpha_{x,u}(t^{-1}))(T) = R_{x,u}(t)$ and $\frac{1}{2\pi} \alpha_{x,u}(T) = F_{x,u}(u_0^{\mu})$. Now, we apply Theorem 1 or 2 of [17].

Case 1. $|u|=1$. Put $F=O_{x,u}/p_{x,u}$. We first consider the case $\mu_{x,u}=0$. Suppose that the assertion of Theorem 2 does not hold. Then, it follows from the first paragraph of Section 3 of [17] that

$F_{x,u}(T) \equiv p_{x,u} \in F(T)$, and Theorem 1 of [17] shows that

$\Xi_{x,u}^{+}(T) \equiv p_{x,u} \in F[T]$ for a sufficiently large $n \in \mathbb{N}$. Putting $g=f_{x,u}$, it follows from (1.3.1) that $(T^{g} - u^{g})(T^{g} - u^{-g}) \equiv p_{x,u}$ in $F[T]$. Considering

the degrees of these polynomials, we deduce $u^{2a} - u^{a+1} \equiv 0 (mod p_{x,u})$ for any $a$ with $1 \leq a < g$ and $(a,g)=1$. This contradicts the fact that $|u^{2a} - u^{a+1}|=1$ for some $a$ with $1 \leq a < g$ and $(a,g)=1$, as is known from the proof of Theorem 1 in Section 3 of the preceding Chapter. Hence, Theorem 2 must hold in the case $\mu_{x,u}=0$.

Next, if $\mu_{x,u} \neq 0$, then Theorem 1 shows that $\chi(-1)=1$ and that $\mu_{x,u} = \text{ord}_p(1+u)$. Applying the above argument to $(1+u)^{-1}F_{x,u}(T)$ and
\((1+u)^{-1} \widetilde{R}_{x,u}^* (T)\) instead of \(F_{x,u}(T)\) and \(\widetilde{R}_{x,u}^* (T)\), we obtain the assertion of Theorem 2.

**Case 2.** \(|u|>1\). In this case, putting \(g=f \gamma p\), we have

\[
\widetilde{R}_{x,u}^* (T) = \sum_{a=0}^{g-1} x(a) u^{-a} \left((T/u)^{g-1}\right) = \sum_{a=0}^{g-1} x(a) u^{-a} \sum_{m=0}^{\infty} (u^{-1}T)^{gm}
\]

and so

\[
(2.1) \quad \widetilde{R}_{x,u}^* (T) = \sum_{m=0}^{\infty} x(m) u^{-m} (T^m + T^{-m}).
\]

In order to apply Theorem 2 of [17], we first prove the following

**Lemma 2.1.** Let \(\mathbb{O}\) be the ring of integers of a finite extension of \(\mathbb{Q}_\gamma\), and let \((a_m)\) and \((b_m)\) be sequences of \(\mathbb{O}\) such that \(\lim_{m \to \infty} a_m = \lim_{m \to \infty} b_m = 0\). If both of the elements \(\sum_{m=0}^{\infty} a_m (T^m + T^{-m})\) and \(\sum_{m=0}^{\infty} b_m (T^m + T^{-m})\) in \(\mathbb{O}[T^{-1}]\) are equal, then we have \(a_m = b_m\) for all \(m \in \mathbb{N}\).

**Proof.** Put \(f(T) = \sum_{m=0}^{\infty} a_m (T^m + T^{-m})\). It is sufficient to show that if \(f(T) = 0\), then we have \(a_m = 0\) for all \(m \in \mathbb{N}\). For each \(c \in \mathbb{Z}_\gamma\), let \(\delta_c\) denote the Dirac measure of mass 1 supported at \(c\). Then, \(\delta_c(T) = T^c\). Let \(\alpha_f \in \mathbb{A}_0\)
be the measure satisfying \( \alpha_f(T)=f(T) \). Then, for any \( n \in \mathbb{N} \) and any integer \( l \), we have 
\[
\alpha_f(l+p^nL_p) = \sum_{m=0}^{\infty} a_m. 
\]
Now, assume that \( a_m \neq 0 \) for some \( m_0 \in \mathbb{N} \). Since \( \lim_{m \to \infty} a_m=0 \), there is an integer \( m_1 \in \mathbb{N} \) such that \( |a_m|<|a_{m_1}| \) for all \( m>m_1 \). Choose an integer \( n \in \mathbb{N} \) with \( p^{n-m_0}>m_1 \). Then, we have \( \alpha_f(m_0+p^nL_p)=0 \). In fact, if \( m_0 \neq 0 \), then 
\[
|\alpha_f(m_0+p^nL_p)| = |a_{m_0}| = |\alpha_{m_0}| = 0, 
\]
and if \( m_0 = 0 \), then 
\[
|\alpha_f(m_0+p^nL_p)| = |2a_0| = 0. 
\]
Thus, we see that if \( \alpha_f=0 \), we have \( a_m=0 \) for all \( m \in \mathbb{N} \). Now, \( \alpha_f=0 \) holds if and only if \( f(T)=0 \). Hence, if \( f(T)=0 \), we have \( a_m=0 \) for all \( m \in \mathbb{N} \).

Let \( \mathcal{V} \) be the torsion subgroup of \( \mathbb{Z}_p^\times \). Lemma 1 of [17] states that there is a rational number \( r \geq 1 \) such that 
\[
\mathcal{V}u_0\mathbb{Z}_p^\times \cap \mathbb{Q}^\times = (1,-1)r^\mathbb{Z}. 
\]

Now, we continue to prove Theorem 1 in the case \( |u|>1 \). From (2.1), Lemma 2.1 and Theorem 2 of [17], it is sufficient to show that there exists an integer \( n \in \mathbb{N} \) prime to \( g \) which does not belong to \( r^\mathbb{Z} \). Indeed, for any \( k \in \mathbb{N} \), the integer \( kg+1 \) is prime to \( g \), but not all of the integers of this form belong to \( r^\mathbb{Z} \).

Case 3. \( |u|<1 \). From Lemma 1.2.2, a direct calculation shows that 
\[
\tilde{\alpha}_{\chi, u^{-1}}(T) = -\chi(-1)\tilde{\alpha}_{\chi, u}(T^{-1}), 
\]
namely 
\[
\tilde{\alpha}_{\chi, u^{-1}} = -\chi(-1)\tilde{\alpha}_{\chi, u}(-1). 
\]
Hence, the case \( |u|<1 \) is reduced to Case 2.
CHAPTER 3. The $p$-adic valuation of the generalized Euler numbers

1. The $p$-adic valuation of the generalized Euler numbers

Let $S = \{p_1, \ldots, p_l\}$ be a finite set of prime numbers distinct from $p$ and, as described in Introduction, we denote by $\Psi$ (resp. $\Phi$) the set of Dirichlet characters of the second kind for $S$ (resp. $p$). Let $\chi$ be a Dirichlet character with conductor $f_{\chi}$. In this section, we assume the condition (0.3) and, by using the methods in [18], we prove Theorem 3.

Fix $\psi \in \Psi$, $\varphi \in \Phi$ and $n \in \mathbb{N}$ arbitrarily, and denote the conductor of $\chi_n \psi \varphi$ simply by $f$. We denote by $Q_p(\chi, \psi, \varphi, u)$ the field generated by $u$ and the values of $\chi, \psi$ and $\varphi$ over $Q_p$, by $O_{\chi, \psi, \varphi, u}$ the ring of integers of $Q_p(\chi, \psi, \varphi, u)$ and by $p_{\chi, \psi, \varphi, u}$ the maximal ideal of $O_{\chi, \psi, \varphi, u}$. Let $e$ denote the ramification index of the extension $Q_p(\chi, \psi, \varphi, u)/Q_p$.

In the case $|u| \neq 1$ and $n \geq 1$, we see from Lemma 1 of [19] and Theorem 1 that for any sufficiently large $N \in \mathbb{N}$,

$$\sum_{\psi, \varphi \in \Phi} e_{\chi, \psi, \varphi}^u \left( \frac{fp_{\chi, \psi, \varphi}^N - 1}{1 - u} \right)$$

where

$$\sum_{\psi, \varphi \in \Phi} e_{\chi, \psi, \varphi}^u \left( \frac{fp_{\chi, \psi, \varphi}^N - 1}{1 - u} \right) \equiv \sum_{a=1}^{f_{\chi}} (a, \varphi = 1) \frac{\psi(a) a^n u^{fp_{\chi, \psi, \varphi}^N - a}}{1 - u} \left( \mod p_{\chi, \psi, \varphi, u} \right)$$

On account of Theorem 1 again, we see
Hence, we obtain (0.4).

In the case $|u| \neq 1$ and $n=0$, since we have $H^0_{\chi \psi \phi} = \sum_{a=0}^{f-1} \chi \psi \phi(u)^a u^{-a-1}$, it is easy to deduce the assertion of Theorem 3.

In the case $|u|=1$, we use the methods in [18]. For that purpose, we first introduce notations and propositions without assuming $|u|=1$.

We put $Z_S = \mathbb{Z} \times \cdots \times \mathbb{Z}$ and $P = P_1 \cdots P_t$. For each $m \in \mathbb{N}$, let $\mu_m$ denote the group of $m$-th root of unity in $\mathbb{Q}$ and put $\mu_S = \bigcup_{n=0}^{m} \mu_n$. We put $k=Q(\chi, u)$ and $k_S=k(\mu_S)$.

For any $k_S$-valued measure $\nu$ on $Z_S$, its Fourier transform $\hat{\nu}: \mu_S \rightarrow k_S$ is defined by $\hat{\nu}(\xi) = \int_{Z_S} \psi(x) d\nu(x)$. If there exists $R(T) \in k_S(T)$ such that $\hat{\nu}(\xi) = R(\xi)$ holds for almost all $\xi \in \mu_S$, we call $\nu$ a rational function measure and $R(T)$ the associated rational function of $\nu$. Any rational function in $k_S(T)$ can occur as the associated rational function of a certain $k_S$-valued rational function measure on $Z_S$ ([18]§2).

Let $\nu_{\chi, u}$ be a rational function measure on $Z_S$ whose associated rational function is $R_{\chi, u}(T)$. We assume that in the case $|u| \neq 1$, $R_{\chi, u}(\xi) = \nu_{\chi, u}(\xi)$ holds for all $\xi \in \mu_S$. For any $\psi \in \Psi$, regarding $\psi$ as a character of $Z_S^\times$, we put

$$\Gamma_{\nu_{\chi, u}}(\psi) = \int_{Z_S} \psi(x) d\nu_{\chi, u}(x).$$
For any $S' \subset S$, we put

$$\Psi_{S'} = (\psi \in \Psi \mid l \not| f_{\chi,\psi} \text{ if } l \not\in S' \text{ and } l \not| f_{\chi,\psi} \text{ if } l \in S-S').$$

**Proposition 3.1.1.** For almost all $\psi \in \Psi_{S'}$, we have

(3.1.1) \[ \Gamma_{\nu,\chi,u}(\psi) = \frac{\Gamma_{\nu,\chi,u}^0(u)}{1 - u \chi,\psi \nu_{\chi,u}(\psi)}. \]

If $|u| \neq 1$, then (3.1.1) holds for all $\psi \in \Psi_{S'}$.

**Proof.** For any $\psi \in \Psi$, choose $n_1, \psi, \ldots, n_t, \psi \in \mathbb{N}$ such that the conductor $f_\psi$ of $\psi$ divides $F_{\psi} = \prod_{i=1}^t p_{n_i,\psi}$. As in the proof of Proposition 2.2 of [18], we have

$$\Gamma_{\nu,\chi,u}(\psi) = \sum_{\xi \in \mu_{F_{\psi}}} a_{\psi}(\xi) v_{\chi,u}(\xi),$$

where $a_{\psi}(\xi) = 1/F_{\psi} \sum_{a=0}^{F_{\psi}-1} \psi(a) \xi^{-a}$. If $\xi$ is a $F_{\psi}$-th root of unity whose order is not divisible by $f_\psi$, then $a_{\psi}(\xi) = 0$. Since $v_{\chi,u}$ is a rational function measure with the associated rational function $R_{\chi,u}(T)$, there is an integer $L \in \mathbb{N}$ divisible only by the primes in $S$ such that

$$v_{\chi,u}(\xi) = R_{\chi,u}(\xi) \text{ for all } \xi \in \mu_{S-L}. \text{ Now, there are only finitely many } \psi \text{ with } f_\psi \not| L. \text{ If } f_\psi \not| L, \text{ then we have } L \not| 1 \text{ for any } F_{\psi}-\text{th root of unity } \xi.$$
whose order is divisible by $f_{\psi}$. Hence,

$$(3.1.2) \quad \Gamma_{\chi, u}^{(\psi)} = \sum_{\xi \in \mu_{F_{\psi}}} a_{\psi}(\xi) R_{\chi, u}(\xi) \mid_{T=1}. $$

Put $g=f_{\chi} F_{\psi}$. Then, a direct calculation shows

$$(3.1.3) \quad \sum_{\xi \in \mu_{F_{\psi}}} a_{\psi}(\xi) R_{\chi, u}(\xi) = \sum_{a=1}^{g-1} \frac{\chi_{\psi}(a) u^{g-a} - \chi_{\psi}(a) u^{-a}}{1 - u^{g}}. $$

In the case $\psi \in \Psi_{S}$, we have

$$(3.1.4) \quad \sum_{\xi \in \mu_{F_{\psi}}} a_{\psi}(\xi) R_{\chi, u}(\xi) = R_{\chi \psi, u}(T). $$

Since $R_{\chi \psi, u}(1) = \frac{u}{1 - u^{\psi} x^{\psi}(y)}$, we see that (3.1.1) holds for any $\psi \in \Psi_{S}$ with $f_{\psi} \neq L$.

If $|u| \neq 1$, (3.1.2) holds for all $\psi \in \Psi$, and so, if $\psi \in \Psi_{S}$, we see from (3.1.4) that (3.1.1) holds. This completes the proof.

**Proposition 3.1.2.** For almost all $\psi \in \Psi$, we have

$$(3.1.5) \quad \text{ord}_{p} (\Gamma_{\chi, u}^{(\psi)}) = \mu_{\chi, u}. $$

If $|u| \neq 1$, then (3.1.5) holds for all $\psi \in \Psi$. 

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Proof. Let $\nu_{x,u}^*$ be the measure on $\mathbb{Z}_S$ obtained by restricting $\nu_{x,u}$ to $\mathbb{Z}_S^\times$ and extending by 0. Let $\phi(x)$ be the characteristic function on $\mathbb{Z}_S^\times$. Then,

$$\phi(x) = \sum_{\xi \in \mu_{PS}} a_\xi x^\xi,$$

where $a_\xi = \frac{1}{\sum_{a=1}^{PS} \xi^{-a}}$. Put

$$R_{x,u}^*(T) = \sum_{\xi \in \mu_{PS}} a_\xi R_{x,u}(\xi T).$$

Then, $\nu_{x,u}^*$ is a rational function measure whose associated rational function is $R_{x,u}^*(T)$. Let $\nu_{x,u}^*(-1)$ be the measure on $\mathbb{Z}_S$ defined by $\nu_{x,u}^*(-1)(X) = \nu_{x,u}^*(-X)$ for any compact and open subset $X$ of $\mathbb{Z}_S$.

Then, $\nu_{x,u}^*(-1)$ is also a rational function measure whose associated rational function is $R_{x,u}^*(T^{-1})$. Put $g = f^P_{xPS}$. Then, a direct calculation shows

$$R_{x,u}^*(T) = \sum_{a=0}^{g-1} \frac{\chi(a) u^{g-a} T^a}{T^a u^g}$$

and in the same way as we have shown Proposition 1.2.1, we see that $R_{x,u}^*(T) \in \mathcal{O}_{x,u}[[T^{-1}]]$. From the remark after Theorem 3.1 of [18], we have, for almost all $\psi \in \Psi$,
Further, in the same way as in the proof of Theorem 1 in the second section of Chapter 1, we see

\[ \mu(R_{\chi,u}^*(T) + R_{\chi,u}^*(T^{-1})) = \mu_{\lambda}^u. \]

Hence (3.1.5) holds for almost all \( \psi \in \Psi \).

In the case \(|u| \neq 1\), (3.1.2) holds for all \( \psi \in \Psi \). Hence, we see from (3.1.3) and Theorem 1 that (3.1.5) holds for all \( \psi \in \Psi \).

PROPOSITION 3.1.3. Suppose that \( \mu_{\lambda}^u \neq 0 \). Let \( \pi \) be a prime of \( O_{\chi,\psi,\varphi,\psi} \). Then, for almost all \( \psi \in \Psi_S \),

(3.1.6) \[ F_{\chi,\psi,u}(T)/\pi \in \mu_{\lambda}^u, u \in (O_{\chi,\psi,u[[T^{-1}]]})^\times. \]

If \(|u| \neq 1\), then (3.1.6) holds for all \( \psi \in \Psi_S \).

Proof. If \( p \) divides \( f_{\chi,\psi} \), then \( p \) divides \( f_{\chi,\psi} \) and Proposition 1.2.3 shows \( F_{\chi,\psi,u}(1) = \frac{u}{f_{\chi,\psi}(u)} \). Hence by Propositions 3.1.1, 3.1.2, we have, for almost all \( \psi \in \Psi_S \),

(3.1.7) \[ \ord_p (F_{\chi,\psi,u}(1)) = \mu_{\lambda}^u, u \]

Hence (3.1.6) holds for all \( \psi \in \Psi_S \) satisfying (3.1.7).
Unless \( p \) divides \( f_\chi \), then \( p \) divides \( f_{\chi \psi} \) for all \( \psi \in \Phi^{-1}(1) \). For any \( \psi \in \Psi \), we have \( \psi \in \Psi_S \) if and only if \( P_\psi \mid f_{\chi \psi} \). Hence, for almost all \( \psi \in \Psi_S \), we have

\[
\text{ord}_p (F_{\chi \psi}, u(1)) = \mu_{\chi \psi}, u.
\]

From Proposition 1.2.4 and Theorem 1, we see that (3.1.8) is equivalent to (3.1.6). Therefore, (3.1.6) holds for almost all \( \psi \in \Psi_S \).

Finally, if \( |u| \neq 1 \), we see from Propositions 3.1.1, 3.1.2 that, for all \( \psi \in \Psi_S \), (3.1.7) or (3.1.8) holds. Hence, (3.1.6) holds for all \( \psi \in \Psi_S \). This completes the proof.

**Proof of Theorem 3 in the case \( |u| = 1 \).** Again, we fix \( \psi \in \Psi \), \( \phi \in \Phi \) and \( n \in \mathbb{N} \), and write \( f \) for the conductor of \( \chi_n \psi \).

We first assume that \( \psi \in \Psi_S \). Suppose that \( n \geq 1 \) in the first place.

**Case 1.** \( \mu_{\chi_n} u = 0 \). From Lemma 1 of [19], we have, for any sufficiently large \( N \in \mathbb{N} \),

\[
\frac{u}{1-u} f_{\chi_n \psi} (u) \equiv \sum_{a=0}^{f_p^n-1} \chi_n \psi (a) \frac{u^{f_p^n-a}}{1-u^{f_p^n}} \pmod{p_{\chi_n \psi}}.
\]

Hence, we have
Therefore, if \( \psi \) satisfies (3.1.6), we have
\[
\left| \frac{u}{1-u^f} h_n^{\psi} \phi(u) \right| = |F| \chi_{\psi}, u(1) = 1 = \frac{u}{1-u^f} h_n^{\psi} \phi(u) \mod p, \phi, u.
\]

and (0.4) holds.

Case 2. \( \mu, u > 0 \). In this case, Theorem 1 states that \( \chi \) is even and that \( \mu = \text{ord}_p (1+u) \), and (0.3) implies \( 2^f p \). Choose \( N_0 \in \mathbb{N} \) arbitrarily. Then, Lemma 1 of [19] shows that, for any sufficiently large \( N \in \mathbb{N} \), we have
\[
\frac{u}{1-u^f} h_n^{\psi} \phi(u) = \sum_{a=1}^{f_p - 1} \chi_n^{\psi}(a) a \frac{u^{f_p - a}}{1-u^{f_p}} \mod p, \phi, u.
\]

Since \( 2^f p \), we have \( |u^{f_p - 2a + 1}| \leq |u + 1| \). If \( u = -1 \), then \( h_n^{\psi} \phi(u) = 0 \), and (0.4) holds. If \( u = -1 \), choose the above \( N_0 \) such that \( p, \phi, u \subset (1+u) \). Then,
\[
\frac{u}{1-u^f} h_n^{\psi} \phi(u) = \sum_{a=1}^{f_p - 1} \chi_n^{\psi}(a) a \frac{u^{f_p - a}}{1-u^{f_p}} \mod p, \phi, u.
\]
Thus, we have shown

\[ \frac{u}{1-u'} \frac{\mu_n}{1+u} \chi_n \psi(u) = \frac{1}{1+u} F, \psi, u(1) (\mod p, \psi, \varphi, \mu). \]

Therefore, if \( \psi \) satisfies (3.1.6), then (0.4) holds.

Next, we consider the case \( n=0 \). If \( p \mid f \psi \), then Propositions 1.2.3, 1.2.4 show that \( F, \psi, u(\varphi(u)) = \frac{u}{1-u} H^0, \psi(u) \). In the case \( \mu_n, u = \infty \), we have \( H^0, \psi(u) = 0 \), and (0.4) holds. In the case \( \mu_n, u = \infty \), if \( \psi \) satisfies (3.1.6), then (0.4) holds. If \( p \not\mid f \psi \), then Theorem 1 and Propositions 3.1.1, 3.1.2 show that \( \text{ord}_p \left( \frac{u}{1-u} H^0, \psi'(u) \right) = \mu_n, u = \mu_n, u \) holds for almost all \( \psi' \in \Psi_S \). Further, we note that such \( \varphi \) is unique in \( \Phi \) if it exists.

Thus, we conclude that (0.4) holds for almost all \( \psi \in \Psi_S \) and for all \( \varphi \in \Phi \) and \( n \in \mathbb{N} \).

Finally, let \( S' \) be a proper subset of \( S \). If \( S' \) is empty, then \( \Psi_S \) is empty or a set consisting of only one element. If \( \Psi_S \) is an infinite set, then each element \( \psi \) of \( \Psi_S \) is expressed as \( \psi = \psi_S, \psi' \), where \( \psi_S \) (resp. \( \psi'_S \)) is a Dirichlet character of the second kind for \( S \) (resp. \( S-S' \)), and it is easy to see that \( \psi'_S \) depends only on the set \( S' \). Hence, putting \( \psi'_S = (\psi'_S,^{-1})_l \in \Psi_S \) and applying the above argument to \( S' \), \( \Psi'_S \), and \( \psi'_S \) instead of \( S \), \( \Psi_S \) and \( \chi \), we deduce that

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(0.4) holds for almost all $\psi \in \Psi_S$, and for all $\varphi \in \Phi$ and $n \in \mathbb{N}$. Since $\Psi = \cup \Psi_S$, we complete the proof of Theorem 3.

Remark. If $X(-1) = 1$ and if $(2, f_x^{p_1 \cdots p_t} \varphi) = 1$, then

$$
(1 - 2^n x_n \varphi(2)) \frac{B_n, x_n \varphi}{n} = \frac{1}{2^n x_n \varphi(-1)}
$$

for all $n \in \mathbb{N}$, $\psi \in \Psi$ and $\varphi \in \Phi$ ([19]). Since $x_n = (x_{\psi}^{-1})_{n-1}$ and $x_{\psi}^{-1}(-1) = -1$, we obtain from Theorems 1, 3 the result (0.5) of Friedman under the restricted condition that $(2, f_x^{p_1 \cdots p_t} \varphi) = 1$.

2. Another proof of Theorem 3

In this section, using the methods in [4] and [20], we give another proof of Theorem 3 in the case $|\psi| = 1$. Note that if $|\psi| \neq 1$, the assertion of Theorem 3 was obtained easily from Lemma 1 of [19].

As in Section 1, let $S = (p_1, \cdots, p_t)$ be a finite set of prime numbers distinct from $p$ and put $\mathbb{Z}_S = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_t}$. Let $q_i = p_i$ or $q_i = 4$ according as $p_t > 2$ or $p_t = 2$ and put $Q = \prod q_i$. For $N = (n_1, \cdots, n_t) \in \mathbb{Z}^t$, put $Q^t = \prod q_i^{n_i}$. For $N = (n_1, \cdots, n_t), H = (m_1, \cdots, m_t) \in \mathbb{Z}^t$, we write $N \geq H$ to mean

$$
n_i \geq m_i$$

for each $1 \leq i \leq t$. We write simply $N \geq 0$ (resp. $N > 0$) for $N \geq (0, \cdots, 0)$ (resp. $N > (1, \cdots, 1)$). We write $N > 0$ to mean $N \geq N_0$ for some fixed $N_0 > 0$. For each $x = (x_1, \cdots, x_t) \in \mathbb{Z}_S$ and $N = (n_1, \cdots, n_t) \geq 0$, we denote by $S_N(x)$ the
unique integer satisfying \(0 \leq S_N(x) < \Omega^n\) and \(S_N(x) \equiv x \pmod{q_i \rho_i l_i} \) for \(1 \leq i \leq l\). We regard each \(\psi \in \Psi\) also as a character of \(\mathbb{Z}_S^\times\). Let \(H\) denote the torsion subgroup of \(\mathbb{Z}_S^\times\). For each \(m \in \mathbb{N}\), let \(\xi_m\) denote an arbitrarily chosen primitive \(m\)-th root of unity. In the case \(|u| = 1\), Theorem 3 can be expressed as in the following

**THEOREM 3'.** Suppose that \(|u| = 1\) and that the condition (0.3) holds. Then, we have

\[
(3.2.1) \quad |H^n_{\chi_{n,\psi}}(u)| = \begin{cases} |1+u| & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}
\]

for almost all \(\psi \in \Psi\) and for all \(\phi \in \Phi\) and \(n \in \mathbb{N}\).

In order to prove this, we use the following

**LEMMA 3.2.1 (Friedman [4]).** Let \(H \in \mathbb{Z}_t^\times\) with \(H > 0\) and \(g \in \mathbb{N}\) with \((g, Q) = 1\). Then, we can choose a complete set \(E\) of representatives of \(H\) modulo its subgroup \((1,-1) = ((1, \cdots, 1), (-1, \cdots, -1))\), \(\eta_0 \in E\) and \(N_0 \in \mathbb{Z}_t^\times\) with \(N_0 > H\), such that for all \(N > N_0\), there exist \(x_1\) and \(x_2\) in \(\mathbb{Z}_S^\times\) satisfying

1. \(S_N(x_1 \eta) = S_{N-H}(x_1 \eta) \equiv 0 \pmod{g}\) for all \(\eta \in E\),
2. \(S_N(x_2 \eta) = S_{N-H}(x_2 \eta) \equiv 0 \pmod{g}\) for all \(\eta \in E - \{\eta_0\}\),
3. \(S_N(x_2 \eta_0) = S_{N-H}(x_2 \eta_0) + \Omega^n_{N-H} \equiv 0 \pmod{g}\),
4. \(S_H(x_1) = S_H(x_2) = 1\).

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Now, we prove Theorem 3'. Without loss of generality, we assume that $\chi$ is of the first kind for each $p_1, \ldots, p_t$ and $p$. For each $\psi \in \Psi$, let $N_\psi \in \mathbb{Z}$ be such that $f_\psi = N_\psi^{\psi}$. Applying induction on $t$, it is sufficient to prove that (3.2.1) holds for all $\psi \in \Psi$ with $N_\psi > 0$, all $\varphi \in \Phi$ and all $n \in \mathbb{N}$.

In the case where $\chi$ is even and $u = -1$, we can deduce $H_n^{\chi \psi \varphi}(u) = 0$ for all $\psi \in \Psi$, $\varphi \in \Phi$ and $n \in \mathbb{N}$ directly from Lemma 1 of [19]. We exclude this case in the rest of this section.

We deduce from Lemma 1 of [19] that

$$
\frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) \equiv \frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) \pmod{p, \chi, \psi, \varphi, u}
$$

holds for all $\varphi \in \Phi$ and $n \in \mathbb{N}$ and that

$$
\frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) \equiv \frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) \pmod{p, \chi, \psi, \varphi, u}
$$

holds for all $\varphi \in \Phi - \{1\}$ and $n \in \mathbb{N}$. If $\chi$ is even, we have

$$
\frac{1}{1+u} \cdot \frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) = \frac{1}{1+u} \cdot \frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) \pmod{p, \chi, \psi, \varphi, u}
$$

for all $\varphi \in \Phi$ and $n \in \mathbb{N}$ and

$$
\frac{1}{1+u} \cdot \frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) = \frac{1}{1+u} \cdot \frac{u}{f_{\chi \psi \varphi} x_n^\varphi}(u) \pmod{p, \chi, \psi, \varphi, u}
$$
for all \( \psi \in \Psi - (1) \) and \( n \in \mathbb{N} \). Hence, it is sufficient to prove that, for
an arbitrarily fixed \( \psi \in \Psi \), \((3.2.1)\) holds for any \( \psi \in \Psi \) with \( N_\psi > 0 \) and \( n=0 \).

We write \( \chi_\psi = d \prod_{i=1}^{N_\psi} q_i^{a_i} \) with \( (d,Q)=1 \) and \( a_i = 0 \) or \( 1 \). We fix \( \psi \in \Psi \) and denote the conductor of \( \chi_\psi \) simply by \( f \). Then, \( f=dQ\prod_{i=1}^{N_\psi} p_i \).

Let \( k \) be the field obtained by adjoining \( u \) and the values of \( \chi \) and \( \psi \) to \( \mathbb{Q} \), and put \( k_N = k(t \prod_{N=1}^{N_\psi} \) for each \( N \in \mathbb{Z}^+ \). We can choose \( M \in \mathbb{Z}^+ \) with \( M > 0 \) such that, for any \( N \geq M \), \( [k_N:k_M] = \frac{N-M}{n-N-M} \) and that the prime ideal of \( k_M \) defined by the valuation \( 1 \) does not split in \( k_N \). Then, denoting by \( T_{N,M} \) the trace map from \( k_N \) to \( k_M \), we have \( |T_{N,M}(x)| < 1 \) for any \( x \in k_N \) with \( |x| < 1 \).

Let \( f' \) be the least positive integer such that \( |u^{-1}| < 1 \) and write \( f' = d_1 \prod_{N=1}^{M_1} \) with \( (d_1,Q)=1 \) and \( M_1 \in \mathbb{Z}^+ \). In what follows, we fix \( M \in \mathbb{Z}^+ \) satisfying \( M > M_1 \) and the property described above. Further, we always suppose that \( N \geq M \) and write simply \( T \) for \( T_{N,M} \). We denote the prime ideal of the completion of \( k_N \) in \( \mathbb{C}_p \) simply by \( p \). For each \( x \in k_S^\times \), \( j \in \mathbb{Z} \) and \( \eta \in H \), we put \( \tau_{x,j,\eta} = S_{N-M}(x \eta) + jQ \eta^{N-M} \), \( A(x,j,\eta) = \psi(x)^{-1} \chi_\psi(\tau_{x,j,\eta}) \) and \( B(x,j,\eta) = u^{-f+1} \chi_\psi(\tau_{x,j,\eta}) \). We first prove the following

**Lemma 3.2.2.** Let \( E \) be a complete set of representatives of \( H \) modulo \((1,-1)\), and suppose that \( N=N_\psi \). Then, for any \( x \in k_S^\times \), we have

\[
T(\psi(x)^{-1} H_0 \chi_\psi(u)) = u^{f-1} \prod_{N=1}^{N_\psi} \sum_{\eta \in E} \sum_{j=0}^{\eta - M - 1} A(x,j,\eta) B(x,j,\eta).
\]
Proof. We have \( \sum_{a=0}^{f-1} x_1^a u^{-a} \). Hence,

\[
\sum_{a=0}^{f-1} x_1^a u^{-a} T(x^{-1}a).
\]

As in the proof of Lemma 1 of [4], we see that \( T(x^{-1}a) \neq 0 \) holds if and only if \( \psi(x^{-1}a)^n \equiv 1 \) and in that case, we have \( T(x^{-1}a) = \psi(x^{-1}a)^n \equiv 1 \) if and only if \( a \equiv \pm nx \) (mod \( QnH \)) for some \( n \in E \). Since the integers \( a \) with \( 0 < a < f \) satisfying the above property are exactly the integers \( \tau x, j, n \) and \( f - \tau x, j, n \) with \( 0 < j < d nH \), we obtain the assertion of this lemma.

We continue to prove Theorem 3'. In the first place, we consider the case \( |1 + x(-1)u| = 1 \), that is, \( x \) is odd or both \( x(-1) = 1 \) and \( |1 + u| = 1 \) hold. We put \( g = dd_1 \) and apply Lemma 3.2.1. Suppose that \( N = n \psi \geq H \) and that \( E, x_1, x_2 \) in \( Z/S \) and \( n_0 \in E \) satisfy the conditions (1)-(4) of Lemma 3.2.1. We prove \( |H_0^0 \chi_{\psi\theta}(u)| = 1 \).

If \( n \neq n_0 \), we have \( x_1^{-1} \tau x_1, j, n \equiv x_2^{-1} \tau x_2, j, n \) (mod \( f_\psi \)) and \( \tau x_1, j, n \) is congruent to \( \tau x_2, j, n \) modulo both \( f \chi_\psi \) and \( f' \). Hence,

\[
A(x_1, j, n)B(x_1, j, n) \equiv A(x_2, j, n)B(x_2, j, n) \pmod{p}.
\]

On the other hand, if \( n = n_0 \), we have \( x_1^{-1} \tau x_1, j, n_0 \equiv x_2^{-1} \tau x_2, j, n_0 \) (mod \( f_\psi \)) and \( \tau x_1, j, n_0 \) is congruent to \( \tau x_2, j, n_0 \) modulo both \( f \chi_\psi \) and \( f' \). Hence,
\[ A(x_1, j-1, n_0)B(x_1, j-1, n_0) \equiv A(x_2, j, n_0)B(x_2, j, n_0) \pmod{p}. \]

Therefore, we see from Lemma 3.2.2 that

\[
(3.2.2) \quad u^{-f+1} \prod_{T=(\psi(x_1)^{-1} \psi(x_2)^{-1})H^0_{X_{\psi\psi}(u)}}
\equiv A(x_1, d\eta-1, n_0)B(x_1, d\eta-1, n_0)-A(x_1, -1, n_0)B(x_1, -1, n_0) \pmod{p}
\equiv A(x_1, -1, n_0)u^{-2\tau_{x_1, -1, n_0}}(\mod{p}).
\]

Now, it is easy to see that \((\tau_{x_1, -1, n_0}, f_{X_{\psi\psi}}) = 1\), and so,

\[ |A(x_1, -1, n_0)| = 1. \] From (0.3), we have \(|u - d\eta| = 1\).

Suppose that \(|H^0_{X_{\psi\psi}(u)}| < 1\). Then, we have \(|u - d\eta| \equiv 0(\mod f')\) and we deduce \(f' = 2\), namely, \(|u + 1| < 1\), which contradicts the assumption that \(|u + 1| = 1\). If \(x\) is even, then we have \(4\tau_{x_1, -1, n_0} \equiv 0(\mod f')\) and we deduce \(f' = 4\). Further, we see easily that

\[
\sum_{a=0}^{2f-1} x_{\psi\psi}(a)u^a \equiv \sum_{a=0}^{2f-1} x_{\psi\psi}(a)u^{-a} \equiv 0(\mod p),
\]

and so

\[
2f-1 \sum_{a=0}^{f-1} x_{\psi\psi}(a)(u^a + u^{-a}) \equiv 0(\mod p).
\]

Since \(f' = 4\), we have \(u^2 \equiv -1(\mod p)\). Hence, we obtain

\[
2(2, f_pQ) = 1. \] Hence, \(H^0_{X_{\psi\psi}}(-1) \equiv 0(\mod p)\). Then, in the same way as

we have deduced (3.2.2), we obtain

\[
-2\tau_{x_1, -1, n_0}(\mod{p}).
\]

namely \(2(-2) \equiv 0(\mod p)\), which is a contradiction. Thus, we conclude
that \(|H^0_{\chi\psi\phi}(u)|=1\).

Finally, suppose that \(x\) is even and that \(|1+u|<1\). Then, (0.3) implies \((f,pQ,2)=1\), and we have \(|(1+u)^{-1}H^0_{\chi\psi\phi}(u)|\leq 1\). We must show that \(|(1+u)^{-1}H^0_{\chi\psi\phi}(u)|=1\). Put \(g=2dp\) and apply Lemma 3.2.1. Suppose that \(N=N\psi=g\) and that \(E, x_1\) and \(x_2\) in \(\mathbb{Z}_S^X\) and \(\eta_0\in E\) satisfy the conditions (1)-(4) of Lemma 3.2.1. We prove \(|(1+u)^{-1}H^0_{\chi\psi\phi}(u)|=1\). We have, for \(\eta\neq\eta_0\),

\[
(3.2.3) \quad S_{N-H}(x_1\eta) - S_{N-H}(x_2\eta) \quad \text{and} \quad \frac{u}{u+1} \equiv 0 \pmod{p}.
\]

In fact, putting \(a=\max(S_{N-H}(x_1\eta) - S_{N-H}(x_2\eta), S_{N-H}(x_2\eta) - S_{N-H}(x_1\eta))\), we have \((u^a-1)/(u+1)=\sum_{j=0}^{a-1} (-u)^j \equiv a \pmod{p}\). Since \(p|g\), we have \(a\equiv 0 \pmod{p}\).

Hence (3.2.3) holds. In the case \(\eta=\eta_0\), we have

\[
(3.2.4) \quad S_{N-H}(x_1\eta_0) - Q\eta^{N-H} S_{N-H}(x_2\eta_0) \quad \text{and} \quad \frac{u}{u+1} \equiv 0 \pmod{p}.
\]

Using a similar method that we have deduced (3.2.2) and taking account of (3.2.3) and (3.2.4), we deduce

\[
u^{-f+1} \frac{1}{\chi^{N-H}} (\psi(x_1)^{-1} - \psi(x_2)^{-1}) H^0_{\chi\psi\phi}(u) / (u+1) = \tau x_1^{-1} - 1, \eta_0^-(u - dQ^N)^{-1} / (u+1) \pmod{p}.
\]

Now,
\[ \frac{2r_{x_1, -1, \eta_0^{-1}}}{(u+1)} = \sum_{f=0}^{2r_{x_1, -1, \eta_0^{-1}}} (-u)^f \equiv 2r_{x_1, -1, \eta_0} \pmod{p}, \]

and since \((p, 2)=1\) and \(r_{x_1, -1, \eta_0} \equiv -Q_n^{N-H} \neq 0 \pmod{p}\), we see

\[ T((\psi(x_1)^{-1} - \psi(x_2)^{-1})H^0_{x, \psi}(u))/(u+1) \neq 0 \pmod{p}. \]

Thus, we obtain \(|(1+u)^{-1}H^0_{x, \psi}(u)| = 1\).
REFERENCES
