有限群のブロックにおける次数方程式およびベーシックセットに関する研究

池田, 和興

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INVESTIGATION ON
DEGREE EQUATIONS AND BASIC SETS
FOR BLOCKS OF FINITE GROUPS
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Preface

Chapter I provides a degree equation which holds for any $p$-block of an arbitrary finite group and the concept of residue sets introduced. We get a sufficient condition for a set of irreducible ordinary characters to be a $p$-block making use of residue sets. Two conjectures are described, while one of them is equivalent to a conjecture of K. Harada. It is shown that the two conjectures hold for cyclic blocks and $PSL(2, q)$. Furthermore Harada's conjecture is proved for $Sp(4, q)$, $G_2(q)$, in non-defining characteristic.

Chapter II is concerned with describing a simple method to distribute irreducible ordinary characters into $\pi$-blocks and showing that there exists a basic set of Brauer characters consisting of the irreducible ordinary characters for every block of the twenty-one sporadic simple groups, their associated covering and automorphism groups.
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Introduction

After L. E. Dickson, various colleagues developed the fascinating theory of modular representations of finite groups. The present subject is the representations over a field of prime characteristic $p$ dividing the order of a group $G$.

There are several methods to distribute the irreducible ordinary characters of $G$ into $p$-blocks. Most famous and available one is to use the central characters. Another one is to use Osima’s Theorem which gives a sufficient condition for a set of the irreducible ordinary characters to be a union of $p$-blocks (see Curtis and Reiner [4] (85.22) or Osima [21]).

Let $G^0$ be the set of $p$-regular elements of $G$. Let $\text{Irr}(G)=\{\chi_1, \ldots, \chi_n\}$ be the irreducible ordinary characters of $G$ and $\{\chi_J\}=\{\chi_J|J \in J\}$ for a subset $J$ of the index set $\{1, \ldots, n\}$.

**Theorem 1.** (Osima) For $J \subseteq \{1, \ldots, n\}$, if $\sum_{J \in J} \chi_J(x) \chi_J(y) = 0$ whenever $x \in G^0$ and $y \in G - G^0$, then $\{\chi_J\}$ is a union of $p$-blocks.

Put $\rho_J = \sum_{J \in J} \chi_J(1) \chi_J$ for $J \subseteq \{1, \ldots, n\}$. In his paper [8], Harada stated the following:

**Conjecture A.** If $\rho_J$ vanishes on $G - G^0$, then $\{\chi_J\}$ is a union of $p$-blocks of $G$.

As in [8], the proof of Conjecture A is reduced to the case where $\{\chi_J\}$ is contained in a single block as follows:

**Conjecture $A'$.** Let $B$ be a $p$-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. For $J \subseteq \{1, \ldots, k\}$, assume that $\rho_J$ vanishes on $G - G^0$. Then $\{\chi_J\} = B$ or $\emptyset$.

The main result of Section 1 asserts that a residue set of a block implies a degree equation of the irreducible ordinary characters of $G$ and that we can find a new conjecture (Conjecture B) which is equivalent to Conjecture $A'$.

Harada also proved the conjecture in case the Sylow $p$-subgroups of
G are cyclic. In [18], Kiyota and Okuyama gave a proof in the case when G is p-solvable. If we use the concept of residue sets, we can verify those results by the same argument and moreover show that Conjecture A holds for the infinite series of the symplectic groups Sp(4, q) and the finite Chevalley groups G2(q) in non-defining characteristic making use of basic sets consisting of the irreducible ordinary characters. We also state an analogous conjecture (Conjecture C) and prove it for cyclic blocks and PSL(2, q). These are the contents of Chapter I. The results are almost contained in Ikeda [14-17].

It is an open question whether a basic set of Brauer characters can always be chosen from among the irreducible ordinary characters. It is well known that, for every block with a cyclic defect group, there exists such a basic set by Brauer and Dade's Theorem. Precisely non-exceptional characters form a basic set and exceptional characters can be expressed as linear combinations of the basic set. For every block of p-solvable groups, there also exists such a basic set, since the irreducible Brauer characters are all liftable by Fong and Swan's Theorem (see Dornhoff [5] 72.1 or Feit [6] X 2.1). Furthermore, in [7], Geck and Hiss showed that, for finite groups of Lie type, the answer to this question is affirmative in all reasonable cases.

The aim of Chapter 2 is to calculate basic sets consisting of the irreducible ordinary characters for the blocks of 21 sporadic simple groups, their associated covering and automorphism groups. The Brauer trees for the cyclic blocks of the sporadic groups are described by Hiss and Lux in [9]. Hence we omit these cases and the blocks of defect 0. We also omit the conjugate blocks, since they have the same expressions.

Concerning notation and terminology we refer to the books of Curtis and Reiner [4], Dornhoff [5], Feit [6] and the ATLAS [3]. Let the order of G be \( p^a q \) such that \((p, q) = 1\) and Z denote the rational integer ring as usual. Let \( \text{GCD}\{z_i\} \) denote the greatest common divisor of \( \{z_i\} \). The notation of this introduction will be frequently used throughout this paper. Other notation will be introduced as needed. All characters are considered for a suitably splitting p-modular system.
Chapter I. Degree Equations and Conjectures

§1. Degree Equations

The purpose of this Section is to obtain a degree equation which holds for any p-block of an arbitrary finite group.

**Theorem 2.** Let $B$ be a p-block of $G$ with defect $d$ which contains the irreducible ordinary characters $\chi_1, \ldots, \chi_k$ and the principal indecomposable characters $\Phi_1, \ldots, \Phi_l$. Let $D = [d_{ij}]$ denote the decomposition matrix of $B$. Then the following assertions hold.

(i) There exist $m_i \in \mathbb{Z}$ ($i = 1, \ldots, k$) which satisfy $[m_1 \ldots m_k] D = [w_1 \ldots w_l]$, where $\Phi_i(1) = p^a u_{s_i}$ ($s = 1, \ldots, l$) with $\text{GCD} \{ \Phi_i(1) \} = p^a u$.

(ii) If we set $\chi_i(1) = p^a u_m + p^{a-d} \varepsilon_i$ ($i = 1, \ldots, k$), then all $\varepsilon_i$ are integers which satisfy $[\varepsilon_1 \ldots \varepsilon_k] D = O$ and $\eta_B = \sum_{i=1}^k \varepsilon_i \chi_i$ vanishes on $G^0$. In particular, we have a degree equation $\eta_B(1) = \sum_{i=1}^k \varepsilon_i \chi_i(1) = 0$.

Proof. (i) Since $D$ has rank $l$ and its invariant factors are all 1, there are integral invertible matrices $X$ and $Y$ such that $D = X \begin{bmatrix} E & \cr & O \end{bmatrix} Y$, where $E$ is the $l \times l$ identity matrix. If we put here $[w_1 \ldots w_l] Y^{-1} [E \ O] X^{-1} = [m_1 \ldots m_k]$, then $m_i \in \mathbb{Z}$ and

$$[m_1 \ldots m_k] D = [m_1 \ldots m_k] X \begin{bmatrix} E & \cr & O \end{bmatrix} Y = [w_1 \ldots w_l]$$

as required.

(ii) As is well-known, $\text{GCD} \{ \chi_i(1) \} = p^{a-d} u$ (see Brauer [1]) and so all $\varepsilon_i$ are in $\mathbb{Z}$. By (i) we have

$$p^{a-d} u [\varepsilon_1 \ldots \varepsilon_k] D = [\chi_1(1) \ldots \chi_k(1)] D - p^a u [m_1 \ldots m_k] D = O.$$ 

Hence for $x \in G^0$,

$$\eta_B(x) = \sum_{i=1}^k \varepsilon_i \chi_i(x) = \sum_{i=1}^k \varepsilon_i \sum_{j=1}^l d_{ij} \phi_j(x) = \sum_{j=1}^l \sum_{i=1}^k \varepsilon_i d_{ij} \phi_j(x) = 0.$$
where \( \{ \varphi_i \} \) are the irreducible Brauer characters of \( B \). This completes the proof of Theorem 2.

We call this \( \{ \varepsilon_i \} \) a residue set (with \( \{ m_k \} \)) associated to \( B \). We now give a sufficient condition for a set of irreducible ordinary characters to be a \( p \)-block making use of residue sets.

**Theorem 3.** Let \( B \) be a \( p \)-block of \( G \) with the irreducible ordinary characters \( \chi_1, \ldots, \chi_k \). For \( J \subseteq \{ 1, \ldots, k \} \), assume that \( \sum_{j \in J} \varepsilon_j \chi_j \) vanishes on \( G^0 \) for every residue set \( \{ \varepsilon_i \} \) associated to \( B \). Then \( \{ \chi_j \} = B \) or \( \emptyset \).

**Proof.** Let \( D \) be the decomposition matrix of \( B \). We consider the vector space \( V = \langle [x_1 \cdots x_k][x_1 \cdots x_k]D = [0 \cdots 0] \rangle \) over the complex field. Since \( D \) is of rank \( l \), \( V \) has a basis with entries in \( \mathbb{Z} \). Let \( \delta = [\delta_1 \cdots \delta_k] \) be an element of \( V \) with \( \delta_i \in \mathbb{Z} \) and \( \varepsilon = [\varepsilon_1 \cdots \varepsilon_k] \) be a residue set with \( \{ m_i \} \) associated to \( B \). Then \( \varepsilon' = [\varepsilon_1 - p^d \delta_1 \cdots \varepsilon_k - p^d \delta_k] \) is a residue set with \( \{ m_i + \delta_i \} \) and \( \delta = \frac{1}{p^d}(\varepsilon - \varepsilon') \). Hence \( V \) is generated by all \( [\varepsilon_1 \cdots \varepsilon_k] \) such that \( \{ \varepsilon_i \} \) are residue sets associated to \( B \). For every \( y \in G - G^0 \), \( [\chi_1(y) \cdots \chi_k(y)] \) is evidently contained in \( V \) by the orthogonality relation and so it is expressed by a linear combination of \( \{ [\varepsilon_1 \cdots \varepsilon_k] \} \). Hence \( \sum_{j \in J} \chi_j(y) \chi_j \) vanishes on \( G^0 \) by our assumption. Thus from Osima's Theorem, we obtain \( \{ \chi_j \} = B \) or \( \emptyset \). The proof is now complete.
§2. Conjecture B

Replacing the hypothesis of Theorem 3 with weaker one, we state the following,

**Conjecture B.** Let $B$ be a $p$-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. For $J \subseteq \{1, \ldots, k\}$, assume that $\sum_{j \in J} \epsilon_j \chi_j(1) = 0$ for every residue set $\{\epsilon_i\}$ associated to $B$. Then $\{\chi_j\} = B$ or $\emptyset$.

It is verified that two conjectures $A'$ and $B$ are equivalent.

**Theorem 4.** Conjecture $A'$ holds if and only if Conjecture $B$ holds.

Proof. First, assume that Conjecture $A'$ holds and $\sum_{j \in J} \epsilon_j \chi_j(1) = 0$ for every residue set $\{\epsilon_i\}$. Then the same argument as in the proof of Theorem 3 implies that $\rho_J(y) = \sum_{j \in J} \chi_j(y) = 0$ for every $y \in G - G^0$. Hence $\{\chi_j\} = B$ or $\emptyset$.

Conversely, suppose that Conjecture $B$ holds and $\rho_J$ vanishes on $G - G^0$. Since $\eta_B = \sum_{i=1}^k \epsilon_i \chi_i$ vanishes on $G^0$ by Theorem 2, we have

$$0 = (\eta_B, \rho_J) = \left( \sum_{i=1}^k \epsilon_i \chi_i, \sum_{j \in J} \chi_j(1) \chi_j \right) = \sum_{j \in J} \epsilon_j \chi_j(1).$$

Hence $\{\chi_j\} = B$ or $\emptyset$ which satisfies to complete the proof.

By Theorem 4, in order to prove Harada's conjecture, it suffices to show that Conjecture $B$ holds for every block of $G$. Many examples show that the following hypothesis makes sense.

**Hypothesis 5.** A basic set for a block can be chosen from the set of irreducible ordinary characters.

Let $B$ be a $p$-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. Under Hypothesis 5, let $\{\chi_{1}, \ldots, \chi_l\}$ be a basic set for $B$ and the other characters are expressed as $\mathbb{Z}$-linear combinations of the basic set on $G^0$ as follows;
\[
\chi_\lambda = a^{\lambda}_1 \chi_1 + \cdots + a^{\lambda}_l \chi_l \quad (\lambda = l + 1, \ldots, k).
\]

Hence the decomposition matrix of \( B \) is of the form

\[
D = \begin{bmatrix}
  d_{11} & \cdots & d_{1l} & \chi_1 \\
  \vdots & & \vdots & \vdots \\
  d_{l1} & \cdots & d_{ll} & \chi_l \\
  \sum_{r=1}^l a^{l+1}_r d_{rl} & \cdots & \sum_{r=1}^l a^{l+1}_r d_{rl} & \chi_{l+1} \\
  \vdots & & \vdots & \vdots \\
  \sum_{r=1}^l a^k_r d_{rl} & \cdots & \sum_{r=1}^l a^k_r d_{rl} & \chi_k
\end{bmatrix}
\]

Then
\[
\begin{align*}
\mathbf{n}_1 &= [-a^{l+1}_1 \cdots -a^{l+1}_l 1 0 0 \cdots 0] \\
\mathbf{n}_2 &= [-a^{l+2}_1 \cdots -a^{l+2}_l 0 1 0 \cdots 0] \\
\mathbf{n}_{k-l} &= [-a^k_1 \cdots -a^k_l 0 0 0 \cdots 1]
\end{align*}
\]

are linearly independent solutions of the equation

\[
[x_1 \cdots x_k] D = [0 \cdots 0].
\]

As in Theorem 2, let \( \mathbf{m}_0 = [m^0_1 \cdots m^0_k] \) be a \( Z \)-solution of the equation

\[
[x_1 \cdots x_k] D = [w_1 \cdots w_k].
\]

Then

\[
[m_1 \cdots m_k] = \mathbf{m}_0 + z_1 \mathbf{n}_1 + \cdots + z_{k-l} \mathbf{n}_{k-l} \quad (z_1, \ldots, z_{k-l} \in Z)
\]

are all of \( Z \)-solutions of (2). We define, for a residue set \( \{ \varepsilon_i \} \) with \( \{ m_i \} \),

\[
\begin{align*}
\chi(1) &= [\chi_1(1) \cdots \chi_k(1)] \\
\varepsilon &= [\varepsilon_1 \cdots \varepsilon_k]
\end{align*}
\]

Since \( \chi_i(1) = p^a u m_i + p^{a-d} u \varepsilon_i \), we have
\[ p^{n-d}ue = \chi(1)^J - p^n u (m_0 + z_1 n_1 + \cdots + z_{k-1} n_{k-1}). \]  

Let \( \cdot \) denote the scalar product of vectors. For \( J \subseteq \{1, \ldots, k\} \) and a vector \( v = [v_1 \cdots v_k] \), let \( v^J \) denote the vector of size \( k \) whose \( i \)-th component is \( v_i \) if \( i \in J \) and 0 otherwise. Then by (3)

\[
(x(1)^J - p^n u m_0^J - p^n u z_1 n_1^J - \cdots - p^n u z_{k-1} n_{k-1}^J) \cdot x(1)^J = p^{n-d} \sum_{j \in J} \epsilon_j \chi_j(1). \tag{4}
\]

If \( \sum_{j \in J} \epsilon_j \chi_j(1) = 0 \) for every residue set \( \{\epsilon_j\} \), then by (4) we have

\[
n_1^J \cdot x(1)^J = \cdots = n_{k-1}^J \cdot x(1)^J = 0. \tag{5}
\]

Since \( \eta(1) = \sum_{i=1}^k \epsilon_i \chi_i(1) = 0 \) by Theorem 2, similarly we have

\[
n_1^{J'} \cdot x(1)^{J'} = \cdots = n_{k-1}^{J'} \cdot x(1)^{J'} = 0, \tag{6}
\]

where \( J' = \{1, \ldots, k\} - J \). Hence the next is proved.

**Lemma 6.** Under Hypothesis 5, if there is no non-empty proper subset \( J \) of \( \{1, \ldots, k\} \) for which (5) or (6) holds, then Conjecture B holds.

**Lemma 7.** Under Hypothesis 5, if a basic set \( \{\chi_1, \ldots, \chi_l\} \) is contained in \( \{\chi_J\} \) or \( \{\chi_J'\} \) and \( \sum_{j \in J} \epsilon_j \chi_j(1) = 0 \) for every residue set \( \{\epsilon_j\} \) associated to \( B \), then \( \{\chi_J\} = B \) or \( \emptyset \).

Proof. This follows from (5) and (6).

**Lemma 8.** Under Hypothesis 5, if the coefficients \( \alpha_i^l \) in the equation (1) are all non-negative, then Conjecture B holds.
Proof. If \( k = l + 1 \), then by (1) we have \( \chi_k(1) = a_1^k \chi_1(1) + \cdots + a_l^k \chi_l(1) \) and the definition of blocks yields that \( a_s^k \neq 0 \) for all \( s = 1, \ldots, l \). Hence the result is easily verified by considering (5) and (6).

Suppose that \( k > l + 1 \). Assume that for a non-empty proper subset \( J, \sum_{j \in J} \varepsilon_j \chi_j(1) = 0 \) for every residue set \( \{ \varepsilon_j \} \). Then by (5)

\[
\mathbf{n}_1^T \cdot \chi(1)^T = \cdots = \mathbf{n}_{k-1}^T \cdot \chi(1)^T = 0.
\]

Since \( a_s^k \) are all non-negative, by exchanging the indices if necessary, we may assume that

\[
a_1^{t+1} = \cdots = a_r^{t+1} = 0 \quad a_{r+1}^{t+1} = \cdots = a_l^{t+1} = 0
\]

for some \( t \) and \( r \) such that \( l < t < k, 1 \leq r < l \). Hence the decomposition matrix \( D \) is of the form

\[
D = \begin{bmatrix}
d_{11} & \cdots & d_{1l} \\
\vdots & \ddots & \vdots \\
d_{l1} & \cdots & d_{ll} \\
\sum_{r=r+1}^l a_r^{t+1} d_{rl} & \cdots & \sum_{r=r+1}^l a_r^{l+1} d_{rl} \\
\vdots & \ddots & \vdots \\
\sum_{r=r+1}^l a_r^{t+1} d_{rl} & \cdots & \sum_{r=r+1}^l a_r^{l+1} d_{rl} \\
\sum_{r=1}^r a_r^{t+1} d_{rl} & \cdots & \sum_{r=1}^r a_r^{l+1} d_{rl}
\end{bmatrix}
\]

Thus we can arrange the rows and the columns as

\[
\begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix}.
\]

This is a contradiction which completes the proof of Lemma 8.

In particular, we deduce

**Corollary 9.** Under Hypothesis 5, if \( l = 1 \) or \( 2 \), then Conjecture B holds.
Proof. If $l = 1$, then it is immediate from Lemma 8. Suppose $l = 2$. By (1) we have $\chi_3(1) = a_1^2 \chi_1(1) + a_2^2 \chi_2(1)$ (3 $\leq \lambda \leq k$). Assume that $\chi_1 \in \{\chi_{rs}\}$ and $\chi_2 \in \{\chi_{rs}\}$. Then either $a_1^2 > 0$, $a_2^2 = 0$, or $a_1^2 = 0$, $a_2^2 > 0$ for each $\lambda$. Hence the result actually follows from Lemma 8. (See Kiyota [19].)

**Corollary 10.** If the irreducible Brauer characters of $B$ are all liftable, then Conjecture B holds.

Proof. Straightforward by Lemma 8.

**Theorem 11.** (Kiyota and Okuyama [18]) If $G$ is $p$-solvable, then Conjecture A holds.

Proof. By Fong and Swan’s Theorem, the irreducible Brauer characters are all liftable. Hence the consequence is clear by Corollary 10.
§3. Conjecture C

Analogously to Section 2, replacing the hypothesis of Theorem 3 with another one, we state the following:

**Conjecture C.** Let $B$ be a $p$-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. Let \( \{\epsilon_i\} \) be a residue set associated to $B$. For $J \subseteq \{1, \ldots, k\}$, assume that $\sum_{j \in J} \epsilon_j \chi_j$ vanishes on $G^0$. Then $\{\chi_j\} = B$ or $\emptyset$.

Suppose that the assumption of Conjecture C holds, i.e. $\sum_{j \in J} \epsilon_j \chi_j$ vanishes on $G^0$ for $J \subseteq \{1, \ldots, k\}$. Then by the linearity independence of the irreducible Brauer characters, we have $\sum_{j \in J} \epsilon_j d_{js} = 0$ for all $s = 1, \ldots, l$, where $D = [d_{ss}]$ is the decomposition matrix of $B$. Since $\chi_j(1) = p^u a_m + p^s u \epsilon_j$, we obtain $\sum_{j \in J} \chi_j(1) d_{js} \equiv 0 \pmod{p^s}$ for all $s = 1, \ldots, l$. Since $\eta_H = \sum_{i=1}^k \epsilon_i \chi_i$ vanishes on $G^0$, for $J' = \{1, \ldots, k\} - J$, similarly we have $\sum_{j \in J'} \chi_j(1) d_{js} \equiv 0 \pmod{p^s}$ for all $s = 1, \ldots, l$. Therefore the next is proved.

**Lemma 12.** If there is no non-empty proper subset $J$ of $\{1, \ldots, k\}$ for which

\[
\sum_{j \in J} \chi_j(1) d_{js} \equiv 0 \pmod{p^s}
\]

or

\[
\sum_{j \in J'} \chi_j(1) d_{js} \equiv 0 \pmod{p^s}
\]

holds for all $s = 1, \ldots, l$, then Conjecture C holds.

**Lemma 13.** If $k = l + 1$, then Conjecture C holds.

Proof. If a non-empty proper subset $J$ satisfies $\sum_{j \in J} \epsilon_j d_{js} = 0$ for all $s$, then $J' = \{1, \ldots, k\} - J$ satisfies $\sum_{j \in J'} \epsilon_j d_{js} = 0$ for all $s$. Hence the rank of $D$ is less than $k - 1 = l$. This is a contradiction and the proof is complete by Lemma 12.
§4. Cyclic Blocks

The following theorem is clear from Harada's paper [8] and Theorem 4. But we here give an another explicit proof.

**Theorem 14.** (Harada) If a defect group of a $p$-block $B$ is cyclic, then Conjecture B holds.

Proof. We use the same notation as in Section 2. Without loss, we may assume that the defect of $B$ is positive. We refer to Dade's Theorem on block structure with cyclic defect groups (see Dornhoff [5] or Feit [6] for detail).

The decomposition matrix $D$ of $B$ is arranged as follows. The first $e$ parts of the rows are non-exceptional characters $\chi_1, \ldots, \chi_e$ and the second $|\Lambda|$ parts of the rows are exceptional characters $\{\chi_\lambda|\lambda \in \Lambda\}$. By Dade's Theorem the principal indecomposable characters are either $\Phi_\iota = \chi_\iota + \sum_{\lambda \in \Lambda} \chi_\lambda$ ($1 \leq j \leq e$) or $\Phi_j = \chi_j + \chi_j$ ($1 \leq i, j \leq e$). For such $j$, $i$ and $J \subseteq \{1, \ldots, k\}$, we set $\{j|j \in J\} = \{j_1, \ldots, j_n\}$ and $\{i|i \in J\} = \{i_1, \ldots, i_t\}$.

The vectors $\mathbf{n}$ in (5) are

$$ n_1 = \left[ \begin{array}{ccccccc} \cdots & 1 & \cdots & -1 & \ldots & 1 & 0 & \ldots & 0 \\ \end{array} \right] $$

$$ n_2 = \left[ \begin{array}{ccccccc} \cdots & 1 & \cdots & -1 & \cdots & 0 & 1 & \cdots & 0 \\ \end{array} \right] $$

$$ \vdots $$

$$ n_{|\Lambda|} = \left[ \begin{array}{ccccccc} \cdots & 1 & \cdots & -1 & \cdots & 0 & 0 & \cdots & 1 \\ \end{array} \right] $$

where the $j$-th component is $-1$ and the $i$-th component is $1$. Assume that $\sum_{\iota \in J} \varepsilon_j \chi_\iota(1) = 0$ for every residue set $\{\varepsilon_j\}$. Then the same argument as in Section 2 yields that the equation (5) is

$$ n_j^J \cdot \chi(1)^J = \cdots = n_{|\Lambda|}^J \cdot \chi(1)^J = 0. \quad (7) $$

Hence we have either $\Lambda \subseteq J$ or $\Lambda \subseteq J'$. Since $\sum_{\iota \in J} \varepsilon_j \chi_\iota(1) = 0$ and $\sum_{\iota \in J'} \varepsilon_j \chi_\iota(1) = 0$, we may assume that $\Lambda \subseteq J'$ by exchanging $J$ for $J'$ if necessary. Hence $J = \{j_1, \ldots, j_n, i_1, \ldots, i_t\}$. We claim that $J = \emptyset$.

By (7) we obtain
\[-\chi_{j_1}(1) - \cdots - \chi_{j_e}(1) + \chi_{i_1}(1) + \cdots + \chi_{i_s}(1) = 0. \quad (8)\]

Since \(\{e_i\}\) is a residue set with \(\{m_i\}\), we get

\[p^a u [m_1 \cdots m_k] D = p^a u [w_1 \cdots w_k] = (\Phi_1(1) \cdots \Phi_l(1)).\]

Thus

\[\Phi_1(1) = \chi_1(1) + \sum_{\lambda \in \Lambda} \chi_\lambda(1) = p^a u m_j + p^a u \sum_{\lambda \in \Lambda} m_{\lambda},\]

\[\Phi_l(1) = \chi_l(1) + \chi_j(1) = p^a u m_i + p^a u m_j.\]

Hence \(\chi_j(1) - p^a u m_j = p^a u \sum_{\lambda \in \Lambda} m_{\lambda} - \sum_{\lambda \in \Lambda} \chi_\lambda(1)\) is independent of \(j\) and so \(\chi_i(1) - p^a u m_i = -(\chi_j(1) - p^a u m_j)\) is independent of \(i\). Therefore \(\varepsilon_j = \cdots = \varepsilon_j\), \(\varepsilon_i = \cdots = \varepsilon_i\) (\(= \varepsilon_i\)) and \(\varepsilon_j = -\varepsilon_i\). By (8) we have

\[-(p^a u m_j + p^{a-d} u \varepsilon_j) - \cdots - (p^a u m_j + p^{a-d} u \varepsilon_j)\]

\[+(p^a u m_i + p^{a-d} u \varepsilon_i) + \cdots + (p^a u m_i + p^{a-d} u \varepsilon_i) = 0.\]

Hence \((\kappa + \tau) \varepsilon_i \equiv 0 \pmod{p^a}\). Since \(0 \leq \kappa + \tau \leq e \leq p - 1\) and \(\varepsilon_i \neq 0 \pmod{p^a}\), we have \(\kappa = \tau = 0\) and thus \(J = 0\) as required. This completes the proof of Theorem 14.

**Theorem 15.** If a defect group of a \(p\)-block \(B\) is cyclic, then Conjecture \(C\) holds.

**Proof.** Without loss, we may assume that the defect \(d\) of \(B\) is positive. We also refer to Dade’s Theorem.

The principal indecomposable characters are either \(\Phi_1 = \chi_j + \sum_{\lambda \in \Lambda} \chi_\lambda (1 \leq j \leq e)\) or \(\Phi_l = \chi_i + \chi_j (1 \leq i, j \leq e)\). Since \(\Phi_1(1) \equiv 0 \pmod{p^a}\) and \(\chi_j(1) \equiv 0 \pmod{p^a}\), we have \(\nu(\chi_j(1)) = \nu(\sum_{\lambda \in \Lambda} \chi_\lambda(1))\) for all \(j = 1, \ldots, e\), where \(\nu\) denotes the \(p\)-adic exponential valuation with \(\nu(p) = 1\). Since \(\sum_{\lambda \in \Lambda} \chi_\lambda(1) = |\Lambda| \chi_j(1)\) and \(|\Lambda| = (p^a-1)/e\), we have \(\nu(\chi_j(1)) = \nu(\chi_\lambda(1)) = a - d \) for all \(j = 1, \ldots, e\) and \(\lambda \in \Lambda\).
If $\Phi_t = \chi_i + \chi_j$, then either $i, j \in J$ or $i, j \in J'$. Assume that $\Phi_x = \chi_j + \sum_{\lambda \in \Lambda} \chi_\lambda$. We claim that either $\{j\} \cup \Lambda \subseteq J$ or $\{j\} \cup \Lambda \subseteq J'$.

If $j \notin J$, then
\[
\nu(\sum_{i \in J} \chi_i(1) d_{ai}) = \nu(|J \cap \Lambda| \chi(1))
\]
\[
\leq \nu(|J \cap \Lambda|) + \nu(\chi(1)) < d + a - d = a.
\]

Hence $J \cap \Lambda = \emptyset$ by Lemma 12 and so $\{j\} \cup \Lambda \subseteq J'$. Consequently we obtain either $\{j\} \cup \Lambda \subseteq J$ or $\{j\} \cup \Lambda \subseteq J'$ as desired. By the definition of blocks, it is impossible to arrange rows and columns of $D$ so that
\[
\begin{bmatrix}
D_1 & O \\
O & D_2
\end{bmatrix}.
\]
Therefore we have $\{\chi_j\} = B$ or $\emptyset$. This completes the proof of Theorem 15.
§5. $PSL(2,q)$

In this Section, we prove Conjectures A and C for $PSL(2,q)$.

**Theorem 16.** If $G = PSL(2,q)$ such that $q$ is a power of a prime, then Conjecture A holds.

Proof. By Theorem 4, it suffices to show that Conjecture B holds for every block. The cases of cyclic blocks and blocks with $l = 1$ or 2 can be omitted by Harada [8] and Corollary 9 since Hypothesis 5 holds for $G$. Hence we consider the following four cases:

(i) $p = 2, \; q \equiv 1 \pmod{4}$,
(ii) $p = 2, \; q \equiv -1 \pmod{4}$,
(iii) $2 = p|q$,
(iv) $2 \neq p|q$.

The decomposition matrices which we use in this proof are determined by Burkhardt in [2]. We use the notation of that paper for the characters.

(i) $p = 2, \; q \equiv 1 \pmod{4}$. For the principal block, a basic set and the expressions of the other characters as linear combinations of the basic set are as follows:

<table>
<thead>
<tr>
<th>BS</th>
<th>1</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$-$1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\eta^G$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus by Lemma 6 we have the result.

(ii) $p = 2, \; q \equiv -1 \pmod{4}$. For the principal block, the irreducible Brauer characters are all liftable. Hence the result follows by Corollary 10.

(iii) $2 = p|q$. The irreducible characters are $1, \alpha, \eta_i^G (1 \leq i \leq q/2 - 1)$ and $\gamma_i^c (1 \leq i \leq q/2)$. Thus $\{\alpha\}$ is a block of defect 0 and only principal
block is a block of positive defect. We claim that we can choose a residue set \(\{\varepsilon_i\}\) which satisfies \(\sum_{J \in J} \varepsilon_i \chi_J(1) \neq 0\) for any non-empty proper subset \(J\) of \(\{1, \ldots, k\}\). The decomposition matrix is as in [2] and so we have such a residue set \(\{\varepsilon_i\}\) with \(\{m_i\}\) as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
& \text{degree} & \varepsilon_i & m_i \\
\hline
1 & 1 & 1 & 0 \\
q/2 - 1 & \eta^G & q + 1 & 1 & 1 \\
& \vdots & \vdots & \vdots & \vdots \\
& \eta^G & q + 1 & 1 & 1 \\
q/2 & \gamma^* & q - 1 & -1 & 1 \\
& \vdots & \vdots & \vdots & \vdots \\
& \gamma^* & q - 1 & -1 & 1 \\
\hline
\end{array}
\]

Hence the result easily follows.

(iii) \(2 \neq p|q\). If \(q \equiv 1 \pmod{4}\), then the irreducible characters are \(1, \eta^G(1 \leq i \leq \frac{1}{4}(q - 5)), \delta^*_1(1 \leq i \leq \frac{1}{4}(q - 1)), \gamma_1\) and \(\gamma_2\). The decomposition matrix of the principal block is as in [2]. Thus we have a residue set \(\{\varepsilon_i\}\) with \(\{m_i\}\) as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
& \text{degree} & \varepsilon_i & m_i \\
\hline
1 & 1 & q + 1 & -1 \\
\frac{1}{4}(q - 5) & \eta^G & q + 1 & q + 1 & 0 \\
& \vdots & \vdots & \vdots & \vdots \\
& \eta^G & q + 1 & q + 1 & 0 \\
\frac{1}{4}(q - 1) & \delta^* & q - 1 & -(q + 1) & 2 \\
& \vdots & \vdots & \vdots & \vdots \\
& \delta^* & q - 1 & -(q + 1) & 2 \\
& \gamma_1 & \frac{1}{2}(q + 1) & \frac{1}{2}(q + 1) & 0 \\
& \gamma_2 & \frac{1}{2}(q + 1) & \frac{1}{2}(q + 1) & 0 \\
\hline
\end{array}
\]

If \(q \equiv -1 \pmod{4}\), then similarly we have the following;
These $\{\varepsilon_i\}$'s also satisfy $\sum_{i \in J} \varepsilon_i \chi(1) \neq 0$ for any non-empty proper subset $J$ of $\{1, \ldots, k\}$ and thus Conjecture B holds. This completes the proof of Theorem 16.

**Theorem 17.** If $G = \text{PSL}(2, q)$ such that $q$ is a power of a prime, then Conjecture C holds.

Proof. The case of cyclic blocks can be omitted by Theorem 15. Hence it suffices to show the following four cases.

(i) $p = 2$, $q \equiv 1 \pmod{4}$,

(ii) $p = 2$, $q \equiv -1 \pmod{4}$,

(iii) $2 = p|q$,

(iv) $2 \neq p|q$.

The decomposition matrices determined by Burkhardt are used as well.

(i) $p = 2$, $q \equiv 1 \pmod{4}$. In this case the irreducible characters of $G$ are $1$, $\alpha$, $\gamma_1$, $\gamma_2$, $\eta^G_i$ ($1 \leq i \leq \frac{1}{4}(q - 5)$) and $\delta^*_i$ ($1 \leq i \leq \frac{1}{4}(q - 1)$). Each degree is $\alpha(1) = q$, $\gamma_1(1) = \gamma_2(1) = \frac{1}{2}(q + 1)$, $\eta^G_i(1) = q + 1$ and $\delta^*_i(1) = q - 1$. Let $P \times C$ be a cyclic subgroup of order $\frac{1}{4}(q - 1)$, where $P$ is a subgroup of order $2^{a-1}$ and $C$ is a subgroup of order $c$, $2 \mid c$. Thus there are only one block of maximal defect and $\frac{1}{4}(c - 1)$ blocks of defect $a - 1$. The other blocks are all of defect 0.

The decomposition matrix of the principal block is

<table>
<thead>
<tr>
<th>degree</th>
<th>$\varepsilon_i$</th>
<th>$m_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{4}(q - 3)$</td>
<td>$\eta^G$</td>
<td>$q + 1$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\frac{1}{4}(q - 3)$</td>
<td>$\delta^*$</td>
<td>$q - 1$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$-\frac{1}{2}(q + 1)$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$\frac{1}{2}(q - 1)$</td>
</tr>
</tbody>
</table>
Hence by Lemma 12, we have \( \{ \chi_f \} = B \) or \( \emptyset \).

The decomposition matrix of the block of defect \( a - 1 \) is

\[
\begin{array}{c|ccc|c}
1 & \varphi_1 & \varphi_2 & \text{degree} \\
\hline
1 & 1 & 0 & 0 & 1 \\
\alpha & 1 & 1 & 1 & 2^a c + 1 \\
\gamma_1 & 1 & 1 & 0 & 2^{a-1} c + 1 \\
\gamma_2 & 1 & 0 & 1 & 2^{a-1} c + 1 \\
2^{a-2} - 1 & \eta^G & 2 & 1 & 1 & 2^a c + 2 \\
\end{array}
\]

Thus clearly \( \{ \chi_f \} = B \) or \( \emptyset \).

(ii) \( p = 2, \ q \equiv -1 \pmod{4} \). In this case there are the irreducible characters \( 1, \alpha, \gamma_1, \gamma_2, \eta^G (1 \leq i \leq \frac{1}{2}(q - 3)) \) and \( \delta_i^* (1 \leq i \leq \frac{1}{2}(q - 3)) \).

Each degree is \( \alpha(1) = q, \gamma_1(1) = \gamma_2(1) = \frac{1}{2}(q - 1), \eta^G(1) = q + 1 \) and \( \delta_i^*(1) = q - 1 \). Let \( P \times C \) be a cyclic subgroup of order \( \frac{1}{2}(q + 1) \), where \( P \) is a subgroup of order \( 2^{a-1} \) and \( C \) is a subgroup of order \( c, \ 2 \nmid c \). There are only one block of maximal defect and \( \frac{1}{2}(c - 1) \) blocks of defect \( a - 1 \).

The other blocks are all of defect 0.

The decomposition matrix of the principal block is

\[
\begin{array}{c|ccc|c}
1 & \varphi_1 & \varphi_2 & \text{degree} \\
\hline
1 & 1 & 0 & 0 & 1 \\
\alpha & 1 & 1 & 1 & 2^a c - 1 \\
\gamma_1 & 0 & 1 & 0 & 2^{a-1} c - 1 \\
\gamma_2 & 0 & 0 & 1 & 2^{a-1} c - 1 \\
2^{a-2} - 1 & \delta^* & 0 & 1 & 1 & 2^a c - 2 \\
\end{array}
\]
The same argument as in (i) yields the result.

The decomposition matrix of the block of defect \( a - 1 \) is

\[
\begin{array}{c|cc}
\phi & \text{degree} \\
\hline
\delta^* & 1 & 2^{a-1}c - 2 \\
\vdots & \vdots & \vdots \\
\delta^* & 1 & 2^{a-1}c - 2
\end{array}
\]

Thus the result is clear.

(iii) \( 2 = p|q \). The irreducible characters are \( 1, \alpha, \eta_i^G (1 \leq i \leq q/2 - 1) \) and \( \gamma_i^* (1 \leq i \leq q/2) \) and their degrees are \( \alpha(1) = q, \eta_i^G = q + 1 \) and \( \gamma_i^*(1) = q - 1 \). Thus \( \{\alpha\} \) is a block of defect 0 and only principal block is a block of positive defect. So we consider the principal block \( B \). The number \( k \) of the irreducible ordinary characters of \( B \) is \( 1 + (q/2 - 1) + q/2 = q \). It is known that there is a one-to-one correspondence between the irreducible Brauer characters of \( B \) and the proper subsets of \( \{1, \ldots, a\} \). Thus the number \( l \) of the irreducible Brauer characters is \( q - 1 \). Hence \( k = l + 1 \) and Lemma 13 asserts the result.

(iv) \( 2 \neq p|q \). If \( q \equiv 1 \pmod{4} \), then the irreducible characters are \( 1, \alpha, \eta_i^G (1 \leq i \leq \frac{1}{2}(q - 5)), \delta_i^* (1 \leq i \leq \frac{1}{2}(q - 1)), \gamma_1 \) and \( \gamma_2 \) and their degrees are \( \eta_i^G (1) = q + 1, \delta_i^* (1) = q - 1, \gamma_1 (1) = \gamma_2 (1) = \frac{1}{2}(q + 1) \). There is only one block \( B \).

Set \( F = \{I = (i_1 \cdots i_s) | 1 \leq i_1 \leq p - 1, \sum_{i=1}^s i_i \equiv 0 \pmod{2}\} = \{(p - 1) \cdots 1\} \). Then it is known that there is a one-to-one correspondence between the irreducible Brauer characters \( \varphi_I \) of \( B \) and the elements \( I \) of \( F \). In particular, the principal Brauer character corresponds to \((0 \cdots 0)\). Thus the number \( l \) is equal to \(|F| = \frac{1}{2}(q - 1)\). Since \( k = \frac{1}{2}(q + 3) \), we have \( k = l + 2 \).

A residue set \( \{e_i\} \) with \( \{m_i\} \) associated to \( B \) is arranged as follows:

\[
egin{align*}
1(1) &= qa_0 + ue_0 \\
\eta_i^G (1) &= qum_i + ue_i (1 \leq i \leq h) \\
\delta_i^* (1) &= qum_i' + ue_i' (1 \leq i \leq h + 1) \\
\gamma_i (1) &= qum_i'' + ue_i'' (1 \leq i \leq 2),
\end{align*}
\]

where \( h = \frac{1}{2}(q - 5) \). The rows and the columns of the decomposition matrix \( D \) of \( B \) is arranged as follows: The first row is 1, the second is
\( \eta_1^G, \ldots, \) the \((h+1)\)-st is \( \eta_h^G \), the \((h+2)\)-nd is \( \delta_1^* \), \ldots, the \((2h+2)\)-nd is \( \delta_{h+1}^* \), the \((2h+3)\)-rd is \( \gamma_1 \) and the last is \( \gamma_2 \). The first column is \((0 \cdots 0)\), from the second \( I = (i_1 \cdots i_a) \neq (0 \cdots 0) \) such that no \( i_r = p - 1 \), the next is \( I = (i_1 \cdots i_a) \) such that exactly one \( i_r = p - 1 \), \ldots and the last is \( I = (i_1 \cdots i_a) \) such that exactly \((a-1)\) \( i_r = p - 1 \). Now we define for \( I = (i_1 \cdots i_a) \),

\[
\sum(I) = \left\{ \frac{1}{2} \sum_{r=1}^a \tau_r (\rho - 1 - i_r) p^{r - 1} \mid \tau_r \in \{1, -1\} \right\}.
\]

If \( \eta_h^G = \sum_{I \in F} d_{i_I} \varphi_I \) is the decomposition of \( \eta_h^G \) into the irreducible Brauer characters, then

\[
d_{i_I} = 1 \text{ for } \{i, \frac{1}{2}(q-1) - i\} \cap \sum(I) \neq \emptyset
\]

\[
= 0 \text{ otherwise.}
\]

If \( \delta_1^* = \sum_{I \in F} d_{i_I} \varphi_I \), then

\[
d_{i_I} = 1 \text{ for } \{i, \frac{1}{2}(q+1) - i\} \cap \sum(I) \neq \emptyset
\]

\[
= 0 \text{ otherwise.}
\]

If \( \gamma_{h/2} = \sum_{I \in F} d_{i_I} \varphi_I \), then

\[
d_I = 1 \text{ for } \frac{1}{4}(q-1) \in \sum(I)
\]

\[
= 0 \text{ otherwise.}
\]

If \( I = (0 \cdots 0) \), then

\[
\left| \left\{ \eta_h^G \mid d_{i_I} = 1 \right\} \right| + 1 = \left| \left\{ \delta_1^* \mid d_{i_I} = 1 \right\} \right| = 2^{a-1}.
\]

If \( I = (i_1 \cdots i_a) \) such that exactly \( t \) \( i_r = p - 1 \) and \( \frac{1}{4}(q-1) \not\in \sum(I) \), then

\[
\left| \left\{ \eta_h^G \mid d_{i_I} = 1 \right\} \right| = \left| \left\{ \delta_1^* \mid d_{i_I} = 1 \right\} \right| = 2^{a-t-1}.
\]

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If \( I = (i_1 \cdots i_a) \) such that exactly \( t \) \( i_j = p - 1 \) and \( \frac{1}{4}(q - 1) \in \Sigma(I) \), then
\[
|\{i_t^G | d_t = 1\}| = 1 = |\{\delta_t^* | d_t = 1\}| = 2^{n - t - 1}.
\]

Then the degrees of the principal indecomposable characters \( \Phi_I \) are
\[
g(2^a - 1), q2^{a-1}, \ldots, g2, \ldots
\]

Hence
\[
(\cdots w_I \cdots) = (2^a - 1 \ 2^a \cdots \ 2^{a-1} \cdots \cdot \cdots \cdot \cdot \cdot)
\]
and \( v = 1 \), where \( \Phi_I(1) = q\nu\Omega_I \).

Next we consider the linear homogeneous equation
\[
(x_0 \ x_1 \cdots x_h \ x'_1 \cdots x'_{h+1} \ x''_1 \ x''_2)D = (0 \cdots 0).
\]

\[(1 \ 1 \cdots 1 \ -1 \cdots -1 \ 0 \ 1) \text{ and } (0 \cdots 0 \ 1 \ -1) \text{ are linearly independent solutions of (10). Furthermore } (-1 \ 0 \cdots 0 \ 1 \ -1)\text{ is a solution of the equation}
\]
\[
(m_0 \ m_1 \cdots m_h \ m'_1 \cdots m'_{h+1} \ m''_1 \ m''_2)D = (\cdots w_I \cdots).
\]

Since the rank of \( D \) is \( l \) and \( k = l + 2 \),
\[
(-1 \ 0 \cdots 0 \ 2 \cdots 2 \ 0 \ 0) + z_1 (1 \ 1 \cdots 1 \ -1 \cdots -1 \ 0 \ 1) + z_2 (0 \cdots 0 \ 1 \ -1)
\]
\((z_1, z_2 \in \mathbb{Z})\) are all of the solutions of (11). Hence \( m_i = z_1 \ (1 \leq i \leq h) \)
and \( m'_i = 2 - z_1 \ (1 \leq i \leq h + 1) \). Therefore by (9) we have \( \epsilon_i = q(z_1 - 1) + 1 \)
\((1 \leq i \leq h)\) and \( \epsilon'_i = q(z_1 - 1) - 1 \ (1 \leq i \leq h + 1) \).

By Lemma 12, we may assume that \( 1, \gamma_1, \gamma_2 \notin \{\chi_J\} \). We claim that
\( \{\chi_J\} = \emptyset \). Suppose that \( n_1 \eta_{i}^G \)'s and \( n_2 \delta_{i}^* \)'s are contained in \( \{\chi_J\} \). Then
by \( \sum_{J \in I} \epsilon_J \chi_J(1) = 0 \),
\[
n_1 \{q(1 - z_1) + 1\}(q + 1) + n_2 \{q(z_1 - 1) - 1\}(q - 1) = 0,
\]
\[
0 \leq n_1 \leq h, \quad 0 \leq n_2 \leq h + 1.
\]

Hence \( n_1 + n_2 \equiv 0 \pmod{q} \), \( 0 \leq n_1 + n_2 \leq 2h + 1 = \frac{1}{2}(q - 3) \) and so
\( n_1 = n_2 = 0 \). We obtain \( \{\chi_J\} = \emptyset \) as required.

If \( q \equiv -1 \pmod{4} \), then the same argument implies the result. This
completes the proof of Theorem 17.
§6. $Sp(4, q)$

**Theorem 18.** If $G$ is the symplectic group $Sp(4, q)$, where $q$ is a power of an odd prime $e$, then Conjecture A holds for every $p$ different from $e$.

Proof. By Theorem 4, it suffices to show that Conjecture B holds for every block. Basic sets of $Sp(4, q)$ are determined by White in [23-25]. We use the notation of those papers for the characters and the blocks. The order of $G = Sp(4, q)$ is $q^4(q^2 + 1)(q + 1)^2(q - 1)^2$, so if $p (\neq e)$ is a prime dividing $|G|$, then $p = 2$ or $p$ divides exactly one of $q^2 + 1$, $q + 1$ or $q - 1$. If $p$ is odd and divides $q^2 + 1$, then the defect group of each block is cyclic. If $p$ is odd and divides one of $q + 1$ or $q - 1$, then blocks with non-maximal defect have cyclic defect groups. In these cases, the result is clear by Harada [8].

(i) $p = 2$. For the blocks $b_1(r)$, $b_2(r)$, $b_3(r, s)$, $b_4(r, s)$, $b_5(r, s)$, $b_6(r)$ and $b_{89}(r)$, we have $l = 1$ or $2$ and for the blocks $b_1(r)$, the irreducible Brauer characters are all liftable. Hence the result follows by Corollaries 9 and 10. For the other blocks, basic sets and the expressions of the other characters as linear combinations of the basic sets are shown in the Tables below. The first row in each Table is a basic set and missing entries are 0.

$b_{11}(r)$

<table>
<thead>
<tr>
<th>BS</th>
<th>$\xi_3$</th>
<th>$\xi_4$</th>
<th>$\xi_{41}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{42}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\xi_{43}$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\xi_{42}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>1</td>
<td>(q \equiv 1 (mod 4))</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>-1</td>
<td>1</td>
<td>(q \equiv 3 (mod 4))</td>
</tr>
</tbody>
</table>

$b_0$ (the principal block)

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For the blocks $b_{111}(r)$ and $b_0$, assume that a subset $J$ satisfies (5) of Section 2 and some character in the basic set is contained in $\{\chi_J\}$. Then the above Table shows that the other characters are also contained in $\{\chi_J\}$. Hence the result follows by Lemma 7.

(ii) $p \neq 2, p|q - 1$. For the blocks $b_3(s, l), b_{111}(s), b_{41}(s)$ and $b_{87}(s)$, we have $l = 1$ or $2$ and for the other blocks, the irreducible Brauer characters are all liftable. Thus Conjecture B holds by Corollaries 9 and 10.

(iii) $p \neq 2, p|q + 1$. For the blocks $b_4(s, l), b_7(s), b_{21}(s)$ and $b_{317}(s)$, we have $l = 1$ or $2$ and for the blocks $b_1$ and $b_2$, the irreducible Brauer characters are all liftable. Hence the result follows by Corollaries 9 and 10. For the principal block $b_0$, a basic set and the expressions of the other characters as linear combinations of the basic set are as follows;

$b_0$ (the principal block)

<table>
<thead>
<tr>
<th>BS</th>
<th>$\chi_4$</th>
<th>$\psi_{\phi}$</th>
<th>$\psi_{\theta}$</th>
<th>$\psi_{\tau}$</th>
<th>$\psi_{\delta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_{\phi}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_{\theta}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_{\tau}$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_{\delta}$</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Evidently, the equation (5) or (6) does not occur for any non-empty proper subset $J$ and the result is clear by Lemma 6. This completes the proof of Theorem 18.
§7. $G_2(q)$

**Theorem 19.** If $G$ is the finite Chevalley group $G_2(q)$, where $q$ is a power of a prime $e$, then Conjecture A holds for every $p$ different from $e$.

Proof. In this case, basic sets are determined by Hiss in [10] and Hiss and Shamash in [11, 12]. We use the notation of those papers for characters and the blocks. The proof is very similar to the case of Theorem 18. The order of $G = G_2(q)$ is $q^6(q - 1)^2(q + 1)^2(q^2 + q + 1)(q^2 - q + 1)$. If $p$ $(\neq e)$ divides $|G|$ and does not divide $q \pm 1$, then the Sylow $p$-subgroups are cyclic and the result follows from Harada’s work. Hence it suffices to consider the case where $p$ divides $q - 1$ or $q + 1$. For the cyclic blocks, the result is also clear by [8]. For the blocks with $l = 1$ or 2, the result follows by Corollary 9. For the blocks such that the irreducible Brauer characters are all liftable, the result is immediate by Corollary 10. Thus we omit these cases. For the other blocks, we display basic sets and the expressions of the other characters in the Tables below. In every case, using the character degrees if necessary, the consequence follows by Lemma 6. Hence Conjecture B holds for all blocks and the proof of Theorem 19 is complete.
\( B_1 \) (the principal block)

\[ p = 2, \; q \equiv 1 \pmod{4} \]

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\( p \neq 2, 3, q \equiv -1 \pmod{p} \)

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\( B_2 \)

\( p \neq 2, q \equiv -1 \pmod{p}, q : \text{odd} \)

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\[ p = 2, \ q \equiv -1 \pmod{3} \]

\[
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X'_{26} & 1 & 1 & \\
X_b & 1 & 1 & \\
X_2 & 1 & -1 & -2 \quad \text{(only if } q \equiv -1 \pmod{4}) \ \\
\hline
\end{array}
\]

\[ p \neq 2, \ 3, \ q \equiv -1 \pmod{p}, \ q \equiv -1 \pmod{3} \]

\[
\begin{array}{|c|c|c|}
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\text{BS} & X_{31} & X_{32} & X_{33} \\
\hline
X_{26} & 1 & -1 & \\
X'_{26} & 1 & 1 & \\
X_2 & 1 & -1 & -2 \\
\hline
\end{array}
\]
Chapter II. Basic Sets of Brauer Characters

§1. Distribution into Blocks

In this Section, a simple method to distribute irreducible ordinary characters of \( G \) into \( \pi \)-blocks is described, where \( \pi \) is a set of primes. The following two Lemmas which are due, essentially, to Lükska [13], can be found in Robinson [22].

**Lemma 20.** Let \( B \) be a \( \pi \)-block of \( G \). Then whenever \( x \in G \) is \( \pi \)-regular and \( y \in G \) is \( \pi \)-singular, \( \sum_{\chi \in B} \chi(x) \chi(y) = 0 \). Conversely, any subset of \( \text{Irr}(G) \) with the above property is a union of \( \pi \)-blocks.

**Lemma 21.** Let \( H \) be a \( \pi \)-block of \( G \), \( \chi, \chi' \in \text{Irr}(G) \) with \( \chi \in H, \chi' \notin H \). Then \( \sum_{g \in G'} \chi(g) \chi'(g) = 0 \), where \( G' \) is the set of \( \pi \)-regular elements of \( G \).

**Theorem 22.** Let \( \{\chi_1, \ldots, \chi_n\} \) be the irreducible ordinary characters of \( G \) and \( \{x_1, \ldots, x_n\} \) the complete set of representatives of \( \pi \)-regular conjugate classes of \( G \). Let \( X = [\chi_i(x_j)] \) be the submatrix of the character table of \( G \). Then by making elementary column operations and interchanging rows, \( X \) can be changed of the form

\[
\begin{bmatrix}
A_1 & O \\
O & A_t
\end{bmatrix}
\]

where each \( A_i \) is of the form \( A_i = \begin{bmatrix} E & \cdot \\
A_i' & A_i''
\end{bmatrix} \) with the identity matrix \( E \) and can not be arranged of the form \( \begin{bmatrix} A_i'' & O \\
O & A_i''
\end{bmatrix} \). Furthermore, each \( A_i \) forms a single \( \pi \)-block of \( G \).

**Proof.** As is well known, \( X \) is of full rank. Hence by making elementary column operations and interchanging rows, it can be changed of
the required form. It suffices to show that $A_1$ forms a single $\pi$-block. Let $A_1$ consist of the irreducible characters $\chi_1, \ldots, \chi_k$ and put

$$X_1 = \begin{bmatrix} \chi_1(x_1) & \cdots & \chi_1(x_r) \\ \vdots & & \vdots \\ \chi_k(x_1) & \cdots & \chi_k(x_r) \end{bmatrix}, \quad X_2 = \begin{bmatrix} \chi_{k+1}(x_1) & \cdots & \chi_{k+1}(x_r) \\ \vdots & & \vdots \\ \chi_n(x_1) & \cdots & \chi_n(x_r) \end{bmatrix}.$$  

Then there exists an invertible matrix $P$ such that

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} A_1 & O \\ O & C \end{bmatrix}.$$

Hence $X_1 P = [A_1 \ O]$. Put

$$\chi_1(y) = [\chi_1(y) \cdots \chi_k(y)], \quad \chi_2(y) = [\chi_{k+1}(y) \cdots \chi_n(y)]$$

for a $\pi$-singular element $y$ of $G$. Then by Lemma 20, we have

$$[\chi_1(y) \chi_2(y)] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = O. \text{ Hence } [\chi_1(y) \chi_2(y)] \begin{bmatrix} A_1 & O \\ O & C \end{bmatrix} = O \text{ and so } \chi_1(y)[A_1 \ O] = O.$$

Since $P$ is invertible, we have $\chi_1(y)X_1 = O$ for every $y$. Therefore again by Lemma 20, $A_1 = \{\chi_1, \ldots, \chi_k\}$ is a union of $\pi$-blocks.

Assume that $A_1$ is not a single $\pi$-block, so that $A_1 = B \cup B'$, where $B$ is a single $\pi$-block and $B' = A_1 - B$. By interchanging columns and rows, let $A_1$ be arranged of the form

$$\begin{bmatrix} E & O \\ S & T \\ O & E \\ U & V \end{bmatrix},$$

where $[E \ O]$ belongs to $B$ and $[O \ E]$ belongs to $B'$. We claim that $T$ and $U$ are zero matrices. Choose an irreducible character $\chi \in B$, and let $\chi_0$ be the class function of $G$ which agrees with $\chi$ on $\pi$-regular elements and vanishes elsewhere. By Lemma 21, $(\chi_0, \chi') = 0$ for every $\chi' \in B'$. Hence $\chi_0$ is a linear combination of characters in $B$. Thus $T = O$ and similar argument implies $U = O$. However this contradicts that $A_1$ can not be arranged of the form $\begin{bmatrix} A_1^n & O \\ O & A_1^m \end{bmatrix}$. Hence $A_1$ is a single $\pi$-block and the proof of Theorem 22 is complete.
§2. Basic Sets for Sporadic Groups

In this Section, we calculate basic sets consisting of the irreducible ordinary characters for the blocks of 21 sporadic groups and their extensions.

The 21 sporadic groups, their orders, Schur multipliers (M) and outer automorphisms (A) are as follows:

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<th>Order</th>
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<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$Suz$</td>
<td>Suzuki</td>
<td>$2^{23} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>$O'N$</td>
<td>O'Nan</td>
<td>$2^{20} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 19 \cdot 31$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$Co_3$</td>
<td>Conway</td>
<td>$2^{10} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Co_2$</td>
<td>Conway</td>
<td>$2^{20} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Fi_{22}$</td>
<td>Fischer</td>
<td>$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>$HN$</td>
<td>Harada-Norton</td>
<td>$2^{10} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 19$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$Ly$</td>
<td>Lyons</td>
<td>$2^{23} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 31 \cdot 13 \cdot 19 \cdot 31$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Th$</td>
<td>Thompson</td>
<td>$2^{23} \cdot 3^{10} \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 19 \cdot 31$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$J_4$</td>
<td>Janko</td>
<td>$2^{24} \cdot 3^{11} \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Theorem 23.** For every block of the sporadic simple groups $M_{11}$, $M_{12}$, $J_1$, $M_{22}$, $J_2$, $M_{23}$, $HS$, $J_3$, $M_{24}$, $M_{27}$, $He$, $Ru$, $Suz$, $O'N$, $Co_3$, $Co_2$, $Fi_{22}$, $HN$, $Ly$, $Th$, $J_4$, their associated covering and automorphism groups, a basic set of Brauer characters can be chosen from among the irreducible ordinary characters.
This is proved by displaying basic sets and the expressions of the other characters as \( \mathbb{Z} \)-linear combinations in the Tables of Appendix. We use the character tables and the notation in the form of ATLAS-style (see Conway et al. [3] for details).

As in the proof of Theorem 22, let

\[
A_i = \begin{bmatrix}
1 & O \\
O & \ddots \\
\vdots & \ddots & \ddots \\
O & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
a_1^l & \cdots & a_i^l & a_{i+1}^l & \cdots & a_k^l
\end{bmatrix}
\begin{bmatrix}
\chi_1 \\
\chi_i \\
\chi_{i+1} \\
\vdots \\
\chi_{k+1}
\end{bmatrix}
\]

If we can choose \( \{\chi_1, \ldots, \chi_l\} \) such that all \( a_i^l \) are rational integers, then \( \{\chi_1, \ldots, \chi_l\} \) is a basic set of Brauer characters for this block and

\[
\chi_1 = a_1^l \chi_1 + \cdots + a_i^l \chi_i \quad (\lambda = l + 1, \ldots, k)
\]
on \( G^0 \). Then the Table in Appendix is displayed as

<table>
<thead>
<tr>
<th>BS</th>
<th>( \chi_1 )</th>
<th>( \cdots )</th>
<th>( \chi_l )</th>
<th>( \chi_{l+1} )</th>
<th>( \cdots )</th>
<th>( \chi_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1^l )</td>
<td>( \cdots )</td>
<td>( a_i^l )</td>
<td>( \cdots )</td>
<td>( a_{i+1}^l )</td>
<td>( \cdots )</td>
<td>( a_k^l )</td>
</tr>
</tbody>
</table>

Missing entries in the Tables are 0.

We describe how to read the Tables. When a character \( \chi \) of \( G \) splits for \( G.2 \), we denote them by \( \chi^+ \) and \( \chi^- \). (In the ATLAS tables, \( \chi^+ \) reads \( \chi^0 \) and \( \chi^- \) reads \( \chi^1 \).) Hence \( \chi^+ \) and \( \chi^- \) have the same values as \( \chi \) on elements of \( G \) and the values on elements of \( G.2 \) outside \( G \) are negatives of each other. A superscript indicates whether the character values on outer elements agree (+), or disagree (−), with those given in the ATLAS.

When 2 characters \( \chi_m \) and \( \chi_n \) of \( G \) fuse for \( G.2 \), we denote it as in the ATLAS by \( \chi_{m,n} \). For a covering group \( t.G.a \) \( (t \geq 3) \), when a character \( \chi_m \) of \( t.G \) fuses with the character whose proxy is \( \chi_{m,1} \), we denote them by \( \chi_{m,n}^0 \) and \( \chi_{m,n}^1 \). When \( \chi_m \) fuses with the character whose proxy is \( \chi_m \), itself, we denote it by \( \chi_m^* \).
First example is the case of the Mathieu group $M_{12}$. The Table in Appendix is

<table>
<thead>
<tr>
<th>Group: $M_{12}$</th>
<th>$G \ 2G$</th>
<th>Prime: 2</th>
<th>Defect: 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2G$</td>
<td>$2G$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BS</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
<th>$\chi_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{15}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{19}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{25}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{26}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

This Table should be read as follows;
For the group $M_{12}$ with defect 2,

<table>
<thead>
<tr>
<th>BS</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
<th>$\chi_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{15}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For the group $M_{12} \cdot 2$ with defect 3,

<table>
<thead>
<tr>
<th>BS</th>
<th>$\chi_{4.5}$</th>
<th>$\chi_{14}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{14}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{15}^+$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\bar{\chi}_{15}$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For the group $2M_{12}$ with defect 3,

<table>
<thead>
<tr>
<th>BS</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
<th>$\chi_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{15}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{19}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{25}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{26}$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

For the group $2M_{12} \cdot 2$ with defect 4,

<table>
<thead>
<tr>
<th>BS</th>
<th>$\chi_{4.5}$</th>
<th>$\chi_{14}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{14}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{15}^+$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{15}^{-}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{19}^+$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{19}^{-}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\chi_{25,26}$</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
In general, the table

\[
\begin{array}{cccc}
\text{BS} & \cdots & \chi_{m_1} & \cdots & \chi_{m_2} & \cdots \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
\chi_{n_1} & \cdots & \text{i} & (\pm 1) & \text{\vdots} & \vdots \\
\chi_{n_2} & \cdots & (\mp 1) & 2e & \text{\vdots} & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

should be read as follows:

For \( G \),

\[
\begin{array}{cccc}
\text{BS} & \cdots & \chi_{m_1} & \cdots & \chi_{m_2} & \cdots \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
\chi_{n_1} & \cdots & \text{i} & \text{\vdots} & \text{\vdots} & \vdots \\
\chi_{n_2} & \cdots & 2e & \text{\vdots} & \text{\vdots} & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

and for \( G.2 \),

\[
\begin{array}{cccccccc}
\text{BS} & \cdots & \chi_{m_1}^+ & \chi_{m_1}^- & \cdots & \chi_{m_2}^+ & \chi_{m_2}^- & \cdots \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
\chi_{n_1}^+ & \cdots & \text{j} & \text{i} - \text{j} & 1 & -1 & \text{\vdots} & \vdots \\
\chi_{n_1}^- & \cdots & \text{i} - \text{j} & \text{j} & -1 & 1 & \text{\vdots} & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{n_2}^+ & \cdots & -1 & 1 & \text{e} & \text{e} & \text{\vdots} & \vdots \\
\chi_{n_2}^- & \cdots & 1 & -1 & \text{e} & \text{e} & \text{\vdots} & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

We also explain the case of the group \( H.S \). The Table in Appendix is

Group: \( H.S \) \( [G.G.2] \) \hspace{1cm} Prime: 3 \hspace{1cm} Defect: 2 2

\[
\begin{array}{ccccccccc}
\text{BS} & \chi_1^+ & \chi_2^+ & \chi_3^+ & \chi_4^+ & \chi_5^+ & \chi_6^+ & \chi_7^+ & \chi_8^+ \\
\chi_4^- & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
\chi_9^- & 1 & 1 & 1 & -1 \\
\end{array}
\]

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This should be read as follows:

For $H.S.

<table>
<thead>
<tr>
<th>BS</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_{10}$</th>
<th>$X_{18}$</th>
<th>$X_{20}$</th>
<th>$X_{23}$</th>
<th>$X_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_4$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{19}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For $H.S.2$, this splits into two blocks;

<table>
<thead>
<tr>
<th>BS</th>
<th>$X_1^+$</th>
<th>$X_2$</th>
<th>$X_{10}$</th>
<th>$X_{18}$</th>
<th>$X_{20}^+$</th>
<th>$X_{23}$</th>
<th>$X_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_4^+$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{19}^+$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BS</th>
<th>$X_1^-$</th>
<th>$X_2^+$</th>
<th>$X_{10}$</th>
<th>$X_{18}$</th>
<th>$X_{20}^+$</th>
<th>$X_{23}$</th>
<th>$X_{24}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_4^-$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{19}^-$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Another example of $M_{22}$ is

Group: $M_{22}$: \[
\begin{bmatrix}
G & G.2 \\
\end{bmatrix}
\]
Prime: 3
Defect: 2 2

<table>
<thead>
<tr>
<th>BS</th>
<th>$X_1$</th>
<th>$X_5$</th>
<th>$X_{10}$</th>
<th>$X_{11}$</th>
<th>$X_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{7^*}$</td>
<td>-1$^0$</td>
<td>-1$^0$</td>
<td>-1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>$X_{36}$</td>
<td>-1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{37}$</td>
<td>-1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{46}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$X_{41}$</td>
<td>-1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{42}$</td>
<td>-1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For 3.$M_{22.2}$, this means

<table>
<thead>
<tr>
<th>BS</th>
<th>$X_1^+$</th>
<th>$X_1^-$</th>
<th>$X_5^+$</th>
<th>$X_5^-$</th>
<th>$X_{10,11}$</th>
<th>$X_{12}$</th>
<th>$X_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_7$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_7$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{36,37}$</td>
<td>-1</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{36,37}$</td>
<td>-1</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{46}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$X_{41}$</td>
<td>-1</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{42}$</td>
<td>-1</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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