On the Structure of the Character Ring of a Finite Group

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Introduction

Throughout this paper $G, Z, Q$ and $C$ denote a finite group, the ring of rational integers, the rational field and the field of complex numbers respectively. For a finite set $X$, we denote the number of elements in $X$ by $|X|$.

Let $\text{Irr}(G) = \{\chi_1(\text{the principal character}), ..., \chi_h\}$ be a full set of irreducible complex characters of $G$. Let $\mathcal{R}(G)$ be the set of generalized characters of $G$. That is,

$$\mathcal{R}(G) = \{\sum_{i=1}^{h} a_i \chi_i \mid a_i \in \mathbb{Z} \ (i = 1, ..., h)\}$$

Then $\mathcal{R}(G)$ forms a commutative associative ring with the identity element $\chi_1$ under addition and multiplication of characters of $G$ and $\mathcal{R}(G)$ is called the character ring of $G$.

In representation theory of a finite group $G$ over $C$, $\mathcal{R}(G)$ is a fundamental ring and in modular representations and integral representations of finite groups, representation rings and Grothendieck rings are treated respectively in place of character rings.([5],[12],[13],[17],[18],[19],[23],[24],[28],[29])

It seems that a typical representatives of the theorems concerning the structure of the character ring of a finite group is Brauer’s induction theorem. Here we state Brauer’s induction theorem in terms of character rings. Let $\mathcal{E}$ be the set of elementary subgroups of $G$ and $\mathcal{R}^l(G)$ be the subring of $\mathcal{R}(G)$ generated by the linear characters. For $H$ an arbitrary subgroup of $G$, induction of characters gives
rise to a \( Z \)-homomorphism \( \text{ind}: R(H) \rightarrow R(G) \).

Then we may state Brauer's induction theorem in the following form.

**Brauer's induction theorem** The \( Z \)-homomorphism \( \text{ind}: \bigoplus_{E \in \mathcal{E}} R^1(E) \rightarrow R(G) \) defined by \( \sum \psi \rightarrow \sum \psi^G \) is surjective, where \( \psi^G \) is the induced character of \( \psi \).

In connection with the above theorem, in [1] B.Banaschewski studied the maximal ideals of \( R(G) \) and obtained a theorem analogous to Brauer's induction theorem. There are many papers concerning induction theorems. ([1],[3],[4],[10],[25],... etc)

There are a few papers concerning the units in a character ring. In [20] A.I.Saksonov treated the units of finite order in a character ring when he studied the isomorphisms of a character ring onto another. However as long as we know, it seems that there is no paper concerning the units in a character ring which are not of finite order. In this paper we will treat the units in \( R(A_n) \) which are not of finite order where \( A_n \) is an alternating group on \( n \) symbols.

Concerning the isomorphisms of a character ring onto another, the theorem that if \( R(G) \cong R(H) \) for two finite groups \( G, H \), then \( G \) and \( H \) have the same character table was proved by D.R.Weidman in [27]. In addition in [20] A.I.Saksonov proved a theorem analogous to Weidman's theorem which is a strengthened version of Weidman's result. In this paper we will also treat the same problem with respect to Brauer character rings.

This paper is composed of four chapters. In chapter 1 we study the units of finite
order in a character ring. In [20] A.I.Saksonov determined the units of finite order in $AR(G)$ where $A$ is a ring of algebraic integers in a finite extension of $Q$ and $AR(G)$ is an $A$-algebra spanned by $\chi_1, ..., \chi_k$. Here we state a theorem which is a generalization of Saksonov's theorem, and give a short proof of this theorem which is different from Saksonov's proof.

In chapter 2 we treat the units in $R(A_n)$ ($n \geq 5$). If we denote the unit group of $R(A_n)$ by $U(R(A_n))$ and the set $\{\psi^2 \mid \psi \text{ is a unit in } R(A_n)\}$ by $U^2(R(A_n))$, then we will construct $c(n)$ units $\psi_1, ..., \psi_{c(n)}$ in $R(A_n)$ which are not of finite order and show that $U^2(R(A_n)) \subseteq \langle \psi_1, ..., \psi_{c(n)} \rangle$ where $\langle \psi_1, ..., \psi_{c(n)} \rangle$ is the subgroup of $U(R(A_n))$ generated by $\psi_1, ..., \psi_{c(n)}$. (Concerning a number $c(n)$, see Definition 2.5 in chapter 2) Here we would like to throw the main emphasis upon the fact that the units in a character ring previously treated are of finite order.

In chapter 3 we define a Brauer character ring $BR(G)$ and consider isomorphisms of a Brauer character ring onto another. We will prove a theorem analogous to the theorems of D.R.Weidman and A.I.Saksonov. This theorem is a generalization of the results of D.R.Weidman and A.I.Saksonov.

In chapter 4 we study isomorphisms of a character ring onto another. We will determine the form of these isomorphisms. This work is an extension of the result of A.I.Saksonov which is cited above.

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Chapter 1. On the units of finite order in a character ring

§1. Introduction

Let \( n \) be the exponent of a finite group \( G \) and let \( \zeta \) be a primitive \( n \)-th root of unity. Then \( K = \mathbb{Q}(\zeta) \subset \mathbb{C} \) is a splitting field for \( G \).

In particular, if \( G \) is a finite abelian group and \( A \) is the ring of algebraic integers in \( K \), then any unit of finite order in the group ring \( AG \) has the form \( \epsilon g \) for some \( g \in G \) and some unit \( \epsilon \in A \). (See p 263, Theorem 37.4 of [6]) This result yields an interesting theorem. That is, if \( G \) and \( H \) are finite abelian groups such that \( ZG \cong ZH \), then \( G \cong H \). (See p 264, Theorem 37.7 of [6]).

We denote the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \) by \( \overline{\mathbb{Q}} \) and the ring of algebraic integers in \( \overline{\mathbb{Q}} \) by \( \mathbb{Z} \).

In this chapter, applying the theory of characters of finite groups, we intend to study the units of finite order in the ring \( \mathbb{Z}R(G) \) where \( \mathbb{Z}R(G) \) is the \( \mathbb{Z} \)-algebra of \( \mathbb{Z} \)-linear combinations of irreducible complex characters of a finite group \( G \).

Afterward we shall show that any unit of finite order in \( \mathbb{Z}R(G) \) has the form \( \epsilon_\chi \) for some linear character \( \chi \) and some unit \( \epsilon \) in \( \mathbb{Z} \) (See Theorem 2.1 ) and then we shall apply this result to conclude that if \( G \) and \( H \) are finite groups such that \( R(G) \cong R(H) \) as rings, then \( G/D(G) \cong H/D(H) \) where \( D(G) \) and \( D(H) \) are commutator subgroups of \( G \) and \( H \) respectively.
The theorems concerning the units of finite order in a character ring are stated in [20] and [30] (See Theorem 1 of [20] and Lemma 6.1 of [30]), and Theorem 2.1 is an extension of these results. We also present a short proof of Theorem 2.1.
§2. A study of units of finite order

We keep the notation in §1 and in addition use the following notation.

\( \chi_1(=1_G), \ldots, \chi_{h-1} \) and \( \chi_h \) denote the irreducible complex characters of \( G \).

For \( \alpha \in \mathbb{C} \), \( \overline{\alpha} \) denotes a conjugate complex number of \( \alpha \) and \( |\alpha| \) an absolute value of \( \alpha \).

For any ring \( B \), \( U_f(B) \) denotes the set of units of finite order in \( B \) and \( \hat{G} \) the group of linear characters of \( G \).

For \( \theta, \eta \in \mathbb{Z}R(G) \), we set

\[ (\theta, \eta) = \frac{1}{|G|} \sum_{g \in G} \theta(g) \overline{\eta(g)}. \]

Then we have the following theorem about the units of finite order in \( \mathbb{Z}R(G) \).

**Theorem 2.1.** \( U_f(\mathbb{Z}R(G)) = U_f(\mathbb{Z}) \times \hat{G} \) (a direct product).

**Proof.** For \( u = \sum_{i=1}^{h} a_i \chi_i \in \mathbb{Z}R(G), a_i \in \mathbb{Z} \), we set \( \bar{u} = \sum_{i=1}^{h} \overline{a_i} \overline{\chi_i} \), where \( \chi_i \) denotes a conjugate character of \( \chi_i \) \( (i = 1, \ldots, h) \). Suppose that \( u \in U_f(\mathbb{Z}R(G)) \).

Then \( u(g) \) is a root of unity for all \( g \in G \). Hence \( |u(g)|^2 = u(g) \overline{u(g)} = 1 \). Therefore we have \( u\bar{u} = 1_G \). From this equation, it follows that

\[ \sum_{i=1}^{h} |a_i|^2 = (u, u) = (u\bar{u}, 1_G) = 1 \] ........................ (2.1)

For any \( \sigma \in G(\overline{Q}/Q) \), we set \( u^\sigma = \sum_{i=1}^{h} a_i^\sigma \chi_i^\sigma \). Since \( \chi_i^\sigma \) is also an irreducible character of \( G \), we have \( u^\sigma \in U_f(\mathbb{Z}R(G)) \).

By the equation of (2.1), we have
\[ \sum_{i=1}^{h} |a_i^\sigma|^2 = 1 \quad \text{for all } \sigma \in G(Q/Q). \]

Hence for each \( i \), \( |a_i^\sigma| \leq 1 \) for all \( \sigma \in G(Q/Q) \). Therefore \( a_i \) is either 0 or a root of unity. \((i = 1, \ldots, h)\). That is, it follows that \( u = \epsilon_i \chi_i \) for some \( i \), where \( \epsilon_i \) is a root of unity. Since \( |\chi_i(1)| = |\epsilon_i^{-1} u(1)| = 1 \), \( \chi_i \) must be a linear character of \( G \). This completes the proof. \( \text{Q.E.D.} \)

As a consequence of Theorem 2.1, we can easily obtain the following corollary.

**Corollary 2.2.** \( U_f(R(G)) = \{ \pm 1 \} \times \hat{G} \) (a direct product)

**Theorem 2.3** If \( R(G) \cong R(H) \) as rings for two finite groups \( G, H \), then we have \( G/D(G) \cong H/D(H) \).

**Proof.** Since \( R(G) \cong R(H) \), we see that \( U_f(R(G)) \cong U_f(R(H)) \). By Corollary 2.2, we have \( \{ \pm 1 \} \times \hat{G} \cong \{ \pm 1 \} \times \hat{H} \). By the fundamental theorem of finite abelian groups, we obtain \( \hat{G} \cong \hat{H} \). Hence we have

\[ G/D(G) \cong \hat{G} \cong \hat{H} \cong H/D(H) \]

This completes the proof. \( \text{Q.E.D.} \)
Chapter 2. A unit group in a character ring of an
alternating group

§1. Introduction

In what follows, $G$ denotes a finite group, $Z$ the ring of rational integers, $Q$ the
field of rational numbers, $C$ the field of complex numbers. In addition we fix the
following notation.

$\mathcal{R}(G) :=$ the character ring of $G$
$U(\mathcal{R}(G)) :=$ the unit group of $\mathcal{R}(G)$
$U_f(\mathcal{R}(G)) :=$ the subgroup of $U(\mathcal{R}(G))$ which consists of units of finite order in
$\mathcal{R}(G)$

$S_n, A_n :=$ a symmetric group and an alternating group on $n$ symbols respectively
for a natural number $n$

In section 2, we will prove that $U(\mathcal{R}(G))$ is finitely generated. Hence a factor
group $U(\mathcal{R}(G))/U_f(\mathcal{R}(G))$ is a free abelian group of finite rank.

In the same section, we also state the results in §6, Ch.VI in [2] which play a
fundamental role in this chapter, and define a non-negative rational integer $c(n)$ for
a natural number $n$. This number is very important. That is, we will construct $c(n)$
units $\psi_1, ..., \psi_{c(n)}$ in $R(A_n)$ in section 3 and show that $U^2(R(A_n)) \subseteq \langle \psi_1, ..., \psi_{c(n)} \rangle$
in section 4 where $U^2(R(A_n)) = \{ \psi^2 \mid \psi \in U(R(A_n)) \}$ and $\langle \psi_1, ..., \psi_{c(n)} \rangle$ is the
subgroup of $U(R(A_n))$ generated by $\psi_1, ..., \psi_{c(n)}$. As a direct consequence of this
result, we obtain that $c(n) = \text{rank of } U(R(A_n))/U_1(R(A_n))$.

In section 5, as an application of the above results, we state some examples such that \{±1\} x \langle \psi_1, ..., \psi_{c(n)} \rangle = U(R(A_n)) by finding generators of $U(R(A_n))$ concretely.
§2. Preliminaries

We first show that $U(R(G))$ is finitely generated.

**Theorem 2.1.** For a finite group $G$, $U(R(G))$ is finitely generated.

**Proof.** Let $\zeta$ be a primitive $|G|$-th root of unity, and let $K = \mathbb{Q}(\zeta)$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $\zeta$. Let us denote the ring of algebraic integers in $K$ by $A$. Let $C_1, ..., C_h$ be a full set of conjugacy classes in $G$ and let $c_1, ..., c_h$ be the representatives of $C_1, ..., C_h$ respectively. Let $u$ be an element of $U(R(G))$.

Then there exists $u' \in R(G)$ such that

$$uu' = \chi_1 \quad \text{(the principal character)}$$

Hence $u(c_i) \cdot u'(c_i) = 1 \quad (i = 1, ..., h)$. If $\chi$ is an irreducible complex character of $G$, then $\chi(c_i) \in A \quad (i = 1, ..., h)$. Therefore $u(c_i) \in A, u'(c_i) \in A \quad (i = 1, ..., h)$. That is, $u(c_i)$ and $u'(c_i)$ are units in $A \quad (i = 1, ..., h)$. We denote a unit group of $A$ by $U(A)$.

Now we define a mapping $\phi$ from $U(R(G))$ to a direct product of $h$ copies of $U(A)$;

$$\phi : U(R(G)) \ni u \rightarrow (u(c_1), ..., u(c_h)) \in U(A) \times \cdots \times U(A) \quad (h \text{ copies})$$

Then it is clear that $\phi$ is a homomorphism and injective. Since $A$ is the ring of algebraic integers in $K$, $U(A)$ is finitely generated by Dirichlet's Theorem. Therefore $U(A) \times \cdots \times U(A)$ is an abelian group which is finitely generated. As $U(R(G))$ is
isomorphic to a subgroup of $U(A) \times \cdots \times U(A), U(R(G))$ is finitely generated. The theorem is proved.

Q.E.D.

There are three irreducible complex characters of $A_3$. We denote them by $\chi_1, \chi_2, \chi_3$. Each $\chi_i$ is a linear character and $\chi_i(x) \in Q(\sqrt{-3})$ for $x \in A_3$. Hence for any $\psi \in R(A_3), \psi(x) \in Q(\sqrt{-3})$ for $x \in A_3$. Since $U(Q(\sqrt{-3})) = \{\pm 1, \pm \rho, \pm \rho^2\}$ where $\rho = (-1 + \sqrt{-3})/2$, by the proof of Theorem 2.1, we can see that any unit in $R(A_3)$ is of finite order. Therefore we have $U(R(A_3)) = U_f(R(A_3)) = \{\pm \chi_1, \pm \chi_2, \pm \chi_3\}$ by Corollary 2.2 in §2, Ch.1.

$A_4$ has four irreducible complex characters $\chi_1, \chi_2, \chi_3, \chi_4$ such that $\chi_1(1) = \chi_2(1) = \chi_3(1) = 1$ and $\chi_4(1) = 3$. For any $x \in A_4, \chi_i(x) \in Q(\sqrt{-3})$ ($i = 1, 2, 3, 4$). Analogously we have $U(R(A_4)) = U_f(R(A_4)) = \{\pm \chi_1, \pm \chi_2, \pm \chi_3\}$.

For a natural number $n \geq 5, A_n$ is a simple group. And so $A_n = D(A_n)$ (the commutator subgroup of $A_n$). Hence $A_n$ has only one linear character $\chi_1$ (i.e. the principal character). By Corollary 2.2 in §2, Ch.1, we have $U_f(R(A_n)) = \{\pm \chi_1\}$.

From now on, we may assume $n \geq 5$, when we consider about $U(R(A_n))$, and we use a notation “$U(R(A_n))/\{\pm 1\}$” in place of “$U(R(A_n))/U_f(R(A_n))$” for simplicity, by identifying $\{\pm 1\}$ with $\{\pm \chi_1\}$.

Now we state the irreducible complex characters of an alternating group $A_n$. The irreducible characters of the symmetric groups which are not self-associated, are also irreducible characters of the alternating groups.

Every self-associated character of the symmetric group $S_n$ is the sum of two ir-
reducible characters of the alternating group $A_n$. These two irreducible characters of $A_n$ take exactly half the values of the character of $S_n$, except for the conjugacy class for which the value of the character of $S_n$ is $\pm 1$. This conjugacy class splits into two for $A_n$, and it is for these conjugacy classes alone that the two irreducible characters of $A_n$ differ, the characteristic values in the two conjugacy classes being interchanged for the second character. Again we repeat these circumstances explicitly. (See p222 of [2])

Let \([m_1, \ldots, m_r], m_1 + \cdots + m_r = n\) be a self-associated frame. In the following way, we can assign to \([m_1, \ldots, m_r]\) a conjugacy class of $S_n$ with cycles of odd lengths $q_1 > q_2 > \cdots > q_k$, $q_1 + q_2 + \cdots + q_k = n$; let $q_1$ be the length of the "hook" consisting of the first row and the first column; $q_1 = 2m_1 - 1$. If this hook is deleted, another self-associated frame remains, from which we determine $q_2$ in the same way; $q_2 = 2(m_2 - 1) - 1 = 2m_2 - 3$. We continue thus until there is nothing left.

Here we use the following notation;

\[(q_1, q_2, \ldots, q_k) := \text{a conjugacy class of } S_n \text{ with cycles of lengths } q_1 > q_2 > \cdots > q_k, q_1 + q_2 + \cdots + q_k = n.\]

Then the following two theorems, which play a fundamental role, are well known (See p 222-223 of [2]).

**Theorem 2.2.** The character of a self-associated representation of $S_n$ which corresponds to a self-associated frame \([m_1, \ldots, m_r], m_1 + \cdots + m_r = n\) is
\((-1)^{\frac{n-k}{2}} = (-1)^{\frac{p-1}{2}}\)

in the conjugacy class \((q_1, q_2, \ldots, q_k)\) which is assigned to \([m_1, \ldots, m_r]\) where \(p = q_1q_2 \cdots q_k\); in all other conjugacy classes it is an even number.

**Theorem 2.3.** (Frobenius's theorem) Let \(\chi\) be a self-associated character of \(S_n\) which corresponds to a self-associated frame \([m_1, \ldots, m_r]\), \(m_1 + \cdots + m_r = n\). Then we have

1. If we consider \(\chi\) as a character of \(A_n\), \(\chi\) is the sum of two irreducible characters \(\chi_1, \chi_2\) of \(A_n\); \(\chi = \chi_1 + \chi_2\).

2. If \((q_1, q_2, \ldots, q_k)\) is a conjugacy class which is assigned to \([m_1, \ldots, m_r]\), then \((q_1, q_2, \ldots, q_k)\) splits into two conjugacy classes \(C', C''\) of \(A_n\). The values of \(\chi_1\) and \(\chi_2\) are

\[
\chi_1 = \frac{\lambda + \sqrt{p\lambda}}{2}, \quad \chi_2 = \frac{\lambda - \sqrt{p\lambda}}{2}
\]

in the two classes \(C', C''\), where \(\lambda = (-1)^{\frac{n-k}{2}} = (-1)^{\frac{p-1}{2}}\) and \(p = q_1q_2 \cdots q_k\).

The values of \(\chi_1\) and \(\chi_2\) are equal in all other conjugacy classes of \(A_n\); \(\chi_1 = \chi_2\).

Let \(\Gamma = [m_1, \ldots, m_r], m_1 + \cdots + m_r = n\) be a self-associated frame. Then we assign to \(\Gamma\) a conjugacy class \(C = (q_1, q_2, \ldots, q_k)\) of \(S_n\) with cycles of odd lengths \(q_1 > q_2 > \cdots > q_k, q_1 + q_2 + \cdots + q_k = n\) \((q_1 = 2m_1 - 1, q_2 = 2m_2 - 3, \ldots)\) and we
set \( p = q_1q_2 \cdots q_k \). In addition, we assume that \( p \equiv 1 (mod.4) \) and \( p \) is not the square of a number (i.e. \( \sqrt{p} \not\in Q \)). Then we state the following two definitions.

**Definition 2.4.** In the above situation we call \( \Gamma \) a self-associated frame of real type and we also say that \((\Gamma, C, p)\) is a triple of a self-associated frame of real type \( \Gamma \).

**Definition 2.5.** For a natural number \( n \) we define a non-negative integer \( c(n) \) as follows

\[
c(n) := \text{the number of self-associated frames of real type such that } [m_1, ..., m_r], m_1 + \cdots + m_r = n.
\]

**Example.** We compute \( c(15) \). There are three self-associated frames; \([8, 1, ..., 1], [5, 4, 3, 2, 1], [4, 4, 4, 3]\). We can assign to \([8, 1, ..., 1], [5, 4, 3, 2, 1], [4, 4, 4, 3]\) conjugacy classes of \( S_{15} \) \((15), (9, 5, 1), (7, 5, 3)\) respectively. And conjugacy classes \((15), (9, 5, 1), (7, 5, 3)\) determine odd numbers \( 15, 9 \times 5 \times 1 = 45, 7 \times 5 \times 3 = 105 \) respectively. \( 45 \equiv 1 (mod.4), 105 \equiv 1 (mod.4) \). Therefore we have \( c(15) = 2 \).

In §4 we will show that the rank of \( U(R(A_n))/{\pm 1} = c(n) \).
§3. Construction of unit elements

In this section we construct unit elements of $\mathbb{R}(A_n)$ which are not of finite order. Let $\Gamma = [m_1, \ldots, m_r], m_1 + \cdots + m_r = n$ be a self-associated frame of real type and let $(q_1, q_2, \ldots, q_k)$ be a conjugacy class of $S_n$ which is assigned to $[m_1, \ldots, m_r]$. We set $p = q_1 q_2 \cdots q_k$. Then $p \equiv 1 \pmod{4}$ and $p$ is not the square of a number. Hence $\mathbb{Q}(\sqrt{p})$ is the real quadratic field. Here we state several lemmata in the above situation.

**Lemma 3.1.** A conjugacy class $(q_1, q_2, \ldots, q_k)$ of $S_n$ consists of $\frac{|S_n|}{p}$ elements.

**Proof.** Since $(q_1, q_2, \ldots, q_k)$ is a conjugacy class with cycles of lengths $q_1 > q_2 > \cdots > q_k$, $q_1 + q_2 + \cdots + q_k = n$, then it consists of

\[
\frac{n!}{q_1 q_2 \cdots q_k} = \frac{|S_n|}{p}
\]

elements (See p 31 of [2]). The lemma is proved. Q.E.D.

**Lemma 3.2.** We set $p = f^2 p_0$, $(p_0: \text{square-free})$. Then we have

(i) $p_0 \equiv 1 \pmod{4}$

(ii) If $\frac{1}{2}(t + u\sqrt{p}), t, u \in \mathbb{Z}$ is an algebraic integer in $\mathbb{Q}(\sqrt{p_0})$, then $t \equiv u \pmod{2}$

(iii) If $e$ is a fundamental unit of $\mathbb{Q}(\sqrt{p_0})$, then the units of $\mathbb{Q}(\sqrt{p_0})$ which take the form of $\frac{1}{2}(t + u\sqrt{p}), t, u \in \mathbb{Z}$, are given by $\pm E^n$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $E = e^e$ for some natural number $e$. 

\[12\]
Proof. It is clear that (i) and (ii) hold. For (iii), for example, see p319 of [26].

Q.E.D.

Definition 3.3. We call a unit $E$ which appears in Lemma 3.2 (iii), a standard unit in $Q(\sqrt{p})(= Q(\sqrt{p^e}))$ for convenience.

Lemma 3.4 There exists a unit of $Q(\sqrt{p})$ which takes the form of

$$\frac{1}{2}(a + b\sqrt{p}) + 1, a, b \in \mathbb{Z}, p \mid a \text{ (i.e. } a \text{ divides by } p), b \neq 0.$$

and of which the norm over $Q$ is equal to 1.

Proof. By Lemma 3.2, there exists a unit $\eta = \frac{1}{2}(t + u\sqrt{p}), t, u \in \mathbb{Z}$ such that $N\eta = 1$ where $N\eta$ denotes the norm of $\eta$ over $Q$. Hence $t^2 - pu^2 = 4$. Thus $t^2 = pu^2 + 4$. If we set $a = pu^2, b = tu$, then we obtain

$$\eta^2 = \frac{1}{4}(t^2 + pu^2 + 2tu\sqrt{p}) = \frac{1}{2}(a + b\sqrt{p}) + 1$$

because a equation $t^2 = pu^2 + 4 = a + 4$ holds. Thus $\frac{1}{2}(a + b\sqrt{p}) + 1$ is the desired unit of $Q(\sqrt{p})$ and so the proof is complete.

Q.E.D.

Now we construct a unit of $R(A_n)$ which is not of finite order.

Let $[m_1, ..., m_r], m_1 + \cdots + m_r = n$ be a self-associated frame of real type and let $(q_1, q_2, ..., q_k)$ be a conjugacy class of $S_n$ which is assigned to $[m_1, ..., m_r]$; $q_1 = \cdots$
2m₁ - 1, q₂ = 2m₂ - 3, ...). Let C', C'' be the two conjugacy classes of Aₙ into which (q₁, q₂, ..., qₖ) splits. We set p = q₁q₂ ··· qₖ. Then p ≡ 1 (mod.4) and p is not the square of a number.

Let \( \frac{1}{2}(a + b\sqrt{p}) + 1, a, b \in Z \) (p\(\mid a, b \neq 0 \)) be the unit of \( Q(\sqrt{p}) \) which is stated in Lemma 3.4. Then we have Theorem 3.5.

**Theorem 3.5.** There exists a unit \( \psi \) of \( R(Aₙ) \) such that

\[
\psi(x) = 1 \text{ for } x \in Aₙ, x \notin C', C''.
\]

\[
\psi(c') = \frac{1}{2}(a + b\sqrt{p}) + 1, \psi(c'') = \frac{1}{2}(a - b\sqrt{p}) + 1
\]

where \( c', c'' \) are the representatives of \( C', C'' \) respectively.

**Proof.** First we note that a self-associated character \( \theta \) of \( Sₙ \) which corresponds to the self-associated frame \([m₁, ..., m_r]\), is the sum of two irreducible characters \( \phi₁, \phi₂ \) of \( Aₙ \).

By Theorem 2.3, we may assume that

\[
\phi₁(c') = \frac{1}{2}(1 + \sqrt{p}), \phi₁(c'') = \frac{1}{2}(1 - \sqrt{p})
\]

\[
\phi₂(c') = \frac{1}{2}(1 - \sqrt{p}), \phi₂(c'') = \frac{1}{2}(1 + \sqrt{p})
\]

\[
\phi₁(x) = \phi₂(x) \in Z \text{ for } x \in Aₙ, x \notin C', C''.
\]

Let \( \chi₁ (\text{the principal character}), ..., \chi_s \) be all other irreducible characters of \( Aₙ \).

Then \( \chiᵢ(c') = \chiᵢ(c'') \in Z \) (i = 1, ..., s). Here we show that the class function \( \psi \)
which is stated in this theorem, is actually written as a linear combination of $\chi_i$ and $\phi_j$ ($i = 1, \ldots, s; j = 1, 2$) with integral coefficients.

Now we pay attention to the fact that $|C'| = |C''| = |A_n|/p$ (See Lemma 3.1) and that

$$(\psi - \chi_1)(x) = 0 \text{ for } x \in A_n, x \notin C', C''$$

$$(\psi - \chi_1)(c') = \frac{1}{2}(a + b\sqrt{p}), (\psi - \chi_1)(c'') = \frac{1}{2}(a - b\sqrt{p}).$$

We denote by $(\lambda, \mu)$ the inner product of two class functions $\lambda, \mu$ of $A_n$. That is,

$$(\lambda, \mu) = \frac{1}{|A_n|} \sum_{g \in A_n} \lambda(g)\overline{\mu(g)}$$

where $\overline{\mu(g)}$ is the conjugate complex number of $\mu(g)$.

Here we compute several inner products as follows

$$(\psi - \chi_1, \chi_i) = \frac{1}{|A_n|} \left\{ |C'|(\psi - \chi_1)(c')\overline{\chi_i(c')} + |C''|(\psi - \chi_1)(c'')\overline{\chi_i(c'')} \right\}$$

$$= \frac{1}{p} \left[ \frac{a + b\sqrt{p}}{2} + \frac{a - b\sqrt{p}}{2} \right] \chi_i(c') = \frac{a}{p} \chi_i(c') \in \mathbb{Z}$$

because $\chi_i(c') = \chi_i(c'') \in \mathbb{Z}$ and $a$ divides by $p$.

$$(\psi - \chi_1, \phi_1) = \frac{1}{|A_n|} \left\{ |C'|(\psi - \chi_1)(c')\overline{\phi_1(c')} + |C''|(\psi - \chi_1)(c'')\overline{\phi_1(c'')} \right\}$$

$$= \frac{1}{p} \left[ \frac{a + b\sqrt{p}}{2} \frac{1 + \sqrt{p}}{2} - \frac{a - b\sqrt{p}}{2} \frac{1 - \sqrt{p}}{2} \right] = \frac{1}{2p}(a + bp) \in \mathbb{Z}$$

because $a \equiv b \ (mod.2), p$ is an odd number and $a$ divides by $p$. Analogously we have

$$(\psi - \chi_1, \phi_2) = \frac{1}{2p}(a - bp) \in \mathbb{Z}.$$

Therefore we obtain
\[
\psi = \chi_1 + \frac{a}{p} \sum_{i=1}^{s} \chi_i(c') \chi_i + \frac{a + bp}{2p} \phi_1 + \frac{a - bp}{2p} \phi_2 \in R(A_n)
\]

Now we denote by \(\psi'\) the class function of \(A_n\) which satisfies
\[
\psi'(x) = 1 \text{ for } x \in A_n, x \notin C', C''
\]
\[
\psi'(c') = \frac{1}{2}(a - b \sqrt{p}) + 1, \psi'(c'') = \frac{1}{2}(a + b \sqrt{p}) + 1
\]

Then we obtain by the same method
\[
\psi' = \chi_1 + \frac{a}{p} \sum_{i=1}^{s} \chi_i(c') \chi_i + \frac{a - bp}{2p} \phi_1 + \frac{a + bp}{2p} \phi_2 \in R(A_n).
\]

By the proof of Lemma 3.4, we can see that \(\eta^2 = \frac{1}{2}(a + b \sqrt{p}) + 1\), \(N \eta = 1\) where \(\eta\) is a unit of \(Q(\sqrt{p})\). Since \(N(\eta^2) = \frac{a + b \sqrt{p}}{2} + 1\)
\[
= \left(\frac{a + b \sqrt{p}}{2} + 1\right) = 1,
\]
we have \(\psi \psi' = \chi_1\). Therefore \(\psi\) is a unit of \(R(A_n)\) which is not of finite order.

This completes the proof of Theorem 3.5. \(\text{Q.E.D.}\)
§4. Units in $R(\mathbb{A}_n)$ ($n \geq 5$)

Let $\Gamma = [m_1, ..., m_r], m_1 + \cdots + m_r = n$ be a self-associated frame of real type and let $(\Gamma, C, p)$ be a triple of $\Gamma$. Let $C', C''$ be the two conjugacy classes of $\mathbb{A}_n$ into which $C$ splits and let $c', c''$ be the representatives of $C', C''$ respectively. Let $E = \frac{1}{2}(t + u\sqrt{p}), \ (t, u \in \mathbb{Z}, tu \neq 0)$ be the standard unit in $Q(\sqrt{p})$. We denote by $N(E)$ the norm of $E$ over $Q$. Then we have the following theorem.

**Theorem 4.1.** In the above situation, we define a class function $\psi$ of $\mathbb{A}_n$ as follows

In case $N(E) = 1$

$\psi(x) = 1$ for $x \in \mathbb{A}_n, x \notin C', C''$

$\psi(c') = E^2, \psi(c'') = E^{-2}$

In case $N(E) = -1$

$\psi(x) = -1$ for $x \in \mathbb{A}_n, x \notin C', C''$

$\psi(c') = E^2, \psi(c'') = E^{-2}$

Then $\psi$ is a unit in $R(\mathbb{A}_n)$ which is not of finite order.

**Proof.** In case $N(E) = 1$, by both Lemma 3.4 and Theorem 3.5 we can see that $\psi$ is a unit in $R(\mathbb{A}_n)$ which is not of finite order and so in case $N(E) = -1$, we prove that $\psi$ is a unit in $R(\mathbb{A}_n)$. Since $N(E) = \frac{1}{4}(t^2 - pu^2) = -1$, we have $t^2 = pu^2 - 4$. Hence we get the following equation.
\[ E^2 = \frac{1}{4}(t^2 + pu^2 + 2tu\sqrt{p}) = \frac{1}{4}(2pu^2 - 4 + 2tu\sqrt{p}) \]

\[ = \frac{1}{2}(a + b\sqrt{p}) - 1 \]

where \( a = pu^2 \) and \( b = tu \ (\neq 0) \).

Therefore we have

\[ (\psi + \chi_1)(x) = 0 \text{ for } x \in A_n, x \notin C', C'' \]

\[ (\psi + \chi_1)(e') = \frac{1}{2}(a + b\sqrt{p}) \]

\[ (\psi + \chi_1)(e'') = \frac{1}{2}(a - b\sqrt{p}), \ p|a, b \neq 0 \]

where \( \chi_1 \) is the principal character of \( A_n \). By the same proof as that of Theorem 3.5 we can prove that \( \psi \) is actually written as a linear combination of irreducible complex characters of \( A_n \) with integral coefficients and that \( \psi \) is a unit in \( R(A_n) \) and so we omit its proof. Thus the proof is complete. Q.E.D.

Let \((\Gamma_1, C_1, p_1), \ldots, (\Gamma_{c(n)}, C_{c(n)}, p_{c(n)})\) be the triples of self-associated frames of real type and let \( \lambda_1, \ldots, \lambda_{c(n)} \) be the characters of self-associated representations of \( S_n \) which correspond to \( \Gamma_1, \ldots, \Gamma_{c(n)} \) respectively. If we consider \( \lambda_i \) as a character of \( A_n \), then \( \lambda_i \) is the sum of two irreducible complex characters \( \phi'_i, \phi''_i \) of \( A_n; \lambda_i = \phi'_i + \phi''_i \ (i = 1, \ldots, c(n)) \). Let \( C'_i, C''_i \) be the two conjugacy classes of \( A_n \) into which \( C_i \) splits, and let \( c'_i, c''_i \) be the representatives of \( C'_i, C''_i \) respectively \((i = 1, \ldots, c(n))\).

We denote by \( E_i \) the standard unit in \( Q(\sqrt{p_i}) \) \((i = 1, \ldots, c(n))\) and we keep these
notations throughout this section. Then we have

**Theorem 4.2.** In the above situation let \( \psi \) be a unit in \( R(\mathbb{A}) \) which is not of finite order such that

\[
\psi(c'_i) = \pm E_i^{j_i}, \quad \psi(c''_i) = \pm E_i^{j_i}, \quad j_i \in \mathbb{Z} \quad (i = 1, \ldots, c(n))
\]

where \( E_i' \) is the conjugate number of \( E_i \) over \( \mathbb{Q} \) and the sign of \( E_i^{j_i} \) is equal to that of \( E_i^{j_i} \).

Then we have \( N(E_i'^j) = 1 \) \((i = 1, \ldots, c(n))\) where \( N(E_i'^j) \) denotes the norm of \( E_i'^j \) over \( \mathbb{Q} \).

**Proof.** Let \( \chi_1 (the \ principal \ character), \ldots, \chi_k \) be the irreducible complex characters of \( \mathbb{A} \) such that \( \{\chi_1, \ldots, \chi_k\} \cup \{\phi_1', \phi_2'' \mid i = 1, \ldots, c(n)\} \) is a full set of irreducible complex characters of \( \mathbb{A} \). Now we assume that \( \psi \) is written as a linear combination of irreducible complex characters of \( \mathbb{A} \) with integral coefficients as follows

\[
\psi = \sum_{i=1}^{c(n)} a_i \phi_i' + \sum_{i=1}^{c(n)} b_i \phi_i'' + \sum_{j=1}^{k} c_j \chi_j, \quad a_i, b_i, c_j \in \mathbb{Z}
\]

If we set

\[
\psi' = \sum_{i=1}^{c(n)} b_i \phi_i' + \sum_{i=1}^{c(n)} a_i \phi_i'' + \sum_{j=1}^{k} c_j \chi_j,
\]

then by Theorem 2.3, we can see that

\[
\psi'(x) = \psi(x) \text{ for } x \in \mathbb{A}, \ x \notin C_1', C_2'' \quad (i = 1, \ldots, c(n))
\]

\[
\psi'(c'_i) = \psi(c''_i) = \pm E_i^{j_i}, \quad \psi'(c'_i) = \psi(c''_i) = \pm E_i^{j_i}
\]
where the sign of $E_i^*E_{i}^{**}$ is equal to that of $E_i^{**}$. Therefore it follows that $(\psi\psi')(x) = \pm 1$ or $(\psi\psi')(x)$ is a unit in an imaginary quadratic field for $x \in A_n, x \notin C_i, C_i^{**}$ ($i = 1, ..., c(n)$) and that

$$(\psi\psi')(c_l^i) = (\psi\psi')(c_l^{**}) = N(E_i^{**}) = \pm 1 \text{ for } i = 1, ..., c(n).$$

Thus we can conclude that $(\psi\psi')$ is a unit in $R(A_n)$. Since $U_f(R(A_n)) = \{\pm \chi_1\}$ ($n \geq 5$), we have $\psi\psi' = \pm \chi_1$. Since $(\psi\psi')(1) = 1$ for the identity element 1 of $A_n$, we have the equation $\psi\psi' = \chi_1$. This implies that $N(E_i^{**}) = 1$ for $i = 1, ..., c(n)$. Thus the proof is complete. Q.E.D.

We assume further that $E_1, ..., E_r$ are the standard units such that $N(E_1) = \cdots = N(E_r) = 1$ and $E_{r+1}, ..., E_{r+s}(= E_c(n))$ are the standard units such that $N(E_j) = -1$ ($j = r+1, ..., r+s = c(n)$).

Then, for each $i \in \{1, ..., r\}$, we set

$$\psi_i(x) = 1 \text{ for } x \in A_n, x \notin C_i, C_i^{**}$$

$$\psi_i(c_l^i) = E_i^2, \quad \psi_i(c_l^{**}) = E_i^{-2}$$

and for each $j \in \{r+1, ..., r+s = c(n)\}$, we set

$$\psi_j(x) = -1 \text{ for } x \in A_n, x \notin C_j, C_j^{**}$$

$$\psi_j(c_l^j) = E_j^2, \quad \psi_j(c_l^{**}) = E_j^{-2}.$$

By Theorem 4.1, it follows that $\psi_1, ..., \psi_{r+s}(= \psi_c(n))$ are units in $R(A_n)$ which are
not of finite order, and we fix these units throughout this section. Then we have

Theorem 4.3. For any unit $\psi$ in $R(\mathbb{A}_n)$ which is not of finite order, we can write

$$\psi^2 = \psi_1^{i_1} \cdots \psi_r^{i_r} \cdot \psi_{r+1}^{j_{r+1}} \cdots \psi_{r+s}^{j_{r+s}}$$

where $i_1, \ldots, i_r, j_{r+1}, \ldots, j_{r+s} \in \mathbb{Z}$.

Proof. Since $N(E_k) = -1$ for $k \in \{r + 1, \ldots, r + s = c(n)\}$, by Theorem 4.2 we have

$$\psi(c'_k) = \pm E_k^{2j_k}, \quad \psi(c''_k) = \pm E_k^{-2j_k} \text{ for some } j_k \in \mathbb{Z}$$

where the sign of $E_k^{2j_k}$ is equal to that of $E_k^{-2j_k}$. Hence we have

$$(\psi^2)(c'_k) = E_k^{4j_k}, \quad (\psi^2)(c''_k) = E_k^{-4j_k}.$$ 

On the other hand, for $h \in \{1, \ldots, r\}$ we have

$$\psi(c'_h) = \pm E_h^{i_h}, \quad \psi(c''_h) = \pm E_h^{-i_h} \text{ for some } i_h \in \mathbb{Z}$$

where the sign of $E_h^{i_h}$ is equal to that of $E_h^{-i_h}$. Therefore if we set

$$\mu = \psi^2 \psi_1^{-i_1} \cdots \psi_r^{-i_r} \psi_{r+1}^{-2j_{r+1}} \cdots \psi_{r+s}^{-2j_{r+s}},$$

then we can see that $\mu(x) = 1$ for $x \in C'_i$ or $x \in C''_i$ ($i = 1, \ldots, r + s = c(n)$).

Thus it follows that $\mu$ is a unit in $R(\mathbb{A}_n)$ which is of finite order. Since $U_f(R(\mathbb{A}_n)) = \{\pm \chi_1\}$ ($n \geq 5$), we have $\mu = \pm \chi_1$. For an identity element 1 of $\mathbb{A}_n$, $\mu(1) = 1$ holds and so we obtain $\mu = \chi_1$.

This implies that

$$\psi^2 = \psi_1^{i_1} \cdots \psi_r^{i_r} \psi_{r+1}^{2j_{r+1}} \cdots \psi_{r+s}^{2j_{r+s}}.$$
Thus the result follows. Q.E.D.

Corollary 4.4. Let \( \psi \) be any unit in \( R(A_n) \). Then \( \psi(x) \) is a real number for all \( x \in A_n \). In particular, \( \psi(x) = \pm 1 \) for \( x \in A_n, x \notin C'_i, C''_i \) (\( i = 1, \ldots, c(n) \)).

Proof. It is clear that \( \psi(x) \) is a real number for \( x \in C'_i \) or \( x \in C''_i \) (\( i = 1, \ldots, c(n) \)).

By Theorem 4.3 we can see that \( (\psi^2)(x) = 1 \) for \( x \in A_n, x \notin C'_i, C''_i \) (\( i = 1, \ldots, c(n) \)).

Thus the result follows. Q.E.D.

We denote the subgroup of \( U(R(A_n)) \) generated by \( \psi_1, \ldots, \psi_{c(n)} \) by \( \langle \psi_1, \ldots, \psi_{c(n)} \rangle \) and the set \( \{ \psi^2 \mid \psi \in U(R(A_n)) \} \) by \( U^2(R(A_n)) \). Then the following theorem is a direct consequence of Theorem 4.3.

Theorem 4.5. \( U^2(R(A_n)) \subseteq \langle \psi_1, \ldots, \psi_{c(n)} \rangle \).

Lemma 4.6. The rank of \( \langle \psi_1, \ldots, \psi_{c(n)} \rangle = c(n) \).

Proof. Suppose that \( \psi_1^e_1 \cdots \psi_{c(n)}^{e_{c(n)}} = x_1 \) (\( e_1, \ldots, e_{c(n)} \in \mathbb{Z} \)). Then we have

\[
1 = x_1(c'_i) = (\psi_1^{e_1} \cdots \psi_{c(n)}^{e_{c(n)}})(c'_i) = (\psi_i(c'_i))^{e_i} = E_i^{2e_i}
\]

Here \( e_i = 0 \) (\( i = 1, \ldots, c(n) \)). Therefore we obtain the rank of \( \langle \psi_1, \ldots, \psi_{c(n)} \rangle = c(n) \).

The lemma is proved. Q.E.D.
The following result is a direct consequence of Theorem 4.5 and Lemma 4.6.

**Theorem 4.7.** The rank of \( U(R(A_n))/\{ \pm 1 \} = c(n) \).

Let \( \Gamma = [m_1, \ldots, m_r], m_1 + \cdots + m_r = n \) be a self-associated frame of real type and let \((\Gamma, C, p)\) be a triple of \( \Gamma \). Let \( C', C'' \) be the two conjugacy classes of \( A_n \) into which \( C \) splits and let \( c', c'' \) be the representatives of \( C', C'' \) respectively.

Let \( \frac{1}{2}(t + u\sqrt{p}) \) \((tu \neq 0)\) be the unit in \( Q(\sqrt{p}) \). Then we have the following theorem.

**Theorem 4.8.** In the above situation, let \( \psi \) be the unit in \( R(A_n) \) such that

\[
\psi(x) = \pm 1 \quad \text{for } x \in A_n, x \notin C', C''
\]

\[
\psi(c') = \frac{1}{2}(t + u\sqrt{p}), \psi(c'') = \frac{1}{2}(t - u\sqrt{p})
\]

Then the following conditions are equivalent.

(i) \( \psi \) is a difference of two irreducible complex characters of \( A_n \).

(ii) \( u = \pm 1 \).

**Proof.** We denote by \( \chi_1 \) the principal character of \( A_n \) and by \((\lambda, \mu)\) the inner product of two class functions \( \lambda, \mu \) of \( A_n \).

(i) \( \Rightarrow \) (ii) Since \( \psi \) is a difference of two irreducible complex characters of \( A_n \) and \( \psi(x) \) \((x \in A_n)\) is a real number, we have
\[ (\psi^2 \chi_1) = (\overline{\psi}, \chi_1) = (\psi, \psi) = 2 \] 

\begin{align*}
\text{On the other hand by Theorem 4.2 } & N(\frac{1}{2}(t + u\sqrt{p})) = 1 \text{ and so we derive } t^2 = pu^2 + 4. \text{ From this formula we get } \\
\left( \frac{t \pm u\sqrt{p}}{2} \right)^2 &= \frac{pu^2 \pm tu\sqrt{p}}{2} + 1 \\
\text{Hence we have} \\
(\psi^2 - \chi_1)(x) &= 0 \text{ for } x \in A_n, x \notin C', C'' \\
(\psi^2 - \chi_1)(c') &= \frac{pu^2 + tu\sqrt{p}}{2}, \quad (\psi^2 - \chi_1)(c'') = \frac{pu^2 - tu\sqrt{p}}{2} \\
\text{By Lemma 3.1 we have } |C'| = |C''| = \frac{1}{p} |A_n|. \text{ Now we calculate an inner product} \\
(\psi^2 - \chi_1, \chi_1). \\
(\psi^2 - \chi_1, \chi_1) &= \frac{1}{|A_n|} \left( \frac{|A_n|}{p} \left( \frac{pu^2 + tu\sqrt{p}}{2} \right) + |A_n| \left( \frac{pu^2 - tu\sqrt{p}}{2} \right) \right) = u^2 \ldots (4.2) \\
\text{Therefore it follows that } (\psi^2, \chi_1) = 1 + u^2. \text{ Hence by the formula (4.1) we have } \\
1 + u^2 &= 2 \text{ and so we get } u = \pm 1. \\
(ii) \Rightarrow (i) & \text{ We assume that } u = \pm 1. \text{ Then by the formula (4.2), we get } (\psi^2 - \chi_1, \chi_1) = 1 \text{ and so we have} \\
(\psi^2, \chi_1) &= (\psi, \overline{\psi}) = (\psi, \psi) = 2 \\
\text{Because } \psi \text{ is a unit in } R(A_n), \text{ it follows that } \psi(1) = \pm 1 \text{ for the identity element 1 of } A_n. \text{ Hence we can see that } \psi \text{ is a difference of two irreducible complex characters of } A_n. \text{ This completes the proof of Theorem 4.8.} \quad \text{Q.E.D.}
\end{align*}
§5. Some examples

Example 1. $U(R(A_{10}))$. We will find the generators of $U(R(A_{10}))$. First we compute $c(10)$. There are two self-associated frames: $[4, 3, 2, 1], [5, 2, 1^3]$. We assign to $[4, 3, 2, 1], [5, 2, 1^3]$ conjugacy classes of $S_{10}$: $(7, 3), (9, 1)$ respectively and conjugacy classes $(7, 3), (9, 1)$ determine odd numbers $7 \times 3 = 21 \equiv 1 (mod. 4), 9 \times 1 = 3^2$ respectively. Therefore we have $c(10) = 1$.

Now we set $\epsilon = \frac{1}{2}(5 + \sqrt{21})$. Then $\epsilon$ is a fundamental unit in $Q(\sqrt{21})$. (At the same time $\epsilon$ is a standard unit in $Q(\sqrt{21})$ and $N(\epsilon)$ (the norm of $\epsilon$ over $Q$) is equal to 1.)

Secondly we prove that there is no unit $\mu$ in $R(A_{10})$ such that

\[
\begin{align*}
\mu(x) &= \pm 1 \quad \text{for } x \in A_{10}, x \notin C', C'' \\
\mu(c') &= \pm \epsilon, \quad \mu(c'') = \pm \epsilon^{-1}
\end{align*}
\]

(5.1)

where $C', C''$ are the conjugacy classes of $A_{10}$ into which the conjugacy class $(7, 3)$ of $S_{10}$ splits, and $c', c''$ are the representatives of $C', C''$ respectively.

Assume by way of contradiction that there is a unit $\mu$ in $R(A_{10})$ which satisfies the equations of (5.1). Let $\lambda$ be a self-associated character of $S_{10}$ which corresponds to the frame $[4, 3, 2, 1]$ and let $\psi_1, \psi_2$ be the two irreducible complex characters of $A_{10}$ into which $\lambda$ splits. By Theorem 4.8 we can see that $\mu$ is a difference of two irreducible complex characters of $A_{10}$ and so we may assume that $\mu = \pm (\psi_1 - \chi)$ for some irreducible complex character $\chi$ of $A_{10}$. Now we can easily compute $deg \lambda = 768$. (See p 78 Theorem 3.9 of [16].) Hence we have $deg \psi_1 = deg \psi_2 = 384$. Since $\mu(1) = \pm (\psi_1(1) - \chi(1)) = \pm (384 - \chi(1)) = \pm 1$, it follows that $\chi(1) = 383$ or

$\vdots$
\( \chi(1) = 385. \) But there is no irreducible complex character \( \chi \) of \( A_{10} \) such that 
\( \chi(1) = 383 \) or \( \chi(1) = 385 \), because 
\[
\frac{|A_{10}|}{\chi(1)} = \frac{10!}{2 \times 383} \notin \mathbb{Z} \quad \text{and} \quad \frac{|A_{10}|}{\chi(1)} = \frac{10!}{2 \times 385} = \frac{10!}{2 \times 5 \times 7 \times 11} \notin \mathbb{Z}.
\]
This contradiction implies that there is no unit \( \mu \) in \( R(A_{10}) \) which satisfies the equations of (5.1).

Let \( \psi \) be the class function of \( A_{10} \) such that 
\[
\psi(x) = 1 \quad \text{for} \ x \in A_{10}, \ x \notin C', C''
\]
\[
\psi(c') = e^2 = \frac{23 + 5\sqrt{21}}{2}, \psi(c'') = e^{-2} = \frac{23 - 5\sqrt{21}}{2}.
\]
Then by Theorem 4.1, it follows that \( \psi \) is a unit in \( R(A_{10}) \). Therefore we have 
\[
U(R(A_{10})) = \{ \pm \psi^i \mid i \in \mathbb{Z} \}. \quad \text{(See the proofs of Theorem 4.2 and Theorem 4.3.)}
\]

Example 2. \( U(R(A_p)) \). Let \( p \) be a prime number such that \( p \equiv 1 \pmod{4} \) and \( c(p) = 1 \). For example 5, 13 and 17 are the prime numbers which satisfy these conditions. Then we will find the generators of \( U(R(A_p)) \). Let \( \epsilon \) be a fundamental unit of \( Q(\sqrt{p}) \), then \( N(\epsilon) = -1 \). (See p 316 Problem 5 of [26].)

There is a self-associated frame; \( \left[ \frac{p+1}{2}, \frac{p+1}{2}, \frac{p+1}{2}, \frac{p+1}{2} \right] \). We assign to this frame a conjugacy class of \( S_p, (p) \). Then the conjugacy class \( (p) \) splits into two conjugacy classes \( C', C'' \) of \( A_p \). Let \( \lambda \) be a self-associated character of \( S_p \) which corresponds to \( \left[ \frac{p+1}{2}, \frac{p+1}{2}, \frac{p+1}{2}, \frac{p+1}{2} \right] \).

When we consider \( \lambda \) as a character of \( A_p \), by Theorem 2.3 we can see that \( \lambda \) is the
sum of two irreducible complex characters $\psi_1, \psi_2$ of $A_5$ such that $\psi_1(c') = \psi_2(c'') = \frac{1}{2}(1 + \sqrt{p})$, $\psi_1(c''') = \psi_2(c''') = \frac{1}{2}(1 - \sqrt{p})$ where $c'$, $c''$ are the representatives of $C', C''$ respectively. Here we note that $\epsilon$ is a standard unit in $Q(\sqrt{p})$. Since $N(\epsilon) = -1$, by Theorem 4.2 there is no unit $\mu$ in $R(A_5)$ such that $\mu(x) = \pm 1$ for $x \in A_5, x \notin C', C''$ and $\mu(c') = \pm \epsilon, \mu(c'') = \pm \epsilon'$ where the sign of $\epsilon$ is equal to that of $\epsilon'$ (the conjugate number of $\epsilon$ over $Q$). Let $\psi$ be the class function of $A_5$ such that $\psi(x) = -1$ for $x \in A_5, x \notin C', C''$ and $\psi(c') = \epsilon^2, \psi(c'') = \epsilon^{-2}$. Then by Theorem 4.1, $\psi$ is a unit in $R(A_5)$ and so we have $U(R(A_5)) = \{\pm \omega^i | i \in Z\}.$

Example 3. We show that there is a unit in $R(A_5)$ which is a difference of two irreducible complex characters of $A_5$. (See Theorem 4.8.) $A_5$ has the following conjugacy classes:

$C_1 = \{1\}, C_2 = \{(12)(34), \ldots\}, C_3 = \{(123), \ldots\}$

$C_4 = \{(12345), \ldots\}, C_5 = \{(13524), \ldots\}.$

Hence $A_5$ has five irreducible complex characters $\chi_1, \ldots, \chi_5$. For the character table of $A_5$, we obtain
<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$\frac{1 + \sqrt{5}}{2}$</td>
<td>$\frac{1 - \sqrt{5}}{2}$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$\frac{1 - \sqrt{5}}{2}$</td>
<td>$\frac{1 + \sqrt{5}}{2}$</td>
</tr>
</tbody>
</table>

From this character table we get the following table for $\chi_4 - \chi_2$ and $\chi_5 - \chi_2$.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_4 - \chi_2$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$\frac{3 + \sqrt{5}}{2}$</td>
<td>$\frac{3 - \sqrt{5}}{2}$</td>
</tr>
<tr>
<td>$\chi_5 - \chi_2$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$\frac{3 - \sqrt{5}}{2}$</td>
<td>$\frac{3 + \sqrt{5}}{2}$</td>
</tr>
</tbody>
</table>

Therefore $\chi_4 - \chi_2$ and $\chi_5 - \chi_2$ are units in $R(A_5)$ which are differences of two irreducible complex characters of $A_5$. Since $c(5) = 1, \frac{3 + \sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^2$ and $\frac{1 + \sqrt{5}}{2}$ is a fundamental unit in $Q(\sqrt{5})$ of which the norm over $Q$ is equal to $-1$, the units in $R(A_5)$ are given by $\pm (\chi_4 - \chi_2)^i$, $i = 0, \pm 1, \pm 2, \cdots$. (See Example 2.)

Finally we note that as for $U(R(A_6))$ we also can prove the same statement.
Chapter 3. On isomorphisms of a Brauer character ring onto another

§1. Introduction

Throughout this chapter $G, Z$ and $Q$ denote a finite group, the ring of rational integers and the rational field respectively. Moreover we write $\mathbb{Z}$ to denote the ring of all algebraic integers in the complex numbers and $\mathbb{Q}$ to denote the algebraic closure of $Q$ in the field of complex numbers. For a finite set $X$, we denote by $|X|$ the number of elements in $X$.

Let $\text{Irr}(G) = \{\chi_1, \ldots, \chi_h\}$ be the complete set of absolutely irreducible complex characters of $G$. Then we can view $\chi_1, \ldots, \chi_h$ as functions from $G$ into the complex numbers. We write $\mathbb{Z}R(G)$ to denote the $\mathbb{Z}$-algebra spanned by $\chi_1, \ldots, \chi_h$. For two finite groups $G$ and $H$, let $\lambda$ be a $\mathbb{Z}$-algebra isomorphism of $\mathbb{Z}R(G)$ onto $\mathbb{Z}R(H)$. Then we can write

$$\lambda(\chi_i) = \sum_{j=1}^{h} a_{ij} \chi'_j \quad (i = 1, \ldots, h)$$

where $a_{ij} \in \mathbb{Z}$ and $\text{Irr}(H) = \{\chi'_1, \ldots, \chi'_h\}$. In this case we write $A$ to denote the $h \times h$ matrix with $(i, j)$-entry equal to $a_{ij}$ and say that $A$ is afforded by $\lambda$ with respect to $\text{Irr}(G)$ and $\text{Irr}(H)$.

As is well known, concerning the isomorphism $\lambda$, we have the following two results, which seem to be most important.
(i) \(|C_G(c_i)| = |C_H(c'_i)|\) \((i = 1, \ldots, h)\) where \(\{c_1, \ldots, c_h\}\) and \(\{c'_1, \ldots, c'_h\}\) are complete sets of representatives of the conjugate classes in \(G\) and \(H\) respectively and \(c_i \sim c'_i\) \((i = 1, \ldots, h)\). (Concerning a symbol \(c_i \mapsto c'_i\), see the notation stated before Lemma 2.4.)

(ii) \(A\) is unitary where \(A\) is the matrix afforded by \(\lambda\) with respect to \(\text{Irr}(G)\) and \(\text{Irr}(H)\).

In this chapter our main objective is to give a necessary and sufficient condition under which the above statements (i) and (ii) hold, concerning an isomorphism \(\lambda\) of a Brauer character ring onto another, and to state a generalization of theorems of Saksonov and Weidman about character tables of finite groups. (See Theorem 2, Corollary 2.1 in [20] and Theorem 3 in [27].)

From now on, when we consider homomorphisms from an algebra to another, unless otherwise specified, we shall only deal with algebra homomorphisms.
§2. Preliminaries

We fix a rational prime number $p$ and use the following notation with respect to a finite group $G$.

$G_0 :=$ the set of all $p$-regular elements of $G$

$Cl(G_0) = \{C_1, \ldots, C_r\} :=$ the complete set of $p$-regular conjugate classes in $G$

$\{c_1, \ldots, c_r\} :=$ a complete set of representatives of $C_1, \ldots, C_r$ respectively

$IBr(G) = \{\phi_1, \ldots, \phi_r\} :=$ the complete set of irreducible Brauer characters of $G$, which can be viewed as functions from $G_0$ into the complex numbers.

For any subring $R$ of the field of complex numbers such that $R \ni 1$, we write $RBR(G)$ to denote the ring of linear combinations of $\phi_1, \ldots, \phi_r$ over $R$. That is, $RBR(G)$ is the $R$-algebra spanned by $\phi_1, \ldots, \phi_r$. In particular we use the notation $BR(G)$ instead of $ZBR(G)$ and say that $BR(G)$ is the Brauer character ring of $G$.

Moreover we add the following notation.

$G(\overline{Q}/Q) :=$ the Galois group of $\overline{Q}$ over $Q$

If $A = (a_{ij})$ is a matrix over $Q$, then for $\sigma \in G(\overline{Q}/Q)$ we write $A^\sigma$ to denote the matrix $(a_{ij}^\sigma)$. We use the common notation $M^*$ for the conjugate transpose of a matrix $M$.

Now we define characteristic class functions on $G_0$. 

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Definition 2.1. We define class functions $f_i$ on $G_\circ$ $(i = 1, \ldots, r)$ as follows

\[ f_i(e) = 1 \quad , \quad f_i(e_j) = 0 \quad (i \neq j) \]

In this case we say that these class functions are the characteristic class functions on $G_\circ$ and that $f_i$ corresponds to $C_i$ or $C_i$ corresponds to $f_i$ $(i = 1, \ldots, r)$.

Now we prove an easy lemma concerning characteristic class functions on $G_\circ$.

Lemma 2.2. Let $\{f_1, \ldots, f_r\}$ be the complete set of characteristic class functions on $G_\circ$. Then we have

\[ f_i \in \overline{QBR}(G) \quad (i = 1, \ldots, r) \]

Proof. Let $\hat{f}_i$ be a characteristic class function of $G$ such that $\hat{f}_i|_{G_\circ} = f_i$ where $\hat{f}_i|_{G_\circ}$ indicates the restriction of $\hat{f}_i$ to $G_\circ$. Then each $\hat{f}_i$ is written as a $\overline{Q}$-linear combination of $\chi_1, \ldots, \chi_h$. That is,

\[ \hat{f}_i = \sum_{j=1}^h (|C_i|/|G|) \omega_j(e) \chi_j \quad (i = 1, \ldots, r) \quad \ldots \quad (2.1) \]

For each absolutely irreducible complex character $\chi_i$ of $G$, $\chi_i|_{G_\circ}$ is written as a $\mathbb{Z}$-linear combination of $\phi_1, \ldots, \phi_r$. That is,

\[ \chi_i|_{G_\circ} = \sum_{j=1}^r d_{ij} \phi_j \quad (i = 1, \ldots, h) \quad \ldots \quad (2.2) \]
where \((d_{ij})\) is the decomposition matrix of \(G\).

By virtue of the formulas (2.1) and (2.2), we can conclude that \(f_i \in \overline{QBR}(G) \quad (i = 1, \ldots, r)\) as required. Q.E.D.

We are given two finite groups \(G\) and \(H\). For \(G\) and \(H\) we assume that there exists an isomorphism \(\lambda\) of \(\overline{ZBR}(G)\) onto \(\overline{ZBR}(H)\). Then it follows that the rank of \(BR(G)\) is the rank of \(BR(H)\) and \(|Cl(G_e)| = |Cl(H_e)|\). We also can extend \(\lambda\) to an isomorphism \(\tilde{\lambda}\) of \(\overline{QBR}(G)\) onto \(\overline{QBR}(H)\) by linearity. By Lemma 2.2 we have \(f_i \in \overline{QBR}(G)\). Here we use the following additional notation.

\[
Cl(H_e) = \{C^i_1 = \{1', \ldots, C_i^r\}
\]

\[
\{c_1 = 1', \ldots, c_i\} = \text{a complete set of representatives of } C_1^i, \ldots, C_i^r \text{ respectively.}
\]

\[
\{f'_1, \ldots, f'_r\} = \text{the complete set of characteristic class functions on } H_e \text{ where } f'_i 
\]
corresponds to \(C^i_1\) \(= 1', \ldots, C_i^r\).

\[
IBr(H) = \{\phi'_1 = 1', \ldots, \phi'_r\}
\]

We now show a lemma which is actually the key step in the proof of Lemma 2.4.

Lemma 2.3. In the above situation, \(\tilde{\lambda}(f_i)\) is a characteristic class function on \(H_e\) \(= 1, \ldots, r\).

Proof. Since \(\overline{QBR}(G)f_i = \overline{Q}f_i \cong \overline{Q}, \overline{QBR}(G)f_i\) is a minimal ideal of \(\overline{QBR}(G)\) and so \(f_i\) is a (central) primitive idempotent \(= 1, \ldots, r\). Since \(\tilde{\lambda}(f_i) \in \overline{QBR}(H)\), we can write
\( \tilde{\lambda}(f_i) = \sum_{j=1}^{r} a_j f_j', \quad a_j \in \mathbb{Q} \) \hspace{1cm} (2.3)

Since \( f_i^2 = f_i \) and \( f_i f_j' = 0 \) \((i \neq j)\), by the formula (2.3) we have

\[ \tilde{\lambda}(f_i) = \sum_{j=1}^{r} a_j^2 f_j' \]. Thus \( a_j^2 = a_j \) \((j = 1, ..., r)\).

Hence \( a_j = 0 \) or \( a_j = 1 \) \((j = 1, ..., r)\). It follows that \( \tilde{\lambda}(f_i) = f_j' \) for some \( j \in \{1, ..., r\} \), because \( f_i \) is a primitive idempotent, hence the result. \hspace{1cm} \text{Q.E.D.}

Now we define a bijection from \( Cl(G_0) \) to \( Cl(H_0) \) through the isomorphism \( \lambda \) as follows. For a \( p \)-regular conjugate class \( C_i \) of \( G \), \( C_i \) corresponds to a characteristic class function \( f_i \) on \( G \). Since by Lemma 2.3 \( \tilde{\lambda}(f_i) \) is also a characteristic class function \( f_i^{i''} \) on \( H \), \( \tilde{\lambda}(f_i) = f_i^{i''} \) corresponds to a \( p \)-regular conjugate class \( C_i^{i''} \) of \( H \).

Here we assign \( C_i^{i''} \) to \( C_i \) \((i = 1, ..., r)\). Thus we get a one-to-one correspondence between \( Cl(G_0) \) and \( Cl(H_0) \):

\[ c_i \in C_i \rightarrow f_i \rightarrow \tilde{\lambda}(f_i) = f_i^{i''} \rightarrow C_i^{i''} \subseteq C_i' \]

where \( i \rightarrow i'' \) \((i = 1, ..., r)\) is a permutation. In this case we write \( C_i \leftrightarrow C_i' \) or \( c_i \leftrightarrow c_i' \) \((i = 1, ..., r)\).

Keeping the above notation, we give the following lemma concerning the Brauer character table of \( G \). This lemma plays a fundamental role in the proof of Theorem 3.1.

**Lemma 2.4.** \( (\phi_i(c_j)) = (\lambda(\phi_i)(c_j^{i''})) \) \(( r \times r \) matrices \) where \( c_j \leftrightarrow c_j' \).
(j = 1, ..., r).

**Proof.** Since \{f_1, ..., f_r\} and \{\phi_1, ..., \phi_r\} are \(\mathcal{Q}\)-bases of \(\mathcal{Q}BR(G)\), we can write

\[ f_i = \sum_{j=1}^{r} a_{ij} \phi_j, \quad a_{ij} \in \mathcal{Q} \quad (i = 1, ..., r) \]  \(\text{(2.4)}\)

If we set \(M = (a_{ij})\), then \(M\) is a regular matrix. By the formula \(2.4\) we get

\[
\begin{pmatrix}
\phi_1 \\
\vdots \\
\phi_r
\end{pmatrix}
= M
\begin{pmatrix}
\phi_1 \\
\vdots \\
\phi_r
\end{pmatrix}
\]  \(\text{(2.5)}\)

On the other hand, by the formula \(2.4\) we get

\[ f_i' = \bar{\lambda}(f_i) = \sum_{j=1}^{r} a_{ij} \lambda(\phi_j) \quad (i = 1, ..., r) \]

Hence we have

\[
\begin{pmatrix}
f_i' \\
\vdots \\
f_r'
\end{pmatrix}
= M
\begin{pmatrix}
\lambda(\phi_1) \\
\vdots \\
\lambda(\phi_r)
\end{pmatrix}
\]  \(\text{(2.6)}\)

Since \(M\) is a regular matrix, \(f_i(c_k) = f_i'(c'_k) = 1\) and \(f_i(c_j) = f_i'(c'_j) = 0 \quad (i \neq j)\)

where \(c_k \sim c'_k \quad (k = 1, ..., r)\), by the formulas \(2.5\) and \(2.6\) we have

\[
\begin{pmatrix}
\phi_1(c_k) \\
\vdots \\
\phi_r(c_k)
\end{pmatrix}
= \begin{pmatrix}
\lambda(\phi_1)(c'_k) \\
\vdots \\
\lambda(\phi_r)(c'_k)
\end{pmatrix}
\]  \(i = 1, ..., r\)

Hence we have \((\phi_i(c_j)) \sim (\lambda(\phi_i)(c'_j))\), thus completing the proof.  \(\text{Q.E.D.}\)
§3. Main theorems

Let $G$ and $H$ be two finite groups with Cartan matrices $C$ and $C'$ respectively. Let $\lambda$ be an isomorphism of $\mathbb{Z}BR(G)$ onto $\mathbb{Z}BR(H)$ and $A = (a_{ij})$ be the matrix afforded by $\lambda$ with respect to $IBr(G) = \{\phi_1, \ldots, \phi_r\}$ and $IBr(H) = \{\phi'_1, \ldots, \phi'_r\}$. We set $Cl(G_s) = \{C_1, \ldots, C_r\}$ and $Cl(H_s) = \{C'_1, \ldots, C'_r\}$ and assume that $c_i \in C_i$, $c'_i \in C'_i$ and $c_i \rightarrow c'_\mu$ where $i \rightarrow i''$ $(i = 1, \ldots, r)$ is a permutation. We write $m$ to denote the vector with $i$-th entry equal to $|C_G(c_i)|$ and $m'$ to denote the vector with $i$-th entry equal to $|C_H(c'_\mu)|$ $(i = 1, \ldots, r)$. Then we have the following two theorems.

**Theorem 3.1.** With the above notation, $m = m'$ iff $A^*CA = C'$. This necessarily happens if $CA = AC'$, in which case $A$ is clearly unitary.

**Proof.** To prove this theorem, we introduce some simplifying notation: Write $P$ to denote the $r \times r$ matrix with $(i, j)$-entry equal to $\phi_i(c_j)$ and similarly write $P'$ for the matrix with $(i, j)$-entry equal to $\phi'_i(c'_\mu)$.

Since $\lambda(\phi_i) = \sum_{k=1}^r a_{ik} \phi'_k$ where $A = (a_{ij})$, by Lemma 2.4 we have

$$\phi_i(c_j) = \lambda(\phi_i)(c'_\mu) = \sum_{k=1}^r a_{ik} \phi'_k(c'_\mu).$$

This implies that $P = AP'$. Also, if $B$ is the diagonal matrix with $(i, i)$-entry equal to $|C_G(c_i)|$, it follows that $P^*CP = B$ by Theorem 60.5 in [8]. Similarly
(P')*C'P' = B', where B' is the diagonal matrix with (i, i)-entry equal to |C_H(c'_\alpha)|.

Here we note that B = B' iff m = m'. Since P* = (P')*A*, we have the two equations

(P')*A*CAP' = B and (P')*C'P' = B'

It is now obvious that B = B' iff A*CA = C'.

Now suppose CA = AC'. Then we show that A is unitary. If we write J = A*CA, then we have (P')*JC'P' = B. Thus (B')^{-1}B = (P')^{-1}(C')^{-1}JC'P'. This is a diagonal matrix with rational entries and this shows that J has rational eigenvalues.

But J has algebraic integer entries, and so must have integer eigenvalues. Thus (B')^{-1}B is a diagonal matrix with positive integer diagonal entries. Also, A is invertible over \mathbb{Z} and thus A* is too. It follows that \text{det}(J) = \text{det}((B')^{-1}B) = 1 and so (B')^{-1}B is the identity matrix I. It follows that J = A*A = I and so A is unitary, as required.

Q.E.D.

**Theorem 3.2.** If CA = AC', then we have

(i) \( \lambda(\phi_i) = \epsilon_i\phi_{i'} \) where the \( \epsilon_i \) are roots of 1 and \( i \rightarrow i' \) \( (i = 1, ... r) \) is a permutation.

(ii) The Brauer character tables of G and H are the same.

**Proof.** (i) Now we pay attention to the fact that if \( \alpha \in \mathbb{Z} \) and \( |\alpha^\sigma| \leq 1 \) (an absolute value) for all \( \sigma \in G(\overline{Q}/Q) \), then \( \alpha = 0 \) or \( \alpha \) is a root of 1.
If we use the same notation as in the proof of Theorem 3.1, then we have $A = P(P')^{-1}$ and so $A$ has entries that lie in a field with an abelian Galois group. Thus $(A^*)^\sigma = (A^\sigma)^*$ for all $\sigma \in G(\mathbb{Q}/\mathbb{Q})$. Since $A$ is unitary by Theorem 3.1, $A^\sigma$ is automatically unitary for all $\sigma \in G(\mathbb{Q}/\mathbb{Q})$. Hence we have the equation with respect to the $i$-th row of $A^\sigma$

$$\sum_{j=1}^r a_{ij}^\sigma \bar{a}_{ij}^\sigma = \sum_{j=1}^r |a_{ij}^\sigma|^2 = 1 \quad (i = 1, \ldots, r)$$

Hence we have $|a_{ij}^\sigma| \leq 1$ for all $\sigma \in G(\mathbb{Q}/\mathbb{Q})$. This implies that $a_{ij} = 0$ or $a_{ij}$ is a root of 1 because of the above attention. Thus it follows that for each $i \in \{1, \ldots, r\}$, there exists $i' \in \{1, \ldots, r\}$ such that $a_{ii'}$ is a root of 1 and $a_{ij} = 0$ ($j \neq i'$). Hence $\lambda(\phi_i) = \epsilon_i \phi_i'$, where $\epsilon_i = a_{ii'}$ is a root of 1 and $i \rightarrow i'$ ($i = 1, \ldots, r$) is a permutation.

(ii) We state a one-to-one correspondence $\mu$ between $IBr(G)$ and $IBr(H)$ through the isomorphism $\lambda$ as follows. By (i) of this theorem, we have $\lambda(\phi_i) = \epsilon_i \phi_i'$ ($i = 1, \ldots, r$) where the $\epsilon_i$ are roots of 1. Here we assign $\phi_i'$ to $\phi_i : \mu(\phi_i) = \phi_i'$ ($i = 1, \ldots, r$). Then we show that $\mu$ can be extended to an isomorphism of $BR(G)$ onto $BR(H)$ by linearity. For $\phi_i, \phi_j \in IBr(G)$, we assume that

$$\phi_i \phi_j = \sum_{k=1}^r m_{ijk} \phi_k, \quad m_{ijk} \geq 0 \in \mathbb{Z}, \phi_k \in IBr(G).$$

Then we have $\lambda(\phi_i) \lambda(\phi_j) = \sum_{k=1}^r m_{ijk} \lambda(\phi_k)$. Hence we get

$$\phi_i' \phi_j' = \sum_{k=1}^r m_{ijk} \epsilon_i^{-1} \epsilon_j^{-1} \epsilon_k \phi_k'.$$

Since $\phi_i' \phi_j'$ is a Brauer character of $H$, it follows that $m_{ijk} \epsilon_i^{-1} \epsilon_j^{-1} \epsilon_k$ is a natural number for $m_{ijk} \neq 0$. Hence $m_{ijk} \epsilon_i^{-1} \epsilon_j^{-1} \epsilon_k = m_{ijk}$ because $\epsilon_i^{-1} \epsilon_j^{-1} \epsilon_k$ is a root of 1. Consequently we have
\[ \phi_r' \phi_r'' = \sum_{k=1}^{n-1} m_{jk} \phi_r'^k. \]

This means that \( \mu \) can be extended to an isomorphism of \( BR(G) \) onto \( BR(H) \).

By lemma 2.4 we have \( (\phi_j(c_j)) = (\phi_r' (c_r'')) \) (\( r \times r \) matrices) where \( c_j \rightarrow c' \rightarrow c'' \) \( j = 1, ..., r \). That is, \( G \) and \( H \) have the same Brauer character table. Thus the result follows. Q.E.D.

**Remark** If the condition \( m = m' \) in Theorem 3.1 holds, then we can easily prove \( |G| = |H| \). But we can give examples such that for two finite groups \( G, H \) with \( |G| \neq |H| \), a matrix \( A \) is unitary where \( A \) is afforded by an isomorphism of \( BR(G) \) onto \( BR(H) \). Actually, such an example is given by taking \( G \) and \( H \) to be any two \( p \)-groups of different orders. Another example can be found in [6]. ( \( p=2, G= \) the symmetric groups \( S_4 \) on 4 symbols and \( H= \) the dihedral group \( D_6 \) of order 12. See the examples of section 91 in [6] )
Chapter 4. On automorphisms of a character ring

§1. Introduction

Throughout this chapter $G$, $Z(G)$ and $C$ denote a finite group, the center of $G$ and the field of complex numbers respectively. For a finite set $X$, we denote the number of elements in $X$ by $|X|$.

Let $Irr(G)$ be the full set of irreducible $C$-characters of $G$ and $X(G)$ be the character ring of $G$. If $S$ is any subring of $C$, we write $SR(G)$ to denote the $S$-algebra of $S$-linear combinations of irreducible $C$-characters of $G$. We denote by $\mathbb{Z}$ the ring of all algebraic integers in $C$.

Suppose $G$ and $H$ are finite groups. Weidman showed that if $R(G)$ is isomorphic to $R(H)$, then $G$ and $H$ have the same character table.

In addition Saksonov proved the following theorem, which is a strengthened version of Weidman's theorem.

Theorem 1.1. (Saksonov) Suppose there exists an $\mathbb{Z}$-algebra isomorphism $\phi$ from $ZR(G)$ onto $ZR(H)$. If $Irr(G) = \{\chi_1, ..., \chi_h\}$ and $Irr(H) = \{\psi_1, ..., \psi_h\}$, then the following hold:

(i) The character tables of $G$ and $H$ are the same.

(ii) $\phi(\chi_i) = \epsilon_i \psi_{i'}$ \hspace{1cm} ($i = 1, ..., h$) \hspace{1cm} where the $\epsilon_i$ are roots of unity and $i \rightarrow i'$ is a permutation.
In this chapter we intend to prove the following theorem.

**Theorem 1.2.** Suppose $G$ and $H$ are finite groups. Then we have

(i) If $u$ is a central element in $G$ and $\tau_u : ZR(G) \to ZR(G)$ is the map defined by $\chi \mapsto (\chi(u)/\chi(1))\chi$ where $\chi \in \text{Irr}(G)$ and $1$ is the identity element of $G$, then $\tau_u$ is an $\mathbb{Z}$-automorphism of $ZR(G)$. Furthermore the map $u \mapsto \tau_u$ is a group isomorphism of $Z(G)$ onto a subgroup $T = \{\tau_u | u \in Z(G)\}$ of $\text{Aut}(ZR(G))$.

(ii) Every $\mathbb{Z}$-isomorphism $\phi : ZR(G) \to ZR(H)$ is the composition of an $\mathbb{Z}$-isomorphism $\theta$ that maps $\text{Irr}(G)$ onto $\text{Irr}(H)$ with an automorphism of $ZR(H)$ of the form $\tau_u$ for some element $u$ in $Z(H)$.

(iii) The full group $A = \text{Aut}(ZR(G))$ is the product of the subgroup $T$ of part (i) above, which is normal with the subgroup $P$ consisting of those automorphisms that map $\text{Irr}(G)$ onto $\text{Irr}(G)$. 
§2. Proof of Theorem 1.2

In order to prove Theorem 1.2 we prove a basic lemma concerning the roots of unity which appear in Saksonov’s Theorem.

Lemma 2.1. Suppose for each character $\chi$ in $\text{Irr}(G)$, there is a root of unity $\epsilon(\chi)$ such that each product $\epsilon(\chi)\epsilon(\psi)$ for $\chi, \psi$ in $\text{Irr}(G)$ is a non-negative integer linear combination of $\epsilon(\xi)\xi$, as $\xi$ runs over $\text{Irr}(G)$. Then there exists $u$ in $\mathbb{Z}(G)$ such that $\epsilon(\chi) = \chi(u)/\chi(1)$ for every character $\chi$ in $\text{Irr}(G)$.

Proof. If we are given $\chi$ and $\psi$ in $\text{Irr}(G)$, then we assume that
\[
\chi \psi = \sum_{\xi \in \text{Irr}(G)} m_\xi \xi \quad \text{and} \quad \epsilon(\chi)\epsilon(\psi) = \sum_{\xi \in \text{Irr}(G)} n_\xi \epsilon(\xi)\xi
\]
where the coefficients $m_\xi$ and $n_\xi$ are non-negative integers. Then it follows easily that $m_\xi = n_\xi$ for all characters $\xi$ in $\text{Irr}(G)$ and thus the map $\phi : \chi \mapsto \epsilon(\chi)\chi$ defines an automorphism of the algebra $\mathcal{C}(G)$. In particular the map $\phi$ permutes the primitive idempotents of this $\mathcal{C}$-algebra (See the proof of Lemma 2.3 of chapter 3.) and so it carries the characteristic class function of the identity to the characteristic class function of some other conjugacy class, say the class $K$. Therefore we have
\[
(1/|G|) \sum_{\chi \in \text{Irr}(G)} \epsilon(\chi)\chi(1)\chi = (1/|C_G(v)|) \sum_{\chi \in \text{Irr}(G)} \chi(v)\chi
\]
where $v$ is an element in $K$. It follows that for each irreducible character $\chi$ in $\text{Irr}(G)$ we have $\chi(1)\epsilon(\chi) = |K|\chi(u)$ where $u = v^{-1}$. Applying this where $\chi$ is the principal
character yields that $|K|$ is a root of unity and so $u$ is a central element in $G$. Thus for every character $\chi$ in $\text{Irr}(G)$, $\epsilon(\chi) = \chi(u)/\chi(1)$ for some element $u$ in $Z(G)$, as claimed.

Q.E.D.

Proof of Theorem 1.2

(i) Suppose $u$ is a central element in $G$. Then for each character $\chi$ in $\text{Irr}(G)$ we denote by $\epsilon(\chi)$ and $T(\chi)$ the root of unity given by $\chi(u)/\chi(1)$ and the irreducible matrix representation of $G$ which affords $\chi$ respectively. We assume further that for $\chi, \psi$ in $\text{Irr}(G)$, $\chi \psi = \sum_{\xi \in \text{Irr}(G)} m_\xi \xi$ where the $m_\xi$ are non-negative integers. Then we show $\epsilon(\xi) = \epsilon(\chi)\epsilon(\psi)$ for $m_\xi \neq 0$.

Indeed $T(\chi)(u) = \text{diag}(\epsilon(\chi), \ldots, \epsilon(\chi))$ and $T(\psi)(u) = \text{diag}(\epsilon(\psi), \ldots, \epsilon(\psi))$ which have diagonals of lengths $\chi(1)$ and $\psi(1)$ respectively. Hence

$$T(\chi)(u) \otimes T(\psi)(u) = \text{diag}(\epsilon(\chi)\epsilon(\psi), \ldots, \epsilon(\chi)\epsilon(\psi))$$

where $T(\chi) \otimes T(\psi)$ is the Kronecker product of $T(\chi)$ and $T(\psi)$. Since $T(\chi) \otimes T(\psi)$ is the representation of $G$ which affords $\chi \psi$, we have $\epsilon(\xi) = \epsilon(\chi)\epsilon(\psi)$ for $m_\xi \neq 0$, as claimed. Therefore we have $\epsilon(\chi)\epsilon(\psi)\psi = \sum_{\xi \in \text{Irr}(G)} m_\xi \epsilon(\xi)\xi$.

Thus the map $\tau_u$ defined by $\chi \mapsto \epsilon(\chi)\chi$ is an $\mathbb{Z}$-automorphism of $\mathbb{Z}R(G)$.

The fact that $Z(G) \cong T$ is easy to prove and so we omit its proof.

(ii) Now we can easily observe that Saksonov's result guarantees that the image of $\text{Irr}(G)$ under $\phi$ satisfies the hypotheses of Lemma 2.1 for $H$. Hence we may write
\( \phi(\chi_i) = \epsilon(\psi_{i'}) \psi_{i'} \), \( \epsilon(\psi_{i'}) = \psi_{i'}(u)/\psi_{i'}(1) \) for some element \( u \) in \( Z(H) \). \((i = 1, \ldots, h)\)

where \( \text{Irr}(G) = \{\chi_1, \ldots, \chi_h\}, \text{Irr}(H) = \{\psi_1, \ldots, \psi_h\} \) and \( i \rightarrow i' \) is a permutation.

Therefore the map \( \tau_u \) defined by \( \psi \rightarrow \epsilon(\psi)\psi \) is an \( \mathbb{Z} \)-automorphism of \( \mathbb{Z}R(H) \) from fact (i) above. If we put \( \theta = \tau_u^{-1} \phi \), then \( \theta(\chi_i) = \tau_u^{-1}(\phi(\chi_i)) = \psi_{i'} \). \((i = 1, \ldots, h)\)

and so \( \theta \) maps \( \text{Irr}(G) \) onto \( \text{Irr}(H) \). Hence we have \( \phi = \tau_u \theta \), as required.

(iii) Fact (iii) follows since fact (ii) tells us that \( A = TP \) and it is clear from fact (ii) that \( A \) induces a permutation action on \( \text{Irr}(G) \) and \( T \) is the kernel of this action.

This completes the proof of the theorem. Q.E.D.
References


