Appell's F_4 with Finite Irreducible Monodromy Group

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https://doi.org/10.11501/3120503

出版情報：九州大学, 1996, 博士（数理学）, 論文博士
バージョン：
権利関係：

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1. Introduction

Appell's hypergeometric series

$$F_4(a,b,c,c';X,Y)=\sum\frac{(a,m+n)(b,m+n)}{(c,m)(c',n)(1,m)(1,n)}X^mY^n$$

with $(a,n)=\Gamma(a+n)/\Gamma(a)$, satisfies the following system of differential equations of rank four ([1]):

$$\begin{cases} X(1-X)z_{XX}-Y^2z_{YY}-2XYz_{XY}+cz_X \\ \qquad -(a+b+1)(Xz_X+Yz_Y)-abz=0 \\ Y(1-Y)z_{YY}-X^2z_{XX}-2XYz_{XY}+c'z_Y \\ \qquad -(a+b+1)(Xz_X+Yz_Y)-abz=0 \end{cases}$$

which we denote by $E_4(a,b,c,c')$.

This is an extension of Gauss' hypergeometric series

$$F(a,b,c;x)=\sum\frac{(a,n)(b,n)}{(c,n)(1,n)}x^n$$

with hypergeometric differential equation (HGD for short)

$$x(1-x)d^2z/dx^2+(c-(a+b+1)x)dz/dx-abz=0,$$

which is of rank two and is denoted by $E(a,b,c)$.

Denote the monodromy group of $E(a,b,c)$ by

$$M(a,b,c),$$

and that of $E_4(a,b,c,c')$ by

$$M_4(a,b,c,c')$$

(see Section 2 for the definitions).

It is known that $M(a,b,c)$ is finite and irreducible if and only if $(1-c, c-a-b, b-a)$ belongs to the Schwarz' list (S-list) ([15],[5]).

As for Appell's F_1 and Lauricella's F_D, Sasaki [12] and Cohen-Wolfart [3] obtained the finiteness conditions of the monodromy groups. (Recently professor Sasaki told the author that Theorem 2 in [13] asserting non-existence of Appell's F_2 with finite irreducible monodromy group is false.)

The singular locus of $E_4(a,b,c,c')$ is $L_X \cup L_Y \cup L_\infty \cup C$, where $L_X = \{X=0\}$, $L_Y = \{Y=0\}$, $C = \{(X-Y)^2 - 2(X+Y)+1=0\}$ and L_∞ is the line at infinity. The differential equation $E_4(a,b,c,c')$ has characteristic exponents $0, 0, 1-c, 1-c$ along L_X. This implies that, at any point $P \in L_X - L_Y \cup L_\infty \cup C$, $E_4(a,b,c,c')$ has a fundamental system $(h_1, h_2, X^{1-c}h_3, X^{1-c}h_4)$ of solutions, where each h_j is holomorphic at P. Similarly $E_4(a,b,c,c')$ has exponents $0, 0, 1-c', 1-c'$ along L_Y, a, a, b, b along L_∞, $0, 0, 0, \varepsilon + 1/2$ along C, where

$$\varepsilon = c + c' - a - b - 1$$

(see [8]).

Since $F_4(a,b,c,c';X,0) = F(a,b,c;X)$ and $F_4(a,b,c,c';0,Y)$ $= F(a,b,c';Y)$, we can show that if $M_4(a,b,c,c')$ is finite and irreducible then so are $M(a,b,c)$ and $M(a,b,c')$ (see Section 3).

In this paper we will prove the following theorem.

THEOREM 1. *$M_4(a,b,c,c')$ is finite irreducible if and only if the following two conditions hold.*

(1) $M(a,b,c)$ and $M(a,b,c')$ are finite irreducible.

(2) The quantity ε is an integer, or at least two of $1-c, 1-c', b-a$ are equivalent to 1/2 modulo $\mathbf{Z}$.

The structure of these finite irreducible monodromy groups are stated in Proposition 4.1, Theorem 7.1 and Theorem 7.2.

Let $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ be a system of fundamental solutions of $E_4(a,b,c,c')$. Then Ψ defines a (multi-valued) mapping of $U := \mathbf{P}^2 - L_X \cup L_Y \cup L_\infty \cup C$ into $\mathbf{P}^3$. Sasaki-Yoshida [14] proved that if $\varepsilon = 0$ then the image $\Psi(U)$ belongs to a smooth quadratic surface. In Section 8, we will verify, in the cases $c = c' = 1/2$ and $(c-a-b, b-a) = (1/n, 1/2)$ or $(1/3, 1/3)$ or $\{c-a-b, b-a\} = \{1/3, 1/4\}$ or $\{1/3, 1/5\}$, that the closure S_Ψ of $\Psi(U)$ is smooth hypersufaces in $\mathbf{P}^3$ and the inverse of Ψ is single valued.

The author thanks to professors J. Kaneko, T. Sasaki and M. Yoshida for valuable advises.

2. Monodromy representations

2.1. $M(a,b,c)$

Assume that $c \notin \mathbf{Z}$ and that $M(a,b,c)$ is irreducible. Put

$$v_1 = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a,b,c;x),$$

$$v_2 = \frac{\Gamma(1+a-c)\Gamma(1+b-c)}{\Gamma(2-c)} x^{1-c} F(1+a-c, 1+b-c, 2-c; x).$$

Then v_1 and v_2 form a system of fundamental solutions of $E(a,b,c)$. Let L_0, L_1 be the loops surrounding 0,1 positively with base point $x_0 = 1/2$. We denote by $V(x_0)$ the set of germs of holomorphic solutions of $E(a,b,c)$. Then for any $L \in \pi_1(\mathbf{C} - \{0,1\}, x_0)$ and $f \in V(x_0)$, the analytic continuation fL_* of f along L is again belongs to $V(x_0)$. We write

$$f(LL')_* = (fL_*)L'_* = fL_*L'_*,$$

if L' is continued after L. This defines a monodromy representation

$$\pi_1(\mathbf{C} - \{0,1\}, x_0) \longrightarrow GL(V(x_0)).$$

For a subset $S \subset \pi_1(\mathbf{C} - \{0,1\}, x_0)$, we denote

$$S_* = \{L_* | L \in S\}.$$

We call

$$M(a,b,c) = M(a,b,c;x_0) = (\pi_1(\mathbf{C} - \{0,1\}, x_0))_*$$

the monodromy group of $E(a,b,c)$.

For $v = {}^t(v_1, v_2)$, we denote by vL_* the analytic continuation ${}^t(v_1L_*, v_2L_*)$ of v along L. Then by use of connection formulas for Gauss' HGD (see, for example, [4]), we have

$$vL_{0*} = G_0 v,$$

$$vL_{1*} = G_1 v$$

where

$$G_0 = \begin{pmatrix} 1 & 0 \\ 0 & e(1-c) \end{pmatrix},$$

$$G_1 = I + \frac{2\sqrt{-1}e((c-a-b)/2)}{\sin \pi c}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} (-\sin \pi a \sin \pi b, \sin \pi(c-a) \sin \pi(c-b)),$$

$$e(x) = exp(2\pi\sqrt{-1}x).$$

Let $N_1(a,b,c;x_0) = N_1(a,b,c)$ be the smallest normal subgroup of $M(a,b,c)$ containing L_{1*}. Then we have

$$M(a,b,c) = N_1(a,b,c) \cdot < L_{\bullet *} > .$$

2.2. $M_4(a,b,c,c')$

The monodromy representations of $E_4(a,b,c,c')$ are first founded by Kaneko [6] and Takano [17]. Here for our convenience, we adopt the monodromy representation in [9].

We assume in this section that $E_4(a,b,c,c')$ is irreducible and that $c, c' \notin \mathbf{Z}$. Recall that $E_4(a,b,c,c')$ is irreducible if and only if none of $a, b, c-a, c-b, c'-a, c'-b, c+c'-a, c+c'-b$ is an integer ([9],[10]). Hence

$$\varphi_1 := \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')} F_4(a,b,c,c';X,Y),$$

$$\varphi_2 := \frac{\Gamma(1+a-c)\Gamma(1+b-c)}{\Gamma(2-c)\Gamma(c')} X^{1-c} F_4(1+a-c, 1+b-c, 2-c, c'; X, Y),$$

$$\varphi_3 := \frac{\Gamma(1+a-c')\Gamma(1+b-c')}{\Gamma(c)\Gamma(2-c')} Y^{1-c'} F_4(1+a-c', 1+b-c', c, 2-c'; X, Y),$$

$$\varphi_4 := \frac{\Gamma(2+a-c-c')\Gamma(2+b-c-c')}{\Gamma(2-c)\Gamma(2-c')} X^{1-c} Y^{1-c'} F_4(2+a-c-c', 2+b-c-c', 2-c, 2-c'; X, Y)$$

form a system of fundamental solutions of $E_4(a,b,c,c')$.

Let δ be a sufficiently small positive number and put $P_0 = (\delta, \delta)$. Recall that $U = \mathbf{P}^2 - L_X \cup L_Y \cup L_\infty \cup C$. Then the fundamental group $\pi_1(U, P_\bullet)$ is generated by the following γ_1,γ_2 and γ_3:

$$\gamma_1 = \{X = \delta e(t) \quad 0 \le t \le 1, Y = \delta\},$$
$$\gamma_2 = \{X = \delta, Y = \delta e(t) \quad 0 \le t \le 1\},$$
$$\gamma_3 = \{X = Y = 1/4 - (1/4 - \delta)e(t) \quad 0 \le t \le 1\}.$$

We denote by $V(P_0)$ the set of germs of holomorphic solutions of $E_4(a,b,c,c')$ at P_0. Then for any $\gamma \in \pi_1(U,P_0)$, γ_* (the analytic continuation along γ) is an element of $GL(V(P_0))$. This defines a monodromy representation

$$\pi_1(U,P_0) \longrightarrow GL(V(P_0)).$$

We denote the image by

$$M_4(a,b,c,c';P_0) = M_4(a,b,c,c')$$

and call it the monodromy group of $E_4(a,b,c,c')$.

Put $\varphi = {}^t(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$, then γ_{j*} $j=1,2,3$ are represented by matricies in the following way.

THEOREM 2.2. *Assume that $E_4(a,b,c,c')$ is irreducible and that $c,c' \notin$ **Z** then we have*

$$\varphi\gamma_{1*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e(1-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e(1-c) \end{pmatrix} \varphi,$$

$$\varphi\gamma_{2*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e(1-c') & 0 \\ 0 & 0 & 0 & e(1-c') \end{pmatrix} \varphi,$$

$$\varphi\gamma_{3*} = \left(I + \frac{e(\varepsilon/2)}{\sin\pi c \sin\pi c'} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (\gamma_{31},\gamma_{32},\gamma_{33},\gamma_{34}) \right) \varphi,$$

where
$\gamma_{31} = \sin\pi a \sin\pi b$, $\gamma_{32} = -\sin\pi(c-a)\sin\pi(c-b)$,
$\gamma_{33} = -\sin\pi(c'-a)\sin\pi(c'-b)$, $\gamma_{34} = \sin\pi(c+c'-a)\sin\pi(c+c'-b)$.

PROOF: By the base change of the monodromy representation in Theorem 7.1 in [9], we obtain the theorem. ∎

Since γ_3 is a loop surrounding C, we denote by

$$N_C(a,b,c,c';P_0) = N_C(a,b,c,c')$$

the smallest normal subgroup of $M_4(a,b,c,c';P_0)$ containing γ_{3*}. Then we have

$$M_4(a,b,c,c') = N_C(a,b,c,c') \cdot < \gamma_{1*}, \gamma_{2*} > .$$

The eigenvalues of γ_{3*} are $1, 1, 1, e(\varepsilon + 1/2)$. Hence if $\varepsilon + 1/2 \in \mathbf{Q} - \mathbf{Z}$ then γ_{3*} is a reflection. So we call $N_C(a, b, c, c')$ the reflection subgroup of $M_4(a, b, c, c')$. The terminology of "reflection subgroup" appeared in Beukers-Heckman [2] for the generalized hypergeometric function ${}_nF_{n-1}$.

3. Restrictions of E_4 to singularities

We assume in this section that $M_4(a, b, c, c')$ is finite and irreducible. Concerning to the characteristic exponents of $E_4(a, b, c, c')$ (see Section 1) we have

Lemma 3.1. *All the parametors a, b, c, c' are (real) rational numbers and none of $1 - c, 1 - c', b - a, \varepsilon + 1/2$ is an integer.*

Proof: Assume $c \in \mathbf{Z}$. Then $E_4(a, b, c, c')$ has a solution with logarithmic factor $\log X$ (Section 2 of [10]). This contradicts to the finiteness of M_4. Hence we have $c \notin \mathbf{Z}$. Similarly we have $c', b - a \notin \mathbf{Z}$. Assume $\varepsilon + 1/2 \in \mathbf{Z}$. Then since γ_{3*} is diagonizable, we have $\gamma_{3*} = I$. Hence E_4 is reducible. This contradiction proves that $\varepsilon + 1/2 \notin \mathbf{Z}$.

Since $c \notin \mathbf{Z}$, at $L_X(= \{X = 0\})$, $E_4(a, b, c, c')$ has solutions h_1, h_2, $X^{1-c}h_3$, $X^{1-c}h_4$ with h_j being holomorphic. Since $(X^{1-c}h_3)\gamma_{1*}^n = X^{1-c}h_3$ for some $n \in \mathbf{Z}$, we must have $1 - c \in \mathbf{Q}$. Similarly, we have $1 - c', \varepsilon + 1/2, a, b \in \mathbf{Q}$. ∎

Lemma 3.2. *$M(a, b, c)$ is finite irreducible.*

Proof: Let $\mathcal{U}$ and $\mathcal{V}$ be a small neighborhoods of X_0 and 0 in $\mathbf{C}$ respectively, where we assume $X_0 \neq 0, 1$. Then the map

$$\{\text{holomorphic solutions of } E_4(a, b, c, c') \text{ in } \mathcal{U} \times \mathcal{V}\}$$
$$\longrightarrow \{\text{holomorphic solutions of } E(a, b, c) \text{ in } \mathcal{U}\}$$

defined by the restriction $f(X, Y) \longmapsto f(X, 0)$ is one-to-one onto (Section 2.1 of [8]). Hence $M(a, b, c)$ must be finite.

Since none of $a, b, c - a, c - b$ is an integer by the assumption of irredicibility of M_4, $M(a, b, c)$ is irreducible. ∎

By the same way we have the following lemma.

Lemma 3.3. *$M(a, b, c')$ is finite irreducible.*

Lemma 3.4. *$M(1 + a - c, 1 + b - c, c')$, $M(1 + a - c', 1 + b - c', c)$, $M(a, 1 + a - c', c)$, $M(b, 1 + b - c', c)$ are finite irreducible.*

Proof: First we note that $1 - c,\ b - a \notin \mathbf{Z}$ by Lemma 3.1.

Since $X^{1-c}f(X,Y)$ is a solution of $E_4(a,b,c,c')$ if and only if $f(X,Y)$ is a solution of $E_4(1+a-c, 1+b-c, 2-c, c')$, we know that $M_4(1+a-c, 1+b-c, 2-c, c')$ is finite irreducible. Then, by Lemma 3.3, $M(1+a-c, 1+b-c, c')$ is finite irreducible.

Since $Y^{-a}f(X/Y, 1/Y)$ is a solution of $E_4(a,b,c,c')$ if and only if $f(X,Y)$ is a solution of $E_4(a, 1+a-c', c, 1+a-b)$, we know that $M_4(a, 1+a-c', c, 1+a-b)$ is finite irreducible. Then, by Lemma 3.2, $M(a, 1+a-c', c)$ is finite irreducible.

$M(1+a-c', 1+b-c', c)$ and $M(b, 1+b-c', c)$ are also finite irreducible by the same way. ∎

4. Proof of "if" part of Theorem 1

Assume the conditions (1) and (2) in Theorem 1. In each case $M_4(a,b,c,c')$ is irreducible. The problem is to show the finiteness of $M_4(a,b,c,c')$. We notice that $a,b,c,c' \in \mathbf{Q}$ by the assertion (1). This implies that γ_{j*} $(j=1,2,3)$ are of finite order.

In Section 4.1, we deal with the case when $\varepsilon(=c+c'-a-b-1)$ is an integer. In Section 4.2, we deal with the case when ε is not an integer.

4.1. Case of $\varepsilon \in \mathbf{Z}$

Assume that $\varepsilon \in \mathbf{Z}$. Let

$$\phi : (x,y) \longrightarrow (X,Y) \qquad X = xy, Y = (1-x)(1-y)$$

be the branched double covering of $\mathbf{C}^2$ onto $\mathbf{C}^2$. The covering ϕ is locally biholomorphic at any point (x,y) with $x \neq y$. We have $\phi(\{x=y\}) = C = \{(X-Y)^2 - 2(X+Y) + 1 = 0\}$. Recall $P_0 = (\delta, \delta)$, $U = \mathbf{C}^2 - L_X \cup L_Y \cup C$. Put $W = \phi^{-1}(U)$ and $P_1 = (x_1, y_1)$ be a point such that $\phi(P_1) = P_0$. It is easily verified that $W = \{(x,y) | xy(1-x)(1-y)(x-y) = 0\}$. We have one to one homomorphism

$$\phi_* : \pi_1(W, P_1) \longrightarrow \pi_1(U, P_0).$$

The image of ϕ_* is a normal subgroup of $\pi_1(U, P_0)$ with index 2. Precicely speaking, we have

$$\pi_1(U, P_0) = \phi_*(\pi_1(W, P_1)) \cdot <\gamma_3>$$

with $\gamma_3^2 \in \phi_*(\pi_1(W, P_1))$. Hence

$$N := (\phi_*(\pi_1(W, P_1)))_*$$

is a normal subgroup of M_4 with

$$M_4 = N \cdot < \gamma_{3*} > .$$

This implies that M_4 is finite if and only if N is finite. The finiteness of N is a direct consequence of the following proposition.

PROPOSITION 4.1. *Assume that $\varepsilon \in \mathbf{Z}$ and that $M_4(a,b,c,c')$ is irreducible. Then*

$$N \simeq M(a,b,c) \otimes M(a,b,c) := \{g \otimes g' | g, g' \in M(a,b,c)\}$$

with $M_4(a,b,c,c') = N \cdot < \gamma_{3*} >$, $N \cap < \gamma_{3*} >= \{1\}$ and $< \gamma_{3*} > \simeq \mathbf{Z}_2$.

PROOF: Put $\varepsilon = n$. Since $M_4(a,b,c,c')$ is irreducible, we have $M_4(a,b,c,c') \simeq M_4(a,b,c,c'-n)$ by Theorem 2.2. Hence it is enough to prove for the case of $\varepsilon = 0$. So we assume $\varepsilon = 0$.

Since $\varepsilon = 0$, we have

$$\phi^*(E_4(a,b,c,c')) = E(a,b,c;x) \cdot E(a,b,c;y)$$

(Section 1 of [7]), and $\{x = y\}$ is an apparent singular locus of $\phi^*(E_4(a,b,c,c'))$.

Since ϕ is locally biholomorphic at P_1, $V(P_0)$ is isomorphic to the space of germs of holomorphic solutions of $\phi^*(E_4)$ at P_1, which is again isomorphic to $V(x_1) \otimes V(y_1)$ where $V(x_1)$ (resp. $V(y_1)$) is the space of germs of solutions of $E(a,b,c)$ at x_1 (resp. y_1). Hence the representation of $\phi_*(\pi_1(W,P_1))$ in $GL(V(P_0))$ is isomorphic to the representation of $\pi_1(W,P_1)$ in $V(x_1) \otimes V(y_1)$, which is again isomorphic to the representation of $\pi_1(\mathbf{C}-\{0,1\},x_1) \times \pi_1(\mathbf{C}-\{0,1\},y_1)$ in $V(x_1) \otimes V(y_1)$. This implies that $N \simeq M(a,b,c) \otimes M(a,b,c)$.

If g and g' ($\in M(a,b,c)$) have eigenvalues (λ,μ) and (λ',μ') then the eigenvalues of $g \otimes g'$ are $\lambda\lambda', \lambda\mu', \mu\lambda', \mu\mu'$. Because $E_4(a,b,c,c')$ has exponents $0, 0, 0, \varepsilon + 1/2$ along C (see Section 1), the eigenvalues of γ_{3*} are $1, 1, 1, -1$. Hence γ_{3*} cannot be contained in $M(a,b,c) \otimes M(a,b,c)$. This implies that $N \cap < \gamma_{3*} >= \{1\}$. ∎

4.2. Case of $\varepsilon \notin \mathbf{Z}$

Assume that ε is not an integer. Recall that $M_4 = N_C \cdot < \gamma_{1*}, \gamma_{2*} >$ (see Section 2.2). Since γ_{1*} and γ_{2*} are of finite order and satisfy $\gamma_{1*}\gamma_{2*} = \gamma_{2*}\gamma_{1*}$, $< \gamma_{1*}, \gamma_{2*} >$ is also of finite order. Hence M_4 is finite if and only if N_C is finite. The finiteness of N_C is a direct concequence of the following two lemmas.

LEMMA 4.2.1. *Assume that $M_4(a,b,c,c')$ is irreducible and that $1-c, 1-c' \equiv 1/2 \bmod \mathbf{Z}$ then*

$$N_C(a,b,c,c') \simeq N_1(a,b,c) \times N_1(a,b,c) \simeq N_1(a,b,c') \times N_1(a,b,c')$$

and $M_4(a,b,c,c')$ is imprimitive.

PROOF: In this case, generators of γ_{j*} of M_4 in Section 2.2 are as follows:

$$\varphi\gamma_{1*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \varphi, \quad \varphi\gamma_{2*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \varphi$$

$$\varphi\gamma_{3*} = \left(I + e((-a-b)/2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}) \right) \varphi$$

where $\gamma_{31} = \gamma_{34} = \sin \pi a \sin \pi b$, $\gamma_{32} = \gamma_{33} = -\cos \pi a \cos \pi b$.

Put

$$\psi_1 = \varphi_1 + \varphi_4, \quad \psi_2 = \varphi_2 + \varphi_3, \quad \psi_3 = \varphi_1 - \varphi_4, \quad \psi_4 = \varphi_2 - \varphi_3$$

and let

$$V_0 := <\psi_1, \psi_2>, \qquad V_1 := <\psi_3, \psi_4>$$

be subspaces of $V = V(P_0)$. Then γ_{1*}, γ_{2*} interchange V_0 and V_1, and γ_{3*} fixes $V_j \quad (j = 0, 1)$ invariant. This means that $M_4(a,b,c,c')$ is imprimitive and that V_0, V_1 are invariant under $(<\gamma_3, \gamma_2\gamma_3\gamma_2^{-1}, \gamma_1\gamma_2>)_*$.

Put

$$g_0 = \gamma_1\gamma_2, \quad g_1 = \gamma_3, \quad g_2 = \gamma_2\gamma_3\gamma_2^{-1}.$$

Then g_{1*} is identity on V_1 and g_{2*} is identity on V_0. Hence we have

$$\begin{aligned} N_C =& (< g_1, g_0 g_1 g_0^{-1}, g_2, g_0 g_2 g_0^{-1} >)_* \\ \simeq& (< g_1, g_0 g_1 g_0^{-1} >)_* \times (< g_2, g_0 g_2 g_0^{-1} >)_*. \end{aligned}$$

The operations of $g_{j*} \quad (j = 0, 1, 2)$ on V_0 and V_1 are as follows:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} g_{0*} = G_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} g_{1*} = G_1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} g_{2*} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} g_{0*} = G_0 \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} g_{2*} = G_1 \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} g_{1*} = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

where

$$G_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G_1 = I + 2e((-a-b)/2)\begin{pmatrix} 1 \\ 1 \end{pmatrix}(\gamma_{31}, \gamma_{32}).$$

Hence

$$(< g_1, g_0 g_1 g_0^{-1} >)_* | V_0 \simeq N_1(a,b,c) \simeq N_1(a,b,c')$$
$$(< g_1, g_0 g_1 g_0^{-1} >)_* | V_1 = \{I\}.$$

$$(< g_2, g_0 g_2 g_0^{-1} >)_* | V_1 \simeq N_1(a,b,c) \simeq N_1(a,b,c')$$
$$(< g_2, g_0 g_2 g_0^{-1} >)_* | V_1 = \{I\}.$$

This proves that

$$N_C(a,b,c,c') \simeq N_1(a,b,c) \times N_1(a,b,c) \simeq N_1(a,b,c') \times N_1(a,b,c').$$

∎

LEMMA 4.2.2. *Assume that $M_4(a,b,c,c')$ is irreducible.*
If $1-c', b-a \equiv 1/2 \bmod \mathbf{Z}$ *then*

$$N_C(a,b,c,c') \simeq N_1(a,b,c) \times N_1(a,b,c).$$

If $1-c, b-a \equiv 1/2 \bmod \mathbf{Z}$ *then*

$$N_C(a,b,c,c') \simeq N_1(a,b,c') \times N_1(a,b,c').$$

In any case, $M_4(a,b,c,c')$ is imprimitive.

PROOF: Assume that $1-c', b-a \equiv 1/2 \bmod \mathbf{Z}$. Another statement under the assumption of $1-c, b-a \equiv 1/2 \bmod \mathbf{Z}$ is proved in the same way. In this case we have

$$\varphi\gamma_{1*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e(1-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e(1-c) \end{pmatrix} \varphi,$$

$$\varphi\gamma_{2*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \varphi,$$

$$\varphi\gamma_{3*} = \left(I - \frac{e((c-2a)/2)}{2\sin \pi c} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}) \right) \varphi,$$

where $\gamma_{31} = \gamma_{33} = \sin 2\pi a$, $\gamma_{32} = \gamma_{34} = \sin 2\pi(c - a)$.

Put

$$\psi_1 = \varphi_1 + \varphi_3,\ \psi_2 = \varphi_2 + \varphi_4,\ \psi_3 = \varphi_1 - \varphi_3,\ \psi_4 = \varphi_2 - \varphi_4$$

and let

$$V_0 := < \psi_1, \psi_2 >, \qquad V_1 := < \psi_3, \psi_4 >$$

be subspaces of $V = V(P_0)$.

Then γ_{2*} interchanges V_0 and V_1, and γ_{1*}, γ_{3*} fix V_j $(j = 0, 1)$ invariant. This means that $M_4(a, b, c, c')$ is imprimitive and that V_0, V_1 are invariant under $(< \gamma_1, \gamma_3, \gamma_2\gamma_3\gamma_2^{-1} >)_*$.

Put

$$g_0 = \gamma_1, \quad g_1 = \gamma_3, \quad g_2 = \gamma_2\gamma_3\gamma_2^{-1}$$

Then g_{1*} is identity on V_1 and g_{2*} is identity on V_0. Hence we have

$$\begin{aligned} N_C =& (< \{g_0^j g_1 g_0^{-j}, \quad g_0^j g_2 g_0^{-j} \quad j \in \mathbf{Z}\} >)_* \\ \simeq& (< \{g_0^j g_1 g_0^{-j} \quad j \in \mathbf{Z}\} >)_* \times (< \{g_0^j g_2 g_0^{-j} \quad j \in \mathbf{Z}\} >)_*. \end{aligned}$$

The operations of g_{0*}, g_{1*}, g_{2*} on V_0 and V_1 are as follows:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} g_{0*} = G_0 \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} g_{1*} = G_1 \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} g_{2*} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} g_{0*} = G_0 \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}, \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} g_{2*} = G_1 \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}, \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} g_{1*} = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$$

where

$$G_0 = \begin{pmatrix}1 & 0\\ 0 & \epsilon(1-c)\end{pmatrix}, \quad G_1 = I - \frac{\epsilon((c-2a)/2)}{\sin \pi c} \begin{pmatrix}1\\ 1\end{pmatrix} (\gamma_{31}, \gamma_{32}).$$

Hence Lemma 4.2.2 holds in the same way as the previous lemma. ∎

5. Proof of "only if" part of Theorem 1

It is sufficient to prove the following lemma.

Lemma 5. *Assume that $M_4(a, b, c, c')$ is finite and irreducible and that $\varepsilon \notin \mathbf{Z}$. Then at least two of $1 - c, 1 - c', b - a$ are equivalent to 1/2 mod $\mathbf{Z}$.*

PROOF: From Lemma 3.2, 3.3 and 3.4 we have

(1) $(1-c, c-a-b, b-a)$ belongs to the S-list,

(2) $(1-c', c'-a-b, b-a)$ belongs to the S-list,

(3) $(1-c', c'-a-b-2(1-c), b-a)$ belongs to the S-list,

(4) $(1-c, c-a-b-2(1-c'), b-a)$ belongs to the S-list,

(5) $(1-c, (c'-a-b)+(b-a)-(1-c), 1-c')$ belongs to the S-list.

Suppose Lemma 5 does not hold. Then by the symmetry, we may assume that

$$1-c = p/k, \ \ 1-c' = p'/k' \quad k, k' \in \{3, 4, 5\}.$$

Put

$$c-a-b = q/m, \ \ c'-a-b = q'/m', \ \ b-a = r/n \quad m, m', n \in \{2, 3, 4, 5\}.$$

We will derive contradictions in any of the following cases.

(Case 1) $k = k' = 4$, p, p' are odd.

The property (4) implies that the denominator of $c-a-b-2(1-c')$ $= q/m - 2p'/k'$ is one of 2,3,4,5. Hence m is eaven. If $m = 4$ then $\varepsilon = q/m - p'/k' \equiv 0$ or $1/2 \bmod \mathbf{Z}$. Since $\varepsilon, \varepsilon + 1/2 \notin \mathbf{Z}$, this is a contradiction. If $m = 2$ then $c-a-b-2(1-c') = q/m - 2p'/k' \in \mathbf{Z}$ and hence (4) does not hold. This is a contradiction.

(Case 2) $k = 4$, $k' = 3$ or 5, p is odd.

The property (5) implies that $k' = 3$. Then (4) implies that the denominator of $c-a-b-2(1-c') = q/m - 2p'/k'$ is 3 and hence $m = 3$. By the same reason, (3) implies that $m' = 4$. Since ε is not an integer, the denominator of $\varepsilon = c-a-b-(1-c') = q/m - p'/k'$ is 3. On the other hand $\varepsilon = c'-a-b-(1-c) = q'/m' - p/k$ has even denominator. This is a contradiction.

(Case 3) k and k' are odd (=3 or 5).

The properties (3) and (4) imply that $m' = k$ and $m = k'$ respectively. Since $\varepsilon = (c-a-b)-(1-c') = (c'-a-b)-(1-c)$ is not an integer, we have $k = k'$ which is the denominator of ε. Then (5) implies that the denominator of $(c'-a-b)-(1-c)+(b-a) = \varepsilon + (b-a)$ is k. Hence $n = k$. This concludes that $k = k' = m = m' = n$.

(Case 3.1) $k = k' = m = m' = n = 3$.

Since $\varepsilon = c'-a-b-(1-c) = (q'-p)/3 \notin \mathbf{Z}$, we have $p \not\equiv q' \bmod 3$. On the other hand (3) implies $c'-a-b-2(1-c) = (q'-2p)/3 \notin \mathbf{Z}$. Hence $p \equiv q' \bmod 3$. This is a contradiction.

(Case 3.2) $k = k' = m = m' = n = 5$.

In order that (1) and (2) hold, there are two cases, that is,

$$p, q, p', q', r \equiv \pm 1 \text{ or } p, q, p', q', r \equiv \pm 2 \bmod 5.$$

Since $\varepsilon = (q' - p)/5 = (q - p')/5$ is not an integer, we have $p \not\equiv q'$, $p' \not\equiv q$ mod 5.

If $p, q, p', q', r \equiv \pm 1$ (and $p \not\equiv q'$) mod 5 then the numerator of $c' - a - b - 2(1 - c) = (q' - 2p)/5$ is congruent to ± 2 mod 5.

If $p, q, p', q', r \equiv \pm 2$ (and $p \not\equiv q'$) mod 5 then the numerator of $c' - a - b - 2(1 - c) = (q' - 2p)/5$ is congruent to ± 1 mod 5.

In any case (3) does not hold. This is a contradiction.

This completes the proof of Lemma 5. ∎

6. Lemmas on $M(a, b, c)$

In this section we denote

$$\lambda = 1 - c, \quad \mu = c - a - b, \quad \nu = b - a$$

and we assume that $M(a, b, c)$ is finite irreducible. Recall that $N_1(a, b, c)$ is the smallest normal subgroup of of $M(a, b, c)$ containing L_{1*} (see Section 2.1). In this section we fix the base v_1, v_2 of $V(x_0)$ and identify L_{x*} and G_x $x = 0, 1$.

Lemma 6.1. *Assume that $\lambda \equiv \nu \equiv 1/2$ mod **Z**. Then $L_{0*} \notin N_1(a, b, c)$.*

Proof: We have $G_0^2 = I$, $(G_0 G_1)^2 = \alpha I$ for some root of unity α. Since $G_0 G_1 G_0^{-1} = \alpha G_1^{-1}$, G_1 and $G_0 G_1 G_0^{-1}$ have the common eigen vectors. This means that N_1 is reducible hence we have $N_1 \neq M(a, b, c)$. This implies $G_0 \notin N_1$. ∎

Lemma 6.2. *Assume that $\lambda \equiv 1/2$, $\mu, \nu \not\equiv 1/2$ mod **Z**. Then $L_{0*} \notin N_1(a, b, c)$.*

Proof: If the denominator of μ is odd (*i.e.* 3 or 5) then the determinant of any $L_* \in N_1$ cannot be $-1 = det(G_0)$. Hence $G_0 \notin N_1$. If the denominator of μ is 4, then direct computations show that the orders of $M(a, b, c)$ and N_1 are 192 and 96 (refer to Shephard-Todd [16]). Hence $G_0 \notin N_1$. ∎

Lemma 6.3. *Assume that $\nu \equiv 1/2$, $\lambda, \mu \not\equiv 1/2$ mod **Z**. If both of the denominators of λ and μ are 5 then $L_{0*} \in N_1$. Otherwise $< L_{0*} > \cap N_1 = \{I\}$.*

Proof: In the first case, we may assume $\lambda = 1/5, \mu = 2/5$. Then by direct calculations we have $(G_0 G_1)^2 = (G_0 G_1^3)^3 = \alpha I$, $\alpha = e(1/10)$. The equality $(G_0G_1)^2 = \alpha I$ implies $\alpha G_0^3 = (G_0G_1G_0^{-1})(G_0^2G_1G_0^{-2}) \in N_1$. The equality $(G_0G_1^3)^3 = \alpha I$ implies $\alpha G_0^2 = (G_0G_1^3G_0^{-1})(G_0^2G_1^3G_0^{-2})$ $(G_0^3G_1^3G_0^{-3}) \in N_1$. Hence $G_0 \in N_1$.

In the case of $(\lambda, \mu) = (1/3, 1/3)$, by direct computations, we know that the orders of $M(a,b,c)$ and $N_1(a,b,c)$ are 72 and 24 (refer to Shephard-Todd [16]). Hence $< G_0 > \cap N_1 = \{I\}$.

In the case of $\{\lambda, \mu\} = \{1/3, 1/4\}, \{1/3, 1/5\}, \{2/5, 1/3\}$, the denominators of λ and μ are relatively prime. Hence we have $< G_0 > \cap N_1 = \{I\}$. ∎

7. STRUCTURE OF FINITE IRREDUCIBLE $M_4(a,b,c,c')$

The structure of M_4 with $\varepsilon \in \mathbf{Z}$ is stated in Proposition 4.1. We will consider finite irreducible $M_4(a,b,c,c')$ with $\varepsilon \notin \mathbf{Z}$. Recall that $M_4(a,b,c,c') = N_C \cdot < \gamma_{1*}, \gamma_{2*} >$ is imprimitive in this case (Lemma 4.2.1, 4.2.2).

THEOREM 7.1. *Assume that $M(a,b,c)$ is finite irreducible and that $\varepsilon \notin \mathbf{Z}$, $c, c' \equiv 1/2 \bmod \mathbf{Z}$. Then $M_4(a,b,c,c') = N_C \cdot < \gamma_{1*}, \gamma_{2*} >$ with $N_C \cap < \gamma_{1*}, \gamma_{2*} >= \{I\}$, $N_C \simeq N_1(a,b,c) \times N_1(a,b,c)$, $< \gamma_{1*}, \gamma_{2*} > \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$ and $M(a,b,c)/N_1 \simeq \mathbf{Z}_2$.*

PROOF: Since $c - a - b \equiv \varepsilon + 1/2 \not\equiv 1/2$, Lemma 6.1 and Lemma 6.2 imply that $L_{0*} \notin N_1$, whence $M(a,b,c)/N_1(a,b,c) \simeq \mathbf{Z}_2$. By Lemma 4.2.1, we have $N_C \simeq N_1(a,b,c) \times N_1(a,b,c)$ and $< \gamma_{1*}, \gamma_{2*} > \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$.

Next we will prove $N_C \cap < \gamma_{1*}, \gamma_{2*} >= \{I\}$. As in the proof of Lemma 4.2.1, $V = V_0 + V_1$. V_0, V_1 are invariant under N_C while γ_{1*}, γ_{2*} interchange V_0 and V_1. Hence $\gamma_{1*}, \gamma_{2*} \notin N_C$. In the proof of Lemma 4.2.1, we have shown that the restrictions of $(\gamma_1\gamma_2)_*$ and N_C to V_0 are L_{0*} and $N_1(a,b,c)$. Since $L_{0*} \notin N_1(a,b,c)$ by Lemma 6.1, 6.2, we have $(\gamma_1\gamma_2)_* \notin N_C$. This proves that $N_C \cap < \gamma_{1*}, \gamma_{2*} >= \{I\}$. ∎

THEOREM 7.2. *Assume that $M(a,b,c)$ is finite irreducible and that $\varepsilon \notin \mathbf{Z}$, $c', b - a \equiv 1/2 \bmod \mathbf{Z}$. Put $c = p/k$ with $(p,k) = 1$.*

(7.2.1) If both of the denominators of $1-c$ and $c-a-b$ are 5, then $\gamma_{1*} \in N_C$, hence we have $M_4(a,b,c,c') = N_C \cdot < \gamma_{2*} >$ with $N_C \cap < \gamma_{2*} >= \{I\}$. And we have $N_C \simeq N_1(a,b,c) \times N_1(a,b,c) = M(a,b,c) \times M(a,b,c)$ and $< \gamma_{2*} > \simeq \mathbf{Z}_2$.

(7.2.2) If the condition of (7.2.1) does not hold, then $M_4(a,b,c,c') = N_C \cdot < \gamma_{1*}, \gamma_{2*} >$ with $N_C \cap < \gamma_{1*}, \gamma_{2*} >= \{I\}$, $N_C \simeq N_1(a,b,c) \times N_1(a,b,c)$, $M(a,b,c)/N_1 \simeq \mathbf{Z}_k$ and $< \gamma_{1*}, \gamma_{2*} > \simeq \mathbf{Z}_k \times \mathbf{Z}_2$.

PROOF: As is shown in the proof of Lemma 4.2.2, $V = V_0 + V_1$ and γ_{2*} interchanges V_0 and V_1 while γ_{1*} and γ_{3*} fix (set theoretically) V_j $j = 0, 1$. Hence any element of $N_{C'}$ also fix V_j. Consequently we have $\gamma_{2*} \notin N_{C'}$. By Lemma 4.2.2, the restrictions of γ_{1*} and $N_{C'}$ to V_j are L_{0*} and $N_1(a, b, c)$ for each $j = 0, 1$.

In case of (7.2.1), by Lemma 6.3, $L_{0*} \in N_1$. This implies $\gamma_{1*} \in N_{C'}$. Hence $M_4(a, b, c, c') = N_{C'} \cdot < \gamma_{1*}, \gamma_{2*} >= N_{C'} \cdot < \gamma_{2*} >$ with $N_{C'} \cap < \gamma_{2*} >= \{I\}$. By Lemma 4.2.2, we have $N_{C'} \simeq N_1(a, b, c) \times N_1(a, b, c) = M(a, b, c) \times M(a, b, c)$ and $< \gamma_{2*} > \simeq \mathbf{Z}_2$.

In case of (7.2.2), by Lemma 6.3, $< L_{0*} > \cap N_1 = \{I\}$. Hence $< \gamma_{1*}, \gamma_{2*} > \cap N_{C'} = \{I\}$. By Lemma 4.2.2, we have $N_{C'} \simeq N_1(a, b, c) \times N_1(a, b, c)$ and $< \gamma_{1*}, \gamma_{2*} > \simeq \mathbf{Z}_k \times \mathbf{Z}_2$. ∎

8. EXAMPLES

We assume in this section that $c = c' = 1/2$ and that $M_4(a, b, c, c')$ is irreducible. We fix the base v_1, v_2 of $V(x_0)$ (see Section 2.1). Recall that

$$V = V_0 + V_1 =< \psi_1, \psi_2 > + < \psi_3, \psi_4 >,$$

where V_0 and V_1 are invariant subspaces of $V = V(P_0)$ under g_{0*}, g_{1*}, g_{2*} (see the proof of Lemma 4.2.1).

Put

$$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4).$$

Then Ψ defines a multi-valued locally biholomorphic mapping of $\mathbf{P}^2 - L_X \cup L_Y \cup L_\infty \cup C$ into $\mathbf{P}^3$. Let S_Ψ be the closure of its image in $\mathbf{P}^3$. In the following examples S_Ψ are smooth hypersurfaces and Ψ^{-1} are defined by meromorphic functions on S_Ψ. The defining functions of S_Ψ and the inverse mapping functions are composed of the invariant (homogeneous) polynomials $\in \mathbf{C}[v_1, v_2]$ under the actions of $M(a, b, c)$. First we prepare the following two lemmas.

LEMMA 8.1. *Assume that $c, c' \equiv 1/2 \mod \mathbf{Z}$.*

(1) If $f(v_1, v_2)$ is an invariant polynomial under the action of $M(a, b, c)$ then $f(\psi_1, \psi_2) + f(\psi_3, \psi_4)$ and $f(\psi_1, \psi_2) f(\psi_3, \psi_4)$ are both invariant under $M_4(a, b, c, c')$.

(2) If $f(v_1, v_2) L_{1*} = f(v_1, v_2)$ and $f(v_1, v_2) L_{0*} = -f(v_1, v_2)$ then $f(\psi_1, \psi_2)^2 + f(\psi_3, \psi_4)^2$ and $f(\psi_1, \psi_2) f(\psi_3, \psi_4)$ are both invariant under $M_4(a, b, c, c')$.

PROOF:

Proof of (1). $f(\psi_1,\psi_2)$ and $f(\psi_3,\psi_4)$ are invariant under g_{0*}, g_{1*}, g_{2*} while $f(\psi_1,\psi_2)\gamma_{2*} = f(\psi_3,\psi_4)$. Hence (1) holds.

Proof of (2). $f(\psi_1,\psi_2)^2 + f(\psi_3,\psi_4)^2$ is invariant from (1). By the proof of Lemma 4.2.1, $f(\psi_1,\psi_2)$ and $f(\psi_3,\psi_4)$ are both invariant under N_C. Since $f(\psi_1,\psi_2)g_{0*} = -f(\psi_1,\psi_2)$, $f(\psi_3,\psi_4)g_{0*} = -f(\psi_3,\psi_4)$ and $f(\psi_1,\psi_2)\gamma_{2*} = f(\psi_3,\psi_4)$, we know that $f(\psi_1,\psi_2)f(\psi_3,\psi_4)$ is invariant under $< \gamma_{1*}, \gamma_{2*} >$. Hence (2) holds. ∎

In Shephard-Todd [16], three invariants

$$f_n(v_1,v_2),\ h_{2n-4}(v_1,v_2),\ t_{3n-6}(v_1,v_2)$$

are considered. Where n denotes the degree of f_n, h_{2n-4} is the Hessian of f_n of degree $2n-4$ and t_{3n-6} is the Jacobian of f_n and h_{2n-4} of degree $3n-6$. For the application to $M_4(a,b,c,c')$, we will calculate the definite formes of them.

We put

$$\begin{aligned} v_1' &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(1+a+b-c)} F(a,b,1+a+b-c;1-x), \\ v_2' &= \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(1+c-a-b)} (1-x)^{c-a-b} F(c-a,c-b,1+c-a-b;1-x). \end{aligned}$$

LEMMA 8.2. *By the analytic continuations along real segment $0 < x < 1$, we have*

$$\begin{aligned} v_1' &= \frac{\pi}{\Gamma(1+a-c)\Gamma(1+b-c)}(v_1 - v_2), \\ v_2' &= -\frac{\pi}{\Gamma(1+a-c)\Gamma(1+b-c)}(\beta v_1 + v_2), \end{aligned}$$

where

$$\beta = -\frac{\sin \pi a \sin \pi b}{\sin \pi(c-a)\sin \pi(c-b)}.$$

PROOF: This follows from the connection formulas for $E(a,b,c)$, given in [4], for example. ∎

In the following examples we put

$$w_1 = \beta^{1/4} v_1,\ w_2 = \beta^{-1/4} v_2.$$

Example 8.3. $c = c' = b - a = 1/2,\ \varepsilon + 1/2\ (= c - a - b) = 1/n.$

In this case, β (in the previous lemma) = 1. Hence

$$P_n(v_1, v_2) := (v_1 - v_2)^n + (v_3 + v_4)^n = constant \cdot (v_1'^n \pm v_2'^n)$$

is invariant under $M(a, b, c)$. Put

$$Q_n(v_1, v_2) = (v_1 - v_2)^n - (v_1 + v_2)^n.$$

Then Q_n is invariant under L_{1*} but $Q_n L_{\bullet *} = -Q_n$. From Lemma 8.1, we know that

$$P_n(\psi_1, \psi_2) + P_n(\psi_3, \psi_4), \quad P_n(\psi_1, \psi_2)P_n(\psi_3, \psi_4),$$

$$Q_n(\psi_1, \psi_2)^2 + Q_n(\psi_3, \psi_4)^2, \quad Q_n(\psi_1, \psi_2)Q_n(\psi_3, \psi_4)$$

are invariant under $M_4(a, b, c, c')$. Since the exponents along L_∞ are $-1/2n, -1/2n, (n-1)/2n, (n-1)/2n$, $P_n(\psi_1, \psi_2) + P_n(\psi_3, \psi_4)$ is constant while other three invariant functions are at most one degree polynomials in X, Y. Since $P_n(\psi_1, \psi_2)$ is invariant under $g_{1*} = \gamma_{3*}$, and $g_{\bullet *} = (\gamma_1\gamma_2)_*$, $P_n(\psi_1, \psi_2)$ has the following form: $P_n(\psi_1, \psi_2) = A_\bullet(X, Y) + A_1(X, Y)(XY)^{1/2}$. Then we have $P_n(\psi_3, \psi_4) = A_0(X, Y)$ $-A_1(X, Y)(XY)^{1/2}$. Hence we know that A_0 is constant $(= 2(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')})^2)$ and $A_1 = 0$. By expanding at $X = 0, Y = 0$, we have

$$Q_n(\psi_1, \psi_2)^2 + Q_n(\psi_3, \psi_4)^2 = 8(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')})^2(X + Y)$$
$$Q_n(\psi_1, \psi_2)Q_n(\psi_3, \psi_4) = 4(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')})^2(X - Y)$$

Thus we have proved that

$$S_\Psi = \{[\psi_1 : \psi_2 : \psi_3 : \psi_4] \in \mathbf{P}^3 | P_n(\psi_1, \psi_2) - P_n(\psi_3, \psi_4) = 0\}$$

which is a smooth hypersurface of degree n, and that Ψ^{-1} is given by

$$X = \frac{(Q_n(\psi_1, \psi_2) + Q_n(\psi_3, \psi_4))^2}{(P_n(\psi_1, \psi_2) + P_n(\psi_3, \psi_4))^2},$$
$$Y = \frac{(Q_n(\psi_1, \psi_2) - Q_n(\psi_3, \psi_4))^2}{(P_n(\psi_1, \psi_2) + P_n(\psi_3, \psi_4))^2}.$$

Recall that $M_4(a, b, c, c')$ is of order $4n^4$ with center of order n.

Example 8.4. $c = c' = 1/2, \ \ b - a = \varepsilon + 1/2 \ (= c - a - b) = 1/3.$

In this case $\beta = (\sqrt{3}-1)/(\sqrt{3}+1)$. $M(a,b,c)$ is the group No.6 in Shephard-Todd's list, the order of which is 48 and the center of which is $\{e(k/4)I | 0 \le k \le 3\}$. There are invariant polynomials $f_4(v_1, v_2)$ and $t_6(v_1, v_2)^2$ of degree 4 and 12 (Shephard-Todd [16]). In order that f_4 shoud be invariant under L_{1*}, f_4 must be of the form $f_4 = v_1'^4 + \alpha v_1' v_2'^3$. In order that f_4 shoud be invariant under L_{0*}, by direct computations, we have

$$f_4(v_1, v_2) = w_1^4 + 2\sqrt{3} w_1^2 w_2^2 - w_2^4.$$

By a constant multiplication, we have

$$t_6(v_1, v_2) = w_1 w_2 (w_1^4 + w_2^4)$$

which satisfies $t_6 L_{1*} = t_6$ and $t_6 L_{0*} = -t_6$. Then

$$f_4(\psi_1, \psi_2) = f_4(\psi_3, \psi_4) = \beta \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')} \right)^4.$$

We also have

$$\begin{aligned} t_6(\psi_1, \psi_2)^2 + t_6(\psi_3, \psi_4)^2 &= 2k(X+Y), \\ t_6(\psi_1, \psi_2) t_6(\psi_3, \psi_4) &= k(X-Y), \end{aligned}$$

where

$$k = \beta^2 \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')} \right)^{10} \left(\frac{\Gamma(1+a-c)\Gamma(1+b-c)}{\Gamma(2-c)\Gamma(c')} \right)^2.$$

Thus we have proved that

$$S_\Psi = \{[\psi_1 : \psi_2 : \psi_3 : \psi_4] \in \mathbf{P}^3 | f_4(\psi_1, \psi_2) - f_4(\psi_3, \psi_4) = 0\}$$

which is a smooth hypersurface of degree 4 and that Ψ^{-1} is given by

$$\begin{aligned} X &= \alpha \frac{(t_6(\psi_1, \psi_2) + t_6(\psi_3, \psi_4))^2}{(f_4(\psi_1, \psi_2) + f_4(\psi_3, \psi_4))^3}, \\ Y &= \alpha \frac{(t_6(\psi_1, \psi_2) - t_6(\psi_3, \psi_4))^2}{(f_4(\psi_1, \psi_2) + f_4(\psi_3, \psi_4))^3} \end{aligned}$$

where

$$\alpha = 2\beta \left(\frac{\Gamma(a)\Gamma(b)\Gamma(2-c)}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c)} \right)^2 = 24\sqrt{3}.$$

Example 8.5. $c = c' = 1/2$, $\varepsilon + 1/2$ $(= c-a-b) = 1/3$, $b-a = 1/4$. In this case $\beta = (\sqrt{3}-\sqrt{2})/(\sqrt{3}+\sqrt{2})$. $M(a,b,c)$ is the group No.14 in

Shephard-Todd's list, the order of which is 144 and the center of which is $\{e(k/6)I | 0 \le k \le 5\}$. There are invariant polynomials $f_6(v_1, v_2)$ and $t_{12}(v_1, v_2)^2$ of degree 6 and 24 (Shephard-Todd [16]).

By direct computations, we have

$$\begin{aligned} f_6(v_1, v_2) &= w_1^6 + 5w_1^4w_2^2 - 5w_1^2w_2^4 - w_2^6, \\ t_{12}(v_1, v_2) & \\ &= w_1w_2(w_1^{10} - \frac{11}{9}w_1^8w_2^2 + \frac{66}{9}w_1^6w_2^4 + \frac{66}{9}w_1^4w_2^6 - \frac{11}{9}w_1^2w_2^8 + w_2^{10}). \end{aligned}$$

The polynomial t_{12} satisfies $t_{12}L_{1*} = t_{12}$ and $t_{12}L_{0*} = -t_{12}$. Then

$$f_6(\psi_1, \psi_2) = f_6(\psi_3, \psi_4) = \beta^{3/2}\left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')}\right)^6.$$

We also have

$$\begin{aligned} t_{12}(\psi_1, \psi_2)^2 + t_{12}(\psi_3, \psi_4)^2 &= 2k(X + Y), \\ t_{12}(\psi_1, \psi_2)t_{12}(\psi_3, \psi_4) &= k(X - Y), \end{aligned}$$

where

$$k = \beta^5\left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')}\right)^{22}\left(\frac{\Gamma(1 + a - c)\Gamma(1 + b - c)}{\Gamma(2 - c)\Gamma(c')}\right)^2.$$

Thus we have proved that

$$S_\Psi = \{[\psi_1 : \psi_2 : \psi_3 : \psi_4] \in \mathbf{P}^3 | f_6(\psi_1, \psi_2) - f_6(\psi_3, \psi_4) = 0\}$$

which is a smooth hypersurface of degree 6 and that Ψ^{-1} is given by

$$\begin{aligned} X &= \alpha\frac{(t_{12}(\psi_1, \psi_2) + t_{12}(\psi_3, \psi_4))^2}{(f_6(\psi_1, \psi_2) + f_6(\psi_3, \psi_4))^4}, \\ Y &= \alpha\frac{(t_{12}(\psi_1, \psi_2) - t_{12}(\psi_3, \psi_4))^2}{(f_6(\psi_1, \psi_2) + f_6(\psi_3, \psi_4))^4} \end{aligned}$$

where

$$\alpha = 4\beta\left(\frac{\Gamma(a)\Gamma(b)\Gamma(2 - c)}{\Gamma(1 + a - c)\Gamma(1 + b - c)\Gamma(c)}\right)^2.$$

Example 8.6. $c = c' = 1/2$, $\varepsilon + 1/2$ $(= c - a - b) = 1/4$, $b - a = 1/3$. In this case $\beta = (\sqrt{2} - 1)/(\sqrt{2} + 1)$. $M(a, b, c)$ is the group No.9 in

Shephard-Todd's list, the order of which is 192 and the center of which is $\{e(k/8)I|0 \le k \le 7\}$. The following polynomial

$$f_6(v_1, v_2) = w_1^6 - 5w_1^4 w_2^2 - 5w_1^2 w_2^4 + w_2^6$$

satisfies $f_6 L_{0*} = f_6$, $f_6 L_{1*} = \sqrt{-1} f_6$. The polynomials h_8 and t_{12}^2 are invariant under $M(a, b, c)$. We have (up to constant multiplications)

$$\begin{aligned} h_8(v_1, v_2) &= w_1^8 + \frac{28}{3} w_1^6 w_2^2 - \frac{14}{3} w_1^4 w_2^4 + \frac{28}{3} w_1^2 w_2^6 + w_2^8, \\ t_{12}(v_1, v_2) & \\ &= w_1 w_2 (w_1^{10} + \frac{11}{9} w_1^8 w_2^2 + \frac{66}{9} w_1^6 w_2^4 - \frac{66}{9} w_1^4 w_2^6 - \frac{11}{9} w_1^2 w_2^8 - w_2^{10}). \end{aligned}$$

The polynomial t_{12} satisfies $t_{12} L_{1*} = t_{12}$ and $t_{12} L_{0*} = -t_{12}$. Then

$$h_8(\psi_1, \psi_2) = h_8(\psi_3, \psi_4) = \beta^2 \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')} \right)^8 .$$

We also have

$$\begin{aligned} t_{12}(\psi_1, \psi_2)^2 + t_{12}(\psi_3, \psi_4)^2 &= 2k(X + Y), \\ t_{12}(\psi_1, \psi_2) t_{12}(\psi_3, \psi_4) &= k(X - Y), \end{aligned}$$

where

$$k = \beta^5 \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')} \right)^{22} \left(\frac{\Gamma(1+a-c)\Gamma(1+b-c)}{\Gamma(2-c)\Gamma(c')} \right)^2 .$$

Thus we have proved that

$$S_\Psi = \{[\psi_1 : \psi_2 : \psi_3 : \psi_4] \in \mathbf{P}^3 | h_8(\psi_1, \psi_2) - h_8(\psi_3, \psi_4) = 0\}$$

which is a smooth hypersurface of degree 8 and that Ψ^{-1} is given by

$$\begin{aligned} X &= \alpha \frac{(t_{12}(\psi_1, \psi_2) + t_{12}(\psi_3, \psi_4))^2}{(h_8(\psi_1, \psi_2) + h_8(\psi_3, \psi_4))^3}, \\ Y &= \alpha \frac{(t_{12}(\psi_1, \psi_2) - t_{12}(\psi_3, \psi_4))^2}{(h_8(\psi_1, \psi_2) + h_8(\psi_3, \psi_4))^3} \end{aligned}$$

where

$$\alpha = 2\beta \left(\frac{\Gamma(a)\Gamma(b)\Gamma(2-c)}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c)} \right)^2 .$$

Example 8.7. $c = c' = 1/2$, $\varepsilon + 1/2$ $(= c - a - b) = 1/3$, $b - a = 1/5$. $M(a, b, c)$ is the group No.21 in Shephard-Todd's list, the order of which is 720 and the center of which is $\{e(k/12)I | 0 \leq k \leq 11\}$. The following polynomial

$$\begin{aligned} f_{12}(v_1, v_2) =& w_1^{12} + \frac{22}{\sqrt{5}} w_1^{10} w_2^2 \\ & - 33 w_1^8 w_2^4 - \frac{44}{\sqrt{5}} w_1^6 w_2^6 - 33 w_1^4 w_2^8 + \frac{22}{\sqrt{5}} w_1^2 w_2^{10} + w_2^{12} \end{aligned}$$

is invariant under $M(a, b, c)$. The polynomial t_{30} satisfies $t_{30} L_{1*} = t_{30}$ and $t_{30} L_{0*} = -t_{30}$.

By the same reason as previous examples, we have

$$S_\Psi = \{[\psi_1 : \psi_2 : \psi_3 : \psi_4] \in \mathbf{P}^3 | f_{12}(\psi_1, \psi_2) - f_{12}(\psi_3, \psi_4) = 0\}$$

which is a smooth hypersurface of degree 12 and that Ψ^{-1} is given by

$$\begin{aligned} X =& \alpha \frac{(t_{30}(\psi_1, \psi_2) + t_{30}(\psi_3, \psi_4))^2}{(f_{12}(\psi_1, \psi_2) + f_{12}(\psi_3, \psi_4))^5}, \\ Y =& \alpha \frac{(t_{30}(\psi_1, \psi_2) - t_{30}(\psi_3, \psi_4))^2}{(f_{12}(\psi_1, \psi_2) + f_{12}(\psi_3, \psi_4))^5} \end{aligned}$$

where

$$\alpha = 8\beta \left(\frac{\Gamma(a)\Gamma(b)\Gamma(2-c)}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c)} \right)^2.$$

Example 8.8. $c = c' = 1/2$, $\varepsilon + 1/2$ $(= c - a - b) = 1/5$, $b - a = 1/3$. $M(a, b, c)$ is the group No.17 in Shephard-Todd's list, the order of which is 1200 and the center of which is $\{e(k/20)I | 0 \leq k \leq 19\}$. The following polynomial

$$\begin{aligned} f_{12}(v_1, v_2) =& w_1^{12} - \frac{22}{\sqrt{5}} w_1^{10} w_2^2 \\ & - 33 w_1^8 w_2^4 + \frac{44}{\sqrt{5}} w_1^6 w_2^6 - 33 w_1^4 w_2^8 - \frac{22}{\sqrt{5}} w_1^2 w_2^{10} + w_2^{12} \end{aligned}$$

satisfies $f_{12} L_{0*} = f_{12}$, $f_{12} L_{1*} = e(1/5) f_{12}$. The polynomial h_{20} is invariant under $M(a, b, c)$ and the polynomial t_{30} satisfies $t_{30} L_{1*} = t_{30}$ and $t_{30} L_{0*} = -t_{30}$.

By the same reason as previous examples, we have

$$S_\Psi = \{[\psi_1 : \psi_2 : \psi_3 : \psi_4] \in \mathbf{P}^3 | h_{20}(\psi_1, \psi_2) - h_{20}(\psi_3, \psi_4) = 0\}$$

which is a smooth hypersurface of degree 20 and that Ψ^{-1} is given by

$$X = \alpha \frac{(t_{30}(\psi_1, \psi_2) + t_{30}(\psi_3, \psi_4))^2}{(h_{20}(\psi_1, \psi_2) + h_{20}(\psi_3, \psi_4))^3},$$

$$Y = \alpha \frac{(t_{30}(\psi_1, \psi_2) - t_{30}(\psi_3, \psi_4))^2}{(h_{20}(\psi_1, \psi_2) + h_{20}(\psi_3, \psi_4))^3}$$

where

$$\alpha = 2\beta \left(\frac{\Gamma(a)\Gamma(b)\Gamma(2-c)}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c)} \right)^2 .$$

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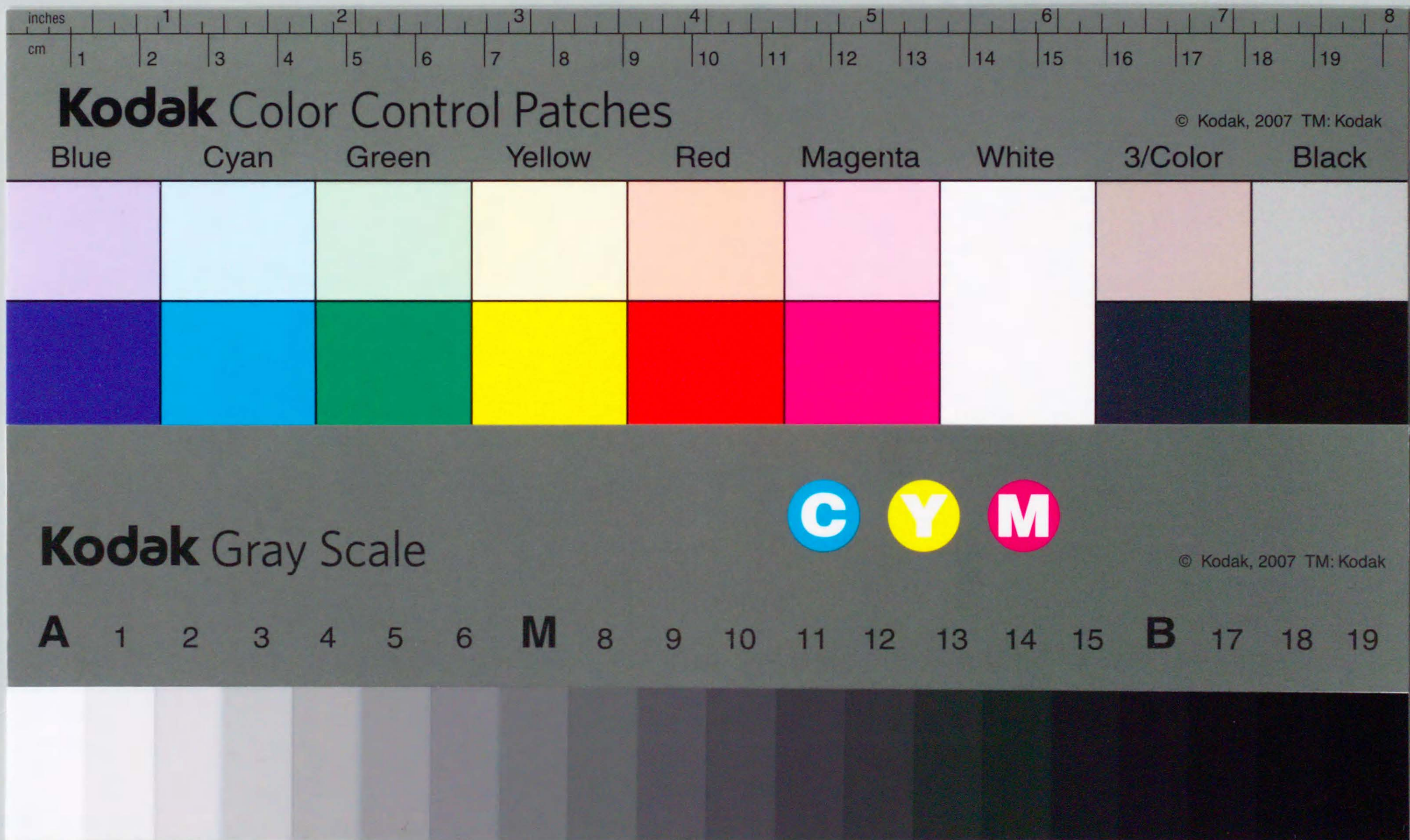
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