STUDIES ON THE EXISTENCE OF H?LDER CONTINUOUS WEAK SOLUTIONS FOR DEGENERATE QUASI-LINEAR ELLIPTIC SYSTEMS

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STUDIES ON THE EXISTENCE OF HÖLDER CONTINUOUS WEAK SOLUTIONS FOR DEGENERATE QUASI- LINEAR ELLIPTIC SYSTEMS

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STUDIES ON THE EXISTENCE OF HÖLDER CONTINUOUS WEAK SOLUTIONS FOR DEGENERATE QUASI-LINEAR ELLIPTIC SYSTEMS

(退化する準線形楕円型偏微分方程式系の
Hölder 連続な弱解の存在についての研究)

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PREFACE

In this paper we study the Dirichlet boundary value problem associated with quasilinear elliptic systems (1.1)-(1.2) of which ellipticity is violated at points where $|\boldsymbol{u}| = 0$. N.N.Ural'tseva [25] showed the existence of Hölder continuous weak solutions for the type (1.1) with only principal part, and yet $a_{ij}^l = a_{ij}$. N.Ikebe-Y.Ohara [14] treated a single equation of the type (1.1) which has a non-negative solution. For the single equation whose solution is not necessarily non-negative, Y.Mizutani [19] showed the existence of a Hölder continuous solution. K.Hayasida-Y.Yokoi [13] considered the continuity of solutions adjacent to the boundary. M.Araki [3] treated a system of the type (1.1) with the coefficients $a_{ij}^l = a_{ij}$, $b_j^l = b_j$ which has non-negative solutions. For the system whose solutions are not necessarily non-negative, Araki-Ikebe-Mizutani [5] showed the existence of Hölder continuous weak solutions. We, here, consider the system whose coefficients $a_{ij}^l(x, \boldsymbol{u})$ and $b_j^l(x, \boldsymbol{u})$ may be different with respect to l, and construct the Hölder continuous weak solutions of the system (1.1)-(1.2). Chapter 1 is devoted to preliminaries. In Section 1, our equations are introduced. In Section 2, main theorem and regularized problems are given. In Section 3, some auxiliary lemmas are prepared to get the Hölder estimates of the solutions. Chapter 2 is devoted to prove the theorem for $0 < \tau \leq 1$. In Section 4, some integral inequalities are prepared to use the auxiliary lemmas in the proof of the theorem. In Section 5, the auxiliary lemmas are applied to the solutions of the regularized problem, and the Hölder estimates of the solutions are obtained by the method which is greatly owed to Ladyzhenskaya-Ural'tseva [15] and N.N. Ural'tseva [25]. In Section 6, the weak solutions of our problem are constructed and the main theorem is proved. Chapter 3 is devoted to prove the theorem for $\tau \geq 1$ in the same way as in Chapter 2.

CONTENTS

CHAPTER I

1. INTRODUCTION

Let Ω be a bounded domain in R^n with $C_{2,\alpha}$-class boundary $\partial\Omega$. We consider the following Dirichlet problem for quasi-linear elliptic systems:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\{a_{ij}^l(x, \boldsymbol{u}(x))u_{x_j}^l(x)\} - \sum_{j=1}^{n} b_j^l(x, \boldsymbol{u}(x))u_{x_j}^l(x)$$

$$-b_0^l(x, \boldsymbol{u}(x)) = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

for $l = 1, \cdots, N$, with nonhomogeneous boundary conditions

$$\boldsymbol{u}(x) = \boldsymbol{\psi}(x) \quad \text{on} \quad \partial\Omega \tag{1.2}$$

where $\boldsymbol{u}(x) \equiv (u^1(x), \cdots, u^N(x)), \boldsymbol{\psi}(x) \equiv (\psi^1(x), \cdots, \psi^N(x)) \in R^N$, $u_{x_j}^l \equiv \partial u^l/\partial x_j$ and the coefficients $a_{ij}^l(x, \boldsymbol{u})$, $b_j^l(x, \boldsymbol{u})$, $b_0^l(x, \boldsymbol{u})$ are all real functions.

The coefficients $a_{ij}^l(x, \boldsymbol{u})$ satisfy the following conditions:

$$a_{ij}^l(x, \boldsymbol{u}) = a_{ji}^l(x, \boldsymbol{u}) \quad \text{for} \quad i, j = 1, \cdots, n,$$

$$C_0|\boldsymbol{u}|^\tau|\boldsymbol{\xi}|^2 \le \sum_{i,j=1}^{n} a_{ij}^l(x, \boldsymbol{u})\xi_i\xi_j \le C_0^{-1}|\boldsymbol{u}|^\tau|\boldsymbol{\xi}|^2 \quad \text{for} \quad \tau > 0, \tag{A$_1$}$$

for any $(x, \boldsymbol{u}, \boldsymbol{\xi}) \in \bar{\Omega} \times R^N \times R^n$ where C_0 is a positive constant, i.e., the systems (1.1) admit degeneracy of their ellipticity at points where $|\boldsymbol{u}| = 0$.

There are two typical types of degenerate quasilinear elliptic equations as follows:

(i) $\nabla(|\nabla u|^{p-2}\nabla u) = f \ \ (p > 1)$, (ii) $\nabla(|u|^\tau\nabla u) + \cdots = f \ \ (\tau > 0)$,

i.e., the equation (i) is degenerate if $p > 2$ and singular if $1 < p < 2$ at points where $|\nabla u| = 0$, and the equation(ii) is degenerate at points where $|u| = 0$ and need to have lower order terms because of the reason: $|u|^\tau\nabla u = 1/(1+\tau)\nabla(|u|^\tau u)$. The equation (i) represents, for example, a steady state of non-Newtonian fluid. The equation (ii), we here consider this type, represents, for example, a steady state of the flow of liquids in porous media where u is the density of the gas or the concentration of the liquid, and the degenerate order τ is a positive number: for example, $\tau = 1$ for thin saturated regions in porous media [22], $\tau = 3$ for thin liquid films spreading under gravity [8], and $\tau = 6$ for radiative heat transfer Marshak waves [16]. A steady state of mathematical model of ecology is represented by a system of the type (ii). Therefore, it is useful to understand the degenerate elliptic equations in order to do their parabolic versions. The author was interested in not only the behavior of solutions in the sense of natural phenomenon, but the intrinsic mathematical interest whether, if the coefficients of equations are more generalized, if the equations have lower order terms, and if the solutions are not necessarily nonnegative, the equation have a Hölder continuous solution (cf. [25, 14, 5]). In Complex Analysis of several values, the potential theory of degenerate elliptic equations is wanted to be studied. In Numerical Analysis, the lack of Hölder continuity of solution leads to difficulties to estimate the precision of approximate solutions. It is an important matter whether the degenerate elliptic system have a Hölder continuous solution or not. There are many results on the Hölder continuity of viscosity solutions for degenerate quasilinear equations, but they are not available to show these for systems. In this paper we shall construct a Hölder continuous weak solution. We first make a series of uniformly elliptic equations which approximate our system (1.1)-(1.2), and show that the solutions of these reguralized equations are Hölder continuous by using the auxiliary lemmas of Ladyzhenskaya-Ural'tseva (cf. [15, 25]). And our solutions are gained as the limit of the series of above solutions. We consider this problem only in the framework of the L_2-theory.

The notations of function spaces are usual ones (cf. [15]). We set a Hölder norm:

$$|\phi|_{\alpha,\mathcal{K}} \equiv \sup_{x\in\mathcal{K}} |\phi(x)| + \sup_{\substack{x,y\in\mathcal{K}\\ x\neq y}} \frac{|\phi(x)-\phi(y)|}{|x-y|^\alpha}$$

for a smooth ϕ defined on a compact set $\mathcal{K}$ of R^n, where $\alpha \in (0, 1)$.

2. ASSUMPTION AND THEOREM

We assume that the boundary $\partial\Omega$ satisfies the following condition:

$$\text{mes}\{K_\rho - \Omega \cap K_\rho\} \geq \theta_o \text{mes} K_\rho \tag{A}$$

for any ball K_ρ of radius ρ ($\leq \rho_o$) with the center on $\partial\Omega$, where ρ_o and θ_o are some positive constants (cf. [15, 25]). In this connection, under the weaker condition than the above, Hayasida-Yokoi [13] showed the Hölder continuity at the boundary for weak solutions of degenerate quasi-linear elliptic equations.

Moreover we assume that the coefficients $a^l_{ij}(x, \boldsymbol{u})$ belong to $C_{1,\alpha}(\Omega \times R^N) \cap C_{0,\alpha}(\bar{\Omega} \times R^N)$ $(0 < \alpha < 1)$, $b^l_j(x, \boldsymbol{u})$ and $b^l_0(x, \boldsymbol{u})$ belog to $C_{0,\alpha}(\Omega \times R^N)$, and they satisfy the following conditions:

$$|b^l_j(x, \boldsymbol{u})| \leq C_1 |\boldsymbol{u}|^{\tau/2}, \quad |b^l_0(x, \boldsymbol{u})| \leq C'_1 \quad \text{for} \quad (x, \boldsymbol{u}) \in \bar{\Omega} \times R^N \tag{A_2}$$

where C_1 and C'_1 are some positive constants, and

$$-\sum_{l=1}^{N} b^l_0(x, \boldsymbol{u}) u^l \leq -C_2 |\boldsymbol{u}|^2 + C_3 \quad \text{for} \quad (x, \boldsymbol{u}) \in \bar{\Omega} \times R^N \tag{A_3}$$

where C_2 and C_3 are some positive constants.

The functions $\psi^l(x)$, where $\boldsymbol{\psi} \equiv (\psi^1(x), \cdots, \psi^N(x))$ from the boundary condition (1.2), belong to $C_{2,\alpha}(\bar{\Omega})$ and satisfy the conditions:

$$|\psi^l(x)| \leq M' \quad \text{on} \quad \partial\Omega \tag{A_4}$$

where M' is a positive constant and $l = 1, \cdots, N$.

Now, we shall define a weak solution of the problem (1.1)-(1.2).

Definition. A vector valued function $\boldsymbol{u}(x) \equiv (u^1(x), \cdots, u^N(x))$ is called a weak solution of the problem (1.1)-(1.2), if the following conditions are fulfilled:

$$|\boldsymbol{u}|^{\tau/2+1}, \quad u^l |\boldsymbol{u}|^{\tau/2} \in W^1_2(\Omega), \tag{1}$$

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\psi}, \tag{2}$$

and the following integral identity holds:

$$\begin{aligned} \int_\Omega [\sum_{i,j=1}^{n} \frac{a^l_{ij}(x, \boldsymbol{u})}{|\boldsymbol{u}|^{\tau/2}} \{(u^l |\boldsymbol{u}|^{\tau/2})_{x_j} - \frac{\tau}{(\tau+2)} \frac{u^l}{|\boldsymbol{u}|} (|\boldsymbol{u}|^{\tau/2+1})_{x_j}\} \varphi^l_{x_i}(x) \\ + \sum_{j=1}^{n} \frac{b^l_j(x, \boldsymbol{u})}{|\boldsymbol{u}|^{\tau/2}} \{(u^l |\boldsymbol{u}|^{\tau/2})_{x_j} - \frac{\tau}{(\tau+2)} \frac{u^l}{|\boldsymbol{u}|} (|\boldsymbol{u}|^{\tau/2+1})_{x_j}\} \varphi^l(x) \\ + b^l_0(x, \boldsymbol{u}) \varphi^l(x)] dx = 0 \end{aligned} \tag{3}$$

for any $\varphi^l(x) \in \overset{\circ}{W}{}^1_2(\Omega)$ and $l = 1, \cdots, N$.

The main result of this paper is the following:

THEOREM. *Under the above conditions, there exists a bounded Hölder continuous weak solution* $\boldsymbol{u}(x)$ *of the Dirichlet problem* (1.1)-(1.2).

In order to prove the theorem, we make ε-regularized approximated systems of our problem (1.1)-(1.2), that is, we consider the following uniformly elliptic systems depending on the parameter $\varepsilon \in [0, 1]$;

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a^l_{\varepsilon ij}(x, \boldsymbol{u}(x)) u^l_{x_j}(x) - \sum_{j=1}^{n} b^l_j(x, \boldsymbol{u}(x)) u^l_{x_j}(x)$$

$$-b^l_0(x, \boldsymbol{u}(x)) = 0 \quad \text{in} \quad \Omega \tag{2.1}$$

where $l = 1, \cdots, N$, with the boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\psi} \tag{2.2}$$

where $a^l_{\varepsilon ij}(x, \boldsymbol{u}(x)) \equiv \varepsilon\delta_{ij} + f_\varepsilon(|\boldsymbol{u}|)a^l_{ij}(x, \boldsymbol{u})$, δ_{ij} is Kronecker's delta, and $f_\varepsilon(t)$ is non-decreasing and twice continuously differentiable function defined for $t \geq 0$;

$$f_\varepsilon(t) \equiv \begin{cases} 0 & \text{if} \quad 0 \leq t < (\varepsilon/4)^{1/\tau}, \\ 1 & \text{if} \quad t > (\varepsilon/2)^{1/\tau}. \end{cases} \tag{2.3}$$

From (2.3) and the condition (A_1), the following estimate is valid;

$$\frac{2C_0}{3}(\varepsilon + |\boldsymbol{u}|^\tau)|\boldsymbol{\xi}|^2 \leq \sum_{i,j=1}^{n} a^l_{\varepsilon ij}(x, \boldsymbol{u})\xi_i\xi_j \leq C_0^{-1}(\varepsilon + |\boldsymbol{u}|^\tau)|\boldsymbol{\xi}|^2 \tag{A_1'}$$

for $(x, \boldsymbol{u}) \in \overline{\Omega} \times R^N$ and any real vector $\boldsymbol{\xi} \equiv (\xi_1, \cdots, \xi_n)$. Now it is already known that for every $\varepsilon > 0$ there exists a solution $\boldsymbol{u}_\varepsilon(x)$ of the ε-regularized problem (2.1)-(2.2), whose components $u^l_\varepsilon(x)$ are in $C_{2,\alpha}(\Omega)$ and satisfy the following estimates:

$$|u^l_\varepsilon(x)| \leq M_1 \quad \text{for} \quad x \in \Omega \tag{2.4}$$

where $l = 1, \cdots, N$ and the constant $M_1 \equiv \max\{\max_{\partial\Omega}|\boldsymbol{\psi}|, (C_3/\min(1, C_2))^{1/2}\}$, that is, the estimates are uniformly bounded with respect to ε. (cf.[15] p.421)

3. AUXILIARY LEMMAS

In this section, we prepare several lemmas to see that a bounded function in $W_2^1(\Omega)$ satisfies the Hölder estimate (cf. [14, 15, 25]). Let K_ρ be an open ball of radius ρ in Ω. For a bounded function $f(x)$ on Ω, we denote by $A_{k,\rho}$ and $B_{k,\rho}$ the sets $\{x \in K_\rho : f(x) > k\}$ and $\{x \in K_\rho : f(x) < k\}$ respectively, and denote by $\mathrm{osc}_{K_\rho} f$ the essential oscillation of $f(x)$ on K_ρ.

LEMMA 3.1. ([15]) *Let $f(x) \in W_2^1(\Omega)$ satisfy the following inequalities for some positive constants M and C;*

$$|f(x)| \le M, \tag{3.1}$$

$$\int_{A_{k,\rho}} |\nabla f(x)|^2 \zeta^2(x)dx \le C \int_{A_{k,\rho}} \{(f(x)-k)^2 |\nabla \zeta(x)|^2 + \zeta^2(x)\}dx \tag{3.2}$$

for any ball $K_\rho \subset \Omega$, for any $\zeta(x) \in C_0^\infty(K_\rho)$ and for any number k such that

$$k \ge \sup_{K_\rho} f - \delta \mathrm{osc}_{K_\rho} f \tag{3.3}$$

where δ is a constant: $0 < \delta < 1$. Moreover, let

$$\mathrm{mes}\{x \in K_{\rho/2} : f(x) \le \sup_{K_\rho} f - \delta \mathrm{osc}_{K_\rho} f\} \ge \gamma \mathrm{mes} K_{\rho/2} \tag{3.4}$$

hold where γ is a constant: $0 < \gamma < 1$. Then, there exists a positive number s depending only on M, C, δ, and γ such that either

$$\mathrm{osc}_{K_{\rho/2}} f \le 2^s \rho \tag{3.5}$$

or

$$\mathrm{osc}_{K_{\rho/4}} f \le (1 - 2^{1-s}) \mathrm{osc}_{K_\rho} f \tag{3.6}$$

holds where the balls $K_{\rho/2}, K_{\rho/4}$ are concentric with K_ρ.

LEMMA 3.1$'$. ([15]) *Let $f(x) \in W_2^1(\Omega)$ satisfy the following inequalities for some positive constants M and C;*

$$|f(x)| \le M, \tag{3.1$'$}$$

$$\int_{B_{k,\rho}} |\nabla f(x)|^2 \zeta^2(x)dx \le C \int_{B_{k,\rho}} \{(f(x)-k)^2 |\nabla \zeta(x)|^2 + \zeta^2(x)\}dx \tag{3.2$'$}$$

for any ball $K_\rho \subset \Omega$, for any $\zeta(x) \in C_0^\infty(K_\rho)$ and for any number k such that

$$k \leq \inf_{K_\rho} f + \delta \mathrm{osc}_{K_\rho} f \tag{3.3$'$}$$

where δ is a constant: $0 < \delta < 1$. Moreover, let

$$\mathrm{mes}\{x \in K_{\rho/2} : f(x) \geq \inf_{K_\rho} f + \delta \mathrm{osc}_{K_\rho} f\} \geq \gamma \mathrm{mes} K_{\rho/2} \tag{3.4$'$}$$

hold where γ is a constant: $0 < \gamma < 1$. Then the same estimate as that of Lemma 3.1 holds.

LEMMA 3.2. ([25]) *Let $f(x) \in W_2^1(\Omega)$ be bounded, i.e., $|f(x)| \leq M$ for some positive constant M. For any positive constants C, δ and γ, it is possible to pick a positive number s depending only on C, δ, γ such that; if $f(x)$ satisfies the inequalities:*

$$\int_{B_{k,\rho} \setminus B_{h,\rho}} |\nabla f(x)|^2 \zeta^2(x) dx \leq C \int_{B_{k,\rho}} \{(f(x) - k)^2 |\nabla \zeta(x)|^2 + \zeta^2(x)\} dx \tag{3.7}$$

for any ball $K_\rho \subset \Omega$, for any $\zeta(x) \in C_0^\infty(K_\rho)$ and for any numbers h, k such that

$$h \in [\inf_{K_\rho} f + \frac{\mathrm{osc}_{K_\rho} f}{2^s}, \inf_{K_\rho} f + \delta \mathrm{osc}_{K_\rho} f], \quad k \in [h, 2h - \inf_{K_\rho} f], \tag{3.8}$$

and

$$\mathrm{mes}\{x \in K_{\rho/2} : f(x) \geq \inf_{K_\rho} f + \delta \mathrm{osc}_{K_\rho} f\} \geq \gamma \mathrm{mes} K_{\rho/2}, \tag{3.9}$$

then either

$$\mathrm{osc}_{K_{\rho/2}} f \leq 2^s \rho \tag{3.10}$$

or

$$\mathrm{osc}_{K_{\rho/4}} f \leq (1 - 2^{1-s}) \mathrm{osc}_{K_\rho} f \tag{3.11}$$

holds where the balls $K_{\rho/2}$, $K_{\rho/4}$ are concentric with K_ρ.

LEMMA 3.3. [15] *Let $\boldsymbol{f}(x) \equiv (f^1(x), \cdots, f^N(x))$ be a vector-valued function on $\Omega_{\rho_o} \equiv K_{\rho_o} \cap \Omega$. Suppose that for any $\Omega_\rho \equiv K_\rho \cap \Omega$ where $\rho \leq \rho_o$, that is, K_ρ is concentric with K_{ρ_o}, there exists a function $f^{l_\rho}(x)$ in the family $\{\pm f^1(x), \cdots, \pm f^N(x)\}$, i.e., $l_\rho \in (1, \cdots, N)$, such that*

$$\mathrm{osc}_{\Omega_\rho} f^{l_\rho} \geq \delta_1 \max_{1 \leq l \leq N} \mathrm{osc}_{\Omega_\rho} f^l \tag{3.12}$$

holds and at least one of the following inequalities holds;

$$\text{osc}_{\Omega_{\rho/4}} f^{l_\rho} \leq c_1 \rho, \tag{3.13}$$

$$\text{osc}_{\Omega_{\rho/4}} f^{l_\rho} \leq \theta \ \text{osc}_{\Omega_\rho} f^{l_\rho}, \tag{3.14}$$

where δ_1, c_1 *and* θ *are some positive constants, also* $\theta < 1$, *and the ball* $K_{\rho/4}$ *is concentric with* K_ρ. *Then, there exists a constant* α' $(0 < \alpha' < 1)$ *depending on* N *and* θ *such that for any positive number* ρ $(\leq \rho_o)$,

$$\text{osc}_{K_\rho} f^l \leq c {\rho_o}^{-\alpha'} \rho^{\alpha'} \quad \textit{for all} \quad l \in (1, \cdots, N) \tag{3.15}$$

holds, where the constant c *depends on* α', N, δ_1, c_1, ρ_o *and* $\max_{1 \leq l \leq N}\{\text{osc}_{\Omega_{\rho_o}} f^l\}$.

For the proofs of these lemmas, we refer to Lemmas 6.1, 6.2 and 7.1 of Chapter 2 in [15] and [25].

Remark. For any ball K_ρ with the center on $\partial\Omega$, Lemmas 3.1, 3.1′, and 3.2 remain also valid on $\Omega_\rho \equiv K_\rho \cap \Omega$, if we assume the following additional conditions:

$$\sup_{\partial\Omega\cap K_\rho} f \leq \sup_{\Omega\cap K_\rho} f - \delta \text{osc}_{\Omega\cap K_\rho} f \qquad \text{(in Lemma 3.1)},$$

$$\inf_{\partial\Omega\cap K_\rho} f \geq \inf_{\Omega\cap K_\rho} f + \delta \text{osc}_{\Omega\cap K_\rho} f \qquad \text{(in Lemmas 3.1}', \text{3.2)},$$

since the above assumptions and the boundary condition (A) in section 2 yield the conditions on the inequalities (3.2),(3.2)′ and (3.7), and the conditions (3.4), (3.4)′ and (3.9) in Lemmas 3.1, 3.1′ and 3.2, respectively.

CHAPTER II

This chapter is concerned with the degenerate order $0 < \tau \leq 1$.

4. INTEGRAL INEQUALITIES

Let us multiply the equation (2.1) by $\varphi(x) \in \overset{\circ}{W}{}_2^1(\Omega)$, and integrate it over Ω, then we have

$$\int_\Omega \{ \sum_{i,j=1}^n a^l_{\varepsilon ij}(x, \boldsymbol{u}_\varepsilon) u^l_{\varepsilon x_j} \varphi_{x_i} + \sum_{j=1}^n b^l_j(x, \boldsymbol{u}_\varepsilon) u^l_{\varepsilon x_j} \varphi + b^l_0(x, \boldsymbol{u}_\varepsilon)\varphi \} dx = 0. \tag{4.1}$$

We prepare the following integral inqualities for $0 < \tau \leq 1$.

LEMMA 4.1. *Let $\boldsymbol{u}_\varepsilon(x) \equiv (u^1_\varepsilon(x), \cdots, u^N_\varepsilon(x))$ be a solution of the ε-regularized problem (2.1)-(2.2) satisfying (2.4). Then, the functions $\pm u^l_\varepsilon(x)$ satisfy the following integral inequalities: for any $\zeta(x) \in C_0^\infty(K_\rho)$ where $K_\rho \subset \Omega$ and for any $k : \inf_{K_\rho} u^l_\varepsilon \leq k \leq \sup_{K_\rho} u^l_\varepsilon$ where $l \in (1, \cdots, N)$,*

$$\begin{aligned} &\inf_{A_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{u^l_\varepsilon > k} |\nabla u^l_\varepsilon|^2 \zeta^2 dx \\ &\leq C_{(1)} \sup_{A_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{u^l_\varepsilon > k} \{(u^l_\varepsilon - k)^2 |\nabla\zeta|^2 + \zeta^2\} dx, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\inf_{B} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{B} |\nabla u^l_\varepsilon|^2 \zeta^2 dx \\ &\leq C_{(2)} \sup_{B_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{u^l_\varepsilon < k} \{(u^l_\varepsilon - k)^2 |\nabla\zeta|^2 + \zeta^2\} dx, \end{aligned} \tag{4.3}$$

where the constants $C_{(1)}$ and $C_{(2)}$ depend only on $\tau, n, N, M_1, C_1, C'_1, C_2$ and C_3. The sets $A_{k,\rho} \equiv \{x \in K_\rho : u^l_\varepsilon > k\}$, $B_{k,\rho} \equiv \{x \in K_\rho : u^l_\varepsilon < k\}$ and the set B denote $B_{k,\rho}$ or $B_{k,\rho} - B_{h,\rho}$ in (4.3).

Proof. For simplicity, $\boldsymbol{u}_\varepsilon(x)$ and $\pm u^l_\varepsilon(x)$ are denoted by $\boldsymbol{u}(x)$ and $u(x)$, respectively. As a test function $\varphi(x)$ in (4.1), we take the following one:

$$\varphi_1(x) \equiv \max\{u(x) - k, 0\}\zeta^2(x) \in \overset{\circ}{W}{}_2^1(\Omega)$$

for $k : \inf_{K_\rho} u \leq k \leq \sup_{K_\rho} u$, then we have

$$\int_{u>k} [\sum_{i,j=1}^n \{ a^l_{\varepsilon ij} u_{x_j} u_{x_i} \zeta^2 + 2a^l_{\varepsilon ij} u_{x_j} (u-k)\zeta\zeta_{x_i} \}$$

$$+\sum_{j=1}^{n} b_j^l u_{x_j}(u-k)\zeta^2 + b_0^l(u-k)\zeta^2]dx = 0. \tag{4.4}$$

Applying the conditions (A_1') and (A_2) to (4.4), we have

$$\frac{2}{3}C_0\int_{u>k}(\varepsilon+|\boldsymbol{u}|^\tau)|\nabla u|^2\zeta^2 dx$$

$$\leq \int_{u>k}\{2C_0^{-1}(\varepsilon+|\boldsymbol{u}|^\tau)|\nabla u||\nabla\zeta|(u-k)\zeta + nC_1|\boldsymbol{u}|^{\tau/2}|\nabla u|(u-k)\zeta^2 + C_1'(u-k)\zeta^2\}dx.$$

Using Young's inequality, we have

$$\int_{u>k}(\varepsilon+|\boldsymbol{u}|^\tau)|\nabla u|^2\zeta^2 dx$$

$$\leq C_4\int_{u>k}[(\varepsilon+|\boldsymbol{u}|^\tau)(u-k)^2|\nabla\zeta|^2 + \{(u-k)^2+(u-k)\}\zeta^2]dx.$$

This yields (4.2), since u is bounded and $0<\tau\leq 1$.

Next, as a test function $\varphi(x)$ in (4.1), we take the following one:

$$\varphi_2(x) \equiv \max\{k-u(x),0\}\zeta^2(x) \in \overset{\circ}{W}{}_2^1(\Omega)$$

for $k : \inf_{K_\rho} u \leq k \leq \sup_{K_\rho} u$, then we have

$$\int_{u<k}[\sum_{i,j=1}^{n}\{-a_{\varepsilon ij}^l u_{x_j}u_{x_i}\zeta^2 + 2a_{\varepsilon ij}^l u_{x_j}(k-u)\zeta\zeta_{x_i}\}$$

$$+\sum_{j=1}^{n} b_j^l u_{x_j}(k-u)\zeta^2 + b_0^l(k-u)\zeta^2]dx = 0. \tag{4.5}$$

Applying the conditions (A_1') and (A_2) to (4.5), we have

$$\frac{2}{3}C_0\int_{u<k}(\varepsilon+|\boldsymbol{u}|^\tau)|\nabla u|^2\zeta^2 dx$$

$$\leq \int_{u<k}\{2C_0^{-1}(\varepsilon+|\boldsymbol{u}|^\tau)|\nabla u||\nabla\zeta|(k-u)\zeta + nC_1|\boldsymbol{u}|^{\tau/2}|\nabla u|(k-u)\zeta^2 + C_1'(k-u)\zeta^2\}dx.$$

Using Young's inequality, we have

$$\int_{u<k}(\varepsilon+|\boldsymbol{u}|^\tau)|\nabla u|^2\zeta^2 dx$$

$$\leq C_5\int_{u<k}[(\varepsilon+|\boldsymbol{u}|^\tau)(k-u)^2|\nabla\zeta|^2 + \{(k-u)^2+(k-u)\}\zeta^2]dx.$$

This yields (4.3) for $B_{k,\rho}$ and $B_{k,\rho}-B_{h,\rho}$, since u is bounded and $0<\tau\leq 1$.

5. HÖLDER ESTIMATES

LEMMA 5.1. *Let $\boldsymbol{u}_\varepsilon(x)$ be a solution of the ε-regularized problem* (2.1)-(2.2) *satisfying* (2.4) *where $\boldsymbol{u}_\varepsilon(x) = (u_\varepsilon^1(x), \cdots, u_\varepsilon^N(x))$ and $u_\varepsilon^l(x) \in C_{2,\alpha}(\Omega)$ for $l = 1, \cdots, N$. Then, there exist some positive constants C_6 and $\beta \in (0, \alpha)$ which are independent of ε such that*

$$|\boldsymbol{u}_\varepsilon(x)|_{\beta,\bar{\Omega}} \leq C_6, \tag{5.1}$$

where $|\cdot|_{\beta,\bar{\Omega}}$ denotes a Hölder norm with exponent β in Ω.

Lemma 5.1 follows from Lemma 3.3 and the following lemma.

LEMMA 5.2. *Let $\boldsymbol{u}_\varepsilon(x)$ be a solution of the ε-regularized problem* (2.1)-(2.2) *satisfying* (2.4), *and consider the following functions from the components of it;*

$$\pm u_\varepsilon^l(x) \qquad for \quad l = 1, \cdots, N. \tag{5.2}$$

Then, for any ball $K_\rho \subset \Omega$, there exists at least one function satisfying the premises of Lemma 3.3 in the family (5.2).

Proof. We fix an arbitrary ball $K_\rho \subset \Omega$. From every pair of functions $\pm u_\varepsilon^l(x)$, we keep only the functions for which the greater parts of these oscillation in K_ρ fall in the domain of positive values; without loss of generality we denote these functions by $u^l(x)$, with regard to which we know that

$$\inf_{K_\rho} u^l + \frac{\omega^l}{2} \quad \geq 0 \tag{5.3}$$

where $\omega^l \equiv \operatorname{osc}_{K_\rho} u^l$ for $l = 1, \cdots, N$. Let $l_M \in (1, \cdots, N)$ be the number such that $\omega^{l_M} = \max_{l=1,\cdots,N} \omega^l$. We indicate a constant q (> 0) determined exclusively by C_0 and τ in the condition (A_1) as below. It is sufficient to consider the following three cases for the family $\{u^1, \cdots, u^N\}$;

Case I : $\quad \forall u^l : \quad |\inf_{K_\rho} u^l| \leq \dfrac{\omega^{l_M}}{2^q} \quad$ for all $\quad l \in (1, \cdots, N)$,

Case II : $\quad \exists u^{l_o} : \quad \inf_{K_\rho} u^{l_o} > \dfrac{\omega^{l_M}}{2^q} \quad$ for some $\quad l_o \in (1, \cdots, N)$, Case III : $\exists u^{l_*}$: $\inf_{K_\rho} u^{l_*} < -\dfrac{\omega^{l_M}}{2^q}$ for some $l_* \in (1, \cdots, N)$.

At first, we consider Case I, in this case the assertion of this lemma is proved for u^{l_M}. The following relations hold;

$$\omega^l \le \omega^{l_M} \quad \text{for all} \quad l \in (1, \cdots, N), \tag{5.4}$$

$$\sup_{K_\rho} |\boldsymbol{u}| \le \sum_{l=1}^{N} |\sup_{K_\rho} u^l - \inf_{K_\rho} u^l + \inf_{K_\rho} u^l|$$

$$\le N\omega^{l_M} + \frac{N\omega^{l_M}}{2^q} = N(1 + \frac{1}{2^q})\omega^{l_M} < 2N\omega^{l_M} \quad (q > 1). \tag{5.5}$$

From (5.4), the premise (3.13) of Lemma 3.3 holds with $\delta_1 = 1$.

Let $k_o \equiv \sup_{K_\rho} u^{l_M} - \omega^{l_M}/4$ and $A_{k,\rho} \equiv \{x \in K_\rho : u^{l_M}(x) > k\}$. For all $k : k_o \le k \le \sup_{K_\rho} u^{l_M}$, we have

$$\inf_{A_{k\rho}} |\boldsymbol{u}| \ge \inf_{A_{k,\rho}} |u^{l_M}| \ge k \ge \inf_{K_\rho} u^{l_M} + \frac{\omega^{l_M}}{2} + \frac{\omega^{l_M}}{4} \ge \frac{\omega^{l_M}}{4}. \tag{5.6}$$

Hence, from (5.5) and (5.6) we have

$$\sup_{K_\rho} |\boldsymbol{u}| \le 8N \inf_{A_{k,\rho}} |\boldsymbol{u}| \quad \text{for} \quad k : k_o \le k \le \sup_{K_\rho} u^{l_M}. \tag{5.7}$$

From (5.7), as well as from inequality (4.2), for $u^{l_M}(x)$ we arrive at the inequality (3.2) of Lemma 3.1 with $C = 8N$ for any $k \ge k_o \equiv \sup_{K_\rho} u^{l_M} - \omega^{l_M}/4$.

Next, let take h and k such that $\omega^{l_M}/2^s \le h \le k \le 3h$ and $B_{k,\rho} \equiv \{x \in K_\rho : u^{l_M}(x) < k\}$ where s is a number indicated in Lemma 3.2. Then we have

$$\inf_{B_{k,\rho} - B_{h,\rho}} |\boldsymbol{u}| \ge \inf_{B_{k,\rho} - B_{h,\rho}} |u^{l_M}| \ge h \ge \frac{\omega^{l_M}}{2^s}. \tag{5.8}$$

Hence, from (5.5) and (5.8) we have

$$\sup_{K_\rho} |\boldsymbol{u}| \le 2^{s+1} N \inf_{B_{k,\rho} - B_{h,\rho}} |\boldsymbol{u}|. \tag{5.9}$$

From (5.9), as well as from inequality (4.3), for $u^{l_M}(x)$ we arrive at the inequality (3.8) of Lemma 3.2 with $C = 2^{s+1}N$ for any $h \ge \omega^{l_M}/2^s$ and $k \in [h, 3h]$.

Moreover, it is apparent that one of the following two relations is true;

$$\text{mes}\{x \in K_{\rho/2} : u^{l_M}(x) \le k_o\} \ge \frac{1}{2}\text{mes}K_{\rho/2}, \tag{a}$$

$$\mathrm{mes}\{x \in K_{\rho/2} : u^{l_M}(x) \geq k_o\} \geq \frac{1}{2}\mathrm{mes}K_{\rho/2}. \qquad (b)$$

If the case (a) is true, the value $k_o \equiv \sup_{K_\rho} u^{l_M} - \omega^{l_M}/4$ obeys postulates (3.3) and (3.4) of Lemma 3.1 with the constants $\delta = 1/4, \gamma = 1/2$, hence the assertion of Lemma 3.1 follows for u^{l_M}. If the case(b) is true, the value k_o obeys postulates (3.9) and (3.10) of Lemma 3.2 with the constants $\delta = 3/4, \gamma = 1/2$, hence the assertion of Lemma 3.2 follows for u^{l_M}. Let s be the larger one of the two numbers dictated by Lemma 3.1 and Lemma 3.2, then Lemma 3.1 and Lemma 3.2 guarantee the premise (3.14) or (3.15) of Lemma 3.3 for $u^{l_M}(x)$.

Secondly, we consider Case II, in this case the assertion of this lemma is proved for u^{l_M} too. The following relations hold:

$$\sup_{K_\rho} |\boldsymbol{u}| \leq \inf_{K_\rho} |\boldsymbol{u}| + N\omega^{l_M} \leq N(\inf_{K_\rho} |\boldsymbol{u}| + \omega^{l_M}), \qquad (5.10)$$

$$\inf_{K_\rho} |\boldsymbol{u}| \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{u}| + \inf_{K_\rho} |u^{l_o}|) \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{u}| + \frac{\omega^{l_M}}{2^q}). \qquad (5.11)$$

Hence, we have, for all $k : \inf_{K_\rho} u^{l_M} \leq k \leq \sup_{K_\rho} u^{l_M}$,

$$\sup_{K_\rho} |\boldsymbol{u}| \leq 2^{q+1}N \inf_{K_\rho} |\boldsymbol{u}| \leq \begin{cases} 2^{q+1}N \inf_{A_{k\rho}} |\boldsymbol{u}|, \\ 2^{q+1}N \inf_{B_{k\rho}} |\boldsymbol{u}|. \end{cases} \qquad (5.12)$$

From (5.12), as well as from inequality (4.2) or (4.3), we arrive at the inequality (3.2) of Lemma 3.1 or $(3.2)'$ of Lemma $3.1'$. Since either condition (a) or (b) is true, the premise (3.14) or (3.15) of Lemma 3.3 is guaranteed for $u^{l_M}(x)$.

Lastly, we consider Case III, in this case the of assertion of this Lemma is proved for u^{l_*}. From the condition (5.3) we have the estimate:

$$\omega^{l_*} \geq \frac{\omega^{l_M}}{2^{q-1}} \geq \frac{\omega^l}{2^{q-1}} \quad \text{for} \quad l = 1, \cdots, N, \qquad (5.13)$$

i.e., u^{l_*} satisfies the premise (3.13) of Lemma 3.3 with $\delta_1 = 2^{-q+1}$.

Let $k_* \equiv \sup_{K_\rho} u^{l_*} - \omega^{l_*}/4 \; (> \omega^{l_*}/4)$ and $A_{k,\rho} \equiv \{x \in K_\rho ; u^{l_*}(x) > k\}$. The following relation holds for $k : k_* \leq k \leq \sup_{K_\rho} u^{l_*}$;

$$\inf_{A_{k,\rho}} |\boldsymbol{u}| \geq \frac{1}{2}(\inf_{A_{k,\rho}} |\boldsymbol{u}| + \inf_{A_{k,\rho}} |u^{l_*}|) \geq \frac{1}{2}(\inf_{A_{k,\rho}} |\boldsymbol{u}| + k_*) \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{u}| + \frac{\omega^{l_M}}{2^{q+1}}). \qquad (5.14)$$

Hence, from (5.10) and (5.14) we have

$$\sup_{K_\rho} |\boldsymbol{u}| \leq 2^{q+2}N \inf_{A_{k,\rho}} |\boldsymbol{u}| \quad \text{for} \quad k : k_* \leq k \leq \sup_{K_\rho} u^{l_*}. \qquad (5.15)$$

Let $k'_* \equiv \inf_{K_\rho} u^{l_*} + \omega^{l_*}/2^{q+1}$ $(< -\omega^{l_*}/2^{q+1})$ and $B_{k,\rho} \equiv \{x \in K_\rho : u^{l_*}(x) < k\}$. The following relation holds for $k : \inf_{K_\rho} u^{l_*} \le k \le k'_*$;

$$\inf_{B_{k,\rho}} |\boldsymbol{u}| \ge \frac{1}{2}(\inf_{B_{k,\rho}} |\boldsymbol{u}| + \inf_{B_{k,\rho}} |u^{l_*}|) \ge \frac{1}{2}(\inf_{B_{k,\rho}} |\boldsymbol{u}| + |k|) \ge \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{u}| + \frac{\omega^{l_*}}{2^{q+1}}). \quad (5.16)$$

Hence, from (5.10) and (5.16) we have

$$\sup_{K_\rho} |\boldsymbol{u}| \le 2^{q+2} N \inf_{B_{k,\rho}} |\boldsymbol{u}| \quad \text{for} \quad k : \inf_{K_\rho} u^{l_*} \le k \le k'_*. \quad (5.17)$$

It is also apparent that one of the following two relations is valid;

$$\text{mes}\{x \in K_{\rho/2} : u^{l_*}(x) \le k_*\} \ge \frac{1}{2}\text{mes}K_{\rho/2}, \quad (a')$$

$$\text{mes}\{x \in K_{\rho/2} : u^{l_*}(x) \ge k'_*\} \ge \frac{1}{2}\text{mes}K_{\rho/2}. \quad (b')$$

From (5.15) or (5.17), as well as from inequality (4.2) or (4.3), we arrive at the inequality (3.2) of Lemma 3.1 or (3.2)$'$ of Lemma 3.1$'$, and hence the premise (3.14) or (3.15) of Lemma 3.3 is guaranteed for $u^{l_*}(x)$. This completes the proof of Lemma 5.2.

Considering Lemma 5.2 and Lemma 3.3, we obtain the interior estimates of $u^l_\varepsilon(x)$ which are uniform on $\varepsilon \in [0, 1]$ for $l = 1, \cdots, N$. Applying the conditions (A) and Remark in Section 3 to the above argument, we obtain the estimates of $u^l_\varepsilon(x)$ in the region adjacent to the boundary. Consequently, we have the estimates:

$$|u^l_\varepsilon|_{\beta,\bar{\Omega}} \le C_7 \quad \text{for} \quad l = 1, \cdots, N \quad (5.18)$$

for some constants C_7 and $\beta \in (0, \alpha)$ which are independent of ε. This proves Lemma 5.1 for $0 < \tau \le 1$.

6. PROOF OF THEOREM FOR $0 < \tau \le 1$

Let $u^l_\varepsilon(x) \in C_{2,\alpha}(\bar{\Omega})$ be the solutions of the ε-regularized problem (2.1)-(2.2) satisfying (2.4), then by virtue of (5.1) we have

$$|u^l_\varepsilon|_{\beta,\bar{\Omega}} \le C_6 \quad \text{for} \quad l = 1, \cdots, N \quad (6.1)$$

where the constants C_6 and β are independent of ε. Since $u^l_\varepsilon(x) - \psi^l(x) \in \overset{\circ}{W}{}^1_2$ (Ω), from (4.1) we have

$$\int_\Omega [\sum_{i,j=1}^n a^l_{\varepsilon ij}(x, \boldsymbol{u}_\varepsilon) u^l_{\varepsilon x_j}(u^l_{\varepsilon x_i} - \psi^l_{x_i})$$

$$+\{\sum_{j=1}^n b^l_j(x, \boldsymbol{u}_\varepsilon) u^l_{\varepsilon x_j} + b^l_0(x, \boldsymbol{u}_\varepsilon)\}(u^l_\varepsilon - \psi^l)]dx = 0.$$

By the use of assumptions (A'_1), (A_2) and (2.4), we have

$$\int_\Omega (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \sum_{l=1}^N |\nabla u^l_\varepsilon|^2 dx \le C_8 \tag{6.2}$$

where a constant C_8 is independent of ε. Consider the following functions:

$$V_\varepsilon(x) \equiv |\boldsymbol{u}_\varepsilon|^{\tau/2+1}, \quad V^l_\varepsilon(x) \equiv u^l_\varepsilon |\boldsymbol{u}_\varepsilon|^{\tau/2} \tag{6.3}$$

for $l = 1, \cdots, N$, and we have the following estimates:

$$|V_{\varepsilon x_j}| = |(1 + \frac{\tau}{2})|\boldsymbol{u}_\varepsilon|^{\tau/2-1} \sum_{r=1}^N u^r_\varepsilon u^r_{\varepsilon x_j}| \le (1 + \frac{\tau}{2})|\boldsymbol{u}_\varepsilon|^{\tau/2} (\sum_{r=1}^N |\nabla u^r_\varepsilon|^2)^{1/2},$$

$$|V^l_{\varepsilon x_j}| = |u^l_{\varepsilon x_j}|\boldsymbol{u}_\varepsilon|^{\tau/2} + \frac{\tau}{2} u^l_\varepsilon |\boldsymbol{u}_\varepsilon|^{\tau/2-2} \sum_{r=1}^N u^r_\varepsilon u^r_{\varepsilon x_j}| \le (1 + \frac{\tau}{2})|\boldsymbol{u}_\varepsilon|^{\tau/2} (\sum_{r=1}^N |\nabla u^r_\varepsilon|^2)^{1/2}.$$

Hence, from (6.2) we deduce the uniform boundedness of the integrals:

$$\int_\Omega \{|V_{\varepsilon x_j}|^2 + \sum_{l=1}^N |V^l_{\varepsilon x_j}|^2\} dx \le C_9 \quad \text{for} \quad j = 1, \cdots, n \tag{6.4}$$

where a constant C_9 is independent of ε. On the basis of the estimates (6.1) and (6.4), there exists a subsequence $\{u^l_{\varepsilon_p}\}$ of $\{u^l_\varepsilon\}$ such that as $\varepsilon_p \to 0$,

$$u^l_{\varepsilon_p} \to \quad u^l \quad \text{in} \quad C_{0,\beta}(\Omega), \tag{6.5}$$

$$(V_{\varepsilon_p})_{x_j} \to V_{x_j} \quad \text{weakly in} \quad L_2(\Omega), \tag{6.6}$$

$$(V^l_{\varepsilon_p})_{x_j} \to V^l_{x_j} \quad \text{weakly in} \quad L_2(\Omega), \tag{6.7}$$

where some $u^l(x) \in C_{0,\beta}(\bar\Omega)$, $V(x) \equiv |\boldsymbol{u}|^{\tau/2+1}$ and $V^l(x) \equiv u^l|\boldsymbol{u}|^{\tau/2}$ for $l = 1, \cdots, N$. Therefore, the functions $u^l(x) \in C_{0,\beta}(\bar\Omega)$ satisfy the identity (3) for any $\varphi^l \in \overset{\circ}{W}{}^1_2(\Omega)$, since the functions $u^l_{\varepsilon_p}$ satisfy the same integral identity as (3) derived from the ε-regularized equations (2.1)-(2.2) and it is possible to pass the limit as $\varepsilon_p \to 0$ in this form. This completes the proof of Theorem for $0 < \tau \le 1$.

CHAPTER III

This chapter is concerned with the degenerate order $\tau \geq 1$.

7. INTEGRAL INEQUALITIES

Let us multiply the equation (2.1) by $\varphi(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ and integrate it over Ω, and we have

$$\int_\Omega \{ \sum_{i,j=1}^n a^l_{\varepsilon ij}(x, \boldsymbol{u}_\varepsilon) u^l_{\varepsilon x_j} \varphi_{x_i} + \sum_{j=1}^n b^l_j(x, \boldsymbol{u}_\varepsilon) u^l_{\varepsilon x_j} \varphi + b^l_0(x, \boldsymbol{u}_\varepsilon)\varphi \} dx = 0 \tag{7.1}$$

where $l = 1, \cdots, N$. From this, we have the following integral inequalities.

LEMMA 7.1. *Let $\boldsymbol{u}_\varepsilon(x) \equiv (u^1_\varepsilon(x), \cdots, u^N_\varepsilon(x))$ be a solution of the ε-regularized problem* (2.1)-(2.2) *satisfying* (2.4). *We consider the following $2N$ functions for $\tau \geq 1$;*

$$U^l(x) \equiv \pm u^l_\varepsilon(x) |u^l_\varepsilon(x)|^{\tau-1} \quad for\ l = 1, \cdots, N. \tag{7.2}$$

Then these functions satisfy the following integral inequalities for any $\zeta(x) \in C_0^\infty(K_\rho)$ where $K_\rho \subset \Omega$, and for any numbers k and h : $\inf_{K_\rho} U^l \leq h \leq k \leq \sup_{K_\rho} U^l$;

(i) *if $k > h \geq 0$, then*

$$\inf_{B_{k,\rho} - B_{h,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{B_{k,\rho} - B_{h,\rho}} |\nabla U^l|^2 \zeta^2 dx$$

$$\leq C_{(1)} \sup_{B_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{B_{k,\rho}} \{ (k - U^l)^2 |\nabla \zeta|^2 + \zeta^2 \} dx, \tag{7.3}$$

(ii) *if $k < 0$, then*

$$\inf_{B_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{B_{k,\rho}} |\nabla U^l|^2 \zeta^2 dx$$

$$\leq C_{(2)} \sup_{B_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{B_{k,\rho}} \{ (k - U^l)^2 |\nabla \zeta|^2 + \zeta^2 \} dx, \tag{7.4}$$

(iii) *if $k > 0$, then*

$$\inf_{A_{k,\rho}} (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{A_{k,\rho}} |\nabla U^l|^2 \zeta^2 dx$$

$$\leq C_{(3)} \sup_{A_{k,\rho}}(\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \int_{A_{k,\rho}} \{(U^l - k)^2 |\nabla\zeta|^2 + \zeta^2\} dx, \tag{7.5}$$

where $A_{k,\rho} \equiv \{x \in K_\rho : U^l(x) > k\}$, $B_{k,\rho} \equiv \{x \in K_\rho : U^l(x) < k\}$, *and the constants* $C_{(1)}$, $C_{(2)}$ *and* $C_{(3)}$ *depend only on* $\tau, n, N, M_1, C_1, C_1', C_2$ *and* C_3.

Proof. For simplicity, $\boldsymbol{u}_\varepsilon(x)$ and $\pm u_\varepsilon^l(x)$ are denoted by $\boldsymbol{u}(x)$ and $u(x)$, respectively. For a number k, we define the number ν such that $k = \nu|\nu|^{\tau-1}$, i.e., $\nu = k|k|^{(1-\tau)/\tau}$, then $k > k'$ implies $\nu > \nu'$ where $k' = \nu'|\nu'|^{\tau-1}$, and vice versa.

Firstly, as a test function $\varphi(x)$ in (7.1), we take the following one:

$$\varphi_1(x) \equiv |\nu|^{2\tau-2} \max\{\nu - u(x), 0\}\zeta^2(x) \in \overset{\circ}{W}{}_2^1(\Omega)$$

where $\nu : \inf_{K_\rho} u \leq \nu \leq \sup_{K_\rho} u$, then we have

$$\int_{u<\nu} [\sum_{i,j=1}^{n} \{-a_{\varepsilon ij}^l u_{x_j} |\nu|^{2\tau-2} u_{x_i} \zeta^2 + 2a_{\varepsilon ij}^l u_{x_j} |\nu|^{2\tau-2}(\nu - u)\zeta\zeta_{x_i}\}$$

$$+ \sum_{j=1}^{n} b_j^l u_{x_j} |\nu|^{2\tau-2}(\nu - u)\zeta^2 + b_0^l |\nu|^{2\tau-2}(\nu - u)\zeta^2] dx = 0. \tag{7.6}$$

Applying the conditions (A_1') and (A_2), we have

$$\frac{2}{3} C_0 \int_{u<\nu} (\varepsilon + |\boldsymbol{u}|^\tau)|\nu|^{2\tau-2} |\nabla u|^2 \zeta^2 dx$$

$$\leq \int_{u<\nu} \{2C_0^{-1}(\varepsilon + |\boldsymbol{u}|^\tau)|\nabla u||\nabla\zeta||\nu|^{2\tau-2}(\nu - u)\zeta$$

$$+ nC_1 |\boldsymbol{u}|^{\tau/2} |\nabla u||\nu|^{2\tau-2}(\nu - u)\zeta^2 + C_1'|\nu|^{2\tau-2}(\nu - u)\zeta^2\} dx.$$

Using Young's inequality, we have

$$\int_{u<\nu} (\varepsilon + |\boldsymbol{u}|^\tau)|\nu|^{2\tau-2} |\nabla u|^2 \zeta^2 dx$$

$$\leq C_5 \int_{u<\nu} [(\varepsilon + |\boldsymbol{u}|^\tau)|\nu|^{2\tau-2}(\nu - u)^2 |\nabla\zeta|^2 + \{|\nu|^{2\tau-2}(\nu - u)^2 + |\nu|^{2\tau-2}(\nu - u)\}\zeta^2] dx. \tag{7.7}$$

where C_5 is a positive constant. By virtue of the inequality:

$$|\nu|^{\tau-1}(\nu - u) \leq 2(\nu|\nu|^{\tau-1} - u|u|^{\tau-1}) \quad \text{if} \quad u < \nu,$$

we have (7.3) from (7.7) provided that $k > h \geq 0$, since u is bounded and $\tau \geq 1$.

Secondly, as a test function $\varphi(x)$ in (7.1), we take the following one:

$$\varphi_2(x) \equiv \max\{\nu|\nu|^{2\tau-2} - u(x)|u(x)|^{2\tau-2}, 0\}\zeta^2(x) \in \overset{\circ}{W}{}_2^1(\Omega)$$

where $\nu : \inf_{K_\rho} u \leq \nu \leq \sup_{K_\rho} u$, then we have

$$\int_{u<\nu} [\sum_{i,j=1}^{n} \{-(2\tau-1)a^l_{\varepsilon ij} u_{x_j} u_{x_i} |u|^{2\tau-2}\zeta^2 + 2a^l_{\varepsilon ij} u_{x_j} (\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2})\zeta\zeta_{x_i}\}$$

$$+ \sum_{j=1}^{n} b^l_j u_{x_j} (\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2})\zeta^2 + b^l_0(\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2})\zeta^2] dx = 0.$$

Applying the conditions (A_1') and (A_2), we have

$$\frac{2}{3}C_0(2\tau-1)\int_{u<\nu}(\varepsilon + |\boldsymbol{u}|^\tau)(|u|^{\tau-1}|\nabla u|)^2\zeta^2 dx$$

$$\leq \int_{u<\nu} \{2C_0^{-1}(\varepsilon + |\boldsymbol{u}|^\tau)|\nabla u||\nabla\zeta|(\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2})\zeta$$

$$+nC_1|\boldsymbol{u}|^{\tau/2}|\nabla u|(\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2})\zeta^2 + C_1'(\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2})\zeta^2\}dx. \quad (7.8)$$

Applying the following inequality:

$$\nu|\nu|^{2\tau-2} - u|u|^{2\tau-2} \leq 2|u|^{\tau-1}(\nu|\nu|^{\tau-1} - u|u|^{\tau-1}) \quad \text{if} \quad u < \nu \leq 0,$$

and using Young's inequality, we have (7.4) from (7.8), since u is bounded and $\tau \geq 1$.

Lastly, as a test function $\varphi(x)$ in (7.1), we take the following one:

$$\varphi_3(x) \equiv |\nu|^{2\tau-2}\max\{u(x) - \nu, 0\}\zeta^2(x) \in \overset{\circ}{W}{}_2^1(\Omega),$$

then we have (7.5) in the same way as the first one.

8. HÖLDER ESTIMATES

LEMMA 8.1. *Let $\boldsymbol{u}_\varepsilon(x) \equiv (u^1_\varepsilon(x), \cdots, u^N_\varepsilon(x))$ be a solution of the ε-regularized problem (2.1)-(2.2) satisfying (2.4) and $u^l_\varepsilon(x) \in C_{2,\alpha}(\bar\Omega)$ for $l = 1, \cdots, N$. Then, there exist some positive constants $\beta \in (0, \alpha)$ and C_6 which are independent of ε such that*

$$|\boldsymbol{u}_\varepsilon(x)|_{\beta,\bar\Omega} \leq C_6 \quad (8.1)$$

where $|\cdot|_{\beta,\Omega}$ *denotes a Hölder norm with exponent* β *in* Ω.

Lemma 8.1 follows from Lemma 3.3 and the following lemma.

LEMMA 8.2. *Let* $\boldsymbol{u}_\varepsilon(x) \equiv (u_\varepsilon^1(x), \cdots, u_\varepsilon^N(x))$ *be a solution of the* ε*-regularized problem* (2.1)-(2.2) *satisfying* (2.4), *and consider the following functions for* $\tau \geq 1$;

$$\pm u_\varepsilon^l(x)|u_\varepsilon^l(x)|^{\tau-1} \qquad for \quad l = 1, \cdots, N. \tag{8.2}$$

Then, for any ball $K_\rho \subset \Omega$, *there exists at least one function satisfying the premises of Lemma* 3.3 *from the family* (8.2).

Proof. We fix an arbitrary ball $K_\rho \subset \Omega$. From every pair of the functions $\pm u_\varepsilon^l|u_\varepsilon^l|^{\tau-1}$, we keep only the function for which the greater part of its oscillation on K_ρ fall in the domain of positive values; without loss of generality we denote this function by $U^l(x)$, with regard to which we know that

$$\inf_{K_\rho} U^l + \frac{\bar{\omega}^l}{2} \quad \geq 0 \tag{8.3}$$

where $\bar{\omega}^l \equiv \mathrm{osc}_{K_\rho} U^l$ for $l = 1, \cdots, N$. Let $l_M \in (1, \cdots, N)$ be the number such that $\bar{\omega}^{l_M} = \max_{l=1,\cdots,N} \bar{\omega}^l$. We indicate a constant q (> 0) determined exclusively by C_0 and τ in the condition (A_1) as below. It is sufficient to consider the following three cases for the family $\{U^1(x), \cdots, U^N(x)\}$;

$$\text{Case I}: \qquad \forall U^l: \quad |\inf_{K_\rho} U^l| \leq \frac{\bar{\omega}^{l_M}}{2^q} \qquad \text{for all} \quad l \in (1, \cdots, N),$$

$$\text{Case II}: \qquad \exists U^{l_o}: \quad \inf_{K_\rho} U^{l_o} > \frac{\bar{\omega}^{l_M}}{2^q} \qquad \text{for some} \quad l_o \in (1, \cdots, N),$$

$$\text{Case III}: \qquad \exists U^{l_*}: \quad \inf_{K_\rho} U^{l_*} < -\frac{\bar{\omega}^{l_M}}{2^q} \qquad \text{for some} \quad l_* \in (1, \cdots, N).$$

For the vector-valued functions $\boldsymbol{u} \equiv (u_\varepsilon^1, \cdots, u_\varepsilon^N)$ and $\boldsymbol{U} \equiv (U^1, \cdots, U^N)$, the following relations hold;

$$|\boldsymbol{u}|^\tau \leq N^{(\tau-1)/2}|\boldsymbol{U}| \quad \text{and} \quad |\boldsymbol{U}| \leq |\boldsymbol{u}|^\tau. \tag{8.4}$$

Firstly, we consider Case I, in this case the assertion of this lemma is proved for U^{l_M}. For U^{l_M}, the premise (3.12) of Lemma 3.3 holds with $\delta_1 = 1$. The following relation holds;

$$\sup_{K_\rho}|\boldsymbol{U}| \le \sum_{l=1}^{N}|\sup_{K_\rho}U^l - \inf_{K_\rho}U^l + \inf_{K_\rho}U^l| \le N\bar{\omega}^{l_M} + N(\frac{\bar{\omega}^{l_M}}{2^q}) < 2N\bar{\omega}^{l_M} \quad (q > 1). \tag{8.5}$$

Let $k_o \equiv \sup_{K_\rho} U^{l_M} - \bar{\omega}^{l_M}/4$ and $A_{k,\rho} \equiv \{x \in K_\rho : U^{l_M}(x) > k\}$. For any $k : k_o \le k \le \sup_{K_\rho} U^{l_M}$, we have

$$\inf_{A_{k\rho}}|\boldsymbol{U}| \ge \inf_{A_{k,\rho}}|U^{l_M}| \ge k_o \ge \bar{\omega}^{l_M}/4. \tag{8.6}$$

Hence, from (8.5) and (8.6) we have, for $k : k_o \le k \le \sup_{K_\rho} U^{l_M}$,

$$\sup_{K_\rho}|\boldsymbol{U}| \le 8N \inf_{A_{k,\rho}}|\boldsymbol{U}|, \quad \text{or} \quad \sup_{K_\rho}|\boldsymbol{u}|^\tau \le 8N^{(\tau+1)/2}\inf_{A_{k,\rho}}|\boldsymbol{u}|^\tau. \tag{8.7}$$

From (8.7), as well as from inequality (7.5), for $U^{l_M}(x)$ we arrive at the inequality (3.2) of Lemma 3.1 with $C = 8N^{(\tau+1)/2}C_{(3)}$ for any $k \ge k_o \equiv \sup_{K_\rho} U^{l_M} - \bar{\omega}^{l_M}/4$.

Next, let us take h and $k : \bar{\omega}^{l_M}/2^s \le h \le k \le 3h$, and $B_{k,\rho} \equiv \{x \in K_\rho; U^{l_M}(x) < k\}$, where s is a number indicated in Lemma 3.2. Then we have

$$\inf_{B_{k,\rho}-B_{h,\rho}}|\boldsymbol{U}| \ge \inf_{B_{k,\rho}-B_{h,\rho}}|U^{l_M}| \ge h \ge \frac{\bar{\omega}^{l_M}}{2^s}. \tag{8.8}$$

Hence, from (8.5) and (8.8) we have

$$\sup_{K_\rho}|\boldsymbol{U}| \le 2^{s+1}N \inf_{B_{k,\rho}-B_{h,\rho}}|\boldsymbol{U}|, \quad \text{or} \quad \sup_{K_\rho}|\boldsymbol{u}|^\tau \le 2^{s+1}N^{(\tau+1)/2}\inf_{B_{k,\rho}-B_{h,\rho}}|\boldsymbol{u}|^\tau. \tag{8.9}$$

From (8.9), as well as from inequality (7.3), for $U^{l_M}(x)$ we arrive at the inequality (3.7) of Lemma 3.2 with $C = 2^{s+1}N^{(\tau+1)/2}C_{(1)}$ for any $h \ge \bar{\omega}^{l_M}/2^s$ and $k \in [h, 3h]$.

Moreover, it is apparent that one of the following two relations is true;

$$\text{mes}\{x \in K_{\rho/2} : U^{l_M}(x) \le k_o\} \ge \frac{1}{2}\text{mes}K_{\rho/2}, \tag{a}$$

$$\text{mes}\{x \in K_{\rho/2} : U^{l_M}(x) \ge k_o\} \ge \frac{1}{2}\text{mes}K_{\rho/2}. \tag{b}$$

If the case (a) is true, the value $k_o \equiv \sup_{K_\rho} U^{l_M} - \bar{\omega}^{l_M}/4$ obeys postulates (3.3) and (3.4) of Lemma 3.1 with the constants $\delta = 1/4, \gamma = 1/2$, hence the assertion of Lemma 3.1 follows for U^{l_M}. If the case (b) is true, the value k_o obeys postulates (3.8) and (3.9) of Lemma 3.2 with the constants $\delta = 3/4, \gamma = 1/2$, hence the assertion of Lemma 3.2 follows for U^{l_M}. Let s be the larger one of the two numbers dictated by Lemma 3.1 and Lemma 3.2, then Lemma 3.1 and Lemma 3.2 guarantee the premise (3.13) or (3.14) of Lemma 3.3 for $U^{l_M}(x)$.

Secondly, we consider Case II, in this case the assertion of this lemma is proved for U^{l_M} too. The following relations hold;

$$\sup_{K_\rho} |\boldsymbol{U}| \leq \inf_{K_\rho} |\boldsymbol{U}| + N^{1/2}\bar{\omega}^{l_M} \leq N^{1/2}(\inf_{K_\rho} |\boldsymbol{U}| + \bar{\omega}^{l_M}), \tag{8.10}$$

$$\inf_{K_\rho} |\boldsymbol{U}| \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{U}| + \inf_{K_\rho} |U^{l_o}|) \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{U}| + \frac{\bar{\omega}^{l_M}}{2^q}). \tag{8.11}$$

Hence, for any k: $\inf_{K_\rho} U^{l_M} \leq k \leq \sup_{K_\rho} U^{l_M}$, we have

$$\sup_{K_\rho} |\boldsymbol{u}|^\tau \leq 2^{q+1} N^{\tau/2} \inf_{K_\rho} |\boldsymbol{u}|^\tau \leq \begin{cases} 2^{q+1} N^{\tau/2} \inf_{A_{k,\rho}} |\boldsymbol{u}|^\tau, \\ 2^{q+1} N^{\tau/2} \inf_{B_{k,\rho}} |\boldsymbol{u}|^\tau. \end{cases} \tag{8.12}$$

From (8.12), as well as from inequality (7.4) or (7.5), we arrive at the inequality (3.2) of Lemma 3.1 or $(3.2)'$ of Lemma $3.1'$. Since either the condition (a) or (b) is true, the premise (3.13) or (3.14) of Lemma 3.3 is guaranteed for $U^{l_M}(x)$.

Lastly, we consider Case III, in this case the assertion of this lemma is proved for U^{l_*}. From the condition (8.3) we have the estimate:

$$\bar{\omega}^{l_*} \geq \frac{\bar{\omega}^{l_M}}{2^{q-1}} \geq \frac{\bar{\omega}^{l}}{2^{q-1}} \quad \text{for all } l = 1, \cdots, N \tag{8.13}$$

i.e., $U^{l_*}(x)$ satisfies the premise (3.12) of Lemma 3.3 with $\delta_1 = 2^{-q+1}$.

Let $k_* \equiv \sup_{K_\rho} U^{l_*} - \omega^{l_*}/4$ $(> \bar{\omega}^{l_*}/4)$ and $A_{k,\rho} \equiv \{x \in K_\rho : U^{l_*}(x) > k\}$. The following relation holds for any $k : k_* \leq k \leq \sup_{K_\rho} U^{l_*}$;

$$\inf_{A_{k,\rho}} |\boldsymbol{U}| \geq \frac{1}{2}(\inf_{A_{k,\rho}} |\boldsymbol{U}| + \inf_{A_{k,\rho}} |U^{l_*}|) \geq \frac{1}{2}(\inf_{A_{k,\rho}} |\boldsymbol{U}| + k_*) \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{U}| + \frac{\bar{\omega}^{l_*}}{4}). \tag{8.14}$$

Hence, from (8.10), (8.13) and (8.14) we have

$$\sup_{K_\rho} |\boldsymbol{u}|^\tau \leq 2^{q+2} N^{\tau/2} \inf_{A_{k,\rho}} |\boldsymbol{u}|^\tau \quad \text{for} \quad k : k_* \leq k \leq \sup_{K_\rho} U^{l_*}. \tag{8.15}$$

Let $k'_* \equiv \inf_{K_\rho} U^{l_*} + \bar{\omega}^{l_*}/2^{q+1}$ $(< -\bar{\omega}^{l_*}/2^{q+1})$ and $B_{k,\rho} \equiv \{x \in K_\rho : U^{l_*}(x) < k\}$. The following relation holds for any $k : \inf_{K_\rho} U^{l_*} \leq k \leq k'_*$;

$$\inf_{B_{k,\rho}} |\boldsymbol{U}| \geq \frac{1}{2}(\inf_{B_{k,\rho}} |\boldsymbol{U}| + \inf_{B_{k,\rho}} |U^{l_*}|) \geq \frac{1}{2}(\inf_{B_{k,\rho}} |\boldsymbol{U}| + |k'_*|) \geq \frac{1}{2}(\inf_{K_\rho} |\boldsymbol{U}| + \frac{\bar{\omega}^{l_*}}{2^{q+1}}). \tag{8.16}$$

Hence, from (8.10), (8.13) and (8.16) we have

$$\sup_{K_\rho} |\boldsymbol{u}|^\tau \leq 2^{2q+1} N^{\tau/2} \inf_{B_{k,\rho}} |\boldsymbol{u}|^\tau \quad \text{for} \quad k : \inf_{K_\rho} U^{l_*} \leq k \leq k'_*. \tag{8.17}$$

It is also apparent that one of the following two relations is valid;

$$\text{mes}\{x \in K_{\rho/2} : U^{l_*}(x) \leq k_*\} \geq \frac{1}{2}\text{mes} K_{\rho/2}, \tag{a'}$$

$$\text{mes}\{x \in K_{\rho/2} : U^{l_*}(x) \geq k'_*\} \geq \frac{1}{2}\text{mes} K_{\rho/2}. \tag{b'}$$

From (8.15) or (8.17), as well as from inequality (7.4) or (7.5), we arrive at the inequality (3.2) of Lemma 3.1 or (3.2)′ of Lemma 3.1′, and hence the premise (3.13) or (3.14) of Lemma 3.3 is guaranteed for $U^{l_*}(x)$. This completes the proof of Lemma 8.2.

Considering Lemma 8.2 and Lemma 3.3, we obtain the interior estimates of $U^l_\varepsilon(x)$ which are uniform on $\varepsilon \in [0, 1]$ for $l = 1, \cdots, N$. Applying the conditions (A) and Remark in Section 3 to the above argument, we obtain the estimates of $U^l_\varepsilon(x)$ in the region adjacent to the boundary. Consequently, we have the estimates:

$$|U^l_\varepsilon|_{\tau\beta,\Omega} \leq C_7 \quad \text{and} \quad |u^l_\varepsilon|_{\beta,\Omega} \leq C'_7 \qquad \text{for} \quad l = 1, \cdots, N \tag{8.18}$$

for some constants β $(0 < \beta < 1)$ and C_7, C'_7 which are independent of ε. This proves Lemma 8.1 for $\tau \geq 1$.

9. PROOF OF THEOREM FOR $\tau \geq 1$

Let $u^l_\varepsilon(x) \in C_{2,\alpha}(\bar{\Omega})$ be the solutions of the ε-regularized problem (2.1)-(2.2) satisfying (2.4), then by virtue of (8.1) we have

$$|u^l_\varepsilon|_{\beta,\bar{\Omega}} \leq C_6 \quad \text{for} \quad l = 1, \cdots, N \tag{9.1}$$

where the constants C_6 and β are independent of ε. Since $u_\varepsilon^l(x) - \psi^l(x) \in \overset{\circ}{W}{}_2^1(\Omega)$, from (7.1) we have

$$\int_\Omega [\sum_{i,j=1}^n a_{\varepsilon ij}^l(x, \boldsymbol{u}_\varepsilon) u_{\varepsilon x_j}^l (u_{\varepsilon x_i}^l - \psi_{x_i}^l)$$

$$+\{\sum_{j=1}^n b_j^l(x, \boldsymbol{u}_\varepsilon) u_{\varepsilon x_j}^l + b_0^l(x, \boldsymbol{u}_\varepsilon)\}(u_\varepsilon^l - \psi^l)] dx = 0.$$

By the use of assumptions (A_1'), (A_2) and (2.4), we have

$$\int_\Omega (\varepsilon + |\boldsymbol{u}_\varepsilon|^\tau) \sum_{l=1}^N |\nabla u_\varepsilon^l|^2 dx \leq C_8 \tag{9.2}$$

where a constant C_8 is independent of ε. Consider the following functions:

$$V_\varepsilon(x) \equiv |\boldsymbol{u}_\varepsilon|^{\tau/2+1}, \quad V_\varepsilon^l(x) \equiv u_\varepsilon^l |\boldsymbol{u}_\varepsilon|^{\tau/2} \tag{9.3}$$

for $l = 1, \cdots, N$, and we have the following estimates:

$$|(V_\varepsilon)_{x_j}| = |(1 + \frac{\tau}{2})|\boldsymbol{u}_\varepsilon|^{\tau/2-1} \sum_{r=1}^N u_\varepsilon^r u_{\varepsilon x_j}^r| \leq (1 + \frac{\tau}{2})|\boldsymbol{u}_\varepsilon|^{\tau/2} (\sum_{r=1}^N |\nabla u_\varepsilon^r|^2)^{1/2},$$

$$|(V_\varepsilon^l)_{x_j}| = |u_{\varepsilon x_j}^l |\boldsymbol{u}_\varepsilon|^{\tau/2} + \frac{\tau}{2} u_\varepsilon^l |\boldsymbol{u}_\varepsilon|^{\tau/2-2} \sum_{r=1}^N u_\varepsilon^r u_{\varepsilon x_j}^r| \leq (1 + \frac{\tau}{2})|\boldsymbol{u}_\varepsilon|^{\tau/2} (\sum_{r=1}^N |\nabla u_\varepsilon^r|^2)^{1/2}.$$

Hence, from (9.2) we deduce the uniform boundedness of the integrals:

$$\int_\Omega \{|(V_\varepsilon)_{x_j}|^2 + \sum_{l=1}^N |(V_\varepsilon^l)_{x_j}|^2\} dx \leq C_9 \quad \text{for} \quad j = 1, \cdots, n \tag{9.4}$$

where a constant C_9 is independent of ε. On the basis of the estimates (9.1) and (9.4), there exists a subsequence $\{u_{\varepsilon_p}^l\}$ of $\{u_\varepsilon^l\}$ such that, as $\varepsilon_p \to 0$,

$$u_{\varepsilon_p}^l \to u^l \quad \text{in} \quad C_{0,\beta}(\bar{\Omega}), \tag{9.5}$$

$$(V_{\varepsilon_p})_{x_j} \to V_{x_j} \quad \text{weakly in} \quad L_2(\Omega), \tag{9.6}$$

$$(V_{\varepsilon_p}^l)_{x_j} \to V_{x_j}^l \quad \text{weakly in} \quad L_2(\Omega), \tag{9.7}$$

where some $u^l(x) \in C_{0,\beta}(\bar{\Omega})$, $V(x) \equiv |\boldsymbol{u}|^{\tau/2+1}$ and $V^l(x) \equiv u^l |\boldsymbol{u}|^{\tau/2}$ for $l = 1, \cdots, N$. Therefore, the functions $u^l(x) \in C_{0,\beta}(\bar{\Omega})$ satisfy the integral identity (3) for any $\varphi^l \in \overset{\circ}{W}{}_2^1(\Omega)$, since the functions $u_{\varepsilon_p}^l$ satisfy the same integral identity as (3) derived from the ε-regularized equations (2.1)-(2.2) and it is possible to pass the limit as $\varepsilon_p \to 0$ in this form. This completes the proof of Theorem for $\tau \geq 1$.

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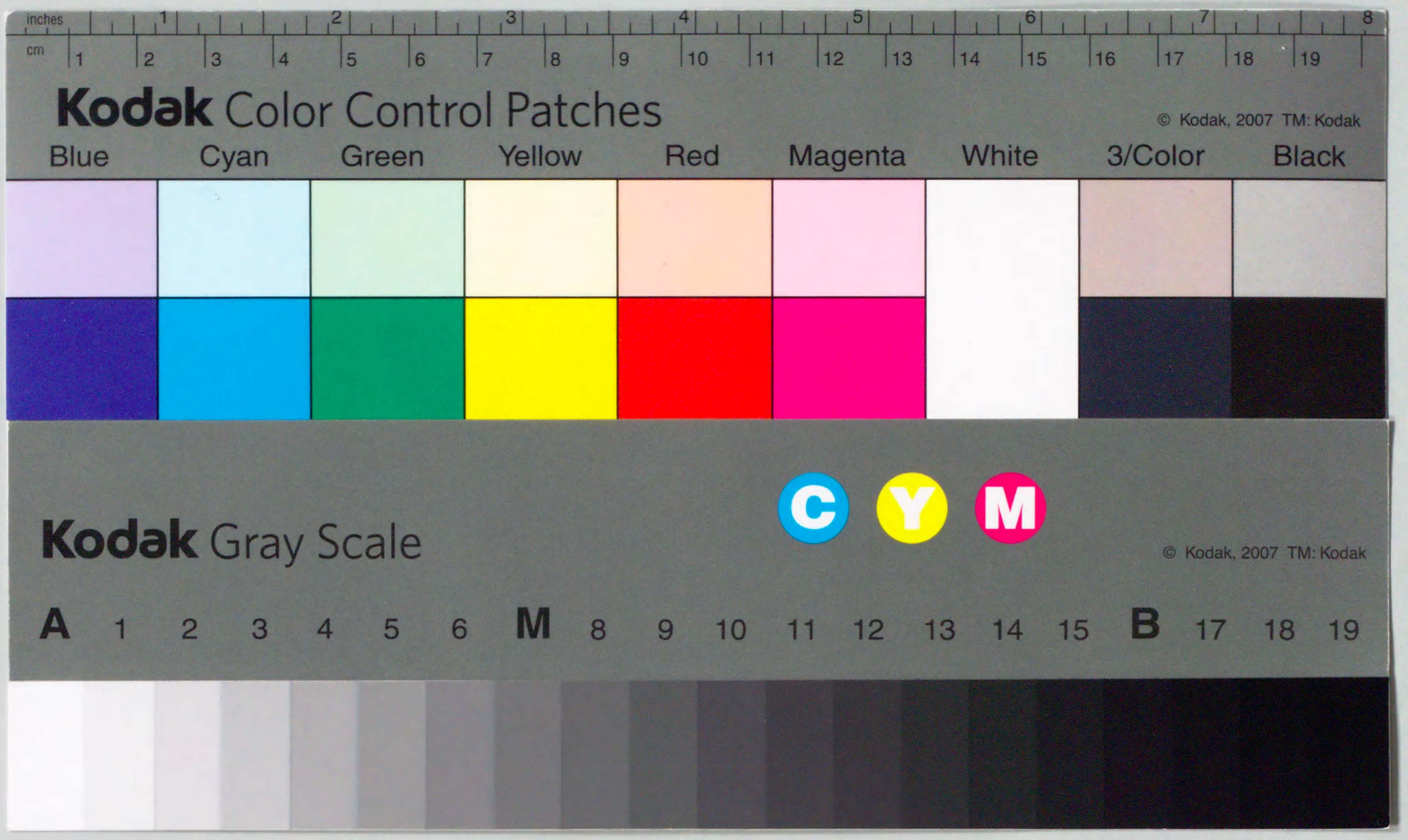
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