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Abstract

The global in time existence of strong solutions to the compressible Navier-Stokes equation around time-periodic parallel flows in $\mathbb{R}^n$, $n \geq 2$, is established under smallness conditions on Reynolds number, Mach number and initial perturbations. Furthermore, it is proved for $n = 2$ that the asymptotic leading part of solutions is given by a solution of one-dimensional viscous Burgers equation multiplied by time-periodic function. In the case $n \geq 3$ the asymptotic leading part of solutions is given by a solution of $n - 1$ -dimensional heat equation with convective term multiplied by time-periodic function.

Mathematics Subject Classification

Keywords. Compressible Navier-Stokes equation, global existence, asymptotic behavior, time-periodic, viscous Burgers equation.

1 Introduction

In this paper we study the stability of solutions around a time-periodic parallel flow to the compressible Navier-Stokes equation with time-periodic external force and time-periodic boundary conditions.

We consider the system of equations

$$\partial_{\tilde{t}} \tilde{\rho} + \operatorname{div}(\tilde{\rho} \tilde{v}) = 0, \quad (1.1)$$

$$\tilde{\rho}(\partial_{\tilde{t}} \tilde{v} + \tilde{v} \cdot \nabla \tilde{v}) - \mu \Delta \tilde{v} - (\mu + \mu') \nabla \operatorname{div} \tilde{v} + \nabla \tilde{P}(\tilde{\rho}) = \tilde{\rho} \tilde{g}, \quad (1.2)$$

in an n dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1} \times (0, \ell)$:

$$\begin{aligned} \Omega_\ell &= \{ \tilde{x} = {}^T(\tilde{x}', \tilde{x}_n); \\ &\quad \tilde{x}' = {}^T(\tilde{x}_1, \dots, \tilde{x}_{n-1}) \in \mathbb{R}^{n-1}, 0 < \tilde{x}_n < \ell \}. \end{aligned}$$

Here, $n \geq 2$; $\tilde{\rho} = \tilde{\rho}(\tilde{x}, \tilde{t})$ and $\tilde{v} = {}^T(\tilde{v}^1(\tilde{x}, \tilde{t}), \dots, \tilde{v}^n(\tilde{x}, \tilde{t}))$ denote the unknown density and velocity at time $\tilde{t} \geq 0$ and position $\tilde{x} \in \Omega_\ell$, respectively; $\tilde{P}$ is the pressure, smooth function of $\tilde{\rho}$, where for given $\rho_* > 0$ we assume

$$\tilde{P}'(\rho_*) > 0;$$

μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$; div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\tilde{x}$. Here and in what follows T denotes the transposition.

In (1.2) $\tilde{g}$ is assumed to have the form

$$\tilde{g} = {}^T(\tilde{g}^1(\tilde{x}_n, \tilde{t}), 0, \dots, 0, \tilde{g}^n(\tilde{x}_n)),$$

with $\tilde{g}^1$ being a τ -periodic function in time, where $\tau > 0$.

The system (1.1)–(1.2) is considered under boundary condition

$$\tilde{v}|_{\tilde{x}_n=0} = \tilde{V}^1(t) \mathbf{e}_1, \quad \tilde{v}|_{\tilde{x}_n=\ell} = 0, \quad (1.3)$$

and initial condition

$$(\tilde{\rho}, \tilde{v})|_{\tilde{t}=0} = (\tilde{\rho}_0, \tilde{v}_0), \quad (1.4)$$

where $\tilde{V}^1$ is a τ -periodic function of time and $\mathbf{e}_1 = {}^T(1, 0, \dots, 0) \in \mathbb{R}^n$.

Under suitable conditions on $\tilde{g}$ and $\tilde{V}^1$, problem (1.1)–(1.3) has smooth time-periodic solution $\bar{u}_p = {}^T(\bar{\rho}_p, \bar{v}_p)$ satisfying

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$$\bar{\rho}_p = \bar{\rho}_p(\tilde{x}_n) \geq \tilde{\rho}_1, \quad \frac{1}{\ell} \int_0^\ell \bar{\rho}_p(\tilde{x}_n) d\tilde{x}_n = \rho_*,$$

$$\bar{v}_p = {}^T(\bar{v}_p^1(\tilde{x}_n, \tilde{t}), 0, \dots, 0), \quad \bar{v}_p^1(\tilde{x}_n, \tilde{t} + \tau) = \bar{v}_p^1(\tilde{x}_n, \tilde{t}),$$

for a positive constant $\tilde{\rho}_1$.

The aim of this paper is to give an asymptotic description of large time behavior of perturbations from $\bar{u}_p$ when Reynolds and Mach numbers are sufficiently small.

To formulate the problem for perturbations, we introduce the following dimensionless variables:

$$\tilde{x} = \ell x, \quad \tilde{t} = \frac{\ell}{V} t, \quad \tilde{v} = V v, \quad \tilde{\rho} = \rho_* \rho, \quad \tilde{P} = \rho_* V^2 P,$$

with

$$\tilde{w} = V w, \quad \tilde{\phi} = \rho_* \gamma^{-2} \phi, \quad \tilde{V}^1 = V V^1, \quad \tilde{\mathbf{g}} = \frac{\mu V}{\rho_* \ell^2} \mathbf{g},$$

where

$$\gamma = \frac{\sqrt{\tilde{P}'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu} \left\{ |\partial_{\tilde{t}} \tilde{V}^1|_{C^0(\mathbb{R})} + |\tilde{\mathbf{g}}^1|_{C^0(\mathbb{R} \times [0, \ell])} \right\} + |\tilde{V}^1|_{C^0(\mathbb{R})} > 0.$$

In this paper we assume $V > 0$. Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0, 1)$; and $g^1(x_n, t)$, $V^1(t)$ are periodic in t with period $T > 0$ defined by

$$T = \frac{V}{\ell} \tau.$$

The time-periodic solution $\bar{u}_p$ is transformed into $u_p = {}^T(\rho_p, v_p)$ satisfying

$$\rho_p = \rho_p(x_n) > 0, \quad \int_0^1 \rho_p(x_n) dx_n = 1,$$

$$v_p = {}^T(v_p^1(x_n, t), 0, \dots, 0), \quad v_p^1(x_n, t + T) = v_p^1(x_n, t).$$

It then follows that the perturbation $u(t) = {}^T(\phi(t), w(t)) \equiv {}^T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t))$ is governed by the following system of equations

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \operatorname{div}(\rho_p w) = f^0, \tag{1.5}$$

$$\begin{aligned} \partial_t w - \frac{\nu}{\rho_p} \Delta w - \frac{\tilde{\nu}}{\rho_p} \nabla \operatorname{div} w + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w \mathbf{e}_1 \\ + \frac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1) \phi \mathbf{e}_1 + \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \phi \right) = \mathbf{f}, \end{aligned} \tag{1.6}$$

$$w|_{\partial\Omega} = 0, \tag{1.7}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{1.8}$$

where f^0 and $\mathbf{f} = {}^T(f^1, \dots, f^n)$ denote nonlinearities, i.e.,

$$f^0 = -\operatorname{div}(\phi w),$$

$$\begin{aligned} \mathbf{f} = & -w \cdot \nabla w + \frac{\nu \phi}{\gamma^2 \rho_p^2} \left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\rho_p \gamma^2} \phi \mathbf{e}_1 \right) - \frac{\nu \phi^2}{\gamma^2 \rho_p^2 (\gamma^2 \rho_p + \phi)} \left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\rho_p \gamma^2} \phi \mathbf{e}_1 \right) \\ & - \frac{\tilde{\nu} \phi}{\rho_p (\gamma^2 \rho_p + \phi)} \nabla \operatorname{div} w + \frac{\phi}{\gamma^2 \rho_p} \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \phi \right) - \frac{1}{2\gamma^4 \rho_p} \nabla (P''(\rho_p) \phi^2) + \tilde{P}_3(\rho_p, \phi, \partial_x \phi), \\ \tilde{P}_3(\rho_p, \phi, \partial_x \phi) = & \frac{\phi^3}{\gamma^4 (\gamma^2 \rho_p + \phi) \rho_p^3} \nabla P(\rho_p) + \frac{\phi \nabla (P''(\rho_p) \phi^2)}{2\gamma^4 \rho_p (\gamma^2 \rho_p + \phi)} \\ & - \frac{\phi^2 \nabla (P'(\rho_p) \phi)}{\gamma^4 \rho_p^2 (\gamma^2 \rho_p + \phi)} - \frac{1}{2\gamma^4 (\gamma^2 \rho_p + \phi)} \nabla (\phi^3 P_3(\rho_p, \phi)), \end{aligned}$$

with

$$P_3(\rho_p, \phi) = \int_0^1 (1 - \theta)^2 P'''(\theta \gamma^{-2} \phi + \rho_p) d\theta.$$

Here, div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x ; ν , ν' and $\tilde{\nu}$ are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \tilde{\nu} = \nu + \nu'.$$

We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively. See [1] for the derivation of (1.5)–(1.8).

In the case g^1 and V^1 do not depend on t , problem (1.1)–(1.3) has a stationary parallel flow. The stability of stationary parallel flows were studied in [5, 6, 7, 11]. It was shown in [6] and [7] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $H^m(\Omega) \cap L^1(\Omega)$ with $m \geq [n/2] + 1$, provided that $Re \ll 1$, $Ma \ll 1$ and density of the parallel flow is sufficiently close to a positive constant. Furthermore, the asymptotic behavior of perturbations from the stationary parallel flow is described by $n - 1$ dimensional linear heat equation in the case $n \geq 3$ ([6]) and by one-dimensional viscous Burgers equation in the case $n = 2$ ([7]).

The case of time-periodic parallel flows was considered in [1, 2] for $Re \ll 1$ and $Ma \ll 1$. In [1, 2] the authors investigated the linearized problem, i.e., (1.5)–(1.8) with $(f^0, \mathbf{f}) = (0, 0)$, which is written as

$$\partial_t u + L(t)u = 0, \quad w|_{x_n=0,1} = 0, \quad u|_{t=s} = u_0. \quad (1.9)$$

Here, $u = {}^T(\phi, w)$ and $L(t)$ is operator of the form

$$\begin{aligned} L(t) = & \begin{pmatrix} v_p^1(t) \partial_{x_1} & \gamma^2 \text{div}(\rho_p \cdot) \\ \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \cdot \right) & -\frac{\nu}{\rho_p} \Delta I_n - \frac{\tilde{\nu}}{\rho_p} \nabla \text{div} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2} \partial_{x_n}^2 v_p^1(t) \mathbf{e}_1 & v_p^1(t) \partial_{x_1} I_n + (\partial_{x_n} v_p^1(t)) \mathbf{e}_1 {}^T \mathbf{e}_n \end{pmatrix}. \end{aligned} \quad (1.10)$$

Note that $L(t)$ satisfies $L(t) = L(t + T)$.

In [1, 2] spectral properties of the solution operator $U(t, s)$ for (1.9) were studied by using Fourier transform with respect to $x' \in \mathbb{R}^{n-1}$. The Fourier transform of (1.9) can be written in the form:

$$\frac{d}{dt} \hat{u} + \hat{L}_{\xi'}(t) \hat{u} = 0, \quad t > s, \quad \hat{u}|_{t=s} = \hat{u}_0, \quad (1.11)$$

where $\hat{u}$ denotes the Fourier transform of u in x' ; and ξ' is dual variable to x' . For each $\xi' \in \mathbb{R}^{n-1}$ and for all $t \geq s$ there exists a unique evolution operator $\hat{U}_{\xi'}(t, s)$ for (1.11).

Since $\hat{L}_{\xi'}(t)$ is T -time periodic, the spectrum of $\hat{U}_{\xi'}(T, 0)$ plays an important role in the study of large time behavior. It was shown in [1] that the spectrum of $\hat{U}_{\xi'}(T, 0)$ satisfies the following inclusion

$$\sigma(\hat{U}_{\xi'}(T, 0)) \subseteq \begin{cases} \{e^{\lambda_{\xi'} T}\} \cup \{|\lambda| < q_1\} & (|\xi'| < r), \\ \{|\lambda| < q_1\} & (|\xi'| \geq r), \end{cases}$$

for a constant $0 < q_1 < 1$ and $0 < r \ll 1$. Here, $e^{\lambda_{\xi'} T}$ is the simple eigenvalue of $\hat{U}_{\xi'}(T, 0)$ and $\lambda_{\xi'} = -i\kappa_0 \xi_1 - \kappa_1 \xi_1^2 - \kappa'' |\xi''|^2 + O(|\xi'|^3)$ with $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, $\kappa'' > 0$ and $\xi' = {}^T(\xi_1, \xi'')$, $\xi'' = (\xi_2, \dots, \xi_{n-1})$.

In [2] spectral properties of $\hat{U}_{\xi'}(t, s)$ were investigated for $|\xi'| < r$ by using the Floquet theory. A family $\{\mathbb{P}(t)\}_{t \in \mathbb{R}}$ of bounded projections on $L^2(\Omega)$ was constructed to represent $\mathbb{P}(t)U(t, s)$ as

$$\mathbb{P}(t)U(t, s) = \mathcal{Q}(t)e^{(t-s)\Lambda} \mathcal{P}(s). \quad (1.12)$$

Here, $e^{t\Lambda} = \mathcal{F}^{-1} \hat{\chi}_1 e^{\lambda_{\xi'} t} \mathcal{F}$ with frequency cut off function $\hat{\chi}_1 : \hat{\chi}_1(\xi') = 1$ ($|\xi'| < r$), $\hat{\chi}_1(\xi') = 0$ ($|\xi'| \geq r$), and $\mathcal{Q}(t) = \mathcal{F}^{-1} \hat{\chi}_1 \widehat{\mathcal{Q}}_{\xi'}(t) \mathcal{F}$ and $\mathcal{P}(t) = \mathcal{F}^{-1} \hat{\chi}_1 \widehat{\mathcal{P}}_{\xi'}(t) \mathcal{F}$ with

$$\widehat{\mathcal{Q}}_{\xi'}(t) : \mathbb{C} \rightarrow L^2(0, 1) \text{ and } \widehat{\mathcal{P}}_{\xi'}(t) : L^2(0, 1) \rightarrow \mathbb{C},$$

expanded as

$$\begin{aligned} \widehat{\mathcal{Q}}_{\xi'}(t) &= \mathcal{Q}^{(0)}(t) + i\xi' \cdot \mathcal{Q}^{(1)}(t) + O(|\xi'|^2), \\ \widehat{\mathcal{P}}_{\xi'}(t) &= \mathcal{P}^{(0)} + i\xi' \cdot \mathcal{P}^{(1)}(t) + O(|\xi'|^2), \end{aligned}$$

for $|\xi'| \leq r$, where $\mathcal{Q}^{(0)}(t)\sigma = \sigma u^{(0)}(\cdot, t)$ ($\sigma \in \mathbb{C}$), $u^{(0)}(\cdot, t) = u^{(0)}(x_n, t)$ is a function T -periodic in t and $\mathcal{P}^{(0)}u = [\phi]$ ($u = {}^T(\phi, w) \in L^2(0, 1)$). One consequence of (1.12) is that

$$\|\partial_{x'}^k \partial_{x_n}^l \mathbb{P}(t)U(t, s)u_0\|_{L^2(\Omega)} \leq C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1(\Omega)},$$

$$\|(I - \mathbb{P}(t))U(t, s)u_0\|_{H^1(\Omega)} \leq e^{-d(t-s)}(\|u_0\|_{H^1 \times L^2} + \|\partial_{x'} w_0\|_{L^2}),$$

$$\|\partial_{x'}^k \partial_{x_n}^l (\mathbb{P}(t)U(t, s)u_0 - \sigma_{t,s}[u_0]u^{(0)}(t))\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{n-1}{4}-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1(\Omega)},$$

for $t-s \geq T$, $s \geq 0$; $k = 0, 1, \dots$, $l = 0, \dots, m$ for $m \geq 2$. Here, $\sigma_{t,s}[u_0] = \sigma_{t,s}(x')[u_0]$ is a function whose Fourier transform in x' is given by

$$\mathcal{F}(\sigma_{t,s}[u_0]) = e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2)(t-s)}[\widehat{\phi}_0(\xi')],$$

where $[\widehat{\phi}_0(\xi')]$ is a quantity given by

$$[\widehat{\phi}_0(\xi')] = \int_0^1 \widehat{\phi}_0(\xi', x_n) dx_n,$$

with $\widehat{\phi}_0$ being the Fourier transform of ϕ_0 in x' and $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, $\kappa'' > 0$ are positive constants depending on ρ_* , l , V , μ , μ' and $\widetilde{P}'(\rho_*)$.

Another consequence of (1.12) is that if $u(t)$ is a solution of

$$\partial_t u + L(t)u = f, \quad u|_{t=0} = u_0,$$

then $\mathbb{P}(t)u(t)$ is represented as

$$\mathbb{P}(t)u(t) = \mathcal{Q}(t) \left(e^{t\Lambda} \mathcal{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda} \mathcal{P}(z)f(z)dz \right). \quad (1.13)$$

In this paper we show the following results. Let u_0 be sufficiently small in $H^m(\Omega) \cap L^1(\Omega)$ for a given $m \geq [n/2] + 1$; and let u_0 satisfy a suitable compatibility condition, then there exists unique solution $u(t)$ of (1.5)–(1.8) in $C([0, \infty); H^m(\Omega))$, provided that $Re \ll 1$, $Ma \ll 1$ and $|1 - \rho_p|_{C^{m+1}([0,1])} \ll 1$. Furthermore, $u(t)$ satisfies

$$\|\partial_{x'}^k u(t)\|_{L^2(\Omega)} \leq O(t^{-\frac{n-1}{4}-\frac{k}{2}}), \quad k = 0, 1,$$

as $t \rightarrow \infty$.

In the case $n = 2$, we show that the asymptotic leading term of perturbation $u(t)$ is described by a solution of one-dimensional viscous Burgers equation, i.e.,

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{3}{4}+\delta}), \quad \forall \delta > 0,$$

as $t \rightarrow \infty$. Here, $u^{(0)} = u^{(0)}(x_2, t)$ is a given time-periodic function; and $\sigma = \sigma(x_1, t)$ is function satisfying

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + \omega_0 \partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2) dx_2, \quad (1.14)$$

with constants $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$ being the same ones as those in λ_{ξ_1} and $\omega_0 \in \mathbb{R}$ determined by the nonlinearity $\mathbf{F}$.

In the case $n \geq 3$, we show that the asymptotic leading term of $u(t)$ is the same one as for the linearized problem and thus it is given by $n-1$ -dimensional heat equation with convective term, i.e.,

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)),$$

as $t \rightarrow \infty$. Here, $\sigma = \sigma(x', t)$ is function satisfying

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) dx_n,$$

with constants $\kappa_0 \in \mathbb{R}$, $\kappa_1, \kappa'' > 0$ being the same ones as those in $\lambda_{\xi'}$; where $\Delta'' = \partial_{x_2}^2 + \dots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

The proof of the main results is given by a combination of various estimates for $\mathbb{P}(t)U(t, s)$ mentioned above and a variant of Matsumura-Nishida energy method ([6, 7], cf. [12]). We decompose the solution $u(t)$

of (1.5)–(1.8) into the $\mathbb{P}(t)$ -part and $(I - \mathbb{P}(t))$ -part. Considering the $\mathbb{P}(t)$ -part, we represent $\mathbb{P}(t)u(t)$ as in (1.13) with $f = {}^T(f^0, \mathbf{f})$ being the nonlinearity given in (1.5) and (1.6). We then combine various estimates on $\mathbb{P}(t)$ and $\mathbb{P}(t)U(t, s)$ to obtain the necessary estimates on $\mathbb{P}(t)u(t)$. On the other hand, $(I - \mathbb{P}(t))u(t)$ can be estimated by a variant of Matsumura-Nishida energy method as in the case of the stationary parallel flow ([7]). However, in contrast to [7], the linearized operator has time-dependent coefficients. Therefore a modification of the argument in [7] is needed for the time-periodic case to acquire the necessary energy estimate. It is worth mentioning that in the case $n = 2$ the asymptotic leading part of $u(t)$ is not described by the linearized problem due to the quadratic nonlinearities $-\operatorname{div}(\phi w)$, $\frac{\nu\phi}{\gamma^2\rho_p^2} \left(-\partial_{x_n}^2 w^1 + \frac{\partial_{x_n}^2 v_p^1}{\rho_p\gamma^2} \phi \right)$ and $-\frac{1}{2\gamma^4\rho_p} \partial_{x_n}(P''(\rho_p)\phi^2)$. This leads to the 1-dimensional Burgers equation (1.14).

Our result is an extension of previous results on the stationary case [5, 6, 7, 11] to the case of time-periodic external force and time-periodic boundary conditions.

Structure of this paper is the following. In Section 2 we introduce basic notations that are used throughout the paper. In Section 3 we state the main results. In Section 4 we present the results on spectral properties of the linearized problem obtained in [2]. In Section 5 we introduce decomposition of solution $u(t)$ to (1.5)–(1.8) based on the spectral properties of $L(t)$ introduced in Section 4. Moreover, we prove the a priori estimate using the estimates on $\mathbb{P}(t)u(t)$ and $(I - \mathbb{P}(t))u(t)$ from subsequent sections 6, 7 and 8. In Section 6 we show estimate for $\mathbb{P}(t)u(t)$ using properties of $\mathbb{P}(t)$ and $\mathbb{P}(t)U(t, s)$. In Section 7 we obtain estimate on $(I - \mathbb{P}(t))u(t)$ using energy method. In Section 8 we estimate the nonlinearities f^0 and $\mathbf{f}$. Finally, in Section 9 we prove the asymptotic behavior of solutions.

2 Notation

In this section we introduce some notations which are used throughout the paper. For a domain E we denote by $L^p(E)$ the usual Lebesgue space on E and its norm is denoted by $\|\cdot\|_{L^p(E)}$ for $1 \leq p \leq \infty$. Let k be a nonnegative integer. $H^k(E)$ denotes the k -th order L^2 Sobolev space on E with norm $\|\cdot\|_{H^k(E)}$. $C_0^k(E)$ stands for the set of all C^k functions which have compact support in E . We denote by $H_0^1(E)$ the completion of $C_0^1(E)$ in $H^1(E)$.

We simply denote by $L^p(E)$ (resp., $H^k(E)$) the set of all vector fields $w = {}^T(w^1, \dots, w^n)$ on E with $w^j \in L^p(E)$ (resp., $H^k(E)$), $j = 1, \dots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(E)}$ (resp., $\|\cdot\|_{H^k(E)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(E)$ and $w = {}^T(w^1, \dots, w^n) \in H^l(E)$, we define $\|u\|_{H^k(E) \times H^l(E)}$ by $\|u\|_{H^k(E) \times H^l(E)} = \|\phi\|_{H^k(E)} + \|w\|_{H^l(E)}$. When $k = l$, we simply write $\|u\|_{H^k(E) \times H^k(E)} = \|u\|_{H^k(E)}$.

In the case $E = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^k(\Omega)$) as L^p (resp., H^k). In particular, we denote the norm of L^p (resp., H^k) by $\|\cdot\|_p$ (resp., $\|\cdot\|_{H^k}$).

In the case $E = (0, 1)$ we denote the norm $|\cdot|_{L^2(0,1)}$ (resp., $|\cdot|_{H^k(0,1)}$) by $|\cdot|_2$ (resp., $|\cdot|_{H^k}$).

The inner product of L^2 is denoted by

$$(f, g) = \int_{\Omega} f(x)g(x) dx, \quad f, g \in L^2.$$

Furthermore, we introduce a weighted inner product $\langle \cdot, \cdot \rangle_{\Omega}$ defined by

$$\langle u_1, u_2 \rangle_{\Omega} = \int_{\Omega} \phi_1 \phi_2 \frac{P'(\rho_p)}{\gamma^4 \rho_p} dx + \int_{\Omega} w_1 w_2 \rho_p dx,$$

for $u_j = {}^T(\phi_j, w_j) \in L^2$, $j = 1, 2$; and for $u_j = {}^T(\phi_j, w_j) \in L^2(0, 1)$, $j = 1, 2$, we also define $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \int_0^1 \phi_1 \bar{\phi}_2 \frac{P'(\rho_p)}{\gamma^4 \rho_p} dx_n + \int_0^1 w_1 \bar{w}_2 \rho_p dx_n.$$

Here, $\bar{g}$ denotes the complex conjugate of g .

Furthermore, for $f \in L^1(0, 1)$ we denote the mean value of f in $(0, 1)$ by $[f]$:

$$[f] = (f, 1) = \int_0^1 f(x_n) dx_n.$$

For $u = {}^T(\phi, w) \in L^1(0, 1)$ with $w = {}^T(w^1, \dots, w^n)$ we define $[u]$ by

$$[u] = [\phi] + [w^1] + \dots + [w^n].$$

We often write $x \in \Omega$ as

$$x = {}^T(x', x_n), \quad x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Partial derivatives of function u in x, x', x_n and t are denoted by $\partial_x u, \partial_{x'} u, \partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$; by $\Delta' = \sum_{i=1}^{n-1} \partial_{x_i}^2$, $\nabla' = (\partial_{x_1}, \dots, \partial_{x_{n-1}})$ and $\text{div}' = \nabla' \cdot$ we denote the Laplacian, gradient and divergence with respect to x' , respectively.

We denote $k \times k$ identity matrix by I_k . In particular, when $k = n + 1$, we simply write I for I_{n+1} . We define $(n + 1) \times (n + 1)$ diagonal matrices Q_j, Q' and $\tilde{Q}$ by

$$Q_j = \text{diag}(0, \dots, 0, \underbrace{1}_{j\text{-th}}, 0, \dots, 0), \quad j = 0, 1, \dots, n,$$

and

$$Q' = \text{diag}(0, 1, \dots, 1, 0), \quad \tilde{Q} = \text{diag}(0, 1, \dots, 1).$$

We then have for $u = {}^T(\phi, w) \in \mathbb{R}^{n+1}$, $w = {}^T(w^1, \dots, w^n) = {}^T(w', w^n)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad Q_j u = \begin{pmatrix} 0 \\ w^j \\ 0 \end{pmatrix}, \quad Q_n u = \begin{pmatrix} 0 \\ 0 \\ w^n \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ w' \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We denote $e'_1 = {}^T(1, 0, \dots, 0) \in \mathbb{R}^{n-1}$. We note that

$$[Q_0 u] = [\phi] \quad \text{for } u = {}^T(\phi, w).$$

For a function $f = f(x')$ ($x' \in \mathbb{R}^{n-1}$), we denote its Fourier transform by $\hat{f}$ or $\mathcal{F}f$:

$$\hat{f}(\xi') = (\mathcal{F}f)(\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} dx'.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$(\mathcal{F}^{-1}f)(x') = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} d\xi'.$$

For closed linear operator A in X we denote the spectrum of A by $\sigma(A)$. We denote the set of all bounded linear operators from X_0 into itself by $L(X_0)$ and denote the norm by $|\cdot|_{L(X_0)}$. For operators A, B we denote $[A, B]$ the commutator, i.e., $[A, B] = AB - BA$. For time interval $[a, b] \subset \mathbb{R}$, we denote the usual Bochner spaces by $L^2(a, b; X)$, $H^m(a, b; X)$, etc., where X denotes a Banach space.

Definition 2.1 For a domain E we define the following function spaces:

$$X_0 = H^1(0, 1) \times L^2(0, 1), \quad H_*^j(E) = \begin{cases} H^{-1}(E) = (H_0^1)^*(E) & \text{for } j = -1, \\ L^2(E) & \text{for } j = 0, \\ H^j(E) \cap H_0^1(E) & \text{for } j \geq 1. \end{cases}$$

Definition 2.2 We introduce the following norms:

$$\|f(t)\|_k = \left(\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \|\partial_t^j f(t)\|_{H^{k-2j}} \right)^{\frac{1}{2}},$$

$$\|Df(t)\|_k = \begin{cases} \|\partial_x f(t)\|_2 & \text{for } k = 0, \\ (\|\partial_x f(t)\|_k^2 + \|\partial_t f(t)\|_{k-1}^2)^{\frac{1}{2}} & \text{for } k \geq 1. \end{cases}$$

Remark 2.3 Let us note that

$$\|Dv\|_{m-1} \leq 2\|v\|_m \quad \text{and} \quad \|v\|_m \leq \|v\|_2 + \|Dv\|_{m-1},$$

for $\|v\|_m < \infty$.

Definition 2.4 Let $m \geq [n/2] + 1$. For $\tau > 0$ we define a function space $Z^m(\tau)$ by

$$Z^m(\tau) = \{u \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0, \tau]; H^{m-2j}), \|u\|_{Z^m(\tau)} < \infty\},$$

where

$$\|u\|_{Z^m(\tau)} = \sup_{0 \leq z \leq \tau} \llbracket u(z) \rrbracket_m + \left(\int_0^\tau \|Dw(z)\|_m^2 dz \right)^{\frac{1}{2}}.$$

3 Main results

In this section we state the main results of this paper.

In the whole article we assume the following regularity for $\tilde{\mathbf{g}}$ and $\tilde{V}^1$.

Assumptions 3.1 For a given integer $m \geq [n/2] + 1$ assume that $\tilde{\mathbf{g}} = {}^T(\tilde{g}^1(\tilde{x}_n, \tilde{t}), 0, \dots, 0, \tilde{g}^n(\tilde{x}_n))$ and $\tilde{V}^1(\tilde{t})$ belong to the following spaces:

$$\tilde{g}^n \in C^{m+1}[0, \ell],$$

and

$$\begin{aligned} \tilde{g}^1 &\in \bigcap_{j=0}^{[\frac{m+1}{2}]} C_{per}^j([0, \tau]; H^{m+1-2j}(0, \ell)), \\ \tilde{V}^1 &\in C_{per}^{[\frac{m+2}{2}]}([0, \tau]). \end{aligned}$$

Furthermore, we assume

$$\tilde{P}(\cdot) \in C^{m+2}(\mathbb{R}).$$

It is straightforward that $\mathbf{g}$ and V^1 belong to similar spaces as $\tilde{\mathbf{g}}$ and $\tilde{V}^1$.

Under Assumptions 3.1 one can see that flow u_p has the following properties (see [1]).

Proposition 3.2 There exists $\delta_0 > 0$ such that if

$$\nu|g^n|_{C^{m+1}([0,1])} \leq \delta_0,$$

then the following assertions hold true(see [1]). The flow $u_p = {}^T(\rho_p(x_n), v_p(x_n, t))$ exists and under Assumptions 3.1, it satisfies

$$v_p \in \bigcap_{j=0}^{[\frac{m+3}{2}]} C_{per}^j(J_T; H^{m+3-2j}(0, 1)), \quad \rho_p \in C^{m+2}[0, 1],$$

and

$$0 < \rho_1 \leq \rho_p(x_n) \leq \rho_2, \quad \int_0^1 \rho_p(x_n) dx_n = 1, \quad v_p(x_n, t) = {}^T(v_p^1(x_n, t), 0),$$

with

$$P'(\rho) > 0 \text{ for } \rho_1 \leq \rho \leq \rho_2,$$

$$|1 - \rho_p|_{C^{k+1}([0,1])} \leq \frac{C}{\gamma^2} \nu(|P''|_{C^{k-1}(\rho_1, \rho_2)} + |g^n|_{C^k([0,1])}), \quad k = 1, \dots, m+1, \quad (3.1)$$

$$|P'(\rho_p) - \gamma^2|_{C^0([0,1])} \leq \frac{C}{\gamma^2} \nu|g^n|_{C^0([0,1])},$$

for some constants $0 < \rho_1 < 1 < \rho_2$.

First, let us introduce the local existence result. To do so, we rewrite (1.5)–(1.8) in the form

$$\partial_t \phi + v \cdot \nabla \phi = -\gamma^2 w \cdot \nabla \rho_p - \rho \operatorname{div} w, \quad (3.2)$$

$$\rho \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w = -\frac{\nu}{\gamma^2 \rho_p} \partial_{x_n}^2 v_p \phi - \nabla (P(\rho) - P(\rho_p)) - \rho(v \cdot \nabla v), \quad (3.3)$$

$$w|_{\partial\Omega} = 0, \quad (3.4)$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \quad (3.5)$$

where $\rho = \rho_p + \gamma^{-2} \phi$ and $v = v_p + w$.

Next, let us mention the compatibility condition for $u_0 = {}^T(\phi_0, w_0)$. We look for a solution $u = {}^T(\phi, w)$ of (3.2)–(3.5) in $\bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \infty); H^{m-2j})$ satisfying $\int_0^t \|\partial_x w(z)\|_{H^m}^2 dz < \infty$ for all $t \geq 0$ with $m \geq [n/2] + 1$. Therefore, we need to require the compatibility condition for the initial value $u_0 = {}^T(\phi_0, w_0)$, which is formulated as follows.

Let $u = {}^T(\phi, w)$ be a smooth solution of (3.2)–(3.5). Then $\partial_t^j u = {}^T(\partial_t^j \phi, \partial_t^j w)$, $j \geq 1$ is inductively determined by

$$\partial_t^j \phi = -v \cdot \nabla \partial_t^{j-1} \phi - \rho \operatorname{div} \partial_t^{j-1} w - \gamma^2 \partial_t^{j-1} w \cdot \nabla \rho_p - \{[\partial_t^{j-1}, v \cdot \nabla] \phi + [\partial_t^{j-1}, \rho] \operatorname{div} w\},$$

and

$$\begin{aligned} \partial_t^j w = & -\rho^{-1} \{-\nu \Delta \partial_t^{j-1} w - \tilde{\nu} \nabla \operatorname{div} \partial_t^{j-1} w + P'(\rho) \nabla \partial_t^{j-1} \rho\} - \rho^{-1} \{\gamma^{-2} [\partial_t^{j-1}, \phi] \partial_t w + [\partial_t^{j-1}, P'(\rho)] \nabla \rho\} \\ & - \rho^{-1} \left\{ \frac{\nu}{\gamma^2 \rho_p} \partial_t^{j-1} (\partial_{x_n}^2 v_p \phi) - \partial_t^{j-1} \nabla P(\rho_p) \right\} - \rho^{-1} \partial_t^{j-1} (\rho(v \cdot \nabla v)). \end{aligned}$$

From these relations we see that $\partial_t^j u|_{t=0} = {}^T(\partial_t^j \phi, \partial_t^j w)|_{t=0}$ is inductively given by $u_0 = {}^T(\phi_0, w_0)$ in the following way:

$$\partial_t^j u|_{t=0} = {}^T(\partial_t^j \phi, \partial_t^j w)|_{t=0} = {}^T(\phi_j, w_j) = u_j,$$

where

$$\phi_j = -v_0 \cdot \nabla \phi_{j-1} - \rho_0 \operatorname{div} w_{j-1} - \gamma^2 w_{j-1} \cdot \nabla \rho_p - \sum_{l=1}^{j-1} \binom{j-1}{l} \{v_l \cdot \nabla \phi_{j-1-l} + \gamma^{-2} \phi_l \operatorname{div} w_{j-1}\},$$

and

$$\begin{aligned} w_j = & -\rho_0^{-1} \{-\nu \Delta w_{j-1} - \tilde{\nu} \nabla \operatorname{div} w_{j-1} + P'(\rho_0) \nabla \rho_{j-1}\} - \rho_0^{-1} \sum_{l=1}^{j-1} \binom{j-1}{l} \{\gamma^{-2} \phi_l w_{j-l} \\ & + a_l(\phi_0; \phi_1, \dots, \phi_l) \nabla \rho_{j-1-l}\} - \rho_0^{-1} \frac{\nu}{\gamma^2 \rho_p} \sum_{l=0}^{j-1} \binom{j-1}{l} \partial_t^{j-1-l} \partial_{x_n}^2 v_p(0) \phi_l + \delta_{1j} \rho_0^{-1} \nabla P(\rho_p) \\ & - \rho_0^{-1} G_{j-1}(\phi_0, w_0, \partial_x w_0; \phi_1, \dots, \phi_{j-1}, w_1, \dots, w_{j-1}, \partial_x w_1, \dots, \partial_x w_{j-1}), \end{aligned}$$

with $v_l = \partial_t^l v_p(0) + w_l$, $\rho_l = \delta_{1l} \rho_p + \gamma^{-2} \phi_l$; and $a_l(\phi_0; \phi_1, \dots, \phi_l)$ is certain polynomial in $\phi_1, \dots, \phi_l$; and analogously. Here, δ_{1j} denotes the Kronecker's delta.

By the boundary condition $w|_{\partial\Omega} = 0$ in (3.4), we necessarily have $\partial_t^j w|_{\partial\Omega} = 0$, and hence,

$$w_j|_{\partial\Omega} = 0.$$

Assume that $u = {}^T(\phi, w)$ is a solution of (3.2)–(3.5) in $\bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \tau_0]; H^{m-2j})$ for some $\tau_0 > 0$. Then from above observation, we need the regularity $u_j = {}^T(\phi_j, w_j) \in H^{m-2j} \times H^{m-2j}$ for $j = 1, \dots, [m/2]$, which, indeed follows from the fact that $u_0 = {}^T(\phi_0, w_0) \in H^m$ with $m \geq [n/2] + 1$. Furthermore, it is necessary to require that $u_0 = {}^T(\phi_0, w_0)$ satisfies the $\hat{m}$ -th order compatibility condition:

$$w_j \in H_0^1 \text{ for } j = 0, \dots, \hat{m} = \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Now, we can apply local solvability result obtained in [8] to show the following assertion.

Proposition 3.3 Let $n \geq 2$, m be an integer satisfying $m \geq [n/2] + 1$ and $M > 0$. Assume that $u_0 = {}^T(\phi_0, w_0) \in H^m$ satisfies the following conditions:

- (a) $\|u_0\|_{H^m} \leq M$ and u_0 satisfies the $\hat{m}$ -th compatibility condition,
- (b) $-\frac{\gamma^2}{4}\rho_1 \leq \phi_0$.

Then there exists a positive number τ_0 depending on M and ρ_1 such that problem (3.2)–(3.5) has a unique solution $u(t)$ on $[0, \tau_0]$ satisfying $u(t) \in Z^m(\tau_0)$.

Remark 3.4 It is straightforward to see that solution $u(t)$ of (3.2)–(3.5) is a solution of (1.5)–(1.8). Condition (b) in previous proposition assures that $\gamma^{-2}\phi_0 + \rho_p > \frac{3}{4}\rho_1$.

We are in a position to state our main results of this paper.

Theorem 3.5 Suppose that $n = 2$ and let m be an integer satisfying $m \geq 2$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ then the following assertions hold true.

There is a positive number ε_0 such that if $u_0 = {}^T(\phi_0, w_0) \in H^m \cap L^1$ satisfies the $\hat{m}$ -th compatibility condition and $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of (1.5)–(1.8) with $n = 2$ in $\bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \infty); H^{m-2j})$ which satisfies

$$\|\partial_x^k u(t)\|_2 = O(t^{-\frac{1}{4} - \frac{k}{2}}), \quad k = 0, 1, \quad (3.6)$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{3}{4} + \delta}), \quad \forall \delta > 0, \quad (3.7)$$

as $t \rightarrow \infty$. Here, $u^{(0)} = u^{(0)}(x_2, t)$ is function given in Lemma 4.9 below; $\sigma = \sigma(x_1, t)$ is function satisfying

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + \omega_0 \partial_{x_1} (\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2) dx_2,$$

with given constants $\kappa_0, \omega_0 \in \mathbb{R}$, $\kappa_1 > 0$.

Theorem 3.6 Suppose that $n \geq 3$ and let m be an integer satisfying $m \geq [n/2] + 1$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ then the following assertions hold true.

There is a positive number ε_0 such that if $u_0 = {}^T(\phi_0, w_0) \in H^m \cap L^1$ satisfies the $\hat{m}$ -th compatibility condition and $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of (1.5)–(1.8) in $\bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \infty); H^{m-2j})$ which satisfies

$$\|\partial_x^k u(t)\|_2 = O(t^{-\frac{n-1}{4} - \frac{k}{2}}), \quad k = 0, 1, \quad (3.8)$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n-1}{4} - \frac{1}{2}} \eta_n(t)), \quad (3.9)$$

as $t \rightarrow \infty$. Here, $\sigma = \sigma(x', t)$ is function satisfying

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) dx_n,$$

with given constants $\kappa_0 \in \mathbb{R}$, $\kappa_1, \kappa'' > 0$; where $\Delta'' = \partial_{x_2}^2 + \dots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

As in [8, 12], the global existence result in Theorem 3.5 and Theorem 3.6 is proved by combining the local existence and the a priori estimate. Next we introduce the a priori estimate.

Proposition 3.7 Let $n \geq 2$ and m be an integer satisfying $m \geq [n/2] + 1$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$, then the following assertion holds true.

There exists number $\varepsilon_1 > 0$ such that if solution $u(t)$ of (1.5)–(1.8) is in $Z^m(\tau)$ and $u(t)$ satisfies $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_1$, then there holds the estimate

$$\|u(t)\|_m^2 \leq C_1 \|u_0\|_{H^m \cap L^1}^2,$$

for a constant $C_1 > 0$ independent of τ .

Remark 3.8 In the proof of Proposition 3.7 we use the estimate (3.1) with $k = m$ only, i.e.,

$$|1 - \rho_p|_{C^{m+1}([0,1])} \leq \frac{C}{\gamma^2} \nu(|P''|_{C^{m-1}(\rho_1, \rho_2)} + |g^n|_{C^m([0,1])}).$$

Moreover, we require the boundedness of ρ_p in $C^{m+1}([0,1])$ only.

The global existence of the solution $u(t)$ follows from Proposition 3.3 and the a priori estimate in Proposition 3.7 in standard manner as follows.

Proof of global existence. Let $n \geq 2$ and let us fix $m \geq [n/2] + 1$ and $\nu \geq \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ such that Proposition 3.7 holds true.

Since $m \geq [n/2] + 1$ we have the Sobolev inequality

$$\|f\|_\infty \leq C_S \|f\|_{H^m}, \text{ for any } f \in H^m(\Omega). \quad (3.10)$$

Let us define $\varepsilon_0 > 0$ as

$$\varepsilon_0 = \min\{\varepsilon_1, \frac{\gamma^2}{4C_S} \rho_1, \frac{\varepsilon_1}{\sqrt{C_1}}, \frac{\gamma^2}{4C_S \sqrt{C_1}} \rho_1\}.$$

Here, ε_1 and C_1 are given by Proposition 3.7.

Let $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$ satisfies $\widehat{m}$ -th compatibility condition. It is easy to see that such u_0 satisfies conditions (a), (b) of Proposition 3.3 and therefore, there exists $\tau_0 > 0$, which is determined by ε_1 , such that the problem (1.5)–(1.8) has a unique solution $u(\cdot) \in Z^m(\tau_0)$.

Since $\varepsilon_0 \leq \varepsilon_1$ we see from Proposition 3.7 that $u(t)$ satisfies

$$\|u(\tau_0)\|_m^2 \leq C_1 \|u_0\|_{H^m \cap L^1}^2 \leq \min\{\varepsilon_1^2, \left(\frac{\gamma^2}{4C_S} \rho_1\right)^2\}. \quad (3.11)$$

Thus, $\|u(\tau_0)\|_{H^m} \leq \varepsilon_1$ and $u(\tau_0)$ satisfies conditions (a) and (b) of Proposition 3.3. Hence, there exists unique extension of solution $u(t)$ of (1.5)–(1.8) on $[\tau_0, 2\tau_0]$ and we get

$$u(\cdot) \in Z^m(2\tau_0).$$

It is straightforward to see that we can use Proposition 3.7 again, to obtain estimate (3.11) for $u(2\tau_0)$, which enables us to extend solution $u(t)$ on $[0, 3\tau_0]$. By repeating this procedure the existence on $[0, \infty)$ is showed. This concludes the proof. $\square$

Proposition 3.7 together with L^2 -decay estimates (3.6) and (3.8) are proved in Sections 4-8. The asymptotic behavior, i.e., (3.7) and (3.9), is proved in Section 9.

4 Spectral properties of the linearized operator

Let us write (1.5)–(1.8) in the form

$$\begin{aligned} \partial_t u + L(t)u &= \mathbf{F}, \\ w|_{\partial\Omega} &= 0, \quad u|_{t=0} = u_0. \end{aligned} \quad (4.1)$$

Here, $u = {}^T(\phi, w)$; $\mathbf{F} = {}^T(f^0, \mathbf{f})$ with $\mathbf{f} = {}^T(f^1, \dots, f^n)$ is the nonlinearity; and $L(t)$ is the operator given in (1.10)

In this section we introduce the spectral properties of the linearized problem, i.e., (4.1) with $\mathbf{F} = 0$. These results were established in [2]. At the end of this section we show regularity improvements for ϕ .

Now, let us consider the linearized problem

$$\partial_t u + L(t)u = 0, \quad t > s, \quad w|_{x_n=0,1} = 0, \quad u|_{t=s} = u_0. \quad (4.2)$$

We introduce space Z_s defined by

$$Z_s = \{u = {}^T(\phi, w); \phi \in C_{loc}([s, \infty); H^1), \partial_{x'}^\alpha w \in C_{loc}([s, \infty); L^2) \cap L_{loc}^2([s, \infty); H_0^1) (|\alpha'| \leq 1), w \in C_{loc}((s, \infty); H_0^1)\}.$$

In [1] we showed that for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in H^1 \cap L^2$ with $\partial_{x'} w_0 \in L^2$ there exists a unique solution $u(t)$ of linear problem (4.2) in Z_s . We denote $U(t, s)$ the evolution operator for (4.2) given by

$$u(t) = U(t, s)u_0.$$

To investigate problem (4.2) we consider the Fourier transform of (4.2). We thus obtain

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0, \quad t > s, \quad \widehat{u}|_{t=s} = \widehat{u}_0. \quad (4.3)$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transforms of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' \in \mathbb{R}^{n-1}$ being the dual variable; $\widehat{L}_{\xi'}(t)$ is an operator on X_0 with domain $D(\widehat{L}_{\xi'}(t)) = H^1(0, 1) \times H_*^2(0, 1)$, which takes the form

$$\begin{aligned} \widehat{L}_{\xi'}(t) = & \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 \rho_p^T \xi' & \gamma^2 \partial_{x_n}(\rho_p \cdot) \\ i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p} \xi'^T \xi' & -i \frac{\tilde{\nu}}{\rho_p} \xi' \partial_{x_n} \\ \partial_{x_n} \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \cdot \right) & -i \frac{\tilde{\nu}}{\rho_p} \xi' \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p} \partial_{x_n}^2 \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1(t))e'_1 & i\xi_1 v_p^1(t)I_{n-1} & \partial_{x_n}(v_p^1(t))e'_1 \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}. \end{aligned}$$

Let us note that $\widehat{L}_{\xi'}(t)$ is sectorial uniformly with respect to $t \in \mathbb{R}$ for each $\xi' \in \mathbb{R}^{n-1}$. As for the evolution operator $\widehat{U}_{\xi'}(t, s)$ for (4.3) we have the following results.

Lemma 4.1 *For each $\xi' \in \mathbb{R}^{n-1}$ and for all $t \geq s$ there exists unique evolution operator $\widehat{U}_{\xi'}(t, s)$ for (4.3) that satisfies*

$$|\widehat{L}_{\xi'}(t)\widehat{U}_{\xi'}(t, s)|_{L(X_0)} \leq C_{t_1 t_2}, \quad t_1 \leq s < t \leq t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha([s, \infty); X_0)$, $\alpha \in (0, 1]$ there exists unique classical solution u of inhomogeneous problem

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = f, \quad t > s, \quad u|_{t=s} = u_0, \quad (4.4)$$

satisfying $u \in C_{loc}([s, \infty); X_0) \cap C^1(s, \infty; X_0) \cap C(s, \infty; H^1(0, 1) \times H_*^2(0, 1))$; and the solution u is given by

$$u(t) = (\phi(t), w(t)) = \widehat{U}_{\xi'}(t, s)u_0 + \int_s^t \widehat{U}_{\xi'}(t, z)f(z)dz.$$

Next, let us introduce *adjoint problem* to

$$\partial_t u + \widehat{L}_{\xi'}(t)u = 0, \quad t > s, \quad u|_{t=s} = u_0.$$

Lemma 4.2 *For each $\xi' \in \mathbb{R}^{n-1}$ and for all $s \leq t$ there exists unique evolution operator $\widehat{U}_{\xi'}^*(s, t)$ for adjoint problem*

$$-\partial_s u + \widehat{L}_{\xi'}^*(s)u = 0, \quad s < t, \quad u|_{s=t} = u_0,$$

on X_0 . Here, $\widehat{L}_{\xi'}^*(s)$ is an operator on X_0 with domain $D(\widehat{L}_{\xi'}^*(s)) = H^1(0, 1) \times H_*^2(0, 1)$, which takes the form

$$\begin{aligned} \widehat{L}_{\xi'}^*(s) = & \begin{pmatrix} -i\xi_1 v_p^1(s) & -i\gamma^2 \rho_p^T \xi' & -\gamma^2 \partial_{x_n}(\rho_p \cdot) \\ -i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p} \xi'^T \xi' & -i \frac{\tilde{\nu}}{\rho_p} \xi' \partial_{x_n} \\ -\partial_{x_n} \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \cdot \right) & -i \frac{\tilde{\nu}}{\rho_p} \xi' \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p} \partial_{x_n}^2 \end{pmatrix} \\ & + \begin{pmatrix} 0 & \frac{\nu \gamma^2}{P'(\rho_p)}(\partial_{x_n}^2 v_p^1(s))^T e'_1 & 0 \\ 0 & -i\xi_1 v_p^1(s)I_{n-1} & 0 \\ 0 & \partial_{x_n}(v_p^1(s))^T e'_1 & -i\xi_1 v_p^1(s) \end{pmatrix}. \end{aligned}$$

Moreover, $\widehat{L}_{\xi'}^*(s)$ satisfies $\langle \widehat{L}_{\xi'}^*(s)u, v \rangle = \langle u, \widehat{L}_{\xi'}(s)v \rangle$ for $s \in \mathbb{R}$ and $u, v \in H^1 \times H_*^2$ and

$$|\widehat{L}_{\xi'}^*(s)\widehat{U}_{\xi'}^*(s,t)|_{L(X_0)} \leq C_{t_1 t_2}, \quad t_1 \leq s < t \leq t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha((-\infty, t]; X_0)$, $\alpha \in (0, 1]$ there exists unique classical solution u of inhomogeneous problem

$$-\frac{d}{ds}u + \widehat{L}_{\xi'}^*(s)u = f, \quad s < t, \quad u|_{s=t} = u_0, \quad (4.5)$$

satisfying $u \in C_{loc}((-\infty, t]; X_0) \cap C^1(-\infty, t; X_0) \cap C(-\infty, t; H^1(0, 1) \times H_*^2(0, 1))$; and the solution u is given by

$$u(s) = (\phi(s), w(s)) = \widehat{U}_{\xi'}^*(s, t)u_0 + \int_s^t \widehat{U}_{\xi'}^*(s, z)f(z)dz.$$

Note that $\widehat{U}_{\xi'}(t, s)$ and $\widehat{U}_{\xi'}^*(s, t)$ are defined for all $t \geq s$ and

$$\widehat{U}_{\xi'}(t+T, s+T) = \widehat{U}_{\xi'}(t, s), \quad \widehat{U}_{\xi'}^*(s+T, t+T) = \widehat{U}_{\xi'}^*(s, t).$$

Lemma 4.3 *There exist positive numbers ν_1 and γ_1 such that if $\nu \geq \nu_1$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_1^2$ then there exists $r_0 > 0$ such that for each ξ' with $|\xi'| \leq r_0$ there hold the following statements.*

(i) *The spectrum of operator $\widehat{U}_{\xi'}(T, 0)$ on $H^1(0, 1) \times H_0^1(0, 1)$ satisfies*

$$\sigma(\widehat{U}_{\xi'}(T, 0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\},$$

with constant $q_0 < \operatorname{Re} \mu_{\xi'} < 1$. Here, $\mu_{\xi'} = e^{\lambda_{\xi'} T}$ is simple eigenvalue of $\widehat{U}_{\xi'}(T, 0)$ and $\lambda_{\xi'}$ has an expansion

$$\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3), \quad (4.6)$$

where $\kappa_0 \in \mathbb{R}$ and $\kappa_1 > 0$, $\kappa'' > 0$.

Moreover, let $\widehat{\Pi}_{\xi'}$ denote the eigenprojection associated with $\mu_{\xi'}$. There holds

$$|\widehat{U}_{\xi'}(t, s)(I - \widehat{\Pi}_{\xi'})u|_{H^1} \leq Ce^{-d(t-s)}|(I - \widehat{\Pi}_{\xi'})u|_{X_0},$$

for $u \in X_0$ and $T \leq t - s$. Here, d is a positive constant depending on r_0 .

(ii) *The spectrum of operator $\widehat{U}_{\xi'}^*(0, T)$ on $H^1(0, 1) \times H_0^1(0, 1)$ satisfies*

$$\sigma(\widehat{U}_{\xi'}^*(0, T)) \subset \{\bar{\mu}_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}.$$

Here, $\bar{\mu}_{\xi'}$ is simple eigenvalue of $\widehat{U}_{\xi'}^*(0, T)$.

Furthermore, let $\widehat{\Pi}_{\xi'}^*$ denote the eigenprojection associated with $\bar{\mu}_{\xi'}$. There holds

$$\langle \widehat{\Pi}_{\xi'} u, v \rangle = \langle u, \widehat{\Pi}_{\xi'}^* v \rangle,$$

for $u, v \in X_0$.

Next, we introduce Floquet theory.

Definition 4.4 *Let $k = 1, 2, \dots$. Let us define spaces Y_{per}^k as*

$$Y_{per}^1 = L_{per}^2([0, T]; X_0),$$

$$Y_{per}^k = \bigcap_{j=0}^{\lfloor \frac{k}{2} \rfloor} H_{per}^j([0, T]; H^{k-2j}(0, 1) \times H^{k-1-2j}(0, 1)), \quad \text{for } k \geq 2.$$

Here, for Banach space X and $j = 0, \dots$ spaces $L_{per}^2([0, T]; X)$ and $H_{per}^j([0, T]; X)$ consist of functions from $L^2([0, T]; X)$ and $H^j([0, T]; X)$, respectively, that are restrictions of T -periodic functions.

Definition 4.5 We define operator $B_{\xi'}$ on space Y_{per}^1 with domain

$$D(B_{\xi'}) = H_{per}^1([0, T]; X_0) \cap L_{per}^2([0, T]; H^1(0, 1) \times H_*^2(0, 1)),$$

in the following way

$$B_{\xi'} v = \partial_t v + \widehat{L}_{\xi'}(\cdot) v,$$

for $v \in D(B_{\xi'})$. Moreover, we define formal adjoint operator $B_{\xi'}^*$ with respect to inner product $\frac{1}{T} \int_0^T \langle \cdot, \cdot \rangle dt$ as

$$B_{\xi'}^* v = -\partial_t v + \widehat{L}_{\xi'}^*(\cdot) v,$$

for $v \in D(B_{\xi'}^*) = D(B_{\xi'})$.

Remark 4.6 Operators $B_{\xi'}$ and $B_{\xi'}^*$ are closed, densely defined on Y_{per}^1 for each fixed $\xi' \in \mathbb{R}^{n-1}$.

Definition 4.7 Let $k \geq 1$. We say that $u = {}^T(\phi, w)$ is k -regular function on time interval $[a, b]$ whenever

$$\begin{aligned} u &\in \bigcap_{j=0}^{\lfloor \frac{k}{2} \rfloor} C^j([a, b]; (H^{k-2j} \times H_*^{k-2j})(\Omega)), \\ \phi &\in \bigcap_{j=0}^{\lfloor \frac{k}{2} \rfloor} H^{j+1}(a, b; H^{k-2j}(\Omega)), \quad w \in \bigcap_{j=0}^{\lfloor \frac{k+1}{2} \rfloor} H^j(a, b; H_*^{k+1-2j}(\Omega)). \end{aligned}$$

Lemma 4.8 There exist positive numbers $\nu_2 \geq \nu_1$ and $\gamma_2 \geq \gamma_1$ such that if $\nu \geq \nu_2$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_2^2$ then there exists $0 < r_1 \leq 1$ such that for each $|\xi'| \leq r_1$ there hold the following statements.

(i) Let $1 \leq k \leq m+1$. There exists $q_1 > 0$ such that spectrum of operator $B_{\xi'}$ on Y_{per}^k satisfies

$$\sigma(B_{\xi'}) \subset \{-\lambda_{\xi'}\} \cup \{\lambda : \operatorname{Re} \lambda \geq q_1\},$$

with $0 \leq -\operatorname{Re} \lambda_{\xi'} \leq \frac{1}{2} q_1$ uniform for all k . Here, $-\lambda_{\xi'}$ is simple eigenvalue of $B_{\xi'}$.

(ii) Let $1 \leq k \leq m+1$. Spectrum of operator $B_{\xi'}^*$ on Y_{per}^k satisfies

$$\sigma(B_{\xi'}^*) \subset \{-\bar{\lambda}_{\xi'}\} \cup \{\lambda : \operatorname{Re} \lambda \geq q_1\}.$$

Here, $-\bar{\lambda}_{\xi'}$ is simple eigenvalue of $B_{\xi'}^*$.

(iii) There exist $u_{\xi'}$ and $u_{\xi'}^*$ eigenfunctions associated with $-\lambda_{\xi'}$ and $-\bar{\lambda}_{\xi'}$, respectively, with the following properties:

$$\langle u_{\xi'}(t), u_{\xi'}^*(t) \rangle = 1,$$

$$u_{\xi'}(t) = u^{(0)}(t) + i\xi' \cdot u^{(1)}(t) + |\xi'|^2 u^{(2)}(\xi', t),$$

$$u_{\xi'}^*(t) = u^{*(0)} + i\xi' \cdot u^{*(1)}(t) + |\xi'|^2 u^{*(2)}(\xi', t),$$

for $t \in \mathbb{R}$. Here, all functions

$$u_{\xi'}, u_{\xi'}^*, u^{(0)}, u^{(0)*}, u^{(1)}, u^{(1)*}, u^{(2)}(\xi'), u^{(2)*}(\xi'),$$

are T -periodic in t , $m+1$ -regular on $[0, T]$ and we have estimate

$$\sup_{z \in J_T} \sum_{j=0}^{\lfloor \frac{m+1}{2} \rfloor} |\partial_z^j u(z)|_{H^{m+1-2j}}^2 + \int_0^T \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} |\partial_z^{j+1} u|_{H^{m+1-2j} \times H^{m-2j}}^2 + |\partial_z^{\lfloor \frac{m+3}{2} \rfloor} Q_0 u|_2^2 + |u|_{H^{m+1} \times H^{m+2}}^2 dz \leq C,$$

for $u \in \{u_{\xi'}, u_{\xi'}^*, u^{(2)}(\xi'), u^{(2)*}(\xi')\}$ and a constant $C > 0$ depending on r_1 .

Let us introduce more properties of $u^{(0)}$.

Lemma 4.9 *Function $u^{(0)}(t)$ satisfies $\partial_t u^{(0)} + \widehat{L}_0(t)u^{(0)} = 0$ and $u^{(0)}(t) = u^{(0)}(t+T)$ for all $t \in \mathbb{R}$. Function $u^{(0)}(t)$ is given as*

$$u^{(0)}(x_n, t) = {}^T(\phi^{(0)}(x_n), w^{(0),1}(x_n, t), 0).$$

Here,

$$\begin{aligned} \phi^{(0)}(x_n) &= \alpha_0 \frac{\gamma^2 \rho_p(x_n)}{P'(\rho_p(x_n))}, & \alpha_0 &= \left[\frac{\gamma^2 \rho_p}{P'(\rho_p)} \right]^{-1}, \\ w^{(0),1}(x_n, t) &= -\frac{1}{\gamma^2} \int_{-\infty}^t e^{-(t-s)\nu A} \nu \frac{\alpha_0 \gamma^2}{P'(\rho_p) \rho_p} (\partial_{x_n}^2 v_p^1(s)) ds, \end{aligned}$$

where A denotes the uniformly elliptic operator on $L^2(0, 1)$ with domain $D(A) = (H^2 \cap H_0^1)(0, 1)$ and

$$Av = -\frac{1}{\rho_p(x_n)} \partial_{x_n}^2 v,$$

for $v \in D(A)$. Moreover, function $w^{(0),1}$ satisfies

$$\partial_t w^{(0),1}(t) - \frac{\nu}{\rho_p(x_n)} \partial_{x_n}^2 w^{(0),1}(t) = -\frac{\nu}{\gamma^2} \frac{\alpha_0 \gamma^2}{P'(\rho_p) \rho_p} (\partial_{x_n}^2 v_p^1(t)), \quad (4.7)$$

for all $t \in \mathbb{R}$ and

$$\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O\left(\frac{1}{\gamma^2}\right).$$

In the rest of this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_2^2$.

Definition 4.10 *We define $\widehat{\chi}_1$ by*

$$\widehat{\chi}_1(\xi') = \begin{cases} 1, & |\xi'| < r_1, \\ 0, & |\xi'| \geq r_1, \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$.

Now, we introduce time-periodic operators based on $u_{\xi'}$ and $u_{\xi'}^*$.

Definition 4.11 *We define operators $\mathcal{P}(t) : L^2(\Omega) \rightarrow L^2(\mathbb{R}^{n-1})$ by*

$$\mathcal{P}(t)u = \mathcal{F}^{-1}\{\widehat{\mathcal{P}}_{\xi'}(t)\widehat{u}\},$$

$$\widehat{\mathcal{P}}_{\xi'}(t)\widehat{u} = \widehat{\chi}_1 \langle \widehat{u}, u_{\xi'}^*(t) \rangle,$$

for $u \in L^2$ and $t \in [0, \infty)$.

We define operators $\mathcal{Q}(t) : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\Omega)$ by

$$\mathcal{Q}(t)\sigma = \mathcal{F}^{-1}\{\widehat{\mathcal{Q}}_{\xi'}(t)\widehat{\sigma}\},$$

$$\widehat{\mathcal{Q}}_{\xi'}(t)\widehat{\sigma} = \widehat{\chi}_1 u_{\xi'}(\cdot, t)\widehat{\sigma},$$

for $t \in [0, \infty)$ and multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})$ by

$$\Lambda\sigma = \mathcal{F}^{-1}\{\widehat{\chi}_1 \lambda_{\xi'} \widehat{\sigma}\},$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

Moreover, we define projections $\mathbb{P}(t)$ and $\mathbb{P}^*(t)$ on $L^2(\Omega)$ as

$$\mathbb{P}(t)u = \mathcal{F}^{-1}\{\widehat{\chi}_1 \langle u, u_{\xi'}^*(t) \rangle u_{\xi'}(\cdot, t)\} = \mathcal{Q}(t)\mathcal{P}(t)u,$$

$$\mathbb{P}^*(t)u = \mathcal{F}^{-1}\{\widehat{\chi}_1 \langle u, u_{\xi'}(t) \rangle u_{\xi'}^*(\cdot, t)\},$$

for $t \in [0, \infty)$ and $u \in L^2$.

We define projection $\Pi^{(0)}(t)$ on $L^2(\Omega)$ as

$$\Pi^{(0)}(t)u = [Q_0 u]u^{(0)}(t),$$

for $t \in [0, \infty)$ and $u \in L^2$.

In terms of $P(t)$ we have the following decomposition of $U(t, s)$.

Lemma 4.12 $\mathbb{P}(t)$ and $\mathbb{P}^*(t)$ satisfies the following:

(i)

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))\mathbb{P}(t)u(t) = \mathcal{Q}(t)[(\partial_t - \Lambda)\mathcal{P}(t)u(t)],$$

for $u \in L^2([0, T]; H^1 \times H_*^2) \cap H^1([0, T]; L^2)$.

(ii)

$$\mathbb{P}(t)U(t, s) = U(t, s)\mathbb{P}(s) = \mathcal{Q}(t)e^{(t-s)\Lambda}\mathcal{P}(s).$$

If $u \in L^1$, then

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l \mathbb{P}(t)U(t, s)u\|_2 \leq C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_1,$$

for $0 \leq 2j+l \leq m+1$, $k=0, \dots$.

(iii) For $u, v \in L^2$ there holds

$$\langle \mathbb{P}(t)u, v \rangle = \langle u, \mathbb{P}^*(t)u \rangle.$$

If $u \in L^2$, then

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l (\mathbb{P}^*(t)u)\|_2 \leq C\|u\|_2,$$

for $0 \leq 2j+l \leq m+1$, $k=0, 1, \dots$.

(iv) $(I - \mathbb{P}(t))U(t, s) = U(t, s)(I - \mathbb{P}(s))$ satisfies

$$\|(I - \mathbb{P}(t))U(t, s)u\|_{H^1} \leq Ce^{-d(t-s)}(\|u\|_{H^1 \times L^2} + \|\partial_{x'} w\|_2),$$

for $t-s \geq T$. Here d is a positive constant.

Next, let us show the asymptotic properties of $U(t, s)$. First, let us define a semigroup $\mathcal{H}(t)$ on $L^2(\mathbb{R}^{n-1})$ associated with a linear heat equation with a convective term:

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0.$$

Definition 4.13 We define operator $\mathcal{H}(t)$ as

$$\mathcal{H}(t)\sigma = \mathcal{F}^{-1}[e^{-(i\kappa_0 \xi_1 + \kappa_1 \xi_1^2 + \kappa''|\xi''|^2)t} \widehat{\sigma}],$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$. Here, κ_0, κ_1 and κ'' are given by (4.6).

Lemma 4.14 There hold the following estimates for $1 \leq p \leq 2$ and $k=0, 1, \dots$.

(i)

$$\|\partial_{x'}^k (\mathcal{H}(t)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \leq Ct^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \quad (4.8)$$

for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(ii) Λ generates uniformly continuous group $\{e^{t\Lambda}\}_{t \in \mathbb{R}}$ and

$$\|\partial_{x'}^k e^{t\Lambda} \sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \quad (4.9)$$

for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(iii) It holds the relation,

$$\mathcal{P}(t)U(t, s) = e^{(t-s)\Lambda} \mathcal{P}(s).$$

Set $\sigma = [Q_0 u]$. Then

$$\|\partial_{x'}^k (e^{(t-s)\Lambda} \mathcal{P}(s)u - \mathcal{H}(t-s)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \leq C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_p, \quad (4.10)$$

for $u \in L^p$. Furthermore, for any $\sigma \in L^p(\mathbb{R}^{n-1})$ there holds

$$\|(e^{(t-s)\Lambda} - \mathcal{H}(t-s))\partial_{x'}^k \sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}. \quad (4.11)$$

Next, we introduce the properties of $\mathcal{Q}(t)$ and $\mathcal{P}(t)$.

Proposition 4.15 $\mathcal{Q}(t)$ has the following properties:

(i)

$$\mathcal{Q}(t+T) = \mathcal{Q}(t), \quad \partial_{x'}^k \mathcal{Q}(t) = \mathcal{Q}(t) \partial_{x'}^k.$$

(ii)

$$\mathcal{Q}(t)\sigma \in \bigcap_{j=0}^{\lfloor \frac{m+1}{2} \rfloor} C_{per}^j(J_T; H^{m+1-2j} \times H_*^{m+1-2j}),$$

$$\tilde{Q}\mathcal{Q}(t)\sigma \in \bigcap_{j=0}^{\lfloor \frac{m+2}{2} \rfloor} H_{per}^j(J_T; H_*^{m+2-2j}),$$

and

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l (\mathcal{Q}(t)\sigma)\|_2 \leq C \|\sigma\|_{L^2(\mathbb{R}^{n-1})}, \quad 0 \leq 2j+l \leq m+1, \quad k=0,1,\dots,$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

(iii)

$$(\partial_t + L(t))(\mathcal{Q}(t)\sigma(t)) = \mathcal{Q}(t)(\partial_t - \Lambda)\sigma(t),$$

for $\sigma \in H_{loc}^1([0, \infty); L^2(\mathbb{R}^{n-1}))$.

(iv) $\mathcal{Q}(t)$ is decomposed as

$$\mathcal{Q}(t) = \mathcal{Q}^{(0)}(t) + \operatorname{div}' \mathcal{Q}^{(1)}(t) + \Delta' \mathcal{Q}^{(2)}(t).$$

Here, $\mathcal{Q}^{(0)}(t)\sigma = (\mathcal{F}^{-1}\{\widehat{\chi}_1 \widehat{\sigma}\})u^{(0)}(\cdot, t)$, $\mathcal{Q}^{(1)}(t)$ and $\mathcal{Q}^{(2)}(t)$ share the same properties given in (i) and (ii) for $\mathcal{Q}(t)$.

Proposition 4.16 $\mathcal{P}(t)$ has the following properties:

(i)

$$\mathcal{P}(t+T) = \mathcal{P}(t), \quad \partial_{x'}^k \mathcal{P}(t) = \mathcal{P}(t) \partial_{x'}^k, \quad \partial_{x_n} \mathcal{P}(t) = 0.$$

(ii)

$$\mathcal{P}(t)u \in \bigcap_{j=0}^{\lfloor \frac{m+1}{2} \rfloor} C_{per}^j(J_T; H^k(\mathbb{R}^{n-1})), \quad \text{for all } k=0,1,\dots,$$

and

$$\|\partial_t^j \partial_{x'}^k (\mathcal{P}(t)u)\|_{L^2(\mathbb{R}^{n-1})} \leq C \|u\|_2, \quad 0 \leq 2j \leq m+1, \quad k=0,1,\dots,$$

for $u \in L^2$.

Moreover,

$$\|\mathcal{P}(t)u\|_{L^2(\mathbb{R}^{n-1})} \leq C \|u\|_p,$$

for $u \in L^p$ and $1 \leq p \leq 2$.

(iii)

$$\mathcal{P}(t)(\partial_t + L(t))u(t) = (\partial_t - \Lambda)(\mathcal{P}(t)u(t)), \quad (4.12)$$

for $u \in L_{loc}^2([0, \infty); H^1 \times H_*^2) \cap H_{loc}^1([0, \infty); L^2)$.

(iv) $\mathcal{P}(t)$ is decomposed as

$$\mathcal{P}(t) = \mathcal{P}^{(0)} + \operatorname{div}' \mathcal{P}^{(1)}(t) + \Delta' \mathcal{P}^{(2)}(t).$$

Here,

$$\mathcal{P}^{(0)}u = \mathcal{F}^{-1}\{\widehat{\chi}_1 \langle \widehat{u}, u^{*(0)} \rangle\} = \mathcal{F}^{-1}\{\widehat{\chi}_1 [Q_0 \widehat{u}]\},$$

$$\mathcal{P}^{(1)}(t)u = \mathcal{F}^{-1}\{\widehat{\chi}_1\langle\widehat{u}, u^{*(1)}(t)\rangle\},$$

$$\mathcal{P}^{(2)}(t)u = \mathcal{F}^{-1}\{-\widehat{\chi}_1\langle\widehat{u}, u^{*(2)}(\xi', t)\rangle\}.$$

$\mathcal{P}^{(p)}(t)$, $p = 0, 1, 2$, share the same properties given in (i) and (ii) for $\mathcal{P}(t)$.

(v) There holds

$$\|\partial_x^k e^{(t-s)\Lambda} \mathcal{P}(s)u\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|u\|_p, \quad (4.13)$$

$$\|\partial_{x'}^k e^{(t-s)\Lambda} \mathcal{P}^{(q)}(s)u\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|u\|_p, \quad q = 0, 1, 2, \quad (4.14)$$

for $u \in L^p$, $1 \leq p \leq 2$ and $k = 0, 1, \dots$.

Remark 4.17 Note that $\Pi^{(0)}(t)$ and $\mathcal{Q}^{(0)}(t)\mathcal{P}^{(0)}$ are not identical operators.

To close this section, we show improvements of regularity for ϕ .

Proposition 4.18 Let $u = {}^T(\phi, w) \in Z^m(\tau)$ for $m \geq [n/2] + 1$ be a solution of (4.1). There holds

$$\phi \in \bigcap_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} C^j([0, \tau]; H^{m+1-2j}). \quad (4.15)$$

Proof. From definition of $Z^m(\tau)$ we have

$$u \in \bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \tau]; H^{m-2j}) \quad \text{and} \quad \sup_{0 \leq z \leq \tau} \|u(z)\|_m < \infty.$$

We write (1.5) as

$$\partial_t \phi = -v_p^1 \partial_{x_1} \phi - \gamma^2 \operatorname{div}(\rho_p w) - \operatorname{div}(\phi w).$$

Taking $\llbracket \cdot \rrbracket_{m-1}$ -norm we obtain

$$\llbracket \partial_t \phi \rrbracket_{m-1} \leq \llbracket v_p^1 \partial_{x_1} \phi \rrbracket_{m-1} + \gamma^2 \llbracket \rho_p w \rrbracket_m + \llbracket \phi w \rrbracket_m.$$

Since $m \geq [n/2] + 1$ we get using Lemma 8.3 (ii) that

$$\llbracket \partial_t \phi \rrbracket_{m-1} \leq C(\llbracket \phi \rrbracket_m) \{ \llbracket v_p^1 \rrbracket_m \llbracket \phi \rrbracket_m + \|\rho_p\|_{H^m} \llbracket w \rrbracket_m + \llbracket \phi \rrbracket_m \llbracket w \rrbracket_m \}.$$

This concludes the proof. $\square$

5 Decomposition of the solution

In this section we decompose solution $u(t)$ of (4.1) and we prove the a priori estimate in Proposition 3.7. We decompose $u(t)$ into several parts based on the spectral properties of $L(t)$. In this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_2^2$ unless further restricted.

Let us first introduce some notation and projection operators. Let $\widehat{\chi}_2$ and $\widehat{\chi}_3$ be defined by

$$\widehat{\chi}_2(\xi') = \mathbf{1}_{[r_1, 1)}(|\xi'|) \quad \text{and} \quad \widehat{\chi}_3(\xi') = \mathbf{1}_{[1, \infty)}(|\xi'|).$$

We then define $[f]_j$, $j = 1, 2, \infty$ by

$$[f]_j = \mathcal{F}^{-1}(\widehat{\chi}_j \widehat{f}), \quad j = 1, 2,$$

$$[f]_\infty = [f]_1 + [f]_2 = \mathcal{F}^{-1}((\widehat{\chi}_1 + \widehat{\chi}_2) \widehat{f}).$$

Next, we define $P_{\infty, j}$, $j = 1, 2, 3$ as

$$P_{\infty,1}(t)u = \mathcal{F}^{-1}(\widehat{P}_{\infty,1}(t)\widehat{u}), \quad \widehat{P}_{\infty,1}(t)\widehat{u} = \widehat{\chi}_1(I - \widehat{\mathcal{Q}}_{\xi'}(t)\widehat{\mathcal{P}}_{\xi'}(t))\widehat{u},$$

$$P_{\infty,j}u = \mathcal{F}^{-1}(\widehat{P}_{\infty,j}\widehat{u}), \quad \widehat{P}_{\infty,j}\widehat{u} = \widehat{\chi}_j\widehat{u} \quad (j = 2, 3).$$

By setting

$$\widetilde{P}_{\infty}(t) = I - \mathbb{P}(t), \quad P_{\infty}^{(0)}(t) = P_{\infty,1}(t) + P_{\infty,2},$$

we get

$$I = \mathbb{P}(t) + \widetilde{P}_{\infty}(t), \quad \widetilde{P}_{\infty}(t) = P_{\infty}^{(0)}(t) + P_{\infty,3}.$$

Using above operators we decompose solution $u(t)$ into

$$u(t) = \mathbb{P}(t)u(t) + \widetilde{P}_{\infty}(t)u(t),$$

with

$$\mathbb{P}(t)u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t),$$

$$\widetilde{P}_{\infty}(t)u(t) = \sigma_{\infty}(t)u^{(0)}(t) + u_{\infty}(t),$$

where

$$\sigma_1(t) = \mathcal{P}(t)u(t), \quad u_1(t) = (\mathcal{Q}(t) - \mathcal{Q}^{(0)}(t))\mathcal{P}(t)u(t),$$

$$\sigma_{\infty}(t) = [Q_0 P_{\infty}^{(0)}(t)u(t)] = [Q_0 P_{\infty}^{(0)}(t)u(t)]_{\infty}, \quad u_{\infty}(t) = P_{\infty}(t)u(t).$$

By P_{∞} we denote the operator defined as

$$P_{\infty}(t) = (I - \Pi^{(0)}(t))P_{\infty}^{(0)}(t) + P_{\infty,3}.$$

Remark 5.1 Notice that $\sigma_1(t)$, $\sigma_{\infty}(t)$ and $u^{(0)}(t)$ are separate functions. Furthermore, notice that

$$\|u_1(t)\|_2 \leq C\|\partial_{x'}\sigma_1(t)\|_2.$$

Next, we derive the equations for σ_1 , σ_{∞} and u_{∞} . We define $\mathcal{M}(t)$ by

$$\mathcal{M}(t) = \widetilde{A} + \widetilde{B}(t),$$

with

$$\widetilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\nu}{\rho_p}\Delta' I_{n-1} - \frac{\widetilde{\nu}}{\rho_p}\nabla' \operatorname{div} \\ 0 & (-\frac{\widetilde{\nu}}{\rho_p}\partial_{x_n} \operatorname{div}', -\frac{\nu}{\rho_p}\Delta') \end{pmatrix}, \quad \widetilde{B}(t) = \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2 \rho_p \operatorname{div}' & 0 \\ \frac{P'(\rho_p)}{\gamma^2 \rho_p} \nabla' & v_p^1(t)\partial_{x_1} I_{n-1} & 0 \\ 0 & 0 & v_p^1(t)\partial_{x_1} \end{pmatrix}.$$

Proposition 5.2 Let $\tau > 0$ and $u(t)$ be a solution of (4.1) in $Z^m(\tau)$. Then there hold

$$\sigma_k \in \bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \tau]; H^l(\mathbb{R}^{n-1})), \quad \int_0^{\tau} \|D\sigma_k(z)\|_m dz < \infty, \quad k = 1, \infty, \quad l = 0, 1, \dots,$$

$$u_1 \in \bigcap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \tau]; H^{m+1-2j}), \quad \int_0^{\tau} \|Dw_1(z)\|_m dz < \infty,$$

$$u_{\infty} \in Z^m(\tau), \quad \int_0^{\tau} \|\partial_t \phi_{\infty}(z)\|_{m-1} dz < \infty.$$

Moreover, σ_1 , σ_{∞} and u_{∞} satisfy

$$\sigma_1(t) = e^{(t-s)\Lambda} \mathcal{P}(s)u_0 + \int_s^t e^{(t-z)\Lambda} \mathcal{P}(z)\mathbf{F}(z)dz, \quad (5.1)$$

$$\partial_t \sigma_\infty + [Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty = [Q_0 P_\infty^{(0)} \mathbf{F}]_\infty, \quad (5.2)$$

$$\partial_t u_\infty + L(t)u_\infty + \mathcal{M}(t)(\sigma_\infty u^{(0)}) - [Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty u^{(0)} = P_\infty \mathbf{F}. \quad (5.3)$$

$$w_\infty|_{x_n=0,1} = 0,$$

$$\sigma_\infty|_{t=0} = \sigma_{\infty,0}, \quad u_\infty|_{t=0} = u_{\infty,0}.$$

Here, $\sigma_{\infty,0} = [Q_0 P_\infty^{(0)}(0)u_0]_\infty$, $u_{\infty,0} = P_\infty u_0$.

Proof. Since $u \in Z^m(\tau)$, the regularity assertions on σ_k ($k = 1, \infty$) and u_1 follow from properties of $\mathcal{P}(t)$, $\mathcal{Q}(t)$ and (4.15). As for u_∞ , we already know that $\mathbb{P}Z^m(\tau) \subset Z^m(\tau)$ and therefore $\tilde{P}_\infty Z^m(\tau) \subset Z^m(\tau)$. Since it is straightforward to see that $P_{\infty,3}Z^m(\tau) \subset Z^m(\tau)$ we have $P_\infty^{(0)}Z^m(\tau) \subset Z^m(\tau)$. Finally, from properties of $u^{(0)}(t)$ we obtain $\Pi^{(0)}P_\infty^{(0)}Z^m(\tau) \subset Z^m(\tau)$. Therefore, $u_\infty \in Z^m(\tau)$. $\int_0^\tau \|\partial_t \phi_\infty(z)\|_{m-1} dz < \infty$ follows in analogous way using (4.15).

As for (5.1), it follows from (4.1) and (4.12) that

$$(\partial_t - \Lambda)\sigma_1(t) = \mathcal{P}(t)\mathbf{F}(t).$$

The rest is standard.

As for (5.2) and (5.3), we first apply $\tilde{P}_\infty(t)$ to (4.1) to get

$$\partial_t(\tilde{P}_\infty u) + L(t)\tilde{P}_\infty u = \tilde{P}_\infty \mathbf{F}. \quad (5.4)$$

Next, we apply $P_\infty^{(0)}(t)$ and $P_{\infty,3}$ to (5.4) to obtain

$$\partial_t(P_\infty^{(0)}u) + L(t)P_\infty^{(0)}u = P_\infty^{(0)}\mathbf{F}, \quad (5.5)$$

$$\partial_t(P_{\infty,3}u) + L(t)P_{\infty,3}u = P_{\infty,3}\mathbf{F}. \quad (5.6)$$

Since $[Q_0 L(t)v] = [Q_0 \mathcal{M}(t)v]$ for $\tilde{Q}v|_{x_2=0,1} = 0$ and $[Q_0 \mathcal{M}(t)v] = [Q_0 \tilde{B}(t)v]$ for any v , we get by applying $[Q_0 \cdot]$ to (5.5)

$$\partial_t \sigma_\infty + [Q_0 \tilde{B}(t)P_\infty^{(0)}u] = [Q_0 P_\infty^{(0)}\mathbf{F}]. \quad (5.7)$$

There holds

$$[Q_0 \tilde{B}(t)P_\infty^{(0)}u] = [Q_0 \tilde{B}(\sigma_\infty u^{(0)}(t) + (I - \Pi^{(0)}(t))P_\infty^{(0)}u)]_\infty = [Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty,$$

and thus (5.2) follows from (5.7).

To obtain (5.3) we use the fact that $\partial_t \Pi^{(0)}(t) + L(t)\Pi^{(0)}(t) = \mathcal{M}(t)\Pi^{(0)}(t)$. Applying $I - \Pi^{(0)}(t)$ to (5.5) gives us

$$\partial_t((I - \Pi^{(0)}(t))P_\infty^{(0)}u) + L(t)(I - \Pi^{(0)}(t))P_\infty^{(0)}u + \mathcal{M}(t)(\sigma_\infty u^{(0)}(t)) - \Pi^{(0)}(t)\tilde{B}(t)(P_\infty^{(0)}u) = (I - \Pi^{(0)}(t))P_\infty^{(0)}\mathbf{F}. \quad (5.8)$$

(5.3) now follows by adding (5.6) and (5.8). This completes the proof. $\square$

Let us state some properties of σ_1 , σ_∞ and u_∞ parts.

Lemma 5.3 *There hold the following inequalities.*

(i)

$$\|\partial_{x'}^k [Q_0 P_\infty^{(0)}u]_\infty\|_2 \leq \|[Q_0 P_\infty^{(0)}u]_\infty\|_2, \quad k = 0, 1, \dots,$$

(ii)

$$\|P_\infty u\|_2 \leq C \|\partial_x P_\infty u\|_2 \quad \text{if } \tilde{Q}u|_{x_n=0,1} = 0.$$

(iii) Let $\tau > 0$ and $u(t)$ be a solution of (4.1) in $Z^m(\tau)$. Then there hold

$$\|\partial_{x'}^k \sigma_1\|_2 \leq C \|\partial_{x'} \sigma_1\|_2, \quad \|\partial_{x'}^k \sigma_\infty\|_2 \leq C \|\partial_{x'} \sigma_\infty\|_2, \quad k = 1, 2, \dots,$$

$$\|\phi_\infty\|_2 \leq C \|\partial_x \phi_\infty\|_2,$$

$$\|w_\infty\|_2 \leq C \|\partial_x w_\infty\|_2,$$

$$\|\Lambda \sigma_1\|_2 \leq C \|\partial_{x'} \sigma_1\|_2.$$

Proof. Inequality (i) is obvious since $\text{supp}(\widehat{\chi}_1 + \widehat{\chi}_2) \subset \{|\xi'| \leq 1\}$. As for (ii), since $\text{supp} \widehat{\chi}_3 \subset \{|\xi'| \geq 1\}$, we see that

$$\|P_{\infty,3} u\|_2 \leq \|\partial_{x'} P_{\infty,3} u\|_2.$$

Since $\widetilde{Q}u|_{x_n=0,1} = 0$, we have $\widetilde{Q}P_\infty^{(0)}u|_{x_n=0,1} = 0$, and hence, $\widetilde{Q}(I - \Pi^{(0)}(t))P_\infty^{(0)}u|_{x_n=0,1} = 0$. By the Poincaré inequality we obtain

$$\|\widetilde{Q}(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2 \leq \|\partial_{x_n} \widetilde{Q}(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2.$$

Furthermore, since $[Q_0(I - \Pi^{(0)}(t))P_\infty^{(0)}u] = 0$, we see from the Poincaré inequality that

$$\|Q_0(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2 \leq \|\partial_{x_n} Q_0(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2.$$

It then follows that

$$\|P_\infty u\|_2 \leq C\{\|\partial_x(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2 + \|\partial_x P_{\infty,3} u\|_2\} \leq C\|\partial_x P_\infty u\|_2.$$

Here, we used $(\partial_x(I - \Pi^{(0)}(t))P_\infty^{(0)}u, \partial_x P_{\infty,3} u) = 0$, which follows from the fact $\widehat{\chi}_1 \widehat{\chi}_3 = \widehat{\chi}_2 \widehat{\chi}_3 = 0$ and the Plancherel theorem.

As for (iii), it follows from the proof of (i) and (ii). □

We prove the a priori estimate in Proposition 3.7 by estimating the following quantities.

Let $u(t)$ be solution of (4.1) in $Z^m(\tau)$ and let $u(t)$ be decomposed as above, i.e.,

$$u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t) + \sigma_\infty(t)u^{(0)}(t) + u_\infty(t).$$

We define $M(t) \geq 0$ by

$$M(t)^2 = M_1(t)^2 + \sup_{0 \leq z \leq t} (1+z)^{\frac{n+1}{2}} E_\infty(z), \quad t \in [0, \tau].$$

Here, $M_1(t)$ and $E_\infty(t)$ are defined as

$$M_1(t) = \sup_{0 \leq z \leq t} (1+z)^{\frac{n-1}{4}} \|\sigma_1(z)\|_2 + \sup_{0 \leq z \leq t} (1+z)^{\frac{n+1}{4}} \{\|\partial_{x'} \sigma_1(z)\|_2 + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \|\partial_z^j \sigma_1(z)\|_2\},$$

and

$$E_\infty(t) = \|u_\infty(t)\|_m^2 + \|\sigma_\infty(t)\|_m^2.$$

Finally, we introduce quantity $D_\infty(t)$ for $u_\infty(t) = {}^T(\phi_\infty(t), w_\infty(t))$:

$$D_\infty(t) = \|\partial_x \phi_\infty(t)\|_{m-1}^2 + \|\partial_t \phi_\infty(t)\|_{m-1}^2 + \|Dw_\infty(t)\|_m^2 + \|D\sigma_\infty(t)\|_m^2.$$

Remark 5.4 From properties of $\mathcal{Q}^{(p)}(t)$, $p = 1, 2$, we see that

$$\|\partial_{x'}^k \partial_{x_n}^l \partial_t^j u_1(t)\|_2 \leq C \|D\sigma_1(t)\|_m, \quad 0 \leq 2j + l \leq m + 1, \quad k = 0, 1, \dots,$$

and there holds

$$\sup_{0 \leq z \leq t} (1 + z)^{\frac{n+1}{4}} \{\|u_1(z)\|_m + \|\partial_x u_1(z)\|_m\} \leq CM_1(t).$$

Therefore, we do not need special estimates for $u_1(t)$.

We show the following estimates for $M_1(t)$ and $E_\infty(t)$.

Proposition 5.5 *There exist positive constants ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$, then the following assertions hold true.*

There exists $\varepsilon_2 > 0$ such that if solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0 \leq z \leq t} \|u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$ for all $t \in [0, \tau]$, then the following estimates hold uniformly for $t \in [0, \tau]$ with $C > 0$ independent of τ .

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}, \quad (5.9)$$

$$E_\infty(t) + \int_0^t e^{-a(t-z)} D_\infty(z) dz \leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{n+1}{2}} M(t)^4 + \int_0^t e^{-a(t-z)} \tilde{R}(z) dz\}. \quad (5.10)$$

Here, $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant; and $\tilde{R}(t)$ is quantity that satisfies

$$\tilde{R}(t) \leq C\{(1+t)^{-\frac{n+1}{2}} M(t)^3 + M(t) D_\infty(t)\}, \quad (5.11)$$

whenever $\sup_{0 \leq z \leq t} \|u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$.

The proof of Proposition 5.5 is given in Sections 6-8. We prove (5.9), (5.10) and (5.11) in Sections 6, 7 and 8, respectively.

Assuming that Proposition 5.5 holds true, we can show the following estimate.

Proposition 5.6 *If $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$, then the following assertion holds true. There exists number $\varepsilon_3 > 0$ such that if solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_3$, then there holds the estimate*

$$M(t) \leq C\|u_0\|_{H^m \cap L^1}, \quad (5.12)$$

for a constant $C > 0$ independent of τ .

As an immediate consequence of (5.12) we see that the a priori estimate in Proposition 3.7 holds true. Moreover, (5.12) provides us with the following decay estimates:

$$\|u(t)\|_m \leq C(1+t)^{-\frac{n-1}{4}} \|u_0\|_{H^m \cap L^1},$$

$$\|\partial_{x'}^k u(t)\|_2 \leq C(1+t)^{-\frac{n-1}{4} - \frac{k}{2}} \|u_0\|_{H^m \cap L^1}, \quad k = 0, 1,$$

and

$$\|u(t) - \sigma_1(t)u^{(0)}(t)\|_2 \leq C(1+t)^{-\frac{n+1}{4}} \|u_0\|_{H^m \cap L^1}, \quad (5.13)$$

for $t \in [0, \tau]$. This proves (3.6) and (3.8).

Proof of Proposition 5.6 If $\sup_{0 \leq z \leq t} \|u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$, then we see from (5.10) and (5.11) that

$$\begin{aligned} E_\infty(t) + \int_0^t e^{-a(t-z)} D_\infty(z) dz &\leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{n+1}{2}} M(t)^4 \\ &+ \int_0^t e^{-a(t-z)} \{(1+z)^{-\frac{n+1}{2}} M(z)^3 + M(z) D_\infty(z)\} dz \} \end{aligned}$$

$$\leq C\{e^{-at}E_\infty(0) + (1+t)^{-\frac{n+1}{2}}M(t)^3 + M(t)\int_0^t e^{-a(t-z)}D_\infty(z)dz\}.$$

Therefore, using continuity of $E_\infty(t)$ and compatibility conditions we obtain

$$(1+t)^{\frac{n+1}{2}}E_\infty(t) + \mathcal{D}(t) \leq C\{\|u_0\|_{H^m}^2 + M(t)^3 + M(t)\mathcal{D}(t)\}, \quad (5.14)$$

with

$$\mathcal{D}(t) = (1+t)^{\frac{n+1}{2}} \int_0^t e^{-a(t-z)}D_\infty(z)dz.$$

It follows from (5.9) and (5.14) that

$$M(t)^2 + \sup_{0 \leq z \leq t} \mathcal{D}(z) \leq C_1\{\|u_0\|_{H^m \cap L^1}^2 + M(t)^3 + M(t) \sup_{0 \leq z \leq t} \mathcal{D}(z)\}, \quad (5.15)$$

whenever $\sup_{0 \leq z \leq t} \|u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$.

In the same way as in [7, Proof of Proposition 5.4] using (5.15) one can show that there exists $\varepsilon_3 > 0$ such that if $\|u_0\|_{H^m \cap L^1} < \varepsilon_3$, then

$$M(t) < 2C_2\|u_0\|_{H^m \cap L^1},$$

for all $t \in [0, \tau]$ with $C_2 > 0$ independent of τ . This concludes the proof. $\square$

6 Estimates on $\sigma_1(t)$

In this section we estimate the $\mathbb{P}(t)$ -part of $u(t)$. Since

$$\mathbb{P}(t)u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t),$$

where $\sigma_1(t) = \mathcal{P}(t)u(t)$ and $u_1(t) = (\mathcal{Q}(t) - \mathcal{Q}^{(0)}(t))\mathcal{P}(t)u(t)$, it is enough to obtain estimates for σ_1 (see Remark 5.4). In this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_2^2$.

Let us first make an observation. Regarding the spectral properties of linearized operator, we expect $\sigma_1(t)$ to be the most slowly decaying part of $u(t)$. Therefore, the most slowly decaying part of the nonlinearity $\mathbf{F}(t, u(t))$ would be given by the terms containing only $\sigma_1(t)^2$. There are two such terms in $\mathbf{F}(t, u(t))$,

$$\frac{\nu\phi}{\gamma^2\rho_p^2} \left(-\partial_{x_n}^2 w^1 + \frac{\partial_{x_n}^2 v_p^1}{\gamma^2\rho_p} \phi \right) \mathbf{e}_1 \quad \text{and} \quad -\frac{1}{2\gamma^4\rho_p} \partial_{x_n} (P''(\rho_p)\phi^2) \mathbf{e}_n.$$

Since $w^{(0),1}$ satisfies (4.7), we can define $\sigma_1^2 \mathbf{F}_1$ with $\mathbf{F}_1 = \mathbf{F}_1(x_n, t)$ as

$$\mathbf{F}_1 = {}^T \left(0, \frac{\phi^{(0)}(x_n)}{\gamma^2\rho_p} \partial_t w^{(0),1}(x_n, t), \frac{1}{2\gamma^4\rho_p(x_n)} \partial_{x_n} \left(P''(\rho_p(x_2)) \{\phi^{(0)}(x_n)\}^2 \right) \right).$$

We thus write

$$\mathbf{F} = \sigma_1^2 \mathbf{F}_1 + \mathbf{F}_2, \quad (6.1)$$

where $\mathbf{F}_2 = \mathbf{F} - \sigma_1^2 \mathbf{F}_1$ contains terms involving u_∞ , its derivatives and terms of order $O(\sigma_1 \partial_{x'} \sigma_1)$ like $\sigma_1 u_1$, and $O(\sigma_1^3)$, but not just $O(\sigma_1^2)$. In particular, we have that $Q_0 \mathbf{F} = Q_0 \mathbf{F}_2$.

First we introduce two lemmas.

Lemma 6.1 *There hold the following relations.*

(i)

$$[Q_0 \mathbf{F}] = -\text{div}'[\phi w'],$$

(ii)

$$\mathcal{P}(t)\mathbf{F}(t) = -\text{div}'[\phi(t)w'(t)]_1 + \text{div}'\mathcal{P}^{(1)}(t)\mathbf{F}(t) + \Delta'\mathcal{P}^{(2)}(t)\mathbf{F}(t).$$

Proof. Since $w|_{x_n=0,1} = 0$, by integration by parts, we have

$$[Q_0 \mathbf{F}] = -\operatorname{div}'[\phi w'].$$

This shows (i). As for (ii), it is straightforward from definition of $\mathcal{P}^{(0)}$ and (i). □

Remark 6.2 Let $\varepsilon_5 > 0$ be number such that

$$C_S \varepsilon_5 \leq \frac{\gamma^2 \rho_1}{4}.$$

Here, $C_S > 0$ comes from Sobolev inequality (3.10). Then whenever $\llbracket u(t) \rrbracket_m \leq \varepsilon_5$, we have

$$\|\phi(t)\|_\infty \leq C_S \llbracket u(t) \rrbracket_m \leq C_S \varepsilon_5 \leq \frac{\gamma^2 \rho_1}{4},$$

and hence,

$$\rho(x, t) = \rho_p(x_2) + \gamma^{-2} \phi(x, t) \geq \rho_1 - \gamma^{-2} \|\phi(t)\|_\infty \geq \frac{3\rho_1}{4} > 0.$$

Therefore, we see that $\tilde{Q}\mathbf{F}(t)$ is smooth whenever $\llbracket u(t) \rrbracket_m \leq \varepsilon_5$.

Using inequality $\|\sigma_1\|_\infty \leq C\|\sigma_1\|_2^{1/2}\|\partial_{x'}\sigma_1\|_2^{1/2}$ (see Lemmas 8.2 (iii) and 5.3 (iii)), it is not difficult to verify the following estimates on nonlinearities. We omit the proof.

Lemma 6.3 Let solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0 \leq z \leq t} \llbracket u(z) \rrbracket_m \leq \varepsilon_5$ and $M(t) \leq 1$ for $t \in [0, \tau]$, then there hold the following estimates for $t \in [0, \tau]$ with $C > 0$ independent of τ .

- (i) $\|\partial_{x'}(\sigma_1^2(t))\|_1 \leq C(1+t)^{-\frac{n}{2}} M(t)^2,$
- (ii) $\|\operatorname{div}'[\phi w'](t)\|_1 \leq C(1+t)^{-\frac{n}{2}} M(t)^2,$
- (iii) $\|[\phi w'](t)\|_1 \leq C(1+t)^{-\frac{n-1}{2}} M(t)^2,$
- (iv) $\|\mathbf{F}(t)\|_1 \leq C(1+t)^{-\frac{n-1}{2}} M(t)^2,$
- (v) $\|\mathbf{F}_2(t)\|_1 \leq C(1+t)^{-\frac{n}{2}} M(t)^2,$
- (vi) $\|\mathbf{F}(t)\|_2 \leq C(1+t)^{-\frac{2n-1}{4}} M(t)^2,$
- (vii) $\|\partial_{x'}(\sigma_1^2(t))\|_2 \leq C(1+t)^{-\frac{2n+1}{4}} M(t)^2.$

Finally, we prove (5.9).

Proposition 6.4 There exists number $\varepsilon_4 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0 \leq z \leq t} \llbracket u(z) \rrbracket_m \leq \varepsilon_4$ and $M(t) \leq 1$ for all $t \in [0, \tau]$, then the estimate

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\},$$

holds uniformly for $t \in [0, \tau]$ with $C > 0$ independent of τ .

Proof. We write (5.1) for $s = 0$ as

$$\sigma_1(t) = e^{t\Lambda} \mathcal{P}(0)u_0 + I(t),$$

where

$$I(t) = \int_0^t e^{(t-z)\Lambda} \mathcal{P}(z)F(z)dz.$$

(4.13) yields

$$\|\partial_{x'}^k e^{t\Lambda} \mathcal{P}(0)u_0\|_2 \leq C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}} \|u_0\|_1,$$

for $k = 0, 1$. Next, we estimate $I(t)$ which we write it as

$$I(t) = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^{\frac{t}{2}} e^{(t-z)\Lambda} \mathcal{P}(z)F(z)dz,$$

$$I_2(t) = \int_{\frac{t}{2}}^t e^{(t-z)\Lambda} \mathcal{P}(z)F(z)dz.$$

By Lemma 6.1 (ii), we have

$$e^{(t-z)\Lambda} \mathcal{P}(z)F(z) = \operatorname{div}' e^{(t-z)\Lambda} \{-[\phi w']_1 + \mathcal{P}^{(1)}F + \nabla' \mathcal{P}^{(2)}F\}(z).$$

It then follows from (4.14) and Lemma 6.3 that

$$\begin{aligned} \|\partial_{x'}^k I_1(t)\|_2 &\leq C \int_0^{\frac{t}{2}} (1+t-z)^{-\frac{n+1}{4}-\frac{k}{2}} \{\|[\phi w'](z)\|_1 + \|F(z)\|_1\} dz \\ &\leq CM(t)^2 \int_0^{\frac{t}{2}} (1+t-z)^{-\frac{n+1}{4}-\frac{k}{2}} (1+z)^{-\frac{n-1}{2}} dz \leq C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}} M(t)^2, \end{aligned}$$

for $k = 0, 1$.

As for $I_2(t)$, using (6.1) and Lemma 6.1 (ii) we write $\mathcal{P}(z)F(z)$ as

$$\mathcal{P}F = -\operatorname{div}'[\phi w']_1 + (\mathcal{P}^{(1)} + \nabla' \mathcal{P}^{(2)}) \cdot \nabla'(\sigma_1)^2 F_1 + (\operatorname{div}' \mathcal{P}^{(1)} + \Delta' \mathcal{P}^{(2)}) F_2.$$

It then follows from (4.14) and Lemma 6.3 that

$$\begin{aligned} \|\partial_{x'}^k I_2(t)\|_2 &\leq C \int_{\frac{t}{2}}^t (1+t-z)^{-\frac{n-1}{4}-\frac{k}{2}} \{\|\operatorname{div}'[\phi w'](z)\|_1 + \|\nabla'(\sigma_1(z))^2\|_1 + \|F_2(z)\|_1\} dz \\ &\leq CM(t)^2 \int_{\frac{t}{2}}^t (1+t-z)^{-\frac{n-1}{4}-\frac{k}{2}} (1+z)^{-\frac{n}{2}} dz \leq C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}} M(t)^2, \end{aligned}$$

for $k = 0, 1$. We thus obtain

$$\sum_{k=0}^1 (1+t)^{\frac{n-1}{4}+\frac{k}{2}} \|\partial_{x'}^k \sigma_1(t)\|_2 \leq C\{\|u_0\|_1 + M(t)^2\}. \quad (6.2)$$

It remains to estimate time derivatives. From (5.1) we see that

$$\partial_t \sigma_1(t) = \Lambda \sigma_1(t) + \mathcal{P}(t)F(t). \quad (6.3)$$

It then follows from Lemma 6.3 (vi) and previous result that

$$\|\partial_t \sigma_1(t)\|_2 \leq C\{\|\partial_{x'} \sigma_1(t)\|_2 + \|\mathcal{P}(t)F(t)\|_2\} \leq C(1+t)^{-\frac{n+1}{4}} \{\|u_0\|_1 + M(t)^2\}.$$

Concerning $\|\partial_t^{j+1} \sigma_1(t)\|_2$ for $j = 1, \dots, [\frac{m}{2}] - 1$, we obtain from (6.3)

$$\|\partial_t^{j+1} \sigma_1(t)\|_2 \leq C\{\|\partial_t^j \sigma_1(t)\|_2 + \|\partial_t^j (\mathcal{P}(t)F(t))\|_2\}.$$

Since

$$\|\partial_t^j \mathbf{F}(t)\|_2 \leq C(1+t)^{-\frac{n+1}{4}} M(t)^2,$$

for $0 \leq 2j \leq m-2$ as we see in Propositions 8.5 (i)–(iii) and 8.6 (i), we find by induction on j , that estimate

$$\|\partial_t^{j+1} \sigma_1(t)\|_2 \leq C_j(1+t)^{-\frac{n+1}{4}} \{\|u_0\|_1 + M(t)^2\}, \quad (6.4)$$

holds for $j = 0, 1, \dots, [\frac{m}{2}] - 1$.

The desired result now follows from (6.2) and (6.4). This completes the proof. $\square$

7 Estimates on $\tilde{P}_\infty u(t)$

In this section we prove estimate (5.10) for σ_∞ and u_∞ by a variant of Matsumura-Nishida energy method as in the case of stationary parallel flow ([7]). Since coefficients of the linearized operator depend on time some extra terms arise in contrast to [7]. We omit the proofs that can be obtained as modification of those in [7]. In this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_2^2$ unless further restricted.

First, let us show the following inequality.

Proposition 7.1 *There exists $\nu_0 \geq \nu_2$ and $\gamma_0 \geq \gamma_2$ such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ the solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies*

$$\frac{d}{dt} \tilde{E}(t) + 2D(t) \leq \tilde{R}(t). \quad (7.1)$$

Here, $\tilde{E}(t)$, $D(t)$ and $\tilde{R}(t)$ are quantities such that

- (i) $\tilde{E}(t) + \|\partial_{x_n}^2 w_\infty(t)\|_{m-2}^2$ is equivalent to $E_\infty(t)$,
- (ii) $D(t)$ is equivalent to $D_\infty(t)$,
- (iii) $\tilde{R}(t)$ satisfies estimate (5.11).

We introduce some quantities. Let $E^{(0)}[\tilde{P}_\infty u]$ and $D^{(0)}[w]$ be defined by

$$E^{(0)}[\tilde{P}_\infty u] = \frac{\alpha_0}{\gamma^2} \|\sigma_\infty\|_2^2 + \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_p)}{\gamma^2 \rho_p}} \phi_\infty \right\|_2^2 + \|\sqrt{\rho_p} w_\infty\|_2^2,$$

for $\tilde{P}_\infty(t) = \sigma_\infty u^{(0)} + u_\infty$ with $u_\infty = {}^T(\phi_\infty, w_\infty)$; and

$$D^{(0)}[w_\infty] = \nu \|\nabla w_\infty\|_2^2 + \tilde{\nu} \|\operatorname{div} w_\infty\|_2^2.$$

Note that,

$$\langle Au(t), u(t) \rangle_\Omega = D^{(0)}[w(t)],$$

for $u = {}^T(\phi, w) \in Z^m(\tau)$, and

$$\langle B(t)u(t), v(t) \rangle_\Omega = -\langle u(t), B(t)v(t) \rangle_\Omega,$$

for $u, v \in Z^m(\tau)$, $\tilde{Q}u|_{x_n=0,1} = \tilde{Q}v|_{x_n=0,1} = 0$. In particular,

$$\langle B(t)u(t), u(t) \rangle_\Omega = 0,$$

for $u \in Z^m(\tau)$, $\tilde{Q}u|_{x_n=0,1} = 0$.

We denote the tangential derivatives $\partial_t^j \partial_{x'}^k$ by $T_{j,k}$:

$$T_{j,k} u = \partial_t^j \partial_{x'}^k u.$$

In this section we often use $\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O(\frac{1}{\gamma^2})$ in calculations (see Lemma 4.9). It is straightforward to see that following lemma holds true.

Lemma 7.2 *There hold the following assertions.*

(i)

$$\|T_{j,k+1}\sigma_\infty\|_2^2 \leq \|T_{j,1}\sigma_\infty\|_2^2, \quad k \geq 0, \quad 2j \leq m,$$

$$\|T_{j,k}\phi_\infty\|_2^2 \leq C\|\partial_x T_{j,k}\phi_\infty\|_2^2, \quad 2j+k \leq m-1,$$

$$\|T_{j,k}w_\infty\|_2^2 \leq C\|\partial_x T_{j,k}w_\infty\|_2^2, \quad 2j+k \leq m-1.$$

(ii)

$$\|[Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty\|_2^2 \leq C(\|\partial_{x'}\sigma_\infty\|_2^2 + \|\partial_{x'}\phi_\infty\|_2^2 + \gamma^4\|\partial_{x'}w_\infty\|_2^2).$$

(iii) If $w_\infty^2|_{x_n=0,1} = 0$, then $[Q_0\tilde{B}u_\infty]_\infty = [Q_0Bu_\infty]_\infty = [v_p^1\partial_{x_1}\phi_\infty + \gamma^2\operatorname{div}(\rho_p w_\infty)]_\infty$.

(iv) If $w_\infty^2|_{x_n=0,1} = 0$ and $2j+k \leq m$, then

$$\|\partial_{x'}^k\partial_t^j[Q_0\tilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty\|_2^2$$

$$\leq C \sum_{i=0}^j (\|\partial_{x'}^p\partial_t^i\sigma_\infty\|_2^2 + \|\partial_{x'}^q\partial_t^i\phi_\infty\|_2^2) + \gamma^4\|\operatorname{div}(\partial_{x'}^r\partial_t^j w_\infty)\|_2^2 + \gamma^4|\partial_{x_n}\rho_p|_\infty^2\|\partial_{x'}^s\partial_t^j w_\infty\|_2^2,$$

for $0 \leq p, q \leq k+1, 0 \leq r, s \leq k$.

We begin with L^2 -energy estimates for tangential derivatives. We set

$$\sigma_* = \sigma_1 + \sigma_\infty, \quad \phi_* = \phi_1 + \phi_\infty, \quad w_* = w_1 + w_\infty,$$

$$u_* = {}^T(\phi_*, w_*) = u_1 + u_\infty.$$

We write $\tilde{Q}\mathbf{F} = {}^T(0, \mathbf{f})$ in the form

$$\tilde{Q}\mathbf{F} = \tilde{\mathbf{F}}_0 + \tilde{\mathbf{F}}_1 + \tilde{\mathbf{F}}_2 + \tilde{\mathbf{F}}_3.$$

Here, $\tilde{\mathbf{F}}_l = {}^T(0, \mathbf{f}_l)$, $l = 0, 1, 2, 3$, with

$$\mathbf{f}_0 = -w \cdot \nabla w - f_1(\rho_p, \phi)\Delta'\sigma_*w^{(0),1}e_1 - f_2(\rho_p, \phi)\nabla(\partial_{x_1}\sigma_*w^{(0),1})$$

$$+ \mathbf{f}_{01}(x_n, t, \phi)\phi\sigma_* + \mathbf{f}_{02}(x_n, t, \phi)\phi\nabla'\sigma_* + \mathbf{f}_{03}(x_n, t, \phi)\phi\phi_*,$$

$$\mathbf{f}_1 = -f_1(\rho_p, \phi)\Delta w_* = -\operatorname{div}(f_1(\rho_p, \phi)\nabla w_*) + {}^T(\nabla w_*)\nabla(f_1(\rho_p, \phi)),$$

$$\mathbf{f}_2 = -f_2(\rho_p, \phi)\nabla\operatorname{div} w_* = -\nabla(f_2(\rho_p, \phi)\operatorname{div} w_*) + (\operatorname{div} w_*)\nabla(f_2(\rho_p, \phi)),$$

$$\mathbf{f}_3 = -f_3(x_n, t, \phi)\phi\nabla\phi_*.$$

Here, ∇w_* denotes the $n \times n$ matrix $(\partial_{x_i} w_*^j)$; $f_1 = \frac{\nu\phi}{\rho_p(\gamma^2\rho_p+\phi)}$; $f_2 = \frac{\tilde{\nu}\phi}{\rho_p(\gamma^2\rho_p+\phi)}$; and $\mathbf{f}_{0l}(x_n, t, \phi)$, $l = 1, 2, 3$ and $f_3(x_n, \phi)$ are smooth functions of x_n, t and ϕ .

Proposition 7.3 *There exists $\nu_3 > \nu_2$ such that for $\nu \geq \nu_3$ the following estimate holds for $0 \leq 2j+k \leq m$:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} E^{(0)}[T_{j,k}\tilde{P}_\infty u] + \frac{1}{2} D^{(0)}[T_{j,k}w_\infty] \\ & \leq R_{j,k}^{(1)} + C\left\{\left(\frac{\nu}{\gamma^4} + \frac{1}{\gamma^4\nu} + \frac{1}{\gamma^2}\right) \sum_{i=0}^j \|T_{i,k}\phi_\infty\|_2^2 + \left(\frac{\nu}{\gamma^4} + \frac{\tilde{\nu}}{\nu\gamma^4} + \frac{1}{\gamma^2}\right) \sum_{i=0}^j \|T_{i,k+1}\sigma_\infty\|_2^2 \right. \\ & \quad \left. + \frac{1}{\gamma^2} \sum_{i=0}^{j-1} \|T_{i,k+1}\phi_\infty\|_2^2 + \frac{1}{\gamma^2} (1 - \delta_{j0}) \|\partial_t T_{j-1,k}\sigma_\infty\|_2^2 + \frac{1}{\nu^2} \sum_{i=0}^{j-1} D^{(0)}[T_{i,k}w_\infty]\right\}, \end{aligned} \tag{7.2}$$

where δ_{j0} denotes Kronecker's delta and $R_{j,k}^{(1)}$ is given by

$$R_{j,k}^{(1)} = \frac{\alpha_0}{\gamma^2} ([Q_0 T_{j,k} \mathbf{F}]_\infty, T_{j,k} \sigma_\infty) - \frac{\alpha_0}{\gamma^2} ([Q_0 T_{j,k} (\mathbb{P} \mathbf{F})]_\infty, T_{j,k} \sigma_\infty) + \tilde{R}_{j,k}^{(1)} - \langle T_{j,k} ([Q_0 \mathbf{F}]_\infty u^{(0)}), T_{j,k} u_\infty \rangle_\Omega \\ - \langle T_{j,k} (\mathbb{P} \mathbf{F}), T_{j,k} u_\infty \rangle_\Omega + \langle T_{j,k} ([Q_0 (\mathbb{P} \mathbf{F})]_\infty u^{(0)}), T_{j,k} u_\infty \rangle_\Omega.$$

Here,

$$\tilde{R}_{j,k}^{(1)} = \langle T_{j,k} \mathbf{F}, T_{j,k} u_\infty \rangle_\Omega,$$

when $2j + k \leq m - 1$, and

$$\tilde{R}_{j,k}^{(1)} = -(T_{j,k}(\phi \operatorname{div} w), T_{j,k} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p}) + \frac{1}{2} (\operatorname{div} (\frac{P'(\rho_p)}{\gamma^4 \rho_p} w), |T_{j,k} \phi_\infty|^2) \\ - (w \nabla T_{j,k}(\sigma_* \phi^{(0)} + \phi_1), T_{j,k} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p}) - ([T_{j,k}, w] \nabla \phi, T_{j,k} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p}) \\ + (T_{j,k} \mathbf{f}_0, T_{j,k} w_\infty \rho_p) + \sum_{l=1}^3 \langle T_{j,k} \mathbf{f}_l, T_{j,k} w_\infty \rho_p \rangle_{-1},$$

when $2j + k = m$. Here and in what follows, for $G = g + \partial_{x_j} \tilde{g}$ with $g, \tilde{g} \in L^2$ and $v \in H_0^1$, $\langle G, v \rangle_{-1}$ denotes

$$\langle G, v \rangle_{-1} = (g, v) - (\tilde{g}, \partial_{x_j} v).$$

Proof. We apply $T_{j,k}$ to (5.2) and (5.3). We then take the inner products of the resulting equations with $T_{j,k} \sigma_\infty$ and $T_{j,k} u_\infty$, respectively. Integration by parts together with symmetric properties of A and B gives us the desired result in the same manner as in [7, Proposition 7.4]. $\square$

We next derive the H^1 -parabolic estimates for w_∞ . We define $J[\tilde{P}_\infty u]$ by

$$J[\tilde{P}_\infty u] = -2 \langle \sigma_\infty u^{(0)} + u_\infty, B \tilde{Q} u_\infty \rangle_\Omega \text{ for } \tilde{P}_\infty u = \sigma_\infty u^{(0)} + u_\infty.$$

A direct computation shows that if $\gamma^2 \geq 1$ then

$$|J[\tilde{P}_\infty u]| \leq \frac{b_0 \gamma^2}{\nu} E^{(0)}[\tilde{P}_\infty] + \frac{1}{2} D^{(0)}[w_\infty],$$

for some constant $b_0 > 0$.

Let b_1 be a positive constant (to be determined later) and define $E^{(1)}[\tilde{P}_\infty u]$ by

$$E^{(1)}[\tilde{P}_\infty u] = \frac{2b_1 \gamma^2}{\nu} E^{(0)}[\tilde{P}_\infty u] + D^{(0)}[w_\infty] + J[\tilde{P}_\infty u].$$

Note that if $b_1 \geq b_0$ then $E^{(1)}[\tilde{P}_\infty u]$ is equivalent to $E^{(0)}[\tilde{P}_\infty u] + D^{(0)}[w_\infty]$.

Proposition 7.4 *There exists $b_1 \geq \max\{b_0, 8C_0\}$ such that if $\nu \geq \nu_3$, $\gamma^2 \geq 1$ and $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \max\{1, \gamma_2^2\}$ then the following estimate holds for $0 \leq 2j + k \leq m - 1$,*

$$\frac{1}{2} \frac{d}{dt} E^{(1)}[T_{j,k} \tilde{P}_\infty u] + \frac{b_1 \gamma^2}{\nu} \frac{3}{4} D^{(0)}[T_{j,k} w_\infty] + \frac{1}{2} \|\sqrt{\rho_p} T_{j,k} \partial_t w_\infty\|_2^2 \leq R_{j,k}^{(2)} \quad (7.3) \\ + C \sum_{i=0}^j \left\{ \left(\frac{\nu^2}{\gamma^4} + \frac{1}{\nu} \right) \|T_{i,k} \phi_\infty\|_2^2 + \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu} \right) \|T_{i,k+1} \sigma_\infty\|_2^2 + \frac{1}{\nu} \|T_{i,k+1} \phi_\infty\|_2^2 \right. \\ \left. + \frac{1}{\nu} (1 - \delta_{j0}) \|\partial_t T_{j-1,k} \sigma_\infty\|_2^2 \right\} + C_0 \frac{\gamma^2}{\nu} \left(1 + \frac{C}{\nu^2} \right) \sum_{i=0}^{j-1} D^{(0)}[T_{i,k} w_\infty].$$

where

$$R_{j,k}^{(2)} = \frac{2b_1 \gamma^2}{\nu} R_{j,k}^{(1)} + C \|T_{j,k} \mathbf{F}\|_2^2.$$

Proof. We apply $T_{j,k}$ to (5.3) and take inner product with $\partial_t T_{j,k} \tilde{Q} u_\infty$ to obtain the desired result in the same manner as in [7, Proposition 7.5]. $\square$

As for the dissipative estimates for x_n -derivatives of ϕ_∞ , we have the following inequality.

Proposition 7.5 *The following estimate holds for $0 \leq 2j + k + l \leq m - 1$:*

$$\frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_p)}{\gamma^2 \rho_p}} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|_2^2 + \frac{1}{2(\nu + \tilde{\nu})} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n}^{l+1} T_{j,k} \phi_\infty \right\|_2^2 \leq R_{j,k,l}^{(3)} + C \frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,l}\|_2^2, \quad (7.4)$$

where

$$R_{j,k,l}^{(3)} = \left| \frac{1}{2} (\operatorname{div} \left(\frac{P'(\rho_p)}{\gamma^4 \rho_p} w \right), |T_{j,k} \partial_{x_n}^{l+1} \phi_\infty|^2) \right| + C \frac{\nu + \tilde{\nu}}{\gamma^4} \|H_{j,k,l}\|_2^2,$$

with

$$\|H_{j,k,l}\|_2^2 \leq C \{ \| [T_{j,k} \partial_{x_n}^{l+1}, w] \cdot \nabla \phi_\infty \|_2^2 + \| T_{j,k} \partial_{x_n}^{l+1} (\widetilde{Q_0 P_\infty} \mathbf{F}) \|_2^2 + \left\| \frac{\gamma^2 \rho_p^2}{\nu + \tilde{\nu}} T_{j,k} \partial_{x_n}^l (Q_n P_\infty \mathbf{F}) \right\|_2^2 \},$$

and

$$\widetilde{Q_0 P_\infty} \mathbf{F} = -\phi \operatorname{div} w - w \cdot \nabla (\sigma_* \phi^{(0)} + \phi_1) - \{Q_0 \mathbb{P} \mathbf{F} + [Q_0 P_\infty^{(0)} \mathbf{F}]_\infty \phi^{(0)}\}$$

Here, $K_{j,k,l}$ is estimated as

$$\begin{aligned} \frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,l}\|_2^2 &\leq C \frac{\nu + \tilde{\nu}}{\gamma^2} \left\{ \frac{\nu^2}{\nu + \tilde{\nu}} \|T_{j,k+1} \partial_{x_n}^l \partial_x w_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\sqrt{\rho_p} T_{j,k} \partial_{x_n}^l \partial_t w_\infty\|_2^2 \right. \\ &\quad + \frac{\nu^2}{\gamma^2} \left(\sum_{q=0}^{l-1} \|T_{j,k+1} \partial_{x_n}^q \partial_x w_\infty\|_2^2 + \sum_{q=0}^l \|T_{j,k} \partial_{x_n}^q \partial_x w_\infty\|_2^2 + \sum_{i=0}^j \|T_{i,k+1} w_\infty\|_2^2 \right) \\ &\quad + \frac{1}{\nu + \tilde{\nu}} \sum_{i=0}^j \sum_{q=0}^l \|T_{i,k+1} \partial_{x_n}^q w_\infty\|_2^2 \\ &\quad \left. + \frac{1}{\gamma^2} \left(\sum_{i=0}^j \sum_{q=0}^{l+1} \|T_{i,k+1} \partial_{x_n}^q \phi_\infty\|_2^2 + \sum_{i=0}^j \|T_{i,k+1} \sigma_\infty\|_2^2 + \sum_{q=0}^l \|\partial_{x_n}^q T_{j,k} \phi_\infty\|_2^2 \right) \right\}. \end{aligned}$$

Proof. We obtain the desired result in the same manner as in [7, Proposition 7.6]. $\square$

The following estimate for the material derivative of ϕ_∞ plays an important role to obtain the dissipative estimate for higher order x_2 -derivatives of w_∞ . We denote the material derivative of ϕ_∞ by $\dot{\phi}_\infty$:

$$\dot{\phi}_\infty = \partial_t \phi_\infty + (v_p + w) \cdot \nabla \phi_\infty.$$

Proposition 7.6 *The following estimates hold for $0 \leq 2j + k + l \leq m - 1$:*

(i)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_p)}{\gamma^2 \rho_p}} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|_2^2 + \frac{1}{4(\nu + \tilde{\nu})} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n}^{l+1} T_{j,k} \phi_\infty \right\|_2^2 \\ + c_0 \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \partial_{x_n}^{l+1} \dot{\phi}_\infty\|_2^2 \leq R_{j,k,l}^{(3)} + C \frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,l}\|_2^2, \end{aligned} \quad (7.5)$$

where c_0 is a positive constant and $R_{j,k,l}^{(3)}$ and $K_{j,k,l}$ satisfy the same estimates as in Proposition 7.5.

(ii) Let $0 \leq q \leq k$ and $0 \leq 2j + k \leq m$. Then

$$\begin{aligned} \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \dot{\phi}_\infty\|_2^2 &\leq C \{ R_{j,k}^{(4)} + D^{(0)} [T_{j,k} w_\infty] + \frac{\nu^2 (\nu + \tilde{\nu})}{\gamma^4} \|T_{j,k} w_\infty\|_2^2 \\ &\quad + \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{i=0}^j \|T_{i,k+1} \sigma_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{i=0}^j \|T_{i,q} \phi_\infty\|_2^2 \}, \end{aligned} \quad (7.6)$$

where $R_{j,k}^{(4)} = \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \widetilde{Q_0 P_\infty} \mathbf{F}\|_2^2$.

Proof. The desired result is obtained in the same manner as in [7, Proposition 7.7]. $\square$

Let us derive the dissipative estimates for σ_∞ .

Proposition 7.7 *Let $\gamma^2/(\nu + \tilde{\nu}) \geq \max\{1, \gamma_2^2\}$, then there holds the following estimate for $0 \leq 2j+k \leq m-1$:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \frac{\nu}{\gamma^2(\nu + \tilde{\nu})} \|T_{j,k} \sigma_\infty\|_2^2 + \frac{\alpha_1}{2(\nu + \tilde{\nu})} \|\nabla' T_{j,k} \sigma_\infty\|_2^2 \leq R_{j,k}^{(5)} \\ & + C \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} (1 - \delta_{j0}) \|\partial_t T_{j-1,k} \sigma_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \left(\frac{\alpha_1}{16} + C \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \sum_{i=0}^{j-1} \|T_{i,k+1} \sigma_\infty\|_2^2 \\ & + C \left\{ \frac{1}{\nu + \tilde{\nu}} \|\sqrt{\rho_p} T_{j,k} \partial_t w_\infty\|_2^2 + \sum_{i=0}^j D^{(0)}[T_{i,k} w_\infty] + \frac{1}{\nu + \tilde{\nu}} \sum_{i=0}^j \left\| \frac{P'(\rho_p)}{\gamma^2} T_{i,p} \partial_{x_n} \phi_\infty \right\|_2^2 \right\}, \end{aligned} \quad (7.7)$$

where $\alpha_1 > 0$ is a constant, p is any integer satisfying $0 \leq p \leq k$, and

$$R_{j,k}^{(5)} = \frac{\nu}{\gamma^2(\nu + \tilde{\nu})} (Q_0 T_{j,k} (P_\infty^{(0)} \mathbf{F}), T_{j,k} \sigma_\infty) - \frac{1}{\nu + \tilde{\nu}} (\operatorname{div}' [\rho_p (-\Delta)^{-1} (\rho_p T_{j,k} Q'(P_\infty \mathbf{F}))]_\infty, T_{j,k} \sigma_\infty).$$

Here, $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $L^2(\Omega)$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. The desired result is obtained in the same manner as in [7, Proposition 7.8]. $\square$

Next, we estimate the higher order derivatives.

Proposition 7.8 *If $\nu \geq 1$ then there holds the following estimate for $0 \leq 2j+k+l \leq m-1$:*

$$\begin{aligned} & \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^{l+2} T_{j,k} w_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_x^{l+1} T_{j,k} \phi_\infty\|_2^2 \leq C R_{j,k,l}^{(6)} + C \left\{ \left(\frac{1}{\nu + \tilde{\nu}} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \sum_{i=0}^j \|T_{i,k+1} \sigma_\infty\|_2^2 \right. \\ & + \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{i=0}^j \|T_{i,k} \phi_\infty\|_{H^l}^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \dot{\phi}_\infty\|_{H^{l+1}}^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_t T_{j,k} w_\infty\|_{H^l}^2 \\ & \left. + \left(\frac{1}{\nu + \tilde{\nu}} + \frac{\nu^2(\nu + \tilde{\nu})}{\gamma^4} \right) \sum_{i=0}^j \|T_{i,k} w_\infty\|_{H^{l+1}}^2 + D^{(0)}[T_{j,k} w_\infty] \right\}, \end{aligned} \quad (7.8)$$

where

$$R_{j,k,l}^{(6)} = \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \widetilde{Q_0 P_\infty \mathbf{F}}\|_{H^{l+1}}^2 + \frac{1}{\nu + \tilde{\nu}} \|T_{j,k} (\tilde{Q} P_\infty \mathbf{F})\|_{H^l}^2.$$

Proof. We use the estimates for the Stokes system. Let ${}^T(\tilde{\phi}, \tilde{w})$ be the solution of the Stokes system

$$\operatorname{div} \tilde{w} = F \quad \text{in } \Omega,$$

$$-\Delta \tilde{w} - \nabla \tilde{\phi} = G \quad \text{in } \Omega,$$

$$\tilde{w}|_{\partial\Omega} = 0.$$

Then for any $l \in \mathbb{Z}$, $l \geq 0$, there exists a constant $C > 0$ such that

$$\|\partial_x^{l+2} \tilde{w}\|_2^2 + \|\partial_x^{l+1} \tilde{\phi}\|_2^2 \leq C \{ \|F\|_{H^{l+1}}^2 + \|G\|_{H^l}^2 + \|\partial_x \tilde{w}\|_2^2 \}, \quad (7.9)$$

(see, e.g., [3][4, Appendix]).

We rewrite (5.3) in the form of Stokes system with $\tilde{w} = T_{j,k} w_\infty$ and $\tilde{\phi} = \frac{P'(\rho_p)}{\nu \gamma^2} T_{j,k} \phi_\infty$. The desired result is then obtained by using (7.9) (cf. [7, Proposition 7.9]). $\square$

At last we estimate the time derivatives of σ_∞ and ϕ_∞ .

Proposition 7.9 (i) If $0 \leq 2j + k \leq m - 1$, then there holds the following estimate:

$$\|\partial_t T_{j,k} \sigma_\infty\|_2^2 \leq C \{R_{j,k}^{(7)} + \sum_{i=0}^j (\|T_{i,k+1} \sigma_\infty\|_2^2 + \|T_{i,k+1} \phi_\infty\|_2^2) + \gamma^4 \|T_{j,k+1} w_\infty\|_2^2\}, \quad (7.10)$$

Here, $R_{j,k}^{(7)} = \|[Q_0 T_{j,k} (P_\infty^{(0)} \mathbf{F})]_\infty\|_2^2$.

(ii) If $0 \leq k + 2j \leq m - 1$ then there holds the following estimate:

$$\|\partial_t^{j+1} \phi_\infty\|_{H^k}^2 \leq C \{R_j^{(8)} + \sum_{i=0}^j (\|\partial_{x'} \partial_t^i \phi_\infty\|_{H^k}^2 + \|\partial_{x'} \partial_t^i \sigma_\infty\|_2^2) + \gamma^4 \|\partial_x \partial_t^j w_\infty\|_{H^k}^2\} \quad (7.11)$$

Here, $R_{j,k}^{(8)} = \|\partial_t^j (Q_0 P_\infty \mathbf{F})\|_{H^k}^2$.

Proof. The estimates (7.10) and (7.11) follow from (5.2) and the first line of (5.3). □

Proposition 7.1 now follows from combination of results in Propositions 7.3–7.9.

Proof of Proposition 7.1. Let us define

$$\begin{aligned} \tilde{E}^{(0)}(t) &= \sum_{\substack{2j+k \leq m \\ 2j \neq m}} E^{(0)}[T_{j,k} \tilde{P}_\infty u(t)], \quad \tilde{E}^{(1)}(t) = \sum_{2j+k \leq m-1} E^{(1)}[T_{j,k} \tilde{P}_\infty u(t)], \\ E^{(2)}(t) &= \sum_{2j+k \leq m-1} \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_p)}{\gamma^2 \rho_p}} \partial_{x_n} T_{j,k} \phi_\infty(t) \right\|_2^2, \quad E^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{\nu}{\gamma^2(\nu + \tilde{\nu})} \|T_{j,k} \sigma_\infty(t)\|_2^2, \end{aligned}$$

and

$$\begin{aligned} \tilde{D}^{(0)}(t) &= \sum_{\substack{2j+k \leq m \\ 2j \neq m}} D^{(0)}[T_{j,k} w_\infty(t)], \\ D^{(1)}(t) &= \sum_{2j+k \leq m-1} \left(\frac{3b_1 \gamma^2}{2\nu(\nu + \tilde{\nu})} D^{(0)}[T_{j,k} w_\infty(t)] + \frac{1}{\nu + \tilde{\nu}} \|\sqrt{\rho_p} T_{j,k} \partial_t w_\infty(t)\|_2^2 \right), \\ D^{(2)}(t) &= \sum_{2j+k \leq m-1} \left(\frac{1}{2(\nu + \tilde{\nu})} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n} T_{j,k} \phi_\infty(t) \right\|_2^2 + \min\{1, 2c_0\} \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \dot{\phi}_\infty(t)\|_{H^1}^2 \right), \\ D^{(3)}(t) &= \sum_{2j+k \leq m-1} \frac{\alpha_1}{\nu + \tilde{\nu}} \|\nabla' T_{j,k} \sigma_\infty(t)\|_2^2. \end{aligned}$$

Let b_l , $l = 2, \dots, 5$, be positive numbers and let us consider

$$\begin{aligned} &\sum_{2j+k \leq m | 2j \neq m} \{2 \times (7.2) + 2b_2 \times (7.6)\} \\ &+ \sum_{2j+k \leq m-1} \left\{ \frac{2}{\nu + \tilde{\nu}} \times (7.3) + 2b_2 \times (7.5)_{l=0} + 2b_3 \times (7.7) + b_4 \times (7.8)_{l=0} \right\} \\ &+ \sum_{2j+k \leq m-1} \frac{1}{(\nu + \tilde{\nu}) \gamma^2} b_5 \times ((7.10) + (7.11)) + 2 \frac{b_6}{\nu + \tilde{\nu}} \times (7.2)_{2j=m}. \end{aligned}$$

Then we obtain

$$\frac{d}{dt} E^{(4)}(t) + D^{(4)}(t) + \frac{1}{(\nu + \tilde{\nu}) \gamma^2} b_5 \sum_{2j+k \leq m-1} (\|\partial_t T_{j,k} \sigma_\infty\|_2^2 + \|\partial_t^{j+1} \phi_\infty(t)\|_k^2) \quad (7.12)$$

$$\leq \sum_{j=1}^8 R^{(j)}(t) + RHS.$$

Here,

$$E^{(4)}(t) = \tilde{E}^{(0)} + \frac{1}{\nu + \tilde{\nu}} \tilde{E}^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t) + \frac{b_6}{\nu + \tilde{\nu}} E^{(0)}[\partial_t^{\lfloor \frac{m}{2} \rfloor} \tilde{P}_\infty u(t)],$$

$$D^{(4)} = \tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t) + b_4 \sum_{2j+k \leq m-1} \left(\frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^2 T_{j,k} w_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_x T_{j,k} \phi_\infty\|_2^2 \right) \\ + \frac{b_6}{\nu + \tilde{\nu}} D^{(0)}[\partial_t^{\lfloor \frac{m}{2} \rfloor} w_\infty],$$

and

$$R^{(1)} = \sum_{2j+k \leq m} R_{j,k}^{(1)}, \quad R^{(p)} = \sum_{2j+k \leq m-1} R_{j,k}^{(p)}, \quad p = 2, 5, 7, 8, \quad R^{(4)} = \sum_{\substack{2j+k \leq m \\ 2j \neq m}} R_{j,k}^{(4)},$$

$$R^{(p)} = \sum_{2j+k \leq m-1} R_{j,k,0}^{(p)}, \quad p = 3, 6,$$

with

$$RHS = C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu} \right) \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1} \|\partial_x T_{j,k} \phi_\infty\|_2^2 + C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu} + b_4 \right) \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1} \|\partial_{x'} T_{j,k} \sigma_\infty\|_2^2 \\ + C \left(\frac{1}{\gamma^2} + \frac{1}{\nu(\nu + \tilde{\nu})} \right) \sum_{2j+k \leq m-2} \|\partial_t T_{j,k} \sigma_\infty\|_2^2 + C \left(\frac{1}{\nu^2} + b_2 + b_3 + b_4 \right) \sum_{2j+k \leq m, 2j \neq m} D^{(0)}[T_{j,k} w_\infty] \\ + \left(\frac{b_1}{4} + b_5 C \right) \frac{\gamma^2}{\nu(\nu + \tilde{\nu})} \sum_{2j+k \leq m-1} D^{(0)}[T_{j,k} w_\infty] + C(b_2 + b_3 + b_4) \sum_{2j+k \leq m-1} \frac{1}{\nu + \tilde{\nu}} \|\sqrt{\rho_p} T_{j,k} \partial_t w_\infty\|_2^2 \\ + \frac{b_3 \alpha_1}{4(\nu + \tilde{\nu})} \sum_{2j+k \leq m-3} \|T_{j,k+1} \sigma_\infty\|_2^2 + C b_3 \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n} T_{j,k} \phi_\infty \right\|_2^2 \\ + C b_4 \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{2j+k \leq m-1} \|T_{j,k} \dot{\phi}_\infty\|_{H^1}^2 + C b_6 \frac{1}{(\nu + \tilde{\nu}) \gamma^2} (\|\partial_t^{\lfloor \frac{m}{2} \rfloor} \phi_\infty\|_2^2 + \|\partial_t^{\lfloor \frac{m}{2} \rfloor} \sigma_\infty\|_2^2).$$

There exists $\nu_0 \geq \max\{1, \nu_3\}$, $\gamma_0 \geq \max\{1, \gamma_2\}$ and $1 > b > 0$ such that if $b_4 < b_3 < b_2$ and $b_6 \leq b_5 \leq b_1$ appropriately with $b_l \leq b$ for $l = 2, \dots, 4$, and $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ we can absorb most of the terms from RHS in the left-hand side of (7.12) to get

$$\frac{d}{dt} E^{(4)}(t) + \frac{1}{2} D^{(4)}(t) + \frac{1}{2} \frac{1}{(\nu + \tilde{\nu}) \gamma^2} b_5 \sum_{2j+k \leq m-1} \|\partial_t T_{j,k} \sigma_\infty\|_2^2 + \frac{1}{2} \frac{1}{(\nu + \tilde{\nu}) \gamma^2} b_5 \sum_{2j \leq m-2} \|\partial_t^{j+1} \phi_\infty(t)\|_2^2 \quad (7.13) \\ \leq C \sum_{j=1}^8 R^{(j)}(t) + C \sum_{2j+k \leq m-2} \|\partial_t T_{j,k} \sigma_\infty\|_2^2 \frac{1}{\nu + \tilde{\nu}}.$$

Next, we estimate higher order derivatives in x_n . For $1 \leq l \leq m-1$, we set

$$E_l^{(4)}(t) = \sum_{2j+k \leq m-1-l} \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_p)}{\gamma^2}} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty(t) \right\|_2^2,$$

and

$$D_l^{(4)}(t) = \sum_{2j+k \leq m-1-l} \left(\frac{1}{2(\nu + \tilde{\nu})} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n}^{l+1} T_{j,k} \phi_\infty(t) \right\|_2^2 + \frac{2c_0(\nu + \tilde{\nu})}{\gamma^4} \|T_{j,k} \partial_{x_n}^{l+1} \dot{\phi}_\infty(t)\|_2^2 \right) \\ + b_7 \sum_{2j+k \leq m-1-l} \left(\frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^{l+2} T_{j,k} w_\infty(t)\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_x^{l+1} T_{j,k} \phi_\infty(t)\|_2^2 \right).$$

We add $2 \times (7.5)$ to $b_7 \times (7.8)$ and sum over $2j + k \leq m - 1 - l$ to obtain

$$\frac{d}{dt} E_l^{(4)}(t) + D_l^{(4)}(t) \leq C R_l^{(9)} + b_7 C \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{2j+k \leq m-1-l} \|T_{j,k} \dot{\phi}_\infty\|_{H^{l+1}}^2 \\ + C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + b_7 \right) \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1-l} \|T_{j,k} \partial_t w_\infty\|_{H^l}^2 + C \left(\frac{1}{\nu^2} + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{\nu^2}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1-l} \|T_{j,k} w_\infty\|_{H^{l+2}}^2 \\ + C \left(b_7 + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1-l} \|T_{j,k+1} \sigma_\infty\|_2^2 + C \frac{\nu + \tilde{\nu}}{\gamma^2} \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1-l} \|T_{j,k} \phi_\infty\|_{H^{l+1}}^2.$$

Here, $R_l^{(9)} = \sum_{2j+k \leq m-1-l} (R_{j,k,l}^{(3)} + R_{j,k,l}^{(6)})$.

Let us sum up to l to get

$$\frac{d}{dt} \sum_{p=1}^l E_p^{(4)}(t) + \sum_{p=1}^l D_p^{(4)}(t) \leq C \sum_{p=1}^l R_p^{(9)} \\ + b_7 C \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{p=1}^l \sum_{2j+k \leq m-1-p} \|T_{j,k} \dot{\phi}_\infty\|_{H^{p+1}}^2 + C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + b_7 \right) \frac{1}{\nu + \tilde{\nu}} \sum_{p=1}^l \sum_{2j+k \leq m-1-p} \|T_{j,k} \partial_t w_\infty\|_{H^p}^2 \\ + C \left(\frac{1}{\nu^2} + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{\nu^2}{\nu + \tilde{\nu}} \sum_{p=1}^l \sum_{2j+k \leq m-1-p} \|T_{j,k} w_\infty\|_{H^{p+2}}^2 + C \left(b_7 + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{1}{\nu + \tilde{\nu}} \sum_{p=1}^l \sum_{2j+k \leq m-1-p} \|T_{j,k+1} \sigma_\infty\|_2^2 \\ + C \frac{\nu + \tilde{\nu}}{\gamma^2} \frac{1}{\nu + \tilde{\nu}} \sum_{p=1}^l \sum_{2j+k \leq m-1-p} \|T_{j,k} \phi_\infty\|_{H^{p+1}}^2.$$

Taking b_7 appropriately small, ν_0 and γ_0 possibly larger (based on l) we obtain

$$\frac{d}{dt} \sum_{p=1}^l E_p^{(4)}(t) + \frac{1}{2} \sum_{p=1}^l D_p^{(4)}(t) \leq C \sum_{p=1}^l R_p^{(9)} + b_7 C \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{2j+k \leq m-2} \|T_{j,k} \dot{\phi}_\infty\|_{H^1}^2 \quad (7.14)$$

$$+ C \frac{1}{\nu^2} \frac{\nu^2}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1} \|T_{j,k} \partial_x^2 w_\infty\|_2^2 + C \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m | 2j \neq m} \|T_{j,k} w_\infty\|_{H^1}^2 + C \left(b_7 + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{1}{\nu + \tilde{\nu}} D^{(0)}[\partial_t^{\lfloor \frac{m}{2} \rfloor} w_\infty] \\ + C \left(\frac{1}{\nu^2} + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{\nu^2}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-2} \|\partial_x^2 T_{j,k} w_\infty\|_2^2 + C \left(\frac{1}{\nu^2} + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \sum_{2j+k \leq m-2} D^{(0)}[T_{j,k} w_\infty] \\ + C \left(b_7 + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-1} \|T_{j,k+1} \sigma_\infty\|_2^2 + C \frac{\nu + \tilde{\nu}}{\gamma^2} \frac{1}{\nu + \tilde{\nu}} \sum_{2j+k \leq m-2} \|\partial_x T_{j,k} \phi_\infty\|_2^2.$$

Now adding $2 \times (7.13)$ together with (7.14) and taking possibly b_7 smaller, ν_0 and γ_0 larger we obtain

$$\frac{d}{dt} (2E^{(4)}(t) + \sum_{p=1}^l E_p^{(4)}(t)) + (D^{(4)}(t) + \sum_{p=1}^l D_p^{(4)}(t)) \quad (7.15) \\ + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \sum_{2j+k \leq m-1} \|\partial_t T_{j,k} \sigma_\infty\|_2^2 + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \sum_{2j \leq m-2} \|\partial_t^{j+1} \phi_\infty(t)\|_2^2$$

$$\leq C(\sum_{j=1}^8 R^{(j)}(t) + \sum_{p=1}^l R_p^{(9)}(t)) + C \sum_{2j+k \leq m-2} \|\partial_t T_{j,k} \sigma_\infty\|_2^2 \frac{1}{\nu + \tilde{\nu}}.$$

To absorb the last term on the right-hand side we use induction on m .

Let $m = 1$ then we have from (7.15) that

$$\begin{aligned} & \frac{d}{dt}(2E_1^{(4)}(t) + \sum_{p=1}^l E_{p,1}^{(4)}(t)) + (D_1^{(4)}(t) + \sum_{p=1}^l D_{p,1}^{(4)}(t)) \\ & + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \|\partial_t \sigma_\infty\|_2^2 \leq C(\sum_{j=1}^8 R^{(j)}(t) + \sum_{p=1}^l R_p^{(9)}(t)). \end{aligned} \quad (7.16)$$

Let $m = 2$ then

$$\begin{aligned} & \frac{d}{dt}(2E^{(4)}(t) + \sum_{p=1}^l E_p^{(4)}(t)) + (D^{(4)}(t) + \sum_{p=1}^l D_p^{(4)}(t)) \\ & + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \sum_{k \leq 1} \|\partial_t T_{0,k} \sigma_\infty\|_2^2 + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \|\partial_t^1 \phi_\infty(t)\|_2^2 \leq C(\sum_{j=1}^8 R^{(j)}(t) + \sum_{p=1}^l R_p^{(9)}(t)) \\ & + C \|\partial_t \sigma_\infty\|_2^2 \frac{1}{\nu + \tilde{\nu}}. \end{aligned} \quad (7.17)$$

By adding $b_8 \gamma^2 \times (7.16)$ to (7.17) with appropriately large $b_8 > 0$ we can absorb $\|\partial_t \sigma_\infty\|_2^2 \frac{1}{\nu + \tilde{\nu}}$ to the left-hand side. It is straightforward to see that this can be done from m to $m + 1$. Therefore, we have

$$\begin{aligned} & C_1 \frac{d}{dt}(2E^{(4)}(t) + \sum_{p=1}^l E_p^{(4)}(t)) + (D^{(4)}(t) + \sum_{p=1}^l D_p^{(4)}(t)) + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \sum_{2j+k \leq m-1} \|\partial_t T_{j,k} \sigma_\infty\|_2^2 \\ & + \frac{1}{(\nu + \tilde{\nu})\gamma^2} b_5 \sum_{2j \leq m-2} \|\partial_t^{j+1} \phi_\infty(t)\|_2^2 \leq C_2(\sum_{j=1}^8 R^{(j)}(t) + \sum_{p=1}^l R_p^{(9)}(t)), \end{aligned} \quad (7.18)$$

with $C_1, C_2 > 0$. The desired estimate (7.1) now follows from (7.18) with $l = m - 1$.

Estimate (5.11) for $\tilde{R}(t)$ is given in Proposition 8.1 (ii) below. This concludes the proof. $\square$

To prove (5.10) we employ the following lemma.

Lemma 7.10 *There exists $\tilde{r}_0 = \tilde{r}_0(\nu, \tilde{\nu}, \gamma)$ such that if $r_1 \leq \tilde{r}_0$, then there holds the estimate*

$$\|[Q_0 P_{\infty,1}(t)u]\|_2 \leq C \|\partial_{x'}(I - \Pi^{(0)}(t))P_{\infty,1}(t)u\|_2.$$

Proof. We set

$$\mathcal{R}(\xi', t) = \widehat{\mathcal{Q}}^{(0)}(t)(i\xi' \widehat{\mathcal{P}}^{(1)}(t) - |\xi'|^2 \widehat{\mathcal{P}}^{(2)}(\xi', t)) + (i\xi' \widehat{\mathcal{Q}}^{(1)}(t) - |\xi'|^2 \widehat{\mathcal{Q}}^{(2)}(\xi', t)) \widehat{\mathcal{P}}_{\xi'}(t).$$

Since

$$\begin{aligned} [Q_0 \widehat{\chi}_1(\xi')(I - \widehat{\mathcal{Q}}_{\xi'}(t) \widehat{\mathcal{P}}_{\xi'}(t))] &= [Q_0 \widehat{\chi}_1(\xi')(I - \widehat{\mathcal{Q}}^{(0)}(t) \widehat{\mathcal{P}}^{(0)} - \mathcal{R}(\xi', t))] \\ &= -[Q_0 \widehat{\chi}_1(\xi') \mathcal{R}(\xi', t)], \end{aligned}$$

we see that

$$[Q_0 \widehat{P_{\infty,1} u}(t)] = [Q_0 \widehat{\chi}_1(\xi')(I - \widehat{\mathcal{Q}}_{\xi'}(t) \widehat{\mathcal{P}}_{\xi'}(t)) \widehat{u}].$$

It then follows that

$$\|[Q_0 \widehat{P_{\infty,1} u}(t)]\|_2 \leq C |\xi'| \|\widehat{\chi}_1 \widehat{u}\|_2$$

$$\leq C|\xi'|(\widehat{\chi}_1|(I - \Pi^{(0)}(t))\widehat{u}|_2 + \widehat{\chi}_1|\Pi^{(0)}(t)\widehat{u}|_2).$$

Since $(P_{\infty,1}(t))^2 = P_{\infty,1}$, we see that

$$|[Q_0\widehat{P_{\infty,1}(t)u}]|_2 \leq C|\xi'|(|(I - \Pi^{(0)}(t))\widehat{P_{\infty,1}(t)u}|_2 + |[Q_0\widehat{P_{\infty,1}(t)u}]|_2),$$

for $|\xi'| \leq r_1$. Therefore, there exists a positive number $\tilde{r}_0$ such that if $r_1 \leq \tilde{r}_0$ then

$$|[Q_0\widehat{P_{\infty,1}(t)u}]|_2 \leq C|\xi'| |(I - \Pi^{(0)}(t))\widehat{P_{\infty,1}u}|_2,$$

for $|\xi'| \leq r_1$, from which we obtain

$$\|[Q_0P_{\infty,1}(t)u]\|_2 \leq C\|\partial_{x'}(I - \Pi^{(0)}(t))P_{\infty,1}(t)u\|_2.$$

This completes the proof. $\square$

Finally, we prove (5.10).

Proof of (5.10). We fix $\nu, \tilde{\nu}, \gamma$ so that inequality (7.1) in Proposition 7.1 holds true and set $r_1 = \min\{r_0, \tilde{r}_0, 1\}$. Then we proceed as in [7, Proof of (5.15)] to obtain

$$\tilde{E}(t) + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_n}^2 w_\infty(t)\|_{m-2}^2 + \int_0^t e^{-\tilde{a}(t-z)} D(z) dz \leq C\{e^{-\tilde{a}t} \tilde{E}(0) + R^{(10)}(t) + \int_0^t e^{-\tilde{a}(t-z)} \tilde{R}(z) dz\}. \quad (7.19)$$

Since

$$R^{(10)}(t) \leq C(1+t)^{-\frac{n+1}{2}} M(t)^4, \quad (7.20)$$

as we show in Proposition 8.1 (i) below, we deduce (5.10) from (7.19), Proposition 7.1 (i) and (7.20). This completes the proof. $\square$

8 Estimates on the nonlinearities

In this section we estimate the nonlinearities, e.g., we prove (5.11) and (7.20). In this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_2^2$.

Proposition 8.1 *There exists number $\varepsilon_6 > 0$ such that if solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0 \leq z \leq t} \|u(z)\|_m \leq \varepsilon_6$ and $M(t) \leq 1$ for all $t \in [0, \tau]$, then the following estimates hold for all $t \in [0, \tau]$ with $C > 0$ independent of τ .*

(i)

$$\|\tilde{Q}P_\infty \mathbf{F}(t)\|_{m-2} \leq C(1+t)^{-\frac{2n-1}{4}} M(t)^2,$$

(ii)

$$\tilde{R}(t) \leq C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{-\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

To show the estimates in Proposition 8.1 we use the following inequalities.

Lemma 8.2 (i) *Let $2 \leq p \leq \infty$ and let j and k be integers satisfying*

$$0 \leq j < k, \quad k > j + n \left(\frac{1}{2} - \frac{1}{p} \right).$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^j f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}^{1-\theta} \|\partial_x^k f\|_{L^2(\mathbb{R}^n)}^\theta,$$

where $\theta = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p})$.

(ii) Let $2 \leq p \leq \infty$ and let j and k be integers satisfying

$$0 \leq j < k, \quad k > j + n \left(\frac{1}{2} - \frac{1}{p} \right).$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^j f\|_{L^p(\Omega)} \leq C \|f\|_{H^k(\Omega)}.$$

(iii) If $f \in H^{n-1}(\Omega)$ and $f = f(x')$ is independent of x_n , then

$$\|f\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_{x'}^{n-1} f\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

Proof. The inequality in (i) is a special case of the Gagliardo-Nirenberg-Sobolev inequality which can be proved using Fourier transform. Inequality in (ii) can be obtained by (i) and the standard extension argument. As for (iii), since

$$\|f\|_{L^p(\Omega)} = \|f\|_{L^p(\mathbb{R}^{n-1})}, \quad 1 \leq p \leq \infty, \quad \|\partial_{x'} f\|_{L^2(\Omega)} = \|\partial_{x'} f\|_{L^2(\mathbb{R}^{n-1})},$$

the inequality is a consequence of (i) with $n = n - 1$, $p = \infty$, $j = 0$ and $k = n - 1$. $\square$

Lemma 8.3 (i) Let m and m_k , $k = 1, \dots, l$ be nonnegative integers and let α_k , $k = 1, \dots, l$ be multi-indices. Suppose that

$$m \geq \left\lceil \frac{n}{2} \right\rceil + 1, \quad 0 \leq |\alpha_k| \leq m_k \leq m + |\alpha_k|, \quad k = 1, \dots, l,$$

and

$$m_1 + \dots + m_l \geq (l - 1)m + |\alpha_1| + \dots + |\alpha_l|.$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^{\alpha_1} f_1 \dots \partial_x^{\alpha_l} f_l\|_2 \leq C \prod_{1 \leq k \leq l} \|f_k\|_{H^{m_k}}.$$

(ii) Let $1 \leq k \leq m$. Suppose that $F(x, t, y)$ is a smooth function on $\Omega \times [0, \infty) \times I$, where I is a compact interval in $\mathbb{R}$. Then for $|\alpha| + 2j = k$ there hold

$$\|[\partial_x^\alpha \partial_t^j, F(x, t, f_1)] f_2\|_2 \leq \begin{cases} C_0(t, f_1(t)) \llbracket f_2 \rrbracket_{k-1} + C_1(t, f_1(t)) \{1 + \| |Df_1| \|_{m-1}^{|\alpha|+j-1}\} \| |Df_1| \|_{m-1} \llbracket f_2 \rrbracket_k, \\ C_0(t, f_1(t)) \llbracket f_2 \rrbracket_{k-1} + C_1(t, f_1(t)) \{1 + \| |Df_1| \|_{m-1}^{|\alpha|+j-1}\} \| |Df_1| \|_m \llbracket f_2 \rrbracket_{k-1}. \end{cases}$$

Here

$$C_0(t, f_1(t)) = \sum_{\substack{(\beta, l) \leq (\alpha, j) \\ (\beta, l) \neq (0, 0)}} \sup_x |(\partial_x^\beta \partial_t^l F)(x, t, f_1(x, t))|,$$

and

$$C_1(t, f_1(t)) = \sum_{\substack{(\beta, l) \leq (\alpha, j) \\ 1 \leq p \leq j + |\alpha|}} \sup_x |(\partial_x^\beta \partial_t^l \partial_y^p F)(x, t, f_1(x, t))|.$$

(iii) Let $m \geq [n/2] + 1$ then there exist constants $C, C' > 0$ such that

$$\|f_1 \cdot f_2\|_{H^m} \leq C \|f_1\|_{H^m} \|f_2\|_{H^m},$$

and when $\llbracket f_1 \rrbracket_m \leq 1$,

$$\llbracket f_1 \cdot f_2 \rrbracket_m \leq C' \llbracket f_1 \rrbracket_m \llbracket f_2 \rrbracket_m.$$

Proof of previous lemma can be found in [9, 10].

We recall that $u(t)$ is decomposed into

$$u = \sigma_1 u^{(0)} + u_1 + \sigma_\infty u^{(0)} + u_\infty,$$

and we write

$$\sigma_* = \sigma_1 + \sigma_\infty, \quad \phi_* = \phi_1 + \phi_\infty, \quad w_* = w_1 + w_\infty,$$

$$u_* = {}^T(\phi_*, w_*) = u_1 + u_\infty.$$

Before investigating the nonlinearities we present some basic estimates.

Lemma 8.4 *Let $u(t) = {}^T(\phi(t), w(t)) = (\sigma_* u^{(0)})(t) + u_*(t)$ be solution of (4.1) in $Z^m(\tau)$. The following estimates hold for all $t \in [0, \tau]$ with $C > 0$ independent of τ .*

(i)

$$\|\sigma_*(t)\|_2 \leq C(1+t)^{-\frac{n-1}{4}} M(t),$$

(ii)

$$\|D\sigma_*(t)\|_{m-1} + \llbracket u_*(t) \rrbracket_m \leq C(1+t)^{-\frac{n+1}{4}} M(t),$$

(iii)

$$\llbracket \phi(t) \rrbracket_m + \llbracket w(t) \rrbracket_m \leq C(1+t)^{-\frac{n-1}{4}} M(t),$$

(iv)

$$\|\sigma_*(t)\|_\infty \leq C(1+t)^{-\frac{n}{4}} M(t),$$

(v)

$$\|u_*(t)\|_\infty \leq C(1+t)^{-\frac{n+1}{4}} M(t),$$

(vi)

$$\|\phi(t)\|_\infty + \|w(t)\|_\infty \leq C(1+t)^{-\frac{n}{4}} M(t).$$

Proof. Estimates (i), (ii) and (iii) immediately follow from definition of $M(t)$. As for (iv), we see from Lemma 8.2 (iii) and Lemma 5.3 (iii) that

$$\|\sigma_*(t)\|_\infty \leq C\|\sigma_*(t)\|_2^{\frac{1}{2}} \|\partial_{x'}^{n-1} \sigma_*(t)\|_2^{\frac{1}{2}} \leq C\|\sigma_*(t)\|_2^{\frac{1}{2}} \|\partial_{x'} \sigma_*(t)\|_2^{\frac{1}{2}} \leq C(1+t)^{-\frac{n}{4}} M(t).$$

This shows (iv). Since $\|u_*(t)\|_\infty \leq C\|u_*(t)\|_{H^m}$ by Lemma 8.2 (ii) we get (v) from (ii). Estimate (vi) now follows from (iv) and (v). This completes the proof. $\square$

First, we consider the estimates on $Q_0 F$.

Proposition 8.5 *Let $u(t)$ be a solution of (4.1) in $Z^m(\tau)$ such that $M(t) \leq 1$ for all $t \in [0, \tau]$. There hold the following estimates for all $t \in [0, \tau]$ with $C > 0$ independent of τ .*

(i)

$$\llbracket \phi \operatorname{div} w \rrbracket_l \leq C \begin{cases} (1+t)^{-\frac{2n+1}{4}} M(t)^2 + (1+t)^{-\frac{n}{4}} M(t) \|Dw_\infty(t)\|_m, & l = m, \\ (1+t)^{-\frac{2n+1}{4}} M(t)^2, & l = m-1, \end{cases}$$

(ii)

$$\llbracket w \cdot \nabla(\sigma_* \phi^{(0)} + \phi_1) \rrbracket_m \leq C(1+t)^{-\frac{2n+1}{4}} M(t)^2,$$

(iii)

$$\llbracket w \cdot \nabla \phi_\infty \rrbracket_{m-1} \leq C(1+t)^{-\frac{2n+1}{4}} M(t)^2,$$

(iv)

$$|(\operatorname{div} \left(\frac{P'(\rho_p)}{\gamma^4 \rho_p} w \right), |\partial_t^j \partial_x^k \phi_\infty|^2)| \leq C(1+t)^{-\frac{n+1}{4}} M(t) D_\infty(t),$$

for $2j+k \leq m$,

(v)

$$\|[\partial_t^j \partial_x^k, w] \cdot \nabla \phi\|_2 \leq C(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) \sqrt{D_\infty(t)},$$

for $2j+k \leq m$,

(vi)

$$\|T_{j,k}(\phi w)\|_2 \leq (1+t)^{-\frac{2n-1}{4}} M(t)^2,$$

for $2j+k \leq m$.

Proof. By Lemma 5.3 (iii) we have

$$\llbracket \partial_{x'} \sigma_*(t) \rrbracket_m \leq \llbracket \partial_{x'} \sigma_*(t) \rrbracket_{m-1} + \llbracket \partial_t \sigma_*(t) \rrbracket_{m-2} \leq \llbracket D \sigma_*(t) \rrbracket_{m-1}.$$

We use this estimate and others that come from properties of $\mathcal{Q}(t)$ and $\mathcal{P}(t)$, e.g.,

$$\|\partial_{x'}^k \sigma_1\|_2 \leq \|\partial_{x'} \sigma_1\|_2, \quad k = 1, \dots,$$

and

$$\llbracket \nabla u_1 \rrbracket_m \leq C \llbracket u_1 \rrbracket_m,$$

together with Lemma 8.3 and Lemma 8.4 to obtain estimates (i)–(vi).

In the case of estimates (i)–(iii), we first use the following expansions and then apply above estimates:

$$\phi \operatorname{div} w = \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* + \sigma_* \phi^{(0)} \operatorname{div} w_* + \phi_* w^{(0),1} \partial_{x_1} \sigma_* + \phi_* \operatorname{div} w_*,$$

$$\begin{aligned} w \cdot \nabla (\sigma_* \phi^{(0)} + \phi_1) &= \sigma_* \phi^{(0)} w^{(0),1} \partial_{x_1} \sigma_* + w'_* \cdot \nabla' \sigma_* \phi^{(0)} + w_*^n \sigma_* \partial_{x_n} \phi^{(0)} \\ &\quad + \sigma_* w^{(0),1} \partial_{x_1} \phi_1 + w_* \cdot \nabla \phi_1, \end{aligned}$$

$$w \cdot \nabla \phi_\infty = \sigma_* w^{(0),1} \partial_{x_1} \phi_\infty + w_* \cdot \nabla \phi_\infty.$$

This concludes the proof. □

Second, we consider $\tilde{Q}F = {}^T(0, \mathbf{f})$. Recall that $\tilde{Q}F$ is written in the form

$$\tilde{Q}F = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3.$$

Here, $\tilde{F}_l = {}^T(0, \mathbf{f}_l)$, $l = 0, 1, 2, 3$, with

$$\mathbf{f}_0 = -w \cdot \nabla w - f_1(\rho_p, \phi) \Delta' \sigma_* w^{(0),1} e_1 - f_2(\rho_p, \phi) \nabla (\partial_{x_1} \sigma_* w^{(0),1})$$

$$+ \mathbf{f}_{01}(x_n, t, \phi) \phi \sigma_* + \mathbf{f}_{02}(x_n, t, \phi) \phi \nabla' \sigma_* + \mathbf{f}_{03}(x_n, t, \phi) \phi \phi_*,$$

$$\mathbf{f}_1 = -f_1(\rho_p, \phi) \Delta w_* = -\operatorname{div} (f_1(\rho_p, \phi) \nabla w_*) + {}^T(\nabla w_*) \nabla (f_1(\rho_p, \phi)),$$

$$\mathbf{f}_2 = -f_2(\rho_p, \phi) \nabla \operatorname{div} w_* = -\nabla (f_2(\rho_p, \phi) \operatorname{div} w_*) + (\operatorname{div} w_*) \nabla (f_2(\rho_p, \phi)),$$

$$\mathbf{f}_3 = -f_3(x_n, \phi) \phi \nabla \phi_*.$$

Here, ∇w_* denotes the $n \times n$ matrix $(\partial_{x_i} w_*^j)$; $f_1 = \frac{\nu \phi}{\rho_p(\gamma^2 \rho_p + \phi)}$; $f_2 = \frac{\tilde{\nu} \phi}{\rho_p(\gamma^2 \rho_p + \phi)}$; and $\mathbf{f}_{0l}(x_n, t, \phi)$, $l = 1, 2, 3$ are smooth functions of x_n , t and ϕ .

Proposition 8.6 *Let $u(t)$ be solution of (4.1) in $Z^m(\tau)$ and assume that $\sup_{0 \leq z \leq t} \llbracket u(z) \rrbracket_m \leq \varepsilon_5$ and $M(t) \leq 1$ for all $t \in [0, \tau]$. The following estimates hold for all $t \in [0, \tau]$ with $C > 0$ independent of τ .*

(i)

$$\llbracket \tilde{Q}F(t) \rrbracket_{m-2} \leq C(1+t)^{-\frac{2n-1}{4}} M(t)^2,$$

(ii)

$$\llbracket \mathbf{f}_0(t) \rrbracket_m \leq C\{(1+t)^{-\frac{2n-1}{4}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) \llbracket Dw_\infty(t) \rrbracket_m\},$$

(iii)

$$\sum_{l=1}^3 \|\mathbf{f}_l(t)\|_{m-1} \leq C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t)\|Dw_\infty(t)\|_m\},$$

(iv)

$$\sum_{l=1}^3 \|T_{j,k}\mathbf{f}_l\|_{H^{-1}} \leq C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t)\|Dw_\infty(t)\|_m\},$$

for $2j+k=m$. Here, we regard $T_{j,k}\mathbf{f}_l$ with $2j+k=m$ as an element in H^{-1} by $(T_{j,k}\mathbf{f}_l)[v] = \langle T_{j,k}\mathbf{f}_l, v \rangle_{-1}$ for $v \in H_0^1$,

Proof. Since $\|u(t)\|_m \leq \varepsilon_5$ we see that $\tilde{Q}\mathbf{F}(t)$ is smooth. Estimates (i)–(iii) can be obtained in similar manner to the proof of Proposition 8.5 and we omit the proof.

Let us prove estimate (iv). Let $2j+k=m$ and let $v \in H_0^1$. If $k \geq 1$ then we see from (iii) that

$$\begin{aligned} |\langle T_{j,k}\mathbf{f}_l, v \rangle_{-1}| &= |-(T_{j,k-1}\mathbf{f}_l, \partial_{x'}v)| \leq \|T_{j,k-1}\mathbf{f}_l\|_2 \|\partial_{x'}v\|_v \\ &\leq C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t)\|Dw_\infty(t)\|_m\} \|v\|_{H_0^1}. \end{aligned}$$

We thus conclude that

$$\|T_{j,k}\mathbf{f}_l\|_{H^{-1}} \leq C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t)\|Dw_\infty(t)\|_m\},$$

in the case $2j+k=m$, $k \geq 1$ and $l=1, 2, 3$.

If $k=0$, i.e., $m=2j$, we write $\langle \partial_t^j \mathbf{f}_1, v \rangle_{-1}$ as

$$\begin{aligned} \langle \partial_t^j \mathbf{f}_1, v \rangle_{-1} &= (\partial_t^j(f_1(\rho_p, \phi)\nabla w_*), \nabla v) + (\partial_t^j(T(\nabla w_*)\partial_{\rho_p}f_1(\rho_p, \phi)\nabla \rho_p), v) \\ &\quad + ([\partial_t^j, T(\nabla w_*)\partial_{\phi}f_1(\rho_p, \phi)]\nabla \phi, v) - (T(\nabla w_*)\partial_{\phi}f_1(\rho_p, \phi)\partial_t^j\phi, \operatorname{div} v) \\ &\quad - (T(\nabla^2 w_*)\partial_{\phi}f_1(\rho_p, \phi)\partial_t^j\phi, v) - (T(\nabla w_*)\nabla_{\rho_p, \phi}\partial_{\phi}f_1(\rho_p, \phi)(\nabla \rho_p + \nabla \phi)\partial_t^j\phi, v) \equiv \sum_{i=1}^6 I_i. \end{aligned}$$

As for I_1 , we have

$$|I_1| \leq \|\partial_t^j(f_1(\rho_p, \phi)\nabla w_*)\|_2 \|\nabla v\|_2.$$

As in the proof of Proposition 8.5 (i) one can estimate $\|\partial_t^j(f_1(\rho_p, \phi)\nabla w_*)\|_2$ to obtain

$$|I_1| \leq C\{(1+t)^{-\frac{2n+1}{4}} M(t)^2 + (1+t)^{-\frac{n}{4}} M(t)\|Dw_\infty(t)\|_m\} \|v\|_{H_0^1}.$$

Similarly, we have

$$|I_2| \leq C\{(1+t)^{-\frac{2n+1}{4}} M(t)^2 + (1+t)^{-\frac{n}{4}} M(t)\|Dw_\infty(t)\|_m\} \|v\|_{H_0^1}.$$

Next, we consider I_3 which we write as follows:

$$I_3 = (\partial_{\phi}f_1(\rho_p, \phi)[\partial_t^j, T(\nabla w_*)]\nabla \phi, v) + ([\partial_t^j, \partial_{\phi}f_1(\rho_p, \phi)](T(\nabla w_*)\nabla \phi), v) \equiv J_1 + J_2.$$

First, we treat J_1 , we have

$$|[\partial_t^j, T(\nabla w_*)]\nabla \phi| \leq C \sum_{l=0}^{j-1} |\partial_t^l \nabla \phi| |\partial_t^{j-l} \partial_x w_*|.$$

Since

$$\frac{1}{2} - \frac{m-1-2l}{n} + \frac{1}{2} - \frac{m-2(j-l)}{n} = 1 - \frac{m-1}{n} < 1,$$

we can find $p_{1l}, p_{2l} \geq 2$ satisfying

$$\frac{1}{p_{1l}} > \frac{1}{2} - \frac{m-1-2l}{n}, \quad \frac{1}{p_{2l}} > \frac{1}{2} - \frac{m-2(j-l)}{n}, \quad \frac{1}{2} \leq \frac{1}{p_{1l}} + \frac{1}{p_{2l}} < 1.$$

Now, we take number $p_{3l} \geq 2$ satisfying $\frac{1}{p_{3l}} = 1 - (\frac{1}{p_{1l}} + \frac{1}{p_{2l}}) > 0$. It then follows from Lemma 8.2 (ii) that

$$\begin{aligned} |(\partial_\phi f_1(\rho_p, \phi)[\partial_t^j, {}^T(\nabla w_*)]\nabla\phi, v)| &\leq C \sum_{l=0}^{j-1} \|\partial_t^l \partial_x \phi\|_{p_{1l}} \|\partial_t^{j-l} \partial_x w_*\|_{p_{2l}} \|v\|_{p_{3l}} \\ &\leq C \sum_{l=0}^{j-1} \|\partial_t^l \partial_x \phi\|_{H^{m-1-2l}} \|\partial_t^{j-l} \partial_x w_*\|_{H^{m-2(j-l)}} \|v\|_{H_0^1} \leq C \llbracket \phi \rrbracket_m \{ \llbracket \partial_x w_1 \rrbracket_m + \llbracket Dw_\infty \rrbracket_m \} \|v\|_{H_0^1}. \end{aligned}$$

Using Lemma 8.4 we conclude that

$$|J_1| \leq C \{ (1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) \llbracket Dw_\infty(t) \rrbracket_m \} \|v\|_{H_0^1}.$$

Second, we estimate J_2 . By Lemma 8.3 (i) we have for $m_k = m-1-2k$ and $m_l = m-1-2l$ that

$$\begin{aligned} \|[\partial_t^j, \partial_\phi f_1(\rho_p, \phi)]({}^T(\nabla w_*)\nabla\phi)\|_2 &\leq C \sum_{k+l=j} \|\partial_t^k (\partial_\phi^2 f_1(\rho_p, \phi) \partial_t \phi)\|_{m_k} \|\partial_t^l ({}^T(\nabla w_*)\nabla\phi)\|_{m_l} \\ &\leq C \llbracket \partial_t \phi \rrbracket_{m-1} \llbracket {}^T(\nabla w_*)\nabla\phi \rrbracket_{m-1}. \end{aligned}$$

Therefore, we obtain

$$|J_2| \leq C \llbracket \partial_t \phi \rrbracket_{m-1} \llbracket {}^T(\nabla w_*)\nabla\phi \rrbracket_{m-1} \|v\|_2,$$

By Lemma 8.3 (ii) we get

$$\begin{aligned} \llbracket {}^T(\nabla w_*)\nabla\phi \rrbracket_{m-1} &\leq C \{ \|\nabla w_*\|_\infty \llbracket \partial_x \phi \rrbracket_{m-1} + \llbracket D\nabla w_* \rrbracket_{m-2} \llbracket \nabla\phi \rrbracket_{m-1} \} \\ &\leq C \{ \|\partial_x w_1\|_{H^m} + \llbracket Dw_\infty \rrbracket_m \} \llbracket \partial_x \phi \rrbracket_{m-1} \leq C \{ (1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) \llbracket Dw_\infty(t) \rrbracket_m \}. \end{aligned}$$

As for $\llbracket \partial_t \phi \rrbracket_{m-1}$, we see from (6.3) and Proposition 7.9 that

$$\begin{aligned} \llbracket \partial_t \phi \rrbracket_{m-1} &\leq C \{ \llbracket \partial_t \sigma_* \rrbracket_{m-1} + \llbracket \partial_t \phi_* \rrbracket_{m-1} \} \leq C \{ \llbracket \Lambda \sigma_1(t) \rrbracket_{m-1} + \llbracket \mathcal{P}(t) \mathbf{F}(t) \rrbracket_{m-1} \\ &\quad + \llbracket \partial_x \phi_\infty \rrbracket_{m-1} + \llbracket \partial_x w_\infty \rrbracket_{m-1} + \llbracket \partial_x \sigma_\infty \rrbracket_{m-1} + \llbracket [Q_0(P_\infty^{(0)} \mathbf{F})]_\infty \rrbracket_{m-1} + \llbracket Q_0 P_\infty \mathbf{F} \rrbracket_{m-1} \}. \end{aligned}$$

Using

$$P_\infty \mathbf{F} = \mathbf{F} - [Q_0 \mathbf{F}]_\infty u^{(0)} - \{ \mathbb{P} \mathbf{F} - [Q_0 \mathbb{P} \mathbf{F}]_\infty u^{(0)} \},$$

and

$$[Q_0 P_\infty^{(0)} \mathbf{F}]_\infty = [Q_0 \mathbf{F}]_\infty - [Q_0 \mathbb{P} \mathbf{F}]_\infty,$$

together with Lemma 8.4 we get

$$\llbracket \partial_t \phi \rrbracket_{m-1} \leq C \{ (1+t)^{-\frac{n+1}{4}} M(t) + \llbracket \mathcal{P}(t) \mathbf{F}(t) \rrbracket_{m-1} + \llbracket Q_0 \mathbf{F} \rrbracket_{m-1} \}.$$

Since $2j = m$, we have $\lceil \frac{m-1}{2} \rceil = \lceil \frac{m-2}{2} \rceil$, and hence, by properties of $\mathcal{P}(t)$,

$$\llbracket \mathcal{P}(t) \mathbf{F}(t) \rrbracket_{m-1} \leq C \llbracket \mathbf{F}(t) \rrbracket_{m-2}.$$

It then follows from Propositions 8.5 (i)–(iii) and 8.6 (i) that

$$\llbracket \mathbf{F}(t) \rrbracket_{m-2} + \llbracket Q_0 \mathbf{F}(t) \rrbracket_{m-1} \leq 2 \llbracket Q_0 \mathbf{F}(t) \rrbracket_{m-1} + \llbracket \tilde{Q} \mathbf{F}(t) \rrbracket_{m-2} \leq C (1+t)^{-\frac{2n-1}{4}} M(t)^2,$$

which implies

$$\llbracket \partial_t \phi \rrbracket_{m-1} \leq C (1+t)^{-\frac{n+1}{4}} M(t).$$

We thus conclude

$$|J_2| \leq C \{ (1+t)^{-\frac{3n+1}{4}} M(t)^3 + (1+t)^{-\frac{n}{2}} M(t)^2 \llbracket Dw_\infty(t) \rrbracket_m \} \|v\|_2.$$

Consequently,

$$|I_3| \leq C\{(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)\|Dw_\infty(t)\|_m\}\|v\|_{H_0^1}.$$

Since

$$|I_4| \leq |(T(\nabla w_*)\partial_\phi f_1(\rho_p, \phi)\partial_t^j \phi, \operatorname{div} v)| \leq C\|\nabla w_*\|_\infty \|\partial_t^j \phi\|_2 \|\nabla v\|_2,$$

and by Lemma 8.3 (i),

$$|I_5| \leq |(T(\nabla^2 w_*)\partial_\phi f_1(\rho_p, \phi)\partial_t^j \phi, v)| \leq C\|\partial_t^j \phi\|_2 \|v\nabla^2 w_*\|_2 \leq C\|\partial_t^j \phi\|_2 \|\nabla w_*\|_{H^m} \|v\|_{H^1},$$

$$|I_6| \leq |(T(\nabla w_*)\nabla_{\rho_p, \phi}\partial_\phi f_1(\rho_p, \phi)(\nabla \rho_p + \nabla \phi)\partial_t^j \phi, v)| \leq C\|\nabla w_*\|_\infty \|\partial_t^j \phi\|_2 \|v\|_{H^1} \|\rho_p + \phi\|_{H^m},$$

we obtain by Lemmas 8.2 (ii) and 8.4,

$$|I_4| + |I_5| + |I_6| \leq C\{(1+t)^{-\frac{n+1}{2}}M(t)^2 + (1+t)^{-\frac{n+1}{4}}M(t)\|Dw_\infty(t)\|_m\}\|v\|_{H_0^1}.$$

Therefore, we arrive at

$$|\langle \partial_t^j \mathbf{f}_1, v \rangle_{-1}| \leq C\{(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)\|Dw_\infty(t)\|_m\}\|v\|_{H_0^1}.$$

This gives

$$\|\partial_t^j \mathbf{f}_1\|_{H^{-1}} \leq C\{(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)\|Dw_\infty(t)\|_m\}.$$

Clearly, one can get the same estimate for $\|\partial_t^j \mathbf{f}_2\|_{H^{-1}}$. Concerning $\|\partial_t^j \mathbf{f}_3\|_{H^{-1}}$, one can write it as

$$\partial_t^j \mathbf{f}_3 = -[\partial_t^j, f_3(x_n, \phi)\phi]\nabla \phi_* - \nabla(f_3(x_n, \phi)\phi\partial_t^j \phi_*) + \nabla(f_3(x_n, \phi)\phi)\partial_t^j \phi_*,$$

and thus the desired estimate is obtained analogously to the one for $\|\partial_t^j \mathbf{f}_1\|_{H^{-1}}$. This completes the proof. $\square$

Proof of Proposition 8.1. Since $\|\tilde{Q}P_\infty \mathbf{F}\|_{m-2} \leq C\|\mathbf{F}\|_{m-2}$, assertion (i) follows from Propositions 8.5 (i)–(iii) and 8.6 (i).

Let us consider $\tilde{R}(t)$. We know that there holds

$$\tilde{R}(t) \leq C\left(\sum_{j=1}^8 R^{(j)}(t) + \sum_{p=1}^{m-1} R_p^{(9)}\right).$$

Let us first show some basic estimates coming from Propositions 8.5, 8.6 and properties of $P(t)$:

$$\|Q_0 \mathbf{F}\|_{m-1} \leq C(1+t)^{-\frac{2n+1}{4}}M(t)^2, \quad (8.1)$$

$$\|\tilde{Q}\mathbf{F}\|_{m-1} \leq C\{(1+t)^{-\frac{2n-1}{4}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)\|Dw_\infty(t)\|_m\}, \quad (8.2)$$

$$\|\mathbb{P}\mathbf{F}\|_{m-1} \leq C\|\mathbf{F}\|_{m-1}, \quad (8.3)$$

Moreover, there holds

$$[Q_0 T_{j,k} \mathbf{F}]_\infty = -[\operatorname{div} T_{j,k}(\phi w)]_\infty = -[\operatorname{div}' T_{j,k}(\phi w')]_\infty, \quad (8.4)$$

since $w \in H_0^1$.

Let us begin with $R_{j,k}^{(1)}$. We write

$$R_{j,k}^{(1)} = \frac{\alpha_0}{\gamma^2}([Q_0 T_{j,k} \mathbf{F}]_\infty, T_{j,k} \sigma_\infty) - \frac{\alpha_0}{\gamma^2}([Q_0 T_{j,k}(\mathbb{P}\mathbf{F})]_\infty, T_{j,k} \sigma_\infty) + \tilde{R}_{j,k}^{(1)} - \langle T_{j,k}([Q_0 \mathbf{F}]_\infty u^{(0)}), T_{j,k} u_\infty \rangle_\Omega$$

$$- \langle T_{j,k}(\mathbb{P}\mathbf{F}), T_{j,k} u_\infty \rangle_\Omega + \langle T_{j,k}([Q_0(\mathbb{P}\mathbf{F})]_\infty u^{(0)}), T_{j,k} u_\infty \rangle_\Omega = \sum_{l=1}^6 I_l.$$

Here,

$$I_3 = \tilde{R}_{j,k}^{(1)} = \langle T_{j,k} \mathbf{F}, T_{j,k} u_\infty \rangle_\Omega,$$

when $2j + k \leq m - 1$, and

$$\begin{aligned} I_3 = \tilde{R}_{j,k}^{(1)} &= -(T_{j,k}(\phi \operatorname{div} w), T_{j,k} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p}) + \frac{1}{2} (\operatorname{div} (\frac{P'(\rho_p)}{\gamma^4 \rho_p} w), |T_{j,k} \phi_\infty|^2) \\ &\quad - (w \nabla T_{jk}(\sigma_* \phi^{(0)} + \phi_1), T_{jk} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p}) - ([T_{j,k}, w] \nabla \phi, T_{j,k} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p}). \\ &\quad + (T_{j,k} \mathbf{f}_0, T_{j,k} w_\infty \rho_p) + \sum_{l=1}^3 \langle T_{j,k} \mathbf{f}_l, T_{j,k} w_\infty \rho_p \rangle_{-1}, \end{aligned}$$

when $2j + k = m$.

We first consider I_3 . If $2j + k \leq m - 1$, then by applying (8.1) and (8.2), we have

$$\sum_{2j+k \leq m-1} |\langle T_{j,k} \mathbf{F}, T_{j,k} u_\infty \rangle_\Omega| \leq C \llbracket \mathbf{F} \rrbracket_{m-1} \llbracket u_\infty \rrbracket_{m-1} \leq C \{ (1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{-\frac{n}{4}} M(t) D_\infty(t) \}.$$

Here, we used relation $a^2 b \leq \frac{1}{2}(a^3 + ab^2)$ and Lemma 8.4 (ii).

In the case $2j + k = m$, we use Lemma 8.4 to calculate

$$|(w \nabla T_{jk}(\sigma_* \phi^{(0)} + \phi_1), T_{jk} \phi_\infty \frac{P'(\rho_p)}{\gamma^4 \rho_p})| \leq C \|w\|_\infty (\llbracket \sigma_* \rrbracket_m + \llbracket \phi_1 \rrbracket_m) \llbracket \phi_\infty \rrbracket_m \leq C (1+t)^{-n} M(t)^3.$$

From above estimate and Propositions 8.5 (i), (iv), (v) and 8.6 (ii), (iv) we see that

$$\sum_{2j+k \leq m} |I_3| \leq C \{ (1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t) \}.$$

We next consider I_5 . If $2j + k \leq m - 1$, then by (8.3) we see that

$$\sum_{2j+k \leq m-1} |\langle T_{j,k}(\mathbb{P} \mathbf{F}), T_{j,k} u_\infty \rangle_\Omega| \leq C \llbracket \mathbb{P} \mathbf{F} \rrbracket_{m-1} \llbracket u_\infty \rrbracket_{m-1} \leq C \llbracket \mathbf{F} \rrbracket_{m-1} \llbracket u_\infty \rrbracket_{m-1}.$$

If $2j + k = m$ and $k \geq 1$, then from properties of $P(t)$ we obtain

$$\sum_{2j+k=m, k \geq 1} |\langle T_{j,k}(\mathbb{P} \mathbf{F}), T_{j,k} u_\infty \rangle_\Omega| \leq \sum_{2j+k=m, k \geq 1} C \|T_{j,k-1}(\mathbb{P} \mathbf{F})\|_2 \|T_{j,k} u_\infty\|_2 \leq C \llbracket \mathbf{F} \rrbracket_{m-1} \llbracket u_\infty \rrbracket_m.$$

In the case $2j = m$, we write

$$|\langle \partial_t^j(\mathbb{P} \mathbf{F}), \partial_t^j u_\infty \rangle_\Omega| \leq C |\langle [\partial_t^j, \mathbb{P}] \mathbf{F}, \partial_t^j u_\infty \rangle_\Omega| + |\langle \mathbb{P} \partial_t^j \mathbf{F}, \partial_t^j u_\infty \rangle_\Omega| \leq C \llbracket \mathbf{F} \rrbracket_{m-2} \llbracket u_\infty \rrbracket_m + |\langle \mathbb{P} \partial_t^j \mathbf{F}, \partial_t^j u_\infty \rangle_\Omega|.$$

To estimate $\langle \mathbb{P} \partial_t^j \mathbf{F}, \partial_t^j u_\infty \rangle_\Omega$, we write it as

$$\langle \mathbb{P} \partial_t^j \mathbf{F}, \partial_t^j u_\infty \rangle_\Omega = \langle \partial_t^j \mathbf{F}, \mathbb{P}^* \partial_t^j u_\infty \rangle_\Omega.$$

Using integration by parts, we have

$$\langle \partial_t^j Q_0 \mathbf{F}, \mathbb{P}^* \partial_t^j u_\infty \rangle_\Omega = (\partial_t^j(\phi w), \nabla \left(Q_0(\mathbb{P}^* \partial_t^j u_\infty) \frac{P'(\rho_p)}{\gamma^4 \rho_p} \right)).$$

Since

$$\|\nabla \left(Q_0(\mathbb{P}^* \partial_t^j u_\infty) \frac{P'(\rho_p)}{\gamma^4 \rho_p} \right)\|_2 \leq C \|\partial_t^j u_\infty\|_2,$$

by properties of $\mathbb{P}^*$, we see from Proposition 8.5 (vi) that

$$|\langle \partial_t^j Q_0 \mathbf{F}, \mathbb{P}^* \partial_t^j u_\infty \rangle_\Omega| \leq C (1+t)^{-\frac{3n}{4}} M(t)^3.$$

Since $\tilde{Q} \mathbb{P}^* \partial_t^j u_\infty \in H_0^1$ one can estimate $|\langle \partial_t^j \tilde{Q} \mathbf{F}, \mathbb{P}^* \partial_t^j u_\infty \rangle_\Omega|$ using Proposition 8.6 (ii), (iv) to obtain

$$\sum_{2j=m} |\langle \partial_t^j \tilde{Q} \mathbf{F}, \mathbb{P}^* \partial_t^j u_\infty \rangle_\Omega| \leq C \{ (1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t) \}.$$

Therefore, we have

$$\sum_{2j+k \leq m} |I_5| \leq C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

As for I_4 , from (8.4) we compute

$$-\langle T_{j,k}([Q_0 \mathbf{F}]_\infty u^{(0)}), T_{j,k} u_\infty \rangle_\Omega = \sum_{i=0}^j \binom{j}{i} \langle [\operatorname{div}' T_{i,k}(\phi w')]_\infty T_{j-i,0} u^{(0)}, T_{j,k} u_\infty \rangle_\Omega.$$

Since

$$\|[\operatorname{div}' T_{i,k}(\phi w')]_\infty T_{j-i,0} u^{(0)}\|_2 \leq C \|T_{i,k}(\phi w')\|_2,$$

we see using Proposition 8.5 (vi) that

$$\sum_{2j+k \leq m} |I_4| \leq C(1+t)^{-\frac{3n}{4}} M(t)^3.$$

As for I_1 , using (8.4) and integration by parts we obtain

$$\frac{\alpha_0}{\gamma^2} ([Q_0 T_{j,k} \mathbf{F}]_\infty, T_{j,k} \sigma_\infty) = -\frac{\alpha_0}{\gamma^2} ([\operatorname{div}' T_{j,k}(\phi w')]_\infty, T_{j,k} \sigma_\infty) \leq C \|T_{j,k}(\phi w')\|_2 \|\nabla' T_{j,k} \sigma_\infty\|_2.$$

and thus by Proposition 8.5 (vi) and Lemma 8.4 (ii) we get

$$\sum_{2j+k \leq m} |I_1| \leq C(1+t)^{-\frac{3n}{4}} M(t)^3.$$

As for I_2 , in the case $1 \leq 2j+k \leq m$ we treat it analogously to I_5 to show

$$|\frac{\alpha_0}{\gamma^2} ([Q_0 T_{j,k}(\mathbb{P} \mathbf{F})]_\infty, T_{j,k} \sigma_\infty)| \leq C \|\mathbf{F}\|_{m-1} \|\partial_x' \sigma_\infty\|_{m-1} + |([Q_0 \mathbb{P} \partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F}]_\infty, \partial_t^{\lfloor \frac{m}{2} \rfloor} \sigma_\infty)|.$$

We further estimate

$$|([Q_0 \mathbb{P} \partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F}]_\infty, \partial_t^{\lfloor \frac{m}{2} \rfloor} \sigma_\infty)| \leq \|\mathcal{P}(\partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F})\|_2 \|\partial_t^{\lfloor \frac{m}{2} \rfloor} \sigma_\infty\|_2.$$

Using Plancherel theorem we have

$$\|\mathcal{P}(t)(\partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F}(t))\|_2 = \|\widehat{\chi}_1 \langle \partial_t^{\lfloor \frac{m}{2} \rfloor} \widehat{\mathbf{F}}(t), u_{\xi'}^*(t) \rangle\|_2.$$

Therefore, using above relation, analogously to I_5 , we estimate

$$\begin{aligned} \|\mathcal{P}(t)(\partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F}(t))\|_2 &\leq \|\mathcal{P}(t)(Q_0 \partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F}(t))\|_2 + \|\mathcal{P}(t)(\tilde{Q} \partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F}(t))\|_2 \\ &\leq C\{\|\partial_t^{\lfloor \frac{m}{2} \rfloor}(\phi w)\|_2 + \|\mathbf{f}_0\|_m + \sum_{l=1}^3 \|\partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{f}_l\|_{H^{-1}}\}. \end{aligned}$$

In the case $j = k = 0$ we see from Lemma 6.1 (ii) and properties of $\mathbb{P}(t)$,

$$|\frac{\alpha_0}{\gamma^2} ([Q_0(\mathbb{P} \mathbf{F})]_\infty, \sigma_\infty)| \leq C \|\partial_x' \sigma_\infty\|_2 (\|\phi w'\|_2 + \|\mathbf{F}\|_2) \leq (1+t)^{-\frac{3n}{4}} M(t)^3.$$

Therefore, using Propositions 8.5 (vi) and 8.6 (ii), (iv) we obtain

$$\sum_{2j+k \leq m} |I_2| \leq C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{-\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

As for I_6 , it can be treated in a way analogous to I_5 in the case $1 \leq 2j+k$, thus we get

$$\sum_{2j+k \leq m} |\langle T_{j,k}([Q_0(\mathbb{P} \mathbf{F})]_\infty u^{(0)}), T_{j,k} u_\infty \rangle_\Omega| \leq C\{\|\mathbf{F}\|_{m-1} \|u_\infty\|_{m-1} + \|\mathcal{P}(\partial_t^{\lfloor \frac{m}{2} \rfloor} \mathbf{F})\|_2 \|\partial_t^{\lfloor \frac{m}{2} \rfloor} u_\infty\|_2\},$$

and

$$\sum_{2j+k \leq m} |I_6| \leq C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

We thus conclude

$$R^{(1)}(t) \leq C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

It is straightforward to show that

$$\begin{aligned} \tilde{R}(t) &\leq C\{R^{(1)}(t) + \|\mathbf{F}\|_{m-1}^2 + \|Q_0(P_\infty^{(0)}\mathbf{F})\|_{m-1}^2 + \|P_\infty\mathbf{F}\|_{m-1}^2 + \|\phi \operatorname{div} w\|_m^2 \\ &\quad + \|w \cdot \nabla(\sigma_*\phi^{(0)} + \phi_1)\|_m^2 + \sum_{2j+k \leq m} \|[\partial_t^j \partial_x^k, w] \cdot \nabla \phi_\infty\|_2^2 \\ &\quad + \sum_{2j+k+l \leq m-1} \left| \operatorname{div} \left(\frac{P'(\rho_p)}{\gamma^4 \rho_p} w \right), |T_{j,k} \partial_{x_n}^{l+1} \phi_\infty|^2 \right| \\ &\quad + \sum_{2j+k \leq m-1} |(Q_0 T_{j,k}(P_\infty^{(0)}\mathbf{F}), T_{j,k} \sigma_\infty)| + \|\tilde{Q}(P_\infty\mathbf{F})\|_{m-1} \|\partial_{x'} \sigma_\infty\|_{m-1} \}. \end{aligned}$$

From definition we have

$$P_\infty^{(0)}(t) = I - \mathbb{P}(t) - P_{\infty,3},$$

which together with

$$|(Q_0 P_{\infty,3} T_{j,k} \mathbf{F}, T_{j,k} \sigma_\infty)| \leq C \|T_{j,k}(\phi w)\|_2 \|D \sigma_\infty\|_{m-1}.$$

gives (analogously to previous computations)

$$\sum_{2j+k \leq m-1} |(Q_0 T_{j,k}(P_\infty^{(0)}\mathbf{F}), T_{j,k} \sigma_\infty)| \leq C(1+t)^{-\frac{3n}{4}} M(t)^3.$$

Since

$$\|P_\infty^{(0)}\mathbf{F}\|_{m-1} + \|P_\infty\mathbf{F}\|_{m-1} \leq C\|\mathbf{F}\|_{m-1},$$

using Propositions 8.5, 8.6 and Lemma 8.4 we obtain the desired estimate (ii) in Proposition 8.1. This completes the proof. $\square$

9 Asymptotic behavior of $\sigma_1(t)$

In this section we show the asymptotic behavior of solutions of (4.1). In the case $n = 2$ we prove that it is described by a solution of a 1-dimensional viscous Burgers equation. In the case $n \geq 3$ we show that the asymptotic behavior is described by a linear heat equation, in fact, asymptotic leading term is the same as for the linearized problem.

In this section we assume that $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$. Let us note that $\sigma_1(t)$ is given by

$$\sigma_1(t) = \mathcal{P}(t)u(t), \quad t \geq 0,$$

where $u(t)$ is a global in time solution of (4.1). Existence of $u(t)$ was proved in Sections 3–8.

First let us treat the case $n = 2$.

Lemma 9.1 *Let $n = 2$ and $\sigma(t)$ is a solution of*

$$\begin{aligned} \partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + \omega_0 \partial_{x_1}(\sigma^2) &= 0, \\ \sigma|_{t=0} &= \sigma_0, \end{aligned} \tag{9.1}$$

where $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$ are the numbers given in (4.6) and $\omega_0 = \frac{1}{T} \int_0^T [\phi^{(0)} w^{(0),1}(z)] - \langle \mathbf{F}_1(z), u^{*(1)}(z) \rangle dz$ and $\sigma_0 = [Q_0 u_0] = [\phi_0]$. Then we can write

$$\sigma(t) = \mathcal{H}(t)\sigma_0 - \omega_0 \int_0^t \mathcal{H}(t-z) \partial_{x_1}(\sigma^2(z)) dz. \tag{9.2}$$

Theorem 9.2 *Let $n = 2$. For any $\delta > 0$ there exists $\varepsilon_7 > 0$ such that if $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_7$, then*

$$\|\sigma_1(t) - \sigma(t)\|_2 \leq C(1+t)^{-\frac{3}{4}+\delta} \|u_0\|_{H^m \cap L^1},$$

for $t \geq 0$.

(3.7) now follows from (5.13) and Theorem 9.2. To prove Theorem 9.2, we employ the following well-known decay properties of $\sigma(t)$.

Lemma 9.3 *Let $n = 2$ and $\sigma(t)$ is a solution of (9.1) with $\|\sigma_0\|_{H^1 \cap L^1} \ll 1$. Then*

$$\|\partial_{x_1}^k \sigma(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|\sigma_0\|_{H^1 \cap L^1} \quad (k = 0, 1),$$

$$\|\sigma(t)\|_\infty \leq C(1+t)^{-\frac{1}{2}} \|\sigma_0\|_{H^1 \cap L^1}.$$

We introduce a quantity. Let $\sigma_1(t)$ and $\sigma(t)$ be solutions of (5.1) for $s = 0$ and (9.1), respectively. We define $N(t)$ by

$$N(t) = \sup_{0 \leq z \leq t} (1+z)^{\frac{3}{4}-\delta} \|\sigma_1(z) - \sigma(z)\|_{H^1}.$$

Theorem 9.2 would then follow if we could show that $N(t) \leq C\|u_0\|_{H^m \cap L^1}$.

Proof of Theorem 9.2. It is obvious that estimate holds for $0 \leq t < 1$. Let us show that it holds for $t \geq 1$. Assume $t \geq 1$. From (5.1) we have that for $s = 0$

$$\sigma_1(t) = e^{t\Lambda} \mathcal{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda} \mathcal{P}(z)\mathbf{F}(z)dz. \quad (9.3)$$

We next rewrite $e^{(t-z)\Lambda} \mathcal{P}(z)\mathbf{F}(z)$. By Lemma 6.1 (ii), we have

$$\begin{aligned} \mathcal{P}(z)\mathbf{F}(z) &= -\partial_{x_1}[\phi w^1]_1 + \partial_{x_1} \mathcal{P}^{(1)}(z)\mathbf{F}(z) + \partial_{x_1}^2 \mathcal{P}^{(2)}(z)\mathbf{F}(z) \\ &= -a_{11}(z)\partial_{x_1}(\sigma_1^2) - \partial_{x_1}([\phi w^1]_1 - [\phi^{(0)}w^{(0),1}\sigma_1^2]_1) \\ &\quad + \partial_{x_1} \mathcal{P}^{(1)}(z)(\sigma_1^2 \mathbf{F}_1(z) + \mathbf{F}_2(z)) + \partial_{x_1}^2 \mathcal{P}^{(2)}(z)\mathbf{F}(z). \end{aligned}$$

Here $a_{11}(z) = [\phi^{(0)}w^{(0),1}(z)]$. Since

$$\mathcal{F}\{\mathcal{P}^{(1)}(z)(\sigma_1^2 \mathbf{F}_1(z))\} = \widehat{\chi}_1 \langle \widehat{(\sigma_1^2)} \mathbf{F}_1(z), u^{*(1)}(z) \rangle = \widehat{\chi}_1 \langle \mathbf{F}_1(z), u^{*(1)}(z) \rangle \widehat{(\sigma_1^2)} = -a_{12}(z)\sigma_1^2,$$

where $a_{12}(z) = -\langle \mathbf{F}_1(z), u^{*(1)}(z) \rangle$.

Using properties of $e^{(t-z)\Lambda}$, we thus arrive at

$$\begin{aligned} e^{(t-z)\Lambda} \mathcal{P}(z)\mathbf{F}(z) &= -a_1(z)e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2(z)) \\ &\quad - e^{(t-z)\Lambda} \partial_{x_1} \{[\phi w^1]_1(z) - [\phi^{(0)}w^{(0),1}(z)]\sigma_1^2(z)\} \\ &\quad + e^{(t-z)\Lambda} h_5(z) + e^{(t-z)\Lambda} h_6(z), \end{aligned}$$

where $a_1(z) = a_{11}(z) + a_{12}(z)$, $\sup_{z \in J_T} |a_1(z)| \leq C$ and

$$h_5(z) = \partial_{x_1} \mathcal{P}^{(1)}(z)\mathbf{F}_2(z) + \partial_{x_1}^2 \mathcal{P}^{(2)}(z)\mathbf{F}_2(z),$$

$$h_6(z) = \partial_{x_1}^2 \mathcal{P}^{(2)}(z)(\sigma_1^2 \mathbf{F}_1(z)).$$

It then follows from (9.2) and (9.3) that

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^6 I_j(t),$$

where

$$I_0(t) = e^{t\Lambda} \mathcal{P}(0)u_0 - \mathcal{H}(t)\sigma_0,$$

$$I_1(t) = - \int_0^t \omega_0 \mathcal{H}(t-z) \partial_{x_1}(\sigma_1^2(z) - \sigma^2(z)) dz,$$

$$I_2(t) = - \int_0^t \omega_0 (e^{(t-z)\Lambda} - \mathcal{H}(t-z)) \partial_{x_1}(\sigma_1^2) dz,$$

$$I_3(t) = - \int_0^t (a_1(z) - \omega_0) e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2) dz,$$

$$I_4(t) = - \int_0^t \partial_{x_1} e^{(t-z)\Lambda} ([\phi w^1]_1(z) - [\phi^{(0)} w^{(0),1}(z)] \sigma_1^2(z)) dz,$$

$$I_j(t) = \int_0^t e^{(t-z)\Lambda} h_j(z) dz, \quad j = 5, 6.$$

Let us show estimates on I_j , $j = 0, \dots, 6$.

As for I_0 , from (4.10) we see

$$\|I_0(t)\|_{H^1} \leq C t^{-\frac{3}{4}} \|u_0\|_{L^1}.$$

Let us consider $I_1(t)$. By Lemma 9.3, (5.12) and the definition of $M(t)$ and $N(t)$, we have

$$\|(\sigma_1^2 - \sigma^2)(z)\|_1 \leq \|(\sigma_1 + \sigma)(z)\|_2 \|(\sigma_1 - \sigma)(z)\|_2 \leq C(1+z)^{-1+\delta} N(t) \|u_0\|_{H^m \cap L^1},$$

for $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_3$. Furthermore, by Lemma 8.2 (iii) we have $\|(\sigma_1 - \sigma)(z)\|_\infty \leq C(1+z)^{-\frac{3}{4}+\delta} N(t)$, and hence,

$$\begin{aligned} \|\partial_{x_1}(\sigma_1^2 - \sigma^2)(z)\|_2 &\leq C \{ \|(\sigma_1 + \sigma)(z)\|_\infty \|\partial_{x_1}(\sigma_1 - \sigma)(z)\|_2 + \|(\sigma_1 - \sigma)(z)\|_\infty \|\partial_{x_1}(\sigma_1 + \sigma)(z)\|_2 \} \\ &\leq C(1+z)^{-\frac{5}{4}+\delta} \|u_0\|_{H^m \cap L^1} N(t). \end{aligned}$$

It then follows from (4.8) that for $k = 0, 1$,

$$\begin{aligned} \|\partial_{x_1}^k I_1(t)\|_2 &\leq C \left\{ \int_0^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}} (1+z)^{-1+\delta} dz + \int_{t-\frac{1}{2}}^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}} (1+z)^{-1+\delta} dz \right. \\ &\quad \left. + \int_{t-\frac{1}{2}}^t (t-z)^{-\frac{k}{2}} (1+z)^{-\frac{5}{4}+\delta} dz \right\} \|u_0\|_{H^m \cap L^1} N(t) \leq C(1+t)^{-\frac{3}{4}+\delta} \|u_0\|_{H^m \cap L^1} N(t). \end{aligned}$$

As for $I_2(t)$, we see from (4.11) that for $k = 0, 1$,

$$\|\partial_{x_1}^k I_2(t)\|_2 \leq C \left\{ \int_0^{t-\frac{1}{2}} (t-z)^{-\frac{5}{4}-\frac{k}{2}} \|\sigma_1^2(z)\|_1 dz + \int_{t-\frac{1}{2}}^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}} \|\partial_{x_1}(\sigma_1^2(z))\|_1 dz + \int_{t-\frac{1}{2}}^t (t-z)^{-\frac{k}{2}} \|\partial_{x_1}(\sigma_1^2(z))\|_2 dz \right\}.$$

From Lemma 6.3 we have

$$\begin{aligned} &\leq C \left\{ \int_0^{t-\frac{1}{2}} (t-z)^{-\frac{5}{4}-\frac{k}{2}} (1+z)^{-\frac{1}{2}} dz + \int_{t-\frac{1}{2}}^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}} (1+z)^{-1} dz + \int_{t-\frac{1}{2}}^t (t-z)^{-\frac{k}{2}} (1+z)^{-\frac{5}{4}} dz \right\} M(t)^2 \\ &\leq C(1+t)^{-\frac{3}{4}} \|u_0\|_{H^m \cap L^1}^2. \end{aligned}$$

As for $I_3(t)$, let us define $b(t) = \int_0^t a_1(z) - \omega_0 dz$. Then $\partial_t b(t) = a_1(t) - \omega_0$ and $b(0) = b(T) = 0$. Since $a_1(t+T) = a_1(t)$ we have $\partial_t b(t+T) = \partial_t b(t)$ and thus $b(t+T) = b(t)$. We arrive at $\sup_{z \in J_T} |b(z)| \leq C$. We write

$$I_3(t) = - \int_0^t \partial_z b(z) e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2) dz = - \left[b(z) e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2(z)) \right]_0^t + \int_0^t b(z) \partial_z \left(e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2(z)) \right) dz$$

$$= -b(t)\partial_{x_1}(\sigma_1^2(t)) - \int_0^t b(z)e^{(t-z)\Lambda}\Lambda\partial_{x_1}(\sigma_1^2(z))dz + \int_0^t b(z)\partial_{x_1}e^{(t-z)\Lambda}\partial_z(\sigma_1^2(z))dz \equiv J_1(t) + J_2(t) + J_3(t).$$

From Lemma 6.3 (vii) we have for $k = 0, 1$,

$$\|\partial_{x_1}^k J_1(t)\|_2 \leq C(1+t)^{-\frac{5}{4}}M(t)^2.$$

We see from (4.9) and Lemma 6.3 that

$$\begin{aligned} \|J_2(t)\|_2 &\leq C\left\{\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{5}{4}}\|\sigma_1^2(z)\|_1 dz + \int_{\frac{t}{2}}^t(1+t-z)^{-\frac{3}{4}}\|\partial_{x_1}(\sigma_1^2(z))\|_1 dz\right\} \\ &\leq C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m \cap L^1}^2. \end{aligned}$$

As for J_3 , using (6.3) we calculate

$$J_3(t) = 2 \int_0^t b(z)\partial_{x_1}e^{(t-z)\Lambda}\sigma_1(z)\Lambda\sigma_1(z)dz + 2 \int_0^t b(z)\partial_{x_1}e^{(t-z)\Lambda}\sigma_1(z)\mathcal{P}(z)\mathbf{F}(z)dz \equiv J_{31} + J_{32}.$$

Using (4.9) and Lemma 6.3 we calculate

$$\begin{aligned} \|J_{31}\|_2 &\leq C \int_0^t (1+t-z)^{-\frac{3}{4}}\|\sigma_1(z)\Lambda\sigma_1(z)\|_1 dz \leq C \int_0^t (1+t-z)^{-\frac{3}{4}}\|\sigma_1(z)\|_2\|\Lambda\sigma_1(z)\|_2 dz \\ &\leq CM(t)^2 \int_0^t (1+t-z)^{-\frac{3}{4}}(1+z)^{-1} dz \leq C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m \cap L^1}^2. \end{aligned}$$

Analogously we obtain for J_{32} that

$$\|J_{32}\|_2 \leq C \int_0^t (1+t-z)^{-\frac{3}{4}}\|\sigma_1(z)\|_2\|\mathbf{F}(z)\|_2 dz \leq C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m \cap L^1}^3.$$

As for $I_4(t)$, we have

$$\begin{aligned} \|[\phi w^1]_1(z) - [\phi^{(0)}w^{(0),1}(z)]\sigma_1^2(z)\|_1 &\leq C\{\|\sigma_1(z)\|_2\|u(z) - \sigma_1(z)u^{(0)}(z)\|_2 + \|u(z) - \sigma_1(z)u^{(0)}(z)\|_2^2\} \\ &\leq C(1+z)^{-1}M(z)^2. \end{aligned}$$

Thus, (4.9) gives us

$$\|\partial_{x_1}^k I_4(t)\|_2 \leq CM(t)^2 \int_0^t (1+t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1} dz \leq C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m \cap L^1}^2.$$

To estimate $I_5(t)$, we write $h_5(z)$ as

$$h_5(z) = \partial_{x_1} \left(\mathcal{P}^{(1)}(z)\mathbf{F}_2(z) + \partial_{x_1}\mathcal{P}^{(2)}(z)\mathbf{F}_2(z) \right).$$

Using (4.14) and Lemma 6.3 (v), we have

$$\|\partial_{x_1}^k I_5(t)\|_2 \leq CM(t)^2 \int_0^t (1+t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1} dz \leq C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m \cap L^1}^2.$$

As for $I_6(t)$, we write $h_6(z)$ as

$$h_6(z) = \begin{cases} \partial_{x_1}^2 \mathcal{P}^{(2)}(z)(\sigma_1^2 \mathbf{F}_1)(z) & \text{for } z \in [0, \frac{t}{2}], \\ \partial_{x_1} \mathcal{P}^{(2)}(z)(\partial_{x_1}(\sigma_1^2) \mathbf{F}_1)(z) & \text{for } z \in [\frac{t}{2}, t]. \end{cases}$$

We see from (4.14) and Lemma 6.3 that

$$\|\partial_{x_1}^k I_6(t)\|_2 \leq C\left\{\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{5}{4}-\frac{k}{2}}\|\sigma_1^2(z)\|_1 dz + \int_{\frac{t}{2}}^t(1+t-z)^{-\frac{3}{4}-\frac{k}{2}}\|\partial_{x_1}(\sigma_1^2)\|_1 dz\right\}M(t)^2$$

$$\leq C \left\{ \int_0^{\frac{t}{2}} (1+t-z)^{-\frac{5}{4}-\frac{k}{2}} (1+z)^{-\frac{1}{2}} dz + \int_{\frac{t}{2}}^t (1+t-z)^{-\frac{3}{4}-\frac{k}{2}} (1+z)^{-1} dz \right\} M(t)^2 \leq C(1+t)^{-\frac{3}{4}} \|u_0\|_{H^m \cap L^1}^2.$$

We thus obtain

$$\|(\sigma_1 - \sigma)(t)\|_{H^1} \leq C(1+t)^{-\frac{3}{4}+\delta} \|u_0\|_{H^m \cap L^1} \{1 + \|u_0\|_{H^m \cap L^1} + \|u_0\|_{H^m \cap L^1}^2 + N(t)\},$$

which yields

$$N(t) \leq \|u_0\|_{H^m \cap L^1} \{1 + \|u_0\|_{H^m \cap L^1} + \|u_0\|_{H^m \cap L^1}^2 + N(t)\}.$$

The desired result now follows by taking $\|u_0\|_{H^m \cap L^1}$ suitably small. This completes the proof. $\square$

Now let us show the asymptotic behavior in cases $n \geq 3$.

Theorem 9.4 *Let $n \geq 3$. There exists $\varepsilon_8 > 0$ such that if $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_8$, then*

$$\|\sigma_1(t) - \mathcal{H}(t)\sigma_0\|_2 \leq C(1+t)^{-\frac{n+1}{4}} \eta_n(t) \|u_0\|_{H^m \cap L^1},$$

where $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$ and $t \geq 0$.

Proof. From (9.3) we see that

$$\sigma_1(t) - \mathcal{H}(t)\sigma_0 = e^{t\Lambda} \mathcal{P}(0)u_0 - \mathcal{H}(t)\sigma_0 + \int_0^t e^{(t-z)\Lambda} \mathcal{P}(z)\mathbf{F}(z)dz.$$

Estimate (4.10) then implies

$$\|e^{t\Lambda} \mathcal{P}(0)u_0 - \mathcal{H}(t)\sigma_0\|_2 \leq Ct^{-\frac{n-1}{4}-\frac{1}{2}} \|u_0\|_{L^1(\Omega)}.$$

By Lemma 6.1 (ii) we have

$$\mathcal{P}(z)\mathbf{F}(z) = -\operatorname{div}'[\phi(z)w'(z)]_1 + \operatorname{div}' \mathcal{P}^{(1)}(z)\mathbf{F}(z) + \Delta' \mathcal{P}^{(2)}(z)\mathbf{F}(z),$$

and thus by using (4.14) and Lemma 6.3 we obtain

$$\begin{aligned} \left\| \int_0^t e^{(t-z)\Lambda} \mathcal{P}(z)\mathbf{F}(z)dz \right\|_2 &\leq C \int_0^t (1+t-z)^{-\frac{n-1}{4}-\frac{1}{2}} (\|\phi w'\|_1 + \|\mathbf{F}(z)\|_1) dz \\ &\leq CM(t)^2 \int_0^t (1+t-z)^{-\frac{n-1}{4}-\frac{1}{2}} (1+z)^{-\frac{n-1}{2}} dz \leq C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \eta_n(t) \|u_0\|_{H^m \cap L^1}. \end{aligned}$$

This concludes the proof. $\square$

(3.9) now follows from (5.13) and Theorem 9.4.

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