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Brezina, Jan
Graduate School of Mathematics, Kyushu University

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Jan Brezina

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow

Jan Březina *

Abstract

The global in time existence of strong solutions to the compressible Navier-Stokes equation around time-periodic parallel flows in $\mathbb{R}^n$, $n \geq 2$, is established under smallness conditions on Reynolds number, Mach number and initial perturbations. Furthermore, it is proved for $n = 2$ that the asymptotic leading part of solutions is given by a solution of one-dimensional viscous Burgers equation multiplied by time-periodic function. In the case $n \geq 3$ the asymptotic leading part of solutions is given by a solution of $n - 1$-dimensional heat equation with convective term multiplied by time-periodic function.

Mathematics Subject Classification

Keywords. Compressible Navier-Stokes equation, global existence, asymptotic behavior, time-periodic, viscous Burgers equation.

1 Introduction

In this paper we study the stability of solutions around a time-periodic parallel flow to the compressible Navier-Stokes equation with time-periodic external force and time-periodic boundary conditions.

We consider the system of equations

$$\begin{align*}
\partial_t \rho \mathbf{v} + \nabla (\rho \mathbf{v} \cdot \mathbf{v}) &= 0, \\
\rho (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \mu \Delta \mathbf{v} - (\mu + \mu') \nabla \text{div} \mathbf{v} + \nabla \tilde{P}(\rho) &= \tilde{\rho} \mathbf{g},
\end{align*}$$

in an $n$ dimensional infinite layer $\Omega_e = \mathbb{R}^{n-1} \times (0, \ell)$:

$$
\Omega_e = \{ \mathbf{x} = (\mathbf{z}', x_n) ; \\
\mathbf{z}' = (\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \in \mathbb{R}^{n-1}, 0 < x_n < \ell \}.
$$

Here, $n \geq 2$; $\tilde{\rho} = \tilde{\rho}(\mathbf{z}, \ell)$ and $\mathbf{v} = (\tilde{v}^1(\mathbf{z}, \tilde{\ell}), \ldots, \tilde{v}^n(\mathbf{z}, \tilde{\ell}))$ denote the unknown density and velocity at time $\tilde{\ell} \geq 0$ and position $\mathbf{z} \in \Omega_e$, respectively; $\tilde{P}$ is the pressure, smooth function of $\tilde{\rho}$, where for given $\rho_\ast > 0$ we assume

$$
\tilde{P}'(\rho_\ast) > 0;
$$

$\mu$ and $\mu'$ are the viscosity coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{\tau} \mu + \mu' \geq 0$; $\text{div} \mathbf{v}$ and $\Delta$ denote the usual divergence, gradient and Laplacian with respect to $\mathbf{z}$. Here and in what follows $^T\cdot$ denotes the transposition.

In (1.2) $\tilde{\mathbf{g}}$ is assumed to have the form

$$
\tilde{\mathbf{g}} = (\tilde{g}^1(\mathbf{z}_n, \tilde{\ell}), 0, \ldots, 0, \tilde{g}^n(\mathbf{z}_n)),
$$

with $\tilde{g}^1$ being a $\tau$-periodic function in time, where $\tau > 0$.

The system (1.1)–(1.2) is considered under boundary condition

$$
\mathbf{v}|_{x_n=0} = \tilde{V}^1(t) e_1, \quad \mathbf{v}|_{x_n=\ell} = 0,
$$

and initial condition

$$
(\rho, \mathbf{v})|_{t=0} = (\tilde{\rho}_0, \tilde{\mathbf{v}}_0),
$$

where $\tilde{V}^1$ is a $\tau$-periodic function of time and $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

Under suitable conditions on $\tilde{\mathbf{g}}$ and $\tilde{V}^1$, problem (1.1)–(1.3) has smooth time-periodic solution $\pi_p = T(\tilde{\pi}_p, \tilde{\mathbf{v}}_p)$ satisfying

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*Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, JAPAN
\[
\begin{align*}
\overline{\rho}_p &= \overline{\rho}_p(\overline{x}_n) \geq \overline{\rho}_1, \\
\overline{\pi}_p &= T(\overline{\pi}^1_p(\overline{x}_n, \overline{t}), 0, \ldots, 0), \\
\overline{\pi}^1_p(\overline{x}_n, \overline{t} + \tau) &= \overline{\pi}^1_p(\overline{x}_n, \overline{t}),
\end{align*}
\]
for a positive constant \(\overline{\rho}_1\).

The aim of this paper is to give an asymptotic description of large time behavior of perturbations from \(\overline{\pi}_p\) when Reynolds and Mach numbers are sufficiently small.

To formulate the problem for perturbations, we introduce the following dimensionless variables:
\[
\begin{align*}
\tilde{x} &= \ell x, \\
\tilde{t} &= \frac{\ell}{V} t, \\
\tilde{v} &= V v, \\
\tilde{\rho} &= \rho \gamma, \\
\tilde{P} &= \rho_u V^2 P,
\end{align*}
\]
with
\[
\begin{align*}
\tilde{\omega} &= V w, \\
\tilde{\phi} &= \rho_u^{-2} \phi, \\
\tilde{V}^1 &= V V^1, \\
\tilde{g} &= \frac{\mu V}{\rho_u \ell^2} g,
\end{align*}
\]
where
\[
\gamma = \frac{V}{V^1}, \\
V = \frac{\rho_u^2}{\mu} \left\{ |\partial_t \tilde{V}^1|_{C^0(\Omega)} + |\tilde{g}^1|_{C^0(\Omega \times [0, \ell])} \right\} + |\tilde{V}^1|_{C^0(\ell)} > 0.
\]

In this paper we assume \(V > 0\). Under this change of variables the domain \(\Omega_\ell\) is transformed into \(\Omega = \mathbb{R}^{n-1} \times (0, 1)\); and \(g^1(x_n, t), V^1(t)\) are periodic in \(t\) with period \(T > 0\) defined by
\[
T = \frac{V}{\ell} \tau.
\]

The time-periodic solution \(\overline{\pi}_p\) is transformed into \(u_p = T(\rho_p, v_p)\) satisfying
\[
\rho_p = \rho_p(x_n) > 0, \\
\int_0^1 \rho_p(x_n) dx_n = 1,
\]
\[
v_p = T(v_p^1(x_n, t), 0, \ldots, 0), \\
v_p^1(x_n, t + T) = v_p^1(x_n, t).
\]

It then follows that the perturbation \(u(t) = T(\phi(t), w(t)) \equiv T(\gamma^2 (\rho(t) - \rho_p), v(t) - v_p(t))\) is governed by the following system of equations
\[
\begin{align*}
\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \text{div} (\rho_p w) &= f^0, \\
\partial_t w - \frac{v_p^1}{\rho_p} \Delta w - \frac{v_p^1}{\rho_p} \nabla \text{div} w + v_p^1 \partial_{x_1} w + (\partial_{x_1} v_p^1) w_n e_1 \\
&+ \frac{v_p^1}{\gamma^2 \rho_p^2} (\partial_{x_1} v_p^1) \phi e_1 + \nabla \left( \frac{P'(\rho_p) \phi}{\gamma^2 \rho_p^2} \right) = f, \\
|w|_{\partial \Omega} &= 0,
\end{align*}
\]
(1.7)
(1.8)
where \(f^0\) and \(f = T(f^1, \ldots, f^n)\) denote nonlinearities, i.e.,
\[
f^0 = -\text{div} (\phi w),
\]
\[
f = -w \cdot \nabla w + \frac{v_p^1}{\gamma^2 \rho_p} \left( -\Delta w + \frac{\partial^2 v_p^1}{\gamma^2 \rho_p^2} \phi e_1 \right) - \frac{v_p^1}{\gamma^2 \rho_p^2} \left( -\Delta w + \frac{\partial^2 v_p^1}{\gamma^2 \rho_p^2} \phi e_1 \right)
\]
\[
- \frac{v_p^1}{\rho_p (\gamma^2 \rho_p + \phi)} \nabla \text{div} w + \frac{\phi}{\gamma^2 \rho_p} \nabla \left( \frac{P'(\rho_p)}{\gamma^2 \rho_p} \phi \right) - \frac{1}{2 \gamma^2 \rho_p} \nabla (P''(\rho_p) \phi^2) + \tilde{P}_3(\rho_p, \phi, \partial_{x_1} \phi),
\]
\[
\tilde{P}_3(\rho_p, \phi, \partial_{x_1} \phi) = \frac{\phi^3}{(\gamma^2 \rho_p + \phi)^2} \nabla P(\rho_p) + \frac{\phi^3}{2 \gamma^2 \rho_p (\gamma^2 \rho_p + \phi)} \nabla (\phi^3 P(\rho_p)),
\]
with
\[
P_3(\rho_p, \phi) = \int_0^1 (1 - \theta)^2 P''(\theta \gamma^{-2} \phi + \rho_p) d\theta.
\]
Here, \(\text{div}, \nabla\) and \(\Delta\) denote the usual divergence, gradient and Laplacian with respect to \(x\); \(\nu, \nu'\) and \(\tilde{\nu}\) are the non-dimensional parameters:

\[
\nu = \frac{\mu}{\rho \varepsilon}, \quad \nu' = \frac{\mu'}{\rho \varepsilon}, \quad \tilde{\nu} = \nu + \nu'.
\]

We note that the Reynolds number \(Re\) and Mach number \(Ma\) are given by \(Re = \nu^{-1}\) and \(Ma = \gamma^{-1}\), respectively. See [1] for the derivation of (1.5)–(1.8).

In the case \(g'\) and \(V'\) do not depend on \(t\), problem (1.1)–(1.3) has a stationary parallel flow. The stability of stationary parallel flows were studied in [5, 6, 7, 11]. It was shown in [6] and [7] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in \(H^m(\Omega) \cap L^1(\Omega)\) with \(m \geq [n/2] + 1\), provided that \(Re \ll 1, Ma \ll 1\) and density of the parallel flow is sufficiently close to a constant. Furthermore, the asymptotic behavior of perturbations from the stationary parallel flow is described by \(n-1\) dimensional linear heat equation in the case \(n \geq 3\) ([6]) and by one-dimensional viscous Burgers equation in the case \(n = 2\) ([7]).

The case of time-periodic parallel flows was considered in [1, 2] for \(Re \ll 1\) and \(Ma \ll 1\). In [1, 2] the authors investigated the linearized problem, i.e., (1.5)–(1.8) with \((f, m) \geq H\), expanded as

Here, for a constant \(0 < q < 1\), \(\xi, \xi', \xi''\) respectively. See [1] for the derivation of (1.5)–(1.8).

In [2] spectral properties of the solution operator \(U(t, s)\) for (1.9) were studied by using Fourier transform with respect to \(x' \in \mathbb{R}^{n-1}\). The Fourier transform of (1.9) can be written in the form:

\[
\begin{aligned}
\partial_t u + L(t)u &= 0, \quad w|_{x_n=0} = 0, \quad u|_{x_n=s} = u_0.
\end{aligned}
\]

Here, \(u = T(\phi, w)\) and \(L(t)\) is operator of the form

\[
L(t) = \begin{pmatrix}
\nu_{p1}(t) & \gamma^2 \text{div}(\rho_p) \\
\nabla (\rho_p/\gamma^2) & -\frac{\gamma}{\rho_p} \Delta I_n - \frac{\nu}{\rho_p} \nabla \text{div}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
\frac{\nu}{\gamma^2} \partial_{x_n} v_{p1}(t) e_1 & v_{p1}(t) \partial_{x_n} I_n + (\partial_{x_n} v_{p1}(t)) e_1^T e_n
\end{pmatrix}.
\]

Note that \(L(t)\) satisfies \(L(t) = L(t + T)\).

In [1, 2] spectral properties of the solution operator \(\tilde{U}_2(T, 0)\) plays an important role in the study of large time behavior. It was shown in [1] that the spectrum of \(\tilde{U}_2(T, 0)\) satisfies the following inclusion

\[
\sigma(\tilde{U}_2(T, 0)) \subseteq \{e^{\lambda T} \cup (|\lambda| < q_1) \quad (|\xi'| < r),
\}
\]

\[
\{ (|\lambda| < q_1) \quad (|\xi'| \geq r),
\}

for a constant \(0 < q_1 < 1\) and \(0 < r \ll 1\). Here, \(e^{\lambda T}\) is the simple eigenvalue of \(\tilde{U}_2(T, 0)\) and \(\lambda = -ik_0 \xi_1 - k_1 \xi_1^2 - k''|\xi'||\xi''| + O(|\xi'|^2)\) with \(k_0 \in \mathbb{R}, k_1 > 0, k'' > 0\) and \(\xi = T(\xi_1, \xi'), \xi'' = (\xi_2, \ldots, \xi_n-1)\).

In [2] spectral properties of \(\tilde{U}_2(t, s)\) were investigated for \(|\xi'| < r\) by using the Floquet theory. A family \(\{P(t)\}_{t \in \mathbb{R}}\) of bounded projections on \(L^2(\Omega)\) was constructed to represent \(P(t)U(t, s)\) as

\[
P(t)U(t, s) = \mathcal{D}(t)e^{(s-t)\mathcal{A}}\mathcal{P}(s).
\]

Here, \(e^{\mathcal{A}} = \mathcal{F}^{-1}\hat{\mathcal{A}}_1 e^{\mathcal{A}}\mathcal{F}\) with frequency cut off function \(\hat{\mathcal{A}}_1 : \hat{\mathcal{A}}_1(\xi') = 1 (|\xi'| < r), \hat{\mathcal{A}}_1(\xi') = 0 (|\xi'| \geq r),\) and \(\mathcal{D}(t) = \mathcal{F}^{-1}\hat{\mathcal{D}}_1 \mathcal{D}(\xi, t)\mathcal{F}\) and \(\mathcal{P}(t) = \mathcal{F}^{-1}\hat{\mathcal{P}}_1 \mathcal{P}(\xi, t)\mathcal{F}\) with

\[
\mathcal{D}(\xi, t) : \mathcal{C} \rightarrow L^2(0, 1) \text{ and } \mathcal{P}(\xi, t) : L^2(0, 1) \rightarrow \mathcal{C},
\]

expanded as

\[
\mathcal{D}(\xi, t) = \mathcal{D}^{(0)}(t) + i\xi' \cdot \mathcal{D}^{(1)}(t) + O(|\xi'|^2),
\]

\[
\mathcal{P}(\xi, t) = \mathcal{P}^{(0)} + i\xi' \cdot \mathcal{P}^{(1)}(t) + O(|\xi'|^2),
\]

3
for $|\xi'| \leq r$, where $\mathcal{Q}^{(0)}(t)\sigma = \sigma u^{(0)}(\cdot, t)$ ($\sigma \in \mathbb{C}$), $u^{(0)}(\cdot, t) = u^{(0)}(x, t)$ is a function $T$-periodic in $t$ and $\mathcal{P}^{(0)}[u] = [\phi]$ ($u = T(\phi, w) \in L^{2}(0, 1)$). One consequence of (1.12) is that

$$\|\partial_{x}^{k}\partial_{x}^{l}P(t)U(t, s)u_{0}\|_{L^{2}(\Omega)} \leq C(1 + t - s)^{-\frac{n+1}{2} - \frac{k}{2}}\|u_{0}\|_{L^{1}(\Omega)},$$

$$\|\partial_{x}^{k}\partial_{x}^{l}P(t)U(t, s)u_{0}\|_{L^{2}(\Omega)} \leq e^{-d(t-s)}\|u_{0}\|_{H^{1}\times L^{2}} + \|\partial_{x}^{2}w_{0}\|_{L^{2}},$$

$$\|\partial_{x}^{k}\partial_{x}^{l}(\mathcal{P}(t)U(t, s)u_{0} - \sigma_{t,s}[u_{0}]u^{(0)}(t))\|_{L^{2}(\Omega)} \leq C(t - s)^{-\frac{n+1}{2} - \frac{k}{2}}\|u_{0}\|_{L^{1}(\Omega)},$$

for $t - s \geq T$, $s \geq 0$; $k = 0, 1, \ldots, l = 0, \ldots, m$ for $m \geq 2$. Here, $\sigma_{t,s}[u_{0}] = \sigma_{t,s}(x')[u_{0}]$ is a function whose Fourier transform in $x'$ is given by

$$\mathcal{F}(\sigma_{t,s}[u_{0}]) = e^{-i\omega_{0}\xi_{1} + \kappa_{1}\xi_{1}^{2} + \kappa''|\xi''|^{2}(t-s)}[\hat{\phi}_{0}(\xi')],$$

where $[\hat{\phi}_{0}(\xi')]$ is a quantity given by

$$[\hat{\phi}_{0}(\xi')] = \int_{0}^{1} \hat{\phi}_{0}(\xi', x_{n}) \, dx_{n},$$

with $\hat{\phi}_{0}$ being the Fourier transform of $\phi_{0}$ in $x'$ and $\kappa_{0} \in \mathbb{R}$, $\kappa_{1} > 0$, $\kappa'' > 0$ are positive constants depending on $\rho_{1}, l, V, \mu, \mu'$ and $\mathcal{P}'(\rho_{s}).$

Another consequence of (1.12) is that if $u(t)$ is a solution of

$$\partial_{t}u + L(t)u = f, \quad u|_{t=0} = u_{0},$$

then $\mathcal{P}(t)u(t)$ is represented as

$$\mathcal{P}(t)u(t) = \mathcal{Q}(t) \left( e^{t\Lambda} \mathcal{P}(0)u_{0} + \int_{0}^{t} e^{(t-z)\Lambda} \mathcal{P}(z)f(z)dz \right),$$

(1.13)

In this paper we show the following results. Let $u_{0}$ be sufficiently small in $H^{m}(\Omega) \cap L^{2}(\Omega)$ for a given $m \geq [n/2] + 1$; and let $u_{0}$ satisfy a suitable compatibility condition, then there exists unique solution $u(t)$ of (1.5)–(1.8) in $C([0, \infty); H^{m}(\Omega))$, provided that $Re \ll 1, Ma \ll 1$ and $|1 - \rho_{p}|C^{m+1([0, 1])} \ll 1$. Furthermore, $u(t)$ satisfies

$$\|\partial_{x}^{k}\partial_{x}^{l}(t)\|_{L^{2}(\Omega)} \leq O(t^{-\frac{n+1}{2} - \frac{k}{2}}), \quad k = 0, 1,$$

as $t \to \infty$.

In the case $n = 2$, we show that the asymptotic leading term of perturbation $u(t)$ is described by a solution of one-dimensional viscous Burgers equation, i.e.,

$$\|u(t) - (\sigma u^{(0)}(t))\|_{2} = O(t^{-\frac{1}{2}+\delta}), \quad \forall \delta > 0,$$

as $t \to \infty$. Here, $u^{(0)}(x, t) = u^{(0)}(x, t)$ is a given time-periodic function; and $\sigma = \sigma(x_{1}, t)$ is function satisfying

$$\partial_{t}\sigma - \kappa_{1}\partial_{x}^{2}\sigma + \kappa_{0}\partial_{x}\sigma + \omega_{0}\partial_{x}^{2}\sigma = 0, \quad \sigma|_{t=0} = \int_{0}^{1} \phi_{0}(x_{1}, x_{2}) \, dx_{2},$$

(1.14)

with constants $\kappa_{0} \in \mathbb{R}$, $\kappa_{1} > 0$ being the same ones as those in $\lambda_{\xi}$, and $\omega_{0} \in \mathbb{R}$ determined by the nonlinearity $F$.

In the case $n \geq 3$, we show that the asymptotic leading term of $u(t)$ is the same one as for the linearized problem and thus it is given by $n - 1$-dimensional heat equation with convective term, i.e.,

$$\|u(t) - (\sigma u^{(0)}(t))\|_{2} = O(t^{-\frac{n+1}{2} - \frac{1}{2}}\eta_{n}(t)),$$

as $t \to \infty$. Here, $\sigma = \sigma(x', t)$ is function satisfying

$$\partial_{t}\sigma - \kappa_{1}\partial_{x}^{2}\sigma - \kappa''\Delta^{n}\sigma + \kappa_{0}\partial_{x}\sigma = 0, \quad \sigma|_{t=0} = \int_{0}^{1} \phi_{0}(x', x_{n}) \, dx_{n},$$

with constants $\kappa_{0} \in \mathbb{R}$, $\kappa_{1}, \kappa'' > 0$ being the same ones as those in $\lambda_{\xi}$; where $\Delta^{n} = \partial_{x}^{2} + \cdots + \partial_{x}^{2}$; and $\eta_{n}(t) = \log(1 + t)$ when $n = 3$ and $\eta_{n}(t) = 1$ when $n \geq 4$.

The proof of the main results is given by a combination of various estimates for $\mathcal{P}(t)U(t, s)$ mentioned above and a variant of Matsumura-Nishida energy method ([6, 7], cf. [12]). We decompose the solution $u(t)$
of (1.5)–(1.8) into the $P(t)$-part and $(I-P(t))$-part. Considering the $P(t)$-part, we represent $P(t)u(t)$ as in (1.13) with $f = T(f^0, f)$ being the nonlinearity given in (1.5) and (1.6). We then combine various estimates on $P(t)$ and $P(t)U(t,s)$ to obtain the necessary estimates on $P(t)u(t)$. On the other hand, $(I-P(t))u(t)$ can be estimated by a variant of Matsumura-Nishida energy method as in the case of the stationary parallel flow ([7]).

However, in contrast to [7], the linearized operator has time-dependent coefficients. Therefore a modification of the argument in [7] is needed for the time-periodic case to acquire the necessary energy estimate. It is worth mentioning that in the case $n = 2$ the asymptotic leading part of $u(t)$ is not described by the linearized problem due to the quadratic nonlinearities $-\text{div}(\phi w, \frac{\nu \phi}{\gamma \rho_p} - \frac{\partial^2 w^1}{\partial \gamma} \phi)$ and $-\frac{1}{\gamma \rho_p} \partial_x (P''(\rho_p) \phi^2)$.

This leads to the 1-dimensional Burgers equation (1.14).

Our result is an extension of previous results on the stationary case [5, 6, 7, 11] to the case of time-periodic external force and time-periodic boundary conditions.

Structure of this paper is the following. In Section 2 we introduce basic notations that are used throughout the paper. In Section 3 we state the main results. In Section 4 we present the results on spectral properties of the linearized problem obtained in [2]. In Section 5 we introduce decomposition of solution $u(t)$ to (1.5)–(1.8) based on the spectral properties of $L(t)$ introduced in Section 4. Moreover, we prove the a priori estimate using the estimates on $P(t)u(t)$ and $(I-P(t))u(t)$ from subsequent sections 6, 7, and 8. In Section 6 we show estimate for $P(t)u(t)$ using properties of $P(t)$ and $P(t)U(t,s)$. In Section 7 we obtain estimate on $(I-P(t))u(t)$ using energy method. In Section 8 we estimate the nonlinearities $f^0$ and $f$. Finally, in Section 9 we prove the asymptotic behavior of solutions.

2 Notation

In this section we introduce some notations which are used throughout the paper. For a Domain $E$ we denote by $L^p(E)$ the usual Lebesgue space on $E$ and its norm is denoted by $\| \cdot \|_{L^p(E)}$ for $1 \leq p \leq \infty$. Let $k$ be a nonnegative integer. $H^k(E)$ denotes the $k$-th order $L^2$ Sobolev space on $E$ with norm $\| \cdot \|_{H^k(E)}$. $C^k_0(E)$ stands for the set of all $C^k$ functions which have compact support in $E$. We denote by $H_0^k(E)$ the completion of $C^k_0(E)$ in $H^k(E)$.

We simply denote by $L^p(E)$ (resp., $H^k(E)$) the set of all vector fields $w = (w^1, \ldots, w^n)$ on $E$ with $w_j \in L^p(E)$ (resp., $H^k(E)$), $j = 1, \ldots, n$, and its norm is also denoted by $\| \cdot \|_{L^p(E)}$ (resp., $\| \cdot \|_{H^k(E)}$). For $u = T(\phi, w)$ with $\phi \in H^k(E)$ and $w = T(w^1, \ldots, w^n) \in H^k(E)$, we define $\| u \|_{H^k(E) \times H^k(E)}$ by $\| u \|_{H^k(E) \times H^k(E)} = \| \phi \|_{H^k(E)} + \| w \|_{H^k(E)}$. When $k = l$, we simply write $\| u \|_{H^l(E) \times H^l(E)} = \| u \|_{H^l(E)}$.

In the case $E = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^k(\Omega)$) as $L^p$ (resp., $H^k$). In particular, we denote the norm of $L^p$ (resp., $H^k$) by $\| \cdot \|_p$ (resp., $\| \cdot \|_{H^k}$).

In the case $E = (0,1)$ we denote the norm $\| \cdot \|_{L^2(0,1)}$ (resp., $\| \cdot \|_{H^k(0,1)}$) by $\| \cdot \|_{2}$ (resp., $\| \cdot \|_{H^k}$).

The inner product of $L^2$ is denoted by

$$\langle f, g \rangle = \int_\Omega f(x)g(x) \, dx, \quad f, g \in L^2.$$ 

Furthermore, we introduce a weighted inner product $\langle \cdot, \cdot \rangle_\Omega$ defined by

$$\langle u_1, u_2 \rangle_\Omega = \int_\Omega \overline{\phi_1 \phi_2} \frac{P''(\rho_p)}{\gamma^4 \rho_p} \, dx + \int_\Omega w_1 w_2 \rho_p \, dx,$$

for $u_j = T(\phi_j, w_j) \in L^2$, $j = 1, 2$; and for $u_j = T(\phi_j, w_j) \in L^2(0,1)$, $j = 1, 2$, we also define $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \int_0^1 \overline{\phi_1 \phi_2} \frac{P''(\rho_p)}{\gamma^4 \rho_p} \, dx_n + \int_0^1 w_1 w_2 \rho_p \, dx_n.$$

Here, $\overline{\gamma}$ denotes the complex conjugate of $\gamma$.

Furthermore, for $f \in L^1(0,1)$ we denote the mean value of $f$ in $(0,1)$ by $[f]$:

$$[f] = (f,1) = \int_0^1 f(x) \, dx_n.$$

For $u = T(\phi, w) \in L^1(0,1)$ with $w = T(w^1, \ldots, w^n)$ we define $[u]$ by

$$[u] = [\phi] + [w^1] + \cdots + [w^n].$$

We often write $x \in \Omega$ as
\[ x = T(x', x_n), \quad x' = T(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}. \]

Partial derivatives of function \( u \) in \( x, \ x', \ x_n \) and \( t \) are denoted by \( \partial_x u, \ \partial_{x'} u, \ \partial_{x_n} u \) and \( \partial_t u \), respectively. We also write higher order partial derivatives of \( u \) in \( x \) as \( \partial_x^\alpha u = (\partial_x^\alpha u; |\alpha| = k) \); by \( \Delta' = \sum_{j=1}^{n-1} \partial_{x_j}^2 \), \( \nabla' = (\partial_{x_1}, \ldots, \partial_{x_{n-1}}) \) and \( \div' = \nabla' \). We denote the Laplacian, gradient and divergence with respect to \( x' \), respectively.

We denote \( k \times k \) identity matrix by \( I_k \). In particular, when \( k = n + 1 \), we simply write \( I \) for \( I_{n+1} \). We define \( (n+1) \times (n+1) \) diagonal matrices \( Q_j, \ \bar{Q}^0 \) and \( \bar{Q} \) by

\[
Q_j = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0), \quad j = 0, 1, \ldots, n,
\]

and

\[
Q' = \text{diag}(0, 1, \ldots, 1, 0), \quad \bar{Q} = \text{diag}(0, 1, \ldots, 1).
\]

We then have for \( u = T(\phi, w) \in \mathbb{R}^{n+1}, \ w = T(w^1, \ldots, w^n) = T(w', w^n), \)

\[
Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad Q_j u = \begin{pmatrix} 0 \\ w^j \\ 0 \end{pmatrix}, \quad Q_n u = \begin{pmatrix} 0 \\ 0 \\ w^n \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ w' \\ 0 \end{pmatrix}, \quad \bar{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.
\]

We denote \( e'_1 = T(1, 0, \ldots, 0) \in \mathbb{R}^{n-1} \). We note that

\[
[Q_0 u] = [\phi] \quad \text{for} \quad u = T(\phi, w).
\]

For a function \( f = f(x') \ (x' \in \mathbb{R}^{n-1}) \), we denote its Fourier transform by \( \hat{f} \) or \( \mathcal{F} f \):

\[
\hat{f}(\xi') = (\mathcal{F} f)(\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} dx'.
\]

The inverse Fourier transform is denoted by \( \mathcal{F}^{-1} \):

\[
(\mathcal{F}^{-1} f)(x') = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} d\xi'.
\]

For closed linear operator \( A \) in \( X \) we denote the spectrum of \( A \) by \( \sigma(A) \). We denote the set of all bounded linear operators from \( X_0 \) into itself by \( L(X_0) \) and denote the norm by \( | \cdot |_{L(X_0)} \). For operators \( A, B \) we denote \( [A, B] \) the commutator, i.e., \( [A, B] = AB - BA \). For time interval \( [a, b] \subset \mathbb{R} \), we denote the usual Bochner spaces by \( L^2(a, b; X), \ H^m(a, b; X) \), etc., where \( X \) denotes a Banach space.

**Definition 2.1** For a domain \( E \) we define the following function spaces:

\[
X_0 = H^1(0, 1) \times L^2(0, 1), \quad H^1_j(E) = \begin{cases} H^{-1}(E) = (H^1_j)^*(E) & \text{for } j = -1, \\ L^2(E) & \text{for } j = 0, \\ H^j(E) \cap H^0_0(E) & \text{for } j \geq 1. \end{cases}
\]

**Definition 2.2** We introduce the following norms:

\[
\|f(t)\|_k = \left( \sum_{j=0}^{\lfloor k \rfloor} \|\partial_x^j f(t)\|_{H^k-2j} \right)^{\frac{1}{2}}.
\]

\[
\|Df(t)\|_k = \begin{cases} \|\partial_x f(t)\|_2 & \text{for } k = 0, \\ \left( \|\partial_x f(t)\|_2^2 + \|\partial_t f(t)\|_{-1}^2 \right)^{\frac{1}{2}} & \text{for } k \geq 1. \end{cases}
\]

**Remark 2.3** Let us note that

\[
\|Dv\|_{m-1} \leq 2[v]_m \quad \text{and} \quad [v]_m \leq \|v\|_2 + \|Dv\|_{m-1},
\]

for \([v]_m < \infty\).
Definition 2.4 Let $m \geq \lceil n/2 \rceil + 1$. For $\tau > 0$ we define a function space $Z^m(\tau)$ by

$$Z^m(\tau) = \{ u \in \bigcap_{j=0}^{[n/2]} C^j([0, \tau]; H^{m-2j}), \| u \|_{Z^m(\tau)} < \infty \},$$

where

$$\| u \|_{Z^m(\tau)} = \sup_{0 \leq z \leq \tau} \| u(z) \|_m + \left( \int_0^\tau \| Dw(z) \|_{m}^2 \, dz \right)^{\frac{1}{2}}.$$

3 Main results

In this section we state the main results of this paper.

In the whole article we assume the following regularity for $\bar{g}$ and $V^1$.

Assumptions 3.1 For a given integer $m \geq \lceil n/2 \rceil + 1$ assume that $\bar{g} = T(\bar{g}^1(\bar{x}_n, \bar{t}), 0, \ldots, 0, \bar{g}^n(\bar{x}_n))$ and $\bar{V}^1(\bar{t})$ belong to the following spaces:

$$\bar{g}^n \in C^{m+1}[0, \ell],$$

and

$$\bar{g}^1 \in \bigcap_{j=0}^{[m/2]} C^j_{\text{per}}([0, \tau]; H^{m+1-2j}(0, \ell)),$$

$$\bar{V}^1 \in C^{[m/2]}_{\text{per}}([0, \tau]).$$

Furthermore, we assume

$$\bar{P}(\cdot) \in C^{m+2}(\mathbb{R}).$$

It is straightforward that $g$ and $V^1$ belong to similar spaces as $\bar{g}$ and $\bar{V}^1$.

Under Assumptions 3.1 one can see that flow $u_p$ has the following properties (see [1]).

Proposition 3.2 There exists $\delta_0 > 0$ such that if

$$\nu \| g^n \|_{C^{m+1}([0, 1])} \leq \delta_0,$$

then the following assertions hold true (see [1]). The flow $u_p = T(\rho_p(x_n), v_p(x_n, t))$ exists and under Assumptions 3.1, it satisfies

$$v_p \in \bigcap_{j=0}^{[m/2]} C^j_{\text{per}}(J_T; H^{m+3-2j}(0, 1)), \quad \rho_p \in C^{m+2}[0, 1],$$

and

$$0 < \rho_1 \leq \rho_p(x_n) \leq \rho_2, \quad \int_0^1 \rho_p(x_n) \, dx_n = 1, \quad v_p(x_n, t) = T(v_p^1(x_n, t), 0),$$

with

$$P'(\rho) > 0 \text{ for } \rho_1 \leq \rho \leq \rho_2,$$

$$|1 - \rho_p|_{C^{k+1}(\mathbb{R})} \leq \frac{C}{\gamma^2} \nu \| P' \|_{C^{k-1}(\rho_1, \rho_2)} + \| g^n \|_{C^{k}(\mathbb{R})}, \quad k = 1, \ldots, m + 1, \quad (3.1)$$

$$|P'(\rho_p) - \gamma^2|_{C^m(\mathbb{R})} \leq \frac{C}{\gamma^2} \nu \| g^n \|_{C^{m}(\mathbb{R})},$$

for some constants $0 < \rho_1 < 1 < \rho_2$. 

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First, let us introduce the local existence result. To do so, we rewrite (1.5)–(1.8) in the form

\[
\partial_t \phi + v \cdot \nabla \phi = -\gamma^2 w \cdot \nabla \rho_p - \rho \text{div } w,
\]

\[
\rho \partial_t w - \nu \Delta w - \nabla \text{div } w = -\frac{\nu}{\gamma^2 \rho_p} \partial^2 x_n v_p \phi - \nabla (P(\rho) - P(\rho_p)) - \rho (v \cdot \nabla v),
\]

\[
w|_{t=0} = 0,
\]

\[
(\phi, w)|_{t=0} = (\phi_0, w_0),
\]

where \( \rho = \rho_p + \gamma^{-2} \phi \) and \( v = v_p + w \).

Next, let us mention the compatibility condition for \( u_0 = T(\phi_0, w_0) \). We look for a solution \( u = T(\phi, w) \) of (3.2)–(3.5) in \( \{z \in \mathbb{R}^m : C^j([0, \infty); H^{m-2j}) \) satisfying \( \int_0^1 \| \partial_t w(z) \|_{H^m}^2 \, \mathrm{d}z < \infty \) for all \( t \geq 0 \) with \( m \geq [n/2] + 1 \). Therefore, we need to require the compatibility condition for the initial value \( u_0 = T(\phi_0, w_0) \), which is formulated as follows.

Let \( u = T(\phi, w) \) be a smooth solution of (3.2)–(3.5). Then \( \partial^j_t u = T(\partial^j_t \phi, \partial^j_t w) \), \( j \geq 1 \) is inductively determined by

\[
\partial^j_t \phi = -v \cdot \nabla \partial^{j-1}_t \phi - \rho \text{div } \partial^{j-1}_t w - \gamma^2 \partial^{j-1}_t w \cdot \nabla \rho_p - \left[ [\partial^{j-1}_t, v \cdot \nabla] + [\partial^{j-1}_t, \rho] \text{div } w \right],
\]

and

\[
\partial^j_t w = -\rho^{-1} \left\{ -\nu \Delta \partial^{j-1}_t w - \nabla \text{div } \partial^{j-1}_t w + P'(\rho) \partial^{j-1}_t \rho \right\} - \rho^{-1} \left\{ [\partial^{j-1}_t, \phi] \partial_t w + [\partial^{j-1}_t, P'(\rho)] \nabla \rho \right\} - \rho^{-1} \left\{ [\partial^{j-1}_t, \rho] \partial_t \nabla P(\rho_p) \right\} - \rho^{-1} \partial_t \nabla (\rho_v \cdot \nabla v).
\]

From these relations we see that \( \partial^j_t w|_{t=0} = T(\partial^j_t \phi, \partial^j_t w)|_{t=0} \) is inductively given by \( u_0 = T(\phi_0, w_0) \) in the following way:

\[
\partial^j_t w|_{t=0} = T(\partial^j_t \phi, \partial^j_t w)|_{t=0} = T(\phi_j, w_j) = u_j,
\]

where

\[
\phi_j = -v_0 \cdot \nabla \phi_{j-1} - \rho_0 \text{div } w_{j-1} - \gamma^2 w_{j-1} \cdot \nabla \rho_p - \sum_{l=1}^{j-1} \left( j - l \right) \nabla \phi_{j-l} + \gamma^{-2} \phi \text{div } w_{j-1},
\]

and

\[
w_j = -\rho_0^{-1} \left\{ -\nu \Delta w_{j-1} - \nabla \text{div } w_{j-1} + P'(\rho_0) \nabla \rho_{j-1} \right\} - \rho_0^{-1} \sum_{l=1}^{j-1} \left( j - l \right) \nabla \phi_{j-l} + \rho_0^{-1} \left\{ \delta_1 \nabla \phi_0 + \delta_2 \nabla P(\rho_0) \right\} - \rho_0^{-1} \left\{ \gamma^{-2} \phi \text{div } w_{j-1} \right\}.
\]

Assume that \( u = T(\phi, w) \) is a solution of (3.2)–(3.5) in \( \{z \in \mathbb{R}^m : C^j([0, \tau_0]; H^{m-2j}) \) for some \( \tau_0 > 0 \). Then from above observation, we need the regularity \( u_j = T(\phi_j, w_j) \in H^{m-2j} \times H^{m-2j} \) for \( j = 1, \ldots, [m/2] \), which, indeed follows from the fact that \( u_0 = T(\phi_0, w_0) \in H^m \) with \( m \geq [n/2] + 1 \). Furthermore, it is necessary to require that \( u_0 = T(\phi_0, w_0) \) satisfies the \( m \)-th order compatibility condition:

\[
w_j \in H_{\delta_1}^{2j} \text{ for } j = 0, \ldots, \tilde{m} = \left[ \frac{m-1}{2} \right].
\]

Now, we can apply local solvability result obtained in [8] to show the following assertion.
Proposition 3.3 Let \( n \geq 2 \), \( m \) be an integer satisfying \( m \geq \lfloor n/2 \rfloor + 1 \) and \( M > 0 \). Assume that \( u_0 = T(\phi_0, w_0) \in H^m \) satisfies the following conditions:

(a) \( \|u_0\|_{H^m} \leq M \) and \( u_0 \) satisfies the \( m \)-th compatibility condition,

(b) \( -\Delta u_0 \leq \phi_0 \).

Then there exists a positive number \( \tau_0 \) depending on \( M \) and \( \rho_1 \) such that problem (3.2)–(3.5) has a unique solution \( u(t) \) on \([0, \tau_0)\) satisfying \( u(t) \in Z^m(\tau_0) \).

Remark 3.4 It is straightforward to see that solution \( u(t) \) of (3.2)–(3.5) is a solution of (1.5)–(1.8). Condition (b) in previous proposition assures that \( \gamma^{-2}\phi_0 + \rho_0 > \frac{3}{4}\rho_1 \).

We are in a position to state our main results of this paper.

Theorem 3.5 Suppose that \( n = 2 \) and let \( m \) be an integer satisfying \( m \geq 2 \). There are positive numbers \( \nu_0 \) and \( \gamma_0 \) such that if \( \nu \geq \nu_0 \) and \( \gamma^2/(\nu + \tilde{v}) \geq \gamma_0^2 \) then the following assertions hold true.

There is a positive number \( \varepsilon_0 \) such that if \( u_0 = T(\phi_0, w_0) \in H^m \cap L^1 \) satisfies the \( m \)-th compatibility condition and \( \|u_0\|_{H^m \cap L^1} \leq \varepsilon_0 \), then there exists a unique global solution \( u(t) = T(\phi(t), w(t)) \) of (1.5)–(1.8) with \( n = 2 \) in \( \bigcap_{j=0}^{\infty} C^j([0, \infty); H^{m-2j}) \) which satisfies

\[
\|\partial^k_x u(t)\|_2 = O(t^{-\frac{k}{2}}), \quad k = 0, 1, \tag{3.6}
\]

\[
\|u(t) - (\sigma u(0))(t)\|_2 = O(t^{-\frac{3}{4} + \delta}), \quad \forall \delta > 0, \tag{3.7}
\]

as \( t \to \infty \). Here, \( u(0) = u(0)(x, t) \) is function given in Lemma 4.9 below; \( \sigma = \sigma(x, t) \) is function satisfying

\[
\partial_t \sigma - \kappa_1 \partial^2_{x_1} \sigma + \kappa_0 \partial_{x_1} \sigma + \omega_0 \partial_{x_1} (\sigma^2) = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x, x, t) \, dx,
\]

with given constants \( \kappa_0, \omega_0 \in \mathbb{R}, \kappa_1 > 0 \).

Theorem 3.6 Suppose that \( n \geq 3 \) and let \( m \) be an integer satisfying \( m \geq \lfloor n/2 \rfloor + 1 \). There are positive numbers \( \nu_0 \) and \( \gamma_0 \) such that if \( \nu \geq \nu_0 \) and \( \gamma^2/(\nu + \tilde{v}) \geq \gamma_0^2 \) then the following assertions hold true.

There is a positive number \( \varepsilon_0 \) such that if \( u_0 = T(\phi_0, w_0) \in H^m \cap L^1 \) satisfies the \( m \)-th compatibility condition and \( \|u_0\|_{H^m \cap L^1} \leq \varepsilon_0 \), then there exists a unique global solution \( u(t) = T(\phi(t), w(t)) \) of (1.5)–(1.8) in \( \bigcap_{j=0}^{\infty} C^j([0, \infty); H^{m-2j}) \) which satisfies

\[
\|\partial^k_x u(t)\|_2 = O(t^{-\frac{k}{2} + \delta}), \quad k = 0, 1, \tag{3.8}
\]

\[
\|u(t) - (\sigma u(0))(t)\|_2 = O(t^{-\frac{n-1}{2} + \delta}), \tag{3.9}
\]

as \( t \to \infty \). Here, \( \sigma = \sigma(x', t) \) is function satisfying

\[
\partial_t \sigma - \kappa_1 \partial^2_{x_1} \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \quad \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) \, dx_n,
\]

with given constants \( \kappa_0 \in \mathbb{R}, \kappa_1, \kappa'' > 0 \); where \( \Delta'' = \partial^2_{x_2} + \cdots + \partial^2_{x_{n-1}} \); and \( \eta_n(t) = \log(1 + t) \) when \( n = 3 \) and \( \eta_n(t) = 1 \) when \( n \geq 4 \).

As in [8, 12], the global existence result in Theorem 3.5 and Theorem 3.6 is proved by combining the local existence and the a priori estimate. Next we introduce the a priori estimate.

Proposition 3.7 Let \( n \geq 2 \) and \( m \) be an integer satisfying \( m \geq \lfloor n/2 \rfloor + 1 \). There are positive numbers \( \nu_0 \) and \( \gamma_0 \) such that if \( \nu \geq \nu_0 \) and \( \gamma^2/(\nu + \tilde{v}) \geq \gamma_0^2 \), then the following assertion holds true.

There exists number \( \varepsilon_1 > 0 \) such that if solution \( u(t) \) of (1.5)–(1.8) is in \( Z^m(\tau) \) and \( u(t) \) satisfies

\[
\|u_0\|_{H^m \cap L^1} \leq \varepsilon_1, \quad \text{then holds the estimate}
\]

\[
\|u(t)\|_m^2 \leq C_1 \|u_0\|_m^2, \tag{1.8}
\]

for a constant \( C_1 > 0 \) independent of \( \tau \).
Remark 3.8 In the proof of Proposition 3.7 we use the estimate (3.1) with \( k = m \) only, i.e.,

\[
|1 - \rho_p|_{C^{m+1}([0,1])} \leq C \gamma^2 \nu |\phi_{m+1}(\rho_p) + |\phi_m|_{C^m([0,1])}.
\]

Moreover, we require the boundedness of \( \rho_p \) in \( C^{m+1}([0,1]) \) only.

The global existence of the solution \( u(t) \) follows from Proposition 3.3 and the a priori estimate in Proposition 3.7 in standard manner as follows.

**Proof of global existence.** Let \( n \geq 2 \) and let us fix \( m \geq [n/2] + 1 \) and \( \nu \geq \nu_0, \gamma^2/(\nu + \delta) \geq \gamma_0^2 \) such that Proposition 3.7 holds true.

Since \( m \geq [n/2] + 1 \) we have the Sobolev inequality

\[
\|f\|_{L^2} \leq C_S\|f\|_{H^m}, \text{ for any } f \in H^m(\Omega).
\]

Let us define \( \varepsilon_0 > 0 \) as

\[
\varepsilon_0 = \min\{\varepsilon_1, \frac{\gamma^2}{4C_S}, \frac{\varepsilon_1}{\sqrt{C_1}}, \frac{\gamma^2}{4C_S\sqrt{C_1}}\).
\]

Here, \( \varepsilon_1 \) and \( C_1 \) are given by Proposition 3.7.

Let \( \|u_0\|_{H^{m+2}} \leq \varepsilon_0 \) satisfies \( \tilde{m} \)-th compatibility condition. It is easy to see that such \( u_0 \) satisfies conditions (a), (b) of Proposition 3.3 and therefore, there exists \( \tau_0 > 0 \), which is determined by \( \varepsilon_1 \), such that the problem (1.5)–(1.8) has a unique solution \( u(\cdot) \in Z^m(\tau_0) \).

Since \( \varepsilon_0 \leq \varepsilon_1 \) we see from Proposition 3.7 that \( u(t) \) satisfies

\[
[u(t)]_{m}^{2} \leq C_1\|u_0\|_{H^{m+2}}^{2} \leq \min\{\varepsilon_1^{2}, \left(\frac{\gamma^2}{4C_S}\right)^{2}\}.
\]

Thus, \( \|u(\tau_0)\|_{H^m} \leq \varepsilon_1 \) and \( u(\tau_0) \) satisfies conditions (a) and (b) of Proposition 3.3. Hence, there exists unique extension of solution \( u(t) \) of (1.5)–(1.8) on \([\tau_0, 2\tau_0]\) and we get

\[
u(\cdot) \in Z^m(2\tau_0).
\]

It is straightforward to see that we can use Proposition 3.7 again, to obtain estimate (3.11) for \( u(2\tau_0) \), which enables us to extend solution \( u(t) \) on \([0, 3\tau_0]\). By repeating this procedure the existence on \([0, \infty)\) is showed. This concludes the proof.

Proposition 3.7 together with \( L^2 \)-decay estimates (3.6) and (3.8) are proved in Sections 4-8. The asymptotic behavior, i.e., (3.7) and (3.9), is proved in Section 9.

4 Spectral properties of the linearized operator

Let us write (1.5)–(1.8) in the form

\[
\partial_t u + L(t)u = F,
\]

\[
w|_{x=\alpha} = 0, \quad u|_{t=0} = u_0.
\]

Here, \( u = T(\phi, w); \quad F = T(f, f) \) with \( f = T(f^1, \cdots, f^m) \) is the nonlinearity; and \( L(t) \) is the operator given in (1.10).

In this section we introduce the spectral properties of the linearized problem, i.e., (4.1) with \( F = 0 \). These results were established in [2]. At the end of this section we show regularity improvements for \( \phi \).

Now, let us consider the linearized problem

\[
\partial_t u + L(t)u = 0, \quad t > s, \quad w|_{x=s, t=0} = 0, \quad u|_{t=s} = u_0.
\]

We introduce space \( Z_s \) defined by

\[
Z_s = \{u = T(\phi, w); \phi \in C_{tloc}([s, \infty); H^1), \quad \partial_{\nu}^\alpha w \in C_{tloc}([s, \infty); L^2) \cap C_{tloc}([s, \infty); H^1) \ (|\alpha| \leq 1), \quad w \in C_{tloc}([s, \infty); H^1)\}.
\]

In [1] we showed that for any initial data \( u_0 = T(\phi_0, w_0) \) satisfying \( u_0 \in H^1 \cap L^2 \) with \( \partial_{\nu} w_0 \in L^2 \) there exists a unique solution \( u(t) \) of linear problem (4.2) in \( Z_s \). We denote \( U(t, s) \) the evolution operator for (4.2) given by

\[
u(t) = U(t, s)u_0.
\]
To investigate problem (4.2) we consider the Fourier transform of (4.2). We thus obtain
\[
\frac{d}{dt} \hat{u} + \hat{L}_\xi'(t)\hat{u} = 0, \quad t > s, \quad \hat{u}|_{t=s} = \hat{u}_0.
\] (4.3)

Here \( \hat{\phi} = \hat{\phi}(\xi', x_n, t) \) and \( \hat{w} = \hat{w}(\xi', x_n, t) \) are the Fourier transforms of \( \phi = \phi(x', x_n, t) \) and \( w = w(x', x_n, t) \) in \( x' \in \mathbb{R}^{n-1} \) with \( \xi' \in \mathbb{R}^{n-1} \) being the dual variable; \( \hat{L}_\xi'(t) \) is an operator on \( X_0 \) with domain \( \text{D}(\hat{L}_\xi'(t)) = H^1(0, 1) \times H^2_\xi(0, 1) \), which takes the form

\[
\hat{L}_\xi'(t) = \begin{pmatrix}
  i\xi_1 v_p^1(t) & i\gamma_1 p^T \xi' & \gamma^2 \partial_x (p_p \cdot) \\
  i\xi_1^2 \frac{p'(p_p)}{\gamma^2 p_p} & \frac{\nu}{p_p} (\xi'|\xi| - \partial_x^2_{n-1}) I_{n-1} + \frac{\nu}{p_p} \xi^T \xi' & -i \frac{\nu}{p_p} \xi' \partial_x n \\
  \partial_x (\frac{p'(p_p)}{\gamma^2 p_p}) & -i \frac{\nu}{p_p} \xi' \partial_x n & \frac{\nu}{p_p} (\xi'|\xi| - \partial_x^2_{n-1}) - \frac{\nu}{p_p} \partial_x^2 n \\
\end{pmatrix} \\
+ \begin{pmatrix}
  0 & 0 & 0 \\
  0 & i\xi_1 v_p^1(t) I_{n-1} & \partial_x (v_p^1(t)) e_1 \\
  0 & 0 & i\xi_1 v_p^1(t) \\
\end{pmatrix}.
\]

Let us note that \( \hat{L}_\xi'(t) \) is sectorial uniformly with respect to \( t \in \mathbb{R} \) for each \( \xi' \in \mathbb{R}^{n-1} \). As for the evolution operator \( \hat{U}_\xi'(t, s) \) for (4.3) we have the following results.

**Lemma 4.1.** For each \( \xi' \in \mathbb{R}^{n-1} \) and for all \( t \geq s \) there exists unique evolution operator \( \hat{U}_\xi'(t, s) \) for (4.3) that satisfies

\[
|\hat{L}_\xi'(t)\hat{U}_\xi'(t, s)|_{L(X_0)} \leq C_{t_1 t_2}, \quad t_1 \leq s < t \leq t_2.
\]

Furthermore, for \( u_0 \in X_0, \ f \in C^\alpha([s, \infty); X_0), \alpha \in (0, 1] \) there exists unique classical solution \( u \) of inhomogeneous problem

\[
\frac{d}{dt} u + \hat{L}_\xi'(t) u = f, \quad t > s, \quad u|_{t=s} = u_0,
\] (4.4)
satisfying \( u \in C_{\text{loc}}([s, \infty); X_0) \cap C^1(s, \infty; X_0) \cap C(s, \infty; H^1(0, 1) \times H^2_\xi(0, 1)) \); and the solution \( u \) is given by

\[
u(t) = (\phi(t), w(t)) = \hat{U}_\xi'(t, s) u_0 + \int_s^t \hat{U}_\xi'(t, z) f(z) dz.
\]

Next, let us introduce adjoint problem to

\[
\partial_t u + \hat{L}_\xi^*(s) u = 0, \quad t > s, \quad u|_{t=s} = u_0.
\]

**Lemma 4.2.** For each \( \xi' \in \mathbb{R}^{n-1} \) and for all \( s \leq t \) there exists unique evolution operator \( \hat{U}_\xi^*(s, t) \) for adjoint problem

\[
-\partial_s u + \hat{L}_\xi^*(s) u = 0, \quad s < t, \quad u|_{s=t} = u_0,
\]
on \( X_0 \). Here, \( \hat{L}_\xi^*(s) \) is an operator on \( X_0 \) with domain \( \text{D}(\hat{L}_\xi^*(s)) = H^1(0, 1) \times H^2_\xi(0, 1) \), which takes the form

\[
\hat{L}_\xi^*(s) = \begin{pmatrix}
  -i\xi_1 v_p^1(s) & -i\gamma_1 p^T \xi' & \gamma^2 \partial_x (p_p \cdot) \\
  -i\xi_1^2 \frac{p'(p_p)}{\gamma^2 p_p} & \frac{\nu}{p_p} (\xi'|\xi| - \partial_x^2_{n-1}) I_{n-1} + \frac{\nu}{p_p} \xi^T \xi' & -i \frac{\nu}{p_p} \xi' \partial_x n \\
  \partial_x (\frac{p'(p_p)}{\gamma^2 p_p}) & -i \frac{\nu}{p_p} \xi' \partial_x n & \frac{\nu}{p_p} (\xi'|\xi| - \partial_x^2_{n-1}) - \frac{\nu}{p_p} \partial_x^2 n \\
\end{pmatrix} \\
+ \begin{pmatrix}
  0 & 0 & 0 \\
  0 & -i\xi_1 v_p^1(s) I_{n-1} & \partial_x (v_p^1(s)) e_1 \\
  0 & 0 & -i\xi_1 v_p^1(s) \\
\end{pmatrix}.
\]

Moreover, \( \hat{L}_\xi^*(s) \) satisfies \( \langle \hat{L}_\xi^*(s) u, v \rangle = \langle u, \hat{L}_\xi^*(s) v \rangle \) for \( s \in \mathbb{R} \) and \( u, v \in H^1 \times H^2_\xi \) and
inhomogeneous problem

Here, for Banach space $X$

Definition 4.4

Lemma 4.3

There exist positive numbers $\nu$ and $\gamma_1$ such that if $\nu \geq \nu_1$ and $\gamma^2/(\nu + \gamma) \geq \gamma_1^2$ then there exists $r_0 > 0$ such that for each $\xi'$ with $|\xi'| \leq r_0$ there hold the following statements.

(i) The spectrum of operator $\hat{U}_{\xi'}(T, 0)$ on $H^1(0, 1) \times H^1_0(0, 1)$ satisfies

$$\sigma(\hat{U}_{\xi'}(T, 0)) \subset \{ \mu_{\xi'} \} \cup \{ \mu : |\mu| \leq q_0 \},$$

with constant $q_0 < \Re \mu_{\xi'} < 1$. Here, $\mu_{\xi'} = e^{\lambda_{\xi'} T}$ is simple eigenvalue of $\hat{U}_{\xi'}(T, 0)$ and $\lambda_{\xi'}$ has an expansion

$$\lambda_{\xi'} = -i\kappa_0 \xi_1 - \kappa_1 \xi_1^2 - \kappa''|\xi'|^2 + O(|\xi'|^3),$$

where $\kappa_0 \in \mathbb{R}$ and $\kappa_1 > 0$, $\kappa'' > 0$.

Moreover, let $\hat{\Pi}_{\xi'}$ denote the eigenprojection associated with $\mu_{\xi'}$. There holds

$$[\hat{U}_{\xi'}(t, s)(I - \hat{\Pi}_{\xi'}) u]|_{H^1} \leq C e^{-d(t-s)} |(I - \hat{\Pi}_{\xi'}) u|_{X_0},$$

for $u \in X_0$ and $T \leq t - s$. Here, $d$ is a positive constant depending on $r_0$.

(ii) The spectrum of operator $\hat{U}_{\xi'}(0, T)$ on $H^1(0, 1) \times H^1_0(0, 1)$ satisfies

$$\sigma(\hat{U}_{\xi'}(0, T)) \subset \{ \overline{\mu}_{\xi'} \} \cup \{ \mu : |\mu| \leq q_0 \}.$$

Here, $\overline{\mu}_{\xi'}$ is simple eigenvalue of $\hat{U}_{\xi'}(0, T)$.

Furthermore, let $\hat{\Pi}_{\xi'}$ denote the eigenprojection associated with $\overline{\mu}_{\xi'}$. There holds

$$\langle \hat{\Pi}_{\xi'} u, v \rangle = \langle u, \hat{\Pi}_{\xi'} v \rangle,$$

for $u, v \in X_0$.

Next, we introduce Floquet theory.

Definition 4.4 Let $k = 1, 2, \ldots,$. Let us define spaces $Y^k_{\text{per}}$ as

$$Y^1_{\text{per}} = L^2_{\text{per}}([0, T]; X_0),$$

$$Y^{k}_{\text{per}} = \bigcap_{j=0}^{k} \bigcap_{j=0}^{k} H^{j}_{\text{per}}([0, T]; H^{k-j}(0, 1) \times H^{k-j}(0, 1)), \text{ for } k \geq 2.$$

Here, for Banach space $X$ and $j = 0, \ldots$ spaces $L^2_{\text{per}}([0, T]; X)$ and $H^j_{\text{per}}([0, T]; X)$ consist of functions from $L^2([0, T]; X)$ and $H^j([0, T]; X)$, respectively, that are restrictions of $T$-periodic functions.
**Definition 4.5** We define operator $B_{\xi'}$ on space $Y_{per}^1$ with domain

$$D(B_{\xi'}) = H_{per}^1([0,T]; X_0) \cap L_{per}^2([0,T]; H^1(0,1) \times H^2(0,1)),$$

in the following way

$$B_{\xi'} v = \partial_t v + \tilde{L}_{\xi'}(\cdot)v,$$

for $v \in D(B_{\xi'}).$ Moreover, we define formal adjoint operator $B_{\xi'}^*$ with respect to inner product $\frac{1}{T} \int_0^T \langle \cdot, \cdot \rangle dt$ as

$$B_{\xi'}^* v = -\partial_t v + \tilde{L}_{\xi'}^*(\cdot)v,$$

for $v \in D(B_{\xi'}^*) = D(B_{\xi'}).$

**Remark 4.6** Operators $B_{\xi'}$ and $B_{\xi'}^*$ are closed, densely defined on $Y_{per}^1$ for each fixed $\xi' \in \mathbb{R}^{n-1}.$

**Definition 4.7** Let $k \geq 1.$ We say that $u = T(\phi,w)$ is $k$-regular function on time interval $[a,b]$ whenever

$$u \in \bigcup_{j=0}^k C^j([a,b]_t; (H^{k-2j} \times H^{k-2j}_*)(\Omega)),$$

$$\phi \in \bigcup_{j=0}^{k+1} H^{j+1}(a,b; H^{k-2j}(\Omega)), w \in \bigcup_{j=0}^{k+1} H^{j}(a,b; H^{k+1-2j}(\Omega)).$$

**Lemma 4.8** There exist positive numbers $\nu_2 \geq \nu_1$ and $\gamma_2 \geq \gamma_1$ such that if $\nu \geq \nu_2$ and $\gamma^2/(\nu + \bar{v}) \geq \gamma_2^2$ then there exists $0 < r_1 \leq 1$ such that for each $|\xi'| \leq r_1$ there hold the following statements.

(i) Let $1 \leq k \leq m + 1.$ There exists $q_1 > 0$ such that spectrum of operator $B_{\xi'}$ on $Y_{per}^k$ satisfies

$$\sigma(B_{\xi'}) \subset \{-\lambda_{\xi'}\} \cup \{\lambda : \Re \lambda \geq q_1\},$$

with $0 \leq -\Re \lambda_{\xi'} \leq \frac{1}{2} q_1$ uniform for all $k.$ Here, $-\lambda_{\xi'}$ is simple eigenvalue of $B_{\xi'}.$

(ii) Let $1 \leq k \leq m + 1.$ Spectrum of operator $B_{\xi'}^*$ on $Y_{per}^k$ satisfies

$$\sigma(B_{\xi'}^*) \subset \{-\overline{\lambda_{\xi'}}\} \cup \{\lambda : \Re \lambda \geq q_1\}.$$

Here, $-\overline{\lambda_{\xi'}}$ is simple eigenvalue of $B_{\xi'}^*.$

(iii) There exist $u_{\xi'}$ and $u_{\xi'}^*,$ eigenfunctions associated with $-\lambda_{\xi'}$ and $-\overline{\lambda_{\xi'}},$ respectively, with the following properties:

$$\langle u_{\xi'}(t), u_{\xi'}^*(t) \rangle = 1,$$

$$u_{\xi'}(t) = u^{(0)}(t) + i\xi' \cdot u^{(1)}(t) + |\xi'|^2 u^{(2)}(\xi', t),$$

$$u_{\xi'}^*(t) = u^{*(0)}(t) + i\xi' \cdot u^{*(1)}(t) + |\xi'|^2 u^{*(2)}(\xi', t),$$

for $t \in \mathbb{R}.$ Here, all functions

$$u_{\xi'}, u_{\xi'}^*, u^{(0)}, u^{(0)*}, u^{(1)}, u^{(1)*}, u^{(2)}(\xi'), u^{(2)*}(\xi'),$$

are $T$-periodic in $t,$ $m + 1$-regular on $[0,T]$ and we have estimate

$$\sup_{z \in J_T} \sum_{j=0}^m |\partial_z^j u(z)|_{H^{m+1-2j}}^2 + \int_T \sum_{j=0}^m |\partial_z^{j+1} u|_{H^{m+1-2j} \times H^{m-2j}}^2 + |\partial_z^{m+1} Q_0 u|_2^2 + |u|_{H^{m+1} \times H^{m+2}}^2 dz \leq C,$$

for $u \in \{u_{\xi'}, u_{\xi'}^*, u^{(2)}(\xi'), u^{(2)*}(\xi')\}$ and a constant $C > 0$ depending on $r_1.$
Let us introduce more properties of \(u^{(0)}\).

**Lemma 4.9** Function \(u^{(0)}(t)\) satisfies \(\partial_t u^{(0)} + \hat{L}_0(t)u^{(0)} = 0\) and \(u^{(0)}(t) = u^{(0)}(t+T)\) for all \(t \in \mathbb{R}\). Function \(u^{(0)}(t)\) is given as

\[
u^{(0)}(x_n, t) = T(\phi^{(0)}(x_n), w^{(0),1}(x_n, t), 0).
\]

Here,

\[
\phi^{(0)}(x_n) = a_0 \frac{\gamma^2 \rho_p(x_n)}{P'(\rho_p(x_n))}, \quad a_0 = \left[ \frac{\gamma^2 \rho_p}{P'P}\right]^{-1},
\]

\[
w^{(0),1}(x_n, t) = -\frac{1}{\gamma} \int_{-\infty}^{t} e^{-(t-s)\nu} \alpha \frac{a_0 \gamma^2}{P'(\rho_p)} (\partial_{x_n} u_1^1(s)) ds,
\]

where \(A\) denotes the uniformly elliptic operator on \(L^2(0, 1)\) with domain \(D(A) = (H^2 \cap H^1_0)(0, 1)\) and

\[
\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O\left(\frac{1}{\gamma^2}\right).
\]

In the rest of this section we assume that \(\nu \geq \nu_2\) and \(\gamma^2/(\nu + \nu) \geq \gamma^2_2\).

**Definition 4.10** We define \(\tilde{\chi}_1\) by

\[
\tilde{\chi}_1(\xi') = \begin{cases} 
1, & |\xi'| < r_1, \\
0, & |\xi'| \geq r_1,
\end{cases}
\]

for \(\xi' \in \mathbb{R}^{n-1}\).

Now, we introduce time-periodic operators based on \(u_{\xi'}\) and \(u_{\xi'}^*\).

**Definition 4.11** We define operators \(\mathcal{P}(t) : L^2(\Omega) \rightarrow L^2(\mathbb{R}^{n-1})\) by

\[
\mathcal{P}(t)u = \mathcal{F}^{-1}\{\mathcal{F}_\xi(t)\tilde{u}\},
\]

\[
\mathcal{F}_\xi(t)\tilde{u} = \tilde{\chi}_1(\tilde{u}, u_{\xi'}(t)),
\]

for \(u \in L^2\) and \(t \in [0, \infty)\).

We define operators \(\mathcal{Q}(t) : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\Omega)\) by

\[
\mathcal{Q}(t)\sigma = \mathcal{F}^{-1}\{\mathcal{F}_\xi(t)\tilde{\sigma}\},
\]

\[
\mathcal{F}_\xi(t)\tilde{\sigma} = \tilde{\chi}_1(u_\xi(\cdot, t)\tilde{\sigma}),
\]

for \(t \in [0, \infty)\) and multiplier \(\Lambda : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})\) by

\[
\Lambda\sigma = \mathcal{F}^{-1}\{\tilde{\chi}_1(\Lambda_{\xi'}\tilde{\sigma})\},
\]

for \(\sigma \in L^2(\mathbb{R}^{n-1})\).

Moreover, we define projections \(\mathbb{P}(t)\) and \(\mathbb{P}^*(t)\) on \(L^2(\Omega)\) as

\[
\mathbb{P}(t)u = \mathcal{F}^{-1}\{\tilde{\chi}_1(\tilde{u}, u_{\xi'}(t))\tilde{u}_{\xi'}(\cdot, t)\} = \mathcal{Q}(t)\mathcal{P}(t)u,
\]

\[
\mathbb{P}^*(t)u = \mathcal{F}^{-1}\{\tilde{\chi}_1(\tilde{u}, u_{\xi'}(t))\tilde{u}_{\xi'}(\cdot, t)\},
\]

for \(t \in [0, \infty)\) and \(u \in L^2\).

We define projection \(\Pi^{(0)}(t)\) on \(L^2(\Omega)\) as

\[
\Pi^{(0)}(t)u = [Q_0 u]u^{(0)}(t),
\]

for \(t \in [0, \infty)\) and \(u \in L^2\).
In terms of $P(t)$ we have the following decomposition of $U(t, s)$.

**Lemma 4.12** $\mathbb{P}(t)$ and $\mathbb{P}^+(t)$ satisfies the following:

(i) 
$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))\mathbb{P}(t)u(t) = \mathcal{D}(t)[(\partial_t - \Lambda)\mathcal{P}(t)u(t)],$$
for $u \in L^2([0, T]; H^1 \times H^2) \cap H^1([0, T]; L^2)$.

(ii) 
$$\mathbb{P}(t)U(t, s) = U(t, s)\mathbb{P}(s) = \mathcal{D}(t)e^{(t-s)\Lambda}\mathcal{P}(s).$$
If $u \in L^1$, then
$$\|\partial_t^j\partial_x^l\partial_{z_n}^m\mathbb{P}(t)U(t, s)u\|_2 \leq C(1 + t - s)^{-\frac{n+1}{2} - \frac{1}{2}}\|u\|_1,$$
for $0 \leq 2j + l \leq m + 1$, $k = 0, \ldots$.

(iii) For $u, v \in L^2$ there holds
$$\langle \mathbb{P}(t)u, v \rangle = \langle u, \mathbb{P}^+(t)v \rangle.$$
If $u \in L^2$, then
$$\|\partial_t^j\partial_x^l\partial_{z_n}^m(\mathbb{P}^+(t)u)\|_2 \leq C\|u\|_2,$$
for $0 \leq 2j + l \leq m + 1$, $k = 0, 1, \ldots$.

(iv) $(I - \mathbb{P}(t))U(t, s) = U(t, s)(I - \mathbb{P}(s))$ satisfies
$$\|\partial_t\mathbb{P}(t)U(t, s)u\|_{H^1} \leq Ce^{-d(t-s)}(\|u\|_{H^1 \times L^2} + \|\partial_x^j u\|_2),$$
for $t - s \geq T$. Here $d$ is a positive constant.

Next, let us show the asymptotic properties of $U(t, s)$. First, let us define a semigroup $\mathcal{H}(t)$ on $L^2(\mathbb{R}^{n-1})$ associated with a linear heat equation with a convective term:

$$\partial_t\sigma - \kappa_1\partial_x^2\sigma - \kappa''\Delta''\sigma + \kappa_0\partial_x\sigma = 0.$$

**Definition 4.13** We define operator $\mathcal{H}(t)$ as

$$\mathcal{H}(t)\sigma = \mathcal{P}^{-1}[e^{-(\kappa_0\xi_1 + \kappa_1\xi_2 + \kappa''\xi''\xi')t}\sigma],$$
for $\sigma \in L^2(\mathbb{R}^{n-1})$. Here, $\kappa_0, \kappa_1$ and $\kappa''$ are given by (4.6).

**Lemma 4.14** There hold the following estimates for $1 \leq p \leq 2$ and $k = 0, 1, \ldots$.

(i) 
$$\|\partial_t^j(\mathcal{H}(t)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \leq C(C)^{\frac{n+1}{2}}(1 + t - s)^{-\frac{n+1}{2} - \frac{1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},$$
for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(ii) $\Lambda$ generates uniformly continuous group $\{e^{t\Lambda}\}_{t \in \mathbb{R}}$ and
$$\|\partial_x^k e^{t\Lambda}\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1 + t)^{-\frac{n+1}{2}}(1 + t - s)^{-\frac{n+1}{2} - \frac{1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},$$
for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(iii) It holds the relation,
$$\mathcal{P}(t)U(t, s) = e^{(t-s)\Lambda}\mathcal{P}(s).$$
Set $\sigma = [Q_0u]$. Then
$$\|\partial_x^k e^{(t-s)\Lambda}\mathcal{P}(s)u - \mathcal{H}(t-s)\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(t-s)^{-\frac{n+1}{2}}(1 + t - s)^{-\frac{k+1}{2}}\|u\|_p,$$
for $u \in L^p$. Furthermore, for any $\sigma \in L^p(\mathbb{R}^{n-1})$ there holds
$$\|e^{(t-s)\Lambda} - \mathcal{H}(t-s)\partial_x^k\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(t-s)^{-\frac{n+1}{2}}(1 + t - s)^{-\frac{k+1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}.$$
Next, we introduce the properties of $\mathcal{Q}(t)$ and $\mathcal{P}(t)$.

**Proposition 4.15** $\mathcal{Q}(t)$ has the following properties:

(i) \[ \mathcal{Q}(t + T) = \mathcal{Q}(t), \quad \partial_t \mathcal{Q}(t) = \mathcal{Q}(t) \partial_x^k. \]

(ii) \[ \mathcal{Q}(t) \sigma \in \bigcap_{j=0}^{[m+1]} C^j_{\text{per}}(J_T; H^{m+1-2j} \times H^m_{\text{per}}), \]

\[ \mathcal{Q}(t) \sigma \in \bigcap_{j=0}^{[m+2]} H^j_{\text{per}}(J_T; H^m_{\text{per}}), \]

and

\[ \| \partial_t^j \partial_x^k (\mathcal{Q}(t) \sigma) \|_2 \leq C \| \sigma \|_{L^1(\mathbb{R}^{n-1})}, \quad 0 \leq 2j + k \leq m + 1, \quad k = 0, 1, \ldots, \]

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

(iii) \[ (\partial_t + L(t))(\mathcal{Q}(t) \sigma(t)) = \mathcal{Q}(t)(\partial_t - \Lambda) \sigma(t), \]

for $\sigma \in H^1_{\text{loc}}([0, \infty); L^2(\mathbb{R}^{n-1}))$.

(iv) $\mathcal{Q}(t)$ is decomposed as

\[ \mathcal{Q}(t) = \mathcal{Q}^{(0)}(t) + \text{div}' \mathcal{Q}^{(1)}(t) + \Delta' \mathcal{Q}^{(2)}(t). \]

Here, $\mathcal{Q}^{(0)}(t) \sigma = (\mathcal{F}^{-1} \{\hat{\chi}_1 \hat{\sigma}\}) u^{(0)}(\cdot, t), \mathcal{Q}^{(1)}(t)$ and $\mathcal{Q}^{(2)}(t)$ share the same properties given in (i) and (ii) for $\mathcal{Q}(t)$.

**Proposition 4.16** $\mathcal{P}(t)$ has the following properties:

(i) \[ \mathcal{P}(t + T) = \mathcal{P}(t), \quad \partial_t \mathcal{P}(t) = \mathcal{P}(t) \partial_x^k, \quad \partial_x^k \mathcal{P}(t) = 0. \]

(ii) \[ \mathcal{P}(t) u \in \bigcap_{j=0}^{[m+1]} C^j_{\text{per}}(J_T; H^k(\mathbb{R}^{n-1})), \]

for all $k = 0, 1, \ldots, \]

and

\[ \| \partial_t^j \partial_x^k (\mathcal{P}(t) u) \|_{L^2(\mathbb{R}^{n-1})} \leq C \| u \|_2, \quad 0 \leq 2j \leq m + 1, \quad k = 0, 1, \ldots, \]

for $u \in L^2$.

Moreover,

\[ \| \mathcal{P}(t) u \|_{L^2(\mathbb{R}^{n-1})} \leq C \| u \|_p, \]

for $u \in L^p$ and $1 \leq p \leq 2$.

(iii) \[ \mathcal{P}(t)(\partial_t + L(t)) u(t) = (\partial_t - \Lambda)(\mathcal{P}(t) u(t)), \]

for $u \in L^2_{\text{loc}}([0, \infty); H^1 \times H^2) \cap H^1_{\text{loc}}([0, \infty); L^2)$.

(iv) $\mathcal{P}(t)$ is decomposed as

\[ \mathcal{P}(t) = \mathcal{P}^{(0)} + \text{div}' \mathcal{P}^{(1)}(t) + \Delta' \mathcal{P}^{(2)}(t). \]

Here,

\[ \mathcal{P}^{(0)} u = \mathcal{F}^{-1} \{\hat{\chi}_1 \langle \hat{u}, u^{(0)} \rangle\} = \mathcal{F}^{-1} \{\hat{\chi}_1 \hat{Q}_0 \hat{u}\}, \]

\[ \mathcal{P}^{(1)} u = \mathcal{F}^{-1} \{\hat{\chi}_1 \hat{u}, u^{(0)} \rangle\} = \mathcal{F}^{-1} \{\hat{\chi}_1 \hat{Q}_0 \hat{u}\}, \]

\[ \mathcal{P}^{(2)} u = \mathcal{F}^{-1} \{\hat{\chi}_1 \hat{u}, u^{(0)} \rangle\} = \mathcal{F}^{-1} \{\hat{\chi}_1 \hat{Q}_0 \hat{u}\}, \]
We then define $f = F^{-1}\{\hat{\chi}_1(\hat{u}, u^{(1)}(t))\}$,

$P^{(2)}(t)u = F^{-1}\{-\hat{\chi}_1(\hat{u}, u^{(2)}(\xi', t))\}$.

$P^{(p)}(t)$, $p = 0, 1, 2$, share the same properties given in (i) and (ii) for $P(t)$.

(v) There holds

$$\|\partial^k_t e^{(t-s)^2} P(s)u\|_{L^2(\mathbb{R}^n-1)} \leq C(1 + t - s)^{-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{2})-\frac{k}{2}}\|u\|_p,$$

(4.13)

$$\|\partial^k_t e^{(t-s)^2} P(q)(s)u\|_{L^2(\mathbb{R}^n-1)} \leq C(1 + t - s)^{-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{2})-\frac{k}{2}}\|u\|_p, q = 0, 1, 2,$$

(4.14)

for $u \in L^p$, $1 \leq p \leq 2$ and $k = 0, 1, \ldots$.

Remark 4.17 Note that $H^{(0)}(t)$ and $P^{(0)}(t)P^{(0)}$ are not identical operators.

To close this section, we show improvements of regularity for $\phi$.

Proposition 4.18 Let $u = T(\phi, w) \in Z^m(\tau)$ for $m \geq [n/2] + 1$ be a solution of (4.1). There holds

$$\phi \in \bigcap_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} C^j([0, \tau]; H^{m+1-2j}).$$

(4.15)

Proof. From definition of $Z^m(\tau)$ we have

$$u \in \bigcap_{j=0}^{\lfloor \frac{n}{2} \rfloor} C^j([0, \tau]; H^{m-2j}) \text{ and } \sup_{0 \leq z \leq \tau} \|u(z)\|_m < \infty.$$

We write (1.5) as

$$\partial_t \phi = -v_{p}^2 \partial_{x_j} \phi - \gamma^2 \text{div}(\rho_p w) - \text{div}(\phi w).$$

Taking $[\cdot]_{m-1}$-norm we obtain

$$[\partial_t \phi]_{m-1} \leq [v_{p}^2 \partial_{x_j} \phi]_{m-1} + \gamma^2 [\rho_p w]_m + [\phi w]_m.$$

Since $m \geq [n/2] + 1$ we get using Lemma 8.3 (ii) that

$$[\partial_t \phi]_{m-1} \leq C([\phi]_m)\{[v_{p}^2]_m[\phi]_m + [\rho_p w]_m + [\phi]_m[w]_m\}.$$

This concludes the proof. \[ \square \]

5 Decomposition of the solution

In this section we decompose solution $u(t)$ of (4.1) and we prove the a priori estimate in Proposition 3.7. We decompose $u(t)$ into several parts based on the spectral properties of $L(t)$. In this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \nu) \geq \gamma^2_2$ unless further restricted.

Let us first introduce some notation and projection operators. Let $\widehat{\chi}_2$ and $\widehat{\chi}_3$ be defined by

$$\widehat{\chi}_2(\xi') = 1_{[r_1, 1]}(|\xi'|) \text{ and } \widehat{\chi}_3(\xi') = 1_{[1, \infty]}(|\xi'|).$$

We then define $[f]_j$, $j = 1, 2, \infty$ by

$$[f]_j = F^{-1}(\widehat{\chi}_j[\widehat{f}]), \quad j = 1, 2,$$

$$[f]_\infty = [f]_1 + [f]_2 = F^{-1}(\widehat{\chi}_2[\widehat{f}]).$$

Next, we define $P_{\infty, j}$, $j = 1, 2, 3$ as
Moreover, Proposition 5.2 with Remark 5.1

By setting

\[ \tilde{P}_\infty(t) = I - P(t), \quad P^{(0)}(t) = P_{\infty,1}(t) + P_{\infty,2}, \]

we get

\[ I = P(t) + \tilde{P}_\infty(t), \quad P_{\infty}(t) = P^{(0)}(t) + P_{\infty,3}. \]

Using above operators we decompose solution \( u(t) \) into

\[ u(t) = P(t)u(t) + \tilde{P}_\infty(t)u(t), \]

with

\[ P(t)u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t), \]

\[ \tilde{P}_\infty(t)u(t) = \sigma_{\infty}(t)u^{(0)}(t) + u_{\infty}(t), \]

where

\[ \sigma_1(t) = \mathcal{P}(t)u(t), \quad u_1(t) = (\mathcal{Q}(t) - \mathcal{Q}^{(0)}(t))\mathcal{P}(t)u(t), \]

\[ \sigma_{\infty}(t) = [Q_0P_{\infty}^{(0)}(t)u(t)] = [Q_0P_{\infty}^{(0)}(t)u(t)]_\infty, \quad u_{\infty}(t) = P_{\infty}(t)u(t). \]

By \( P_{\infty} \) we denote the operator defined as

\[ P_{\infty}(t) = (I - H^{(0)}(t))P^{(0)}(t) + P_{\infty,3}. \]

Remark 5.1 Notice that \( \sigma_1(t) \), \( \sigma_{\infty}(t) \) and \( u^{(0)}(t) \) are separate functions. Furthermore, notice that

\[ \|u_1(t)\|_2 \leq C\|\partial_x \sigma_1(t)\|_2. \]

Next, we derive the equations for \( \sigma_1, \sigma_{\infty} \) and \( u_{\infty} \). We define \( \mathcal{M}(t) \) by

\[ \mathcal{M}(t) = \tilde{A} + \tilde{B}(t), \]

with

\[ \tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_p} \Delta' I_{n-1} - \frac{\nu}{\rho_p} \nabla^p \text{div}' & 0 \\ 0 & \left( -\frac{\nu}{\rho_p} \partial_x^{(p)} \text{div}', -\frac{\nu}{\rho_p} \Delta' \right) \end{pmatrix}, \quad \tilde{B}(t) = \begin{pmatrix} v_1(t)\partial_{x_1}^{(p)} & \gamma^2 \rho_p \text{div}' & 0 \\ 0 & v_1(t)\partial_{x_1}^{(p)} I_{n-1} & 0 \\ 0 & 0 & v_1(t)\partial_{x_1}^{(p)} \end{pmatrix}. \]

Proposition 5.2 Let \( \tau > 0 \) and \( u(t) \) be a solution of (4.1) in \( Z^m(\tau) \). Then there hold

\[ \sigma_k \in \bigcap_{j=0}^{n/2} C^j([0,\tau];H^l(\mathbb{R}^{n-1})), \quad \int_0^\tau \|D\sigma_k(z)\|_m dz < \infty, \quad k = 1, \infty, l = 0, 1, \ldots, \]

\[ u_1 \in \bigcup_{j=0}^{n/2} C^j([0,\tau];H^{m+1-2j}), \quad \int_0^\tau \|Dw_1(z)\|_m dz < \infty, \]

\[ u_{\infty} \in Z^m(\tau), \quad \int_0^\tau \|\theta_t \phi_{\infty}(z)\|_{m-1} dz < \infty. \]

Moreover, \( \sigma_1, \sigma_{\infty} \) and \( u_{\infty} \) satisfy

\[ \sigma_1(t) = e^{(t-s)\Lambda} \mathcal{P}(s)u_0 + \int_s^t e^{(t-z)\Lambda} \mathcal{P}(z)F(z)dz, \quad (5.1) \]
\[ \partial_t \sigma_\infty + \left[ Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \right]_\infty = \left[ Q_0 P_\infty^{(0)} F \right]_\infty, \quad (5.2) \]

\[ \partial_t u_\infty + L(t) u_\infty + \mathcal{M}(t)(\sigma_\infty u^{(0)}) - \left[ Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty) \right]_\infty u^{(0)} = P_\infty F. \quad (5.3) \]

\[ \sigma_\infty |_{t=0} = \sigma_{\infty,0}, \quad u_\infty |_{t=0} = u_{\infty,0}. \]

**Proof.** Since \( u \in Z^m(\tau) \), the regularity assertions on \( \sigma_k \) \((k = 1, \infty)\) and \( u_1 \) follow from properties of \( \mathcal{P}(t) \), \( \mathcal{Q}(t) \) and (4.15). As for \( u_\infty \), we already know that \( PZ^m(\tau) \subset Z^m(\tau) \) and therefore \( P_\infty Z^m(\tau) \subset Z^m(\tau) \). It is straightforward to see that \( P_\infty Z^m(\tau) \subset Z^m(\tau) \) and finally, from properties of \( u^{(0)}(t) \) we obtain \( \Pi^{(0)} P_\infty^{(0)} Z^m(\tau) \subset Z^m(\tau) \). Therefore, \( u_\infty \in Z^m(\tau) \). \( \int_0^\tau [\partial_t \phi(z)]_{m-1} dz < \infty \) follows in analogous way using (4.15).

As for (5.1), it follows from (4.1) and (4.12) that

\[ (\partial_t - \Lambda) \sigma_1(t) = \mathcal{P}(t) F(t). \]

The rest is standard.

As for (5.2) and (5.3), we first apply \( \tilde{P}_\infty(t) \) to (4.1) to get

\[ \partial_t (\tilde{P}_\infty u) + L(t) \tilde{P}_\infty u = \tilde{P}_\infty F. \quad (5.4) \]

Next, we apply \( P_\infty^{(0)}(t) \) and \( P_{\infty,3} \) to (5.4) to obtain

\[ \partial_t (P_\infty^{(0)} u) + L(t) P_\infty^{(0)} u = P_\infty F, \quad (5.5) \]

\[ \partial_t (P_{\infty,3} u) + L(t) P_{\infty,3} u = P_{\infty,3} F. \quad (5.6) \]

Since \( [Q_0 L(t) v] = [Q_0 \mathcal{M}(t) v] \) for \( \tilde{Q} v |_{x=0,1} = 0 \) and \( [Q_0 \mathcal{M}(t) v] = [Q_0 \tilde{B}(t) v] \) for any \( v \), we get by applying \( [Q_0] \) to (5.5)

\[ \partial_t \sigma_\infty + [Q_0 \tilde{B}(t) P_\infty^{(0)} u] = [Q_0 P_\infty^{(0)} F]. \quad (5.7) \]

There holds

\[ [Q_0 \tilde{B}(t) P_\infty^{(0)} u] = [Q_0 \tilde{B}(\sigma_\infty u^{(0)}(t) + (I - \Pi^{(0)}(t))) P_\infty^{(0)} u] \]

\[ = [Q_0 \tilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty, \]

and thus (5.2) follows from (5.7).

To obtain (5.3) we use the fact that \( \partial_t \Pi^{(0)}(t) + L(t) \Pi^{(0)}(t) = \mathcal{M}(t) \Pi^{(0)}(t) \). Applying \( I - \Pi^{(0)}(t) \) to (5.5) gives us

\[ \partial_t ((I - \Pi^{(0)}(t)) P_\infty^{(0)} u) + L(t)(I - \Pi^{(0)}(t)) P_\infty^{(0)} u + \mathcal{M}(t)(\sigma_\infty u^{(0)}(t)) - \Pi^{(0)}(t) \tilde{B}(t) (P_\infty^{(0)} u) = (I - \Pi^{(0)}(t)) P_\infty^{(0)} F. \quad (5.8) \]

(5.3) now follows by adding (5.6) and (5.8). This completes the proof.

\[ \square \]

Let us state some properties of \( \sigma_1, \sigma_\infty \) and \( u_\infty \) parts.

**Lemma 5.3** There hold the following inequalities.

(i)

\[ \| \partial_x^k [Q_0 P_\infty^{(0)} u]_\infty \|_2 \leq \| [Q_0 P_\infty^{(0)} u]_\infty \|_2, \quad k = 0, 1, \ldots, \]

(ii)

\[ \| P_\infty u \|_2 \leq C \| \partial_x P_\infty u \|_2 \quad \text{if} \quad \tilde{Q} u |_{x=0,1} = 0. \]
(iii) Let $\tau > 0$ and $u(t)$ be a solution of (4.1) in $Z^m(\tau)$. Then there hold
\[
\|D_k^p \sigma_1\|_2 \leq C\|D_k^p \sigma_1\|_2, \quad \|D_k^p \sigma_\infty\|_2 \leq C\|D_k^p \sigma_\infty\|_2, \quad k = 1, 2, \ldots,
\]
\[
\|\phi_\infty\|_2 \leq C\|\partial_x \phi_\infty\|_2,
\]
\[
\|w_\infty\|_2 \leq C\|\partial_x w_\infty\|_2.
\]
\[
\|\Lambda \sigma_1\|_2 \leq C\|\partial_x \sigma_1\|_2.
\]

**Proof.** Inequality (i) is obvious since $\text{supp}(\hat{\chi}_1 + \hat{\chi}_2) \subset \{|\xi'| \leq 1\}$. As for (ii), since $\text{supp} \hat{\chi}_3 \subset \{|\xi'| \geq 1\}$, we see that
\[
\|P_{\infty, 3} u\|_2 \leq \|\partial_x P_{\infty, 3} u\|_2.
\]

Since $\hat{Q} u|_{x_n = 0, 1} = 0$, we have $\hat{Q} P_{\infty}^{(0)} u|_{x_n = 0, 1} = 0$, and hence, $\hat{Q}(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u|_{x_n = 0, 1} = 0$. By the Poincaré inequality we obtain
\[
\|\hat{Q}(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u\|_2 \leq \|\partial_x, \hat{Q}(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u\|_2.
\]

Furthermore, since $[Q_0(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u] = 0$, we see from the Poincaré inequality that
\[
\|Q_0(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u\|_2 \leq \|\partial_x, Q_0(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u\|_2.
\]

It then follows that
\[
\|P_{\infty} u\|_2 \leq C\|\partial_x(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u\|_2 + \|\partial_x P_{\infty, 3} u\|_2 \leq C\|\partial_x P_{\infty} u\|_2.
\]

Here, we used $(\partial_x(I - \Pi^{(0)}(t)) P_{\infty}^{(0)} u, \partial_x P_{\infty, 3} u) = 0$, which follows from the fact $\hat{\chi}_1 \hat{\chi}_3 = \hat{\chi}_2 \hat{\chi}_3 = 0$ and the Plancherel theorem.

As for (iii), it follows from the proof of (i) and (ii).

We prove the a priori estimate in Proposition 3.7 by estimating the following quantities. Let $u(t)$ be solution of (4.1) in $Z^m(\tau)$ and let $u(t)$ be decomposed as above, i.e.,
\[
u(t) = \sigma_1(t) u^{(0)}(t) + u_1(t) + \sigma_\infty(t) u^{(0)}(t) + u_\infty(t).
\]

We define $M(t) \geq 0$ by
\[
M(t)^2 = M_1(t)^2 + \sup_{0 \leq z \leq t} (1 + z)^{\frac{m+1}{2}} E_\infty(z), \quad t \in [0, \tau].
\]

Here, $M_1(t)$ and $E_\infty(t)$ are defined as
\[
M_1(t) = \sup_{0 \leq z \leq t} (1 + z)^{\frac{m+1}{2}} \|\sigma_1(z)\|_2 + \sup_{0 \leq z \leq t} (1 + z)^{\frac{m+1}{2}} \{\|D_k^p \sigma_1(z)\|_2 + \sum_{j=1}^{[\frac{m}{2}]} \|D_j^p \sigma_1(z)\|_2\},
\]
and
\[
E_\infty(t) = \|u_\infty(t)\|_m^2 + \|\sigma_\infty(t)\|_m^2.
\]

Finally, we introduce quantity $D_\infty(t)$ for $u_\infty(t) = T(\phi_\infty(t), w_\infty(t))$,
\[
D_\infty(t) = \|\partial_x \phi_\infty(t)\|_{m-1}^2 + \|\partial_t \phi_\infty\|_{m-1}^2 + \|D w_\infty(t)\|_m^2 + \|D \sigma_\infty(t)\|_m^2.
\]
Remark 5.4 From properties of $\mathcal{Q}^{(p)}(t)$, $p = 1, 2$, we see that
\[ \|\partial^k_z \partial^l_{\tau} \partial^l_{\nu} u_1(t)\| \leq C\|D\sigma_1(t)\|_m, \quad 0 \leq 2j + l \leq m + 1, \quad k = 0, 1, \ldots, \]
and there holds
\[ \sup_{0 \leq z \leq t} (1 + z)^{\frac{\nu + 1}{4}} \{\|u_1(z)\|_m + \|\partial_z u_1(z)\|_m\} \leq CM_1(t). \]
Therefore, we do not need special estimates for $u_1(t)$.

We show the following estimates for $M_1(t)$ and $E_\infty(t)$.

**Proposition 5.5** There exist positive constants $\nu_0$ and $\gamma_0$ such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$, then the following assertions hold true.

There exists $\varepsilon_2 > 0$ such that if solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0 \leq z \leq t} \{u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$ for all $t \in [0, \tau]$, then the following estimates hold uniformly for $t \in [0, \tau]$ with $C > 0$ independent of $\tau$.

\[ M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}, \quad (5.9) \]
\[ E_\infty(t) + \int_0^t e^{-a(t-z)} D_\infty(z)dz \leq C\{e^{-at}E_\infty(0) + (1 + t)^{-\frac{\nu + 1}{4}} M(t)^4 + \int_0^t e^{-a(t-z)} \tilde{R}(z)dz\}. \quad (5.10) \]

Here, $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant; and $\tilde{R}(t)$ is quantity that satisfies
\[ \tilde{R}(t) \leq C\{(1 + t)^{-\frac{\nu + 1}{4}} M(t)^3 + M(t)D_\infty(t)\}, \quad (5.11) \]
whenever $\sup_{0 \leq z \leq t} \{u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$.

The proof of Proposition 5.5 is given in Sections 6-8. We prove (5.9), (5.10) and (5.11) in Sections 6, 7 and 8, respectively.

Assuming that Proposition 5.5 holds true, we can show the following estimate.

**Proposition 5.6** If $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$, then the following assertion holds true. There exists number $\varepsilon_3 > 0$ such that if solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_3$, then there holds the estimate

\[ M(t) \leq C\|u_0\|_{H^m \cap L^1}, \quad (5.12) \]
for a constant $C > 0$ independent of $\tau$.

As an immediate consequence of (5.12) we see that the a priori estimate in Proposition 3.7 holds true. Moreover, (5.12) provides us with the following decay estimates:

\[ \|u(t)\|_m \leq C(1 + t)^{-\frac{\nu + 1}{4}} \|u_0\|_{H^m \cap L^1}, \]
\[ \|\partial^k_z u(t)\|_2 \leq C(1 + t)^{-\frac{\nu + 1}{4} - \frac{k}{2}} \|u_0\|_{H^m \cap L^1}, \quad k = 0, 1, \]
and

\[ \|u(t) - \sigma_1(t)u^{(0)}(t)\|_2 \leq C(1 + t)^{-\frac{\nu + 1}{4}} \|u_0\|_{H^m \cap L^1}, \quad (5.13) \]
for $t \in [0, \tau]$. This proves (3.6) and (3.8).

**Proof of Proposition 5.6** If $\sup_{0 \leq z \leq t} \{u(z)\|_m \leq \varepsilon_2$ and $M(t) \leq 1$, then we see from (5.10) and (5.11) that
\[ E_\infty(t) + \int_0^t e^{-a(t-z)} D_\infty(z)dz \leq C\{e^{-at}E_\infty(0) + (1 + t)^{-\frac{\nu + 1}{4}} M(t)^4 \]
\[ + \int_0^t e^{a(t-z)} \{e^{-a(t-z)} M(z)^3 + M(z)D_\infty(z)\}dz\]
\[ \leq C\{e^{-at}E_\infty(0) + (1 + t)^{-\frac{\nu}{2}} M(t)^4 + M(t) \int_0^t e^{-a(t-z)} D_\infty(z)dz\}. \]

Therefore, using continuity of \( E_\infty(t) \) and compatibility conditions we obtain

\[ (1 + t)^{\frac{\nu}{2}} E_\infty(t) + \mathcal{D}(t) + \leq C\{\|u_0\|_{H^m} + M(t)^3 + M(t)\mathcal{D}(t)\}, \tag{5.14} \]

with

\[ \mathcal{D}(t) = (1 + t)^{\frac{\nu}{2}} \int_0^t e^{-a(t-z)} D_\infty(z)dz. \]

It follows from (5.9) and (5.14) that

\[ M(t)^2 + \sup_{0 \leq z \leq t} \mathcal{D}(z) \leq C_1\{\|u_0\|_{H^m \cap L^1} + M(t)^3 + M(t) \sup_{0 \leq z \leq t} \mathcal{D}(t)\}, \tag{5.15} \]

whenever \( \sup_{0 \leq z \leq t} [u(z)]_m \leq \varepsilon_2 \) and \( M(t) \leq 1 \).

In the same way as in [7, Proof of Proposition 5.4] using (5.15) one can show that there exists \( \varepsilon_3 > 0 \) such that if \( \|u_0\|_{H^m \cap L^1} < \varepsilon_3 \), then

\[ M(t) < 2C_2\|u_0\|_{H^m \cap L^1}, \]

for all \( t \in [0, \tau] \) with \( C_2 > 0 \) independent of \( \tau \). This concludes the proof. \[ \square \]

6 Estimates on \( \sigma_1(t) \)

In this section we estimate the \( \mathbb{P}(t) \)-part of \( u(t) \). Since

\[ \mathbb{P}(t)u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t), \]

where \( \sigma_1(t) = \mathcal{P}(t)u(t) \) and \( u_1(t) = (\mathcal{Q}(t) - \mathcal{Q}^{(0)}(t)) \mathcal{P}(t)u(t) \), it is enough to obtain estimates for \( \sigma_1 \) (see Remark 5.4). In this section we assume that \( \nu > \nu_2 \) and \( \gamma^2/(\nu + \nu_2) \geq \gamma_2^2 \).

Let us first make an observation. Regarding the spectral properties of linearized operator, we expect \( \sigma_1(t) \) to be the most slowly decaying part of \( u(t) \). Therefore, the most slowly decaying part of the nonlinearity \( F(t, u(t)) \) would be given by the terms containing only \( \sigma_1(t)^2 \). There are two such terms in \( F(t, u(t)) \),

\[ \frac{\nu\phi}{\gamma^2\rho_p^2}\left(-\partial_{x_n} w^1 + \frac{\partial_x^2 v_1^1}{\gamma^2\rho_p}\phi\right)e_1 \quad \text{and} \quad -\frac{1}{2\gamma^2\rho_p}\partial_{x_n}\left(P''(\rho_p)\phi^2\right)e_n. \]

Since \( w^{(0,1)} \) satisfies (4.7), we can define \( \sigma_1^2 F_1 \) with \( F_1 = F_1(x_n, t) \) as

\[ F_1 = T\left(0, \frac{\phi^{(0)}(x_n)}{\gamma^2\rho_p}\partial_{x_n} w^{(0,1)}(x_n, t), \frac{1}{2\gamma^2\rho_p(x_n)}\partial_{x_n}\left(P''(\rho_p(x_n))\{\phi^{(0)}(x_n)\}^2\right)\right). \]

We thus write

\[ F = \sigma_1^2 F_1 + F_2, \tag{6.1} \]

where \( F_2 = F - \sigma_1^2 F_1 \) contains terms involving \( u_\infty, \) its derivatives and terms of order \( O(\sigma_1 \partial_{x'} \sigma_1) \) like \( \sigma_1 u_1 \), and \( O(\sigma_1^2) \), but not just \( O(\sigma_1^2) \). In particular, we have that \( Q_0 F = Q_0 F_2 \).

First we introduce two lemmas.

Lemma 6.1 There hold the following relations.

(i) \[ [Q_0 F] = -\text{div}'[\phi w'], \]

(ii) \[ \mathcal{P}(t)F(t) = -\text{div}'[\phi(t)w'(t)] + \text{div}'\mathcal{P}^{(1)}(t)F(t) + \Delta' \mathcal{P}^{(2)}(t)F(t). \]
Proof. Since \( w|_{x_n=0,1} = 0 \), by integration by parts, we have
\[
[Q_0 F] = -\text{div}'[\phi w'].
\]
This shows (i). As for (ii), it is straightforward from definition of \( \mathcal{D}^{(0)} \) and (i).

\[ \square \]

Remark 6.2 Let \( \varepsilon_5 > 0 \) be number such that
\[
C_\varepsilon \varepsilon_5 \leq \frac{\gamma^2 \rho_1}{4}.
\]
Here, \( C_\varepsilon > 0 \) comes from Sobolev inequality (3.10). Then whenever \([u(t)]_m \leq \varepsilon_5\), we have
\[
\| \phi(t) \|_\infty \leq C_\varepsilon [u(t)]_m \leq \frac{\gamma^2 \rho_1}{4},
\]
and hence,
\[
\rho(x,t) = \rho_2(x_2) + \gamma^{-2} \phi(x,t) \geq \rho_1 - \gamma^{-2} \| \phi(t) \|_\infty \geq \frac{3\rho_1}{4} > 0.
\]

Therefore, we see that \( \tilde{Q} F(t) \) is smooth whenever \([u(t)]_m \leq \varepsilon_5\).

Using inequality \( \| \sigma_1 \|_\infty \leq C \| \sigma_1 \|_2^2 \| \partial_x \sigma_1 \|_2^2 \) (see Lemmas 8.2 (iii) and 5.3 (iii)), it is not difficult to verify the following estimates on nonlinearities. We omit the proof.

Lemma 6.3 Let solution \( u(t) \) of (4.1) in \( Z^m(\tau) \) satisfies \( \sup_{0 \leq z \leq \varepsilon} |u(z)|_m \leq \varepsilon_5 \) and \( M(t) \leq 1 \) for \( t \in [0,\tau] \), then there hold the following estimates for \( t \in [0,\tau] \) with \( C > 0 \) independent of \( \tau \).

(i) \[
\| \partial_x^\varepsilon (\sigma_1^2(t)) \|_1 \leq C (1 + t)^{-\frac{3}{2}} M(t)^2,
\]
(ii) \[
\| \text{div}'[\phi w'](t) \|_1 \leq C (1 + t)^{-\frac{3}{2}} M(t)^2,
\]
(iii) \[
\| [\phi w'](t) \|_1 \leq C (1 + t)^{-\frac{n+2}{2}} M(t)^2,
\]
(iv) \[
\| F(t) \|_1 \leq C (1 + t)^{-\frac{n+2}{2}} M(t)^2,
\]
(v) \[
\| F_2(t) \|_1 \leq C (1 + t)^{-\frac{3}{4}} M(t)^2,
\]
(vi) \[
\| F(t) \|_2 \leq C (1 + t)^{-\frac{2n+1}{4}} M(t)^2,
\]
(vii) \[
\| \partial_x^\varepsilon (\sigma_1^2(t)) \|_2 \leq C (1 + t)^{-\frac{2n+1}{2}} M(t)^2.
\]

Finally, we prove (5.9).

Proposition 6.4 There exists number \( \varepsilon_4 > 0 \) such that if a solution \( u(t) \) of (4.1) in \( Z^m(\tau) \) satisfies
\[
\sup_{0 \leq z \leq \varepsilon} |u(z)|_m \leq \varepsilon_4 \quad \text{and} \quad M(t) \leq 1 \quad \text{for all} \quad t \in [0,\tau],
\]
then the estimate
\[
M_1(t) \leq C \{ \| u_0 \|_1 + M(t)^2 \},
\]
holds uniformly for \( t \in [0,\tau] \) with \( C > 0 \) independent of \( \tau \).
Proof. We write (5.1) for \( s = 0 \) as
\[
\sigma_1(t) = e^{t\Lambda} \mathcal{P}(0) u_0 + I(t),
\]
where
\[
I(t) = \int_0^t e^{(t-z)\Lambda} \mathcal{P}(z) F(z) dz.
\]
(4.13) yields
\[
\|\partial_x^k e^{t\Lambda} \mathcal{P}(0) u_0\|_2 \leq C(1 + t)^{-\frac{n-1}{2} - \frac{k}{2}} \|u_0\|_1,
\]
for \( k = 0, 1 \). Next, we estimate \( I(t) \) which we write as
\[
I(t) = I_1(t) + I_2(t),
\]
where
\[
I_1(t) = \int_0^t \int_0^1 e^{(t-z)\Lambda} \mathcal{P}(z) F(z) dz,
\]
\[
I_2(t) = \int_t^1 e^{(t-z)\Lambda} \mathcal{P}(z) F(z) dz.
\]
By Lemma 6.1 (ii), we have
\[
e^{(t-z)\Lambda} \mathcal{P}(z) F(z) = \text{div}' e^{(t-z)\Lambda} \{ -[\phi w]'_1 + \mathcal{P}(1) F + \nabla' \mathcal{P}(2) F \}(z).
\]
It then follows from (4.14) and Lemma 6.3 that
\[
\|\partial_x^k I_1(t)\|_2 \leq C \int_0^t (1 + t - z)^{-\frac{n+1}{4} - \frac{k}{2}} \{[\text{div}' w]'_1 + \| F(z) \|_1 \} dz
\leq C M(t)^2 \int_0^t (1 + t - z)^{-\frac{n+1}{4} - \frac{k}{2}} (1 + z)^{-\frac{n+1}{2}} dz \leq C(1 + t)^{-\frac{n+1}{4} - \frac{k}{2}} M(t)^2,
\]
for \( k = 0, 1 \).
As for \( I_2(t) \), using (6.1) and Lemma 6.1 (ii) we write \( \mathcal{P}(z) F(z) \) as
\[
\mathcal{P} F = -\text{div}' [\phi w]'_1 + (\mathcal{P}(1) + \nabla' \mathcal{P}(2)) \cdot \nabla'(\sigma_1)^2 F_1 + (\text{div}' \mathcal{P}(1) + \Delta' \mathcal{P}(2)) F_2.
\]
It then follows from (4.14) and Lemma 6.3 that
\[
\|\partial_x^k I_2(t)\|_2 \leq C \int_t^1 (1 + t - z)^{-\frac{n+1}{4} - \frac{k}{2}} \{[\text{div}' w]'_1 + \| \nabla'(\sigma_1)^2 \|_1 + \| F_2(z) \|_1 \} dz
\leq C M(t)^2 \int_t^1 (1 + t - z)^{-\frac{n+1}{4} - \frac{k}{2}} (1 + z)^{-\frac{n+1}{2}} dz \leq C(1 + t)^{-\frac{n+1}{4} - \frac{k}{2}} M(t)^2,
\]
for \( k = 0, 1 \). We thus obtain
\[
\sum_{k=0}^1 (1 + t)^{-\frac{n+1}{4} + \frac{k}{2}} \|\partial_x^k \sigma_1(t)\|_2 \leq C \{\| u_0 \|_1 + M(t)^2 \}.
\]
(6.2)
It remains to estimate time derivatives. From (5.1) we see that
\[
\partial_t \sigma_1(t) = \Delta \sigma_1(t) + \mathcal{P}(t) F(t).
\]
(6.3)
It then follows from Lemma 6.3 (vi) and previous result that
\[
\|\partial_t \sigma_1(t)\|_2 \leq C \{\|\partial_x \sigma_1(t)\|_2 + \| \mathcal{P}(t) F(t)\|_2 \} \leq C(1 + t)^{-\frac{n+2}{4}} \{\| u_0 \|_1 + M(t)^2 \}.
\]
Concerning \( \|\partial_t^{j+1} \sigma_1(t)\|_2 \) for \( j = 1, \ldots, [\frac{n+1}{2}] - 1 \), we obtain from (6.3)
\[
\|\partial_t^{j+1} \sigma_1(t)\|_2 \leq C \{\|\partial_x \sigma_1(t)\|_2 + \|\partial_t^j (\mathcal{P}(t) F(t))\|_2 \}.
\]
Since
\[\|\partial_t^j F(t)\|_2 \leq C(1 + t)^{-\frac{m+1}{2}} M(t)^2,\]
for \(0 \leq 2j \leq m - 2\) as we see in Propositions 8.5 (i)–(iii) and 8.6 (i), we find by induction on \(j\), that estimate
\[\|\partial_t^{j+1} \sigma_1(t)\|_2 \leq C_j (1 + t)^{-\frac{m+1}{2}} \{\|u_0\|_1 + M(t)^2\},\]
holds for \(j = 0, 1, \ldots, \left[\frac{m}{2}\right] - 1\).

The desired result now follows from (6.2) and (6.4). This completes the proof. \(\square\)

## 7 Estimates on \(\overline{P}_\infty u(t)\)

In this section we prove estimate (5.10) for \(\sigma_\infty\) and \(u_\infty\) by a variant of Matsumura-Nishida energy method as in the case of stationary parallel flow ([7]). Since coefficients of the linearized operator depend on time some extra terms arise in contrast to [7]. We omit the proofs that can be obtained as modification of those in [7]. In this section we assume that \(\nu \geq \nu_2\) and \(\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2\) unless further restricted.

First, let us show the following inequality.

**Proposition 7.1** There exists \(\nu_0 \geq \nu_2\) and \(\gamma_0 \geq \gamma_2\) such that if \(\nu \geq \nu_0\) and \(\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2\) the solution \(u(t)\) of (4.1) in \(Z^m(\tau)\) satisfies
\[
\frac{d}{dt} \overline{E}(t) + 2D(t) \leq \tilde{R}(t).
\]

Here, \(E(t)\), \(D(t)\) and \(\tilde{R}(t)\) are quantities such that

(i) \(\overline{E}(t) + \|\partial^2_x w_\infty(t)\|_{m-2}^2\) is equivalent to \(E_\infty(t)\),
(ii) \(D(t)\) is equivalent to \(D_\infty(t)\),
(iii) \(\tilde{R}(t)\) satisfies estimate (5.11).

We introduce some quantities. Let \(E^{(0)}[\overline{P}_\infty u]\) and \(D^{(0)}[w]\) be defined by
\[
E^{(0)}[\overline{P}_\infty u] = \frac{\alpha_0}{\gamma^2} \|\sigma_\infty\|_2^2 + \frac{1}{\gamma^2} \sqrt{\frac{P'(p_\infty)}{\gamma^2 p_\infty}} \|\phi_\infty\|_2^2 + \|\sqrt{p_\infty} w_\infty\|_2^2,
\]
for \(\overline{P}_\infty u = \sigma_\infty u^{(0)} + u_\infty\) with \(u_\infty = T(\phi_\infty, w_\infty)\); and
\[
D^{(0)}[w_\infty] = \nu \|\nabla w_\infty\|_2^2 + \tilde{\nu} \|\mathrm{div} w_\infty\|_2^2.
\]

Note that,
\[
\langle Au(t), u(t) \rangle_\Omega = D^{(0)}[w(t)],
\]
for \(u = T(\phi, w) \in Z^m(\tau)\), and
\[
\langle B(t)u(t), v(t) \rangle_\Omega = -\langle u(t), B(t)v(t) \rangle_\Omega,
\]
for \(u, v \in Z^m(\tau)\), \(\tilde{Q}u|_{x_n=0,1} = \tilde{Q}v|_{x_n=0,1} = 0\). In particular,
\[
\langle B(t)u(t), u(t) \rangle_\Omega = 0,
\]
for \(u \in Z^m(\tau)\), \(\tilde{Q}u|_{x_n=0,1} = 0\).

We denote the tangential derivatives \(\partial^j_x \partial^k_x\) by \(T_{j,k}\):
\[
T_{j,k} u = \partial^j_x \partial^k_x u.
\]

In this section we often use \(\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O(\frac{1}{\sqrt{t}})\) in calculations (see Lemma 4.9). It is straightforward to see that following lemma holds true.

**Lemma 7.2** There hold the following assertions.
We write \( e \) and \( f \)

\[
(iii) \quad \text{If } w^2 \big|_{x=0,1} = 0, \text{ then } [Q_0 \bar{B} u_\infty] \infty = [Q_0 B u_\infty] \infty = [v^j \partial_x \phi + \gamma^2 \text{div} (\rho_p w_\infty)] \infty.
\]

\[
(iv) \quad \text{If } w^2 \big|_{x=0,1} = 0 \text{ and } 2j + k \leq m, \text{ then}
\]

\[
\| \partial_x \rho \partial_t \bar{B} (\sigma w^0 + u_\infty) \|_2 \leq C \left( \| \partial_x \sigma \|_2^2 + \| \partial_x \phi \|_2^2 + \gamma^4 \| \partial_x w_\infty \|_2^2 \right),
\]

for \( 0 \leq p, q \leq k + 1, 0 \leq r, s \leq k \).

We begin with \( L^2 \)-energy estimates for tangential derivatives. We set

\[
\sigma_* = \sigma + \sigma_\infty, \quad \phi_* = \phi + \phi_\infty, \quad w_* = w_1 + w_\infty,
\]

\[
u_\infty = \nu_1 + \nu_\infty.
\]

We write \( \tilde{Q} F = T(0, f) \) in the form

\[
\tilde{Q} F = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3.
\]

Here, \( \tilde{F}_l = T(0, f_l) \), \( l = 0, 1, 2, 3 \), with

\[
f_0 = -w \cdot \nabla w - f_1(\rho_p, \phi) \Delta \sigma_\infty w^{(0,1)} e_1 - f_2(\rho_p, \phi) \nabla (\partial_x \sigma_\infty w^{(0,1)})
\]

\[
+ f_{01}(x, t, \phi) \phi \sigma_* + f_{02}(x, \phi) \nabla \phi \sigma_* + f_{03}(x, t, \phi) \phi \phi_\infty,
\]

\[
f_1 = -f_1(\rho_p, \phi) \Delta w_* = -\text{div} (f_1(\rho_p, \phi) \nabla w_*) + T(\nabla w_*) \nabla (f_1(\rho_p, \phi)),
\]

\[
f_2 = -f_2(\rho_p, \phi) \nabla \text{div} w_* = -\nabla (f_2(\rho_p, \phi) \text{div} w_*) + (\text{div} w_*) \nabla (f_2(\rho_p, \phi)),
\]

\[
f_3 = -f_3(x, \phi) \phi \nabla \phi_\infty.
\]

Here, \( \nabla w_* \) denotes the \( n \times n \) matrix \( (\partial_x w_*)^T \); \( f_1 = \frac{\nu_\phi}{\rho_p (\gamma p + \phi)} \); \( f_2 = \frac{\bar{\nu}_\phi}{\rho_p (\gamma p + \phi)} \); and \( f_{0l}(x, \phi) \), \( l = 1, 2, 3 \) and \( f_3(x, \phi) \) are smooth functions of \( x, t \) and \( \phi \).

**Proposition 7.3** There exists \( \nu_3 > \nu_2 \) such that for \( \nu \geq \nu_3 \) the following estimate holds for \( 0 \leq 2j + k \leq m \):

\[
\frac{1}{2} \frac{d}{dt} L^0(\sigma^0) (T_{j,k} \bar{u}_\infty) + \frac{1}{2} D^0(T_{j,k} w_\infty) \leq R_{j,k}^{(1)}(\sigma^0) + C \left\{ \frac{\nu}{\gamma^4} + \frac{1}{\gamma^2} \sum_{i=0}^{j-1} \| T_{i,k} \sigma \|_2^2 + \frac{\nu + \bar{\nu}}{\gamma^4} + \frac{1}{\gamma^4} \sum_{i=0}^{j-1} \| T_{i,k+1} \sigma \|_2^2 \right\}
\]

\[
+ \frac{1}{\gamma^4} \sum_{i=0}^{j-1} \| T_{i,k+1} \sigma \|_2^2 + \frac{1}{\gamma^4} (1 - \delta_{j0}) \| \partial_T T_{j-1,k} \sigma \|_2^2 + \frac{1}{\gamma^4} \sum_{i=0}^{j-1} D^0(T_{i,k} w_\infty)
\]

where \( \delta_{j0} \) denotes Kronecker’s delta and \( R_{j,k}^{(1)} \) is given by

\[
\text{(7.2)}
\]
\[ R_{j,k}^{(1)} = \frac{\alpha_0}{\gamma^2} ([Q_0 T_{j,k} F]_{\infty}, T_{j,k} \sigma_{\infty}) - \frac{\alpha_0}{\gamma} ([Q_0 T_{j,k} (P F)]_{\infty}, T_{j,k} \sigma_{\infty}) + \tilde{R}_{j,k}^{(1)} - \langle T_{j,k}, ([Q_0 F]_{\infty} u^{(0)}), T_{j,k} u_{\infty} \rangle_{\Omega} \]

Here,

\[ \tilde{R}_{j,k}^{(1)} = \langle T_{j,k} F, T_{j,k} u_{\infty} \rangle_{\Omega}, \]

when \(2j + k \leq m - 1\), and

\[ \tilde{R}_{j,k}^{(1)} = -(T_{j,k} (\phi \text{div} w), T_{j,k} \phi_{\infty}) P'(\rho_p) \gamma^4 \rho_p + \frac{1}{2} (\text{div} \left( \frac{P'(\rho_p)}{\gamma^4 \rho_p} w \right), |T_{j,k} \phi_{\infty}|^2) \]

\[-(w \nabla T_{j,k} (\sigma \phi^{(0)} + \phi_1), T_{j,k} \phi_{\infty}) P'(\rho_p) \gamma^4 \rho_p - ([T_{j,k}, w] \nabla \phi, T_{j,k} \phi_{\infty} P'(\rho_p) \gamma^4 \rho_p) \]

\[ + (T_{j,k} f_0, T_{j,k} w_{\infty} \rho_p) + \sum_{i=1}^{3} (T_{j,k} f_i, T_{j,k} w_{\infty} \rho_p) \]

when \(2j + k = m\). Here and in what follows, for \(G = g + \partial_x \tilde{g}\) with \(g, \tilde{g} \in L^2\) and \(v \in H_0^1\), \(\langle G, v \rangle_{-1}\) denotes \(\langle g, v \rangle - (\tilde{g}, \partial_x v)\).

**Proof.** We apply \(T_{j,k}\) to (5.2) and (5.3). We then take the inner products of the resulting equations with \(T_{j,k} \sigma_{\infty}\) and \(T_{j,k} u_{\infty}\), respectively. Integration by parts together with symmetric properties of \(A\) and \(B\) gives us the desired result in the same manner as in [7, Proposition 7.4].

We next derive the \(H^1\)-parabolic estimates for \(u_{\infty}\). We define \(\tilde{J}[\tilde{P}_u]\) by

\[ \tilde{J}[\tilde{P}_u] = -2\langle \sigma_{\infty} u^{(0)} + u_{\infty}, B \tilde{Q} u_{\infty} \rangle_{\Omega} \]

for \(\tilde{P}_u = \sigma_{\infty} u^{(0)} + u_{\infty}\).

A direct computation shows that if \(\gamma^2 \geq 1\) then

\[ |J[\tilde{P}_u]| \leq \frac{b_0 \gamma^2}{\nu} E^0[\tilde{P}_u] + \frac{1}{2} D^0[w_{\infty}], \]

for some constant \(b_0 > 0\).

Let \(b_1\) be a positive constant (to be determined later) and define \(E^{(1)}[\tilde{P}_u]\) by

\[ E^{(1)}[\tilde{P}_u] = \frac{2b_1 \gamma^2}{\nu^2} E^0[\tilde{P}_u] + D^0[w_{\infty}] + J[\tilde{P}_u]. \]

Note that if \(b_1 \geq b_0\) then \(E^{(1)}[\tilde{P}_u]\) is equivalent to \(E^0[\tilde{P}_u] + D^0[w_{\infty}]\).

**Proposition 7.4** There exists \(b_1 \geq \max\{b_0, 8C_0\}\) such that if \(\nu \geq \nu_3\), \(\gamma^2 \geq 1\) and \(\frac{\gamma^2}{\nu} \geq \max\{1, \gamma_2^2\}\) then the following estimate holds for \(0 \leq 2j + k \leq m - 1\),

\[ \frac{1}{2} \frac{d}{dt} E^{(1)}[T_{j,k} \tilde{P}_u] + \frac{b_1 \gamma^2}{\nu} D^0[T_{j,k} w_{\infty}] + \frac{1}{2} \| T_{j,k} \phi_{\infty} \|^2 \leq R_{j,k}^{(2)} \]

\[ + C \sum_{i=0}^{j} \left\{ \left( \frac{\nu^2}{\gamma^2} + \frac{1}{\nu} \right) \| T_{i,k} \phi_{\infty} \|^2 + \left( \frac{\nu + 1}{\gamma^2} + \frac{1}{\nu} \right) \| T_{i,k+1} \sigma_{\infty} \|^2 \right\} + \frac{1}{\nu} (1 - \delta_j) \| \partial_t T_{j-1,k} \| \| T_{j,k} \sigma_{\infty} \|^2 \right\} + C_0 \frac{\gamma^2}{\nu} (1 + \frac{C_1}{\nu^2}) \sum_{i=0}^{j-1} D^0[T_{i,k} w_{\infty}]. \]

where

\[ R_{j,k}^{(2)} = \frac{2b_1 \gamma^2}{\nu} R_{j,k}^{(1)} + C \| T_{j,k} F \|^2. \]

**Proof.** We apply \(T_{j,k}\) to (5.3) and take inner product with \(\partial_t T_{j,k} \tilde{Q} u_{\infty}\) to obtain the desired result in the same manner as in [7, Proposition 7.5].
As for the dissipative estimates for $x_n$-derivatives of $\phi_\infty$, we have the following inequality.

**Proposition 7.5** The following estimate holds for $0 \leq 2j + k + l \leq m - 1$:

\[
\frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \frac{P'(\rho_p)}{\gamma^2 \rho_p} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|^2 + \frac{1}{2(\nu + \bar{\nu})} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n}^{l+1} T_{j,k} \phi_\infty \right\|^2 \leq R_{j,k,l}^{(3)} + C \frac{\nu + \bar{\nu}}{\gamma^4} \left\| K_{j,k,l} \right\|^2_2,
\]

where

\[
R_{j,k,l}^{(3)} = \frac{1}{2} \left( \text{div} \left( \frac{P'(\rho_p)}{\gamma^2 \rho_p} w \right), \left| T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right|^2 \right) + C \frac{\nu + \bar{\nu}}{\gamma^4} \left\| H_{j,k,l} \right\|^2_2,
\]

with

\[
\left\| H_{j,k,l} \right\|^2_2 \leq C \left\{ \left\| T_{j,k} \partial_{x_n}^{l+1} w \right\|^2 + \left\| T_{j,k} \partial_{x_n}^{l+1} (Q_0 P_\infty F) \right\|^2 + \left\| \frac{\gamma^2 \rho_p^2}{\nu + \bar{\nu}} T_{j,k} \partial_{x_n}^{l+1} (Q_0 P_\infty F) \right\|^2 \right\},
\]

and

\[
Q_0 P_\infty F = -\phi \text{div} w - w \cdot \nabla (\sigma_\star \phi(0) + \phi_1) - \{ Q_0 P F + [Q_0 P_\infty F]_\infty \phi(0) \}
\]

Here, $K_{j,k,l}$ is estimated as

\[
\frac{\nu + \bar{\nu}}{\gamma^4} \left\| K_{j,k,l} \right\|^2_2 \leq C \frac{\nu + \bar{\nu}}{\gamma^4} \left( \frac{\nu^2}{\nu + \bar{\nu}} \left\| T_{j,k+1} \partial_{x_n}^{l+1} \partial_{x,j} \right\|^2 + \frac{1}{\nu + \bar{\nu}} \left\| \rho_p T_{j,k} \partial_{x_n}^{l+1} \partial_{x,j} \right\|^2 \right)
\]

\[
+ \frac{\nu^2}{\gamma^4} \sum_{q=0}^{l-1} \left\| T_{j,k+1} \partial_{x_n}^{l+1} \partial_{x,q} \right\|^2 + \sum_{q=0}^{l-2} \left\| T_{j,k+1} \partial_{x_n}^{l+1} \partial_{x,q} \right\|^2 + \sum_{i=0}^{j} \left\| T_{i,k+1} \partial_{x_n}^{l+1} \right\|^2)
\]

\[
+ \frac{1}{\nu + \bar{\nu}} \left( \sum_{i=0}^{j} \sum_{q=0}^{i-1} \left\| T_{i,k+1} \partial_{x_n}^{l+1} \phi_\infty \right\|^2 + \sum_{q=0}^{j} \left\| T_{i,k+1} \partial_{x_n}^{l+1} \sigma_\infty \right\|^2 + \sum_{q=0}^{j} \left\| \partial_{x_q} T_{i,k} \phi_\infty \right\|^2 \right).
\]

**Proof.** We obtain the desired result in the same manner as in [7, Proposition 7.6].

The following estimate for the material derivative of $\phi_\infty$ plays an important role to obtain the dissipative estimate for higher order $x_2$-derivatives of $w_\infty$. We denote the material derivative of $\phi_\infty$ by $\dot{\phi}_\infty$:

\[
\dot{\phi}_\infty = \partial_t \phi_\infty + (v_p + w) \cdot \nabla \phi_\infty.
\]

**Proposition 7.6** The following estimates hold for $0 \leq 2j + k + l \leq m - 1$:

(i) \[
\frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \frac{P'(\rho_p)}{\gamma^2 \rho_p} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|^2 + \frac{1}{4(\nu + \bar{\nu})} \left\| \frac{P'(\rho_p)}{\gamma^2} \partial_{x_n}^{l+1} T_{j,k} \phi_\infty \right\|^2 \leq R_{j,k,l}^{(1)} + D^{(0)} \left[ T_{j,k} w_\infty \right] + C \frac{\nu + \bar{\nu}}{\gamma^4} \left\| K_{j,k,l} \right\|^2_2,
\]

where $c_0$ is a positive constant and $R_{j,k,l}^{(1)}$ and $K_{j,k,l}$ satisfy the same estimates as in Proposition 7.5.

(ii) Let $0 \leq q \leq k$ and $0 \leq 2j + k + l \leq m$. Then

\[
\frac{\nu + \bar{\nu}}{\gamma^4} \left\| T_{j,k} \phi_\infty \right\|^2 + \sum_{i=0}^{j} \left\| T_{i,k+1} \sigma_\infty \right\|^2 + \frac{\nu + \bar{\nu}}{\gamma^4} \sum_{i=0}^{j} \left\| T_{i,q} \phi_\infty \right\|^2 \leq C \left\{ R_{j,k,l}^{(1)} + D^{(0)} \right\} + \frac{\nu^2(\nu + \bar{\nu})}{\gamma^4} \left\| T_{j,k} w_\infty \right\|^2
\]

\[
+ \frac{\nu + \bar{\nu}}{\gamma^4} \left( \sum_{i=0}^{j} \left\| T_{i,k+1} \partial_{x_n}^{l+1} \phi_\infty \right\|^2 \right)
\]

\[
where R_{j,k,l}^{(1)} = \frac{\nu + \bar{\nu}}{\gamma^4} \left\| T_{j,k} Q_0 P_\infty F \right\|^2_2.
\]
Proof. The desired result is obtained in the same manner as in [7, Proposition 7.7].

Let us derive the dissipative estimates for \( \sigma_\infty \).

Proposition 7.7 Let \( \gamma^2/(\nu+\tilde{\nu}) \geq \max\{1, \gamma^2_H\} \), then there holds the following estimate for \( 0 \leq 2j+k \leq m-1 \):

\[
\frac{1}{2} \frac{d}{dt} \gamma^2(\nu+\tilde{\nu}) \| T_{j,k}\sigma_\infty \|^2 + \frac{\alpha_1}{2(\nu+\tilde{\nu})} \| \nabla' T_{j,k}\sigma_\infty \|^2 \leq R_{j,k}^{(5)}
\]

where \( \alpha_1 > 0 \) is a constant, \( p \) is any integer satisfying \( 0 \leq p \leq k \), and

\[
R_{j,k}^{(5)} = \frac{\nu}{\gamma^2(\nu+\tilde{\nu})}(Q_0 T_{j,k}(P_\infty F), T_{j,k}\sigma_\infty) - \frac{1}{\nu+\tilde{\nu}}(\text{div}'[\rho_p(-\Delta)^{-1}(\rho_p T_{j,k}Q' P_\infty F))]_\infty, T_{j,k}\sigma_\infty).
\]

Here, \( (-\Delta)^{-1} \) is the inverse of \( -\Delta \) on \( L^2(\Omega) \) with domain \( D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega) \).

Proof. The desired result is obtained in the same manner as in [7, Proposition 7.8].

Next, we estimate the higher order derivatives.

Proposition 7.8 If \( \nu \geq 1 \) then there holds the following estimate for \( 0 \leq 2j+k+l \leq m-1 \):

\[
\frac{\nu^2}{\nu+\tilde{\nu}} \| \partial_x^{l+2} T_{j,k}w_\infty \|^2 + \frac{1}{\nu+\tilde{\nu}} \| \partial_x^{l+1} T_{j,k}\phi_\infty \|^2 \leq CR_{j,k,l}^{(6)} + C\{(\frac{\nu+\tilde{\nu}}{\gamma^4}) \sum_{i=0}^{j} \| T_{i,k}\phi_\infty \|^2_{H^{l+1}} + \frac{\nu+\tilde{\nu}}{\gamma^4} \sum_{i=0}^{j} \| T_{i,k}w_\infty \|^2_{H^{l+1}} + \frac{\nu^2}{\nu+\tilde{\nu}} \| \partial_x T_{j,k}w_\infty \|^2_{H^{l+1}} + D(0)[T_{j,k}w_\infty] \},
\]

where

\[
R_{j,k,l}^{(6)} = \frac{\nu+\tilde{\nu}}{\gamma^4} \| T_{j,k}Q_0 P_\infty F \|^2_{H^{l+1}} + \frac{1}{\nu+\tilde{\nu}} \| T_{j,k}(Q P_\infty F) \|^2_{H^{l+1}}.
\]

Proof. We use the estimates for the Stokes system. Let \( T(\tilde{\phi}, \tilde{w}) \) be the solution of the Stokes system

\[
\text{div} \tilde{w} = F \quad \text{in } \Omega,
\]

\[-\Delta \tilde{w} - \nabla \tilde{\phi} = G \quad \text{in } \Omega,
\]

\[
\tilde{w}|_{\partial \Omega} = 0.
\]

Then for any \( l \in \mathbb{Z} \), \( l \geq 0 \), there exists a constant \( C > 0 \) such that

\[
\| \partial_x^{l+2} \tilde{w} \|_2^2 + \| \partial_x^{l+1} \tilde{\phi} \|_2^2 \leq C \{ \| F \|^2_{H^{l+1}} + \| G \|^2_{H^{l+1}} + \| \partial_x \tilde{w} \|_2^2 \},
\]

(see, e.g., [3][4, Appendix]).

We rewrite (5.3) in the form of Stokes system with \( \tilde{w} = T_{j,k}w_\infty \) and \( \tilde{\phi} = \frac{P'(\rho_p)}{\rho_p} T_{j,k}\phi_\infty \). The desired result is then obtained by using (7.9) (cf. [7, Proposition 7.9]).

At last we estimate the time derivatives of \( \sigma_\infty \) and \( \phi_\infty \).
Proposition 7.9  

(i) If $0 \leq 2j + k \leq m - 1$, then there holds the following estimate:

$$\|\partial_t T_{j,k} \sigma_\infty \|_2^2 \leq C \{ R^{(7)}_{j,k} + \sum_{i=0}^j \|T_{i,k+1} \sigma_\infty \|_2^2 + \|T_{i,k+1} \phi_\infty \|_2^2 \} + \gamma^4 \|T_{j,k+1} w_\infty \|_2^2 \}$$  

(7.10)

Here, $R^{(7)}_{j,k} = \|[Q_0 T_{j,k} (P_0^\infty F)]_{\sigma_\infty} \|_2^2$.

(ii) If $0 \leq k + 2j \leq m - 1$ then there holds the following estimate:

$$\|\partial_t^{j+1} \phi_\infty \|_{H^k}^2 \leq C \{ R^{(8)}_j + \sum_{i=0}^j \|\partial_x \partial_t^i \phi_\infty \|_{H^k} + \|\partial_x \partial_t^i \sigma_\infty \|_{H^k}^2 \} + \gamma^4 \|\partial_x \partial_t^i w_\infty \|_{H^k}^2 \}$$  

(7.11)

Here, $R^{(8)}_j = \|\partial_t^j (Q_0 P_0^\infty F) \|_{H^k}^2$.

**Proof.** The estimates (7.10) and (7.11) follow from (5.2) and the first line of (5.3). \qed

Proposition 7.1 now follows from combination of results in Propositions 7.3–7.9.

**Proof of Proposition 7.1.** Let us define

$$\tilde{E}^{(0)}(t) = \sum_{2j+k \leq m \atop 2j \neq m} E^{(0)}[T_{j,k} \tilde{P}_\infty u(t)], \quad \tilde{E}^{(1)}(t) = \sum_{2j+k \leq m-1} E^{(1)}[T_{j,k} \tilde{P}_\infty u(t)],$$

$$E^{(2)}(t) = \sum_{2j+k \leq m-1} \frac{1}{\gamma^2} \| \frac{P_r(p_r)}{\gamma^2 p_r} \partial_x T_{j,k} \phi_\infty(t) \|_2^2, \quad E^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{\nu}{\gamma^2(\nu + \tilde\nu)} \|T_{j,k} \sigma_\infty(t)\|_2^2,$$

and

$$\tilde{D}^{(0)}(t) = \sum_{2j+k \leq m \atop 2j \neq m} D^{(0)}[T_{j,k} w_\infty(t)],$$

$$D^{(1)}(t) = \sum_{2j+k \leq m-1} \left( \frac{3b_1 \gamma^2}{2(\nu + \tilde\nu)} \right) \tilde{D}^{(0)}[T_{j,k} w_\infty(t)] + \frac{1}{\nu + \tilde\nu} \| \sqrt{p_r} T_{j,k} \partial_t w_\infty(t) \|_2^2,$$

$$D^{(2)}(t) = \sum_{2j+k \leq m-1} \left( \frac{1}{2(\nu + \tilde\nu)} \right) \| \frac{P_r(p_r)}{\gamma^2} \partial_x T_{j,k} \phi_\infty(t) \|_2^2 + \min \{1, 2c_0\} \frac{\nu + \tilde\nu}{\gamma^4} \|T_{j,k} \phi_\infty(t)\|_{H^1}^2,$$

$$D^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{\alpha_1}{\nu + \tilde\nu} \| \nabla T_{j,k} \sigma_\infty(t) \|_2^2.$$

Let $b_l$, $l = 2, \ldots, 5$, be positive numbers and let us consider

$$\sum_{2j+k \leq m \atop 2j \neq m} \{ 2 \times (7.2) + 2b_2 \times (7.6) \}$$

$$+ \sum_{2j+k \leq m-1} \left\{ \frac{2}{\nu + \tilde\nu} \times (7.3) + 2b_2 \times (7.5)_{l=0} + 2b_3 \times (7.7) + b_4 \times (7.8)_{l=0} \right\}$$

$$+ \sum_{2j+k \leq m-1} \frac{1}{\nu + \tilde\nu} b_5 \times ((7.10) + (7.11)) + 2 \frac{b_6}{\nu + \tilde\nu} \times (7.2)_{2j=m}.$$

Then we obtain

$$\frac{d}{dt} E^{(4)}(t) + D^{(4)}(t) + \frac{1}{\nu + \tilde\nu} b_5 \sum_{2j+k \leq m-1} (\|\partial_t T_{j,k} \sigma_\infty \|_2^2 + \|\partial_t^{j+1} \phi_\infty(t) \|_{H^k}^2)$$

(7.12)
\[ \leq \sum_{j=1}^{8} R^{(j)}(t) + \text{RHS}. \]

Here,
\[
E^{(4)}(t) = \tilde{E}^{(0)} + \frac{1}{\nu + \tilde{\nu}} \tilde{E}^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t) + \frac{b_6}{\nu + \tilde{\nu}} E^{(0)}[\partial_t^{\frac{\tilde{\nu}}{2}} \tilde{P}_\infty u(t)],
\]
\[
D^{(4)} = \tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t) + b_4 \sum_{2j + k \leq m - 1} \left( \frac{\nu^2}{\nu + \tilde{\nu}} \| \partial_t^2 T_{j,k} \phi_\infty \|^2_2 + \frac{1}{\nu + \tilde{\nu}} \| \partial_x T_{j,k} \phi_\infty \|^2_2 \right)
\]
\[+ \frac{b_6}{\nu + \tilde{\nu}} D^{(0)}[\partial_t^{\frac{\tilde{\nu}}{2}} w_\infty], \]

and
\[
R^{(1)} = \sum_{2j + k \leq m} R^{(1)}_{j,k}, \quad R^{(p)} = \sum_{2j + k \leq m - 1} R^{(p)}_{j,k}, \quad p = 2, 5, 7, 8, \quad R^{(4)} = \sum_{2j + k \leq m} R^{(4)}_{j,k},
\]
\[
R^{(p)} = \sum_{2j + k \leq m - 1} R^{(p)}_{j,k,0}, \quad p = 3, 6,
\]

with
\[
\text{RHS} = C \left( \frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu + \tilde{\nu}} \sum_{2j + k \leq m - 1} \| \partial_t T_{j,k} \phi_\infty \|^2_2 \right) + C \left( \frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu + \tilde{\nu}} \sum_{2j + k \leq m - 1} \| \partial_x T_{j,k} \sigma_\infty \|^2_2 \right)
\]
\[+ C \left( \frac{1}{\nu} + \frac{1}{\nu + \tilde{\nu}} \right) \sum_{2j + k \leq m - 2} \| \partial_t T_{j,k} \sigma_\infty \|^2_2 + C \left( \frac{1}{\nu^2} + b_2 + b_3 + b_4 \right) \sum_{2j + k \leq m-1} \| D^{(0)}[T_{j,k} w_\infty] \|^2_2
\]
\[+ \left( \frac{b_4}{4} + b_5 C \right) \frac{\gamma^2}{\nu + \tilde{\nu}} \sum_{2j + k \leq m - 1} D^{(0)}[T_{j,k} w_\infty] + C(b_2 + b_3 + b_4) \sum_{2j + k \leq m - 1} \| \sqrt{\rho} T_{j,k} \partial_x w_\infty \|^2_2
\]
\[+ \frac{b_4 \alpha_1}{4(\nu + \tilde{\nu})} \sum_{2j + k \leq m - 3} \| T_{j,k+1} \phi_\infty \|^2_2 + Cb_3 \frac{1}{\nu + \tilde{\nu}} \sum_{2j + k \leq m - 1} \| \frac{P'(\rho)}{\gamma^2} \partial_t T_{j,k} \phi_\infty \|^2_2
\]
\[+ C b_4 \frac{\nu + \tilde{\nu}}{\gamma^4} \sum_{2j + k \leq m - 1} \| T_{j,k} \partial_x \phi_\infty \|^2_2 + Cb_6 \frac{1}{(\nu + \tilde{\nu}) \gamma^2} \left( \| \partial_t^{\frac{\tilde{\nu}}{2}} \phi_\infty \|^2_2 + \| \partial_t^{\frac{\tilde{\nu}}{2}} \sigma_\infty \|^2_2 \right).
\]

There exists \( \nu_0 \geq \max\{1, \nu_1 \}, \gamma_0 \geq \max\{1, \gamma_2 \} \) and \( 1 > b > 0 \) such that if \( b_4 < b_3 < b_2 \) and \( b_6 \leq b_5 \leq b_1 \) appropriately with \( b_1 \leq b \) for \( l = 2, \ldots, 4, \) and \( \nu \geq \nu_0 \) and \( \gamma^2 / (\nu + \tilde{\nu}) \geq \gamma_0^2 \) we can absorb most of the terms from RHS in the left-hand side of (7.12) to get
\[
\frac{d}{dt} E^{(4)}(t) + \frac{1}{2} D^{(4)}(t) + \frac{1}{2} \left( \frac{\nu + \tilde{\nu}}{\gamma^2} \right) b_5 \sum_{2j + k \leq m - 1} \| \partial_t T_{j,k} \sigma_\infty \|^2_2 + \frac{1}{2} \left( \frac{\nu + \tilde{\nu}}{\gamma^2} \right) b_5 \sum_{2j + k \leq m-2} \| \partial_t^{l+1} \phi_\infty(t) \|^2_2 \quad (7.13)
\]
\[\leq C \sum_{j=1}^{8} R^{(j)}(t) + C \sum_{2j + k \leq m - 2} \| \partial_t T_{j,k} \sigma_\infty \|^2_2 \frac{1}{\nu + \tilde{\nu}}. \]

Next, we estimate higher order derivatives in \( x_n \). For \( 1 \leq l \leq m - 1 \), we set
\[
E^{(4)}_{l+1} = \sum_{2j + k \leq m - 1 - l} \frac{1}{\gamma^2} \| \frac{P'(\rho)}{\gamma^2} T_{j,k} \partial_x^{l+1} \phi_\infty(t) \|^2_2,
\]
and
\[ D^{(4)}_t = \sum_{2+j+k \leq m-1-l} \left( \frac{1}{2(\nu + \nu')} \left( \frac{1}{\gamma^2} \left( \frac{1}{\nu} \left[ \frac{\partial^2(P(h^1_\nu) T_{j,k} \phi_{\infty}(t))^2}{\gamma^2} + \frac{2c_0(\nu + \nu')}{\gamma^2} \right] + \frac{1}{\nu + \nu'} \right) \right) \right) + b_7 \sum_{2+j+k \leq m-1-l} \left( \frac{\nu^2}{\nu + \nu'} \left[ \frac{\partial^2 T_{j,k} w_{\infty}(t)^2}{\gamma^2} + \frac{1}{\nu + \nu'} \left[ \frac{\partial^2 T_{j,k} \phi_{\infty}(t)^2}{\gamma^2} \right] \right) \right). \]

We add \(2 \times (7.5)\) to \(b_7 \times (7.8)\) and sum over \(2j + k \leq m - 1 - l\) to obtain

\[ \frac{d}{dt} E^{(4)}_t + D^{(4)}_t \leq C R^{(9)}_t + b_7 C \frac{\nu + \nu'}{\gamma^2} \sum_{2+j+k \leq m-1-l} ||T_{j,k} \phi_{\infty}||^2_{H^1}. \]

Here, \(R^{(9)}_t = \sum_{2+j+k \leq m-1-l} R^{(3)}_{j,k,l} + R^{(6)}_{j,k,l}.\)

Let us sum up to \(l\) to get

\[ \frac{d}{dt} \sum_{p=1}^l E^{(4)}_p(t) + \sum_{p=1}^l D^{(4)}_p(t) \leq C \sum_{p=1}^l R^{(9)}_p \]

\[ + b_7 C \frac{\nu + \nu'}{\gamma^2} \sum_{p=1}^l \sum_{2+j+k \leq m-1-p} ||T_{j,k} \phi_{\infty}||^2_{H^{l+1}} + C \frac{(\nu + \nu')}{\gamma^2} \sum_{p=1}^l \sum_{2+j+k \leq m-1-p} ||T_{j,k} \phi_{\infty}||^2_{H^{l+1}}. \]

Taking \(b_7\) appropriately small, \(\nu_0\) and \(\gamma_0\) possibly larger (based on \(l\)) we obtain

\[ \frac{d}{dt} \sum_{p=1}^l E^{(4)}_p(t) + \sum_{p=1}^l D^{(4)}_p(t) \leq C \sum_{p=1}^l R^{(9)}_p + b_7 C \frac{\nu + \nu'}{\gamma^2} \sum_{2+j+k \leq m-2} ||T_{j,k} \phi_{\infty}||^2_{H^1}. \]

(7.14)

\[ + \frac{1}{\nu + \nu'} \sum_{2+j+k \leq m-1} \left( \frac{\nu^2}{\gamma^2} \left[ \frac{\partial^2 T_{j,k} w_{\infty}}{\gamma^2} \right] + \frac{1}{\nu + \nu'} \left( \frac{\partial^2 T_{j,k} \phi_{\infty}}{\gamma^2} \right) \right) + b_7 \sum_{2+j+k \leq m-1} \left( \frac{\nu + \nu'}{\gamma^2} \right) \left( \frac{D^{(0)}[\partial^2_{T_{j,k} w_{\infty}}]}{\gamma^2} \right) \]

\[ + C \frac{(\nu + \nu')}{\gamma^2} \sum_{2+j+k \leq m-2} \left( \frac{\partial^2 T_{j,k} w_{\infty}}{\gamma^2} \right) + C \frac{(\nu + \nu')}{\gamma^2} \sum_{2+j+k \leq m-2} \left( \frac{\partial^2 T_{j,k} \phi_{\infty}}{\gamma^2} \right) \]

Now adding \(2 \times (7.13)\) together with (7.14) and taking possibly \(b_7\) smaller, \(\nu_0\) and \(\gamma_0\) larger we obtain

\[ \frac{d}{dt} (2E^{(4)}_t) + \sum_{p=1}^l E^{(4)}_p(t) + (D^{(4)}_t) + \sum_{p=1}^l D^{(4)}_p(t) \]

(7.15)
It then follows that

\[ \frac{d}{dt} (2E_1^{(4)}(t) + \sum_{p=1}^{l} E_{p,1}^{(4)}(t)) + (D_1^{(4)}(t) + \sum_{p=1}^{l} D_{p,1}^{(4)}(t)) \]

\[ + \frac{1}{(\nu + \tilde{\nu})^{\gamma^2}} b_5 \| \partial_t \sigma_\infty \|_2^2 \leq C \left( \sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)} \right). \]  

(7.16)

To absorb the last term on the right-hand side we use induction on \( m \).

Let \( m = 1 \) then we have from (7.15) that

\[ C_1 \frac{d}{dt} \left( 2E_1^{(4)}(t) + \sum_{p=1}^{l} E_{p,1}^{(4)}(t) \right) + (D_1^{(4)}(t) + \sum_{p=1}^{l} D_{p,1}^{(4)}(t)) \]

\[ + \frac{1}{(\nu + \tilde{\nu})^{\gamma^2}} b_5 \| \partial_t T_{0,k} \sigma_\infty \|_2^2 \leq C_2 \left( \sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)} \right) \]

(7.17)

By adding \( b_8 \gamma \times (7.16) \) to (7.17) with appropriately large \( b_8 > 0 \) we can absorb \( \| \partial_t \sigma_\infty \|_2^2 \frac{1}{\nu + \tilde{\nu}} \) to the left-hand side. It is straightforward to see that this can be done from \( m \) to \( m + 1 \). Therefore, we have

\[ C_1 \frac{d}{dt} \left( 2E_1^{(4)}(t) + \sum_{p=1}^{l} E_{p}^{(4)}(t) \right) + (D_1^{(4)}(t) + \sum_{p=1}^{l} D_{p}^{(4)}(t)) \]

\[ + \frac{1}{(\nu + \tilde{\nu})^{\gamma^2}} b_5 \| \partial_t T_{0,k} \sigma_\infty \|_2^2 \leq C_2 \left( \sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)} \right), \]

(7.18)

with \( C_1, C_2 > 0 \). The desired estimate (7.1) now follows from (7.18) with \( l = m - 1 \).

Estimate (5.11) for \( \tilde{R}(t) \) is given in Proposition 8.1 (ii) below. This concludes the proof. □

To prove (5.10) we employ the following lemma.

**Lemma 7.10** There exists \( \tilde{r}_0 = \tilde{r}_0(\nu, \tilde{\nu}, \gamma) \) such that if \( r_1 \leq \tilde{r}_0 \), then there holds the estimate

\[ \| [Q_0 P_{\infty,1}(t) u] \|_2 \leq C \| \partial_{\xi'} (I - \Pi^{(0)}(t)) P_{\infty,1}(t) u \|_2. \]

**Proof.** We set

\[ \hat{R}(\xi', t) = \hat{\mathcal{D}}^{(0)}(t) (i \xi' \hat{\mathcal{P}}^{(1)}(t) - |\xi'|^2 \hat{\mathcal{P}}^{(2)}(\xi', t)) + (i \xi' \hat{\mathcal{D}}^{(1)}(t) - |\xi'|^2 \hat{\mathcal{D}}^{(2)}(\xi', t)) \hat{\mathcal{P}}^{\xi'}(t). \]

Since

\[ [Q_0 \hat{\mathcal{H}_1}(\xi')(I - \hat{\mathcal{D}}^{(0)}(t) \hat{\mathcal{P}}^{(0)}(\xi', t))] = [Q_0 \hat{\mathcal{H}_1}(\xi')(I - \hat{\mathcal{D}}^{(0)}(t) \hat{\mathcal{P}}^{(0)} - \hat{R}(\xi', t))] = -[Q_0 \hat{\mathcal{H}_1}(\xi') \hat{R}(\xi', t)], \]

we see that

\[ [Q_0 \hat{P}_{\infty,1} u(t)] = [Q_0 \hat{\mathcal{H}_1}(\xi')(I - \hat{\mathcal{D}}^{(0)}(t) \hat{\mathcal{P}}^{(0)}(\xi', t))] \hat{u}. \]

It then follows that

\[ \| [Q_0 P_{\infty,1}(t) u] \|_2 \leq C \| \hat{\mathcal{H}_1} \hat{u} \|_2 \]
Lemma 8.2

Since \( P_{\infty,1}(t)^2 = P_{\infty,1} \), we see that
\[
||Q_0\dot{P}_{\infty,1}(t)u||_2 \leq C|\xi'|(\|P_{\infty,1}(t)\|_2 + ||\dot{Q}_0\dot{P}_{\infty,1}(t)u||_2),
\]
for \(|\xi'| \leq r_1\). Therefore, there exists a positive number \( \tilde{r}_0 \) such that if \( \tilde{r}_1 \leq \tilde{r}_0 \) then
\[
||Q_0\dot{P}_{\infty,1}(t)u||_2 \leq C|\xi'|(I - \Pi(0))\dot{P}_{\infty,1}u||_2,
\]
for \(|\xi'| \leq r_1\), from which we obtain
\[
||Q_0\dot{P}_{\infty,1}(t)u||_2 \leq C\|\partial_{x'}(I - \Pi(0))\dot{P}_{\infty,1}(t)u||_2.
\]
This completes the proof. \( \square \)

Finally, we prove (5.10).

Proof of (5.10). We fix \( \nu, \tilde{\nu}, \gamma \) so that inequality (7.1) in Proposition 7.1 holds true and set \( r_1 = \min\{r_0, \tilde{r}_0, 1\} \). Then we proceed as in [7, Proof of (5.15)] to obtain
\[
\tilde{E}(t) + \frac{\nu^2}{\nu + \nu'} \| D_{\nu,\infty}(t) \|_{m-2}^2 + \int_0^t e^{-\tilde{a}(t-z)} D(z) dz \leq C\{ e^{-\tilde{a}t} \tilde{E}(0) + R(10)(t) + \int_0^t e^{-\tilde{a}(t-z)} \tilde{R}(z) dz \}. \tag{7.19}
\]
Since
\[
R(10)(t) \leq C(1 + t)^{-\frac{n+1}{2}} M(t)^4, \tag{7.20}
\]
as we show in Proposition 8.1 (i) below, we deduce (5.10) from (7.19), Proposition 7.1 (i) and (7.20). This completes the proof. \( \square \)

8 Estimates on the nonlinearities

In this section we estimate the nonlinearities, e.g., we prove (5.11) and (7.20). In this section we assume that \( \nu \geq \nu_2 \) and \( \gamma^2/\nu' \geq \gamma_2^2 \).

Proposition 8.1 There exists number \( \varepsilon_6 > 0 \) such that if solution \( u(t) \) of (4.1) in \( Z^m(\tau) \) satisfies \( \sup_{0 \leq z \leq \tau} [u(z)]_m \leq \varepsilon_6 \) and \( M(t) \leq 1 \) for all \( t \in [0, \tau] \), then the following estimates hold for all \( t \in [0, \tau] \) with \( C > 0 \) independent of \( \tau \).

(i) \[
\| QP_{\infty} F(t) \|_{m-2} \leq C(1 + t)^{-\frac{n-1}{2}} M(t)^2,
\]

(ii) \[
\tilde{R}(t) \leq C\{ (1 + t)^{-\frac{n-1}{2}} M(t)^3 + (1 + t)^{-\frac{n-1}{2}} M(t) D_{\infty}(t) \}.
\]
To show the estimates in Proposition 8.1 we use the following inequalities.

Lemma 8.2 (i) Let \( 2 \leq p \leq \infty \) and let \( j \) and \( k \) be integers satisfying
\[
0 \leq j < k, \quad k > j + n\left(\frac{1}{2} - \frac{1}{p}\right).
\]
Then there exists a constant \( C > 0 \) such that
\[
\| \partial_{x'}^j f \|_{L^p(\mathbb{R}^n)} \leq C\| f \|_{L^{2}(\mathbb{R}^n)}^{\frac{1}{2}} \| \partial_{x'}^k f \|_{L^p(\mathbb{R}^n)}^{\frac{1}{2}},
\]
where \( \theta = \frac{1}{k}(j + \frac{n}{2} - \frac{3}{p}) \).
(ii) Let \(2 \leq p \leq \infty\) and let \(j\) and \(k\) be integers satisfying
\[
0 \leq j < k, \quad k > j + n \left(\frac{1}{2} - \frac{1}{p}\right).
\]
Then there exists a constant \(C > 0\) such that
\[
\|\partial_j^l f\|_{L^p(\Omega)} \leq C\|f\|_{H^k(\Omega)}.
\]

(iii) If \(f \in H^{n-1}(\Omega)\) and \(f = f(x')\) is independent of \(x_n\), then
\[
\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|\partial_x^m f\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_x^{m-1} f\|_{L^2(\Omega)}^{\frac{1}{2}}.
\]

Proof. The inequality in (i) is a special case of the Galiardo-Niremberg-Sobolev inequality which can be proved using Fourier transform. Inequality in (ii) can be obtained by (i) and the standard extension argument. As for (iii), since
\[
\|f\|_{L^p(\Omega)} = \|f\|_{L^p(\mathbb{R}^{n-1})}, \quad 1 \leq p \leq \infty, \quad \|\partial_x^m f\|_{L^2(\Omega)} = \|\partial_x f\|_{L^2(\mathbb{R}^{n-1})},
\]
the inequality is a consequence of (i) with \(n = n - 1, p = \infty, j = 0\) and \(k = n - 1\). \(\square\)

Lemma 8.3 (i) Let \(m\) and \(m_k, k = 1, \ldots, l\) be nonnegative integers and let \(\alpha_k, k = 1, \ldots, l\) be multi-indexes. Suppose that
\[
m \geq \left\lceil \frac{n}{2} \right\rceil + 1, \quad 0 \leq |\alpha_k| \leq m_k \leq m + |\alpha_k|, \quad k = 1, \ldots, l,
\]
and
\[
m_1 + \cdots + m_l \geq (l - 1)m + |\alpha_1| + \cdots + |\alpha_l|.
\]
Then there exists a constant \(C > 0\) such that
\[
\|\partial_x^{\beta_1} f_1 \cdots \partial_x^{\beta_l} f_l\|_2 \leq C \prod_{1 \leq k \leq l} \|f_k\|_{H^{m_k}}.
\]

(ii) Let \(1 \leq k \leq m\). Suppose that \(F(x, t, y)\) is a smooth function on \(\Omega \times [0, \infty) \times I\), where \(I\) is a compact interval in \(\mathbb{R}\). Then for \(|\alpha| + 2j = k\) there hold
\[
\|\partial_x^\alpha \partial_t^j F(x, t, f_1)\|_{2} \leq \begin{cases} 
C_0(t, f_1(t))\|f_2\|_{k-1} + C_1(t, f_1(t))\{1 + \|DF_1\|_{m-1}^{\alpha+j-1}\} \|DF_1\|_{m-1}\|f_2\|_{k}, \\
C_0(t, f_1(t))\|f_2\|_{k-1} + C_1(t, f_1(t))\{1 + \|DF_1\|_{m-1}^{\alpha+j-1}\} \|DF_1\|_{m}\|f_2\|_{k-1}.
\end{cases}
\]

Here
\[
C_0(t, f_1(t)) = \sum_{(\beta, l) \leq (\alpha, j)} \sup_x |(\partial_x^\beta \partial_t^l F)(x, t, f_1(x, t))|,
\]
and
\[
C_1(t, f_1(t)) = \sum_{(\beta, l) \leq (\alpha, j)} \sup_x |(\partial_x^\beta \partial_t^l \partial_y^p F)(x, t, f_1(x, t))|.
\]

(iii) Let \(m \geq \lfloor n/2 \rfloor + 1\) then there exist constants \(C, C' > 0\) such that
\[
\|f_1 \cdot f_2\|_{H^m} \leq C\|f_1\|_{H^m}\|f_2\|_{H^m},
\]
and when \([f_1]_m \leq 1\),
\[
[f_1 \cdot f_2]_m \leq C'[f_1]_m[f_2]_m.
\]
Proof of previous lemma can be found in [9, 10]. We recall that $u(t)$ is decomposed into

$$u = u_1 + u_1 + \sigma u_0 + u_{\infty},$$

and we write

$$\sigma = \sigma_1 + \sigma_\infty, \quad \phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty,$$

$$u_\infty = T(\phi_\infty, w_\infty) = u_1 + u_{\infty}. $$

Before investigating the nonlinearities we present some basic estimates.

**Lemma 8.4** Let $u(t) = T(\phi(t), w(t)) = (\sigma u_0(t)) + u_\infty(t)$ be solution of (4.1) in $F(\tau)$. The following estimates hold for all $t \in [0, \tau]$ with $C > 0$ independent of $\tau$.

(i) \[ \|\sigma_\infty(t)\|_2 \leq C(1 + t)^{-\frac{n-1}{2}} M(t), \]

(ii) \[ \|D\sigma_\infty(t)\|_{m-1} + \|\sigma_\infty(t)\|_m \leq C(1 + t)^{-\frac{n+1}{2}} M(t), \]

(iii) \[ \|\phi(t)\|_m + \|w(t)\|_m \leq C(1 + t)^{-\frac{n+1}{2}} M(t), \]

(iv) \[ \|\sigma_\infty(t)\|_{\infty} \leq C(1 + t)^{-\frac{n}{2}} M(t), \]

(v) \[ \|\sigma_\infty(t)\|_{\infty} \leq C(1 + t)^{-\frac{n+1}{2}} M(t), \]

(vi) \[ \|\phi(t)\|_{\infty} + \|w(t)\|_{\infty} \leq C(1 + t)^{-\frac{n}{2}} M(t). \]

**Proof.** Estimates (i), (ii) and (iii) immediately follow from definition of $M(t)$. As for (iv), we see from Lemma 8.2 (iii) and Lemma 5.3 (iii) that

\[ \|\sigma_\infty(t)\|_{\infty} \leq C\|\sigma_\infty(t)\|_{\infty} \|D\sigma_\infty(t)\|_{\frac{1}{2}} \leq C\|\sigma_\infty(t)\|_2 \|\sigma_\infty(t)\|_{\frac{1}{2}} \leq C(1 + t)^{-\frac{n}{2}} M(t). \]

This shows (iv). Since $\|\sigma_\infty(t)\|_{\infty} \leq C\|u_\infty(t)\|_{L^m}$ by Lemma 8.2 (ii) we get (v) from (ii). Estimate (vi) now follows from (iv) and (v). This completes the proof.

First, we consider the estimates on $Q_0 F$.

**Proposition 8.5** Let $u(t)$ be a solution of (4.1) in $F(\tau)$ such that $M(t) \leq 1$ for all $t \in [0, \tau]$. There hold the following estimates for all $t \in [0, \tau]$ with $C > 0$ independent of $\tau$.

(i) \[ \|\phi \text{div} w\|_{l} \leq C \left\{ \begin{array}{ll}
(1 + t)^{-\frac{n+1}{2}} M(t)^2 + (1 + t)^{-\frac{n}{2}} M(t) \|Dw_\infty(t)\|_{m}, & l = m, \\
(1 + t)^{-\frac{n+1}{2}} M(t)^2, & l = m - 1, 
\end{array} \right. \]

(ii) \[ \|w \cdot \nabla (\sigma_\infty \phi_0 + \phi_\infty)\|_m \leq C(1 + t)^{-\frac{n+1}{2}} M(t)^2, \]

(iii) \[ \|w \cdot \nabla \phi_\infty\|_{m-1} \leq C(1 + t)^{-\frac{n+1}{2}} M(t)^2, \]

(iv) \[ |(\text{div} \left( \frac{P_t(p_p)}{\gamma p_p} w \right), |\partial_t \partial_2^2 \phi_\infty^2|)| \leq C(1 + t)^{-\frac{n+1}{2}} M(t) D_\infty(t), \]

for $2j + k \leq m$. 

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\[(v)\quad \|\left[\frac{\partial_i^k}{\partial_x^k}, w\right] \cdot \nabla \phi\|_2 \leq C(1 + t)^{-\frac{5}{2}} M(t)^2 + (1 + t)^{-\frac{n-1}{4}} M(t) \sqrt{D_\infty(t)},\]
for \(2j + k \leq m,\)
\[(vi)\quad \|T_{j,k}(\phi w)\|_2 \leq (1 + t)^{-\frac{n+1}{4}} M(t)^2,\]
for \(2j + k \leq m.\)

**Proof.** By Lemma 5.3 (iii) we have
\[\|\partial^k \sigma_\ast(t)\|_m \leq \|\partial^k \sigma_\ast(t)\|_{m-1} + \|\partial \sigma_\ast(t)\|_{m-2} \leq \|\sigma_\ast(t)\|_{m-1}.\]
We use this estimate and others that come from properties of \(\mathcal{Q}(t)\) and \(\mathcal{P}(t),\) e.g.,
\[\|\partial^k \sigma_\ast\|_2 \leq \|\partial^k \sigma_\ast\|_2, \quad k = 1, \ldots,\]
and
\[\|\nabla u_1\|_m \leq C[u_1]_m,\]
together with Lemma 8.3 and Lemma 8.4 to obtain estimates (i)–(vi).

In the case of estimates (i)–(iii), we first use the following expansions and then apply above estimates:
\[
\phi \text{div} w = \sigma_\ast \phi^{(0)} (w) : 1 \partial_{x_1} \sigma_\ast + \sigma_\ast \phi^{(0)} \text{div} w_* + \phi_* \text{div} w_*,
\]
\[
w \cdot \nabla (\sigma_\ast \phi^{(0)} + \phi_1) = \sigma_\ast \phi^{(0)} (w) : 1 \partial_{x_1} \sigma_\ast + \sigma_\ast \phi^{(0)} \nabla \sigma_\ast + w_* \nabla \sigma_\ast \phi^{(0)} + w_* \phi_1 \nabla \phi_1 + w_* \nabla \phi_1,
\]
\[
w \cdot \nabla \phi_\infty = \sigma_\ast \phi^{(0)} (w) : 1 \partial_{x_1} \phi_\infty + w_* \nabla \phi_\infty.
\]
This concludes the proof.

Second, we consider \(\tilde{Q} F = T(0, f).\) Recall that \(\tilde{Q} F\) is written in the form
\[
\tilde{Q} F = \tilde{F}_0 + \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3.
\]
Here, \(\tilde{F}_l = T(0, f_l),\) \(l = 0, 1, 2, 3,\) with
\[
f_0 = -w \cdot \nabla w - f_1(\rho_p, \phi) \Delta \sigma_* w^{(0)} : 1 e_1 - f_2(\rho_p, \phi) \nabla (\partial_{x_1} \sigma_* w^{(0)} : 1)
\]
\[+ f_{01}(x_n, t, \phi) \sigma_* + f_{02}(x_n, \phi) \nabla \sigma_* + f_{03}(x_n, t, \phi) \phi_*,
\]
\[
f_1 = -f_1(\rho_p, \phi) \Delta w_* - \text{div} (f_1(\rho_p, \phi) \nabla w_*) + T(\nabla w_*) \nabla (f_1(\rho_p, \phi)),
\]
\[
f_2 = -f_2(\rho_p, \phi) \text{div} w_* - \nabla (f_2(\rho_p, \phi) \text{div} w_*) + (\text{div} w_*) \nabla (f_2(\rho_p, \phi)),
\]
\[
f_3 = -f_3(x_n, \phi) \phi \nabla \phi_*.
\]
Here, \(\nabla w_*\) denotes the \(n \times n\) matrix \((\partial_{x_i} w_*);\) \(f_1 = \frac{\nu}{\rho_p (\gamma + \rho_p + \sigma)};\) \(f_2 = \frac{\nu}{\rho_p (\gamma + \rho_p + \sigma)};\) and \(f_{0l}(x_n, t, \phi), l = 1, 2, 3\) and \(f_3(x_n, \phi)\) are smooth functions of \(x_n, t\) and \(\phi.\)

**Proposition 8.6** Let \(u(t)\) be solution of (4.1) in \(\mathbb{Z}^n(\tau)\) and assume that \(\sup_{0 \leq z \leq \tau} \|u(z)\|_m \leq \varepsilon_5\) and \(M(t) \leq 1\) for all \(t \in [0, \tau].\) The following estimates hold for all \(t \in [0, \tau]\) with \(C > 0\) independent of \(\tau.\)

(i) \[\|\tilde{Q} F(t)\|_{m-2} \leq C(1 + t)^{-\frac{n-1}{2}} M(t)^2,\]

(ii) \[\|f_0(t)\|_m \leq C((1 + t)^{-\frac{n-1}{2}} M(t)^2 + (1 + t)^{-\frac{n-1}{2}} M(t)) \|Dw_\infty(t)\|_m,\]
we can find

Since

First, we treat

As in the proof of Proposition 8.5 (i) one can estimate

in the case 2

We thus conclude that

Proof. Since \( [v(t)]_m \leq \varepsilon_5 \) we see that \( \mathcal{Q}F(t) \) is smooth. Estimates (i)–(iii) can be obtained in similar manner to the proof of Proposition 8.5 and we omit the proof.

Let us prove estimate (iv). Let \( 2j + k = m \) and let \( v \in H^1_\delta \). If \( k \geq 1 \) then we see from (iii) that

\[
\| (T_{j,k} f, v) \|_{H^{-1}} = | - (T_{j,k-1} f, \partial_x v) | \leq \| T_{j,k-1} f \|_2 \| \partial_x v \|_v \leq C \{(1 + t)^{-\frac{2}{3}} M(t)^2 + (1 + t)^{-\frac{n+1}{2}} M(t) \| Dw_\infty(t) \|_m \} \| v \|_{H^1_\delta}.
\]

We thus conclude that

\[
\| T_{j,k} f \|_{H^{-1}} \leq C \{(1 + t)^{-\frac{2}{3}} M(t)^2 + (1 + t)^{-\frac{n+1}{2}} M(t) \| Dw_\infty(t) \|_m \},
\]

in the case \( 2j + k = m, k \geq 1 \) and \( l = 1, 2, 3 \).

If \( k = 0, \) i.e., \( m = 2j \), we write \( \langle \partial^2_{j} f, v \rangle_{-1} \) as

\[
\langle \partial^2_{j} f, v \rangle_{-1} = (\partial^2_{j} (f_1 (\rho, \phi) \nabla w_\ast), \nabla v) + (\partial^2_{j} (T (\nabla w_\ast) \partial_\rho f_1 (\rho, \phi) \nabla \rho), v) + (\partial^2_{j} (T (\nabla w_\ast) \partial_\phi f_1 (\rho, \phi) \nabla \phi, v) + \nabla (\nabla w_\ast) \partial_\rho f_1 (\rho, \phi) \nabla v + \nabla (\nabla w_\ast) \partial_\phi f_1 (\rho, \phi) \nabla \phi, v) \equiv \sum_{i=1}^{6} I_i.
\]

As for \( I_1 \), we have

\[
| I_1 | \leq \| \partial^2_{j} (f_1 (\rho, \phi) \nabla w_\ast) \|_2 \| \nabla v \|_2.
\]

As in the proof of Proposition 8.5 (i) one can estimate \( \| \partial^2_{j} (f_1 (\rho, \phi) \nabla w_\ast) \|_2 \) to obtain

\[
| I_1 | \leq C \{(1 + t)^{-\frac{2n+1}{2}} M(t)^2 + (1 + t)^{-\frac{3n+1}{2}} M(t) \| Dw_\infty(t) \|_m \} \| v \|_{H^1_\delta}.
\]

Similarly, we have

\[
| I_2 | \leq C \{(1 + t)^{-\frac{2n+1}{2}} M(t)^2 + (1 + t)^{-\frac{3n+1}{2}} M(t) \| Dw_\infty(t) \|_m \} \| v \|_{H^1_\delta}.
\]

Next, we consider \( I_3 \) which we write as follows:

\[
I_3 = (\partial_\rho f_1 (\rho, \phi) [\partial^2_{j} (T (\nabla w_\ast)) \nabla \phi, v) + (\partial^2_{j} (\partial_\phi f_1 (\rho, \phi) [\partial^2_{j} (\nabla w_\ast) \nabla \phi, v) \equiv J_1 + J_2.
\]

First, we treat \( J_1 \), we have

\[
| [\partial^2_{j} (\nabla w_\ast) \nabla \phi) | \leq C \sum_{l=0}^{j-1} | \partial^2_{j} \nabla \phi | \partial^{j-l}_{l} \partial_x w_\ast |.
\]

Since

\[
\frac{1}{2} - \frac{m - 1 - 2l}{n} + \frac{1}{2} - \frac{m - 2(j - l)}{n} = 1 - \frac{m - 1}{n} < 1,
\]

we can find \( p_{1l}, p_{2l} \geq 2 \) satisfying

\[
\frac{1}{p_{1l}} > \frac{1}{2} - \frac{m - 1 - 2l}{n}, \quad \frac{1}{p_{2l}} > \frac{1}{2} - \frac{m - 2(j - l)}{n}, \quad \frac{1}{2} \leq \frac{1}{p_{1l}} + \frac{1}{p_{2l}} < 1.
\]
Now, we take number $p_{3l} \geq 2$ satisfying $\frac{1}{p_{3l}} = 1 - (\frac{1}{p_{31}} + \frac{1}{p_{32}}) > 0$. It then follows from Lemma 8.2 (ii) that

$$
|\partial_t f_1 (\rho, \phi)| \leq C \sum_{l=0}^{j-1} \|\partial_t^l \partial_x \phi\|_{p_{31}} \|\partial_t^j \partial_x w_*\|_{p_{32}} \|v\|_{p_{31}}
$$

$$
\leq C \sum_{l=0}^{j-1} \|\partial_t^l \partial_x \phi\|_{H^{m-1-l}} \|\partial_t^j \partial_x w_*\|_{H^{m-2(j-l)}} \|v\|_{H^1} \leq C[\phi]_m \{[\partial_x w_1]_m + \|Dw_\infty\|_m\} \|v\|_{H^1}.
$$

Using Lemma 8.4 we conclude that

$$
|J_1| \leq C((1 + t)^{-\frac{3}{2}} M(t)^2 + (1 + t)^{-\frac{n+1}{2}} M(t)\|Dw_\infty(t)\|_m) \|v\|_{H^1}.
$$

Second, we estimate $J_2$. By Lemma 8.3 (i) we have for $m_k = m - 1 - 2k$ and $m_l = m - 1 - 2l$ that

$$
\|\partial_t^i \partial_x f_1 (\rho, \phi)\|_{L^2} \leq C \sum_{k+1+l=j} \|\partial_t^k \partial_x f_1 (\rho, \phi)\|_{m_k} \|\partial_t^j \partial_x w_*\|_{m_l}
$$

$$
\leq C[\partial_t \phi]_{m-1} \|\partial_x w_*\|_{m-1}.\]

Therefore, we obtain

$$
|J_2| \leq C[\partial_t \phi]_{m-1} \|\partial_x w_*\|_{m-1} \|v\|_{L^2}.
$$

By Lemma 8.3 (ii) we get

$$
\|\partial_x w_*\|_{m-1} \leq C\{\|\partial_x w_*\|_{\infty} \|\partial_x \phi\|_{m-1} + \|D\partial_x w_*\|_{m-2} \|\partial_x \phi\|_{m-1}\}
$$

$$
\leq C\{\|\partial_x w_*\|_{H^m} + \|Dw_\infty\|_m\} \|\partial_x \phi\|_{m-1} \leq C((1 + t)^{-\frac{3}{2}} M(t)^2 + (1 + t)^{-\frac{n+1}{2}} M(t)\|Dw_\infty(t)\|_m).\}
$$

As for $[\partial_t \phi]_{m-1}$, we see from (6.3) and Proposition 7.9 that

$$
[\partial_t \phi]_{m-1} \leq C\{[\partial_t \sigma_*]_{m-1} + [\partial_t \phi_*]_{m-1}\} \leq C\{[A]_m(t)]_{m-1} + [\partial_t \sigma_*]_{m-1} \}
$$

$$
+ [\partial_t \phi_*]_{m-1} + [\partial_x w_*]_{m-1} + [\partial_x \sigma_*]_{m-1} + [Q_\infty(P_{\infty} F)]_{\infty} m-1 + [Q_\infty P_{\infty} F]_{m-1}.\}
$$

Using

$$
P_\infty F = F - [Q_\infty F]_\infty u^{(0)} - \{P F - [Q_\infty P F]_\infty u^{(0)}\},
$$

and

$$
[Q_\infty P_{\infty} F]_\infty = [Q_\infty F]_\infty - [Q_\infty P F]_\infty,
$$

together with Lemma 8.4 we get

$$
[\partial_t \phi]_{m-1} \leq C((1 + t)^{-\frac{n+1}{2}} M(t) + \|\partial_t \sigma_*]_{m-1} + [Q_\infty F]_{m-1}).\}
$$

Since $2j = m$, we have $[m-1] = [m-2]$, and hence, by properties of $\partial_t \sigma_*$,

$$
[\partial_t \sigma_*]_{m-1} \leq C[F(t)]_{m-2}.\}
$$

It then follows from Propositions 8.5 (i)–(iii) and 8.6 (i) that

$$
[F(t)]_{m-2} + [Q_\infty F(t)]_{m-1} \leq 2[F(t)]_{m-1} + [Q F(t)]_{m-2} \leq C(1 + t)^{-\frac{n+1}{2}} M(t)^2,
$$

which implies

$$
[\partial_t \phi]_{m-1} \leq C(1 + t)^{-\frac{n+1}{2}} M(t).\}
$$

We thus conclude

$$
|J_2| \leq C((1 + t)^{-\frac{n+1}{2}} M(t)^3 + (1 + t)^{-\frac{n+1}{2}} M(t)^2 \|Dw_\infty(t)\|_m) \|v\|_{L^2}.
$$

Consequently,

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\[ |I_3| \leq C \{(1 + t)^{-\frac{2}{4}} M(t)^2 + (1 + t)^{-\frac{n+1}{n-1}} M(t) \| D w_{\infty}(t) \|_m \}\|v\|_{H^1_0}. \]

Since

\[ |I_4| \leq \|T(\nabla w_*)\partial_\phi f_1(\rho_p, \phi)\partial_j^I \phi, \text{div} v \| \leq C \| \nabla w_* \|_\infty \| \partial_j^I \phi \|_2 \| \nabla v \|_2, \]

and by Lemma 8.3 (i),

\[ |I_5| \leq \|T(\nabla^2 w_*)\partial_\phi f_1(\rho_p, \phi)\partial_j^I \phi, v \| \leq C \| \partial_j^I \phi \|_2 \| \nabla^2 w_* \|_2 \leq C \| \partial_j^I \phi \|_2 \| \nabla w_* \|_H^m \| v \|_{H^1}, \]

\[ |I_6| \leq \|T(\nabla w_*)\nabla_\rho_\phi \partial_\phi f_1(\rho_p, \phi)\nabla_\phi \partial_j^I \phi, v \| \leq C \| \nabla w_* \|_\infty \| \partial_j^I \phi \|_2 \| v \|_{H^1} \| \rho_p + \phi \|_H^m, \]

we obtain by Lemmas 8.2 (ii) and 8.4,

\[ |I_4| + |I_5| + |I_6| \leq C \{(1 + t)^{-\frac{2}{4}} M(t)^2 + (1 + t)^{-\frac{n+1}{n-1}} M(t) \| D w_{\infty}(t) \|_m \}\|v\|_{H^1_0}. \]

Therefore, we arrive at

\[ \| \partial_j^I f_1 \|_{H^1} \leq C \{(1 + t)^{-\frac{2}{4}} M(t)^2 + (1 + t)^{-\frac{n+1}{n-1}} M(t) \| D w_{\infty}(t) \|_m \}. \]

Clearly, one can get the same estimate for \( \| \partial_j^I f_2 \|_{H^1} \). Concerning \( \| \partial_j^I f_3 \|_{H^1} \), one can write it as

\[ \partial_j^I f_3 = -[\partial_j^I f_3(x_n, \phi) \nabla \phi_* - \nabla (f_3(x_n, \phi) \partial_j^I \phi_*), \nabla (f_3(x_n, \phi) \partial_j^I \phi*)], \]

and thus the desired estimate is obtained analogously to the one for \( \| \partial_j^I f_1 \|_{H^1} \). This completes the proof.

\[ \square \]

**Proof of Proposition 8.1.** Since \( \tilde{Q}P_{\infty}F \|_{m-2} \leq C \| F \|_{m-2} \), assertion (i) follows from Propositions 8.5 (i)–(iii) and 8.6 (i).

Let us consider \( \tilde{R}(t) \). We know that there holds

\[ \tilde{R}(t) \leq C \left( \sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{m-1} R^{(0)}_p \right). \]

Let us first show some basic estimates coming from Propositions 8.5, 8.6 and properties of \( P(t) \):

\[ [Q_0 F]_{m-1} \leq C (1 + t)^{-\frac{2n+1}{n-1}} M(t)^2, \tag{8.1} \]

\[ [\tilde{Q} F]_{m-1} \leq C \{(1 + t)^{-\frac{2n+1}{n-1}} M(t)^2 + (1 + t)^{-\frac{n+1}{n-1}} M(t) \| D w_{\infty}(t) \|_m \}, \tag{8.2} \]

\[ [PF]_{m-1} \leq C [F]_{m-1}, \tag{8.3} \]

Moreover, there holds

\[ [Q_0 T_{j,k} F]_{\infty} = -[\text{div} T_{j,k}(\phi w)]_{\infty} = -[\text{div} 'T_{j,k}(\phi' w)]_{\infty}, \tag{8.4} \]

since \( w \in H^1_0 \).

Let us begin with \( R^{(1)}_{j,k} \). We write

\[ R^{(1)}_{j,k} = \frac{\alpha_0}{\gamma_2} ([Q_0 T_{j,k} F]_{\infty}, T_{j,k} \sigma_{\infty}) - \frac{\alpha_0}{\gamma_2} ([Q_0 T_{j,k} (P F)]_{\infty}, T_{j,k} \sigma_{\infty}) + \tilde{R}^{(1)}_{j,k} - (T_{j,k}([Q_0 F]_{\infty} u^{(0)}), T_{j,k} u_{\infty}) \Omega \]

\[ - (T_{j,k} (P F), T_{j,k} u_{\infty}) \Omega + (T_{j,k}([Q_0 (P F)]_{\infty} u^{(0)}), T_{j,k} u_{\infty}) \Omega = \sum_{l=1}^{6} I_l. \]

Here,

\[ I_3 = \tilde{R}^{(1)}_{j,k} = (T_{j,k} F, T_{j,k} u_{\infty}) \Omega, \]
when \(2j + k \leq m - 1\), and

\[
I_3 = I_{3,j,k}^{(1)} = -(T_{j,k}(\phi \text{div } u), T_{j,k}\phi\phi_{\infty}\frac{P'(\rho_p)}{\gamma^4\rho_p}) + \frac{1}{2}(\text{div} (\frac{P'(\rho_p)}{\gamma^4\rho_p} w), |T_{j,k}\phi\phi_{\infty}|^2)
\]

\[
- (w\nabla T_{j,k}(\sigma, \phi^{(0)} + \phi), T_{j,k}\phi\phi_{\infty}\frac{P'(\rho_p)}{\gamma^4\rho_p}) - ([T_{j,k}, w]\nabla \phi, T_{j,k}\phi\phi_{\infty}\frac{P'(\rho_p)}{\gamma^4\rho_p}),
\]

\[
+ (T_{j,k}f_0, T_{j,k}w_{\infty}\rho_p) + \sum_{l=1}^{3} (T_{j,k}f_l, T_{j,k}w_{\infty}\rho_{pl}) - 1.
\]

when \(2j + k = m\).

We first consider \(I_3\). If \(2j + k \leq m - 1\), then by applying (8.1) and (8.2), we have

\[
\sum_{2j + k \leq m - 1} |(T_{j,k}F, T_{j,k}u_{\infty})| \leq C \|F\|_{m-1}\|u_{\infty}\|_{m-1} \leq C\{(1 + t)^{-\frac{7}{3}} M(t)^3 + (1 + t)^{-\frac{3}{4}} M(t)D_\infty(t)\}.
\]

Here, we used relation \(a^2b \leq \frac{1}{2}(a^3 + ab^2)\) and Lemma 8.4 (ii).

In the case \(2j + k = m\), we use Lemma 8.4 to calculate

\[
|(w\nabla T_{j,k}(\sigma, \phi^{(0)} + \phi), T_{j,k}\phi\phi_{\infty}\frac{P'(\rho_p)}{\gamma^4\rho_p})| \leq C\|w\|_\infty\{(\|\sigma\|_m + \|\phi_{1}\|_m)\|\phi_{\infty}\|_m \leq C(1 + t)^{-n} M(t)^3.
\]

From above estimate and Propositions 8.5 (i), (iv), (v) and 8.6 (ii), (iv) we see that

\[
\sum_{2j + k \leq m} |I_3| \leq C\{(1 + t)^{-\frac{7}{3}} M(t)^3 + (1 + t)^{-\frac{3}{4}} M(t)D_\infty(t)\}.
\]

We next consider \(J_5\). If \(2j + k \leq m - 1\), then by (8.3) we see that

\[
\sum_{2j + k \leq m - 1} |(T_{j,k}(P_F), T_{j,k}u_{\infty})| \leq C\|P_F\|_{m-1}\|u_{\infty}\|_{m-1} \leq C\|F\|_{m-1}\|u_{\infty}\|_{m-1}.
\]

If \(2j + k = m\) and \(k \geq 1\), then from properties of \(P(t)\) we obtain

\[
\sum_{2j + k = m, k \geq 1} |(T_{j,k}(P_F), T_{j,k}u_{\infty})| \leq \sum_{2j + k = m, k \geq 1} C\|T_{j,k-1}(P_F)\|_2\|T_{j,k}u_{\infty}\|_2 \leq C\|F\|_{m-1}\|u_{\infty}\|_{m}.
\]

In the case \(2j = m\), we write

\[
|\langle \partial^j u_{\infty} \rangle| \leq C\{\|\partial^j u_{\infty} \| + \|P\partial^j u_{\infty} \| \} \leq C\|F\|_{m-2}\|u_{\infty}\|_{m} + \|P\partial^j u_{\infty} \| \}
\]

To estimate \(\langle P\partial^j F, \partial^j u_{\infty} \rangle\), we write it as

\[
\langle P\partial^j F, \partial^j u_{\infty} \rangle = \langle \partial^j F, P\partial^j u_{\infty} \rangle.
\]

Using integration by parts, we have

\[
\langle \partial^j Q_0 F, P\partial^j u_{\infty} \rangle = \langle \partial^j (\phi u), \nabla \left( Q_0 (P\partial^j u_{\infty}) \frac{P'(\rho_p)}{\gamma^4\rho_p} \right) \rangle.
\]

Since

\[
\|\nabla \left( Q_0 (P\partial^j u_{\infty}) \frac{P'(\rho_p)}{\gamma^4\rho_p} \right) \|_2 \leq C\|\partial^j u_{\infty}\|_2,
\]

by properties of \(P\), we see from Proposition 8.5 (vi) that

\[
|\langle \partial^j Q_0 F, P\partial^j u_{\infty} \rangle| \leq C(1 + t)^{-\frac{7}{3}} M(t)^3.
\]

Since \(\tilde{Q}P\partial^j u_{\infty} \in H^1_0\), one can estimate \(\|\partial^j \tilde{Q} F, \tilde{P}\partial^j u_{\infty} \|\) using Proposition 8.6 (ii), (iv) to obtain

\[
\sum_{2j = m} |\langle \partial^j \tilde{Q} F, \tilde{P}\partial^j u_{\infty} \rangle| \leq C\{(1 + t)^{-\frac{7}{3}} M(t)^3 + (1 + t)^{-\frac{3}{4}} M(t)D_\infty(t)\}.
\]

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Therefore, we have
\[\sum_{2j+k\leq m} |I_5| \leq C\{(1 + t)^{-\frac{2p}{p-1}} M(t)^3 + (1 + t)^{-\frac{p+1}{p-1}} M(t) D_{\infty}(t)\}.
\]

As for \(I_4\), from (8.4) we compute
\[-\langle T_{j,k}(\langle Q_0 F \rangle_\infty u^{(0)}), T_{j,k}u_\infty \rangle_\Omega = \sum_{i=0}^{j} \sum_{i=0}^{j} |\langle \text{div}' T_{i,k}(\phi w') \rangle_\infty T_{j-i,0}u^{(0)}, T_{j,k}u_\infty \rangle_\Omega.
\]

Since
\[\|\langle \text{div}' T_{i,k}(\phi w') \rangle_\infty T_{j-i,0}u^{(0)}\|_2 \leq C\|T_{i,k}(\phi w')\|_2,
\]
we see using Proposition 8.5 (vi) that
\[\sum_{2j+k\leq m} |I_4| \leq C(1 + t)^{-\frac{2p}{p-1}} M(t)^3.
\]

As for \(I_1\), using (8.4) and integration by parts we obtain
\[\frac{\alpha_0}{\gamma^\frac{p}{2}} (\langle Q_0 T_{j,k} F \rangle_\infty, T_{j,k} \sigma_\infty) = -\frac{\alpha_0}{\gamma^\frac{p}{2}} (\langle \text{div}' T_{j,k}(\phi w') \rangle_\infty, T_{j,k} \sigma_\infty) \leq C\|T_{j,k}(\phi w')\|_2 \|
\text{\nabla}' T_{j,k} \sigma_\infty\|_2,
\]
and thus by Proposition 8.5 (vi) and Lemma 8.4 (ii) we get
\[\sum_{2j+k\leq m} |I_1| \leq C(1 + t)^{-\frac{2p}{p-1}} M(t)^3.
\]

As for \(I_2\), in the case \(1 \leq 2j + k \leq m\) we treat it analogously to \(I_5\) to show
\[\frac{\alpha_0}{\gamma^\frac{p}{2}} (\langle Q_0 T_{j,k} (\mathcal{P} F) \rangle_\infty, T_{j,k} \sigma_\infty) \leq C\|F\|_{m-1} \|
\partial_x' \sigma_\infty\|_{m-1} + \|
\langle Q_0 \partial_t^{[\frac{m}{2}]F}_0, \partial_t^{[\frac{m}{2}]\sigma_\infty}\|_2.
\]

We further estimate
\[\|
\langle [Q_0 \partial_t^{[\frac{m}{2}]F}_0, \partial_t^{[\frac{m}{2}]\sigma_\infty}\|_2 \|
\partial_t^{[\frac{m}{2}]\sigma_\infty\|_2.
\]

Using Plancherel theorem we have
\[\|\mathcal{P}(t)(\partial_t^{[\frac{m}{2}]F}(t))\|_2 \leq \|\tilde{\chi}_1 (\partial_t^{[\frac{m}{2}]\tilde{F}(t)}, u_0^{(t)}(t))\|_2.
\]

Therefore, using above relation, analogously to \(I_5\), we estimate
\[\|\mathcal{P}(t)(\partial_t^{[\frac{m}{2}]F}(t))\|_2 \leq \|\mathcal{P}(t)(Q_0 \partial_t^{[\frac{m}{2}]F}(t))\|_2 + \|\mathcal{P}(t)(\langle Q_0 \partial_t^{[\frac{m}{2}]F}(t)\|_2
\]
\[\leq C\|\partial_t^{[\frac{m}{2}]\phi w}\|_2 + [F_0]_{m} + \sum_{l=1}^{3} \|\partial_t^{[\frac{m}{2}]f_l\|_H^{-1}.
\]

In the case \(j = k = 0\) we see from Lemma 6.1 (ii) and properties of \(\mathcal{P}(t)\),
\[\frac{\alpha_0}{\gamma^\frac{p}{2}} (\langle Q_0 (\mathcal{P} F) \rangle_\infty, \sigma_\infty) \leq C\|\partial_x' \sigma_\infty\|_2 (\|\phi w'\|_2 + \|F\|_2) \leq (1 + t)^{-\frac{2p}{p-1}} M(t)^3.
\]

Therefore, using Propositions 8.5 (vi) and 8.6 (ii), (iv) we obtain
\[\sum_{2j+k\leq m} |I_2| \leq C\{(1 + t)^{-\frac{2p}{p-1}} M(t)^3 + (1 + t)^{-\frac{p+1}{p-1}} M(t) D_{\infty}(t)\}.
\]

As for \(I_6\), it can be treated in a way analogous to \(I_5\) in the case \(1 \leq 2j + k\), thus we get
\[\sum_{2j+k\leq m} |\langle T_{j,k}(\langle Q_0 (\mathcal{P} F) \rangle_\infty u^{(0)}), T_{j,k}u_\infty \rangle_\Omega| \leq C\|\langle F \rangle_{m-1} \|u_\infty\|_{m-1} + \|\mathcal{P}(\partial_t^{[\frac{m}{2}]F})\|_2 \|
\partial_t^{[\frac{m}{2}]\sigma_\infty\|_2,
\]
and

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\[
\sum_{2j+k \leq m} |I_6| \leq C\{(1 + t)^{-\frac{3\sigma}{2}} M(t)^3 + (1 + t)^{-\frac{\sigma+1}{2}} M(t)D_\infty(t)\}.
\]

We thus conclude
\[
R^{(1)}(t) \leq C\{(1 + t)^{-\frac{3\sigma}{2}} M(t)^3 + (1 + t)^{-\frac{\sigma+1}{2}} M(t)D_\infty(t)\}.
\]

It is straightforward to show that
\[
\tilde{R}(t) \leq C\{R^{(1)}(t) + \|F\|_{m-1}^2 + \|Q_0(P_\infty^{(0)} F)\|_{m-1}^2 + \|P_\infty F\|_{m-1}^2 + \|\phi \div w\|_m^2
\]
\[
+ \|w \cdot \nabla (\sigma_\ast \phi(0) + \phi_1)\|_m^2 + \sum_{2j+k \leq m} \|\|\partial^j_\sigma \phi_k, w \cdot \nabla \phi_\infty\|_2^2
\]
\[
+ \sum_{2j+k \leq m} \left| (\partial^j_\sigma \partial^k_\phi, w) \cdot [T_{j,k} \partial^{l+1}_x \phi_\infty]^2 \right|
\]
\[
+ \sum_{2j+k \leq m-1} \|Q_0 T_{j,k}(P_\infty^{(0)} F), T_{j,k} \sigma_\infty\| + \|Q(P_\infty F)\|_{m-1} \|\partial_x \sigma_\infty\|_{m-1} \}.
\]

From definition we have
\[
P_\infty^{(0)}(t) = I - P(t) - P_\infty,3,
\]
which together with
\[
\|Q_0 P_\infty,3 T_{j,k} F, T_{j,k} \sigma_\infty\| \leq C \|T_{j,k}(\phi w)\|_2 \|\|D \sigma_\infty\|_{m-1}.
\]
gives (analogously to previous computations)
\[
\sum_{2j+k \leq m-1} \|Q_0 T_{j,k}(P_\infty^{(0)} F), T_{j,k} \sigma_\infty\| \leq C(1 + t)^{-\frac{3\sigma}{2}} M(t)^3.
\]

Since
\[
\|P_\infty^{(0)} F\|_{m-1} + \|P_\infty F\|_{m-1} \leq C \|F\|_{m-1},
\]
using Propositions 8.5, 8.6 and Lemma 8.4 we obtain the desired estimate (ii) in Proposition 8.1. This completes the proof.

\[
\square
\]

9 Asymptotic behavior of \(\sigma_1(t)\)

In this section we show the asymptotic behavior of solutions of (4.1). In the case \(n = 2\) we prove that it is described by a solution of a 1-dimensional viscous Burgers equation. In the case \(n \geq 3\) we show that the asymptotic behavior is described by a linear heat equation, in fact, asymptotic leading term is the same as for the linearized problem.

In this section we assume that \(\nu \geq \nu_0\) and \(\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2\). Let us note that \(\sigma_1(t)\) is given by
\[
\sigma_1(t) = \mathcal{R}(t) w(t), \quad t \geq 0,
\]
where \(w(t)\) is a global in time solution of (4.1). Existence of \(w(t)\) was proved in Sections 3–8.

First let us treat the case \(n = 2\).

Lemma 9.1 Let \(n = 2\) and \(\sigma(t)\) is a solution of
\[
\partial_t \sigma - \kappa_1 \partial^2_x \sigma + \kappa_0 \partial_x \sigma + \omega_0 \partial_z (\sigma^2) = 0,
\]
where \(\kappa_0 \in \mathbb{R}, \kappa_1 > 0\) are the numbers given in (4.6) and \(\omega_0 = \frac{1}{L} \int_0^T \varphi(0) w^{(0),1}(z) - F(1)(z, u^{*(1)}(z))dz\) and \(\sigma_0 = |Q_0 \varphi_0| = |\varphi_0|\). Then we can write
\[
\sigma(t) = \mathcal{H}(t) \sigma_0 - \omega_0 \int_0^t \mathcal{H}(t - z) \partial_x (\sigma^2(z))dz.
\]

(9.2)
Theorem 9.2 Let \( n = 2 \). For any \( \delta > 0 \) there exists \( \varepsilon_7 > 0 \) such that if \( \| u_0 \|_{H^{1}\cap L^1} \leq \varepsilon_7 \), then
\[
\| \sigma_1(t) - \sigma(t) \|_2 \leq C(1 + t)^{-\frac{1}{2} + \frac{1}{2} \delta} \| u_0 \|_{H^{1}\cap L^1},
\]
for \( t \geq 0 \).

(3.7) now follows from (5.13) and Theorem 9.2. To prove Theorem 9.2, we employ the following well-known decay properties of \( \sigma(t) \).

Lemma 9.3 Let \( n = 2 \) and \( \sigma(t) \) is a solution of (9.1) with \( \| \sigma_0 \|_{H^{1}\cap L^1} \ll 1 \). Then
\[
\| \partial^k_{x_1} \sigma(t) \|_2 \leq C(1 + t)^{-\frac{1}{2} + \frac{1}{2} \delta} \| \sigma_0 \|_{H^{1}\cap L^1} \quad (k = 0, 1),
\]
\[
\| \sigma(t) \|_\infty \leq C(1 + t)^{-\frac{1}{2}} \| \sigma_0 \|_{H^{1}\cap L^1}.
\]

We introduce a quantity. Let \( \sigma_1(t) \) and \( \sigma(t) \) be solutions of (5.1) for \( s = 0 \) and (9.1), respectively. We define \( N(t) \) by
\[
N(t) = \sup_{0 \leq s \leq t} (1 + z)^{\frac{3}{2} - \delta} \| \sigma_1(z) - \sigma(z) \|_{H^{1}}.
\]

Theorem 9.2 would then follow if we could show that \( N(t) \leq C \| u_0 \|_{H^{1}\cap L^1} \).

Proof of Theorem 9.2. It is obvious that estimate holds for \( 0 \leq t < 1 \). Let us show that it holds for \( t \geq 1 \).

Assume \( t \geq 1 \). From (5.1) we have that for \( s = 0 \)
\[
\sigma_1(t) = e^{tA} \mathcal{P}(0) u_0 + \int_0^t e^{(t-z)A} \mathcal{P}(z) F(z)dz.
\] (9.3)

We next rewrite \( e^{(t-z)A} \mathcal{P}(z) F(z) \). By Lemma 6.1 (ii), we have
\[
\mathcal{P}(z) F(z) = -\partial_{x_1} [\phi w^1]_1 + \partial_{x_1} \mathcal{P}^{(1)}(z) F(z) + \partial_{x_1} \mathcal{P}^{(2)}(z) F(z)
\]
\[
= -a_{11}(z) \partial_{x_1} (\sigma_1^2) - \partial_{x_1} ([\phi w^1]_1 - [\phi^{(0)} w^{(0,1)}]_1 \sigma_1^2)
\]
\[
+ \partial_{x_1} \mathcal{P}^{(1)}(z)(\sigma_1 F_1(z) + F_2(z)) + \partial_{x_1} \mathcal{P}^{(2)}(z) F(z).
\]

Here \( a_{11}(z) = [\phi^{(0)} w^{(0,1)}(z)] \). Since
\[
\mathcal{P} \{ \mathcal{P}^{(1)}(z)(\sigma_1^2 F_1(z)) \} = \bar{\chi}_{\frac{1}{2}}(\sigma_1^2) F_1(z), u^{*(1)}(z) = \bar{\chi}_{\frac{1}{2}}(F_1(z), u^{*(1)}(z))(\sigma_1^2) = -a_{12}(z) \sigma_1^2,
\]
where \( a_{12}(z) = -\langle F_1(z), u^{*(1)}(z) \rangle \).

Using properties of \( e^{(t-z)A} \), we thus arrive at
\[
e^{(t-z)A} \mathcal{P}(z) F(z) = -a_1(z)e^{(t-z)A} \partial_{x_1} (\sigma_1^2(z))
\]
\[
- e^{(t-z)A} \partial_{x_1} [\phi w^1]_1(z) - [\phi^{(0)} w^{(0,1)}(z)] \sigma_1^2(z)
\]
\[
+ e^{(t-z)A} h_5(z) + e^{(t-z)A} h_6(z),
\]

where \( a_1(z) = a_{11}(z) + a_{12}(z), \sup_{z \in \mathbb{R}} \| a_1(z) \| \leq C \) and
\[
h_5(z) = \partial_{x_1} \mathcal{P}^{(1)}(z) F_2(z) + \partial_{x_1} \mathcal{P}^{(2)}(z) F_2(z),
\]
\[
h_6(z) = \partial_{x_1} \mathcal{P}^{(2)}(z)(\sigma_1 F_1(z)).
\]

It then follows from (9.2) and (9.3) that
\[
\sigma_1(t) - \sigma(t) = \sum_{j=0}^6 I_j(t),
\]
where
\[ I_0(t) = e^{LA} \mathcal{P}(0)u_0 - \mathcal{H}(t)\sigma_0, \]
\[ I_1(t) = -\int_0^t \omega_0 \mathcal{H}(t - z) \partial_{x_1}(\sigma_1^2(z) - \sigma^2(z))dz, \]
\[ I_2(t) = -\int_0^t \omega_0 (e^{(t-z)A} - \mathcal{H}(t-z)) \partial_{x_1}(\sigma_1^2)dz, \]
\[ I_3(t) = -\int_0^t (a_1(z) - \omega_0) e^{(t-z)A} \partial_{x_1}(\sigma_1^2)dz, \]
\[ I_4(t) = -\int_0^t \partial_{x_1} e^{(t-z)A} ([\phi w^1]_1(z) - [\phi(0) w(0),1(z)]\sigma_1^2(z))dz, \]
\[ I_j(t) = \int_0^t e^{(t-z)A} h_j(z)dz, \quad j = 5, 6. \]

Let us show estimates on \( I_j, j = 0, \ldots, 6. \)

As for \( I_0, \) from (4.10) we see

\[ \| I_0(t) \|_{H^1} \leq Ct^{-\frac{3}{4}} \| u_0 \|_{L^1}. \]

Let us consider \( I_1(t). \) By Lemma 9.3, (5.12) and the definition of \( M(t) \) and \( N(t), \) we have

\[ \| (\sigma_1^2 - \sigma^2)(z) \| \leq \| (\sigma_1 + \sigma)(z) \| \| (\sigma_1 - \sigma)(z) \| \leq C(1 + z)^{-1+\delta} N(t) \| u_0 \|_{H^{m+L}1}, \]

for \( \| u_0 \|_{H^{m+L}1} \leq \varepsilon_3. \) Furthermore, by Lemma 8.2 (iii) we have \( \| (\sigma_1 - \sigma)(z) \|_{\infty} \leq C(1 + z)^{-\frac{3}{4}+\delta} N(t), \) and hence,

\[ \| \partial_{x_1}(\sigma_1^2 - \sigma^2)(z) \| \leq C (\| (\sigma_1 + \sigma)(z) \|_{\infty} \| \partial_{x_1}(\sigma_1 - \sigma)(\sigma_1 + \sigma)(z) \|_{\infty} \| \partial_{x_1}(\sigma_1 + \sigma)(z) \|_{2}) \]

\[ \leq C(1 + z)^{-\frac{3}{4}+\delta} \| u_0 \|_{H^{m+L}1} N(t). \]

It then follows from (4.8) that for \( k = 0, 1, \)

\[ \| \partial_{x_1}^k I_1(t) \| \leq C \left( \int_0^{\frac{1-t}{2}} (t-z)^{-\frac{3}{4}+\frac{k}{2}} (1 + z)^{-1+\delta}dz + \int_{\frac{1-t}{2}}^{1} (t-z)^{-\frac{3}{4}+\frac{k}{2}} (1 + z)^{-1+\delta}dz \right. \]

\[ + \int_{\frac{1-t}{2}}^{1} (t-z)^{-\frac{3}{4}+\frac{k}{2}} (1 + z)^{-\frac{3}{4}+\delta}dz \| \| u_0 \|_{H^{m+L}1} N(t) \leq C(1 + t)^{-\frac{3}{4}+\delta} \| u_0 \|_{H^{m+L}1} N(t). \]

As for \( I_2(t), \) we see from (4.11) that for \( k = 0, 1, \)

\[ \| \partial_{x_1}^k I_2(t) \| \leq C \left( \int_0^{\frac{1-t}{2}} (t-z)^{-\frac{3}{4}+\frac{k}{2}} \| \sigma_1^2(z) \|_1 dz \right. \]

\[ + \int_{\frac{1-t}{2}}^{1} (t-z)^{-\frac{3}{4}+\frac{k}{2}} \| \partial_{x_1}(\sigma_1^2(z)) \|_1 dz \right) \leq C(1 + t)^{-\frac{3}{4}+\delta} \| u_0 \|_{H^{m+L}1} N(t). \]

From Lemma 6.3 we have

\[ \leq C \left( \int_0^{\frac{1-t}{2}} (t-z)^{-\frac{3}{4}+\frac{k}{2}} (1 + z)^{-\frac{3}{4}+\delta}dz \right. \]

\[ + \int_{\frac{1-t}{2}}^{1} (t-z)^{-\frac{3}{4}+\frac{k}{2}} (1 + z)^{-1}dz \right) \| \partial_{x_1}(\sigma_1^2(z)) \|_1 dz \]

\[ \leq C(1 + t)^{-\frac{3}{4}+\delta} \| u_0 \|_{H^{m+L}1} N(t). \]

As for \( I_3(t), \) let us define \( b(t) = \int_0^t a_1(z) - \omega_0dz. \) Then \( \partial_t b(t) = a_1(t) - \omega_0 \) and \( b(0) = b(T) = 0. \) Since \( a_1(t + T) = a_1(t) \) we have \( \partial_t b(t + T) = \partial_t b(t) \) and thus \( b(t + T) = b(t). \) We arrive at \( \sup_{z \in J_T} |b(z)| \leq C. \) We write

\[ I_3(t) = -\int_0^t \partial_{x_1} e^{(t-z)A} \partial_{x_1}(\sigma_1^2(z))dz = -\left[ b(z)e^{(t-z)A} \partial_{x_1}(\sigma_1^2(z)) \right]_0^t \]

\[ + \int_0^t b(z) e^{(t-z)A} \partial_{x_1}(\sigma_1^2(z))dz \]
\[-b(t)\partial_{x_1}(\sigma_1^2(t)) - \int_0^t b(z)e^{(t-z)\Lambda} \Lambda \partial_{x_1}(\sigma_1^2(z))dz + \int_0^t b(z)e^{(t-z)\Lambda} \partial_5(\sigma_1^2(z))dz \equiv J_1(t) + J_2(t) + J_3(t).\]

From Lemma 6.3 (vii) we have for \(k = 0, 1,\)
\[
\|\partial_{x_1}^k J_1(t)\|_2 \leq C(1 + t)^{-\frac{k}{2}}M(t)^2.
\]

We see from (4.9) and Lemma 6.3 that
\[
\|J_2(t)\|_2 \leq C\{\int_0^\frac{t}{2} (1 + t - z)^{-\frac{3}{2}} \|\sigma_1^2(z)\|_1dz + \int_\frac{t}{2}^t (1 + t - z)^{-\frac{3}{2}} \|\partial_{x_1} \sigma_1^2(z)\|_1dz\}
\leq C(1 + t)^{-\frac{3}{2}}\|u_0\|^2_{H^{1\cap L^1}}.
\]

As for \(J_3,\) using (6.3) we calculate
\[
J_3(t) = 2\int_0^t b(z)\partial_{x_1} e^{(t-z)\Lambda} \sigma_1(z) \Lambda \sigma_1(z)dz + 2\int_0^t b(z)\partial_{x_1} e^{(t-z)\Lambda} \sigma_1(z) \mathcal{P}(z) F(z)dz \equiv J_{31} + J_{32}.
\]

Using (4.9) and Lemma 6.3 we calculate
\[
\|J_{31}\|_2 \leq C\{\int_0^t (1 + t - z)^{-\frac{3}{2}} \|\sigma_1(z) \Lambda \sigma_1(z)\|_1dz \leq C\{\int_0^t (1 + t - z)^{-\frac{3}{2}} \|\sigma_1(z)\|_2 \Lambda \sigma_1(z)\|_2dz
\leq CM(t)^2\{\int_0^t (1 + t - z)^{-\frac{3}{2}} (1 + z)^{-1}dz \leq C(1 + t)^{-\frac{3}{2}} \log(1 + t)\|u_0\|^3_{H^{1\cap L^1}}.
\]

Analogously we obtain for \(J_{32}\) that
\[
\|J_{32}\|_2 \leq C\{\int_0^t (1 + t - z)^{-\frac{3}{2}} \|\sigma_1(z)\|_2 \|F(z)\|_2dz \leq C(1 + t)^{-\frac{3}{2}} \log(1 + t)\|u_0\|^3_{H^{1\cap L^1}}.
\]

As for \(I_4(t),\) we have
\[
\|\phi w^1_1(z) - [\phi^{(0)} w^{(0),1}](z) \sigma_1^2(z)\|_1 \leq C\{\|\sigma_1(z)\|_2 \|u(z) - \sigma_1(z)u^{(0)}(z)\|_2 + \|u(z) - \sigma_1(z)u^{(0)}(z)\|_2^2\}
\leq C(1 + z)^{-1}M(z)^2.
\]

Thus, (4.9) gives us
\[
\|\partial_{x_1}^k I_4(t)\|_2 \leq CM(t)^2\{\int_0^t (1 + t - z)^{-\frac{3}{2}} \|\phi w^1_1(z) - [\phi^{(0)} w^{(0),1}](z) \sigma_1^2(z)\|_1dz \leq C(1 + t)^{-\frac{3}{2}} \log(1 + t)\|u_0\|^2_{H^{1\cap L^1}}.
\]

To estimate \(I_5(t),\) we write \(h_5(z)\) as
\[
h_5(z) = \partial_{x_1} \left( \mathcal{P}^{(1)}(z) F_2(z) + \partial_{x_1} \mathcal{P}^{(2)}(z) F_2(z) \right).
\]

Using (4.14) and Lemma 6.3 (v), we have
\[
\|\partial_{x_1}^k I_5(t)\|_2 \leq CM(t)^2\{\int_0^t (1 + t - z)^{-\frac{3}{2}} \|\phi w^1_1(z) - [\phi^{(0)} w^{(0),1}](z) \sigma_1^2(z)\|_1dz \leq C(1 + t)^{-\frac{3}{2}} \log(1 + t)\|u_0\|^2_{H^{1\cap L^1}}.
\]

As for \(I_6(t),\) we write \(h_6(z)\) as
\[
h_6(z) = \begin{cases} 
\partial_{x_1} \mathcal{P}^{(2)}(z)(\sigma_1^2 F_1)(z) & \text{for } z \in [0, \frac{t}{2}], \\
\partial_{x_1} \mathcal{P}^{(2)}(z)(\partial_{x_1} (\sigma_1^2) F_1)(z) & \text{for } z \in [\frac{t}{2}, t].
\end{cases}
\]

We see from (4.14) and Lemma 6.3 that
\[
\|\partial_{x_1}^k I_6(t)\|_2 \leq C\{\int_0^\frac{t}{2} (1 + t - z)^{-\frac{3}{2}} \|\sigma_1^2(z)\|_1dz + \int_\frac{t}{2}^t (1 + t - z)^{-\frac{3}{2}} \|\partial_{x_1} \sigma_1^2\|_1dz\} M(t)^2
\]

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This concludes the proof. □

and thus by using (4.14) and Lemma 6.3 we obtain

By Lemma 6.1 (ii) we have

Estimate (4.10) then implies

where

Theorem 9.4

\[ \|u_0\|_{H^{m\cap L^1}} \leq C(1 + t)^{-\frac{n+1}{2} - \frac{1}{2}} \eta(t) \|u_0\|_{H^{m\cap L^1}} + N(t) \]

\[ N(t) \leq \|u_0\|_{H^{m\cap L^1}} \{ 1 + \|u_0\|_{H^{m\cap L^1}} + \|u_0\|^2_{H^{m\cap L^1}} + N(t) \} \]

The desired result now follows by taking \( \|u_0\|_{H^{m\cap L^1}} \) suitably small. This completes the proof. □

Now let us show the asymptotic behavior in cases \( n \geq 3 \).

**Theorem 9.4** Let \( n \geq 3 \). There exists \( \varepsilon_8 > 0 \) such that if \( \|u_0\|_{H^{m\cap L^1}} \leq \varepsilon_8 \), then

\[ \|\sigma_1(t) - \mathcal{H}(t)\sigma_0\|_{L^2} \leq C(1 + t)^{-\frac{n+1}{2} - \frac{1}{2}} \eta(t) \|u_0\|_{H^{m\cap L^1}} \]

where \( \eta(t) = \log(1 + t) \) when \( n = 3 \) and \( \eta(t) = 1 \) when \( n \geq 4 \) and \( t \geq 0 \).

**Proof.** From (9.3) we see that

\[ \sigma_1(t) - \mathcal{H}(t)\sigma_0 = e^{tA} \mathcal{P}(0)u_0 - \mathcal{H}(t)\sigma_0 + \int_0^t e^{(t-z)A} \mathcal{P}(z)F(z)dz. \]

Estimate (4.10) then implies

\[ \|e^{tA} \mathcal{P}(0)u_0 - \mathcal{H}(t)\sigma_0\|_{L^2} \leq Ct^{-\frac{n+1}{2} - \frac{1}{2}} \|u_0\|_{L^2(\Omega)}. \]

By Lemma 6.1 (ii) we have

\[ \mathcal{P}(z)F(z) = -\text{div}'[\phi(z)u'(z)]_1 + \text{div}' \mathcal{P}^{(1)}(z)F(z) + \Delta' \mathcal{P}^{(2)}(z)F(z), \]

and thus by using (4.14) and Lemma 6.3 we obtain

\[ \| \int_0^t e^{(t-z)A} \mathcal{P}(z)F(z)dz \|_{L^2} \leq C \int_0^t (1 + t - z)^{-\frac{n+1}{2} - \frac{1}{2}} (\|\phi u'(z)\|_{L^2} + \|F(z)\|_{L^2})dz \]

\[ \leq CM(t)^2 \int_0^t (1 + t - z)^{-\frac{n+1}{2} - \frac{1}{2}} (1 + z)^{-\frac{n+1}{2}} dz \leq C(1 + t)^{-\frac{n+1}{2} - \frac{1}{2}} \eta(t) \|u_0\|_{H^{m\cap L^1}}. \]

This concludes the proof. □

(3.9) now follows from (5.13) and Theorem 9.4.

**References**


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