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https://hdl.handle.net/2324/24552

出版情報：MI Preprint Series. 2012-9, 2012-08-09. 九州大学大学院数理学研究院
バージョン：

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

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MI 2012-9

(Received August 9, 2012)

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Abstract

The linearized problem around a time-periodic parallel flow of the compressible Navier-Stokes equation in an infinite layer is investigated. By using the Floquet theory, spectral properties of the evolution operator associated with the linearized problem are studied in detail. The Floquet representation of low frequency part of the evolution operator, which plays an important role in the study of the nonlinear problem, is obtained.

Mathematics Subject Classification

Keywords. Compressible Navier-Stokes equation, Floquet theory, asymptotic behavior,time-periodic, spectral analysis .

1 Introduction

In this paper we study spectral properties of the linearized operator around a time-periodic solution to the compressible Navier-Stokes equation with time-periodic external force and time-periodic boundary conditions.

We consider the system of equations

$$\partial_{\widetilde{t}}\widetilde{\rho} + \operatorname{div}(\widetilde{\rho}\widetilde{v}) = 0, \tag{1.1}$$

$$\widetilde{\rho}(\partial_{\widetilde{t}}\widetilde{v} + \widetilde{v}\cdot\nabla\widetilde{v}) - \mu\Delta\widetilde{v} - (\mu+\mu')\nabla\operatorname{div}\widetilde{v} + \nabla\widetilde{P}(\widetilde{\rho}) = \widetilde{\rho}\widetilde{\boldsymbol{g}}, \tag{1.2}$$

in an n dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1}\times(0,\ell)$:

$$\begin{aligned}\Omega_\ell \;=\; & \{\widetilde{x} = {}^T(\widetilde{x}', \widetilde{x}_n)\,;\\ & \widetilde{x}' = {}^T(\widetilde{x}_1,\ldots,\widetilde{x}_{n-1}) \in \mathbb{R}^{n-1},\, 0<\widetilde{x}_n<\ell\}.\end{aligned}$$

Here $n\geq 2$; $\widetilde{\rho} = \widetilde{\rho}(\widetilde{x},\widetilde{t})$ and $\widetilde{v} = {}^T(\widetilde{v}^1(\widetilde{x},\widetilde{t}),\ldots,\widetilde{v}^n(\widetilde{x},\widetilde{t}))$ denote the unknown density and velocity at time $\widetilde{t}\geq 0$ and position $\widetilde{x}\in\Omega_\ell$, respectively; $\widetilde{P}$ is the pressure, smooth function of $\widetilde{\rho}$, where for given $\rho_*>0$ we assume $\widetilde{P}'(\rho_*)>0$; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu>0$, $\frac{2}{n}\mu+\mu'\geq 0$; div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\widetilde{x}$. Here and in what follows ${}^T\cdot$ denotes the transposition.

In (1.2) $\widetilde{\boldsymbol{g}}$ is assumed to have the form

$$\widetilde{\boldsymbol{g}} = {}^T(\widetilde{g}^1(\widetilde{x}_n,\widetilde{t}),0,\ldots,0,\widetilde{g}^n(\widetilde{x}_n)),$$

with $\widetilde{g}^1$ being a τ-periodic function in time, where $\tau>0$.

The system (1.1)–(1.2) is considered under boundary condition

$$\widetilde{v}|_{\widetilde{x}_n=0} = \widetilde{V}^1(t)\boldsymbol{e}_1, \quad \widetilde{v}|_{\widetilde{x}_n=\ell} = 0, \tag{1.3}$$

and initial condition

$$(\widetilde{\rho},\widetilde{v})|_{\widetilde{t}=0} = (\widetilde{\rho}_0,\widetilde{v}_0), \tag{1.4}$$

where $\widetilde{V}^1$ is a τ-periodic function of time and $\boldsymbol{e}_1 = {}^T(1,0,\ldots,0)\in\mathbb{R}^n$.

Under suitable conditions on $\widetilde{\boldsymbol{g}}$ and $\widetilde{V}^1$, problem (1.1)–(1.3) has smooth time-periodic solution $\overline{u}_p = {}^T(\overline{\rho}_p,\overline{v}_p)$ satisfying

$$\overline{\rho}_p = \overline{\rho}_p(\widetilde{x}_n) \geq \widetilde{\rho}_1, \quad \frac{1}{\ell}\int_0^\ell \overline{\rho}_p(\widetilde{x}_n)\,d\widetilde{x}_n = \rho_*,$$

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$$\overline{v}_p = {}^T(\overline{v}_p^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0), \quad \overline{v}_p^1(\widetilde{x}_n, \widetilde{t} + \tau) = \overline{v}_p^1(\widetilde{x}_n, \widetilde{t}),$$

for a positive constant $\widetilde{\rho}_1$.

Our main concern is asymptotic description of large time behavior of perturbations from $\overline{u}_p$ when Reynolds and Mach numbers are sufficiently small. For this purpose we consider the linearized problem in this paper.

To formulate the problem for perturbations, we introduce the following dimensionless variables:

$$\widetilde{x} = \ell x, \quad \widetilde{t} = \frac{\ell}{V} t, \quad \widetilde{v} = V v, \quad \widetilde{\rho} = \rho_* \rho, \quad \widetilde{P} = \rho_* V^2 P,$$

with

$$\widetilde{w} = V w, \quad \widetilde{\phi} = \rho_* \gamma^{-2} \phi, \quad \widetilde{V}^1 = V V^1, \quad \widetilde{\boldsymbol{g}} = \frac{\mu V}{\rho_* \ell^2} \boldsymbol{g},$$

where

$$\gamma = \frac{\sqrt{\widetilde{P}'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu} \left\{ |\partial_t \widetilde{V}^1|_{C^0(\mathbb{R})} + |\widetilde{g}^1|_{C^0(\mathbb{R}\times[0,\ell])} \right\} + |\widetilde{V}^1|_{C^0(\mathbb{R})}.$$

In this paper we assume $V > 0$. Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0,1)$ and $g^1(x_n, t)$, $V^1(t)$ are periodic in t with period $T > 0$ defined by

$$T = \frac{V}{\ell} \tau.$$

The time-periodic solution $\overline{u}_p$ is transformed into $u_p = {}^T(\rho_p, v_p)$ satisfying

$$\rho_p = \rho_p(x_n) > 0, \quad \int_0^1 \rho_p(x_n)\, dx_n = 1,$$

$$v_p = {}^T(v_p^1(x_n, t), 0, \ldots, 0), \quad v_p^1(x_n, t + T) = v_p^1(x_n, t).$$

It then follows that the perturbation $u(t) = {}^T(\phi(t), w(t)) \equiv {}^T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t))$ is governed by the following system of equations

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\,(\rho_p w) = f^0, \tag{1.5}$$

$$\begin{aligned} \partial_t w - \tfrac{\nu}{\rho_p} \Delta w - \tfrac{\widetilde{\nu}}{\rho_p} \nabla \mathrm{div}\, w + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w^n\, \boldsymbol{e}_1 & \\ + \tfrac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1) \phi\, \boldsymbol{e}_1 + \nabla \left(\tfrac{P'(\rho_p)}{\gamma^2 \rho_p} \phi \right) &= \boldsymbol{f}, \end{aligned} \tag{1.6}$$

$$w|_{x_n=0} = w|_{x_n=1} = 0, \tag{1.7}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{1.8}$$

where f^0 and $\boldsymbol{f} = {}^T(f^1, \cdots, f^n)$ denote nonlinearities. Here div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x; ν, ν' and $\widetilde{\nu}$ are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \widetilde{\nu} = \nu + \nu'.$$

We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively. Since our concern in this paper is analysis of solutions to the linearized problem, i.e. problem (1.5)–(1.8) with $(f^0, \boldsymbol{f}) = (0, 0)$, we do not write down the exact form of $(f^0, \boldsymbol{f})$. See [1] for the derivation of (1.5)–(1.8) and the exact form of $(f^0, \boldsymbol{f})$.

In case g^1 and V^1 do not depend on t, problem (1.1)–(1.3) has a stationary parallel flow. The stability of stationary parallel flows were studied in [3, 4, 5]. It was shown in [3] and [4] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $H^m(\Omega) \cap L^1(\Omega)$ with $m \geq [n/2] + 1$, provided that $Re \ll 1$, $Ma \ll 1$ and density of the parallel flow is sufficiently close to a positive constant. Furthermore, the asymptotic behavior is described by $n-1$ dimensional linear heat equation in the case $n \geq 3$ ([3]) and by one-dimensional viscous Burgers equation in the case $n = 2$ ([4]).

The case of time-periodic parallel flows was considered in [1]. We investigated the linearized problem, i.e. (1.5)–(1.8) with $(f^0, \boldsymbol{f}) = (0, 0)$, which is written as

$$\partial_t u + L(t) u = 0, \quad w|_{x_n=0,1} = 0, \quad u|_{t=s} = u_0. \tag{1.9}$$

Here $u = {}^T(\phi, w)$ and $L(t)$ is operator of the form

$$L(t) = \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_p\,\cdot) \\ \nabla\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\,\cdot\right) & -\frac{\nu}{\rho_p}\Delta I_n - \frac{\tilde{\nu}}{\rho_p}\nabla \mathrm{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}\partial_{x_n}^2 v_p^1(t)\boldsymbol{e}_1 & v_p^1(t)\partial_{x_1} I_n + (\partial_{x_n} v_p^1(t))\boldsymbol{e}_1{}^T\boldsymbol{e}_n \end{pmatrix}. \tag{1.10}$$

Note that $L(t)$ satisfies $L(t) = L(t+T)$.

It was shown in [1] that if $Re \ll 1$ and $Ma \ll 1$, then the solution operator $U(t,s)$ for (1.9) satisfies

$$\begin{aligned} \|\partial_{x'}^k \partial_{x_n}^l U(t,s)u_0\|_{L^2} &\le C\{(t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1(\mathbb{R}^{n-1};H^1(0,1)\times L^2(0,1))} \\ &\quad + e^{-d(t-s)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_{L^2})\}, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} \|\partial_{x'}^k \partial_{x_n}^l (U(t,s)u_0 - \sigma_{t,s}[u_0]u^{(0)}(t))\|_{L^2} &\le C\{(t-s)^{-\frac{n-1}{4}-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1(\mathbb{R}^{n-1};H^1(0,1)\times L^2(0,1))} \\ &\quad + e^{-d(t-s)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_{L^2})\}, \end{aligned} \tag{1.12}$$

for $t-s \ge 4T, s \ge 0,\ k,l = 0,1$, where $u^{(0)}(t) = u^{(0)}(x_n,t)$ is a function T-periodic in t and $\sigma_{t,s}[u_0] = \sigma_{t,s}(x')[u_0]$ is a function whose Fourier transform in x' is given by

$$\mathscr{F}(\sigma_{t,s}[u_0]) = e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2)(t-s)}[\widehat{\phi}_0(\xi')],$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$ and $\xi'' = (\xi_2, \dots, \xi_{n-1})$. Here $[\widehat{\phi}_0(\xi')]$ is a quantity given by

$$[\widehat{\phi}_0(\xi')] = \int_0^1 \widehat{\phi}_0(\xi', x_n)\,dx_n,$$

with $\widehat{\phi}_0$ being the Fourier transform of ϕ_0 in x' and $\kappa_0 \in \mathbb{R},\ \kappa_1 > 0, \kappa'' > 0$ are positive constants depending on ρ_*, l, V, μ, μ' and $\widetilde{P}'(\rho_*)$.

These results suggest that the asymptotic behavior of solutions of the nonlinear problem (1.5)–(1.8) is expected to be similar to that in the case of stationary parallel flows.

The purpose of this paper is to study more detailed spectral properties of $U(t,s)$, which will be useful to analyze the asymptotic behavior of solutions of the nonlinear problem.

To study spectral properties of $U(t,s)$, we consider Fourier transform of (1.9) that can be written in the form:

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0,\ t > s,\ \ \widehat{u}|_{t=s} = \widehat{u}_0, \tag{1.13}$$

where $\widehat{u}$ denotes the Fourier transform of u in x' and ξ' is dual variable to x'. For each $\xi' \in \mathbb{R}^{n-1}$ and for all $t \ge s$ there exists a unique evolution operator $\widehat{U}_{\xi'}(t,s)$ for (1.13).

Since $\widehat{L}_{\xi'}(t)$ is T-time periodic, the spectrum of $\widehat{U}_{\xi'}(T,0)$ plays an important role to the study of large time behavior. It was shown in [1] that the spectrum of $\widehat{U}_{\xi'}(T,0)$ satisfies the following inclusion

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subseteq \begin{cases} \{e^{\lambda_{\xi'}T}\} \cup \{|\lambda| < q_1\} & (|\xi'| < r), \\ \{|\lambda| < q_1\} & (|\xi'| \ge r), \end{cases}$$

for a constant $0 < q_1 < 1$ and $0 < r \ll 1$. Here $e^{\lambda_{\xi'}T}$ is the simple eigenvalue of $\widehat{U}_{\xi'}(T,0)$ and $\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3)$ with $\kappa_0 \in \mathbb{R},\ \kappa_1 > 0, \kappa'' > 0$ and $\xi' = {}^T(\xi_1, \xi'')$. As a result one can obtain (1.11) and (1.12).

In this paper more detailed analysis is made for the spectral properties of $\widehat{U}_{\xi'}(T,0)$ $(|\xi'| < r)$. We develop a Floquet analysis for $L(t)$ and construct a family of time-periodic projections associated with the eigenspaces for the eigenvalues $e^{\lambda_{\xi'}T}$. The main results of this paper are summarized as follows. We assume that $\boldsymbol{g}^1 \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j(\mathbb{R}; H^{m-2j}(0,1))$, $g^n \in C^m[0,1]$ and $V^1 \in C^{[\frac{m+1}{2}]}(\mathbb{R})$ for a given integer $m \ge 2$. Note that under these assumptions we have $v_p \in \bigcap_{j=0}^{[\frac{m+2}{2}]} C^j(\mathbb{R}; H^{m+2-2j}(0,1))$ and $\rho_p \in C^{m+1}[0,1]$.

Then for the Reynolds and Mach numbers small one can construct a family $\{P(t)\}_{t\in\mathbb{R}}$ of bounded projections on $L^2(\Omega)$ along the Floquet theory; and by $P(t)$ we represent $P(t)U(t,s)$ as

$$P(t)U(t,s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s). \tag{1.14}$$

Here, $e^{t\Lambda} = \mathscr{F}^{-1}\widehat{\chi}_1 e^{\lambda_{\xi'}t}\mathscr{F}$ with frequency cut off function $\widehat{\chi}_1 : \widehat{\chi}_1(\xi') = 1$ ($|\xi'| < r_1$), $\widehat{\chi}_1(\xi') = 0$ ($|\xi'| \geq r_1$), and $\mathscr{Q}(t) = \mathscr{F}^{-1}\widehat{\chi}_1\widehat{\mathscr{Q}}_{\xi'}(t)\mathscr{F}$ and $\mathscr{P}(t) = \mathscr{F}^{-1}\widehat{\chi}_1\widehat{\mathscr{P}}_{\xi'}(t)\mathscr{F}$ with

$$\widehat{\mathscr{Q}}_{\xi'}(t) : \mathbb{C} \to L^2(0,1) \text{ and } \widehat{\mathscr{P}}_{\xi'}(t) : L^2(0,1) \to \mathbb{C},$$

expanded as

$$\widehat{\mathscr{Q}}_{\xi'}(t) = \mathscr{Q}^{(0)}(t) + \xi' \cdot \mathscr{Q}^{(1)}(t) + O(|\xi'|^2),$$
$$\widehat{\mathscr{P}}_{\xi'}(t) = \mathscr{P}^{(0)} + \xi' \cdot \mathscr{P}^{(1)}(t) + O(|\xi'|^2),$$

for $|\xi'| \leq r_1$, where $\mathscr{Q}^{(0)}(t)\sigma = \sigma u^{(0)}(\cdot, t)$ ($\sigma \in \mathbb{C}$), $\mathscr{P}^{(0)}u = [\phi]$ ($u = {}^T(\phi, w) \in L^2(0,1)$). We study boundedness properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$. One consequence of (1.14) is that we improve (1.11) and (1.12) as

$$\|\partial_{x'}^k \partial_{x_n}^l P(t)U(t,s)u_0\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1(\Omega)},$$

$$\|(I - P(t))U(t,s)u_0\|_{H^1(\Omega)} \leq e^{-d(t-s)}(\|u_0\|_{H^1 \times L^2} + \|\partial_{x'} w_0\|_{L^2}),$$

$$\|\partial_{x'}^k \partial_{x_n}^l (P(t)U(t,s)u_0 - \sigma_{t,s}[u_0]u^{(0)}(t))\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{n-1}{4}-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1(\Omega)}.$$

Another consequence of (1.14) is that if $u(t)$ is a solution of

$$\partial_t u + L(t)u = f, \ u|_{t=0} = u_0,$$

then $P(t)u(t)$ is represented as

$$P(t)u(t) = \mathscr{Q}(t)\left(e^{t\Lambda}\mathscr{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)f(z)dz\right).$$

This formula is useful for the analysis of nonlinear interaction of solutions of (1.5) –(1.8); and, in particular, it plays an important role in studying the 2-dimensional nonlinear problem.

Structure of this paper is following. In Section 2 we introduce basic notations that are used throughout the paper. In Section 3 we state the main results. In Section 4 we establish existence and regularity of solutions of inhomogeneous problem (1.13) for fixed $\xi' \in \mathbb{R}^{n-1}$. In Sections 5 and 6 we study spectral properties associated with (1.13). In Section 7 we present the proofs of main results. Section 8 is Appendix. It contains proofs of some proclaims from Sections 5 and 6.

2 Notation

In this section we introduce some notations which are used throughout the paper. For a domain E we denote by $L^2(E)$ the usual Lebesgue space on E and its norm is denoted by $\|\cdot\|_{L^2(E)}$. Let m be a nonnegative integer. $H^m(E)$ denotes the m-th order L^2 Sobolev space on E with norm $\|\cdot\|_{H^m(E)}$. $C_0^m(E)$ stands for the set of all C^m functions which have compact support in E. We denote by $H_0^1(E)$ the completion of $C_0^1(E)$ in $H^1(E)$.

We simply denote by $L^2(E)$ (resp., $H^m(E)$) the set of all vector fields $w = {}^T(w^1, \dots, w^n)$ on E with $w^j \in L^2(E)$ (resp., $H^m(E)$), $j = 1, \dots, n$, and its norm is also denoted by $\|\cdot\|_{L^2(E)}$ (resp., $\|\cdot\|_{H^m(E)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(E)$ and $w = {}^T(w^1, \dots, w^n) \in H^m(E)$, we define $\|u\|_{H^k(E)\times H^m(E)}$ by $\|u\|_{H^k(E)\times H^m(E)} = \|\phi\|_{H^k(E)} + \|w\|_{H^m(E)}$. When $k = m$, we simply write $\|u\|_{H^k(E)\times H^k(E)} = \|u\|_{H^k(E)}$.

In the case $E = \Omega$ we denote the norm $\|\cdot\|_{L^2(\Omega)}$ (resp., $\|\cdot\|_{L^1(\Omega)}$) by $\|\cdot\|_2$ (resp., $\|\cdot\|_1$).

In the case $E = (0,1)$ we abbreviate $L^2(0,1)$ (resp., $L^1(0,1)$, $H^m(0,1)$) as L^2 (resp., L^1, H^m). In particular, we denote the norm of L^2 (resp., H^m) by $|\cdot|_2$ (resp., $|\cdot|_{H^m}$). The inner product of L^2 is denoted by

$$(f, g) = \int_0^1 f(x_n)\overline{g(x_n)}\, dx_n, \quad f, g \in L^2.$$

Here $\overline{g}$ denotes the complex conjugate of g. For $u_j = {}^T(\phi_j, w_j) \in L^2$ with $w_j = {}^T(w_j^1, \dots, w_j^n)$ $(j = 1, 2)$, we also define a weighted inner product $\langle u_1, u_2 \rangle$ by

$$\langle u_1, u_2 \rangle = \int_0^1 \phi_1 \overline{\phi}_2 \frac{P'(\rho_p)}{\gamma^4 \rho_p} \, dx_n + \int_0^1 w_1 \overline{w}_2 \rho_p \, dx_n.$$

Furthermore, for $f \in L^1$ we denote the mean value of f in $(0, 1)$ by $[f]$:

$$[f] = (f, 1) = \int_0^1 f(x_n) \, dx_n.$$

For $u = {}^T(\phi, w) \in L^1$ with $w = {}^T(w^1, \dots, w^n)$ we define $[u]$ by

$$[u] = [\phi] + [w^1] + \cdots + [w^n].$$

We often write $x \in \Omega$ as

$$x = {}^T(x', x_n), \; x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Partial derivatives of function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; \, |\alpha| = k)$.

We denote $k \times k$ identity matrix by I_k. In particular, when $k = n + 1$, we simply write I for I_{n+1}. We define $(n + 1) \times (n + 1)$ diagonal matrices Q_j, Q' and $\widetilde{Q}$ by

$$Q_j = \mathrm{diag}\,(0, \dots, 0, \underbrace{1}_{j\text{-}th}, 0, \dots, 0), \quad j = 0, 1, \dots, n,$$

and

$$Q' = \mathrm{diag}\,(0, 1, \dots, 1, 0), \; \widetilde{Q} = \mathrm{diag}\,(0, 1, \dots, 1).$$

We then have for $u = {}^T(\phi, w) \in \mathbb{R}^{n+1}$, $w = {}^T(w^1, \dots, w^n) = {}^T(w', w^n)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad Q_j u = \begin{pmatrix} 0 \\ w^j \\ 0 \end{pmatrix}, \quad Q_n u = \begin{pmatrix} 0 \\ 0 \\ w^n \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ w' \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We denote $\boldsymbol{e}_1' = {}^T(1, 0, \dots, 0) \in \mathbb{R}^{n-1}$. We note that

$$[Q_0 u] = [\phi] \;\; \text{for } u = {}^T(\phi, w).$$

For a function $f = f(x')$ $(x' \in \mathbb{R}^{n-1})$, we denote its Fourier transform by $\widehat{f}$ or $\mathscr{F} f$:

$$\widehat{f}(\xi') = (\mathscr{F} f)(\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} \, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x') = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} \, d\xi'.$$

For closed linear operator A in X we denote resolvent set of A by $\rho(A)$. We denote the spectrum of A by $\sigma(A)$. For $\lambda \in \rho(A)$ we denote the resolvent operator by $R(\lambda; A)$:

$$R(\lambda; A) = (\lambda I - A)^{-1}.$$

For bounded linear operator A we denote the spectral radius of A by $r(A)$. We denote the set of all bounded linear operators from X_1 to X_2 by $L(X_1, X_2)$, and if $X_1 = X_2$, we simply write $L(X_1)$ instead of $L(X_1, X_1)$. The operator norm is denoted by $| \cdot |_{L(X_1, X_2)}$.

For operators A, B we denote $[A, B]$ the commutator, i.e. $[A, B] = AB - BA$.

For time interval $[a, b] \subset \mathbb{R}$, we denote the usual Bochner spaces by $L^2(a, b; X)$, $H^m(a, b; X)$, etc., where X denotes a Banach space.

In this paper we frequently work with T-time-periodic functions. Instead of T-time-periodic functions on $\mathbb{R}$ we work with their restrictions on interval $[0, T]$. We denote the fundamental interval $[0, T]$ by J_T, i.e.:

$$J_T = [0, T].$$

For any Bochner space $W(J_T; X)$ we denote by $W_{per}(J_T; X)$ the space of restrictions of T-periodic functions, e.g. $H^1_{per}(J_T; X)$ consists of functions from $H^1(J_T; X)$ that are restrictions of T-periodic functions.

Definition 2.1 *We define the following function spaces:*

$$X_0 = H^1 \times L^2, \quad H_*^j = \begin{cases} H^{-1} = (H_0^1)^* & \text{for } j = -1, \\ L^2 & \text{for } j = 0, \\ H^j \cap H_0^1 & \text{for } j \geq 1. \end{cases}$$

Definition 2.2 *Let $1 \leq k \leq m$. Let us define spaces Y_{per}^k as*

$$Y_{per}^1 \equiv Y_{per} = L_{per}^2(J_T; X_0),$$

$$Y_{per}^k = \bigcap_{j=0}^{[\frac{k}{2}]} H_{per}^j(J_T; H^{k-2j} \times H^{k-1-2j}), \text{ for } 2 \leq k \leq m.$$

Remark 2.3 *Note that for $k \geq 3$ we have:*

$$\bigcap_{j=0}^{[\frac{k}{2}]} H^j(J_T; H^{k-2j} \times H^{k-1-2j}) \hookrightarrow \bigcap_{j=1}^{[\frac{k}{2}]} C^{j-1}(J_T; H^{k+1-2j} \times H^{k-2j}).$$

3 Main Results

In this section we state the main results of this paper.

In the whole article we assume the following regularity for $\widetilde{\boldsymbol{g}}$ and $\widetilde{V}^1$.

Assumptions 3.1 *For a given integer $m \geq 2$, $\widetilde{\boldsymbol{g}} = {}^T(\widetilde{g}^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0, \widetilde{g}^n(\widetilde{x}_n))$ and $\widetilde{V}^1(\widetilde{t})$ belong to the following spaces:*

$$\widetilde{g}^n \in C^m[0, \ell],$$

and

$$\widetilde{g}^1 \in \bigcap_{j=0}^{[\frac{m}{2}]} C_{per}^j([0, \tau]; H^{m-2j}(0, \ell)),$$

$$\widetilde{V}^1 \in C_{per}^{[\frac{m+1}{2}]}([0, \tau]).$$

Furthermore, we assume

$$\widetilde{P}(\cdot) \in C^{m+1}(\mathbb{R}).$$

It is straightforward that $\boldsymbol{g}$ and V^1 belong to similar spaces as $\widetilde{\boldsymbol{g}}$ and $\widetilde{V}^1$.

Under Assumptions 3.1 one can see that flow u_p has the following properties (see [1]).

Proposition 3.2 *Let*

$$\nu |g^n|_{C^m([0,1])} \leq C, \quad |P|_{C^{m+1}} \leq C,$$

for a suitable constant $C > 0$ (see [1]). Then flow $u_p = {}^T(\rho_p(x_n), v_p(x_n, t))$ exists and under Assumptions 3.1, it satisfies

$$v_p \in \bigcap_{j=0}^{[\frac{m+2}{2}]} C_{per}^j(J_T; H^{m+2-2j}(0,1)), \quad \rho_p \in C^{m+1}[0,1],$$

and

$$0 < \rho_1 \leq \rho_p(x_n) \leq \rho_2, \ \int_0^1 \rho_p(x_n) dx_n = 1, \ v_p(x_n, t) = {}^T(v_p^1(x_n, t), 0),$$

with

$$P'(\rho) > 0 \text{ for } \rho_1 \leq \rho \leq \rho_2,$$

$$|1 - \rho_p|_{C^{m+1}([0,1])} \leq \frac{C}{\gamma^2} \nu (|P''|_{C^{m-1}(\rho_1, \rho_2)} + |g^n|_{C^m([0,1])}), \quad |P'(\rho_p) - \gamma^2|_{C^0([0,1])} \leq \frac{C}{\gamma^2} \nu |g^n|_{C^0([0,1])},$$

and

$$\frac{\rho_p P'(\rho_p)}{\gamma^2} \geq a_0, \tag{3.1}$$

with some constants $0 < \rho_1 < 1 < \rho_2$ *and* $a_0 > 0$.

Let us consider homogeneous problem

$$\partial_t u + L(t)u = 0,\ t > s,\ w|_{x_n=0,1} = 0,\ u|_{t=s} = u_0. \tag{3.2}$$

Here, $L(t)$ is the operator given in (1.10). We introduce the space Z_s defined by

$$Z_s = \{u = {}^T(\phi, w); \phi \in C_{loc}([s,\infty); H^1),\ \partial_{x'}^{\alpha'} w \in C_{loc}([s,\infty); L^2) \cap L^2_{loc}([s,\infty); H^1_0)\ (|\alpha'| \leq 1),\ w \in C_{loc}((s,\infty); H^1_0)\}.$$

In [1] we showed that for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in (H^1 \cap L^2)(\Omega)$ with $\partial_{x'} w_0 \in L^2(\Omega)$ there exists a unique solution $u(t)$ of linear problem (3.2) in Z_s. We denote $U(t,s)$ the evolution operator for (3.2) given by

$$u(t) = U(t,s)u_0.$$

To investigate problem (3.2) we consider the Fourier transform of (3.2). We then obtain

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0,\ t > s,\quad \widehat{u}|_{t=s} = \widehat{u}_0. \tag{3.3}$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transforms of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' \in \mathbb{R}^{n-1}$ being the dual variable; $\widehat{L}_{\xi'}(t)$ is an operator on X_0 with domain $D(\widehat{L}_{\xi'}(t)) = H^1 \times H^2_*$, which takes the form

$$\widehat{L}_{\xi'}(t) = \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 \rho_p {}^T\xi' & \gamma^2 \partial_{x_n}(\rho_p\ \cdot\ ) \\ i\xi' \frac{P'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \frac{\tilde{\nu}}{\rho_p}\xi'{}^T\xi' & -i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\ \cdot\right) & -i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' & i\xi_1 v_p^1(t) I_{n-1} & \partial_{x_n}(v_p^1(t))\boldsymbol{e}_1' \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}.$$

For each $t \in \mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$, $\widehat{L}_{\xi'}(t)$ is sectorial on X_0. As for the evolution operator $\widehat{U}_{\xi'}(t,s)$ for (3.3) we have the following results.

Lemma 3.3 *For each* $\xi' \in \mathbb{R}^{n-1}$ *and for all* $t \geq s$ *there exists unique evolution operator* $\widehat{U}_{\xi'}(t,s)$ *for* (3.3) *that satisfies*

$$|\widehat{L}_{\xi'}(t)\widehat{U}_{\xi'}(t,s)|_{L(X_0)} \leq C_{t_1 t_2},\ t_1 \leq s < t \leq t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha([s,\infty); X_0), \alpha \in (0,1]$ *there exists unique classical solution* u *of inhomogeneous problem*

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = f,\ t > s,\quad u|_{t=s} = u_0, \tag{3.4}$$

satisfying $u \in C_{loc}([s,\infty); X_0) \cap C^1(s,\infty; X_0) \cap C(s,\infty; H^1 \times H^2_*)$*; and the solution* u *is given by*

$$u(t) = (\phi(t), w(t)) = \widehat{U}_{\xi'}(t,s)u_0 + \int_s^t \widehat{U}_{\xi'}(t,z)f(z)dz.$$

Since $\widehat{L}_{\xi'}(t)$ is sectorial uniformly with respect to $t \in \mathbb{R}$ for each $\xi' \in \mathbb{R}^{n-1}$, Lemma 3.3 can be shown by standard theory (see, e.g. [7, 9]).

Let us introduce *adjoint problem* to

$$\partial_t u + \widehat{L}_{\xi'}(t)u = 0,\ t > s,\quad u|_{t=s} = u_0.$$

Lemma 3.4 *For each $\xi' \in \mathbb{R}^{n-1}$ and for all $s \leq t$ there exists unique evolution operator $\widehat{U}^*_{\xi'}(s,t)$ for adjoint problem*

$$-\partial_s u + \widehat{L}^*_{\xi'}(s)u = 0,\ s < t,\ u|_{s=t} = u_0,$$

on X_0. Here, $\widehat{L}^*_{\xi'}(s)$ is an operator on X_0 with domain $D(\widehat{L}^*_{\xi'}(s)) = H^1 \times H^2_*$, which takes the form

$$\widehat{L}^*_{\xi'}(s) = \begin{pmatrix} -i\xi_1 v_p^1(s) & -i\gamma^2 \rho_p {}^T\xi' & -\gamma^2 \partial_{x_n}(\rho_p\ \cdot\ ) \\ -i\xi' \frac{P'(\rho_p)}{\gamma^2\rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p}\xi'{}^T\xi' & i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ -\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\ \cdot\right) & i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \frac{\nu\gamma^2}{P'(\rho_p)}(\partial_{x_n}^2 v_p^1(s))^T \boldsymbol{e}_1' & 0 \\ 0 & -i\xi_1 v_p^1(s) I_{n-1} & 0 \\ 0 & \partial_{x_n}(v_p^1(s))^T \boldsymbol{e}_1' & -i\xi_1 v_p^1(s) \end{pmatrix}.$$

Moreover, $\widehat{L}^*_{\xi'}(s)$ satisfies $\langle \widehat{L}_{\xi'}(s)u, v\rangle = \langle u, \widehat{L}^*_{\xi'}(s)v\rangle$ for $s \in \mathbb{R}$ and $u, v \in H^1 \times H^2_*$ and

$$|\widehat{L}^*_{\xi'}(s)\widehat{U}^*_{\xi'}(s,t)|_{L(X_0)} \leq C_{t_1 t_2},\ t_1 \leq s < t \leq t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha((-\infty, t]; X_0), \alpha \in (0,1]$ there exists unique classical solution u of inhomogeneous problem

$$-\frac{d}{ds}u + \widehat{L}^*_{\xi'}(s)u = f,\ s < t,\quad u|_{s=t} = u_0, \tag{3.5}$$

satisfying $u \in C_{loc}((-\infty, t]; X_0) \cap C^1(-\infty, t; X_0) \cap C(-\infty, t; H^1 \times H^2_*)$; and the solution u is given by

$$u(s) = (\phi(s), w(s)) = \widehat{U}^*_{\xi'}(s,t)u_0 + \int_s^t \widehat{U}^*_{\xi'}(s,z)f(z)dz.$$

Since $\widehat{L}^*_{\xi'}(s)$ is sectorial uniformly in s on X_0, Lemma 3.4 is obtained in the same way as Lemma 3.3.

Note that $\widehat{U}_{\xi'}(t,s)$ and $\widehat{U}^*_{\xi'}(s,t)$ are defined for all $t \geq s$ and

$$\widehat{U}_{\xi'}(t+T, s+T) = \widehat{U}_{\xi'}(t,s),\ \widehat{U}^*_{\xi'}(s+T, t+T) = \widehat{U}^*_{\xi'}(s,t).$$

The results of the following proposition were established in [1] on space X_0.

Proposition 3.5 *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that for each ξ' with $|\xi'| \leq r_0$ there hold the following statements.*

(i) The spectrum of operator $\widehat{U}_{\xi'}(T,0)$ on $H^1 \times H^1_0$ satisfies

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}, \tag{3.6}$$

with constant $q_0 < \operatorname{Re}\mu_{\xi'} < 1$. Here, $\mu_{\xi'} = e^{\lambda_{\xi'}T}$ is simple eigenvalue of $\widehat{U}_{\xi'}(T,0)$ and $\lambda_{\xi'}$ has an expansion

$$\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3), \tag{3.7}$$

where $\kappa_0 \in \mathbb{R}$ and $\kappa_1 > 0$, $\kappa'' > 0$.

Moreover, let $\widehat{\Pi}_{\xi'}$ denote the eigenprojection associated with $\mu_{\xi'}$. There holds

$$|\widehat{U}_{\xi'}(t,s)(I - \widehat{\Pi}_{\xi'})u|_{H^1} \leq Ce^{-d(t-s)}|(I - \widehat{\Pi}_{\xi'})u|_{X_0},$$

for $u \in X_0$ and $T \leq t - s$. Here, d is a positive constant depending on r_0.

(ii) The spectrum of operator $\widehat{U}^*_{\xi'}(0,T)$ on $H^1 \times H^1_0$ satisfies

$$\sigma(\widehat{U}^*_{\xi'}(0,T)) \subset \{\overline{\mu}_{\xi'}\} \cup \{\mu : |\mu| \le q_0\}.$$

Here, $\overline{\mu}_{\xi'}$ is simple eigenvalue of $\widehat{U}^*_{\xi'}(0,T)$.

Furthermore, let $\widehat{\Pi}^*_{\xi'}$ denote the eigenprojection associated with $\overline{\mu}_{\xi'}$. There holds

$$\langle \widehat{\Pi}_{\xi'} u, v\rangle = \langle u, \widehat{\Pi}^*_{\xi'} v\rangle,$$

for $u, v \in X_0$.

Next we introduce Floquet theory.

Definition 3.6 *We define operator $B_{\xi'}$ on space Y_{per} with domain*

$$D(B_{\xi'}) = H^1_{per}(J_T; X_0) \cap L^2_{per}(J_T; H^1 \times H^2_*),$$

in the following way

$$B_{\xi'} v = \partial_t v + \widehat{L}_{\xi'}(\cdot) v,$$

for $v \in D(B_{\xi'})$. Moreover, we define formal adjoint operator $B^*_{\xi'}$ with respect to inner product $\frac{1}{T}\int_0^T \langle \cdot, \cdot\rangle dt$ as

$$B^*_{\xi'} v = -\partial_t v + \widehat{L}^*_{\xi'}(\cdot) v,$$

for $v \in D(B^*_{\xi'}) = D(B_{\xi'})$.

Remark 3.7 *Operators $B_{\xi'}$ and $B^*_{\xi'}$ are closed, densely defined on Y_{per} for each fixed $\xi' \in \mathbb{R}^{n-1}$.*

Definition 3.8 *Let $k \ge 1$. We say that $u = {}^T(\phi, w)$ is k-regular function on time interval $[a,b]$ whenever*

$$u \in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} C^j([a,b]; (H^{k-2j} \times H^{k-2j}_*)(\Omega)),$$
$$\phi \in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} H^{j+1}(a,b; H^{k-2j}(\Omega)), \ \ w \in \bigcap_{j=0}^{\left[\frac{k+1}{2}\right]} H^{j}(a,b; H^{k+1-2j}_*(\Omega)).$$

Proposition 3.9 *There exist positive numbers ν_0 and γ_0 such that if $\nu \ge \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \ge \gamma_0^2$ then there exists $0 < r_1 \le 1$ such that for each $|\xi'| \le r_1$ there hold the following statements.*

(i) Let $1 \le k \le m$. There exists $q_1 > 0$ such that spectrum of operator $B_{\xi'}$ on Y^k_{per} satisfies

$$\sigma(B_{\xi'}) \subset \{-\lambda_{\xi'}\} \cup \{\lambda : \mathrm{Re}\,\lambda \ge q_1\},$$

with $0 \le -\mathrm{Re}\,\lambda_{\xi'} \le \frac{1}{2} q_1$ uniform for all k. Here, $-\lambda_{\xi'}$ is the simple eigenvalue of $B_{\xi'}$.

(ii) Let $1 \le k \le m$. Spectrum of operator $B^*_{\xi'}$ on Y^k_{per} satisfies

$$\sigma(B^*_{\xi'}) \subset \{-\overline{\lambda}_{\xi'}\} \cup \{\lambda : \mathrm{Re}\,\lambda \ge q_1\}.$$

Here, $-\overline{\lambda}_{\xi'}$ is the simple eigenvalue of $B^*_{\xi'}$.

(iii) There exist $u_{\xi'}$ and $u^*_{\xi'}$ eigenfunctions associated with $-\lambda_{\xi'}$ and $-\overline{\lambda}_{\xi'}$, respectively, with the following properties:

$$\langle u_{\xi'}(t), u^*_{\xi'}(t)\rangle = 1,$$

$$u_{\xi'}(t) = u^{(0)}(t) + i\xi' \cdot u^{(1)}(t) + |\xi'|^2 u^{(2)}(\xi', t),$$

$$u^*_{\xi'}(t) = u^{*(0)} + i\xi' \cdot u^{*(1)}(t) + |\xi'|^2 u^{*(2)}(\xi', t),$$

for $t \in J_T$. Here, all functions

$$u_{\xi'}, u^*_{\xi'}, u^{(0)}, u^{(0)*}, u^{(1)}, u^{(1)*}, u^{(2)}(\xi'), u^{(2)*}(\xi'),$$

are m-regular on J_T and we have estimate

$$\sup_{z \in J_T} \sum_{j=0}^{\left[\frac{m}{2}\right]} |\partial_z^j u(z)|^2_{H^{m-2j}} + \int_0^T \sum_{j=0}^{\left[\frac{m-1}{2}\right]} |\partial_z^{j+1} u|^2_{H^{m-2j} \times H^{m-1-2j}} + |\partial_z^{\left[\frac{m+2}{2}\right]} Q_0 u|_2^2 + |u|^2_{H^m \times H^{m+1}} dz \le C,$$

for $u \in \{u_{\xi'}, u^*_{\xi'}, u^{(2)}(\xi'), u^{(2)*}(\xi')\}$ and a constant $C > 0$ depending on r_1.

Let ν_0, γ_0 and r_1 are given by Proposition 3.9. In the rest of this section we assume that $\nu \ge \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \ge \gamma_0^2$.

Definition 3.10 *We define $\widehat{\chi}_1$ by*

$$\widehat{\chi}_1(\xi') = \begin{cases} 1, & |\xi'| < r_1, \\ 0, & |\xi'| \ge r_1, \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$.

Now, we introduce time-periodic projection based on $u_{\xi'}$ and $u^*_{\xi'}$.

Definition 3.11 *We define operators $\mathscr{P}(t) : L^2(\Omega) \to L^2(\mathbb{R}^{n-1})$ by*

$$\mathscr{P}(t)u = \mathscr{F}^{-1}\{\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u}\},$$

$$\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u} = \widehat{\chi}_1 \langle \widehat{u}, u^*_{\xi'}(t) \rangle,$$

for $u \in L^2(\Omega)$ and $t \in [0, \infty)$.

We define operators $\mathscr{Q}(t) : L^2(\mathbb{R}^{n-1}) \to L^2(\Omega)$ by

$$\mathscr{Q}(t)\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1 \widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma}\},$$

$$\widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma} = u_{\xi'}(\cdot, t)\widehat{\sigma},$$

for $t \in [0, \infty)$ and multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ by

$$\Lambda\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1 \lambda_{\xi'} \widehat{\sigma}\},$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

Moreover, we define projection $P(t)$ on $L^2(\Omega)$ as

$$P(t)u = \mathscr{Q}(t)\mathscr{P}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1 \langle \widehat{u}, u^*_{\xi'}(t) \rangle u_{\xi'}(\cdot, t)\},$$

for $t \in [0, \infty)$ and $u \in L^2(\Omega)$.

One can see that $P(t)^2 = P(t)$ and by Proposition 3.9 the following estimates hold uniformly in t, $t \in [0, \infty)$;

$$\|\mathscr{P}(t)u\|_{H^2(\mathbb{R}^{n-1})} \le C\|u\|_{L^2(\Omega)},$$
$$\|P(t)u\|_{H^2(\Omega)} \le C\|u\|_{L^2(\Omega)},$$

for $u \in L^2(\Omega)$.

As for $\mathscr{Q}(t)$, one can see from Proposition 3.9 that

$$\|\mathscr{Q}(t)\sigma\|_{H^2(\Omega)} \le C\|\sigma\|_{L^2(\mathbb{R}^{n-1})},$$

uniformly in $t \in [0, \infty)$.

Clearly, Λ is bounded linear operator on $L^2(\mathbb{R}^{n-1})$. It then folllows that Λ generates uniformly continuous group $\{e^{t\Lambda}\}_{t\in\mathbb{R}}$. Furthermore, if $\sigma \in (L^1\cap L^2)(\Omega)$, then

$$\|\partial_{x'}^k e^{t\Lambda}\sigma\|_{L^2(\mathbb{R}^{n-1})} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},\ 1\le p\le 2.$$

In terms of $P(t)$ we have the following decomposition of $U(t,s)$.

Theorem 3.12 *$P(t)$ satisfies the following:*

(i)

$$P(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))P(t)u(t) = \mathscr{Q}(t)[(\partial_t-\Lambda)\mathscr{P}(t)u(t)],$$

for $u\in L^2(J_T;(H^1\times H^2_*)(\Omega))\cap H^1(J_T;L^2(\Omega))$.

(ii)

$$P(t)U(t,s) = U(t,s)P(s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s).$$

If $u\in L^1(\Omega)$, then

$$\|\partial_t^j\partial_{x'}^k\partial_{x_n}^l P(t)U(t,s)u\|_{L^2(\Omega)} \le C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_{L^1(\Omega)},$$

for $0\le 2j+l\le m$, $k=0,\dots,m$.

(iii) $(I-P(t))U(t,s) = U(t,s)(I-P(s))$ *satisfies*

$$\|(I-P(t))U(t,s)u\|_{H^1(\Omega)} \le Ce^{-d(t-s)}(\|u\|_{(H^1\times L^2)(\Omega)} + \|\partial_{x'}w\|_{L^2(\Omega)}),$$

for $t-s\ge T$. Here d is a positive constant.

Let us consider the following inhomogeneous problem:

$$\partial_t u + L(t)u = f(t),\ t>0,\ u|_{t=0} = u_0. \tag{3.8}$$

One can show that if $u_0\in(H^1\times H^1_0)(\Omega)$ and $f\in L^2_{loc}([0,\infty);(H^1\times L^2)(\Omega))$, then there exists unique $u(t) = {}^T(\phi(t),w(t))$,

$$u\in C_{loc}([0,\infty);(H^1\times H^1_0)(\Omega)),\ \phi\in H^1_{loc}([0,\infty);L^2(\Omega)),\ w\in\bigcap_{j=0}^{1}H^j_{loc}([0,\infty);H^{2-2j}_*(\Omega)), \tag{3.9}$$

that satisfies (3.8).

Theorem 3.13 *Let $u_0\in(H^1\times H^1_0)(\Omega)$, $f\in L^2_{loc}([0,\infty);(H^1\times L^2)(\Omega))$ and let $u(t) = {}^T(\phi(t),w(t))$ is unique solution of* (3.8) *in the class* (3.9). *Then*

(i) $\mathscr{P}(t)u(t)$ satisfies

$$\mathscr{P}(t)u(t) = e^{t\Lambda}\mathscr{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)f(z)dz,\ t\in[0,\infty).$$

(ii) $u_\infty(t) = {}^T(\phi_\infty(t),w_\infty(t)) = (I-P(t))u(t)$ *belongs to class* (3.9) *and satisfies*

$$\partial_t u_\infty + L(t)u_\infty = (I-P(t))f,\ t>0,\ u_\infty|_{t=0} = (I-P(0))u_0.$$

Moreover, let $1\le k\le m$ and $u(t)$ is k-regular function locally on $[0,\infty)$. Then $P(t)u(t) = \mathscr{Q}(t)\mathscr{P}(t)u(t)$ satisfies

$$P(t)u(t)\in\bigcap_{j=0}^{[\frac{k}{2}]}C^j_{loc}([0,\infty);(H^{m-2j}\times H^{m-2j}_*)(\Omega)),$$

$$Q_0P(t)u(t)\in\bigcap_{j=0}^{[\frac{k}{2}]}H^{j+1}_{loc}([0,\infty);H^{m-2j}(\Omega)),\ \widetilde{Q}P(t)u(t)\in\bigcap_{j=0}^{[\frac{k+1}{2}]}H^j_{loc}([0,\infty);H^{m+1-2j}_*(\Omega)).$$

Remark 3.14 *Let $1 \le k \le m$ and $u_0 \in H^k(\Omega)$, $f \in \bigcap_{j=0}^{[\frac{k}{2}]} H^j_{loc}([0,\infty);(H^{k-2j}\times H^{k-1-2j})(\Omega))$. If u_0 and f satisfy a suitable compatibility condition, then one can show that (3.8) has unique solution $u(t) = {}^T(\phi(t), w(t))$, which is k-regular locally on $[0,\infty)$. Due to Theorem 3.13 one can see that $u_\infty(t) = {}^T(\phi_\infty(t), w_\infty(t)) = (I - P(t))u(t)$ is also k-regular locally on $[0,\infty)$.*

To complete our main results we show some asymptotic properties of $U(t,s)$. First let us define a semigroup $\mathscr{H}(t)$ on $L^2(\mathbb{R}^{n-1})$ associated with a linear heat equation with a convective term:

$$\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma - \kappa''\Delta''\sigma + \kappa_0\partial_{x_1}\sigma = 0.$$

Definition 3.15 *We define operator $\mathscr{H}(t)$ as*

$$\mathscr{H}(t)\sigma = \mathscr{F}^{-1}[e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)t}\widehat{\sigma}],$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$. Here, κ_0, κ_1 and κ'' are given by (3.7).

Theorem 3.16 *There hold the following estimates for $1 \le p \le 2$.*

(i)

$$\|\partial_{x'}^k(\mathscr{H}(t)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \le Ct^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},$$

for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(ii) It holds the relation,

$$\mathscr{P}(t)U(t,s) = e^{(t-s)\Lambda}\mathscr{P}(s).$$

Set $\sigma = [Q_0u]$. Then

$$\|\partial_{x'}^k(e^{(t-s)\Lambda}\mathscr{P}(s)u - \mathscr{H}(t-s)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_{L^p(\Omega)},$$

for $u \in L^p(\Omega)$. Furhermore, for any $\sigma \in L^p(\mathbb{R}^{n-1})$ there holds

$$\|(e^{(t-s)\Lambda} - \mathscr{H}(t-s))\partial_{x'}^k\sigma\|_{L^2(\mathbb{R}^{n-1})} \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},$$

for $k = 0, 1, \ldots$.

(iii) It holds the relation,

$$P(t)U(t,s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s).$$

Furthermore,

$$\|\partial_{x'}^k(\mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s)u - \mathscr{Q}^{(0)}(t)\mathscr{H}(t-s)\sigma)\|_{L^2(\Omega)} \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_{L^p(\Omega)},$$

for $u \in L^p(\Omega)$.

Remark 3.17 *Combining Theorems 3.12 (iii) and 3.16 (iii) we see that asymptotic leading part of $U(t,s)u$ is represented by $\mathscr{Q}^{(0)}(t)\mathscr{H}(t-s)\sigma$, where $\sigma = [Q_0u]$.*

Theorems 3.12, 3.13 and 3.16 follow from the properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$ introduced below. Proofs are given in Section 7. In the rest of this section we introduce properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$.

Theorem 3.18 *$\mathscr{Q}(t)$ has the following properties:*

(i)

$$\mathscr{Q}(t+T) = \mathscr{Q}(t),\ \partial_{x'}^k\mathscr{Q}(t) = \mathscr{Q}(t)\partial_{x'}^k.$$

(ii)

$$\mathscr{Q}(t)\sigma \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j_{per}(J_T; (H^{m-2j} \times H_*^{m-2j})(\Omega)),$$

$$\widetilde{Q}\mathscr{Q}(t)\sigma \in \bigcap_{j=0}^{[\frac{m+1}{2}]} H^j_{per}(J_T; H_*^{m+1-2j}(\Omega)),$$

and

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l (\mathscr{Q}(t)\sigma)\|_{L^2(\Omega)} \le C\|\sigma\|_{L^2(\mathbb{R}^{n-1})},\ 0 \le 2j+l \le m,\ k = 0, 1, \dots,$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

(iii)

$$(\partial_t + L(t))(\mathscr{Q}(t)\sigma(t)) = \mathscr{Q}(t)(\partial_t - \Lambda)\sigma(t),$$

for $\sigma \in H^1_{loc}([0,\infty); L^2(\mathbb{R}^{n-1}))$.

(iv) $\mathscr{Q}(t)$ *is decomposed as*

$$\mathscr{Q}(t) = \mathscr{Q}^{(0)}(t) + \mathrm{div}'\mathscr{Q}^{(1)}(t) + \Delta'\mathscr{Q}^{(2)}(t).$$

Here, $\mathscr{Q}^{(0)}(t)\sigma = (\mathscr{F}^{-1}\{\widehat{\chi}_1\widehat{\sigma}\})u^{(0)}(\cdot,t)$, $\mathscr{Q}^{(1)}(t)$ *and* $\mathscr{Q}^{(2)}(t)$ *share the same properties given in (i) and (ii) for* $\mathscr{Q}(t)$.

Theorem 3.19 $\mathscr{P}(t)$ *has the following properties:*

(i)

$$\mathscr{P}(t+T) = \mathscr{P}(t),\ \partial_{x'}^k \mathscr{P}(t) = \mathscr{P}(t)\partial_{x'}^k,\ \partial_{x_n}\mathscr{P}(t) = 0.$$

(ii)

$$\mathscr{P}(t)u \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j_{per}(J_T; H^k(\mathbb{R}^{n-1})),\ \textit{for all } k = 0, 1, \dots,$$

and

$$\|\partial_t^j \partial_{x'}^k (\mathscr{P}(t)u)\|_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{L^2(\Omega)},\ 0 \le 2j \le m,\ k = 0, 1, \dots,$$

for $u \in L^2(\Omega)$.

Moreover,

$$\|\mathscr{P}(t)u\|_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{L^p(\Omega)},$$

for $u \in L^p(\Omega)$ *and* $1 \le p \le 2$.

(iii)

$$\mathscr{P}(t)(\partial_t + L(t))u(t) = (\partial_t - \Lambda)(\mathscr{P}(t)u(t)),$$

for $u \in L^2_{loc}([0,\infty); (H^1 \times H^2_*)(\Omega)) \cap H^1_{loc}([0,\infty); L^2(\Omega))$.

(iv) $\mathscr{P}(t)$ *is decomposed as*

$$\mathscr{P}(t) = \mathscr{P}^{(0)} + \mathrm{div}'\mathscr{P}^{(1)}(t) + \Delta'\mathscr{P}^{(2)}(t).$$

Here, $\mathscr{P}^{(0)}u = \mathscr{F}^{-1}\{\widehat{\chi}_1[Q_0\widehat{u}]\}$, $\mathscr{P}^{(1)}(t)$ *and* $\mathscr{P}^{(2)}(t)$ *share the same properties given in (i) and (ii) for* $\mathscr{P}(t)$.

(v) *There holds*

$$\|\partial_{x'}^k e^{(t-s)\Lambda}\mathscr{P}^{(q)}(s)u\|_{L^2(\mathbb{R}^{n-1})} \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|u\|_{L^p(\Omega)},\ q = 0, 1, 2,$$

$$\|\partial_{x'}^k((\mathscr{Q}(t) - \mathscr{Q}^{(0)}(t))e^{(t-s)\Lambda}\mathscr{P}(s)u)\|_{L^2(\Omega)} \le C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_{L^p(\Omega)},$$

for $u \in L^p(\Omega)$ *and* $1 \le p \le 2$.

Proofs of Theorem 3.18 and Theorem 3.19 are given in Section 7.

4 Regularity of solutions

In this section we study regularity of solutions of (3.4) for bounded frequencies.

First, we show relation between $\widehat{U}_{\xi'}(t,s)$ and $\widehat{U}^*_{\xi'}(s,t)$.

Proposition 4.1 *There holds*

$$\langle \widehat{U}_{\xi'}(t,s)u_0, u_0^*\rangle = \langle u_0, \widehat{U}^*_{\xi'}(s,t)u_0^*\rangle, \tag{4.1}$$

for all $s \le t$, $\xi' \in \mathbb{R}^{n-1}$ *and* $u_0, u_0^* \in X_0$.

Proof. We have relations

$$\partial_t \widehat{U}_{\xi'}(t,s)u_0 = -\widehat{L}_{\xi'}(t)\widehat{U}_{\xi'}(t,s)u_0,\ t > s,\ u_0 \in X_0,$$

and

$$\partial_s \widehat{U}^*_{\xi'}(s,t)u_0^* = \widehat{L}^*_{\xi'}(s)\widehat{U}^*_{\xi'}(s,t)u_0^*,\ s < t,\ u_0^* \in X_0.$$

Let us take $s < \tau < t$. Then for all $u_0, u_0^* \in X_0$ we have

$$\partial_\tau \langle \widehat{U}_{\xi'}(\tau,s)u_0, \widehat{U}^*_{\xi'}(\tau,t)u_0^*\rangle = \langle \partial_\tau \widehat{U}_{\xi'}(\tau,s)u_0, \widehat{U}^*_{\xi'}(\tau,t)u_0^*\rangle + \langle \widehat{U}_{\xi'}(\tau,s)u_0, \partial_\tau \widehat{U}^*_{\xi'}(\tau,t)u_0^*\rangle$$

$$= -\langle \widehat{L}_{\xi'}(\tau)\widehat{U}_{\xi'}(\tau,s)u_0, \widehat{U}^*_{\xi'}(\tau,t)u_0^*\rangle + \langle \widehat{U}_{\xi'}(\tau,s)u_0, \widehat{L}^*_{\xi'}(\tau)\widehat{U}^*_{\xi'}(\tau,t)u_0^*\rangle = 0.$$

Therefore, $\langle \widehat{U}_{\xi'}(\tau,s)u_0, \widehat{U}^*_{\xi'}(\tau,t)u_0^*\rangle$ is independent of $\tau \in (s,t)$, i.e.

$$\langle \widehat{U}_{\xi'}(\tau_1,s)u_0, \widehat{U}^*_{\xi'}(\tau_1,t)u_0^*\rangle = \langle \widehat{U}_{\xi'}(\tau_2,s)u_0, \widehat{U}^*_{\xi'}(\tau_2,t)u_0^*\rangle,$$

for $s < \tau_2 \le \tau_1 < t$ and $u_0, u_0^* \in X_0$. Taking $\tau_1 \to t$ and $\tau_2 \to s$ we get from strong continuity of evolution operators that

$$\langle \widehat{U}_{\xi'}(t,s)u_0, u_0^*\rangle = \langle \widehat{U}_{\xi'}(t,s)u_0, \widehat{U}^*_{\xi'}(t,t)u_0^*\rangle = \langle \widehat{U}_{\xi'}(s,s)u_0, \widehat{U}^*_{\xi'}(s,t)u_0^*\rangle = \langle u_0, \widehat{U}^*_{\xi'}(s,t)u_0^*\rangle,$$

for $s \le t$ and $u_0, u_0^* \in X_0$. □

Now, let us study regularity of solutions of (3.4). To do so, it is convenient to write (3.4) in the following form

$$\partial_t \phi + i\xi_1 v_p^1 \phi + i\xi'\gamma^2(\rho_p w') + \gamma^2 \partial_{x_n}(\rho_p w^n) = f^0, \tag{4.2}$$

$$\begin{aligned} &\partial_t w' + \tfrac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)w' - i\tfrac{\widetilde{\nu}}{\rho_p}\xi'(i\xi' w' + \partial_{x_n} w^n) \\ &\quad + i\xi' \tfrac{\widetilde{P}'(\rho_p)}{\gamma^2\rho_p}\phi + i\xi_1 v_p^1 w' + (\partial_{x_n} v_p^1)w^n + \tfrac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1)\phi = f', \end{aligned} \tag{4.3}$$

$$\partial_t w^n + \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)w^n - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}(i\xi' w' + \partial_{x_n} w^n) + \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\phi\right) + i\xi_1 v_p^1 w^n = f^n, \tag{4.4}$$

for $t > s$, $s \in \mathbb{R}$ and

$$u|_{t=s} = u_0 = {}^T(\phi_0, w_0).$$

Here, $u(\cdot) \in H^1 \cap H^2_*$, $u_0 \in X_0$ and $f = {}^T(f^0, f', f^n) \in X_0$.

In the rest of this section we always assume $|\xi'| \le M$ for some arbitrary, fixed $M < \infty$.

Definition 4.2 *Let* $-\infty < s < b$, $f \in L^2(s,b;X_0)$ *and* $u_0 \in H^1 \times H^1_0$. *We call a function* $u \in C([s,b];H^1 \times H^1_0) \cap L^2(s,b;H^1 \times H^2_*) \cap H^1(s,b;X_0)$ *solution of* (3.4) *if* $u(s) = u_0$ *and equation* (3.4) *is satisfied for a.a.* $t \in (s,b]$.

Let us study inhomogeneous problem (3.4) with regard to solutions introduced above.

Theorem 4.3 *Let $-\infty < s < b < \infty$, $f \in L^2(s,b;X_0)$ and $u_0 \in H^1 \times H_0^1$. There exists $u = {}^T(\phi, w)$ unique solution of* (3.4)*. Moreover, solution u can be written as*

$$u(t) = \widehat{U}_{\xi'}(t,s)u_0 + \int_s^t \widehat{U}_{\xi'}(t,z)f(z)dz, \tag{4.5}$$

and ϕ satisfies

$$\partial_t \partial_{x_n}\phi + \partial_t(P(x_n,t,s))\partial_{x_n}\phi = h - H[u], \tag{4.6}$$

$$\partial_{x_n}\phi(x_n,t) = e^{-P(x_n,t,s)}\partial_{x_n}\phi_0(x_n) + \int_s^t e^{-P(x_n,t,z)}\{h(x_n,z) - H[u](x_n,z)\}dz, \tag{4.7}$$

where

$$P(x_n,t,s) = \int_s^t \{i\xi_1 v_p^1(x_n,z) + \frac{\gamma^2}{\nu+\widetilde{\nu}}\left(\frac{\rho_p P'(\rho_p)}{\gamma^2}\right)(x_n)\}dz;$$

and $\mathrm{Re}\, P(x_n,t,s) \geq a_0 \frac{\gamma^2}{\nu+\widetilde{\nu}}(t-s)$ *thanks to* (3.1)*;*

$$h = \partial_{x_n} f^0 + \frac{\gamma^2\rho_p^2}{\nu+\widetilde{\nu}} f^n,$$

$$H[u] = i\gamma^2\partial_{x_n}\rho_p\xi'\cdot w' + i\gamma^2\frac{\nu}{\nu+\widetilde{\nu}}\rho_p\xi'\cdot\partial_{x_n}w' + i\xi_1\partial_{x_n}v_p^1\phi + \gamma^2[\partial_{x_n}^2, \rho_p]w^n$$

$$+\frac{\gamma^2\rho_p^2}{\nu+\widetilde{\nu}}\{\partial_z w^n + \frac{\nu}{\rho_p}|\xi'|^2 w^n + \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\right)\phi + i\xi_1 v_p^1 w^n\}.$$

Furthermore, there exists $a_1 > 0$ s.t. solution u satisfies

$$\begin{aligned} &|u(t)|_{H^1}^2 + \int_s^t e^{-a(t-z)}(|u|_{H^1\times H^2}^2 + |\partial_z u|_{X_0}^2)dz \\ &\qquad \leq C\{e^{-a(t-s)}|u_0|_{H^1}^2 + \int_s^t e^{-a(t-z)}|f|_{X_0}^2 dz + \int_s^t e^{-a(t-z)}|u|_2^2 dz\}, \end{aligned} \tag{4.8}$$

for $t \in [s,b]$ and $0 \leq a \leq a_1$. Here, $C = C(M)$ is independent of t.

We have an immediate corollary of (4.8).

Corollary 4.4 *Let $f \in L^2(s,b;X_0)$ and $u_0 \in H^1 \times H_0^1$. Solution u of* (3.4) *satisfies an estimate*

$$|u(t)|_{H^1}^2 + \int_s^t |u(z)|_{H^1\times H^2}^2 + |\partial_z u(z)|_{X_0}^2 dz \leq C(|u_0|_{H^1}^2 + |f|_{L^2(s,b:X_0)}^2), \tag{4.9}$$

for all $t \in [s,b]$. Here C depends on $b - s \in (0,\infty)$ and M.

Proof of Theorem 4.3. Let us first show the existence of solution. Set

$$E_{\xi'}(t) = \begin{pmatrix} \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) + \frac{\widetilde{\nu}}{\rho_p}\xi'^T\xi' + i\xi_1 v_p^1(t) I_{n-1} & -i\frac{\widetilde{\nu}}{\rho_p}\xi'\partial_{x_n} + (\partial_{x_n}v_p^1(t))\boldsymbol{e}_1' \\ -i\frac{\widetilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 + i\xi_1 v_p^1(t) \end{pmatrix},$$

$$F_{\xi'}(t) = \begin{pmatrix} i\xi'\frac{P'(\rho_p)}{\gamma^2\rho_p} + \frac{\nu\partial_{x_n}^2 v_p^1(t)}{\gamma^2\rho_p^2}\boldsymbol{e}_1' \\ \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\ \cdot\right) \end{pmatrix}.$$

Lemma 4.5 *The following statements hold true.*

(i) Let $\phi \in C([s,b];H^1)$*,* $w_0 \in H^1_0$ *and* $\widetilde{f} \in L^2(s,b;L^2)$ *then there exists unique*

$$w \in C([s,b];H^1_0) \cap L^2(s,b;H^2_*) \cap H^1(s,b;L^2),$$

that satisfies

$$\partial_t w + E_{\xi'}(t)w = \widetilde{f} - F_{\xi'}(t)\phi, \quad w|_{t=s} = w_0,$$

and

$$|w(t)|^2_{H^1} + \int_s^t (|w|^2_{H^2} + |\partial_z w|^2_2)dz \\ \leq C_b\{|w_0|^2_{H^1} + \int_s^t |\widetilde{f}|^2_2 dz + \int_s^t |\phi|^2_{H^1} dz\},$$

for $t \in [s,b]$*.*

(ii) Let $w \in L^2(s,b;H^2)$*,* $\phi_0 \in H^1$ *and* $f^0 \in L^2(s,b;H^1)$*. Then there exists unique*

$$\phi \in C([s,b];H^1) \cap H^1(s,b;H^1),$$

that satisfies

$$\partial_t \phi + i\xi_1 v_p^1(t)\phi = f^0 - \gamma^2(i\xi' \cdot \rho_p w' + \partial_{x_n}(\rho_p w^n)), \quad \phi|_{t=s} = \phi_0,$$

and

$$|\phi(t)|^2_{H^1} + \int_s^t |\partial_z \phi|^2_{H^1} dz \leq C_b\{|\phi_0|^2_{H^1} + \int_s^t |f^0|^2_{H^1} dz + \int_s^t |w|^2_{H^2} dz\},$$

for $t \in [s,b]$*.*

Proof. (i) is proved by standard parabolic theory .

As for (ii), $\phi(t)$ is given by

$$\phi(t) = e^{-i\xi_1 V_p^1(t,s)}\phi_0 + \int_s^t e^{-i\xi_1 V_p^1(t,z)}(f^0 - \gamma^2(i\xi' \cdot \rho_p w' + \partial_{x_n}(\rho_p w^n)))(z)dz,$$

where $V_p^1(t,s) = \int_s^t v_p^1(z)dz$. This gives us an estimate

$$|\phi(t)|^2_{H^1} \leq C_b\{|\phi_0|^2_{H^1} + \int_s^t |f^0|^2_{H^1} dz + \int_s^t |w|^2_{H^2} dz\}, \tag{4.10}$$

for $t \in [s,b]$. We also have

$$\partial_t \phi = -i\xi_1 v_p^1(t)\phi + f^0 - \gamma^2(i\xi' \cdot \rho_p w' + \partial_{x_n}(\rho_p w^n)).$$

This, together with (4.10), gives the desired estimate on $\int_s^t |\partial_z \phi|^2_{H^1} dz$.

□

We continue the proof of Theorem 4.3. By Lemma 4.5 we can define

$$u_{(n)} = (\phi_{(n)}, w_{(n)}) \in C([s,b] : H^1 \times H^1_0) \cap L^2(s,b;H^1 \times H^2_*) \cap H^1(s,b;X_0),$$

$n = 0, 1, \ldots$ as follows:

- $$w_{(0)} = 0, \quad \phi_{(0)} \text{ is solution of}$$

$$\partial_t \phi_{(0)} + i\xi_1 v_p^1(t)\phi_{(0)} = f^0, \quad \phi_{(0)}|_{t=s} = \phi_0.$$

- For $n \geq 1$, $w_{(n)}$ is solution of

$$\partial_t w_{(n)} + E_{\xi'}(t) w_{(n)} = \widetilde{f} - F_{\xi'}(t)\phi_{(n-1)}, \quad w_{(n)}|_{t=s} = w_0,$$

and $\phi_{(n)}$ is solution of

$$\partial_t \phi_{(n)} + i\xi_1 v_p^1(t)\phi_{(n)} = f^0 - \gamma^2(i\xi' \cdot \rho_p w'_{(n)} + \partial_{x_n}(\rho_p w^n{}_{(n)})), \quad \phi_{(n)}|_{t=s} = \phi_0.$$

By using Lemma 4.5 one can obtain

$$|(u_{(n+1)} - u_{(n)})(t)|^2_{H^1} + \int_s^t (|(w_{(n+1)} - w_{(n)})(z)|^2_{H^2} + |\partial_z(u_{(n+1)} - u_{(n)})(z)|^2_{X_0})dz$$
$$\leq C_0 \frac{(C_b t)^n}{n!}, \quad n = 1, 2, \ldots.$$

It then follows that $\{u^{(n)}\}$ converges to function u in space

$$[C([s,b]; H^1) \cap H^1(s,b; H^1)] \times [C([s,b]; H^1_0) \cap L^2(s,b; H^2) \cap H^1(s,b; L^2)];$$

and the function u is solution of (3.4).

Next, we show that there holds estimate (4.8). First, we introduce some notations

$$E_0[u] = \frac{1}{\gamma^2}\left|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}\phi\right|_2^2 + |\sqrt{\rho_p} w|_2^2,$$

$$D_{\xi'}[w] = |\xi'|^2|w|_2^2 + |\partial_{x_n} w|_2^2, \quad \widetilde{D}_{\xi'}[w] = \nu D_{\xi'}[w] + \widetilde{\nu}|i\xi' w' + \partial_{x_n} w^n|_2^2.$$

In the following calculations we use $|\xi'| \leq M$ and Poincaré inequality

$$|w|_2 \leq |\partial_{x_n} w|_2,$$

whenever it is convenient and without pointing it out. Moreover, C denotes generic constant that may depend on ν and γ. Constants C_j $(j = 1, \ldots, 5)$ are independent of ν and γ.

First, we take $\langle \cdot, \cdot \rangle$-inner product of (3.4) with u, after integrating by parts we take the real part to get

$$\frac{1}{2}\partial_t E_0[u] + \widetilde{D}_{\xi'}[w] = \mathrm{Re}\,\{\langle f, u\rangle - ((\partial_{x_n} v_p^1)\rho_p \widehat{w}^n, w^1) - (\frac{\nu}{\gamma^2\rho_p^2}(\partial^2_{x_n} v_p^1)\rho_p\widehat{\phi}, w^1)\}.$$

Using Hölder inequality we get

$$\frac{1}{2}\partial_t E_0[u] + \frac{1}{2}\widetilde{D}_{\xi'}[w] \leq C(E_0[u] + |f|_2^2). \tag{4.11}$$

Second, we take $\langle \cdot, \cdot \rangle$-inner product of (3.4) with $\widetilde{Q}\partial_t u$, after integrating by parts we take the real part to get

$$\begin{aligned} |\sqrt{\rho_p}\partial_t w|_2^2 + \tfrac{1}{2}\partial_t \widetilde{D}_{\xi'}[w] = \mathrm{Re}\,\{&(\widetilde{Q}f, \rho_p\partial_t w) - i(\xi' \tfrac{P'(\rho_p)}{\gamma^2\rho_p}\phi, \rho_p\partial_t w') - (\partial_{x_n}\left(\tfrac{P'(\rho_p)}{\gamma^2\rho_p}\phi\right), \rho_p\partial_t w^n) \\ &-i(\xi_1 v_p^1 w, \rho_p\partial_t w) + ((\partial_{x_n} v_p^1)w^n, \rho_p\partial_t w^1) - (\tfrac{\nu}{\gamma^2\rho_p^2}(\partial^2_{x_n} v_p^1)\phi, \rho_p\partial_t w^1)\}. \end{aligned} \tag{4.12}$$

We treat the third item on the righthand side of (4.12) to get

$$\begin{aligned} -(\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\phi\right), \rho_p\partial_t w^n) &= -(\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\right)\phi, \rho_p\partial_t w^n) - (\frac{P'(\rho_p)}{\gamma^2\rho_p}\partial_{x_n}\phi, \rho_p\partial_t w^n) \\ &= -(\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\right)\phi, \rho_p\partial_t w^n) + (\phi, \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2}\right)\partial_t w^n) + (\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\partial_t w^n). \end{aligned} \tag{4.13}$$

Next, we rewrite $(\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\partial_t w^n)$ as

$$(\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\partial_t w^n) = \frac{\partial}{\partial t}(\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) - (\partial_t\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n).$$

Substituting relation (4.2) for $\partial_t\phi$ we get

$$(\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\partial_t w^n) = \frac{\partial}{\partial t}(\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) + (i\xi_1 v_p^1\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) + (i\xi'\gamma^2\rho_p w', \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) \quad (4.14)$$

$$+(\gamma^2\partial_{x_n}(\rho_p w^n), \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) - (f^0, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n).$$

Substituting (4.13) and (4.14) into (4.12) and using Hölder inequality on the resulting equation we get

$$\frac{1}{2}|\sqrt{\rho_p}\partial_t w|_2^2 + \frac{1}{2}\partial_t \widetilde{D}_{\xi'}[w] - \frac{\partial}{\partial t}\mathrm{Re}\,(\phi, \frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) \le C_1\frac{\gamma^2}{\nu}\widetilde{D}_{\xi'}[w] + C\{E_0[u] + |f|_2^2\}. \quad (4.15)$$

We see that

$$\begin{aligned} &\widetilde{D}_{\xi'}[w] - 2\mathrm{Re}\,(\phi, \tfrac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) \\ &\ge \nu|\partial_{x_n} w|_2^2 - (\tfrac{\nu}{2}|\partial_{x_n} w|_2^2 + \tfrac{b_0}{\nu}|\sqrt{\tfrac{P'(\rho_p)}{\gamma^2\rho_p}}\phi|_2^2) = \tfrac{\nu}{2}|\partial_{x_n} w|_2^2 - \tfrac{b_0}{\nu}|\sqrt{\tfrac{P'(\rho_p)}{\gamma^2\rho_p}}\phi|_2^2. \end{aligned} \quad (4.16)$$

Now adding $2(1+\frac{b_1\gamma^2}{\nu})\times$(4.11) to $2\times$(4.15) with $b_1 > \max\{4C_1, 2b_0\}$ suitably large we get

$$\begin{aligned} &(1+\tfrac{b_1\gamma^2}{\nu})\partial_t E_0[u] + \partial_t\widetilde{D}_{\xi'}[w] - \tfrac{\partial}{\partial t}2\mathrm{Re}\,(\phi, \tfrac{P'(\rho_p)}{\gamma^2}\partial_{x_n} w^n) + (1+\tfrac{b_1\gamma^2}{2\nu})\widetilde{D}_{\xi'}[w] + |\sqrt{\rho_p}\partial_t w|_2^2 \\ &\le C\{E_0[u] + |f|_2^2\}. \end{aligned} \quad (4.17)$$

Third, we differentiate (4.2) with respect to x_n to get

$$\partial_t\partial_{x_n}\phi + i\xi_1\partial_{x_n}(v_p^1\phi) + i\xi'\cdot\gamma^2\partial_{x_n}(\rho_p w') + \gamma^2\partial_{x_n}^2(\rho_p w^n) = \partial_{x_n} f^0. \quad (4.18)$$

Now, we multiply (4.4) by $\frac{\gamma^2\rho_p^2}{\nu+\widetilde{\nu}}$ and add together with (4.18), this gives us (4.6). Next, we multiply (4.6) by $e^{P(x_n,t,s)}$ and by integrating the resulting equation in time we obtain (4.7).

We take $(\cdot,\cdot)$-inner product of (4.6) with $\frac{P'(\rho_p)}{\gamma^4\rho_p}\partial_{x_n}\phi$ and take the real part to get

$$\frac{\partial}{\partial t}\frac{1}{2\gamma^2}|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}\partial_{x_n}\phi|_2^2 + \frac{1}{\nu+\widetilde{\nu}}|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\phi|_2^2 = \mathrm{Re}\,(h - H[u], \frac{P'(\rho_p)}{\gamma^4\rho_p}\partial_{x_n}\phi).$$

Using Hölder inequality we get

$$\frac{\partial}{\partial t}\frac{1}{2\gamma^2}|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}\partial_{x_n}\phi|_2^2 + \frac{1}{\nu+\widetilde{\nu}}\frac{3}{4}|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\phi|_2^2 \le C\frac{\nu+\widetilde{\nu}}{\gamma^4}|\frac{1}{\rho_p}(h - H[u])|_2^2. \quad (4.19)$$

Here

$$C\frac{\nu+\widetilde{\nu}}{\gamma^4}|\frac{1}{\rho_p}(h-H[u])|_2^2 \le C\{E_0[u] + |f|_{X_0}\} + C_2\{\frac{1}{\nu+\widetilde{\nu}}|\sqrt{\rho_p}\partial_t w|_2^2 + (1+\frac{\nu+\widetilde{\nu}}{\nu}|\partial_{x_n}\rho_p|_2^2)\widetilde{D}_{\xi'}[w]\}.$$

Fourth, estimating $\partial_{x_n}^2 w$ in L^2-norm from (4.3) and (4.4) we get

$$|\partial_{x_n}^2 w|_2^2 \le C\{E_0[u] + |f|_2^2\} + C_3\{\frac{1}{\nu^2}|\sqrt{\rho_p}\partial_t w|_2^2 + \frac{1}{(\nu+\widetilde{\nu})^2}|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\phi|_2^2 + \frac{1+\widetilde{\nu}}{\nu^2}\widetilde{D}_{\xi'}[w]\}. \quad (4.20)$$

Fifth, we take L^2-norm of (4.2) and by using Hölders inequality we get

$$|\partial_t\phi|_2^2 \le C\{E_0[u] + |f|_2^2\} + C_4\frac{\gamma^4}{\nu}\widetilde{D}_{\xi'}[w]. \quad (4.21)$$

Sixth, we take L^2-norm of (4.18) and by using Hölders inequality we get

$$|\partial_t\partial_{x_n}\phi|_2^2 \le C\{E_0[u] + |f|_{X_0}^2\} + C_5\{|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}\phi|_2^2 + \frac{\gamma^4}{\nu}\widetilde{D}_{\xi'}[w] + \gamma^4|\partial_{x_n}^2 w|_2^2\}. \quad (4.22)$$

By a suitable linear combination of (4.11), (4.17), (4.19)–(4.22) it is possible to find constant $a_1 > 0$ such that after adding $a_1 \times E_0[u]$ on both sides of the suitable linear combination we get

$$\frac{\partial}{\partial t}E_1[u] + a_1 E_1[u] + \widetilde{C}(|u|_{H^1\times H^2} + |\partial_t u|^2_{X_0}) \le C(E_0[u] + |f|^2_{X_0}),$$

where $\widetilde{C} > 0$ and

$$c_1|u|_{H^1} \le E_1[u] \le c_2|u|_{H^1},\ c_1, c_2 > 0,$$

using (4.16).

Finally, multiplying by $e^{a_1 t}$ and integrating with respect to t we get

$$\begin{aligned} E_1[u](t) + \int_s^t e^{-a_1(t-z)}(|u|_{H^1\times H^2} + |\partial_z u|^2_{X_0})dz \\ \le e^{-a_1(t-s)}E_1[u_0] + C\int_s^t e^{-a_1(t-z)}(|f|^2_{X_0} + E_0[u])dz, \quad \text{for } t \ge s. \end{aligned} \tag{4.23}$$

This concludes the proof of (4.8).

Using Gronwall inequality on (4.8) we obtain (4.9) and consequently the uniqueness of solution u. Next, we prove the variation of constants formula (4.5).

Lemma 4.6 *Let $u = (\phi, w)$ is solution of* (3.4).

(i) If, in addition, $w_0 \in H^2_*$ and $\widetilde{Q}f \in \bigcap_{j=0}^1 H^j(s, b; H^{1-2j})$, then

$$w \in \bigcap_{j=0}^1 C^j([s,b]; H^{2-2j}_*) \cap H^1(s, b; H^1_0),$$

and we have an estimate

$$\begin{aligned} |\partial^2_{x_n} w(t)|^2_2 + |\partial_t w(t)|^2_2 + \int_s^t e^{-a(t-z)}|\partial_z w|^2_{H^1}dz \\ \le C\{e^{-a(t-s)}(|\widetilde{Q}f(0)|^2_2 + |u_0|^2_{H^1\times H^2}) + \int_s^t e^{-a(t-z)}(|\widetilde{Q}f|^2_2 + |\partial_z\widetilde{Q}f|^2_{H^{-1}} + |\phi|^2_{H^1} + |\partial_z\phi|^2_2)dz\}, \end{aligned} \tag{4.24}$$

for $t \in [s, b]$ and every $a \ge 0$.

(ii) If, in addition to the assumptions in (i), it holds $f^0 \in C([s,b]; H^1)$, then

$$\phi \in C^1([s,b]; H^1).$$

Proof. As for (i), we give an outline of a proof.

$$\partial_t w + \widetilde{A}_{\xi'}(t)w = F, \quad w|_{t=s} = w_0, \quad F = \widetilde{Q}f - \widetilde{B}_{\xi'}(t)\phi. \tag{4.25}$$

Since $\widetilde{B}_{\xi'}(t)\phi \in C([s,b]; L^2)$, we have $F \in C([s,b]; L^2)$. Furthermore,

$$\partial_t(\widetilde{B}_{\xi'}(t)\phi) = \partial_t\widetilde{B}_{\xi'}(t)\phi + \widetilde{B}_{\xi'}(t)\partial_t\phi \in L^2(s, b; L^2),$$

thus we have $\partial_t F \in L^2(s, b; H^{-1})$. Considering ∂_t(4.25) one can show that

$$\partial_t w \in C([s,b]; L^2) \cap L^2(s, b; H^1_0) \cap H^1(s, b; H^{-1}).$$

It then follows from (4.25) that

$$w \in C([s,b]; H^2_*) \cap L^2(s, b; H^3_*) \cap H^1(s, b; H^1_0).$$

Estimate (4.24) follows in a standard manner.

As for (ii),

$$\partial_t\phi = -i\xi_1 v^1_p(t)\phi + f^0 - \gamma^2(i\xi'\rho_p w' + \partial_{x_n}(\rho_p w^n)) \in C([s,b]; H^1).$$

This completes the proof of Lemma 4.6. □

Let us continue the proof of (4.5). Let $u_0 \in H^1 \times H^1_0$ and $f \in L^2(s,b;X_0)$. Take $\{u_{0,n}\}$ and $\{f_n\}$ such that

$$u_{0,n} \in H^1 \times H^2_*, \quad f_n \in C^1([s,b];X_0),$$

and

$$u_{0,n} \to u_0 \text{ in } H^1 \times H^1_0, \quad f_n \to f \text{ in } L^2(s,b;X_0).$$

Then the existence part of Theorem 4.3 and Lemma 4.6 gives us unique functions

$$u_n = (\phi_n, w_n) \in C^1([s,b];X_0), w_n \in C([s,b];H^2_*),$$

satisfying

$$\partial_t u_n + \widehat{L}_{\xi'}(t)u_n = f_n, \; s < t, \; u_n|_{t=s} = u_{0,n},$$

and from (4.9) we get

$$\begin{aligned} u_n \to u \text{ in } C([s,b];H^1 \times H^1_0), \\ \phi_n \to \phi \text{ in } H^1(s,b;H^1), \\ w_n \to w \text{ in } L^2(s,b;H^2) \cap H^1(s,b;L^2). \end{aligned}$$

On the other hand, each u_n is represented as

$$u_n(t) = \widehat{U}_{\xi'}(t,s)u_{0,n} + \int_s^t \widehat{U}_{\xi'}(t,z)f_n(z)dz,$$

(see, e.g. [7, 8, 9]). Taking limit $n \to \infty$ we get

$$u(t) = \widehat{U}_{\xi'}(t,s)u_0 + \int_s^t \widehat{U}_{\xi'}(t,z)f(z)dz.$$

This concludes the proof of Theorem 4.3.

□

Next, we show results on higher regularity of solutions.

Lemma 4.7 *Let $1 \le k \le m$ and $-\infty < s < b < \infty$. The following statements hold true.*

(i) Let $\phi_0 \in H^{k+1}$, $f \in L^2(s,b;H^{k+1} \times H^k)$. Moreover, let u be solution of (3.4). If u satisfies

$$u \in L^2(s,b;H^k \times H^{k+1}), \quad w \in H^1(s,b;H^k),$$

then

$$\phi \in C([s,b];H^{k+1}).$$

Furthermore, we have an estimate

$$\begin{aligned} |\partial_{x_n}^{k+1}\phi(t)|_2^2 \le C\{e^{-a(t-s)}|\phi_0|^2_{H^{k+1}} + \int_s^t e^{-a(t-z)}|f(z)|^2_{H^{k+1}\times H^k}dz \\ + \int_s^t e^{-a(t-z)}(|u(z)|^2_{H^k \times H^{k+1}} + |\partial_z w(z)|^2_{H^k})dz\}, \end{aligned}$$

for $t \in [s,b]$ and $0 \le a < a_0\frac{\gamma^2}{\nu+\tilde{\nu}}$.

(ii) Let $f \in L^2(s,b;H^{k+1} \times H^k)$. Moreover, let u be solution of (3.4). If u satisfies

$$u \in L^2(s,b;H^{k+1}), \quad \partial_t w \in L^2(s,b;H^k),$$

then

$$\partial_t \phi \in L^2(s,b;H^{k+1}),$$

and we have an estimate

$$\int_s^t e^{-a(t-z)}|\partial_{x_n}^{k+1}\partial_z\phi|_2^2 dz \le C\int_s^t e^{-a(t-z)}\{|f|_{H^{k+1}\times H^k}^2 + |u|_{H^{k+1}}^2 + |\partial_z w|_{H^k}^2\}dz,$$

for $t \in [s,b]$ and every $a \ge 0$.

(iii) Let $f \in L^2(s,b;H^k)$. Moreover, let u be solution of (3.4). If u satisfies

$$u \in L^2(s,b;H^{k+1}), \quad \partial_t w \in L^2(s,b;H^k),$$

then

$$w \in L^2(s,b;H^{k+2}),$$

and we have an estimate

$$\int_s^t e^{-a(t-z)}|\partial_{x_n}^{k+2}w|_2^2 dz \le C\int_s^t e^{-a(t-z)}\{|\partial_{x_n}^k \widetilde{Q}f|_2^2 + |u|_{H^{k+1}}^2 + |\partial_z\partial_{x_n}^k w|_2^2\}dz,$$

for $t \in [s,b]$ and every $a \ge 0$.

In particular, if $f \in C([s,b];H^k)$, $u \in C([s,b];H^{k+1})$ and $\partial_t w \in C([s,b];H^k)$ then

$$w \in C([s,b];H^{k+2}).$$

(iv) Let $w_0 \in H_*^k$. Moreover, let $w \in L^2(s,b;H_*^{k+1}) \cap H^1(s,b;H_*^{k-1})$ be a function satisfying

$$w(s) = w_0,$$

then

$$|\partial_{x_n}^k w(t)|_2^2 \le C\{e^{-a(t-s)}|w_0|_{H^k}^2 + \int_s^t e^{-a(t-z)}(|w|_{H^{k+1}}^2 + |\partial_z w|_{H^{k-1}}^2)dz\},$$

for $t \in [s,b]$ and every $a \ge 0$.

Proof. (i) follows from taking L^2-norm of $\partial_{x_n}^k$(4.7) and using Hölder inequality.

As for (ii), take $(\cdot,\cdot)$ inner product of $\partial_{x_n}^k$(4.6) with $\partial_{x_n}^{k+1}\partial_t\phi$ to get

$$|\partial_{x_n}^{k+1}\partial_t\phi|_2^2 \le C\{|\phi|_{H^{k+1}}^2 + |f|_{H^{k+1}\times H^k}^2 + |u|_{H^k\times H^{k+1}}^2 + |\partial_t w|_{H^k}^2\}.$$

Finally, as for (iii), $\partial_{x_n}^k$(4.3) and $\partial_{x_n}^k$(4.4) read as follows:

$$\partial_{x_n}^{k+2}w' = \frac{\rho_p}{\nu}\{-[\partial_{x_n}^k, \frac{\nu}{\rho_p}]\partial_{x_n}^2 w' + \partial_{x_n}^k(\frac{\nu}{\rho_p}|\xi'|^2 w') + \partial_t\partial_{x_n}^k w' - \partial_{x_n}^k(\frac{\widetilde{\nu}}{\rho_p}i\xi'(i\xi'\cdot w' + \partial_{x_n}w^n))$$
$$+i\xi'\partial_{x_n}^k(\frac{P'(\rho_p)}{\gamma^2\rho_p}\phi) + \partial_{x_n}^k(i\xi_1 v_p^1 w') + \partial_{x_n}^k(\frac{\nu\partial_{x_n}^2 v_p^1}{\gamma^2\rho_p^2}\phi)\boldsymbol{e}_1' + \partial_{x_n}^k(\partial_{x_n}v_p^1 w^n)\boldsymbol{e}_1' - \partial_{x_n}^k f'\},$$

and

$$\partial_{x_n}^{k+2}w^n = \frac{\rho_p}{\widetilde{\nu}+\nu}\{-[\partial_{x_n}^k, \frac{\widetilde{\nu}+\nu}{\rho_p}]\partial_{x_n}^2 w^n + \partial_{x_n}^k(\frac{\nu}{\rho_p}|\xi'|^2 w^n) + \partial_t\partial_{x_n}^k w^n - \partial_{x_n}^k(\frac{\widetilde{\nu}}{\rho_p}i\xi'\cdot\partial_{x_n}w')$$
$$+\partial_{x_n}^{k+1}(\frac{P'(\rho_p)}{\gamma^2\rho_p}\phi) + \partial_{x_n}^k(i\xi_1 v_p^1 w^n) - \partial_{x_n}^k f^n\}.$$

Desired estimate in (iii) now easily follows.

As for (iv), there exists an extension operator $E = E_{k+1}$, such that $E[v] = v$ a.e. on $(0,1)$ and $E[v] \in H^{k+1}(\mathbb{R})$ for $v \in H_*^{k+1}(0,1)$. Moreover, there holds

$$|E[v]|_{H^l(\mathbb{R})} \le C|v|_{H^l(0,1)}, \ 0 \le l \le k+1,$$

where $C > 0$ is independent of $v \in H^{k+1}(0,1)$. Thus for any $a \ge 0$ we have

$$e^{a(t-s)}|\partial_{x_n}^k w(t)|_{L^2(0,1)}^2 \le e^{a(t-s)}|\partial_{x_n}^k Ew(t)|_{L^2(\mathbb{R})}^2$$

$$\le C\{|\partial_{x_n}^k E[w_0]|_{L^2(\mathbb{R})}^2 + \int_s^t e^{az}(a|\partial_{x_n}^k E[w](z)|_{L^2(\mathbb{R})}^2 + |\partial_{x_n}^{k+1} E[w](z)|_{L^2(\mathbb{R})}^2 + |\partial_{x_n}^{k-1}\partial_z E[w](z)|_{L^2(\mathbb{R})}^2)dz\},$$

$$\le C\{|w_0|_{H^k(0,1)}^2 + \int_s^t e^{az}(|w(z)|_{H^{k+1}(0,1)} + |\partial_z w(z)|_{H^{k-1}(0,1)})dz\}.$$

□

Theorem 4.8 *Let $-\infty < s < b < \infty$, $u_0 \in H^2 \times H_*^2$ and $f \in \bigcap_{j=0}^{1} H^j(s,b;H^{2-2j} \times H^{1-2j})$. There exists solution u of* (3.4) *and u satisfies*

$$u \in \bigcap_{j=0}^{1} C^j([s,b];H^{2-2j} \times H_*^{2-2j}),$$

$$\phi \in \bigcap_{j=0}^{1} H^{j+1}(s,b;H^{2-2j}), \quad w \in \bigcap_{j=0}^{1} H^j(s,b;H_*^{3-2j}).$$

Furthermore, there exists $a_2 > 0$ and the following estimate holds

$$\begin{aligned} &\sum_{j=0}^{1} |\partial_t^j u(t)|_{H^{2-2j}}^2 + \int_s^t e^{-a(t-z)}(|\partial_z u|_{H^2\times H^1}^2 + |\partial_z^2\phi|_2^2 + |u|_{H^2\times H^3}^2)dz \\ &\le C\{e^{-a(t-s)}(|u_0|_{H^2}^2 + |f(0)|_2^2) + \int_s^t e^{-a(t-z)}\sum_{j=0}^{1}|\partial_z^j f|_{H^{2-2j}\times H^{1-2j}}^2 dz + \int_s^t e^{-a(t-z)}|u|_2^2 dz\}, \end{aligned} \tag{4.26}$$

for $t \in [s,b]$ and $0 \le a < a_2$; C is uniform for $a \in [0,(a_2-\delta)](\delta > 0)$.

Proof. From Theorem 4.3 and Lemma 4.6 (i) we get that there exists unique solution u and it satisfies (4.8) and (4.24). Since by (4.2)

$$\partial_t\phi = -i\xi_1 v_p^1\phi - i\xi'\gamma^2\rho_p w' - \gamma^2\partial_{x_n}(\rho_p w^n) + f^0,$$

we have

$$|\partial_t\phi(t)|_2^2 \le C\{|u(t)|_{L^2\times H^1}^2 + e^{-a(t-s)}|f^0(0)|_2^2 + \int_s^t e^{-a(t-z)}(|f^0|_2^2 + |\partial_z^0 f|_2^2)dz\},$$

and

$$\int_s^t e^{-a(t-z)}|\partial_z^2\phi|_2^2 dz \le C\int_s^t e^{-a(t-z)}\{|\phi|_2^2 + |\partial_z u|_{L^2\times H^1}^2 + |\partial_z f^0|_2^2\}dz,$$

for every $a \ge 0$. Combining these two estimates with (4.8), (4.24) and estimates from Lemma 4.7 (i)–(iii) with $k = 1$ we obtain (4.26) for $a_2 = \min\{a_0\frac{\gamma^2}{\nu+\tilde{\nu}}, a_1\}$. □

In the same way as we obtained Theorems 4.3 and 4.8 one can obtain the following result for solutions of *adjoint problem* (3.5).

Theorem 4.9 *Let $-\infty < b < t < \infty$ and $1 \le k \le 2$. Let $u_0 \in H^k \times H_*^k$ and $f \in \bigcap_{j=0}^{[\frac{k}{2}]} H^j(b,t;H^{k-2j} \times H^{k-1-2j})$. There exists $u(s) = {}^T(\phi(s),w(s))$ that satisfies* (3.5) *for a.a. $s \in [b,t)$, $u(t) = u_0$ and*

$$u \in \bigcap_{j=0}^{[\frac{k}{2}]} C^j([b,t];H^{k-2j} \times H_*^{k-2j}),$$

$$\phi \in \bigcap_{j=0}^{[\frac{k}{2}]} H^{j+1}(b,t;H^{k-2j}), \quad w \in \bigcap_{j=0}^{[\frac{k+1}{2}]} H^{j}(b,t;H_*^{k+1-2j}).$$

Furthermore, the following estimate holds

$$\sum_{j=0}^{[\frac{k}{2}]} |\partial_s^j u(s)|^2_{H^{k-2j}} + \int_s^t e^{a(s-z)}(\sum_{j=0}^{[\frac{k-1}{2}]} |\partial_z^{j+1} u|^2_{H^{k-2j}\times H^{k-1-2j}} + |\partial_z^{[\frac{k+2}{2}]}\phi|^2_{H^{k-2[\frac{k}{2}]}} + |u|^2_{H^k\times H^{k+1}})dz$$

$$\le C\{e^{a(s-t)}(|u_0|^2_{H^k} + \sum_{j=0}^{[\frac{k}{2}]-1} |\partial_s^j f(0)|^2_{H^{k-2(j+1)}}) + \int_s^t e^{a(s-z)} \sum_{j=0}^{[\frac{k}{2}]} |\partial_z^j f|^2_{H^{k-2j}\times H^{k-1-2j}} dz + \int_0^t e^{a(s-z)}|u|_2^2 dz\},$$

for $s \in [b,t]$ *and* $0 \le a < a_2$*;* C *is uniform for* $a \in [0,(\widetilde{a}-\delta)]$*,* $\delta > 0$*. Here,* $\sum_{j=0}^{[\frac{k}{2}]-1} |\partial_s^j f(0)|^2_{H^{k-2(j+1)}} = 0$ *when* $k = 1$*.*

Moreover, u *can be written as*

$$u(s) = \widehat{U}^*_{\xi'}(s,t)u_0 + \int_s^t \widehat{U}^*_{\xi'}(s,z)f(z)dz.$$

5 Spectral theory for $\widehat{U}_{\xi'}(T,0)$ and $\widehat{U}^*_{\xi'}(0,T)$

In this section we study spectral properties of operators $\widehat{U}_{\xi'}(T,0)$ and $\widehat{U}^*_{\xi'}(0,T)$. At the end of this section we give a proof of Proposition 3.5.

Lemma 5.1 *Following assertions hold true.*

(i) Let $u^{(0)}(t)$ *be defined as*

$$u^{(0)}(x_n,t) = {}^T(\phi^{(0)}(x_n), \frac{1}{\gamma^2}w^{(0),1}(x_n,t), 0).$$

Here

$$\phi^{(0)}(x_n) = \alpha_0 \frac{\gamma^2\rho_p(x_n)}{P'(\rho_p(x_n))}, \qquad \alpha_0 = \left[\frac{\gamma^2\rho_p}{P'(\rho_p)}\right]^{-1},$$

$$w^{(0),1}(x_n,t) = -\int_{-\infty}^t e^{-(t-z)\nu A}\nu\frac{\alpha_0\gamma^2}{P'(\rho_p)\rho_p}(\partial_{x_n}^2 v_p^1(z))\,dz,$$

where A *denotes the uniformly elliptic operator on* $L^2(0,1)$ *with domain* $D(A) = H^2_*(0,1)$ *and*

$$Av = -\frac{1}{\rho_p(x_n)}\partial_{x_n}^2 v, \tag{5.1}$$

for $v \in D(A)$*.*

Function $u^{(0)}(t)$ *satisfies* $\partial_t u^{(0)} + \widehat{L}_0(t)u^{(0)} = 0$ *and* $u^{(0)}(t) = u^{(0)}(t+T)$ *for all* $t \in \mathbb{R}$*.*

(ii) Let $u^{(0)*}$ *be defined as*

$$u^{(0)*}(x_n) = {}^T(\frac{\gamma^2}{\alpha_0}\phi^{(0)}(x_n), 0, 0).$$

Function $u^{(0)*}$ *satisfies* $-\partial_s u^{(0)*} + \widehat{L}_0^*(s)u^{(0)*} = 0$ *for all* $s \in \mathbb{R}$*. Moreover, there holds*

$$\langle u^{(0)}(t), u^{(0)*}\rangle = 1,$$

for $t \in \mathbb{R}$*.*

Proof. (i) and (ii) are obtained by straightforward computation (see also [1]). □

Next, let us introduce result proved in [1, Theorem 5.11 and Theorem 5.16 (ii)].

Proposition 5.2 *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that for each ξ' with $|\xi'| \leq r_0$ there hold the following statements.*

The spectrum of operator $\widehat{U}_{\xi'}(T,0)$ on X_0 satisfies

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\},$$

for a constant q_0 with $\frac{3}{2}q_0 \leq \operatorname{Re}\mu_{\xi'} \leq 1$. Here, $\mu_{\xi'} = e^{\lambda_{\xi'}T}$ is simple eigenvalue of $\widehat{U}_{\xi'}(T,0)$ on X_0 with associated eigenvector $u^{(0)}_{\xi'} \in X_0$; $u^{(0)}_{\xi'}|_{\xi'=0} = u^{(0)}(0)$. Moreover, $\lambda_{\xi'}$ has the expansion (3.7).

Next lemma shows exponential decay of solution operator $\widehat{U}_{\xi'}(t,0)$ on X_0.

Lemma 5.3 *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that following statements hold for each ξ' with $|\xi'| \leq r_0$.*

Let $\widehat{\Pi}_{\xi'}$ denote the eigenprojections associated with $\mu_{\xi'}$. There exists constant $d > 0$ such that $u(t)$ satisfies

$$|\widehat{U}_{\xi'}(t,s)u_0|^2_{H^1} \leq Ce^{-a(t-s)}|u_0|^2_{X_0},\ T \leq t-s, \tag{5.2}$$

for $u_0 \in (I-\widehat{\Pi}_{\xi'})X_0$ and $0 \leq a \leq d$. Here, d depends on r_0.

Proof. Let the assumptions of Proposition 5.2 be satisfied. Therefore, there holds

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subset \{\mu : |\mu| \ \leq q_0\},$$

on $(I-\widehat{\Pi}_{\xi'})X_0$ for $q_0 < \operatorname{Re}\mu_{\xi'} < 1$.

This means that the spectral radius of $\widehat{U}_{\xi'}(T,0)$ on $(I-\widehat{\Pi}_{\xi'})X_0$ satisfies

$$r(\widehat{U}_{\xi'}(T,0)|_{X_0}) \leq q_0 < 1.$$

From general theory there holds

$$r(\widehat{U}_{\xi'}(T,0)|_{(I-\widehat{\Pi}_{\xi'})X_0}) = \lim_{n\to\infty} |[\widehat{U}_{\xi'}(T,0)(I-\widehat{\Pi}_{\xi'})]^n|^{\frac{1}{n}}_{L(X_0)}.$$

Thus there exist $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ there holds

$$|[\widehat{U}_{\xi'}(T,0)(I-\widehat{\Pi}_{\xi'})]^n|^{\frac{1}{n}}_{L(X_0)} \leq q_0 + \varepsilon < 1.$$

Let $\frac{d}{2} = \frac{1}{T}\log\frac{1}{q_0+\varepsilon}$. Since $q_0+\varepsilon < 1$ we have $d > 0$ and

$$|[\widehat{U}_{\xi'}(T,0)(I-\widehat{\Pi}_{\xi'})]^n|^{\frac{1}{n}}_{L(X_0)} \leq e^{-\frac{d}{2}T},$$

for $n \geq N_0$.

Taking γ_0 suitably smaller and $|\xi'| < 1$, we get from [1, Lemma 5.17] that

$$\begin{aligned} &|\widehat{U}_{\xi'}(\tau,\zeta)u_0|_{X_0} \leq C|u_0|_{X_0}, \\ &|\partial_{x_n}\widetilde{Q}\widehat{U}_{\xi'}(\tau,\zeta)u_0|_2 \leq C(\tau-\zeta)^{-\frac{1}{2}}|u_0|_{X_0}, \end{aligned} \tag{5.3}$$

for $0 < \tau - \zeta \leq 2T$.

For $t-s > 0$ let $N = \left[\frac{t-s}{T}\right] - 1$ and for $u_0 \in (I-\widehat{\Pi}_{\xi'})X_0$ we write

$$\widehat{U}_{\xi'}(t,s)u_0 = \widehat{U}_{\xi'}(t,s+NT)[\widehat{U}_{\xi'}(T,0)(I-\widehat{\Pi}_{\xi'})]^N u_0.$$

From (5.3) we see that $|\widehat{U}_{\xi'}(t,s+NT)v|^2_{H^1} \leq \frac{C}{T^{\frac{1}{2}}}|v|^2_{X_0}$ and thus we get an estimate

$$|\widehat{U}_{\xi'}(t,s)u_0|^2_{H^1} \leq |\widehat{U}_{\xi'}(t,s+NT)|^2_{L(X_0,H^1)}|[\widehat{U}_{\xi'}(T,0)(I-\widehat{\Pi}_{\xi'})]^N u_0|^2_{X_0}$$

$$\leq Ce^{-dNT}|u_0|^2_{X_0} \leq Ce^{-d(t-s)}|u_0|^2_{X_0},$$

for $t-s \geq (N_0+1)T$ and $u_0 \in (I-\widehat{\Pi}_{\xi'})X_0$.

We use (5.3) for $T \leq t-s < (N_0+1)T$ repeatedly to get

$$|\widehat{U}_{\xi'}(t,s)u_0|^2_{H^1} \leq \frac{C}{T^{\frac{1}{2}}}|u_0|^2_{X_0} \leq Ce^{-d(t-s)}|u_0|^2_{X_0}.$$

Taking constant $C > 0$ suitably large we obtain (5.2) for all $t-s \geq T$.

□

Next let us treat spectral properties of $\widehat{U}^*_{\xi'}(0,T)$.

Lemma 5.4 *There exists constant $\gamma_0 > 0$ such that if $\gamma \geq \gamma_0$ then spectrum of operator $\widehat{U}^*_0(0,T)$ on $H^1 \times H^1_0$ satisfies*

$$\sigma(\widehat{U}^*_0(0,T)) \subset \{1\} \cup \{|\mu| < \widetilde{q}_0 < 1\}. \tag{5.4}$$

Here, 1 is the simple eigenvalue of $\widehat{U}^*_0(0,T)$ with associated eigenvector $u^{(0)*} \in H^1 \times H^1_0$.

Lemma 5.5 *There exist positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ then there exists $r_0 > 0$ such that following statements hold for each ξ' with $|\xi'| \leq r_0$.*

The spectrum of operator $\widehat{U}^*_{\xi'}(0,T)$ on $H^1 \times H^1_0$ satisfies

$$\sigma(\widehat{U}^*_{\xi'}(0,T)) \subset \{\overline{\mu}_{\xi'}\} \cup \{|\mu| < q_0\}, \tag{5.5}$$

for a constant q_0 with $\frac{3}{2}q_0 \leq \mathrm{Re}\,\overline{\mu}_{\xi'} \leq 1$. Here, $\overline{\mu}_{\xi'}$ is the simple eigenvalue of $\widehat{U}^*_{\xi'}(0,T)$ with associated eigenvector $u^{(0)*}_{\xi'} \in H^1 \times H^1_0$; $u^{(0)*}_{\xi'}|_{\xi'=0} = u^{(0)*}$.

Proofs of Lemmas 5.4 and 5.5 are given in Appendix.

The following lemma is direct consequence of estimates in Lemmas 3.3 and 3.4.

Lemma 5.6 *Let $\xi' \in \mathbb{R}^{n-1}$. Operators $\widehat{U}_{\xi'}(T,0)$ and $\widehat{U}^*_{\xi'}(0,T)$ are bounded from X_0 to $H^1 \times H^2_*$, i.e.*

$$|\widehat{U}_{\xi'}(T,0)v|_{H^1\times H^2} \leq C|v|_{X_0},$$

$$|\widehat{U}^*_{\xi'}(0,T)v|_{H^1\times H^2} \leq C|v|_{X_0},$$

for all $v \in X_0$ and a constant C is bounded for ξ' bounded.

Now, we are ready to prove Proposition 3.5.

Proof of Proposition 3.5 Let the assumptions of Proposition 5.2, Lemmas 5.3 and 5.5 are satisfied.

As for (i), we showed (3.6) on X_0 in Proposition 5.2 together with (3.7). Let $\mu \in \{\mu : |\mu| > q_0\} \backslash \{\mu_{\xi'}\}$. Then for $u, v \in X_0$ we have:

$$(\mu - \widehat{U}_{\xi'}(T,0))^{-1}u = v \Leftrightarrow u = (\mu - \widehat{U}_{\xi'}(T,0))v \Leftrightarrow \mu v = u + \widehat{U}_{\xi'}(T,0)v.$$

By (5.3) we have

$$|\widehat{U}_{\xi'}(T,0)v|_{H^1} \leq C|v|_{X_0}.$$

Boundedness of the inverse operator on X_0 reads as

$$|v|_{X_0} \leq C|u|_{X_0}.$$

Therefore

$$|\mu(\mu - \widehat{U}_{\xi'}(T,0))^{-1}u|_{H^1} \leq |u|_{H^1} + |\widehat{U}_{\xi'}(T,0)v|_{H^1} \leq C(|u|_{H^1} + |u|_{X_0}).$$

We see that $\widehat{U}_{\xi'}(T,0)v \in H^1 \times H^2_*$ for $v \in X_0$ which together with $u \in H^1 \times H^1_0$ gives us $v \in H^1 \times H^1_0$. Thus we proved (3.6) on $H^1 \times H^1_0$. Moreover, $u^{(0)}_{\xi'} \in H^1 \times H^2_*$. The rest of (i) follows from Proposition 5.2 and Lemma 5.3.

As for (ii), it was proved in Lemma 5.5. Relation of $\widehat{\Pi}_{\xi'}$ and $\widehat{\Pi}^*_{\xi'}$ comes from Proposition 4.1 and definition of eigenprojection.

□

6 Spectral properties of $B_{\xi'}$ and $B^*_{\xi'}$

In this section we study spectral properties of operators $B_{\xi'}$ and $B^*_{\xi'}$. At the end of this section we give a proof of Proposition 3.9.

Unless stated otherwise $\nu \geq \nu_0$, $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ and $|\xi'| \leq r_0$, where ν_0, γ_0 and r_0 are given by Proposition 3.5. Based on Proposition 3.5 we introduce the following definition. We remind notation $J_T = [0,T]$.

Definition 6.1 *We define function $v^{(0)}_{\xi'}(t)$ as*

$$v^{(0)}_{\xi'}(t) = e^{-\lambda_{\xi'} t} u^{(0)}_{\xi'}(t),$$

where $u^{(0)}_{\xi'}(t) = \widehat{U}_{\xi'}(t,0)u^{(0)}_{\xi'}$.

Lemma 6.2 *Function $v^{(0)}_{\xi'} \in C_{per}(J_T; H^1 \times H^1_0) \cap L^2_{per}(J_T; H^1 \times H^2_*) \cap H^1_{per}(J_T; X_0)$ and it satisfies the following equation*

$$\partial_t v + \widehat{L}_{\xi'}(t)v = -\lambda_{\xi'} v,$$

for a.a. $t \in J_T$.

Proof. Proof is obtained by simple computation from properties of operator $\widehat{U}_{\xi'}(t,s)$ and Theorem 4.3 since $u^{(0)}_{\xi'} \in H^1 \times H^2_*$.

□

In next theorem we show regularity results for $(\lambda - B_{\xi'})^{-1}$ on Y^k_{per}.

Theorem 6.3 *There exists $0 < r_1 \leq r_0$ and $q_1 > 0$ such that for each $|\xi'| \leq r_1$ there hold $0 \leq -\mathrm{Re}\,\lambda_{\xi'} \leq \frac{q_1}{2}$ and the following statements.*

Let $1 \leq k \leq m$ and $\lambda \in \{\lambda : \mathrm{Re}\,\lambda < q_1\} \backslash \{-\lambda_{\xi'}\}$. For every $f \in Y^k_{per}$ it holds

$$(\lambda - B_{\xi'})^{-1} f \in \bigcap_{j=0}^{[\frac{k}{2}]} C^j_{per}(J_T; H^{k-2j} \times H^{k-2j}_*),$$

$$\phi \in \bigcap_{j=0}^{[\frac{k}{2}]} H^{j+1}_{per}(J_T; H^{k-2j}), \quad w \in \bigcap_{j=0}^{[\frac{k+1}{2}]} H^j_{per}(J_T; H^{k+1-2j}_*).$$

Furthermore, there holds an estimate

$$\begin{aligned} &\sum_{j=0}^{[\frac{k}{2}]} |\partial_t^j u(t)|^2_{H^{k-2j}} + \int_0^t \sum_{j=0}^{[\frac{k-1}{2}]} |\partial_z^{j+1} u|^2_{H^{k-2j} \times H^{k-1-2j}} + |\partial_z^{[\frac{k+2}{2}]} \phi|^2_{H^{k-2[\frac{k}{2}]}} + |u|^2_{H^k \times H^{k+1}} dz \\ &\qquad \leq C\{ \sum_{j=0}^{[\frac{k}{2}]-1} |\partial_t^j f(0)|^2_{H^{k-2(j+1)}} + \int_0^T \sum_{j=0}^{[\frac{k}{2}]} |\partial_z^j f|^2_{H^{k-2j} \times H^{k-1-2j}} dz\}, \end{aligned} \tag{6.1}$$

for $t \in J_T$ uniformly in $\lambda \in \{\lambda : |\lambda| = \frac{3}{4} q_1\}$. Here, $u = {}^T(\phi, w) = (\lambda - B_{\xi'})^{-1} f$ and $\sum_{j=0}^{[\frac{k}{2}]-1} |\partial_t^j f(0)|^2_{H^{k-2(j+1)}} = 0$ when $k = 1$.

Proof. Let $\lambda \in \mathbb{C}$. Let us first motivate our proof. We want to find u which satisfies

$$(\lambda - B_{\xi'})^{-1} f = u.$$

In other words, we want to find function u which is solution to

$$\begin{aligned} \lambda u - \partial_t u - \widehat{L}_{\xi'}(t) u = f, &\text{ for a.a. } t \in J_T, \\ u(0) = u(T). & \end{aligned} \tag{6.2}$$

Multiplying both sides of (6.2) by $-e^{-\lambda t}$ we obtain

$$\partial_t(e^{-\lambda t} u) + \widehat{L}_{\xi'}(t)(e^{-\lambda t} u) = -e^{-\lambda t} f, \text{ for a.a. } t \in J_T.$$

Therefore, we formally write u as

$$u(t) = e^{\lambda t}\widehat{U}_{\xi'}(t,0)u(0) - \int_0^t \widehat{U}_{\xi'}(t,z)e^{\lambda(t-z)}f(z)dz, \text{ for } t \in J_T.$$

Condition $u(0) = u(T)$ gives us that relation

$$(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))u(0) = -\int_0^T \widehat{U}_{\xi'}(T,z)e^{-\lambda z}f(z)dz,$$

has to be satisfied. Such relation is satisfied if it holds

$$u(0) = -(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}\int_0^T \widehat{U}_{\xi'}(T,z)e^{-\lambda z}f(z)dz. \tag{6.3}$$

Therefore, we see that $u = (\lambda - B_{\xi'})^{-1}f$ is a solution of (6.2) with the initial condition (6.3).

Let us first show the case $k = 1$. Let $f \in Y_{per}$. From Theorem 4.3 we see that function v defined as

$$v(t) = -\int_0^t \widehat{U}_{\xi'}(t,z)e^{-\lambda z}f(z)dz,$$

satisfies

$$v \in C(J_T; H^1 \times H_0^1).$$

In particular, we have

$$v(T) \in H^1 \times H_0^1,$$

and thus for $e^{-\lambda T} \notin \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}$ we get by Proposition 3.5 (i) that

$$u_0 = (e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}v(T) \in H^1 \times H_0^1.$$

Finally, by using Theorem 4.3 we see that function u defined as

$$u(t) = e^{\lambda t}\widehat{U}_{\xi'}(t,0)u_0 - \int_0^t \widehat{U}_{\xi'}(t,z)e^{\lambda(t-z)}f(z)dz,$$

satisfies (6.2) and $u(0) = u(T)$. Moreover, estimate (6.1) comes from estimate (4.8) applied to u and $v(T)$. Thus u has desired regularity which concludes the proof for $k = 1$.

For the rest of the proof of Theorem 6.3 let us suppose that

$$\lambda \in \{\lambda : \operatorname{Re}\lambda < \frac{-\ln q_0}{T}\}\backslash\{-\lambda_{\xi'}\},$$

unless further restricted.

Let us now show the case $k = 2$. Let $f \in Y^2_{per}$ then by the previous case $k = 1$ we have that $u = (\lambda - B_{\xi'})^{-1}f$ exists and $u(0) \in H^1 \times H_0^1$. We know that $u(0)$ satisfies

$$(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))u(0) = -\int_0^T \widehat{U}_{\xi'}(T,z)e^{-\lambda z}f(z)dz.$$

From Theorem 4.8 we get that

$$-\int_0^T \widehat{U}_{\xi'}(T,z)e^{-\lambda z}f(z)dz \in H^2 \times H_*^2.$$

By Lemma 5.6 we have $\widehat{U}_{\xi'}(T,0)u(0) \in H^1 \times H_*^2$ and therefore we obtain

$$u(0) \in H^1 \times H_*^2,$$

and

$$|u(0)|^2_{H^2} \leq |e^{\lambda T}\widehat{U}_{\xi'}(T,0))u(0)|^2_{H^2} + |\int_0^T \widehat{U}_{\xi'}(T,z)e^{\lambda(T-z)}f(z)dz|^2_{H^2}$$

$$\leq C|u(0)|^2_{X_0} + C\{|f(0)|_2^2 + \int_0^T \sum_{j=0}^1 |\partial_z^j f|^2_{H^{2-2j}\times H^{1-2j}}dz\} \leq C\{|f(0)|_2^2 + \int_0^T \sum_{j=0}^1 |\partial_z^j f|^2_{H^{2-2j}\times H^{1-2j}}dz\}.$$

Next let us show that also

$$Q_0u(0) = \phi(0) \in H^2.$$

In a similar way to obtaining (4.7) we obtain from (6.2) the following formula

$$\partial_{x_n}\phi(x_n,t) = e^{-P_\lambda(x_n,t,0)}\partial_{x_n}\phi(x_n,0) + \int_0^t e^{-P_\lambda(x_n,t,z)}\{-h(x_n,z) - H[u](x_n,z) + \lambda\frac{\gamma^2\rho_p^2}{\nu+\widetilde{\nu}}w^n(x_n,z)\}dz, \quad (6.4)$$

where

$$P_\lambda(x_n,t,z) = P(x_n,t,z) - \lambda(t-z).$$

$P(x_n,t,z)$, h and $H[u]$ were defined in the statement of Theorem 4.3. Let us take $0 < q_1$ such that

$$Tq_1 < \min\{-\ln q_0,\ \inf_{x_n} \operatorname{Re} P(x_n,T,0)\}.$$

Thanks to (3.1) we see that $\operatorname{Re} P(x_n,t,z) \geq \frac{\gamma^2}{\nu+\widetilde{\nu}}a_0(t-z)$ and thus $q_1 > 0$. Let us take $0 < r_1 \leq r_0$ suitably small so that $|\lambda_{\xi'}| \leq \frac{q_1}{2}$. For the rest of the proof of Theorem 6.3 let us suppose that

$$\lambda \in \{\lambda : \operatorname{Re}\lambda < q_1\}\backslash\{-\lambda_{\xi'}\},$$

and $|\xi'| \leq r_1$.

Since $\phi(0) = \phi(T)$ and $1 - e^{-P_\lambda(x_n,T,0)} \neq 0$ we have

$$\partial_{x_n}\phi(x_n,0) = (1 - e^{-P_\lambda(x_n,T,0)})^{-1}\int_0^T e^{-P_\lambda(x_n,T,z)}\{-h(x_n,z) - H[u](x_n,z) + \lambda\frac{\gamma^2\rho_p^2}{\nu+\widetilde{\nu}}w^n(x_n,z)\}dz. \quad (6.5)$$

Integrating by parts the term containing $\partial_z w^n$ in $H[u](x_n,z)$ on the righthand side of (6.5) we obtain

$$|\phi(0)|_{H^2}^2 \leq C\{|w(0)|_{H^1}^2 + |w(T)|_{H^1}^2 + \int_0^T |f|_{H^2\times H^1}^2 + |u|_{H^1\times H^2}^2 dz\} \leq C\int_0^T |f|_{H^2\times H^1}^2 dz.$$

Therefore, we showed that

$$u(0) \in H^2 \times H_*^2,$$

which together with $f \in Y_{per}^2$ allows us to use Theorem 4.8 to finish the proof of the case $k = 2$.

Let $3 \leq k \leq m$. Let us assume that for $k-1$ Theorem 6.3 holds and we show that it holds for k. By the assumption of induction we have

$$u = (\lambda - B_{\xi'})^{-1}f \in \bigcap_{j=0}^{[\frac{k-1}{2}]} C_{per}^j(J_T; H^{k-1-2j} \times H_*^{k-1-2j}),$$

$$\phi \in \bigcap_{j=0}^{[\frac{k-1}{2}]} H_{per}^{j+1}(J_T; H^{k-1-2j}), \quad w \in \bigcap_{j=0}^{[\frac{k}{2}]} H_{per}^j(J_T; H_*^{k-2j}).$$

Next, let us consider smoothness of $\partial_t u$. To do so, we use the following lemma.

Lemma 6.4 *Let $f \in Y_{per}^3$ and $u = (\lambda - B_{\xi'})^{-1}f$. Then function $\widetilde{u}$ defined as*

$$\widetilde{u} = (\lambda - B_{\xi'})^{-1}(\partial_t f + (\partial_t \widehat{L}_{\xi'}(t))u),$$

satisfies

$$\widetilde{u}(t) = \partial_t u(t),$$

for a.a. $t \in J_T$.

Proof of Lemma 6.4 is given in Appendix. Let us continue the proof of Theorem 6.3.

Set

$$g(t) = \partial_t f + (\partial_t \widehat{L}_{\xi'}(t))u.$$

Since $k \geq 3$ we see that it always holds $f \in Y^3_{per}$ and thus by Lemma 6.4 we have that $\widetilde{u}$ defined as

$$\widetilde{u} = (\lambda - B_{\xi'})^{-1} g,$$

satisfies

$$\widetilde{u} = \partial_t u.$$

Moreover, we have

$$g \in Y^{k-2}_{per}.$$

Therefore, by the assumption of induction we get

$$\partial_t u \in \bigcap_{j=0}^{[\frac{k-2}{2}]} C^j_{per}(J_T; H^{k-2-2j} \times H^{k-2-2j}_*),$$

$$\partial_t \phi \in \bigcap_{j=0}^{[\frac{k-2}{2}]} H^{j+1}_{per}(J_T; H^{k-2-2j}), \quad \partial_t w \in \bigcap_{j=0}^{[\frac{k-1}{2}]} H^j_{per}(J_T; H^{k-1-2j}_*).$$

Thus

$$u \in \bigcap_{j=1}^{[\frac{k}{2}]} C^j_{per}(J_T; H^{k-2j} \times H^{k-2j}_*),$$

$$\phi \in \bigcap_{j=1}^{[\frac{k}{2}]} H^{j+1}_{per}(J_T; H^{k-2j}), \quad w \in \bigcap_{j=1}^{[\frac{k+1}{2}]} H^j_{per}(J_T; H^{k+1-2j}_*),$$

and by the assumption of induction for $k-1$ we already know that

$$u \in C_{per}(J_T; H^{k-1} \times H^{k-1}_*), \ \phi \in H^1_{per}(J_T; H^{k-1}), \ w \in L^2_{per}(J_T; H^k_*).$$

Moreover, we have an estimate

$$|u(t)|^2_{H^{k-1}} + \sum_{j=1}^{[\frac{k}{2}]} |\partial_t^j u(t)|^2_{H^{k-2j}} + \int_0^t |\partial_z u|^2_{H^{k-1} \times H^{k-1}} + \sum_{j=1}^{[\frac{k-1}{2}]} |\partial_z^{j+1} u|^2_{H^{k-2j} \times H^{k-1-2j}}$$

$$+ |\partial_z^{[\frac{k+2}{2}]} \phi|^2_{H^{k-2[\frac{k}{2}]}} + |u|^2_{H^{k-1} \times H^k} dz$$

$$\leq C\{ \sum_{j=0}^{[\frac{k}{2}]-1} |\partial_t^j f(0)|^2_{H^{k-2(j+1)}} + \int_0^T \sum_{j=0}^{[\frac{k}{2}]} |\partial_z^j f(z)|^2_{H^{k-2j} \times H^{k-1-2j}} dz\},$$

for $t \in J_T$.

To conclude the proof it remains to show

$$u \in C(J_T; H^k \times H^k_*),$$

and

$$\phi \in H^1(J_T; H^k), \quad w \in L^2(J_T; H^{k+1}_*),$$

together with (6.1).

First, since

$$f \in L^2(J_T; H^k \times H^{k-1}), \ u \in L^2(J_T; H^{k-1} \times H^k), \ \partial_t w \in L^2(J_T; H^{k-1}),$$

we see from (6.5) that

$$\partial_{x_n}\phi(0) \in H^{k-1},$$

and

$$|\phi(0)|^2_{H^k} \le C\{\int_0^T |f|^2_{H^k\times H^{k-1}} + |u|^2_{H^{k-1}\times H^k} + |\partial_z w|^2_{H^{k-1}} dz\}.$$

It is straightforward to see from (6.4) that $\partial_{x_n}\phi \in C(J_T; H^{k-1})$, which gives us

$$\phi \in C(J_T; H^k),$$

and from (6.4)

$$|\partial^k_{x_n}\phi(t)|^2_2 \le C\{\int_0^T |f|^2_{H^k\times H^{k-1}} + |u|^2_{H^{k-1}\times H^k} + |\partial_z w|^2_{H^{k-1}} dz\}.$$

Second, since

$$\partial_t w \in L^2(J_T; H^{k-1}),\ u \in L^2(J_T; H^k),\ f \in L^2(J_T; H^{k-1}),$$

we obtain by Lemma 4.7 (iii) that

$$w \in L^2(J_T; H^{k+1}),$$

and

$$\int_0^t |\partial^{k+1}_{x_n} w|^2_2 dz \le C\int_0^t |\partial^{k-1}_{x_n}\widetilde{Q}f|^2_2 + |u|^2_{H^k} + |\partial_z\partial^{k-1}_{x_n} w|^2_2 dz.$$

Third, since

$$\partial_t w \in C(J_T; H^{k-2}),\ u \in C(J_T; H^{k-1}),\ f \in C(J_T; H^{k-2}),$$

we obtain by Lemma 4.7 (iii) that

$$w \in C(J_T; H^k).$$

Moreover, since $u \in C^1_{per}(J_T; L^2) \cap C_{per}(J_T; H^1 \times H^2_*)$ we know that (6.2) is satisfied even for $t = 0$ and thus from (6.2) we obtain

$$|\partial^k_{x_n} w(0)|^2_2 \le C\{|u(0)|^2_{H^{k-1}} + |\partial_t\partial^{k-2}_{x_n} w(0)|^2_2 + |\partial^{k-2}_{x_n}\widetilde{f}(0)|^2_2\}.$$

This together with estimate from Lemma 4.7 (iv)

$$|\partial^k_{x_n} w(t)|^2_2 \le C\{|w(0)|^2_{H^k} + \int_0^t (|w|^2_{H^{k+1}} + |\partial_z w|^2_{H^{k-1}}) dz\},$$

gives us

$$|\partial^k_{x_n} w(t)|^2_2 \le C\{|u(0)|^2_{H^{k-1}} + |\partial_t\partial^{k-2}_{x_n} w(0)|^2_2 + |\partial^{k-2}_{x_n}\widetilde{f}(0)|^2_2 + \int_0^t (|w|^2_{H^{k+1}} + |\partial_z w|^2_{H^{k-1}}) dz\}.$$

Fourth, since

$$f \in L^2(J_T; H^k \times H^{k-1}),\ u \in L^2(J_T; H^k),\ \partial_t w \in L^2(J_T; H^{k-1}),$$

we obtain by Lemma 4.7 (ii) that

$$\partial_t\phi \in L^2(J_T; H^k),$$

and

$$\int_0^t |\partial^k_{x_n}\partial_z\phi|^2_2 dz \le C\int_0^t |f|^2_{H^k\times H^{k-1}} + |u|^2_{H^k} + |\partial_z w|^2_{H^{k-1}} dz.$$

This completes the proof of Theorem 6.3. □

Next lemma shows that $-\lambda_{\xi'}$ is simple eigenvalue of $B_{\xi'}$.

Lemma 6.5 *$-\lambda_{\xi'}$ is a simple eigenvalue of $B_{\xi'}$ on Y_{per}.*

Proof. From Lemma 6.2 it is straightforward that $-\lambda_{\xi'}$ is an eigenvalue of $B_{\xi'}$ on Y_{per}.

To show that $-\lambda_{\xi'}$ is simple we first need to show that if $u \in D(B_{\xi'})$ satisfies $u \neq 0$ and $B_{\xi'}u + \lambda_{\xi'}u = 0$ then there exists a constant $C(\xi') \in \mathbb{R}$ such that $u(t) = C(\xi')v_{\xi'}^{(0)}(t)$. Since $u \in D(B_{\xi'})$ we have $u(0) = u(T)$, i.e. $e^{\lambda_{\xi'}T}u(0) = e^{\lambda_{\xi'}T}u(T)$. From $B_{\xi'}u + \lambda_{\xi'}u = 0$ we have that

$$e^{\lambda_{\xi'}t}u(t) = \widehat{U}_{\xi'}(t,0)u(0)$$

and thus

$$e^{\lambda_{\xi'}T}u(0) = e^{\lambda_{\xi'}T}u(T) = \widehat{U}_{\xi'}(T,0)u(0).$$

Since $e^{\lambda_{\xi'}T}$ is a simple eigenvalue of $\widehat{U}_{\xi'}(T,0)$ by Proposition 3.5, we conclude that such $C \in \mathbb{R}$ exists thanks to uniqueness of the solutions.

Second we need to show that there does not exist $u \in D(B_{\xi'})$ such that $(B_{\xi'} + \lambda_{\xi'})u = -v_{\xi'}^{(0)}$. This can be rewritten in a form

$$B_{\xi'}(e^{\lambda_{\xi'}t}u) = -e^{\lambda_{\xi'}t}v_{\xi'}^{(0)}.$$

Thus we have

$$e^{\lambda_{\xi'}t}u(t) = \widehat{U}_{\xi'}(t,0)u(0) - \int_0^t \widehat{U}_{\xi'}(t,s)e^{\lambda_{\xi'}s}v_{\xi'}^{(0)}(s)ds.$$

Since $\widehat{U}_{\xi'}(t,s)e^{\lambda_{\xi'}s}v_{\xi'}^{(0)}(s) = u_{\xi'}^{(0)}(t)$ we have for $t = T$ that

$$e^{\lambda_{\xi'}T}u(T) = \widehat{U}_{\xi'}(T,0)u(0) - Tu_{\xi'}^{(0)}(T).$$

Once again $u(0) = u(T)$ gives us

$$(e^{\lambda_{\xi'}T} - \widehat{U}_{\xi'}(T,0))u(0) = -Tu_{\xi'}^{(0)}(0) = -Tu_{\xi'}^{(0)}.$$

Since $e^{\lambda_{\xi'}T}$ is simple there does not exist $u(0) \in X_0$ that would satisfy the equation above. Therefore we showed that $-\lambda_{\xi'}$ is a simple eigenvalue of $B_{\xi'}$ on Y_{per}.

□

In the previous lemma we showed that $B_{\xi'}$ has simple eigenvalue $-\lambda_{\xi'}$ on Y_{per}. Next theorem says that $-\lambda_{\xi'}$ is a simple eigenvalue of $B_{\xi'}$ on Y_{per}^m.

Theorem 6.6 *For each ξ' with $|\xi'| \leq r_1$ there hold the following statements.*

For any eigenfunction $u_e = {}^T(\phi_e, w_e)$ of $B_{\xi'}$ associated with $-\lambda_{\xi'}$ there holds

$$\begin{aligned} u_e &\in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} C_{per}^j(J_T; H^{k-2j} \times H_*^{k-2j}), \\ \phi_e &\in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} H_{per}^{j+1}(J_T; H^{k-2j}), \ \ w_e \in \bigcap_{j=0}^{\left[\frac{k+1}{2}\right]} H_{per}^j(J_T; H^{k+1-2j}), \end{aligned} \tag{6.6}$$

for all $1 \leq k \leq m$ and we have an estimate

$$\sum_{j=0}^{\left[\frac{k}{2}\right]} |\partial_t^j u_e(t)|_{H^{k-2j}}^2 + \int_0^t \sum_{j=0}^{\left[\frac{k-1}{2}\right]} |\partial_z^{j+1} u_e|_{H^{k-2j}\times H^{k-1-2j}}^2 + |\partial_z^{\left[\frac{k+2}{2}\right]}\phi_e|_2^2 + |u_e|_{H^k\times H^{k+1}}^2 dz \leq C_k, \tag{6.7}$$

for $t \in J_T$.

Proof. Let $|\xi'| \leq r_1$ where r_1 was given in Theorem 6.3. From Lemma 6.2 we see that there exists $u_e \in D(B_{\xi'})$ eigenfunction of $B_{\xi'}$ associated with $-\lambda_{\xi'}$. From Lemma 6.5 we know that $-\lambda_{\xi'}$ is the simple eigenvalue and therefore there exists constant $C(\xi') \neq 0$ such that $u_e = Cv_{\xi'}^{(0)}$. Now, we see from Lemma 6.2 that u_e satisfies (6.6) for $k = 1$.

Since u_e is an eigenfunction it satisfies

$$\begin{aligned} \partial_t u_e + \widehat{L}_{\xi'}(t)u_e &= -\lambda_{\xi'}u_e, \\ u_e(0) &= u_e(T). \end{aligned} \tag{6.8}$$

For $\lambda \in \{\lambda : |\lambda| = \frac{3}{4}q_1\}$ we can rewrite (6.8) in equivalent form as

$$(\lambda - B_{\xi'})^{-1} f = u_e, \tag{6.9}$$

where $f = (\lambda + \lambda_{\xi'})u_e$. Now, estimate (6.7) is easily obtained by Gronwall's inequality from (6.1) in the case $k = 1$. Therefore, we proved Theorem 6.6 in the case $k = 1$.

Let $2 \le k \le m$. Let us assume that for $k-1$ Theorem 6.6 holds and we show that it also holds for k. By induction assumption we have

$$u_e \in \bigcap_{j=0}^{\left[\frac{k-1}{2}\right]} C^j_{per}(J_T; H^{k-1-2j} \times H^{k-1-2j}_*),$$

$$\phi_e \in \bigcap_{j=0}^{\left[\frac{k-1}{2}\right]} H^{j+1}_{per}(J_T; H^{k-1-2j}),\ \ w_e \in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} H^j_{per}(J_T; H^{k-2j}).$$

In a similar manner as we got (4.7) we get from (6.8) that

$$\partial_{x_n}\phi_e(x_n,t) = e^{-P_{\lambda_{\xi'}}(x_n,t,0)}\partial_{x_n}\phi_e(x_n,0) - \int_0^t e^{-P_{\lambda_{\xi'}}(x_n,t,z)}\{\lambda_{\xi'}\frac{\gamma^2\rho_p^2(x_n)}{\nu+\widetilde{\nu}}w_e^n(x_n,z) + H[u_e](x_n,z)\}dz, \tag{6.10}$$

where

$$P_{\lambda_{\xi'}}(x_n,t,z) = P(x_n,t,z) + \lambda_{\xi'}(t-z).$$

$P(x_n,t,z)$ and $H[u]$ were defined in the statement of Theorem 4.3.

At $t = T$ we read (6.10) as

$$\partial_{x_n}\phi_e(x_n,T) = e^{-P_{\lambda_{\xi'}}(x_n,T,0)}\partial_{x_n}\phi_e(x_n,0) - \int_0^T e^{-P_{\lambda_{\xi'}}(x_n,T,z)}\{\lambda_{\xi'}\frac{\gamma^2\rho_p^2(x_n)}{\nu+\widetilde{\nu}}w_e^n(x_n,z) + H[u_e](x_n,z)\}dz,$$

and from $\phi_e(0) = \phi_e(T)$ we get

$$\partial_{x_n}\phi_e(x_n,0) = -(1-e^{-P_{\lambda_{\xi'}}(x_n,T,0)})^{-1}\int_0^T e^{-P_{\lambda_{\xi'}}(x_n,T,z)}\{\lambda_{\xi'}\frac{\gamma^2\rho_p^2(x_n)}{\nu+\widetilde{\nu}}w_e^n(x_n,z) + H[u_e](x_n,z)\}dz, \tag{6.11}$$

since $1 - e^{-P_{\lambda_{\xi'}}(x_n,T,0)} \neq 0$.

After integrating by parts the term containing $\partial_z w_e$ in $H[u_e](x_n,z)$ on the right hand side of (6.11) we get the following estimate

$$|\phi_e(0)|^2_{H^k} \le C\{|w_e|_{L^2(J_T;H^k)} + |\phi_e|_{L^2(J_T;H^{k-1})} + |w_e(0)|_{H^{k-1}}\}.$$

Since $w_e \in C_{per}(J_T; H^{k-1}_*)$ we see from (6.10) that we have

$$\phi_e(0) \in H^k, \phi_e \in C_{per}(J_T; H^k).$$

We thus showed that

$$u_e \in Y^k_{per}.$$

Since $u_e \in Y^k_{per}$ we have

$$f = (\lambda + \lambda_{\xi'})u_e \in Y^k_{per},$$

and from (6.9) it is obvious that we can use Theorem 6.3 and Gronwall's inequality to conclude the proof of Theorem 6.6 for k. □

We have an immediate corollary of Theorem 6.3 for $\xi' = 0$.

Corollary 6.7 *For every $f \in Y_{per}$ there holds*

$$\sup_{t\in J_T}|(\lambda - B_0)^{-1}f(t)|_{H^1} + |(\lambda - B_0)^{-1}f|_{L^2(J_T;H^1\times H^2)} + |\partial_t(\lambda - B_0)^{-1}f|_{L^2(J_T;X_0)} \le C|f|^2_{L^2(J_T;X_0)},$$

uniformly in $\lambda \in \{\lambda : |\lambda| = \frac{3}{4}q_1\}$.

Let $B_{\xi'}$ be denoted by

$$B_{\xi'} = B_0 + \sum_{j=1}^{n-1} \xi_j B_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k B_{jk}^{(2)}.$$

Here

$$\widehat{B}_0(t) = \partial_t I_{n+1} + \begin{pmatrix} 0 & 0 & \gamma^2 \partial_{x_n}(\rho_p \,\cdot\,) \\ \frac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' & -\frac{\nu}{\rho_p}\partial_{x_n}^2 I_{n-1} & (\partial_{x_n} v_p^1(t))\boldsymbol{e}_1' \\ \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2 \rho_p}\,\cdot\right) & 0 & -\frac{\nu+\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix},$$

and

$$B_j^{(1)}(t) = i \begin{pmatrix} v_p^1(t)\delta_{1j} & \gamma^2 \rho_p{}^T\boldsymbol{e}_j' & 0 \\ \frac{P'(\rho_p)}{\gamma^2 \rho_p}\boldsymbol{e}_j' & v_p^1(t)\delta_{1j} I_{n-1} & -\frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}\boldsymbol{e}_j' \\ 0 & -\frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}{}^T\boldsymbol{e}_j' & v_p^1(t)\delta_{1j} \end{pmatrix}, \quad B_{jk}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_p}\delta_{jk} I_{n-1} + \frac{\widetilde{\nu}}{\rho_p}\boldsymbol{e}_j'{}^T\boldsymbol{e}_k' & 0 \\ 0 & 0 & \frac{\nu}{\rho_p}\delta_{jk} \end{pmatrix},$$

for $j,k = 1,\dots,n-1$. Here and it what follows, δ_{jk} denotes Kronecker's delta.

Lemma 6.8 *There exists $0 < r_1$ such that for each $|\xi'| \le r_1$ there hold the following statements.*
Let $\lambda \in \{\lambda : |\lambda| = \frac{3}{4}q_1\}$. Then $(\lambda - B_{\xi'})^{-1}$ is expanded as

$$(\lambda - B_{\xi'})^{-1} = (\lambda - B_0)^{-1} \sum_{N=0}^{\infty} \{(\sum_{j=1}^{n-1} \xi_j B_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k B_{jk}^{(2)})(\lambda - B_0)^{-1}\}^N,$$

in $L(Y_{per})$.

Proof. Let $|\xi'| \le r_1$ where r_1 was given in Theorem 6.3. Let us rewrite $(\lambda - B_{\xi'})$ as

$$(\lambda - B_{\xi'}) = \left(I - (\sum_{j=1}^{n-1} \xi_j B_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k B_{jk}^{(2)})(\lambda - B_0)^{-1} \right)(\lambda - B_0).$$

For $f \in Y_{per}$ there hold

$$\begin{aligned} |B_j^{(1)}(t)(\lambda - B_0(t))^{-1} f(t)|_{X_0} &\le C|(\lambda - B_0(t))^{-1} f(t)|_{H^1}, \\ |B_{jk}^{(2)}(\lambda - B_0(t))^{-1} f(t)|_{X_0} &\le C|\widetilde{Q}(\lambda - B_0(t))^{-1} f(t)|_2, \end{aligned} \tag{6.12}$$

for a.a. $t \in J_T$. Using Corollary 6.7, we take r_1 suitably smaller so that for $|\xi'| \le r_1$ there exists the Neumann series expansion of $(\lambda - B_{\xi'})^{-1}$ on Y_{per}. □

Now, let us consider the *adjoint problem.*

Definition 6.9 *We define function $v_{\xi'}^{(0)*}(s)$ as*

$$v_{\xi'}^{(0)*}(s) = e^{-\overline{\lambda}_{\xi'}(T-s)} \widehat{U}_{\xi'}^*(s,T) u_{\xi'}^{(0)*}.$$

Analogously to Lemma 6.2 we see that function $v_{\xi'}^{(0)*} \in C_{per}(J_T; H^1 \times H_0^1) \cap L^2_{per}(J_T; H^1 \times H_*^2) \cap H^1_{per}(J_T; X_0)$ and it satisfies

$$-\partial_s v + \widehat{L}_{\xi'}^*(s) v = -\overline{\lambda}_{\xi'} v,$$

for a.a. $s \in J_T$.
We have the analogue of Theorem 6.3 for $(\lambda - B_{\xi'}^*)^{-1}$.

Theorem 6.10 *There exists $0 < r_1 \le r_0$ and $q_1 > 0$ such that for each $|\xi'| \le r_1$ there hold $0 \le -\mathrm{Re}\,\overline{\lambda}_{\xi'} < \frac{q_1}{2}$ and the following statements.*
Let $1 \le k \le m$ and $\lambda \in \{\lambda : \mathrm{Re}\,\lambda < q_1\} \backslash \{-\overline{\lambda}_{\xi'}\}$. For every $f \in Y_{per}^k$ it holds

$$(\lambda - B^*_{\xi'})^{-1} f \in \bigcap_{j=0}^{[\frac{k}{2}]} C^j_{per}(J_T; H^{k-2j} \times H^{k-2j}_*),$$

$$\phi \in \bigcap_{j=0}^{[\frac{k}{2}]} H^{j+1}_{per}(J_T; H^{k-2j}), \quad w \in \bigcap_{j=0}^{[\frac{k+1}{2}]} H^j_{per}(J_T; H^{k+1-2j}_*).$$

Furthermore, there holds an estimate

$$\sum_{j=0}^{[\frac{k}{2}]} |\partial_s^j u(s)|^2_{H^{k-2j}} + \int_s^T \sum_{j=0}^{[\frac{k-1}{2}]} |\partial_z^{j+1} u|^2_{H^{k-2j} \times H^{k-1-2j}} + |\partial_z^{[\frac{k+2}{2}]} \phi|^2_{H^{k-2[\frac{k}{2}]}} + |u|^2_{H^k \times H^{k+1}} dz$$
$$\leq C\{ \sum_{j=0}^{[\frac{k}{2}]-1} |\partial_s^j f(T)|^2_{H^{k-2(j+1)}} + \int_0^T \sum_{j=0}^{[\frac{k}{2}]} |\partial_z^j f|^2_{H^{k-2j} \times H^{k-1-2j}} dz\},$$

for $s \in J_T$ uniformly in $\lambda \in \{\lambda : |\lambda| = \frac{3}{4} q_1\}$. Here $u = {}^T(\phi, w) = (\lambda - B^*_{\xi'})^{-1} f$ and $\sum_{j=0}^{[\frac{k}{2}]-1} |\partial_t^j f(T)|^2_{H^{k-2(j+1)}} = 0$ when $k = 1$.

Proof. Proof follows the same steps as the proof of Theorem 6.3. In the case $k = 1$ we use Theorem 4.9 ($k = 1$) and Proposition 3.5 (ii). In the case $k = 2$ we use Theorem 4.9 ($k = 2$) and Lemma 5.6. The rest of the proof is analogous to the proof of Theorem 6.3. □

Using Proposition 3.5 (ii) and Theorem 6.10 we can show the following analogues of Lemma 6.5 and Theorem 6.6 for the *adjoint problem*.

Theorem 6.11 *For each ξ' with $|\xi'| \leq r_1$ there hold the following statements.*
$-\overline{\lambda}_{\xi'}$ is the simple eigenvalue of $B^*_{\xi'}$ on Y^m_{per} and for any eigenfunction $u_e = {}^T(\phi_e, w_e)$ of $B_{\xi'}$ associated with $-\lambda_{\xi'}$ there holds

$$u_e \in \bigcap_{j=0}^{[\frac{k}{2}]} C^j_{per}(J_T; H^{k-2j} \times H^{k-2j}_*),$$
$$\phi_e \in \bigcap_{j=0}^{[\frac{k}{2}]} H^{j+1}_{per}(J_T; H^{k-2j}), \; w_e \in \bigcap_{j=0}^{[\frac{k+1}{2}]} H^j_{per}(J_T; H^{k+1-2j}),$$

for all $1 \leq k \leq m$ and we have an estimate

$$\sum_{j=0}^{[\frac{k}{2}]} |\partial_s^j u_e(s)|^2_{H^{k-2j}} + \int_s^T \sum_{j=0}^{[\frac{k-1}{2}]} |\partial_z^{j+1} u_e|^2_{H^{k-2j} \times H^{k-1-2j}} + |\partial_z^{[\frac{k+2}{2}]} \phi_e|^2_2 + |u_e|^2_{H^k \times H^{k+1}} dz \leq C_k,$$

for $t \in J_T$.

Proof of Proposition 3.9 (i), (ii). As for (i), Lemma 6.5 and Theorem 6.6 show that $-\lambda_{\xi'}$ is simple eigenvalue of $B_{\xi'}$ on Y^m_{per}. Let $f \in Y^k_{per}$ and $\lambda \in \{\lambda : \mathrm{Re}\,\lambda < q_1\} \backslash \{-\lambda_{\xi'}\}$. Then by Theorem 6.3

$$(\lambda - B_{\xi'})^{-1} f \in Y^k_{per}.$$

This concludes the proof of (i).

As for (ii), Theorem 6.11 shows that $-\overline{\lambda}_{\xi'}$ is the simple eigenvalue of $B^*_{\xi'}$ on Y^m_{per}. Let $f \in Y^k_{per}$ and $\lambda \in \{\lambda : \mathrm{Re}\,\lambda < q_1\} \backslash \{-\overline{\lambda}_{\xi'}\}$. Then by Theorem 6.10

$$(\lambda - B_{\xi'})^{-1} f \in Y^k_{per}.$$

This concludes the proof of (ii). □

Let $B^*_{\xi'}$ be denoted by

$$B^*_{\xi'} = B^*_0 + \sum_{j=1}^{n-1} \xi_j B_j^{(1)*} + \sum_{j,k=1}^{n-1} \xi_j\xi_k B_{jk}^{(2)*}.$$

Here

$$\widehat{B}^*_0(s) = -\partial_s I_{n+1} + \begin{pmatrix} 0 & \frac{\nu\gamma^2}{P'(\rho_p)}(\partial^2_{x_n} v^1_p(s))^T \boldsymbol{e}'_1 & -\gamma^2\partial_{x_n}(\rho_p\ \cdot\ ) \\ 0 & -\frac{\nu}{\rho_p}\partial^2_{x_n} I_{n-1} & 0 \\ -\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\ \cdot\ \right) & \partial_{x_n} v^1_p(s)^T \boldsymbol{e}'_1 & -\frac{\nu+\widetilde{\nu}}{\rho_p}\partial^2_{x_n} \end{pmatrix},$$

and

$$B_j^{(1)*}(s) = -i \begin{pmatrix} v^1_p(s)\delta_{1j} & \gamma^2\rho_p{}^T\boldsymbol{e}'_j & 0 \\ \frac{P'(\rho_p)}{\gamma^2\rho_p}\boldsymbol{e}'_j & v^1_p(s)\delta_{1j} I_{n-1} & \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}\boldsymbol{e}'_j \\ 0 & \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}{}^T\boldsymbol{e}'_j & v^1_p(s)\delta_{1j} \end{pmatrix},\ B_{jk}^{(2)*} = B_{jk}^{(2)},\ j,k = 1,\dots,n-1.$$

There holds following result analogous to Lemma 6.8.

Lemma 6.12 *There exists $0 < r_1$ such that for each $|\xi'| \le r_1$ there hold the following statements. Let $\lambda \in \{\lambda : |\lambda| = \frac{3}{4}q_1\}$. Then $(\lambda - B^*_{\xi'})^{-1}$ is expanded as*

$$(\lambda - B^*_{\xi'})^{-1} = (\lambda - B^*_0)^{-1}\sum_{N=0}^{\infty}\{(\sum_{j=1}^{n-1}\xi_j B_j^{(1)*} + \sum_{j,k=1}^{n-1}\xi_j\xi_k B_{jk}^{(2)*})(\lambda - B^*_0)^{-1}\}^N,$$

in $L(Y_{per})$.

In the rest of this section let $|\xi'| \le r_1$ where $r_1 > 0$ is such that all previous results in this section hold true. We introduce eigenprojections for $B_{\xi'}$ and $B^*_{\xi'}$.

Definition 6.13 *The eigenprojections for $B_{\xi'}$ and $B^*_{\xi'}$ associated with $-\lambda_{\xi'}$ and $-\overline{\lambda}_{\xi'}$, respectively, are defined as follows*

$$\Pi(\xi') = \frac{1}{2\pi i}\int_\Gamma (\lambda - B_{\xi'})^{-1} d\lambda,$$

$$\Pi^*(\xi') = \frac{1}{2\pi i}\int_\Gamma (\lambda - B^*_{\xi'})^{-1} d\lambda,$$

where $\Gamma = \{\lambda : |\lambda| = \frac{3}{4}q_1\}$ and Γ is positively oriented.

Definition 6.14 *We define functions $\widetilde{u}_{\xi'}$ and $\widetilde{u}^*_{\xi'}$ in the following way:*

$$\widetilde{u}_{\xi'} = \Pi(\xi')u^{(0)} \quad \textit{and} \quad \widetilde{u}^*_{\xi'} = \Pi(\xi')u^{(0)*}.$$

Proposition 6.15 *The following statements holds true.*

(i) $\widetilde{u}_{\xi'}$ and $\widetilde{u}^*_{\xi'}$ are eigenfunctions of $B_{\xi'}$ and $B^*_{\xi'}$ for eigenvalues $-\lambda_{\xi'}$ and $-\overline{\lambda}_{\xi'}$, respectively.

(ii) $\widetilde{u}_{\xi'}$ and $\widetilde{u}^*_{\xi'}$ can be expanded as

$$\widetilde{u}_{\xi'} = u^{(0)} + i\xi'\cdot\widetilde{u}^{(1)} + |\xi'|^2\widetilde{u}^{(2)}(\xi'),$$

$$\widetilde{u}^*_{\xi'} = u^{*(0)} + i\xi'\cdot\widetilde{u}^{*(1)} + |\xi'|^2\widetilde{u}^{*(2)}(\xi'),$$

and

$$\widetilde{u}_{\xi'}, \widetilde{u}^*_{\xi'}, u^{(0)}, u^{(0)*}, \widetilde{u}^{(1)}, \widetilde{u}^{(1)*}, \widetilde{u}^{(2)}(\xi'), \widetilde{u}^{(2)*}(\xi'),$$

have the regularity (6.6) *with $k = m$. Moreover, we have estimate*

$$\sup_{z\in J_T}\sum_{j=0}^{[\frac{m}{2}]}|\partial_z^j u(z)|^2_{H^{m-2j}}+\int_0^T\sum_{j=0}^{[\frac{m-1}{2}]}|\partial_z^{j+1}u|^2_{H^{m-2j}\times H^{m-1-2j}}+|\partial_z^{[\frac{m+2}{2}]}Q_0u|_2^2+|u|^2_{H^m\times H^{m+1}}dz\le C,$$

for a constant $C>0$ depending on r_1 and $u\in\{\widetilde{u}_{\xi'},\widetilde{u}^*_{\xi'},\widetilde{u}^{(2)}(\xi'),\widetilde{u}^{(2)*}(\xi')\}$.

(iii)

$$\langle\widetilde{u}_{\xi'}(t),\widetilde{u}^*_{\xi'}(t)\rangle=C(\xi')\ge\frac{1}{2},$$

for $t\in J_T$, where $C(\xi')$ does not depend on t.

Proof. (i) is obvious from definition.

As for (ii), from Lemma 6.8 and definition of $\Pi(\xi')$ we see that $\widetilde{u}_{\xi'}$ can be expanded as

$$\widetilde{u}_{\xi'}=u^{(0)}+i\xi'\cdot\widetilde{u}^{(1)}+|\xi'|^2\widetilde{u}^{(2)}(\xi'),$$

where

$$u^{(0)}=\frac{1}{2\pi i}\int_\Gamma(\lambda-B_0)^{-1}u^{(0)}d\lambda,\quad \widetilde{u}^{(1)}_j=\frac{1}{2\pi i}\int_\Gamma(\lambda-B_0)^{-1}(-i)B_j^{(1)}(\lambda-B_0)^{-1}u^{(0)}d\lambda,$$

and

$$\widetilde{u}^{(2)}(\xi')=\frac{1}{2\pi i}\int_\Gamma(\lambda-B_0)^{-1}R^{(2)}(\lambda,\xi')d\lambda,$$

with

$$|\xi'|^2R^{(2)}(\lambda,\xi')=\sum_{j,k=1}^{n-1}\xi_j\xi_kB^{(2)}_{jk}(\lambda-B_0)^{-1}u^{(0)}+\sum_{N=2}^{\infty}\{(\sum_{j=1}^{n-1}\xi_jB_j^{(1)}+\sum_{j,k=1}^{n-1}\xi_j\xi_kB^{(2)}_{jk})(\lambda-B_0)^{-1}\}^Nu^{(0)}.\quad(6.13)$$

Regularity (6.6) with $k=m$ for $\widetilde{u}_{\xi'},\widetilde{u}^*_{\xi'}$ follows from (i) and Theorem 6.6. Since $\widetilde{u}_{\xi'}|_{\xi'=0}=u^{(0)}$, we obtain by Theorem 6.6 that $u^{(0)}$ and $u^{(0)*}$ has the regularity (6.6) with $k=m$. Next, we show the regularity for functions $\widetilde{u}^{(1)}=(\widetilde{u}^{(1)}_1,\dots,\widetilde{u}^{(1)}_{n-1})$ and $\widetilde{u}^{(2)}(\xi')$. From Theorem 6.6 we have

$$u^{(0)}\in\bigcap_{j=0}^{[\frac{m}{2}]}C^j_{per}(J_T;H^{m-2j}\times H^{m-2j}_*).$$

Since $(\lambda-B^{(0)})^{-1}u^{(0)}=\frac{1}{\lambda}u^{(0)}$, we see that

$$B_j^{(1)}(\lambda-B^{(0)})^{-1}u^{(0)}=\frac{1}{\lambda}B_j^{(1)}u^{(0)}\in Y^m_{per}.$$

So Theorem 6.3 gives us the desired regularity of $\widetilde{u}^{(1)}$. Regularity of $\widetilde{u}^{(2)}(\xi')$ simply follows from the regularity of $\widetilde{u}_{\xi'},u^{(0)}$ and $\widetilde{u}^{(1)}$. Estimates uniform in $|\xi'|\le r_1$ follow from Theorem 6.6 and (6.12), (6.13). Results for $\widetilde{u}^*_{\xi'}$ and its expansion holds analogously from Lemma 6.12 and Theorem 6.10. This concludes (ii).

As for (iii), we calculate

$$-\langle\lambda_{\xi'}\widetilde{u}_{\xi'},\widetilde{u}^*_{\xi'}\rangle=\langle\partial_t\widetilde{u}_{\xi'}+\widehat{L}_{\xi'}(t)\widetilde{u}_{\xi'},\widetilde{u}^*_{\xi'}\rangle=\langle\partial_t\widetilde{u}_{\xi'},\widetilde{u}^*_{\xi'}\rangle+\langle\widetilde{u}_{\xi'},\widehat{L}^*_{\xi'}(t)\widetilde{u}^*_{\xi'}\rangle$$

$$=\partial_t\langle\widetilde{u}_{\xi'},\widetilde{u}^*_{\xi'}\rangle-\langle\widetilde{u}_{\xi'},\partial_t\widetilde{u}^*_{\xi'}\rangle+\langle\widetilde{u}_{\xi'},\widehat{L}^*_{\xi'}(t)\widetilde{u}^*_{\xi'}v\rangle=\partial_t\langle\widetilde{u}_{\xi'},\widetilde{u}_{\xi'}\rangle-\langle\widetilde{u}_{\xi'},\overline{\lambda}_{\xi'}\widetilde{u}^*_{\xi'}\rangle.$$

Therefore

$$\partial_t\langle\widetilde{u}_{\xi'}(t),\widetilde{u}^*_{\xi'}(t)\rangle=0,$$

for $t\in J_T$ and the inner product is independent of time. From expansions, estimates in (ii) and Lemma 5.1 we see that

$$\langle\widetilde{u}_{\xi'}(t),\widetilde{u}^*_{\xi'}(t)\rangle=\langle u^{(0)}(t),u^{*(0)}\rangle+O(\xi')=1+O(\xi'),$$

uniformly for $t\in J_T$. (iii) is proved by taking $r_1>0$ smaller if necessary.

□

Proof of Proposition 3.9 (iii). Thanks to Proposition 6.15 (iii) we can define functions $u_{\xi'}$ and $u^*_{\xi'}$ as follows

$$u_{\xi'} = \widetilde{u}_{\xi'}, \quad u^*_{\xi'} = \frac{1}{\langle \widetilde{u}_{\xi'}, \widetilde{u}^*_{\xi'} \rangle} \widetilde{u}^*_{\xi'}.$$

Properties of $u_{\xi'}$ and $u^*_{\xi'}$ follow from Proposition 6.15. This concludes the proof.

□

7 Proofs of main theorems

In this section we give proofs of Theorems 3.12 - 3.16. First let us deduce properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$ in Theorems 3.18 and 3.19.

Proof of Theorem 3.18 All properties of $\mathscr{Q}(t)$ are obtained by straightforward calculations from properties of $u_{\xi'}(\cdot, t)$ stated in Proposition 3.9 (iii). Expansion of $\mathscr{Q}(t)$ follows from the one of $u_{\xi'}(\cdot, t)$ as

$$\mathscr{Q}^{(1)}(t)\sigma = (\mathscr{F}^{-1}\{\widehat{\chi}_1 \widehat{\sigma}\}) u^{(1)}(\cdot, t),$$

$$\mathscr{Q}^{(2)}(t)\sigma = \mathscr{F}^{-1}\{-\widehat{\chi}_1 \widehat{\sigma} u^{(2)}(\xi', \cdot, t)\}.$$

□

Proof of Theorem 3.19 All properties of $\mathscr{P}(t)$ are obtained by straightforward calculations from properties of $u^*_{\xi'}(\cdot, t)$ stated in Proposition 3.9 (iii). Expansion of $\mathscr{P}(t)$ follows from the one of $u^*_{\xi'}(\cdot, t)$ as

$$\mathscr{P}^{(0)} u = \mathscr{F}^{-1}\{\widehat{\chi}_1 \langle \widehat{u}, u^{*(0)} \rangle\} = \mathscr{F}^{-1}\{\widehat{\chi}_1 [Q_0 \widehat{u}]\},$$

$$\mathscr{P}^{(1)}(t) u = \mathscr{F}^{-1}\{\widehat{\chi}_1 \langle \widehat{u}, u^{*(1)}(t) \rangle\},$$

$$\mathscr{P}^{(2)}(t) u = \mathscr{F}^{-1}\{-\widehat{\chi}_1 \langle \widehat{u}, u^{*(2)}(\xi', t) \rangle\}.$$

As for (v), since $\lambda_{\xi'} = -i\kappa_0 \xi_1 - \kappa_1 \xi_1^2 - \kappa'' |\xi''|^2 + O(|\xi'|^3)$, we see from properties of Fourier transform and $\mathscr{P}(t)$ that for $q = 0, 1, 2$ we can calculate

$$\|e^{(t-s)\Lambda} \mathscr{P}^{(q)}(s) \partial^k_{x'} u\|_2^2 \le C \int_{\mathbb{R}^{n-1}} \widehat{\chi}_1(\xi')^2 |\xi'|^{2k} e^{-(\kappa_1 |\xi_1|^2 + \kappa'' |\xi''|^2)(t-s)} |\langle \widehat{u}, u^{*(q)}(s) \rangle|^2 d\xi'.$$

Let $1 \le p \le 2$, then

$$|\langle \widehat{u}, u^{*(q)}(s) \rangle|^2 \le C |\widehat{u}(\xi')|_p^2.$$

Furthermore, let $2 \le s \le \infty$ and $\frac{1}{p} + \frac{1}{s} = 1$, then

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \widehat{\chi}_1(\xi')^2 |\xi'|^{2k} e^{-(\kappa_1 |\xi_1|^2 + \kappa'' |\xi''|^2)(t-s)} |\widehat{u}(\xi')|_p^2 d\xi' \\ &\le C \|\widehat{\chi}_1(\xi')^2 |\xi'|^{2k} e^{-(\kappa_1 |\xi_1|^2 + \kappa'' |\xi''|^2)(t-s)}\|_{L^{\frac{s}{s-2}}(\mathbb{R}^{n-1})} \| |\widehat{u}|_p \|^2_{L^s(\mathbb{R}^{n-1})} \\ &\le C \begin{cases} \|u\|_p^2, \\ (t-s)^{-(n-1)(\frac{1}{p} - \frac{1}{2}) - k} \|u\|_p^2, \end{cases} \le C(1 + t - s)^{-(n-1)(\frac{1}{p} - \frac{1}{2}) - k} \|u\|_p^2, \end{aligned} \tag{7.1}$$

for $1 \le p \le 2$. The second estimate in (v) is obtained in the same way as (7.1).

□

Proof of Theorem 3.12 As for (i), it follows from computation below:

$$\mathscr{F}\{P(t)(\partial_t + L(t))u(t)\} = \widehat{\chi}_1 \langle (\partial_t + \widehat{L}_{\xi'}(t)) u(t), u^*_{\xi'}(t) \rangle u_{\xi'} = \widehat{\chi}_1 \partial_t (\langle u(t), u^*_{\xi'}(t) \rangle u_{\xi'})$$

$$-\widehat{\chi}_1 \langle u(t), \partial_t u^*_{\xi'}(t) \rangle u_{\xi'} + \widehat{\chi}_1 \langle u(t), \widehat{L}^*_{\xi'}(t) u^*_{\xi'}(t) \rangle u_{\xi'} - \widehat{\chi}_1 \langle u(t), u^*_{\xi'}(t) \rangle \partial_t u_{\xi'}$$

$$= \widehat{\chi}_1 \partial_t (\langle u(t), u^*_{\xi'}(t) \rangle u_{\xi'}) + \widehat{\chi}_1 \langle u(t), u^*_{\xi'}(t) \rangle (-\partial_t - \lambda_{\xi'}) u_{\xi'}$$

$$= \widehat{\chi}_1(\partial_t + \widehat{L}_{\xi'}(t))(\langle u(t), u^*_{\xi'}(t)\rangle u_{\xi'}) = \mathscr{F}\{(\partial_t + L(t))P(t)u(t)\},$$

and

$$\mathscr{F}\{P(t)(\partial_t + L(t))u(t)\} = \widehat{\chi}_1\langle(\partial_t + \widehat{L}_{\xi'}(t))u(t), u^*_{\xi'}(t)\rangle u_{\xi'} = \widehat{\chi}_1\partial_t\langle u(t), u^*_{\xi'}(t)\rangle u_{\xi'}$$

$$-\widehat{\chi}_1\langle u(t), \partial_t u^*_{\xi'}(t)\rangle u_{\xi'} + \widehat{\chi}_1\langle u(t), \widehat{L}^*_{\xi'}(t)u^*_{\xi'}(t)\rangle u_{\xi'}$$

$$= \widehat{\chi}_1(\partial_t\langle u(t), u^*_{\xi'}(t)\rangle - \lambda_{\xi'}\langle u(t), u^*_{\xi'}(t)\rangle)u_{\xi'} = \mathscr{F}\{\mathscr{Q}(t)(\partial_t - \Lambda)\mathscr{P}(t)u(t)\}.$$

As for (ii), since $U(t,s)$ satisfies $\partial_t U(t,s) + L(t)U(t,s) = 0$, we have from (i) that

$$0 = (\partial_t + L(t))P(t)U(t,s) = \mathscr{Q}(t)[(\partial_t - \Lambda)\mathscr{P}(t)U(t,s)].$$

Therefore,

$$\mathscr{P}(t)U(t,s) = e^{(t-s)\Lambda}\mathscr{P}(s)U(s,s) = e^{(t-s)\Lambda}\mathscr{P}(s).$$

We proved the equation in (ii).

Next, let us show the estimate in (ii). Using Theorem 3.18 we see that

$$\|\partial_t^j\partial_{x'}^k\partial_{x_n}^l P(t)U(t,s)u\|_2 \le C\sum_{p=0}^{j}\|(\partial_t^{j-p}\partial_{x_n}^l\mathscr{Q}(t))\partial_t^j e^{(t-s)\Lambda}\mathscr{P}(s)\partial_{x'}^k u\|_2 \le C\|e^{(t-s)\Lambda}\mathscr{P}(s)\partial_{x'}^k u\|_2.$$

Theorem 3.19 (v) concludes the proof of (ii).

As for (iii), let $\tau \in [0,T)$ is such that $s+\tau$ is integer multiple of T. Using (ii) we get

$$(I - P(t))U(t,s)u = U(t,s)(I - P(s))u = U(t,s+\tau)U(s+\tau,s)(I - P(s))u$$

$$= U(t,s+\tau)(I - P(s+\tau))U(s+\tau,s)u.$$

Since $P(t) = P(t+T)$ we get

$$U(t,s)(I - P(s))u = U(t,s+\tau)(I - P(0))U(s+\tau,s)u. \tag{7.2}$$

From definition we have relation $P(0)v = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle\widehat{v}, u^*_{\xi'}(0)\rangle u_{\xi'}(0)\}$.

Since $u_{\xi'}(t)$ is eigenfunction for simple eigenvalue $-\lambda_{\xi'}$ there holds true that $u_{\xi'}(0)$ is eigenfunction of $\widehat{U}_{\xi'}(T,0)$ for $\mu_{\xi'} = e^{\lambda_{\xi'}T}$ (see the proof of Lemma 6.5). Analogous result holds true for $u^*_{\xi'}(0)$.

Therefore, there holds

$$\widehat{\Pi}_{\xi'}u_{\xi'}(0) = u_{\xi'}(0),\ \widehat{\Pi}^*_{\xi'}u^*_{\xi'}(0) = u^*_{\xi'}(0).$$

Since the eigenspace is one-dimensional there exists $C(\xi',\cdot) \in \mathbb{C}$ such that

$$\widehat{\Pi}_{\xi'}v = C(\xi',v)u_{\xi'}(0),\ v \in X_0.$$

Taking inner product with $u^*_{\xi'}(0)$ we express $C(\xi',\cdot)$ as

$$C(\xi',v) = \langle\widehat{\Pi}_{\xi'}v, u^*_{\xi'}(0)\rangle = \langle v, \widehat{\Pi}^*_{\xi'}u^*_{\xi'}(0)\rangle = \langle v, u^*_{\xi'}(0)\rangle.$$

Thus we arrive at relation

$$\widehat{\chi}_1\widehat{\Pi}_{\xi'}v = \widehat{\chi}_1\langle v, u^*_{\xi'}(0)\rangle u_{\xi'}(0) = \widehat{P}(0)v.$$

Let us decompose $U(t,s+\tau)(I - P(0))$ as follows

$$U(t,s+\tau)(I-P(0))v = \mathscr{F}^{-1}(\widehat{\chi}_1\widehat{U}_{\xi'}(t,s+\tau)(I - \widehat{\Pi}_{\xi'})\widehat{v} + (1-\widehat{\chi}_1)\widehat{U}_{\xi'}(t,s+\tau)\widehat{v}),$$

where $v = U(s+\tau,s)u$.

Using estimate in Proposition 3.5 (i) we obtain

$$\|\mathscr{F}^{-1}(\widehat{\chi}_1\widehat{U}_{\xi'}(t,s+\tau)(I-\widehat{\Pi}_{\xi'})\widehat{v})\|_{H^1(\Omega)} \le e^{-d(t-s-\tau)}\|v\|_{H^1(\Omega)\times L^2(\Omega)}, \tag{7.3}$$

for a positive constant $d > 0$ and $t - s - \tau > T$.

Moreover, we obtain from [1, Propositions 4.1, 4.2 and Theorem 3.1]

$$\|\mathscr{F}^{-1}((1-\widehat{\chi}_1)\widehat{U}_{\xi'}(t,s+\tau)\widehat{v})\|_{H^1(\Omega)} \le Ce^{-d(t-s-\tau)}\{\|v\|_{H^1(\Omega)\times L^2(\Omega)} + \|\partial_{x'}\widetilde{Q}v\|_{L^2(\Omega)}\}, \tag{7.4}$$

for $t-s-\tau \ge T$ and a positive constant d.

Now we are ready to show the exponential decay estimate. Using (7.2), (7.3), (7.4) and [1, Theorem 3.1] we have

$$\|U(t,s)(I-P(s))u\|_{H^1(\Omega)} = \|U(t,s+\tau)(I-P(0))U(s+\tau,s)u\|_{H^1(\Omega)}$$

$$\le Ce^{d(t-s-\tau)}(\|U(s+\tau,s)u\|_{H^1(\Omega)\times L^2(\Omega)} + \|\partial_{x'}\widetilde{Q}U(s+\tau,s)u\|_{L^2(\Omega)})$$

$$\le Ce^{d(t-s)}(\|u\|_{H^1(\Omega)\times L^2(\Omega)} + \|\partial_{x'}w\|_{L^2(\Omega)}),$$

for $t-s \ge T+\tau$. Using [1, Theorem 3.1] one can obtain

$$\|U(t,s)(I-P(s))u\|_{H^1(\Omega)} \le C(\|u\|_{H^1(\Omega)\times L^2(\Omega)} + \|\partial_{x'}w\|_{L^2(\Omega)}),$$

for $T \le t-s \le T+\tau$. This completes the proof.

□

Proof of Theorem 3.13 Equations in (i) and (ii) can be obtain analogously as in the proof of Theorem 3.12. Regularity follows from Theorems 3.18 and 3.19.

□

Proof of Theorem 3.16 (i) is well-known so we omit the proof. As for (ii), from Theorem 3.19 we have a relation

$$e^{(t-s)\Lambda}\mathscr{P}(s) = e^{(t-s)\Lambda}\mathscr{P}^{(0)} + \sum_{j=1}^{n-1}\partial_{x_j}e^{(t-s)\Lambda}\mathscr{P}_j^{(1)}(s) + \Delta' e^{(t-s)\Lambda}\mathscr{P}^{(2)}(s),$$

and $e^{(t-s)\Lambda}\mathscr{P}^{(0)}u = e^{(t-s)\Lambda}\mathscr{F}^{-1}\{[\widehat{\chi}_1 Q_0\widehat{u}]\}$.

We see that

$$\mathscr{F}\{e^{(t-s)\Lambda}\mathscr{P}^{(0)}u - \mathscr{H}(t-s)\sigma\} = (\widehat{\chi}_1 - 1)e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)(t-s)}\widehat{\sigma}$$

$$+\widehat{\chi}_1(e^{\lambda_{\xi'}(t-s)} - e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)(t-s)})\widehat{\sigma}.$$

Since $\lambda_{\xi'} + (i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2) = O(|\xi'|^3)$, we have

$$|e^{\lambda_{\xi'}(t-s)} - e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)(t-s)}| = |e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)(t-s)}(e^{(\lambda_{\xi'}+i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)(t-s)} - 1)|$$

$$\le C|\xi'|^3(t-s)e^{-\frac{\min\{\kappa_1,\kappa''\}}{4}|\xi'|^2(t-s)}.$$

Using above estimate we obtain

$$\|\partial_{x'}^k(e^{(t-s)\Lambda}\mathscr{P}^{(0)}u - \mathscr{H}(t-s)\sigma)\|_2 \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|\sigma\|_p,$$

$1 \le p \le 2$, cf. (7.1). Combining this with estimates from Theorem 3.19 (v) we conclude the first estimate in (ii). The second estimate in (ii) follows analogously.

As for (iii), since

$$\mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s)u - \mathscr{Q}^0(t)\mathscr{H}(t-s)\sigma$$
$$= (\mathscr{Q}(t) - \mathscr{Q}^0(t))e^{(t-s)\Lambda}\mathscr{P}(s)u + \mathscr{Q}^0(t)(e^{(t-s)\Lambda}\mathscr{P}(s)u - \mathscr{H}(t-s)\sigma),$$

the estimate follows from (ii) and Theorem 3.19 (v).

□

8 Appendix

Proof of Lemma 5.4. Since $u^{(0)*}(x_n) \in H^1 \times H^2_*$ is T-time periodic solution of (3.5), we see that 1 is eigenvalue of monodromy operator $\widehat{U}^*_0(0,T)$. As the next step we show some decay estimates for $\widehat{U}^*_0(s,0)$.

We decompose $\widehat{L}^*_0(s)$ as

$$\widehat{L}^*_0(s) = \widehat{L}^*_1 + \widehat{C}^*(s),$$

$$\widehat{L}^*_1 = \widehat{L}^*_0(s) - \widehat{C}^*(s), \quad \widehat{C}^*(s) = \begin{pmatrix} 0 & \frac{\nu\gamma^2}{P'(\rho_p)}(\partial^2_{x_n} v^1_p(s))^T \boldsymbol{e}'_1 & 0 \\ 0 & 0 & 0 \\ 0 & \partial_{x_n} v^1_p(s)^T \boldsymbol{e}'_1 & 0 \end{pmatrix}.$$

Here, note that $\langle \widehat{L}_1 u, v\rangle = \langle u, \widehat{L}^*_1 v\rangle$, $u, v \in H^1 \times H^2_*$ (see [1]). By properties of solutions of $\widehat{L}_1 u = 0$, $\widetilde{Q}u|_{x_n=0,1} = 0$ and $\widehat{L}^*_1 u = 0$, $\widetilde{Q}u|_{x_n=0,1} = 0$ it is easy to show that function ${}^T(\phi^{(0)}, 0, 0)$ is an eigenfunction for simple eigenvalue 0 of both $\widehat{L}_1$ and $\widehat{L}^*_1$. Moreover, one can show that there exists constant $\gamma_0 > 0$ such that if $\gamma \geq \gamma_0$ then

$$\sigma(-\widehat{L}_1) \cup \sigma(-\widehat{L}^*_1) \subset \{\mu \in \mathbb{C} : \pi \geq |\arg(\mu + \eta_0)| > \theta_0\},$$

for $\eta_0 > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ and there hold the same decay estimates for semigroups generated by $\widehat{L}_1$ and $\widehat{L}^*_1$, e.g.

$$\begin{aligned} &|e^{(s-t)\widehat{L}^*_1} f|_{X_0} \leq Ce^{d_0(s-t)}|f|_{X_0}, \\ &|\partial_{x_n} \widetilde{Q} e^{(s-t)\widehat{L}^*_1} f|_2 \leq \frac{C}{(t-s)^{\frac{1}{2}}} e^{d_0(s-t)}|f|_{X_0}, \end{aligned} \tag{8.1}$$

for a positive constant d_0, $s < t$ and functions $f \in \{g \in X_0 : [Q_0 g] = 0\}$. (See [1, Lemma 5.6, (5.5), (5.6)])

Operator $\widehat{\Pi}^*_0$ defined as

$$\widehat{\Pi}^*_0 u = \langle u, u^{(0)}(0)\rangle u^{(0)*},$$

satisfies by Propositions 4.1 and 5.2

$$\widehat{U}^*_0(0,T)\widehat{\Pi}^*_0 = \widehat{\Pi}^*_0 \widehat{U}^*_0(0,T) = \widehat{\Pi}^*_0.$$

Next let us show that

$$|\widehat{U}^*_0(s,0)u_0|_{X_0} \leq Ce^{ds}|u_0|_{X_0}, \tag{8.2}$$

and

$$|s|^{\frac{1}{2}}|\partial_{x_n}\widetilde{Q}\widehat{U}^*_0(s,0)u_0|_2 \leq Ce^{ds}|u_0|_{X_0}, \tag{8.3}$$

for $u_0 \in X_0$ satisfying $\widehat{\Pi}^*_0 u_0 = 0$ and a positive constant d.

To do so we first introduce operator $\widehat{\Pi}^*_0(s)$,

$$\widehat{\Pi}^*_0(s)u = \langle u, u^{(0)}(s)\rangle u^{(0)*}, \; s \in \mathbb{R}.$$

Let $u(s) = {}^T(\phi(s), w(s)) = \widehat{U}^*_0(s,0)u_0$ and $\widehat{\Pi}^*_0 u_0 = 0$. Since

$$\begin{aligned} \partial_s(\widehat{\Pi}^*_0(s)u(s)) &= \partial_s\langle \widehat{U}^*_0(s,0)u_0, u^{(0)}(s)\rangle u^{(0)*} \\ &= (\langle \widehat{L}^*_{\xi'}(s)\widehat{U}^*_0(s,0)u_0, u^{(0)}(s)\rangle + \langle \widehat{U}^*_0(s,0)u_0, -\widehat{L}_{\xi'}(s)u^{(0)}(s)\rangle)u^{(0)*} = 0, \end{aligned}$$

we have that

$$\widehat{\Pi}^*_0(s)u(s) = 0, \tag{8.4}$$

for all $s \leq 0$. Condition (8.4) is equivalent to

$$[\phi(s)] = -(w^1(s), \frac{1}{\alpha_0}w^{(0),1}(s)\rho_p). \tag{8.5}$$

From (8.5) we obtain by differentiation in s that

$$\partial_s[\phi(s)] = -(\partial_s w^1(s), \frac{1}{\alpha_0}w^{(0),1}(s)\rho_p) - (w^1(s), \frac{1}{\alpha_0}\partial_s w^{(0),1}(s)\rho_p).$$

Using equation for u and $w^{(0),1}$ we get

$$\partial_s[\phi(s)] = (w^1(s), \frac{\nu\gamma^2}{P'(\rho_p)}\partial_{x_n}^2 v_p^1(s)).$$

Since $w^1(s) = e^{s\nu A}w_0^1$ we have

$$|w^1(s)|_2 \le Ce^{\frac{\nu}{2}s}|w_0^1|_2,\ |\partial_{x_n} w^1(s)|_2 \le C\frac{1}{|s|^{\frac{1}{2}}}e^{\frac{\nu}{2}s}|w_0^1|_2,$$

where A is defined by (5.1). Therefore, we get

$$|[\phi(s)]| \le Ce^{\frac{\nu}{2}s}|w_0^1|_2,\ |\partial_s[\phi(s)]| \le Ce^{\frac{\nu}{2}s}|w_0^1|_2.$$

Let us decompose $\phi(s)$ as

$$\phi(s) = [\phi(s)] + \phi_2(s).$$

Thus, $[\phi_2(s)] = 0$ and

$$-\partial_s \begin{pmatrix} \phi_2(s) \\ w'(s) \\ w^n(s) \end{pmatrix} + \widehat{L}_1^* \begin{pmatrix} \phi_2(s) \\ w'(s) \\ w^n(s) \end{pmatrix} = \begin{pmatrix} f^0(s) \\ 0 \\ f^n(s) \end{pmatrix}. \tag{8.6}$$

Here, $f(s) = {}^T(f^0(s), 0, f^n(s))$ and

$$\begin{pmatrix} f^0(s) \\ f^n(s) \end{pmatrix} = \begin{pmatrix} -\frac{\nu\gamma^2\partial_{x_n}^2 v_p^1(s)}{P'(\rho_p)}w^1(s) + \partial_s[\phi(s)] \\ -\partial_{x_n} v_p^1(s)w^1(s) + [\phi(s)]\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\right) \end{pmatrix}.$$

Solution $u_2 = {}^T(\phi_2, w)$ of (8.6) can be expressed as

$$u_2(s) = e^{s\widehat{L}_1^*}\widetilde{u}_0 + \int_s^0 e^{(s-z)\widehat{L}_1^*}f(z)dz.$$

Here, $\widetilde{u}_0 = {}^T(\phi_0 - [\phi_0], w_0)$. From (8.6) and computations above it is straightforward to see

$$[Q_0 f] = [f^0] = 0,\ |f(s)|_{X_0} \le C\frac{1}{|s|^{\frac{1}{2}}}e^{\frac{\nu}{2}s}|w_0^1|_2,$$

and thus, by (8.1),

$$|u_2(s)|_{X_0} \le Ce^{d_0 s}|\widetilde{u}_0|_{X_0} + \int_s^0 e^{d_0(s-z)}|f(z)|_{X_0}dz \le Ce^{ds}|u_0|_{X_0},$$

with $0 \le d < \frac{1}{2}\min\{\frac{\nu}{2}, d_0\}$. Now it is easy to see that (8.2) holds true for $u_0 \in X_0$, $\widehat{\Pi}_0^* u_0 = 0$. (8.3) is proved in analogous way.

Now, let us show that 1 is simple. First, we show that for any $v \in X_0$, $v \ne 0$ that satisfies

$$(\widehat{U}_0^*(0,T) - 1)v = 0, \tag{8.7}$$

there exists constant $C \in \mathbb{R}$ such that $v = Cu^{(0)*}$. Let us decompose v using $\widehat{\Pi}_0^*$ as

$$v = Cu^{(0)*} + v_1,\ (I - \widehat{\Pi}_0^*)v_1 = v_1.$$

Since (8.7) is equivalent to the fact that $\widehat{U}_0^*(s,T)v$ is time-periodic solution, $\widehat{U}_0^*(s,T)v_1$ must also be time-periodic solution. But since

$$\widehat{\Pi}_0^* v_1 = 0,$$

by exponentional decay estimate (8.2) and periodicity we get $v_1 = 0$.

Furthermore, let us suppose that there exists $v \in X_0$ such that

$$(\widehat{U}_0^*(0,T) - 1)v = u^{(0)*}.$$

Since $Q'u^{(0)*} = 0$ we have $Q'\widehat{U}_0^*(0,T)v = Q'v$. Because $Q'\widehat{U}_0^*(s,t) = e^{-(t-s)A}Q'$ is not time periodic we have $Q'v = 0$. Note that $Q'\widehat{U}_0^*(s,T)v = 0$ for all $s \in J_T$. Taking $[Q_0\cdot]$ on both sides we get

$$[Q_0\widehat{U}_0^*(0,T)v]-[Q_0v]=\frac{\gamma^2}{\alpha_0}.$$

Since $Q'v=0$ we have that $\phi=Q_0\widehat{U}_0^*(s,T)v$ satisfies

$$-\partial_s\phi-\gamma^2\partial_{x_n}(\rho_p w^n)=0,$$

and therefore $\partial_s[\phi]=0$. So $[\phi(T)]=[\phi(0)]$, that is $[Q_0\widehat{U}_0^*(0,T)v]-[Q_0v]=0$. Since $0\neq\frac{\gamma^2}{\alpha_0}$ we showed that v can not exist. This proves that 1 is simple eigenvalue.

Now, we show (5.4). Let us compute spectral radius of $\widehat{U}_0^*(0,T)$ on $(I-\widehat{\Pi}_0^*)(H^1\times H_0^1)$. Using (8.2) and (8.3) we have

$$r(\widehat{U}_0^*(0,T)|_{(I-\widehat{\Pi}_0^*)(H^1\times H_0^1)})=r((I-\widehat{\Pi}_0^*)\widehat{U}_0^*(0,T)(I-\widehat{\Pi}_0^*))=\lim_{n\to\infty}|((I-\widehat{\Pi}_0^*)\widehat{U}_0^*(0,T)(I-\widehat{\Pi}_0^*))^n|^{\frac{1}{n}}_{L(X_0,H^1\times H_0^1)}$$

$$=\lim_{n\to\infty}|(I-\widehat{\Pi}_0^*)\widehat{U}_0^*(0,nT)(I-\widehat{\Pi}_0^*)|^{\frac{1}{n}}_{L(X_0,H^1\times H_0^1)}\le\lim_{n\to\infty}(C(1+\frac{1}{\sqrt{nT}}))^{\frac{1}{n}}e^{-dT}=e^{-dT}<1.$$

Therefore, we showed (5.4).

□

Proof of Lemma 5.5. To investigate spectrum of $\widehat{U}_{\xi'}^*(0,T)$ we follow the same steps as in [1, Proposition 5.10]. Let us denote $\widehat{M}_{\xi'}^*(s)=\widehat{L}_{\xi'}^*(s)-\widehat{L}_1^*$. We define

$$\widehat{V}_{\xi'}^*(s,t)=\widehat{U}_{\xi'}^*(s,t)-\widehat{U}_0^*(s,t).$$

Since

$$\widehat{U}_{\xi'}^*(s,t)u_0=e^{(s-t)\widehat{L}_1^*}u_0-\int_s^t e^{(s-z)\widehat{L}_1^*}\widehat{M}_{\xi'}^*(z)\widehat{U}_{\xi'}^*(z,t)u_0dz,$$

it is straightforward to see that $v(s)=\widehat{V}_{\xi'}^*(s,t)u_0$ is the solution of

$$-\partial_s v+\widehat{L}_0^*(s)v=-(\widehat{M}_{\xi'}(s)-\widehat{M}_0(s))\widehat{U}_{\xi'}^*(s,t)u_0,\ v|_{s=t}=0. \tag{8.8}$$

Let us define operator $\widehat{S}_{\xi'}^*V$ as

$$(\widehat{S}_{\xi'}^*V)(s)u_0=-\int_s^T\widehat{U}_0^*(s,z)(\widehat{M}_{\xi'}(z)-\widehat{M}_0(z))V(z)u_0dz,\ s\in J_T,$$

for $u_0\in H^1\times H_0^1$. Here $V\in L(H^1\times H_0^1,C(J_T;H^1\times H_0^1))$ is defined as

$$V:u_0\in H^1\times H_0^1\to V(\cdot)u_0\in C(J_T;H^1\times H_0^1).$$

We first estimate $\widehat{U}_0^*(s,t)u_0$. Observe that

$$|e^{(s-t)\widehat{L}_1*}f|_{X_0}\le Ce^{d_0(t-s)}|f-[Q_0f]u^{(0)*}|_{X_0}+|[Q_0f]u^{(0)*}|_{X_0}. \tag{8.9}$$

Here, we used (8.1) and the fact $e^{s\widehat{L}_1^*}u^{(0)*}=u^{(0)*}$. Since

$$\widehat{U}_0^*(s,t)u_0=e^{(s-t)\widehat{L}_1^*}u_0-\int_s^t e^{(s-z)\widehat{L}_1^*}\widehat{M}_0^*(z)\widehat{U}_0^*(z,t)u_0dz$$
$$=e^{(s-t)\widehat{L}_1^*}u_0-\int_s^t e^{(s-z)\widehat{L}_1^*}\widehat{M}_0^*(z)e^{(z-t)\nu A}w_0^1\boldsymbol{e}_1dz,$$

we see from (8.9) that

$$|\widehat{U}_0^*(s,t)u_0|_{X_0}\le Ce^{d_0(s-t)}|(u_0-[Q_0u_0]u^{(0)*})|_{X_0}+|[Q_0u_0]e^{(s-t)\widehat{L}_1^*}u^{(0)*}|_{X_0}$$
$$+\int_s^t e^{d_0(s-z)}|(\widehat{M}_0^*(z)e^{(z-t)\nu A}w_0^1\boldsymbol{e}_1-[Q_0\widehat{M}_0^*(z)e^{(z-t)\nu A}w_0^1\boldsymbol{e}_1]u^{(0)*})|_{X_0}dz$$
$$+\int_s^t|e^{(s-z)\widehat{L}_1^*}u^{(0)*}[Q_0\widehat{M}_0^*(z)e^{(z-t)\nu A}w_0^1\boldsymbol{e}_1]|_{X_0}dz\le C\{e^{d(s-t)}|u_0|_{X_0}+|u_0|_2\},$$

for a positive constant d and $u_0 \in X_0$. Similarly we can obtain estimate

$$|\widehat{U}_0^*(s,t)u_0|_{X_0} + (t-s)^{\frac{1}{2}}|\partial_{x_n}\widetilde{Q}\widehat{U}_0^*(s,t)u_0|_2 \le C|u_0|_{X_0}, \tag{8.10}$$

for $u_0 \in X_0$.

From (8.10) we see that $\widehat{S}_{\xi'}^* : L(H^1 \times H_0^1, C(J_T; H^1 \times H_0^1)) \to L(H^1 \times H_0^1, C(J_T; H^1 \times H_0^1))$, and $\widehat{U}_{\xi'}^*(\cdot, T)$ and $\widehat{V}_{\xi'}^*(\cdot, T)$ satisfy

$$\widehat{U}_{\xi'}^*(\cdot, T),\ \widehat{V}_{\xi'}^*(\cdot, T) \in L(H^1 \times H_0^1, C(J_T; H^1 \times H_0^1)).$$

Therefore, we conclude from (8.8) that

$$(I - S_{\xi'}^*)\widehat{V}_{\xi'}^*(\cdot, T) = \widehat{S}_{\xi'}^*\widehat{U}_0^*(\cdot, T).$$

By using estimate in Theorem 4.9 we can show that there exists $r_0 > 0$ such that for $|\xi'| \le r_0$ there holds $|\widehat{S}_{\xi'}^*|_{L(L(H^1\times H_0^1, C(J_T;H^1\times H_0^1)))} < 1$ and therefore using Neumann series expansion of $\widehat{S}_{\xi'}^*$ we get formula

$$\widehat{V}_{\xi'}^*(0,T) = \sum_{N=1}^{\infty} \widehat{S}_{\xi'}^N \widehat{U}_0^*(0,T), \tag{8.11}$$

and $|V_{\xi'}^*(0,T)|_{L(H^1\times H_0^1)} = O(\xi')$ (see the proof of [1, Proposition 5.10]).

Let $q_0 = \frac{3\widetilde{q}_0+1}{4}$. We see from Lemma 5.4 that there holds resolvent estimate

$$|(\mu - \widehat{U}_0^*(0,T))^{-1}|_{L(H^1\times H_0^1)} \le C,$$

with $C > 0$ uniform with respect to $\mu \in \{|\mu| > q_0\} \cap \{|\mu - 1| \ge \frac{1-\widetilde{q}_0}{4}\}$. We calculate

$$(\mu - \widehat{U}_{\xi'}^*(0,T)) = (\mu - \widehat{U}_0^*(0,T)) - \widehat{V}_{\xi'}^*(0,T) = [I - \widehat{V}_{\xi'}^*(0,T)(\mu - \widehat{U}_0^*(0,T))^{-1}](\mu - \widehat{U}_0^*(0,T)).$$

Therefore, for r_0 suitably smaller we see from (8.11) that there holds

$$(\mu - \widehat{U}_{\xi'}^*(0,T))^{-1} = (\mu - \widehat{U}_0^*(0,T))^{-1}\sum_{N=0}^{\infty}[\widehat{V}_{\xi'}^*(0,T)(\mu - \widehat{U}_0^*(0,T))^{-1}]^N,$$

for $\mu \in \{|\mu| > q_0\} \cap \{|\mu - 1| \ge \frac{1-\widetilde{q}_0}{4}\}$. We see that $\{|\mu - 1| = \frac{1-\widetilde{q}_0}{4}\}$ belongs to resolvent set of $\widehat{U}_{\xi'}^*(0,T)$ for $|\xi'| \le r_0$. In particular,

$$\widehat{\Pi}_{\xi'}^* = \frac{1}{2\pi i}\int_{\{|\mu-1| = \frac{1-\widetilde{q}_0}{4}\}} (\mu - \widehat{U}_{\xi'}^*(0,T))^{-1} d\mu,$$

is the eigenprojection for the eigenvalues lying inside the circle $\{|\mu - 1| = \frac{1-\widetilde{q}_0}{4}\}$. The continuity of $(\mu - \widehat{U}_{\xi'}^*(0,T))^{-1}$ in (μ, ξ') then implies that dim Range $\widehat{\Pi}_{\xi'}^* =$ dim Range $\widehat{\Pi}_0^* = 1$. Therefore, we see from Lemma 5.5 that $\sigma(\widehat{U}_{\xi'}^*(0,T)) \cap \{\mu : |\mu - 1| \le \frac{1-\widetilde{q}_0}{4}\}$ consists of only one simple eigenvalue, say $\theta_{\xi'}$. Thus we showed (5.5) with $\theta_{\xi'}$.

Let us denote $u_{\xi'}^{(0)*}$ the associated eigenfunction satisfying $u_{\xi'}^{(0)*}|_{\xi'=0} = u^{(0)*}$. We show that $\theta_{\xi'} = \overline{\mu}_{\xi'}$. Since

$$u_{\xi'}^{(0)}|_{\xi'=0} = u^{(0)}(0) \text{ and } u_{\xi'}^{(0)*}|_{\xi'=0} = u^{(0)*},$$

and $u_{\xi'}^{(0)}$, $u_{\xi'}^{(0)*}$ depend continuously on ξ' we get from Lemma 5.1 that

$$\langle u_{\xi'}^{(0)}, u_{\xi'}^{(0)*}\rangle = 1 + O(\xi').$$

Taking $r_0 > 0$ suitably smaller we have $\langle u_{\xi'}^{(0)}, u_{\xi'}^{(0)*}\rangle \ne 0$ for $|\xi'| \le r_0$ and we compute

$$\mu_{\xi'}\langle u_{\xi'}^{(0)}, u_{\xi'}^{(0)*}\rangle = \langle \mu_{\xi'} u_{\xi'}^{(0)}, u_{\xi'}^{(0)*}\rangle = \langle \widehat{U}_{\xi'}(T,0)u_{\xi'}^{(0)}, u_{\xi'}^{(0)*}\rangle$$

$$= \langle u_{\xi'}^{(0)}, \widehat{U}_{\xi'}^*(0,T)u_{\xi'}^{(0)*}\rangle = \overline{\theta}_{\xi'}\langle u_{\xi'}^{(0)}, u_{\xi'}^{(0)*}\rangle,$$

i.e.

$$\theta_{\xi'} = \overline{\mu}_{\xi'}.$$

This concludes the proof.

□

Proof of Lemma 6.4. Let us denote

$$g(t) = \partial_t f + (\partial_t \widehat{L}_{\xi'}(t))u.$$

Since $f \in Y^3_{per}$ it is easy to calculate that $g \in Y_{per}$. Therefore, by Theorem 6.3 in the case $k = 1$ we see that $\widetilde{u} = (\lambda - B_{\xi'})^{-1} g$ satisfies

$$\widetilde{u}(t) = e^{\lambda t}\widehat{U}_{\xi'}(t,0)u_1 - \int_0^t e^{\lambda(t-s)}\widehat{U}_{\xi'}(t,s)g(s)ds,$$

where

$$u_1 = -(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}\int_0^T e^{-\lambda s}\widehat{U}_{\xi'}(T,s)g(s)ds,$$

since $\widetilde{u}(0) = \widetilde{u}(T)$.

For all $v \in X_0$ we calculate

$$-\langle \int_0^t e^{\lambda(t-s)}\widehat{U}_{\xi'}(t,s)g(s)ds, v\rangle = -\int_0^t \langle g(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds.$$

Furthermore, there holds

$$\begin{aligned} &-\langle \partial_s \widehat{L}_{\xi'}(s)u(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle \\ &= \langle -\lambda u(s) + \partial_s u(s) + \widehat{L}_{\xi'}(s)u(s), \widehat{L}^*_{\xi'}(s)e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle - \tfrac{\partial}{\partial s}\langle \widehat{L}_{\xi'}(s)u(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle, \end{aligned} \tag{8.12}$$

and

$$-\langle \partial_s f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle = -\frac{\partial}{\partial s}\langle f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle + \langle f(s), (-\overline{\lambda} + \widehat{L}^*_{\xi'}(s))e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle, \tag{8.13}$$

for a.a. $0 < s < t$ and for all $v \in X_0$. First we show (8.12). Let us for now fix $t \in (0,T]$. Take $h \in C_0^\infty(0,t)$ such that supp $h \subset [\delta_0, t-\delta_0]$, $\delta_0 > 0$. Next we define mollified functions $(\rho_\delta * u) \in H^1(0,t;H^1 \times H^2_*)$ with $0 < \delta < \delta_0$. Now we can compute

$$-\int_0^t \langle \partial_s \widehat{L}_{\xi'}(s)(\rho_\delta * u)(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds = \int_0^t \langle \widehat{L}_{\xi'}(s)(\rho_\delta * \partial_s u)(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds$$

$$+\int_0^t \langle \widehat{L}_{\xi'}(s)(\rho_\delta * u)(s), \partial_s(e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v)\rangle h(s)ds + \int_0^t \langle \widehat{L}_{\xi'}(s)(\rho_\delta * u)(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h'(s)ds$$

$$= \int_0^t \langle (\rho_\delta * \partial_s u)(s), \widehat{L}^*_{\xi'}(s)e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds + \int_0^t \langle \widehat{L}_{\xi'}(s)(\rho_\delta * u)(s), (-\overline{\lambda} + \widehat{L}^*_{\xi'}(s))e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds$$

$$+\int_0^t \langle \widehat{L}_{\xi'}(s)(\rho_\delta * u)(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h'(s)ds.$$

Since

$$|\widehat{L}^*_{\xi'}(s)\widehat{U}^*_{\xi'}(s,t)v|_{X_0} \le C(t-s)^{-1}|v|_{X_0} \le C\delta_0^{-1}|v|_{X_0},$$

for $s \in$ supp h, above computation is rigorous and we can take limit in δ. Taking $\delta \to 0$ we obtain

$$-\int_0^t \langle \partial_s \widehat{L}_{\xi'}(s)u(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds = \int_0^t \langle \partial_s u(s) - \lambda u(s) + \widehat{L}_{\xi'}(s)u(s), \widehat{L}^*_{\xi'}(s)e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds$$

$$+\int_0^t \langle \widehat{L}_{\xi'}(s)u(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h'(s)ds,$$

for all $h \in C_0^\infty(0,t)$. Therefore, we see that (8.12) holds for a.a. $0 < s < t$ and for all $v \in X_0$. Above computation is valid for any $t \in (0,b]$. Second, we show (8.13). Let us fix $t \in (0,b]$ and take $h \in C_0^\infty(0,t)$. We compute

$$-\int_0^t \langle \partial_s f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds = \int_0^t \langle f(s), \partial_s(e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v)\rangle h(s)ds + \int_0^t \langle f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h'(s)ds$$

$$= \int_0^t \langle f(s), (-\overline{\lambda} + \widehat{L}^*_{\xi'}(s))e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds - \int_0^t \frac{\partial}{\partial s}\langle f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle h(s)ds,$$

for all $h \in C_0^\infty(0,t)$. Therefore, we see that (8.13) holds for a.a. $0 < s < t$ and for all $v \in X_0$. Above computation is valid for any $t \in (0,b]$.

Next, using (8.12) and (8.13) we calculate

$$-\int_0^t \langle g(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds = \int_0^t \langle \partial_s u(s) - \lambda u(s) + \widehat{L}_{\xi'}(s)u(s) + f(s), \widehat{L}^*_{\xi'}(s)e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds$$

$$-\int_0^t \frac{\partial}{\partial s}\langle \widehat{L}_{\xi'}(s)u(s) + f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds - \int_0^t \langle f(s), \overline{\lambda}e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds$$

$$= -\int_0^t \frac{\partial}{\partial s}\langle \widehat{L}_{\xi'}(s)u(s) + f(s), e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds - \int_0^t \langle f(s), \overline{\lambda}e^{-\overline{\lambda}(s-t)}\widehat{U}^*_{\xi'}(s,t)v\rangle ds$$

$$= -\int_0^t \frac{\partial}{\partial s}\langle e^{\lambda(t-s)}\widehat{U}_{\xi'}(t,s)(\widehat{L}_{\xi'}(s)u(s) - \lambda u(s) + f(s)), v\rangle ds$$

$$= -\langle \widehat{L}_{\xi'}(t)u(t) - \lambda u(t) + f(t), v\rangle + \langle e^{\lambda t}\widehat{U}_{\xi'}(t,0)(\widehat{L}_{\xi'}(0)u_0 - \lambda u_0 + f(0)), v\rangle,$$

for all $v \in X_0$. Therefore, we obtained

$$-\int_0^t e^{-\lambda(s-t)}\widehat{U}_{\xi'}(t,s)g(s)ds = -\widehat{L}_{\xi'}(t)u(t) + \lambda u(t) - f(t) + e^{\lambda t}\widehat{U}_{\xi'}(t,0)(\widehat{L}_{\xi'}(0)u_0 - \lambda u_0 + f(0)).$$

Using this relation at time $t = T$ we get

$$u_1 = (e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}e^{-\lambda T}\{-\widehat{L}_{\xi'}(T)u(T) + \lambda u(T) - f(T) + e^{\lambda T}\widehat{U}_{\xi'}(T,0)(\widehat{L}_{\xi'}(0)u_0 - \lambda u_0 + f(0))\}.$$

Using relation

$$(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}\widehat{U}_{\xi'}(T,0) = -I + e^{-\lambda T}(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1},$$

we obtain

$$u_1 = (e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}e^{-\lambda T}\{-\widehat{L}_{\xi'}(T)u(T) + \lambda u(T) - f(T)\} - (\widehat{L}_{\xi'}(0)u_0 - \lambda u_0 + f(0))$$

$$+ e^{-\lambda T}(e^{-\lambda T} - \widehat{U}_{\xi'}(T,0))^{-1}(\widehat{L}_{\xi'}(0)u_0 - \lambda u_0 + f(0)).$$

Finally, by periodicity in time we get

$$u_1 = -f(0) - \widehat{L}_{\xi'}(0)u_0 + \lambda u_0,$$

and thus

$$\widetilde{u}(t) = e^{\lambda t}\widehat{U}_{\xi'}(t,0)u_1 + \partial_t u(t) - e^{\lambda t}\widehat{U}_{\xi'}(t,0)u_1 = \partial_t u(t),$$

for a.a. $t \in J_T$.

□

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