九州大学学術情報リポジトリ
Kyushu University Institutional Repository

The Extended Elastoplastic Constitutive Equation with Tangential Stress Rate Effect

Hashiguchi, Koichi
Laboratory of Agricultural Machinery, Faculty of Agriculture, Kyushu University

Okayasu, Takafumi
Laboratory of Agricultural Machinery, Faculty of Agriculture, Kyushu University

Tsutsumi, Seiichiro
Laboratory of Agricultural Machinery, Faculty of Agriculture, Kyushu University

https://doi.org/10.5109/24209

出版情報：九州大学大学院農学研究院紀要. 42 (1/2), pp.225-235, 1997-12. Kyushu University
バージョン：
権利関係：

KYUSHU UNIVERSITY

J. Fac. Agr., Kyushu Univ., **42** (1・2), 225-235 (1997)

The Extended Elastoplastic Constitutive Equation with Tangential Stress Rate Effect

Koichi Hashiguchi, Takafumi Okayasu and Seiichiro Tsutsumi

Laboratory of Agricultural Machinery, Faculty of Agriculture,
Kyushu University 46-01, Fukuoka 812, Japan

(Received August 20, 1997 and accepted August 25, 1997)

The extension of the elastoplastic constitutive equation so as to describe the plastic stretching due to the stress rate component tangential to the yield or loading surface has been one of the most pressing problems in the elastoplasticity. To this aim, various models have been proposed in the past. However, a pertinent model applicable to a general loading process has not previously been proposed. In this article, the elastoplastic constitutive equation extended so as to describe a plastic stretching due to a stress rate component tangential to a yield or loading surface is formulated keeping a single and smooth yield surface. It would be a pertinent one which fulfills the mechanical requirements for elastoplastic constitutive equation and which is applicable to an arbitrary loading process. Based on this equation, a constitutive equation of metals with the isotropic-kinematic hardening is formulated.

INTRODUCTION

The plastic stretching is independent of the stress rate component tangential to the yield or loading surface, called the *tangential stress rate*, in the traditional elastoplastic constitutive equation with a single and smooth plastic potential surface. For metals the single crystal grains have multislip system and thus the plastic stretching would be dependent of the tangential stress rate. They may be observed to some extent even for a polycrystalline metals in which an infinite number of slip systems are activated, because some single crystal grains in the material influence the deformation behavior in terms of macroscopic viewpoint. They cannot be neglected in the process that the loading path abruptly changes or deviates severely from the proportional loading as observed in the plastic instability phenomena with a localization of deformation. The extension of the constitutive equation so as to describe these dependencies pertinently would be one of the most fundamental but unsolved problems in elastoplasticity at present.

The author (Hashiguchi, 1993a, 1993b, 1997) has described the mechanical requirements for constitutive equations describing an irreversible deformation, *i.e.* the irreversibility condition, the continuity condition, the work rate-stiffness relaxation and the smoothness condition. In this article, the elastoplastic constitutive equation extended so as to describe a plastic stretching due to a stress rate component tangential to a yield or loading surface is formulated keeping a single and smooth (regular) yield surface for the steady development of the elastoplasticity in physical and mathematical aspects. It would be a pertinent one fulfilling the above-mentioned requirements and is applicable to an arbitrary loading process including unloading, reloading and reverse loading processes. Based on this equation, the constitutive equation of metals with the von Mises yield condition obeying the isotropic-kinematic hardening is formulated.

CONSTITUTIVE EQUATION

Let it be assumed that the stretching $\boldsymbol{D}$ is additively decomposed into the elastic stretching $\boldsymbol{D}^e$ and the plastic stretching $\boldsymbol{D}^p$ which is further decomposed into the stretchings $\boldsymbol{D}^p_n$ and $\boldsymbol{D}^p_t$ caused by the stress rate component normal and tangential, respectively, to the yield or loading surface, while they are called the *normal-* and the *tangential-plastic stretching*, respectively. That is,

$$\left.\begin{aligned}\boldsymbol{D} &= \boldsymbol{D}^e + \boldsymbol{D}^p,\\ \boldsymbol{D}^p &= \boldsymbol{D}^p_n + \boldsymbol{D}^p_t,\end{aligned}\right\} \tag{1}$$

provided that the elastic stretching $\boldsymbol{D}^e$ is linearly related to the stress rate as

$$\boldsymbol{D}^e = \boldsymbol{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}, \tag{2}$$

where $\boldsymbol{E}$ is the elastic modulus, a function of the stress and plastic internal-state variables in general, $(\ )^{-1}$ designating the inverse and $\boldsymbol{\sigma}$ is the stress, $(^\circ)$ denoting a proper corotational rate (e.g. Dafalias, 1985, Zbib and Aifantis, 1988).

Assume that the evolution (isotropic and anisotropic hardening/softening) of the yield surface

$$f(\boldsymbol{\sigma}, \boldsymbol{H}_i) = 0 \tag{3}$$

is independent of the tangential-plastic stretching $\boldsymbol{D}^p_t$ but dependent only of the normal-plastic stretching $\boldsymbol{D}^p_n$. Then, it holds that

$$\overset{\circ}{\boldsymbol{H}}_i = \boldsymbol{O} \text{ for } \boldsymbol{D}^p_n = \boldsymbol{O}, \tag{4}$$

where $\boldsymbol{H}_i$ $(i = 1, 2,..., n)$ denoting collectively scalar- or tensor-valued plastic internal state variables.

Assume the associated flow rule for the normal-plastic stretching:

$$\boldsymbol{D}^p_n = \lambda \boldsymbol{N}, \tag{5}$$

where

$$\boldsymbol{N} \equiv \frac{\partial f}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|. \tag{6}$$

λ is a proportionality factor, a function of the stress, plastic internal state variables and the stress rate or the stretching in degree one.

The substitution of Eq. (5) into the consistency condition of the yield condition (3), *i.e.*

$$\mathrm{tr}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\right) + \sum_{i=1}^{n} \mathrm{tr}\left(\frac{\partial f}{\partial \boldsymbol{H}_i}\overset{\circ}{\boldsymbol{H}}_i\right) = 0 \tag{7}$$

leads to

$$\boldsymbol{D}^p_n = \frac{\mathrm{tr}\,(\boldsymbol{N}\overset{\circ}{\boldsymbol{\sigma}})}{M_p}\boldsymbol{N}, \tag{8}$$

where

$$M_p \equiv -\sum_{i=1}^{n} \mathrm{tr}\left(\frac{\partial f}{\partial \boldsymbol{H}_i}\boldsymbol{h}_i\right) \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|. \tag{9}$$

$\boldsymbol{h}_i$ is a function of the stress, plastic internal state variables and $\boldsymbol{N}$ in degree one, which is related to $\overset{\circ}{\boldsymbol{H}}_i$ as

$$\overset{\circ}{\boldsymbol{H}}_i = \lambda \boldsymbol{h}_i. \tag{10}$$

Let the tangential-plastic stretching $\boldsymbol{D}_t^p$ be written as

$$\boldsymbol{D}_t^p = \boldsymbol{\xi}\,(\overset{\circ}{\boldsymbol{\sigma}}_t, \boldsymbol{\sigma}, \boldsymbol{H}_i)\,, \tag{11}$$

where $\boldsymbol{\xi}$ is the second-order tensor function of stress rate component tangential to a yield or loading surface, i.e. the tangential stress rate denoted as $\overset{\circ}{\boldsymbol{\sigma}}_t$, stress and plastic internal state variables in general. $\overset{\circ}{\boldsymbol{\sigma}}_t$ is described as

$$\left.\begin{aligned} \overset{\circ}{\boldsymbol{\sigma}}_t &\equiv \overset{\circ}{\boldsymbol{\sigma}} - \overset{\circ}{\boldsymbol{\sigma}}_n, \\ \overset{\circ}{\boldsymbol{\sigma}}_n &\equiv \mathrm{tr}(\boldsymbol{N}\overset{\circ}{\boldsymbol{\sigma}})\,\boldsymbol{N}\,. \end{aligned}\right\} \tag{12}$$

Here, it should be noted that the tangential-plastic stretching $\boldsymbol{D}_t^p$ has to fulfill the irreversibility *condition*

$$\boldsymbol{D}_t^p \neq -\boldsymbol{D}_t^{p\,\prime} \ \ \text{for}\ \overset{\circ}{\boldsymbol{\sigma}}_t = -\overset{\circ}{\boldsymbol{\sigma}}_t{}'\,, \tag{13}$$

where tangential-plastic stretching induced by stress rates $\overset{\circ}{\boldsymbol{\sigma}}_t$ and $\overset{\circ}{\boldsymbol{\sigma}}_t{}'$ which have the same magnitude but opposite directions to each other be denoted as $\boldsymbol{D}_t^p$ and $\boldsymbol{D}_t^{p\,\prime}$, respectively. Thus, needless to say, $\boldsymbol{D}_t^p$ has to be nonlinearly related to the tangential-stress rate (so-called *rate-nonlinearlity*) at least.

The stretching is given from Eqs. (1), (2) and (7) with Eq. (11) as

$$\boldsymbol{D} = \boldsymbol{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}\,(\boldsymbol{N}\overset{\circ}{\boldsymbol{\sigma}})}{M_p}\boldsymbol{N} + \boldsymbol{\xi}\,, \tag{14}$$

while its inverse relation, i.e. the analytical expression of the stress rate in terms of the stretching cannot be derived because of the rate-nonlinearity of Eq. (14).

Let the loading criterion (Hill, 1958, 1967, Hashiguchi, 1994) be modified as follows:

$$\left.\begin{aligned} &\boldsymbol{D}_n^p \neq \boldsymbol{O} : f = 0 \text{ and } \mathrm{tr}\,(\boldsymbol{NED}) > 0, \\ &\boldsymbol{D}_n^p = \boldsymbol{O} : \text{others}, \end{aligned}\right\} \tag{15}$$

while the tangential-plastic stretching $\boldsymbol{D}_t^p$ occurs always by the tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}_t$ so that the continuity condition (Hashiguchi, 1993a, 1997) is fulfilled. The condition $f = 0$ in Eq. (15) is unnecessary in the subloading surface model fulfilling the smoothness condition including the smooth elastic-plastic transition.

Let the *work rate-stiffness relaxation* (Hashiguchi, 1993a) be modified as

$$\mathrm{tr}\,(\boldsymbol{DED}_n^p) \geq 0 \text{ for } \mathrm{tr}\,(\boldsymbol{NED}) \geq 0\,. \tag{16}$$

A concrete example of the tensor $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi} = S_t \,\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|^n \boldsymbol{N}\,, \tag{17}$$

where S_t is the material constant or function of stress and plastic internal variables, and n is the material constant. $(\ )^*$ denotes the deviatoric part. The constitutive equation (14) with Eq. (17) fulfills clearly the inequality (16), while it holds that $\mathrm{tr}\,(\boldsymbol{DED}^p) = \mathrm{tr}\,(\boldsymbol{DED}_t^p) < 0$ for $\mathrm{tr}\,(\boldsymbol{NED}) < 0$.

CONSTITUTIVE EQUATION OF METALS

Based on Eq. (14) with Eq. (17), let a constitutive equation of metals be formulated in this section.

Now, adopt the von Mises yield condition with the isotropic-kinematic hardening:

$$f(\widehat{\boldsymbol{\sigma}}) = F(H) = 0, \tag{18}$$

where

$$f(\widehat{\boldsymbol{\sigma}}) = \sqrt{\frac{3}{2}}\,\|\widehat{\boldsymbol{\sigma}}^*\|, \tag{19}$$

$$\overset{\circ}{\boldsymbol{\alpha}} = K_1\widehat{\boldsymbol{\sigma}} - K_2\boldsymbol{\alpha}\ , \tag{20}$$

$$K_1 = k_1\,\|\boldsymbol{D}^p_n\|,\ K_2 = k_2\,\|\boldsymbol{D}^p_n\|, \tag{21}$$

$$F = F_0\,[1 + h_1\{1 - \exp(-h_2 H)\}], \tag{22}$$

$$\dot{H} = \sqrt{\frac{2}{3}}\,\|\boldsymbol{D}^p_n\|. \tag{23}$$

and

$$\widehat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha}\ , \tag{24}$$

$$\widehat{\boldsymbol{\sigma}}_m \equiv \frac{1}{3}\,\mathrm{tr}\,\widehat{\boldsymbol{\sigma}},\ \widehat{\boldsymbol{\sigma}}^* \equiv \widehat{\boldsymbol{\sigma}} - \widehat{\sigma}_m\boldsymbol{I}, \tag{25}$$

k_1 and k_2 are material constants, while K_1 and K_2 are generally scalar functions of the plastic stretching in degree one, the stress and plastic internal-state variables. F_0 is the initial value of F, h_1 and h_2 are material constants.

The plastic modulus M_p is given from Eq. (9) as

$$M_p = \mathrm{tr}\,(\boldsymbol{N}\boldsymbol{a}) + F'\,h \Big/ \left\|\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\| (> 0), \tag{26}$$

where

$$\boldsymbol{N} = \frac{\widehat{\boldsymbol{\sigma}}^*}{\|\widehat{\boldsymbol{\sigma}}^*\|}, \tag{27}$$

$$\boldsymbol{a} \equiv \overset{\circ}{\boldsymbol{\alpha}} / \lambda = (k_1\widehat{\boldsymbol{\sigma}} - k_2\boldsymbol{\alpha}), \tag{28}$$

$$F' \equiv \frac{dF}{dH} = F_0\,h_1\,h_2 \exp(-h_2 H)(> 0), \tag{29}$$

$$h \equiv \dot{H}/\lambda = \sqrt{\frac{2}{3}}, \tag{30}$$

$$\left\|\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\| = \sqrt{\frac{2}{3}}\ . \tag{31}$$

The stretching is given from Eq. (14) with Eqs. (11), (17), (26) and (28) as

$$\boldsymbol{D} = \boldsymbol{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}\,(\boldsymbol{N}\overset{\circ}{\boldsymbol{\sigma}})}{\mathrm{tr}\,\{\boldsymbol{N}(k_1\widehat{\boldsymbol{\sigma}} - k_2\boldsymbol{\alpha})\} + \boldsymbol{F}'}\,\boldsymbol{N} + \boldsymbol{\xi}\,. \tag{32}$$

In what follows, let the elastic property be given by the Hooke's type

$$E_{ijkl} = L\,\delta_{ij}\,\delta_{kl} + G\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}), \tag{33}$$

where L and G are the material parameters corresponding to the Lame's constant and the elastic shear modulus, respectively, for the elastic body, and δ_{ij} is the Kronecker's delta.

Let the relation for the magnitude of plastic stretching versus the direction of stress rate in the π -plane be evaluated by the following scalar variable μ versus α.

$$\left.\begin{aligned} \alpha &\equiv \cos^{-1}\{\mathrm{tr}\,(\frac{\boldsymbol{\sigma}^*}{\|\boldsymbol{\sigma}^*\|}\frac{\mathring{\boldsymbol{\sigma}}^*}{\|\mathring{\boldsymbol{\sigma}}^*\|})\}, \\ \mu &\equiv \frac{\|\boldsymbol{D}^p\|/\|\mathring{\boldsymbol{\sigma}}^*\|}{(\|\boldsymbol{D}^p\|/\|\mathring{\boldsymbol{\sigma}}^*\|)_{pro}}, \end{aligned}\right\} \tag{34}$$

$(\ )_{pro}$ denoting the value of the quantity inside the bracket in the proportional loading. That is, α denotes the direction of stress rate from that of stress in the preceding proportional loading, and μ the ratio of the magnitude of plastic stretching to that of stress rate compared with the ratio in the proportional loading. The variable μ for Eq. (32) for the isotropic hardening, *i.e.* $k_1 = k_2 = 0$ is expressed as

$$\mu = \begin{cases} \dfrac{\|\boldsymbol{D}_n^p + \boldsymbol{D}_t^p\|/\|\mathring{\boldsymbol{\sigma}}^*\|}{(\|\boldsymbol{D}_n^p\|/\|\mathring{\boldsymbol{\sigma}}^*\|)_{pro}} = \dfrac{\left\|\dfrac{\mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}}^*)}{F'}\boldsymbol{N} + S_t\boldsymbol{\xi}\right\|/\|\mathring{\boldsymbol{\sigma}}^*\|}{\left(\left\|\dfrac{\mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}}^*)}{F'}\boldsymbol{N}\right\|/\|\mathring{\boldsymbol{\sigma}}^*\|\right)_{pro}} = \left\|\cos\alpha\boldsymbol{N} + S_tF'\dfrac{\boldsymbol{\xi}}{\|\mathring{\boldsymbol{\sigma}}^*\|}\right\| & \text{for } \alpha \leq 90^\circ \\[2ex] \dfrac{\|\boldsymbol{D}_t^p\|/\|\mathring{\boldsymbol{\sigma}}^*\|}{(\|\boldsymbol{D}_n^p\|/\|\mathring{\boldsymbol{\sigma}}^*\|)_{pro}} = \dfrac{\|S_t\boldsymbol{\xi}\|/\|\mathring{\boldsymbol{\sigma}}^*\|}{\left(\left\|\dfrac{\mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}}^*)}{F'}\boldsymbol{N}\right\|/\|\mathring{\boldsymbol{\sigma}}^*\|\right)_{pro}} = \left\|S_tF'\dfrac{\boldsymbol{\xi}}{\|\mathring{\boldsymbol{\sigma}}^*\|}\right\| & \text{for } \alpha > 90^\circ \end{cases} \tag{35}$$

which reduces to

$$\mu = \begin{cases} \cos\alpha + S_tF'\sin\alpha & \text{for } \alpha \leq 90^\circ \\ S_tF'\sin\alpha & \text{for } \alpha > 90^\circ \end{cases} \tag{36}$$

for Eq. (17) with $n = 1$. Eq. (36) is depicted in Fig. 1 where the measured values (Ito *et al.*, 1992) are shown by the circles. A good agreement between theoretical curve and

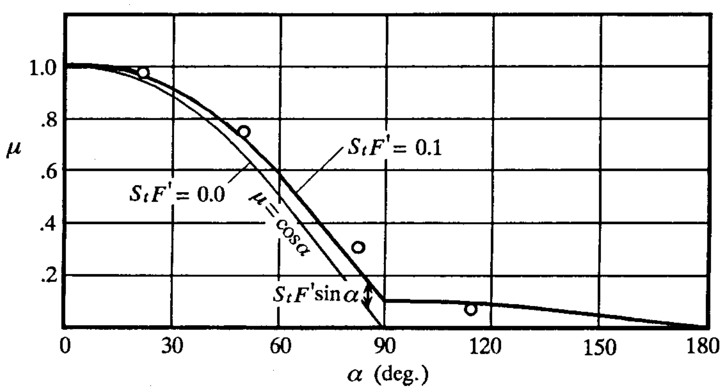

Fig. 1. The variation of the magnitude of plastic stretching versus the direction of the stress rate.

measured values is observed. The shape-transformation of yield surface which protrudes in the pre-stressed direction and inversely becomes flat behind that direction (cf. e.g. Ohashi *et al.*, 1975, Wu and Yeh, 1991) could be described by Eq. (32) without introducing complex yield functions with forth-order tensors (cf. e.g. Mazilu and Meyers, 1985).

Needless to say, the direction of plastic stretching is independent of the stress rate and is normal to the yield or loading surface, although it deviates up to 20° in the test result of Ito *et al.* (1992).

REVIEW OF EXISTING MODELS

The mechanical features of existing models for describing the plastic stretching due to the tangential stress rate are examined below.

(a) *Intersection of yield surfaces* (Koiter, 1953, Bland, 1957, Mandel, 1965, Hill, 1966, Sewell, 1973, 1974)

This model aims at describing the above-mentioned dependencies and the plastic stretching induced for a range covering more than half the stress rate space by incorporating plural smooth yield surfaces which intersect each other and obey the associated flow rule. It cannot, however, always describe the dependencies since a stress does not necessarily exist at the intersecting point of yield surfaces. The probability that a stress exists at the intersecting point of yield surfaces would be small, unless an infinite number of yield surfaces are introduced. While a practical calculation of deformation by this model was performed by Sewell (1973, 1974), it was restricted to base states of uniaxial stress. A computational practical extension of this model to general stress states is not obvious as was indicated by Christoffersen and Hutchinson (1979).

(b) *Corner theory* (Hill, 1967b, Christoffersen and Hutchinson, 1979, Ito, 1979, Gotoh, 1986, Tomita *et al.*, 1986, Goya *et al.*, 1991, 1995)

This theory also aims at describing the aforementioned dependencies and the plastic stretching induced for a wide range of stress rate space by assuming the existence of a corner or a cone on the yield surface which geometrically induces a singularity in the field of the normal vectors of the yield surface. However, the evolutional rule of the cone due to plastic stretching has not been presented yet and perhaps cannot be rationally formulated, especially if the stress rate has a direction with an angle larger than 90° from the outward central axis of the cone which contracts if the tip of the cone is assumed to move with the current stress point. Thus, this model is not applicable to the general loading process which includes unloading, reloading and reverse loading in various directions, although a modification of this model has been attempted by Tomita *et al.* (1986) and also by Goya *et al.* (1991, 1995), incorporating kinematic hardening. Furthermore, if an infinitesimal cone is assumed, finitely different stretchings would be predicted for identical stress rates given in the situation where the stress exists just on the corner (elastoplastic state) and in the other situation where the stress rate exists in the infinitesimally inside the corner (elastic state), violating the continuity condition. Moreover, constitutive equations based on this approach take quite complicated forms, thus creating inconvenience in analytical or numerical calculations. Besides, it should be noted that, while some researchers supported the corner formulation, others failed to conform it, as reviewed by Hecker (1972) or Ikegami (1979).

(c) *Hypoelasticity* (Budiansky, 1959, Rudnicki and Rice, 1975, Storen and Rice, 1975, Dorris and Nemat-Nasser, 1982, Lehmann, 1982, Hashiguchi, 1989b, Zbib, 1991, Papamichos *et al.*, 1993, Vermeer, 1993)

A stretching due to the stress rate component tangential to a single smooth yield surface is introduced in addition to the plastic stretching obeying the associated flow rule, where additional stretching is related linearly to the stress rate so that this approach falls within the framework of hypoelasticity (Truesdell, 1955). The J_2-deformation theory of Storen and Rice (1975) is identical with the deformation theory of Hencky (1924) in a differential form. It should be noted that the additional stretching cannot be regarded to be plastic but is regarded to be elastic, although the proponents of this approach called it "plastic component" in their articles, since it is related linearly to the stress rate without any loading criterion. If the additional stretching is formulated to be plastic by introducing a special loading criterion, it leads to a violation of the continuity condition. This situation is similar to the so-called hypoelastic models (e.g. Stutz 1973, Romano, 1974, Davis and Mullenger, 1978, Dragusin, 1981) criticized by Mroz (1980). Ultimately, this approach is inadequate for describing the elastoplastic deformation.

(d) *Flow rule with a stress rate* (Mroz, 1966, Wang *et al.*, 1990, Hashiguchi, 1993a)

The associated flow rule with a single smooth yield surface is extended by introducing the stress rate tensor in degree zero in addition to the outward-normal tensor of the yield surface. The relation between the stress rate and the stretching becomes high-order nonlinear so that it can describe irreversible deformation mathematically. However, it does not generally fulfill the work rate-stiffness relaxation (16) as described below.

Mroz (1966) proposed the following flow rule by using the representation theorem for the isotropic tensor function of two tensor variables (Rivlin and Ericksen, 1955), i.e. the outward-normal $\boldsymbol{N}$ of the yield surface and the deviatoric stress rate $\mathring{\boldsymbol{\sigma}}^*$ and choosing only two simple terms.

$$\boldsymbol{D}^p = \mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}}^*)\,(a\boldsymbol{N} + b\frac{\mathring{\boldsymbol{\sigma}}^*}{\|\mathring{\boldsymbol{\sigma}}^*\|}), \tag{38}$$

where

$$\left.\begin{aligned} \dot{\sigma}_m &\equiv \frac{1}{3}\,\mathrm{tr}\,\mathring{\boldsymbol{\sigma}},\ \mathring{\boldsymbol{\sigma}}^m \equiv \dot{\sigma}_m \boldsymbol{I}, \\ \mathring{\boldsymbol{\sigma}}^* &\equiv \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{\sigma}}^m. \end{aligned}\right\} \tag{39}$$

a and b are scalar functions of $\boldsymbol{N}$ and $\mathring{\sigma}^*$ in degree zero, and $(\dot{\ })$stands for a material-time derivative. Eqn (38) fulfills the continuity condition only in case of $\mathrm{tr}\boldsymbol{N} = 0$ leading to $\mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}}^*) = \mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}})$. It is unable to describe the influence of the mean stress rate on the direction of the plastic stretching, which is required for plastically pressure-dependent materials.

The author (Hashiguchi, 1993a, 1994) assumed the flow rule

$$\boldsymbol{D}^p = \lambda\ (\boldsymbol{N} + S_t^m\frac{\mathring{\boldsymbol{\sigma}}_t^m}{\|\mathring{\boldsymbol{\sigma}}_n\|} + S_t^*\frac{\mathring{\boldsymbol{\sigma}}_t^*}{\|\mathring{\boldsymbol{\sigma}}_n\|}), \tag{40}$$

where λ is a positive proportionality factor, S_t^m and S_t^* are material parameters, and

$$\left.\begin{aligned} \mathring{\boldsymbol{\sigma}}_t &\equiv \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{\sigma}}_n = \mathring{\boldsymbol{\sigma}}_t^m + \mathring{\boldsymbol{\sigma}}_t^*, \\ \mathring{\boldsymbol{\sigma}}_n &\equiv \mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}})\,\boldsymbol{N}, \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned}\dot{\boldsymbol{\sigma}}_t^m &\equiv \dot{\boldsymbol{\sigma}}^m - \dot{\boldsymbol{\sigma}}_n^m = \dot{\boldsymbol{\sigma}}_m \{\boldsymbol{I} - (\mathrm{tr}\,\boldsymbol{N})\,\boldsymbol{N}\},\\ \mathring{\boldsymbol{\sigma}}_t^* &\equiv \mathring{\boldsymbol{\sigma}}^* - \mathring{\boldsymbol{\sigma}}_n^*,\end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned}\dot{\boldsymbol{\sigma}}_n^m &\equiv \mathrm{tr}\,(\boldsymbol{N}\dot{\boldsymbol{\sigma}}^m)\,\boldsymbol{N} = \dot{\boldsymbol{\sigma}}_m\,(\mathrm{tr}\,\boldsymbol{N})\,\boldsymbol{N},\\ \mathring{\boldsymbol{\sigma}}_n^* &\equiv \mathrm{tr}\,(\boldsymbol{N}\mathring{\boldsymbol{\sigma}}^*)\,\boldsymbol{N}.\end{aligned}\right\} \tag{43}$$

S_t^m is set to be zero for plastically-incompressible materials. The constitutive equation obtained by substituting the flow rule (40) into the consistency condition of the yield surface fulfills the continuity condition. However, the plastic stretching due to the tangential stress rate is not described.

Wang *et al.* (1990) proposed a complicated flow rule for sands, in which the direction of plastic stretching is dependent of the direction of the deviatoric stress rate. However, it fulfills neither the continuity condition nor the work rate-stiffness relaxation.

(e) *Double sliding model* (Spencer, 1964, 1982, Mandel, 1966, Mandel and Luque, 1970, Hashiguchi, 1971, Mehrabadi and Cowin, 1978, 1981, Anand, 1983, Nemat-Nasser, 1983)

This is the model extended from the slip-line theory by assuming that the velocities of particles along the stress characteristics lines depend on the stress rate. It would be difficult for this model to be extended to describe the three dimensional deformation. The more substantial defect of this model is that the derived constitutive equation is rate-linear taking the same form as the one of Rudnicki and Rice (1975) and thus falling within the framework of the hypoelasticity.

(f) *Slip theory* (Batdorf and Budiansky, 1949, Pande and Sharma, 1983, Bazant and Prat, 1988)

One of the more sophisticated flow theories which could in principle be used to explore a general loading behavior with an unloading-reloading is the slip theory of Batdorf and Budiansky (1949). However, the slip theory, which is the simplest of the physical theories, is already too complicated to serve as a constitutive law in calculations of this sort, even when computers are employed.

Besides, it should be noted that the stretching is expressed analytically in terms of the stress rate but an analytical expression of the stress rate in terms of the stretching cannot be derived in the extended models described in this section except for hypoelastic equations, although the latter expression is convenient for ordinary finite element programming based on the displacement method.

CONCLUDING REMARKS

The elastoplastic constitutive equation (14) was proposed in this article, which describes the plastic stretching caused by the stress rate normal and tangential to the yield or loading surface. In this equation a single smooth (regular) yield surface is kept without incorporating plural yield surfaces or a corner of the yield surface. This equation has the rather simple form compared with the corner theories (Christoffersen and Hutchinson, 1979, Ito *et al.*, 1979 etc.) applied to analyses in plastic instability problems. It may contribute to the steady development of elastoplasticity. Furthermore, the constitutive equation may be applicable to the prediction of the deformation for the general loading process including cyclic loading by incorporating the subloading surface model (Hashiguchi, 1980, 1989a) which fulfills the smoothness condition. Needless to say, it results in a high-order nonlinearity of the stress rate-stretching relation, while the conventional elastoplastic constitutive equation is bilinear. Therefore, the inverse expression, *i.e.* the expression of the stress rate in terms of the stretching cannot be derived. Besides, the dependency of the direction of plastic stretching on the stress rate cannot be described, while the dependency of the magnitude of plastic stretching on the stress rate could be described realistically by Eq. (14). Modifications on these points are required if they should not be ignored.

REFERENCES

Anand, L. 1983 Plane deformation of ideal granular materials, *J. Mech. Phys. Solids*: **31**, 105-122

Bazant, Z. P. and Prat, P. C. 1988 Microplane model for brittle-plastic material: I. Theory, *J. Eng. Mech.* (ASCE), **114**: 1672-1702

Bland, D. R. 1957 The associated flow rule of plasticity, *J. Mech. Phys. Solids*, **6**: 71-78

Batdorf, S. B. and Budiansky, B. 1949 A Mathematical theory of plasticity based on the concept of slip, *NACA, TC1871*: 1-31

Budiansky, B. 1959 A reassessment of deformation theories of plasticity, *J. Appl. Mech.* (ASME), **26**: 259-264

Christoffersen, J. and Hutchinson, J. W. 1979 A class of phenomenological corner theories of plasticity, *J. Mech. Phys. Solids*, **27**: 465-487

Dafalias, Y. F. 1985 The plastic spin, *J. Appl. Mech.* (ASME), **52**: 865-871

Davis, R. O. and Mullenger, G. 1978 A rate-type constitutive model for soil with a critical state, *Int. J. Numer. Anal. Meth. Geomech.*, **2**: 255-282

Dorris, J. F. and Nemat-Nasser, S. 1982 A plasticity model for flow of granular materials under triaxial stress states, *Int. J. Solids Struct.*, **18**: 497-531

Dragusin, L. 1981 Hypo-elastic model for soils, *Int. J. Eng. Sci.*: **19**: 511-522

Gotoh, M. 1986 A class of plastic constitutive relations with vertex effect, *Int. J. Solids Struct.*, **21**: 1101-1163

Goya, M. and Itoh, K. 1991 An expression of elastic-plastic constitutive laws incorporating vertex formulation and kinematic hardening, *J. Appl. Mech.* (ASME), **58**: 617-622

Goya, M., Miyagi, K., Itoh, K., Sueyoshi, T, and Itomura, S. 1995 Comparison between numerical and analytical prediction of shear localization of sheets subjected to biaxial tension, Atluri, S. N., Yagawa, G. and Cruse, T. A. (eds.), *Proc. Int. Conf. Comput. Mech.*, Springer, Berlin, pp. 1396-1401

Hashiguchi, K. 1971 Theories on a velocity field for general dilatable plastic soils -concept of double slippage-, *Trans. JSCE*, **3** (Part 1): 72-73

Hashiguchi, K. 1980 Constitutive equations of elastoplastic materials with elastic-plastic transition, *J. Appl. Mech.* (ASME), **47**: 266-272

Hashiguchi, K. 1989a Subloading surface model in unconventional plasticity, *Int. J. Solids Structures*, **25**: 917-945

Hashiguchi, K. 1989b Subloading surface model with tangential plasticity, Khan, A. S. and Tkuda, M. (eds.), *Proc. 2nd Int. Symp. Plasticity*, Tsu, Pergamon Press, Oxford, pp. 451-454

Hashiguchi, K. 1993a Fundamental requirements and formulation of elastoplastic constitutive equations with tangential plasticity, *Int. J. Plasticity*, **9**: 525-549

Hashiguchi, K. 1993b Mechanical requirements and structures of cyclic plasticity models, *Int. J. Plasticity*, **9**, 721-748

Hashiguchi, K. 1994 On the loading criterion, *Int. J. Plasticity*, **10**, 871-878

Hashiguchi, K. 1997 The extended flow rule in plasticity, *Int. J. Plasticity*, **13**: 37-58

Hecker, S. S. 1972 Experimental investigation of corners in yield surface, *Acta Mech.*, **13**: 69-72

Hencky, H. 1924 Zur Theorie plastischer Deformation und der hierdurch im Material herforgerufenen Nachspannungen, *Z.A.M.M.*, **4**: 323-334

Hill, R., 1958 A general theory of uniqueness and stability in elastic-plastic solids, *J. Mech. Phys. Solids*, **6**: 236-249

Hill, R. 1966 Generalized constitutive relations for incremental deformation of metal crystals, *J. Mech.*

Phys. Solids. **14**: 95-102

Hill, R. 1967a On the classical constitutive relations for elastic/plastic solids, *Recent Progress in Appl. Mech.* (The Folke Odqvist Volume), John-Wiley & Sons, Chichester, pp. 241-249

Hill, R. 1967b The essential structures of constitutive laws for metal composites and crystals, *J. Mech. Phys. Solids*, **15**: 79-95

Ikegami, K. 1979 Experimental plasticity on the anisotropy of metals, *Proc. Euromech. Colloquium*, **115**: 201-242

Ito, K. 1979 New flow rule for elastic-plastic solids based on KBW model with a view to lowering the buckling stress of plates and shells, *Tech. Report Tohoku Univ.*, **44**: 199-232

Ito, K., Goya, M. and Takahashi, H. 1992 An expression of elastic-plastic constitutive law incorporating a stress increment dependence (evolutional equation of stress increment dependency parameters on stress path), in Benallal, A., Billardon, R. and Marquis, D. (eds.), *Proc. Int. Seminar on Mechanics of Materials: Multiaxial Plasticity*, Cachan, pp. 689-694

Koiter, W. T. 1953 Stress-strain relations, uniqueness and variational theories for elastic-plastic materials with singular yield surfaces, *Quart. Appl. Math.*, **11**: 350-354

Lehmann, Th. 1982 Some considerations and experimental results concerning elastic-plastic stress-strain relations, *Ing.-Arch.*, **52**: 391-403

Mandel, J. 1965 Generalisation de la Theorie de Plasticite de W. T. Koiter, *Int. J. Solids Struct.*, **1**: 273-

Mandel, J. 1966 Sur les equations d'ecoulement des sols Ideaux en deformation plane et le concept du double glissement, *J. Mech. Phys. Solids*, **14**: 303-308

Mandel, G. and Luque, R. F. 1970 Fully developed plastic shear flow of granular materials, *Geotechnique*, **20**: 277-307

Mazilu, P. and Meyers, A. 1985 Yield surface description of isotropic materials after cold prestrain, *Ing.-Arch.*, **55**: 213-220

Mehrabadi, M. M. and Cowin, S. C. 1978 Initial planar deformation of dilatant granular materials, *J. Mech. Phys. Solids*, **26**: 269-284

Mehrabadi, M. M. and Cowin, S. C. 1981 On the double-sliding free rotating model for the deformation of granular materials, *J. Mech. Phys. Solids*, **29**: 269-282

Mroz, Z. 1966 On forms of constitutive laws for elastic-plastic solids, *Archiwum Mechaniki Stosowanej*, **18**: 1-34

Mroz, Z. 1980 On hypoelasticity and plasticity of approaches to constitutive model of inelastic behavior of soils, *Int. J. Numer. Anal. Meth. Geomech.*, **4**: 45-55

Nemat-Nasser, S. 1983 On finite plastic flow of crystalline solids and geomaterials, *J. Appl. Mech.* (ASME), **50**: 1114-1126

Ohashi, Y., Kawashima, K. and Yokouchi, T. 1975 Anisotropy due to plastic deformation of initially isotropic mild steel and its analytical formulation, *J. Mech. Phys. Sloids*, **23**: 277-294

Pande, G. N. and Sharma, K. G. 1983 Multilaminate model of clay -A numerical evaluation of the influence of rotation of the principal stress axes, *Int. J. Numer. Anal. Meth. Geomech.*, **7**: 397-418

Papamichos, E., Vardoulakis, I. and Han, C. 1993 Noncoaxial flow theory of plasticity: shear failure prediction in sand, Kolymbas, D. (ed.), *Modern Approaches to Plasticity*, Elsevier, Amsterdam, pp. 585-598

Rivlin, R. and Ericksen, I. L. 1955 Stress-deformation relations for isotropic materials, *J. Rat. Mech. Anal.*, **4**: 332-425

Romano, M. 1974 Continuum theory for granular media with a critical state, *Arch. Mech.*, **26**: 1011-1028

Rudnicki, J. W. and Rice, J. R. 1975 Conditions for localization of deformation in pressure-sensitive dilatant materials, *J. Mech. Phys. Solids*, **23**: 371-394

Sewell, M. J. 1973 A yield-surface corner lowers the buckling stress of an elastic-plastic plate under compression, *J. Mech. Phys. Solids*, **21**: 19-45

Sewell, M. J. 1974 A plastic flow at a yield vertex, *J. Mech. Phys. Solids*, **22**: 469-490

Spencer, A. J. M. 1964 A Theory of kinematics of ideal soil under plane strain conditions, *J. Mech. Phys. Solids*, **12**: 337-351

Spencer, A. J. M. 1982 Deformation of ideal granular materials, in Hopkins, H. G. and Sewell, M. J. (eds.), *Mechanics of Solids*, Oxford Univ. Press, Oxford, pp. 607-652

Storen, S. and Rice, J. R. 1975 Localized necking in thin sheet, *J. Mech. Phys. Solids*, **23**: 421-441

Stutz, P. 1973 Comportement elastoplastique des milieux pulverulents, *Sci. et Tech. de l'Armement*, **47**:

475-499

Tomita, Y., Shindoh, A., Kim, Y.-S. and Michiura, K. 1986 Deformation behavior of elastic-plastic tubes under extended pressure and axial load, *Int. J. Mech. Sci.*, **20**: 263-275

Truesdell, C. 1955 Hypo-elasticity, *J. Rat. Mech. Anal.*, **4**: 83-133

Vermeer, P. 1993 Upgrading of soil models by Hencky's theory of plasticity, Kolymbas, D. (ed.), *Modern Approaches to Plasticity*, Elsevier, Amsterdam, pp. 71-82

Wang, Z.-L., Dafalias, Y. F. and Shen, C.-K. 1990 Bounding surface hypoplasticity model for sand, *J. Eng. Mech.* (ASCE), **116**: 983-1001

Wu, H. C. and Yeh, W. C. 1991 On the experimental determination of yield surfaces and some results of annealed 304 stainless steel, *Int. J. Plasticity*, **7**: 803-826.

Zbib, H. M. and Aifantis, E. C. 1988 On the concept of relative and plastic spins and its implications to large deformation theories. Part I: Hypoelasticity and vertex-type plasticity, *Acta Mech.*, **75**: 15-33.

Zbib, H. M. 1991 On the mechanics of large inelastic deformations: noncoaxiality, axial effects in torsion and localization, *Acta Mech.*, **87**: 179-196