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The extension of the elastoplastic constitutive equation so as to describe the plastic stretching due to the stress rate component tangential to the yield or loading surface has been one of the most pressing problems in the elastoplasticity. To this aim, various models have been proposed in the past. However, a pertinent model applicable to a general loading process has not previously been proposed. In this article, the elastoplastic constitutive equation extended so as to describe a plastic stretching due to a stress rate component tangential to a yield or loading surface is formulated keeping a single and smooth yield surface. It would be a pertinent one which fulfills the mechanical requirements for elastoplastic constitutive equation and which is applicable to an arbitrary loading process. Based on this equation, a constitutive equation of metals with the isotropic-kinematic hardening is formulated.

INTRODUCTION

The plastic stretching is independent of the stress rate component tangential to the yield or loading surface, called the *tangential stress rate*, in the traditional elastoplastic constitutive equation with a single and smooth plastic potential surface. For metals the single crystal grains have multislip system and thus the plastic stretching would be dependent of the tangential stress rate. They may be observed to some extent even for a polycrystalline metals in which an infinite number of slip systems are activated, because some single crystal grains in the material influence the deformation behavior in terms of macroscopic viewpoint. They cannot be neglected in the process that the loading path abruptly changes or deviates severely from the proportional loading as observed in the plastic instability phenomena with a localization of deformation. The extension of the constitutive equation so as to describe these dependencies pertinently would be one of the most fundamental but unsolved problems in elastoplasticity at present.

The author (Hashiguchi, 1993a, 1993b, 1997) has described the mechanical requirements for constitutive equations describing an irreversible deformation, *i.e.* the irreversibility condition, the continuity condition, the work rate-stiffness relaxation and the smoothness condition. In this article, the elastoplastic constitutive equation extended so as to describe a plastic stretching due to a stress rate component tangential to a yield or loading surface is formulated keeping a single and smooth (regular) yield surface for the steady development of the elastoplasticity in physical and mathematical aspects. It would be a pertinent one fulfilling the above-mentioned requirements and is applicable to an arbitrary loading process including unloading, reloading and reverse loading processes. Based on this equation, the constitutive equation of metals with the von Mises yield condition obeying the isotropic-kinematic hardening is formulated.

CONSTITUTIVE EQUATION

Let it be assumed that the stretching $\mathbf{D}$ is additively decomposed into the elastic stretching $\mathbf{D}^e$ and the plastic stretching $\mathbf{D}^p$ which is further decomposed into the stretchings $\mathbf{D}_n^p$ and $\mathbf{D}_t^p$ caused by the stress rate component normal and tangential, respectively, to the yield or loading surface, while they are called the *normal*- and the *tangential-plastic stretching*, respectively. That is,

$$\left. \begin{aligned} \mathbf{D} &= \mathbf{D}^e + \mathbf{D}^p, \\ \mathbf{D}^p &= \mathbf{D}_n^p + \mathbf{D}_t^p, \end{aligned} \right\} \quad (1)$$

provided that the elastic stretching $\mathbf{D}^e$ is linearly related to the stress rate as

$$\mathbf{D}^e = \mathbf{E}^{-1} \dot{\boldsymbol{\sigma}}, \quad (2)$$

where $\mathbf{E}$ is the elastic modulus, a function of the stress and plastic internal-state variables in general, $(\)^{-1}$ designating the inverse and $\boldsymbol{\sigma}$ is the stress, $(\dot{\ })$ denoting a proper corotational rate (e.g. Dafalias, 1985, Zbib and Aifantis, 1988).

Assume that the evolution (isotropic and anisotropic hardening/softening) of the yield surface

$$f(\boldsymbol{\sigma}, \mathbf{H}_i) = 0 \quad (3)$$

is independent of the tangential-plastic stretching $\mathbf{D}_t^p$ but dependent only of the normal-plastic stretching $\mathbf{D}_n^p$. Then, it holds that

$$\dot{\mathbf{H}}_i = \mathbf{0} \text{ for } \mathbf{D}_n^p = \mathbf{0}, \quad (4)$$

where $\mathbf{H}_i$ ($i = 1, 2, \dots, n$) denoting collectively scalar- or tensor-valued plastic internal state variables.

Assume the associated flow rule for the normal-plastic stretching:

$$\mathbf{D}_n^p = \lambda \mathbf{N}, \quad (5)$$

where

$$\mathbf{N} \equiv \frac{\partial f}{\partial \boldsymbol{\sigma}} / \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|. \quad (6)$$

λ is a proportionality factor, a function of the stress, plastic internal state variables and the stress rate or the stretching in degree one.

The substitution of Eq. (5) into the consistency condition of the yield condition (3), i.e.

$$\text{tr} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}} \right) + \sum_{i=1}^n \text{tr} \left(\frac{\partial f}{\partial \mathbf{H}_i} \dot{\mathbf{H}}_i \right) = 0 \quad (7)$$

leads to

$$\mathbf{D}_n^p = \frac{\text{tr}(\mathbf{N} \dot{\boldsymbol{\sigma}})}{M_p} \mathbf{N}, \quad (8)$$

where

$$M_p \equiv - \sum_{i=1}^n \text{tr} \left(\frac{\partial f}{\partial \mathbf{H}_i} \mathbf{h}_i \right) / \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|. \quad (9)$$

$\mathbf{h}_i$ is a function of the stress, plastic internal state variables and $\mathbf{N}$ in degree one, which is related to $\dot{\mathbf{H}}_i$ as

$$\dot{\mathbf{H}}_i = \lambda \mathbf{h}_i. \quad (10)$$

Let the tangential-plastic stretching $\mathbf{D}_t^p$ be written as

$$\mathbf{D}_t^p = \boldsymbol{\xi}(\dot{\boldsymbol{\sigma}}_t, \boldsymbol{\sigma}, \mathbf{H}_i), \quad (11)$$

where $\boldsymbol{\xi}$ is the second-order tensor function of stress rate component tangential to a yield or loading surface, i.e. the tangential stress rate denoted as $\dot{\boldsymbol{\sigma}}_t$, stress and plastic internal state variables in general. $\dot{\boldsymbol{\sigma}}_t$ is described as

$$\left. \begin{aligned} \dot{\boldsymbol{\sigma}}_t &\equiv \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_n, \\ \dot{\boldsymbol{\sigma}}_n &\equiv \text{tr}(\mathbf{N}\dot{\boldsymbol{\sigma}})\mathbf{N}. \end{aligned} \right\} \quad (12)$$

Here, it should be noted that the tangential-plastic stretching $\mathbf{D}_t^p$ has to fulfill the irreversibility condition

$$\mathbf{D}_t^p \neq -\mathbf{D}_t^{p'} \text{ for } \dot{\boldsymbol{\sigma}}_t = -\dot{\boldsymbol{\sigma}}_t', \quad (13)$$

where tangential-plastic stretching induced by stress rates $\dot{\boldsymbol{\sigma}}_t$ and $\dot{\boldsymbol{\sigma}}_t'$ which have the same magnitude but opposite directions to each other be denoted as $\mathbf{D}_t^p$ and $\mathbf{D}_t^{p'}$, respectively. Thus, needless to say, $\mathbf{D}_t^p$ has to be nonlinearly related to the tangential-stress rate (so-called *rate-nonlinearity*) at least.

The stretching is given from Eqs. (1), (2) and (7) with Eq. (11) as

$$\mathbf{D} = \mathbf{E}^{-1} \dot{\boldsymbol{\sigma}} + \frac{\text{tr}(\mathbf{N}\dot{\boldsymbol{\sigma}})}{M_p} \mathbf{N} + \boldsymbol{\xi}, \quad (14)$$

while its inverse relation, i.e. the analytical expression of the stress rate in terms of the stretching cannot be derived because of the rate-nonlinearity of Eq. (14).

Let the loading criterion (Hill, 1958, 1967, Hashiguchi, 1994) be modified as follows:

$$\left. \begin{aligned} \mathbf{D}_n^p &\neq \mathbf{0} : f = 0 \text{ and } \text{tr}(\mathbf{NED}) > 0, \\ \mathbf{D}_n^p &= \mathbf{0} : \text{others,} \end{aligned} \right\} \quad (15)$$

while the tangential-plastic stretching $\mathbf{D}_t^p$ occurs always by the tangential stress rate $\dot{\boldsymbol{\sigma}}_t$ so that the continuity condition (Hashiguchi, 1993a, 1997) is fulfilled. The condition $f = 0$ in Eq. (15) is unnecessary in the subloading surface model fulfilling the smoothness condition including the smooth elastic-plastic transition.

Let the *work rate-stiffness relaxation* (Hashiguchi, 1993a) be modified as

$$\text{tr}(\mathbf{DED}_n^p) \geq 0 \text{ for } \text{tr}(\mathbf{NED}) \geq 0. \quad (16)$$

A concrete example of the tensor $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi} = S_t \parallel \dot{\boldsymbol{\sigma}}_t^* \parallel^n \mathbf{N}, \quad (17)$$

where S_t is the material constant or function of stress and plastic internal variables, and n is the material constant. $(\)^*$ denotes the deviatoric part. The constitutive equation (14) with Eq. (17) fulfills clearly the inequality (16), while it holds that $\text{tr}(\mathbf{DED}^p) = \text{tr}(\mathbf{DED}_t^p) < 0$ for $\text{tr}(\mathbf{NED}) < 0$.

CONSTITUTIVE EQUATION OF METALS

Based on Eq. (14) with Eq. (17), let a constitutive equation of metals be formulated in this section.

Now, adopt the von Mises yield condition with the isotropic-kinematic hardening:

$$f(\hat{\boldsymbol{\sigma}}) = F(H) = 0, \quad (18)$$

where

$$f(\hat{\boldsymbol{\sigma}}) = \sqrt{\frac{3}{2}} \|\hat{\boldsymbol{\sigma}}^*\|, \quad (19)$$

$$\dot{\boldsymbol{\alpha}} = K_1 \hat{\boldsymbol{\sigma}} - K_2 \boldsymbol{\alpha}, \quad (20)$$

$$K_1 = k_1 \|\mathbf{D}_p\|, \quad K_2 = k_2 \|\mathbf{D}_p\|, \quad (21)$$

$$F = F_0 [1 + h_1 \{1 - \exp(-h_2 H)\}], \quad (22)$$

$$\dot{H} = \sqrt{\frac{2}{3}} \|\mathbf{D}_p\|. \quad (23)$$

and

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha}, \quad (24)$$

$$\hat{\boldsymbol{\sigma}}_m \equiv \frac{1}{3} \text{tr} \hat{\boldsymbol{\sigma}}, \quad \hat{\boldsymbol{\sigma}}^* \equiv \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}_m \mathbf{I}, \quad (25)$$

k_1 and k_2 are material constants, while K_1 and K_2 are generally scalar functions of the plastic stretching in degree one, the stress and plastic internal-state variables. F_0 is the initial value of F , h_1 and h_2 are material constants.

The plastic modulus M_p is given from Eq. (9) as

$$M_p = \text{tr}(\mathbf{N}\mathbf{a}) + F' h / \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\| (> 0), \quad (26)$$

where

$$\mathbf{N} = \frac{\hat{\boldsymbol{\sigma}}^*}{\|\hat{\boldsymbol{\sigma}}^*\|}, \quad (27)$$

$$\mathbf{a} \equiv \dot{\boldsymbol{\alpha}} / \lambda = (k_1 \hat{\boldsymbol{\sigma}} - k_2 \boldsymbol{\alpha}), \quad (28)$$

$$F' \equiv \frac{dF}{dH} = F_0 h_1 h_2 \exp(-h_2 H) (> 0), \quad (29)$$

$$h \equiv \dot{H} / \lambda = \sqrt{\frac{2}{3}}, \quad (30)$$

$$\left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\| = \sqrt{\frac{2}{3}}. \quad (31)$$

The stretching is given from Eq. (14) with Eqs. (11), (17), (26) and (28) as

$$\mathbf{D} = \mathbf{E}^{-1} \dot{\boldsymbol{\sigma}} + \frac{\text{tr}(\mathbf{N}\dot{\boldsymbol{\sigma}})}{\text{tr}\{\mathbf{N}(k_1 \hat{\boldsymbol{\sigma}} - k_2 \boldsymbol{\alpha})\} + F'} \mathbf{N} + \boldsymbol{\xi}. \quad (32)$$

In what follows, let the elastic property be given by the Hooke's type

$$E_{ijkl} = L \delta_{ij} \delta_{kl} + G (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (33)$$

where L and G are the material parameters corresponding to the Lamé's constant and the

elastic shear modulus, respectively, for the elastic body, and δ_{ij} is the Kronecker's delta.

Let the relation for the magnitude of plastic stretching versus the direction of stress rate in the π -plane be evaluated by the following scalar variable μ versus α .

$$\left. \begin{aligned} \alpha &\equiv \cos^{-1} \left\{ \text{tr} \left(\frac{\boldsymbol{\sigma}^*}{\|\boldsymbol{\sigma}^*\|} \frac{\dot{\boldsymbol{\sigma}}^*}{\|\dot{\boldsymbol{\sigma}}^*\|} \right) \right\}, \\ \mu &\equiv \frac{\|\mathbf{D}^p\| / \|\dot{\boldsymbol{\sigma}}^*\|}{(\|\mathbf{D}^p\| / \|\dot{\boldsymbol{\sigma}}^*\|)_{pro}}, \end{aligned} \right\} \quad (34)$$

$(\)_{pro}$ denoting the value of the quantity inside the bracket in the proportional loading. That is, α denotes the direction of stress rate from that of stress in the preceding proportional loading, and μ the ratio of the magnitude of plastic stretching to that of stress rate compared with the ratio in the proportional loading. The variable μ for Eq. (32) for the isotropic hardening, *i.e.* $k_1 = k_2 = 0$ is expressed as

$$\mu = \begin{cases} \frac{\|\mathbf{D}_n^p + \mathbf{D}_t^p\| / \|\dot{\boldsymbol{\sigma}}^*\|}{(\|\mathbf{D}_n^p\| / \|\dot{\boldsymbol{\sigma}}^*\|)_{pro}} = \frac{\left\| \frac{\text{tr}(\mathbf{N}\dot{\boldsymbol{\sigma}}^*)}{F'} \mathbf{N} + S_t \boldsymbol{\xi} \right\| / \|\dot{\boldsymbol{\sigma}}^*\|}{\left(\left\| \frac{\text{tr}(\mathbf{N}\dot{\boldsymbol{\sigma}}^*)}{F'} \mathbf{N} \right\| / \|\dot{\boldsymbol{\sigma}}^*\| \right)_{pro}} = \left\| \cos \alpha \mathbf{N} + S_t F' \frac{\boldsymbol{\xi}}{\|\dot{\boldsymbol{\sigma}}^*\|} \right\| \text{ for } \alpha \leq 90^\circ \\ \frac{\|\mathbf{D}_t^p\| / \|\dot{\boldsymbol{\sigma}}^*\|}{(\|\mathbf{D}_n^p\| / \|\dot{\boldsymbol{\sigma}}^*\|)_{pro}} = \frac{\|S_t \boldsymbol{\xi}\| / \|\dot{\boldsymbol{\sigma}}^*\|}{\left(\left\| \frac{\text{tr}(\mathbf{N}\dot{\boldsymbol{\sigma}}^*)}{F'} \mathbf{N} \right\| / \|\dot{\boldsymbol{\sigma}}^*\| \right)_{pro}} = \left\| S_t F' \frac{\boldsymbol{\xi}}{\|\dot{\boldsymbol{\sigma}}^*\|} \right\| \text{ for } \alpha > 90^\circ \end{cases} \quad (35)$$

which reduces to

$$\mu = \begin{cases} \cos \alpha + S_t F' \sin \alpha & \text{for } \alpha \leq 90^\circ \\ S_t F' \sin \alpha & \text{for } \alpha > 90^\circ \end{cases} \quad (36)$$

for Eq. (17) with $n = 1$. Eq. (36) is depicted in Fig. 1 where the measured values (Ito *et al.*, 1992) are shown by the circles. A good agreement between theoretical curve and

Fig. 1. The variation of the magnitude of plastic stretching versus the direction of the stress rate.

measured values is observed. The shape-transformation of yield surface which protrudes in the pre-stressed direction and inversely becomes flat behind that direction (cf. e.g. Ohashi *et al.*, 1975, Wu and Yeh, 1991) could be described by Eq. (32) without introducing complex yield functions with forth-order tensors (cf. e.g. Mazilu and Meyers, 1985).

Needless to say, the direction of plastic stretching is independent of the stress rate and is normal to the yield or loading surface, although it deviates up to 20° in the test result of Ito *et al.* (1992).

REVIEW OF EXISTING MODELS

The mechanical features of existing models for describing the plastic stretching due to the tangential stress rate are examined below.

(a) Intersection of yield surfaces (Koiter, 1953, Bland, 1957, Mandel, 1965, Hill, 1966, Sewell, 1973, 1974)

This model aims at describing the above-mentioned dependencies and the plastic stretching induced for a range covering more than half the stress rate space by incorporating plural smooth yield surfaces which intersect each other and obey the associated flow rule. It cannot, however, always describe the dependencies since a stress does not necessarily exist at the intersecting point of yield surfaces. The probability that a stress exists at the intersecting point of yield surfaces would be small, unless an infinite number of yield surfaces are introduced. While a practical calculation of deformation by this model was performed by Sewell (1973, 1974), it was restricted to base states of uniaxial stress. A computational practical extension of this model to general stress states is not obvious as was indicated by Christoffersen and Hutchinson (1979).

(b) Corner theory (Hill, 1967b, Christoffersen and Hutchinson, 1979, Ito, 1979, Gotoh, 1986, Tomita *et al.*, 1986, Goya *et al.*, 1991, 1995)

This theory also aims at describing the aforementioned dependencies and the plastic stretching induced for a wide range of stress rate space by assuming the existence of a corner or a cone on the yield surface which geometrically induces a singularity in the field of the normal vectors of the yield surface. However, the evolutionary rule of the cone due to plastic stretching has not been presented yet and perhaps cannot be rationally formulated, especially if the stress rate has a direction with an angle larger than 90° from the outward central axis of the cone which contracts if the tip of the cone is assumed to move with the current stress point. Thus, this model is not applicable to the general loading process which includes unloading, reloading and reverse loading in various directions, although a modification of this model has been attempted by Tomita *et al.* (1986) and also by Goya *et al.* (1991, 1995), incorporating kinematic hardening. Furthermore, if an infinitesimal cone is assumed, finitely different stretchings would be predicted for identical stress rates given in the situation where the stress exists just on the corner (elastoplastic state) and in the other situation where the stress rate exists in the infinitesimally inside the corner (elastic state), violating the continuity condition. Moreover, constitutive equations based on this approach take quite complicated forms, thus creating inconvenience in analytical or numerical calculations. Besides, it should be noted that, while some researchers supported the corner formulation, others failed to

conform it, as reviewed by Hecker (1972) or Ikegami (1979).

(c) Hypoelasticity (Budiansky, 1959, Rudnicki and Rice, 1975, Storen and Rice, 1975, Dorris and Nemat-Nasser, 1982, Lehmann, 1982, Hashiguchi, 1989b, Zbib, 1991, Papamichos *et al.*, 1993, Vermeer, 1993)

A stretching due to the stress rate component tangential to a single smooth yield surface is introduced in addition to the plastic stretching obeying the associated flow rule, where additional stretching is related linearly to the stress rate so that this approach falls within the framework of hypoelasticity (Truesdell, 1955). The J_2 -deformation theory of Storen and Rice (1975) is identical with the deformation theory of Hencky (1924) in a differential form. It should be noted that the additional stretching cannot be regarded to be plastic but is regarded to be elastic, although the proponents of this approach called it "plastic component" in their articles, since it is related linearly to the stress rate without any loading criterion. If the additional stretching is formulated to be plastic by introducing a special loading criterion, it leads to a violation of the continuity condition. This situation is similar to the so-called hypoelastic models (e.g. Stutz 1973, Romano, 1974, Davis and Mullenger, 1978, Dragusin, 1981) criticized by Mroz (1980). Ultimately, this approach is inadequate for describing the elastoplastic deformation.

(d) Flow rule with a stress rate (Mroz, 1966, Wang *et al.*, 1990, Hashiguchi, 1993a)

The associated flow rule with a single smooth yield surface is extended by introducing the stress rate tensor in degree zero in addition to the outward-normal tensor of the yield surface. The relation between the stress rate and the stretching becomes high-order nonlinear so that it can describe irreversible deformation mathematically. However, it does not generally fulfill the work rate-stiffness relaxation (16) as described below.

Mroz (1966) proposed the following flow rule by using the representation theorem for the isotropic tensor function of two tensor variables (Rivlin and Ericksen, 1955), i.e. the outward-normal $\mathbf{N}$ of the yield surface and the deviatoric stress rate $\dot{\boldsymbol{\sigma}}^*$ and choosing only two simple terms.

$$\mathbf{D}^p = \text{tr}(\mathbf{N} \dot{\boldsymbol{\sigma}}^*) \left(a \mathbf{N} + b \frac{\dot{\boldsymbol{\sigma}}^*}{\|\dot{\boldsymbol{\sigma}}^*\|} \right), \quad (38)$$

where

$$\left. \begin{aligned} \dot{\boldsymbol{\sigma}}_m &\equiv \frac{1}{3} \text{tr} \dot{\boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\sigma}}^*_m \equiv \dot{\boldsymbol{\sigma}}_m \mathbf{I}, \\ \dot{\boldsymbol{\sigma}}^* &\equiv \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_m \end{aligned} \right\} \quad (39)$$

a and b are scalar functions of $\mathbf{N}$ and $\dot{\boldsymbol{\sigma}}^*$ in degree zero, and $(\cdot)$ stands for a material-time derivative. Eqn (38) fulfills the continuity condition only in case of $\text{tr} \mathbf{N} = 0$ leading to $\text{tr}(\mathbf{N} \dot{\boldsymbol{\sigma}}^*) = \text{tr}(\mathbf{N} \dot{\boldsymbol{\sigma}})$. It is unable to describe the influence of the mean stress rate on the direction of the plastic stretching, which is required for plastically pressure-dependent materials.

The author (Hashiguchi, 1993a, 1994) assumed the flow rule

$$\mathbf{D}^p = \lambda \left(\mathbf{N} + S_t^m \frac{\dot{\boldsymbol{\sigma}}_t^m}{\|\dot{\boldsymbol{\sigma}}_n\|} + S_t^* \frac{\dot{\boldsymbol{\sigma}}_t^*}{\|\dot{\boldsymbol{\sigma}}_n\|} \right), \quad (40)$$

where λ is a positive proportionality factor, S_t^m and S_t^* are material parameters, and

$$\left. \begin{aligned} \dot{\boldsymbol{\sigma}}_t &\equiv \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_n = \dot{\boldsymbol{\sigma}}_t^m + \dot{\boldsymbol{\sigma}}_t^*, \\ \dot{\boldsymbol{\sigma}}_n &\equiv \text{tr}(\mathbf{N} \dot{\boldsymbol{\sigma}}) \mathbf{N}, \end{aligned} \right\} \quad (41)$$

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_t^m &\equiv \dot{\boldsymbol{\sigma}}^m - \dot{\boldsymbol{\sigma}}_n^m = \dot{\boldsymbol{\sigma}}_m \{ \mathbf{I} - (\text{tr } \mathbf{N}) \mathbf{N} \}, \\ \dot{\boldsymbol{\sigma}}_t^* &\equiv \dot{\boldsymbol{\sigma}}^* - \dot{\boldsymbol{\sigma}}_n^*,\end{aligned}\quad (42)$$

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_n^m &\equiv \text{tr} (\mathbf{N} \dot{\boldsymbol{\sigma}}^m) \mathbf{N} = \dot{\boldsymbol{\sigma}}_m (\text{tr } \mathbf{N}) \mathbf{N}, \\ \dot{\boldsymbol{\sigma}}_n^* &\equiv \text{tr} (\mathbf{N} \dot{\boldsymbol{\sigma}}^*) \mathbf{N}.\end{aligned}\quad (43)$$

S_t^m is set to be zero for plastically-incompressible materials. The constitutive equation obtained by substituting the flow rule (40) into the consistency condition of the yield surface fulfills the continuity condition. However, the plastic stretching due to the tangential stress rate is not described.

Wang *et al.* (1990) proposed a complicated flow rule for sands, in which the direction of plastic stretching is dependent of the direction of the deviatoric stress rate. However, it fulfills neither the continuity condition nor the work rate-stiffness relaxation.

(e) Double sliding model (Spencer, 1964, 1982, Mandel, 1966, Mandel and Luque, 1970, Hashiguchi, 1971, Mehrabadi and Cowin, 1978, 1981, Anand, 1983, Nemat-Nasser, 1983)

This is the model extended from the slip-line theory by assuming that the velocities of particles along the stress characteristics lines depend on the stress rate. It would be difficult for this model to be extended to describe the three dimensional deformation. The more substantial defect of this model is that the derived constitutive equation is rate-linear taking the same form as the one of Rudnicki and Rice (1975) and thus falling within the framework of the hypoelasticity.

(f) Slip theory (Batdorf and Budiansky, 1949, Pande and Sharma, 1983, Bazant and Prat, 1988)

One of the more sophisticated flow theories which could in principle be used to explore a general loading behavior with an unloading-reloading is the slip theory of Batdorf and Budiansky (1949). However, the slip theory, which is the simplest of the physical theories, is already too complicated to serve as a constitutive law in calculations of this sort, even when computers are employed.

Besides, it should be noted that the stretching is expressed analytically in terms of the stress rate but an analytical expression of the stress rate in terms of the stretching cannot be derived in the extended models described in this section except for hypoelastic equations, although the latter expression is convenient for ordinary finite element programming based on the displacement method.

CONCLUDING REMARKS

The elastoplastic constitutive equation (14) was proposed in this article, which describes the plastic stretching caused by the stress rate normal and tangential to the yield or loading surface. In this equation a single smooth (regular) yield surface is kept without incorporating plural yield surfaces or a corner of the yield surface. This equation has the rather simple form compared with the corner theories (Christoffersen and Hutchinson, 1979, Ito *et al.*, 1979 etc.) applied to analyses in plastic instability problems. It may contribute to the steady development of elastoplasticity. Furthermore, the constitutive equation may be applicable to the prediction of the deformation for the general loading process including cyclic loading by incorporating the subloading surface model (Hashiguchi, 1980, 1989a) which fulfills the smoothness condition. Needless to

say, it results in a high-order nonlinearity of the stress rate-stretching relation, while the conventional elastoplastic constitutive equation is bilinear. Therefore, the inverse expression, *i.e.* the expression of the stress rate in terms of the stretching cannot be derived. Besides, the dependency of the direction of plastic stretching on the stress rate cannot be described, while the dependency of the magnitude of plastic stretching on the stress rate could be described realistically by Eq. (14). Modifications on these points are required if they should not be ignored.

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