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Hashiguchi, Koichi
Laboratory of Mechanics for Bio -Production, Faculty of Agriculture, Kyushu University

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Constitutive Equations of Soils Based on the Subloading Surface Concept

Koichi Hashiguchi

Laboratory of Mechanics for Bio-Production, Faculty of Agriculture,
Kyushu University, Fukuoka 812, Japan
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Various cyclic plasticity models have been proposed in the past. Among them the subloading surface model (Hashiguchi, 1989) only is regraded to have the pertinent structure adaptable to the prediction of cyclic loading behavior of materials as has been revealed by the author (Hashiguchi, 1993b). The constitutive equation of soils is formulated by introducing the subloading surface model and formulating the evolution rule of the rotational hardening for the description of the induced anisotropy.

INTRODUCTION

A prediction of inelastic deformation of materials for cyclic loadings has been required for practical problems in engineering. The conventional theory of plasticity premises that the interior of the yield surface is an elastic domain so that it is unable to predict the mechanical ratchetting behavior for cyclic loading with a stress amplitude within a yield surface, predicting only an elastic deformation after a first cycle. Therefore, various cyclic plasticity models have been proposed in the last quarter of century. Among them the multi (Iwan, 1967; Mroz, 1967), the two (Dafalias and Popov, 1975 ; Krieg, 1975), the single (Dafalias and Popov, 1977), the infinite (Mroz et *al.*, 1981), the initial subloading (Hashiguchi and Ueno, 1977 ; Hashiguchi, 1980) and the subloading (Hashiguchi, 1989) surface models are welllknown. The nonlinear kinematic hardening model (Armstrong and Frederick, 1966 ; Chaboche and Rousselier, 1981) has the identical property with that of the multi surface model as has been revealed by Marquis (1979), Chaboche and Rousselier (1983) and Ohno and Wang (1993). The bounding surface model with a radial mapping of Dafalias and Herrmann (1980) falls within the framework of the initial subloading surface model. It was clarified by the author (Hachiguchi, 1993) that only the subloading surface model is able to predict realistically the cyclic loading behavior of materials, satisfying the fundamental requirements for cyclic plasticity model, i. e. the continuity ***condition in the large*** and ***in the small,*** the *Masing* ***effect*** and the *work* ***rate-stiffness relaxation.*** Besides, the induced anisotropy of metals has been conveniently described by the kinematic hardening after Edelman and Drucker (1951), Ishlinski (1954) and Prager (1955, 1956). However, the kinematic hardening may not be able to describe the anisotropy of soils pertinently since the yield surface may not move in such a way that it goes out from the origin of the stress space.

In this article, the subloading surface model is applied to soils, determining concretely material functions included in this model and formulating the evolution rule of the *rotational hardening* for the description of the induced anisotropy. The positive sign of stress (rate) and stretching stands for extension throughout this article.

SUBLOADING SURFACE MODEL WITH ROTATIONAL HARDENING

The subloading surface model (Hashiguchi, 1989) premises on the existence of the subloading surface which expands/contracts within the yield surface in the conventional sense, renamed the normal-yield surface, passing always through a current stress point in not only a loading but also an unloading process and keeping a similarity to the normal-yield surface. Besides, it is assumed that the subloading surface approaches asymptotically to the normal-yield surface in a loading process, causing an decrease of plastic modulus. This model involves three internal variables, that is, the isotropic hardening/softening variable H and the two back stresses, i. e. the first back stress : the kinematic hardening variable a and the second back stress : the similarity center $\boldsymbol{s}$ of the normal-yield and the subloading surfaces.

The stretching $\boldsymbol{D}$ (symmetric part of velocity gradient) is additively decomposed into the elastic stretching $\boldsymbol{D}^e$ and the plastic stretching $\boldsymbol{D}^p$ ad usual, i. e.

$$\boldsymbol{D}=\boldsymbol{D}^e+\boldsymbol{D}^p, \tag{1}$$

where the elastic stretching is given by

$$\boldsymbol{D}^e=\boldsymbol{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}. \tag{2}$$

$\boldsymbol{\sigma}$ is a stress, ($^\circ$) indicates the corotational rate. $\boldsymbol{E}$ is the elastic modulus given in the Hooke's type as

$$E_{ijkl}=(K-\frac{2}{3}G)\delta_{ij}\delta_{kl}+G(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{3}$$

where K and G are the bulk modulus and the shear modulus, respectively, and δ_{ij} is the Kronecker's delta.

The normal-yield surface with the isotropic-kinematic-rotational hardening is described as

$$f(\hat{p},\ \hat{\chi})=F(H) \tag{4}$$

with

$$f(\hat{p},\hat{\chi})=\hat{p}g(\hat{\chi}), \tag{5}$$

where

$$\hat{\boldsymbol{\sigma}}\equiv\boldsymbol{\sigma}-\boldsymbol{\alpha}, \tag{6}$$

$$\hat{\boldsymbol{p}}\equiv-\frac{1}{3}\mathrm{tr}\hat{\boldsymbol{\sigma}},\ \hat{\boldsymbol{\sigma}}^*\equiv\hat{\boldsymbol{\sigma}}+\hat{p}\boldsymbol{I}, \tag{7}$$

$$\hat{\boldsymbol{\eta}}\equiv\hat{\boldsymbol{q}}-\boldsymbol{\beta},\ \hat{\boldsymbol{q}}\equiv\frac{\hat{\boldsymbol{\sigma}}^*}{\hat{p}}, \tag{8}$$

$$\hat{\chi}\equiv\frac{\|\hat{\boldsymbol{\eta}}\|}{\hat{m}}. \tag{9}$$

$\widehat{m}$ is the function of the material constant ϕ and

$$\sin 3\widehat{\theta} \equiv -\sqrt{6}\left(\frac{\widehat{s}}{\widehat{r}}\right)^3, \tag{10}$$

i. e.

$$\widehat{m} = f_m(\phi, \sin 3\widehat{\theta}), \tag{11}$$

where

$$\widehat{r} \equiv \|\widehat{\boldsymbol{\eta}}\| = \mathrm{tr}^{1/2}\widehat{\boldsymbol{\eta}}^2, \ \ \widehat{s} \equiv \mathrm{tr}^{1/3}\widehat{\boldsymbol{\eta}}^3. \tag{12}$$

$\widehat{r}/\widehat{p} = \widehat{m}$ represents a conical surface in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$, the apex of which is located in the point $(\alpha_1, \alpha_2, \alpha_3)$ and the central axis is described as $\widehat{q}_1/\beta_1 = \widehat{q}_2/\beta_2 = \widehat{q}_3/\beta_3$. The scalar H $\boldsymbol{\alpha}$ is the kinematic hardening variable, i. e. the back stress, while the rotational hardening is described by the additional plastic internal state variable $\boldsymbol{\beta}$. It should be noted that f is a homogeneous function of $\widehat{\boldsymbol{\sigma}}$ in degree one, satisfying the Eule $(\partial f(\widehat{p}, \widehat{\chi})/\partial\widehat{\boldsymbol{\sigma}} \cdot \widehat{\boldsymbol{\sigma}}) = f(\widehat{p}, \widehat{\chi})$.

$$\overset{\circ}{\boldsymbol{\alpha}} = K_1\widehat{\boldsymbol{\sigma}} - K_2\boldsymbol{\alpha}, \tag{13}$$

where K_1 and K_2 are material functions of a stress, plastic internal state variables and $\boldsymbol{D}^p$ in degree one.

For the formulation of the evolution equation of the rotational hardening variable $\boldsymbol{\beta}$, let it be assumed that

1) There exits a limit surface for the rotation of the normal-yield surface.
2) For the monotonic-proportional loading with $\boldsymbol{\sigma}^*/p = \mathrm{const.}$, the so-called anisotropic consolidation, $\boldsymbol{\beta}$ approaches to $\widehat{\boldsymbol{q}}(=\widehat{\boldsymbol{\sigma}}^*/\widehat{p})$.

Hence, let $\overset{\circ}{\boldsymbol{\beta}}$ be assumed as

$$\overset{\circ}{\boldsymbol{\beta}} = b_1\|\boldsymbol{D}^{p*}\|\,\|\widehat{\boldsymbol{\eta}}\|^{b_2}\widehat{\boldsymbol{\eta}}_b, \tag{14}$$

where

$$D_v^p \equiv \mathrm{tr}\boldsymbol{D}^p, \ \boldsymbol{D}^{p*} \equiv \boldsymbol{D}^p - \frac{1}{3}D_v^p\boldsymbol{I}, \tag{15}$$

$$\widehat{\boldsymbol{\eta}}_b \equiv \boldsymbol{\eta}_b - \boldsymbol{\beta}, \ \boldsymbol{\eta}_b \equiv m_b\,\widehat{\boldsymbol{t}}, \tag{16}$$

$$\widehat{\boldsymbol{t}} \equiv \frac{\widehat{\boldsymbol{\eta}}}{\|\widehat{\boldsymbol{\eta}}\|}, \tag{17}$$

m_b is given by the function f_m Of the material constant ϕ_b and Sin3 $\widehat{\theta}$, i. e.

$$m_b = f_m(\phi_b, \sin 3\widehat{\theta}). \tag{18}$$

The surface described by $\widehat{\boldsymbol{\eta}}_b = \boldsymbol{O}$, i. e. $\boldsymbol{\beta} = \boldsymbol{\eta}_b$ leading to $\overset{\circ}{\boldsymbol{\beta}} = \boldsymbol{O}$ is called the *limit surface for rotation* of yield surface.

In what follows, let the subloading surface concept be introduced.

The subloading surface is given by the similarity to the normal-yield surface (4) as

$$f(\bar{p},\ \bar{\chi}) = RF(H) \tag{19}$$

with

$$f(\bar{p},\bar{\chi}) = \bar{p}g(\bar{\chi}), \tag{20}$$

where

$$\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}}, \tag{21}$$

$$\bar{p} \equiv -\frac{1}{3}\mathrm{tr}\,\bar{\boldsymbol{\sigma}},\ \ \bar{\boldsymbol{\sigma}}^* \equiv \bar{\boldsymbol{\sigma}} + \bar{p}\boldsymbol{I}, \tag{22}$$

$$\bar{\boldsymbol{\eta}} \equiv \bar{\boldsymbol{Q}} - \boldsymbol{\beta},\ \ \bar{\boldsymbol{Q}} \equiv \frac{\bar{\boldsymbol{\sigma}}^*}{\bar{p}}, \tag{23}$$

$$\bar{\chi} \equiv \frac{\|\bar{\boldsymbol{\eta}}\|}{\bar{m}}. \tag{24}$$

$\bar{m}$ is given by the function f_m of ϕ and

$$\sin 3\bar{\theta} \equiv -\sqrt{6}\left(\frac{\bar{s}}{\bar{r}}\right)^3 \tag{25}$$

i. e.

$$\bar{m} = f_m(\phi, \sin 3\bar{\theta}), \tag{26}$$

where

$$\bar{r} \equiv \|\bar{\boldsymbol{\eta}}\| = \mathrm{tr}^{1/2}\bar{\boldsymbol{\eta}}^2,\ \ \bar{s} \equiv \mathrm{tr}^{1/3}\bar{\boldsymbol{\eta}}^3 \tag{27}$$

$\bar{\boldsymbol{\alpha}}$ on the subloading sueface is the conjugate point of $\boldsymbol{\alpha}$ on the normal-yield surface, and $R(0 \leq R \leq 1)$ is the ratio of the size of the subloading surface to that of the normal -yield surface, while $R=O$ and 1 correspond to the purely elastic and the normal-yield state, respectively. It holds that

$$\left.\begin{aligned} &\bar{\boldsymbol{\sigma}} = R\hat{\boldsymbol{\sigma}},\ \bar{p} = R\hat{p},\ \bar{\boldsymbol{\sigma}}^* = R\hat{\boldsymbol{\sigma}}^*,\ \ \bar{\boldsymbol{s}} = R\hat{\boldsymbol{s}}, \\ &\bar{\boldsymbol{Q}} = \hat{\boldsymbol{Q}},\ \bar{\boldsymbol{\eta}} = \hat{\boldsymbol{\eta}},\ \bar{\theta} = \hat{\theta},\ \bar{m} = \hat{m},\ \bar{\chi} = \hat{\chi}, \end{aligned}\right\} \tag{28}$$

where

$$\bar{\boldsymbol{s}} \equiv \boldsymbol{s} - \bar{\boldsymbol{\alpha}},\ \hat{\boldsymbol{s}} \equiv \boldsymbol{s} - \boldsymbol{\alpha}. \tag{29}$$

The variable R is calculated from eqn (19) with the substitution of

$$\bar{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}} + R\hat{\boldsymbol{s}} \tag{30}$$

obtained from eqn $(28)_4$, and then $\bar{\boldsymbol{\alpha}}$ is calculated by the equation $\bar{\boldsymbol{\alpha}} = \boldsymbol{s} - R\hat{\boldsymbol{s}}$.

Let the evolution rule of the similarity-center s of the normal-yield and the subloading surface be formulated below.

The following inequality must hold since the similarity-center has to exist inside the normal-yield surface.

$$f(\hat{p}_s, \hat{\chi}_s) \leq \ \mathrm{F(H)}, \tag{31}$$

with

$$f(\hat{p}_s, \hat{\chi}_s) = \hat{p}_s g(\hat{\chi}_s) \tag{32}$$

where

$$\hat{\boldsymbol{s}} \equiv \boldsymbol{s} - \boldsymbol{\alpha}, \tag{33}$$

$$\hat{p}_s \equiv -\frac{1}{3}\mathrm{tr}\,\hat{\boldsymbol{s}},\ \ \hat{\boldsymbol{s}}^* \equiv \hat{\boldsymbol{s}} + \hat{p}_s \boldsymbol{I}, \tag{34}$$

$$\hat{\boldsymbol{\eta}}_s \equiv \hat{\boldsymbol{q}}_s - \boldsymbol{\beta},\ \ \hat{\boldsymbol{q}}_s \equiv \frac{\hat{\boldsymbol{\sigma}}_s^*}{\hat{p}_s}, \tag{35}$$

$$\hat{\chi}_s \equiv \frac{\|\hat{\boldsymbol{\eta}}_s\|}{\hat{m}_s}, \tag{36}$$

$\hat{m}_s$ is the function f_m of ϕ and

$$\sin 3\,\hat{\theta}_s \equiv -\sqrt{6}\left(\frac{\hat{s}_s}{\hat{r}_s}\right)^3, \tag{37}$$

i. e.

$$\hat{m}_s = f_m(\phi,\ \sin 3\,\hat{\theta}_s), \tag{38}$$

where

$$\hat{r}_s \equiv \mathrm{tr}^{1/2}\hat{\boldsymbol{\eta}}_s^2,\ \ \hat{s}_s \equiv \mathrm{tr}^{1/3}\hat{\boldsymbol{\eta}}_s^3. \tag{39}$$

Eqn (31) leads in the differential form to

$$\mathrm{tr}\left[\frac{\partial f(\hat{p}_s, \hat{\chi}_s)}{\partial \hat{\boldsymbol{s}}}\left(\mathring{\boldsymbol{s}} - \mathring{\boldsymbol{\alpha}} + \frac{1}{F}\left\{\mathrm{tr}\left(\frac{\partial f(\hat{p}_s, \hat{\chi}_s)}{\partial \boldsymbol{\beta}}\mathring{\boldsymbol{\beta}}\right) - \dot{F}\right\}\hat{\boldsymbol{s}}\right)\right] \leq 0 \ \text{ for } f(\hat{p}_s, \hat{\chi}_s) = F(H), \tag{40}$$

where $(\,\dot{}\,)$ indicates a material-time derivative. Eqn (31) or (40) is called the enclosing condition of similarity-center. Let the following equation be assumed, which satisfies the inequality (40).

$$\mathring{\boldsymbol{s}} - \mathring{\boldsymbol{\alpha}} + \frac{1}{F}\left\{\mathrm{tr}\left(\frac{\partial f(\hat{p}_s, \hat{\chi}_s)}{\partial \boldsymbol{\beta}}\mathring{\boldsymbol{\beta}}\right) - \dot{F}\right\}\hat{\boldsymbol{s}} = c\|\boldsymbol{D}^p\|\tilde{\boldsymbol{\sigma}} \tag{41}$$

from which the evolution rule of the similarity-center is obtained as follows :

$$\mathring{\boldsymbol{s}} = c\|\boldsymbol{D}^p\|\tilde{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\alpha}} + \frac{1}{F}\left\{\dot{F} - \mathrm{tr}\left(\frac{\partial f(\hat{p}_s, \hat{\chi}_s)}{\partial \boldsymbol{\beta}}\mathring{\boldsymbol{\beta}}\right)\right\}\hat{\boldsymbol{s}}, \tag{42}$$

where c is a material function and

$$a - a - s\,. \tag{43}$$

The associated flow rule is adopted :

$$\boldsymbol{D}^p = \lambda\overline{\boldsymbol{N}}\ (\lambda > 0), \tag{44}$$

where λ is the positive proportionality factor, and the second-order tensor $\overline{\boldsymbol{N}}$ is the normalized outward normal of the subloading surface, i. e.

$$\overline{\boldsymbol{N}} \equiv \frac{\partial f(\overline{p},\ \overline{\chi})}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\overline{p},\ \overline{\chi})}{\partial \boldsymbol{\sigma}} \right\|. \tag{45}$$

Let the loading criterion (Hill, 1958, 1967 ; Hashiguchi, 1994a) be assumed as

$$\left.\begin{aligned} \boldsymbol{D}^p \neq 0 &: \mathrm{tr}(\overline{\boldsymbol{N}}\boldsymbol{E}\boldsymbol{D}) > 0, \\ \boldsymbol{D}^p = \boldsymbol{O} &: \mathrm{tr}(\overline{\boldsymbol{N}}\boldsymbol{E}\boldsymbol{D}) \leq 0. \end{aligned}\right\} \tag{46}$$

A differentiation of eqn (19) leads to

$$\frac{\partial f(\overline{p},\ \overline{\chi})}{\partial \overline{\sigma}_{ij}}(\mathring{\sigma}_{ij} - \mathring{\alpha}_{ij}) + \frac{\partial f(\overline{p},\ \overline{\chi})}{\partial \beta_{ij}} \mathring{\beta}_{ij} = \dot{R}F + R\dot{F}. \tag{47}$$

It holds from eqns (13)-(18) and (28) that

$$\mathring{\boldsymbol{\alpha}} = K_1 \frac{\boldsymbol{\sigma}}{R} - K_2 \boldsymbol{\alpha}, \tag{48}$$

$$\mathring{\boldsymbol{\beta}} = b_1 \|\boldsymbol{D}^{p*}\| \, \|\overline{\boldsymbol{\eta}}\|^{b_2} \, \overline{\boldsymbol{\eta}}_b, \tag{49}$$

where

$$\overline{\boldsymbol{\eta}}_b (= \hat{\boldsymbol{\eta}}_b) = \boldsymbol{\eta}_b - \boldsymbol{\beta}, \quad \overline{\boldsymbol{\eta}}_b = \overline{m}_b \, \overline{\boldsymbol{t}}, \tag{50}$$

$$\overline{\boldsymbol{t}} \equiv \frac{\overline{\boldsymbol{\eta}}}{\|\overline{\boldsymbol{\eta}}\|}, \tag{51}$$

$$\overline{m}_b = f_m(\phi_b, \sin 3\overline{\theta}). \tag{52}$$

It is assumed that $\boldsymbol{R}$ increases monotonically in the plastic loading process, satisfying

$$\left.\begin{aligned} R=0 &: \dot{R} = +\infty, \\ 0<R<1 &: \dot{R} > 0, \\ R=1 &: \dot{R} = 0, \\ R>1 &: \dot{R} < 0, \end{aligned}\right\} (\boldsymbol{D}^p \neq \boldsymbol{O}). \tag{53}$$

Then, the evolution equation of R in the plastic loading process is given by

$$\dot{R} = U \|\boldsymbol{D}^p\|, \tag{54}$$

where U is the monotonically-decreasing function of $\boldsymbol{R}$, satisfying

$$\left.\begin{aligned} R=0 &: U = +\infty, \\ 0<R<1 &: U > 0, \\ R=1 &: U = 0, \\ R>1 &: U < 0, \end{aligned}\right\}. \tag{55}$$

Examples of the function U are as follows :

$$U = u_1(1/R^{m_1} - 1), \tag{56 a}$$

$$U = -u_2 \ln R, \tag{56 b}$$

where u_1, u_2 and m_1 are material constants.

It holds from eqns (30) that

$$\mathring{\overline{\boldsymbol{\sigma}}} = \mathring{\boldsymbol{\sigma}} - R\mathring{\boldsymbol{\alpha}} - (1-R)\mathring{\boldsymbol{s}} + \dot{R}\hat{\boldsymbol{s}}. \tag{57}$$

Substituting eqns (48), (49), (54) and (57) with eqn (44) to the extended consistency condition (47) for the subloading surface, λ is obtained as

$$\lambda \equiv \frac{\langle \mathrm{tr}(\bar{\boldsymbol{N}}\boldsymbol{E}\boldsymbol{D})\rangle}{D+\mathrm{tr}(\bar{\boldsymbol{N}}\boldsymbol{E}\bar{\boldsymbol{N}})}\left(=\langle\frac{\mathrm{tr}(\bar{\boldsymbol{N}}\mathring{\boldsymbol{\sigma}})}{D}\rangle\right), \tag{58}$$

where

$$D \equiv \mathrm{tr}(\bar{\boldsymbol{N}}\bar{\boldsymbol{a}})+\frac{\mathrm{tr}(\bar{\boldsymbol{N}}\bar{\boldsymbol{\sigma}})}{RF}\left\{RF'h-\mathrm{tr}\left(\frac{\partial f(\bar{p},\ \bar{\chi})}{\partial \boldsymbol{\beta}}\boldsymbol{b}\right)+UF\right\}, \tag{59}$$

$$\bar{\boldsymbol{a}} \equiv \frac{\mathring{\bar{\boldsymbol{\alpha}}}}{\lambda}=R\boldsymbol{a}+(1-R)\boldsymbol{z}-U\hat{\boldsymbol{s}}, \tag{60}$$

$$h \equiv \frac{\dot{H}}{\lambda},\ F' \equiv \frac{dF}{dH}, \tag{61}$$

$$\boldsymbol{a} \equiv \frac{\mathring{\boldsymbol{\alpha}}}{\lambda} \equiv \frac{1}{\lambda}(K_1\hat{\boldsymbol{\sigma}}-K_2\boldsymbol{\alpha}), \tag{62}$$

$$\boldsymbol{b} \equiv \frac{\mathring{\boldsymbol{\beta}}}{\lambda}=b_1\|\bar{\boldsymbol{N}}^*\|\,\|\bar{\boldsymbol{\eta}}\|^{b_2}\bar{\boldsymbol{\eta}}_b, \tag{63}$$

$$\boldsymbol{z} \equiv \frac{\mathring{\boldsymbol{s}}}{\lambda}=c\tilde{\boldsymbol{\sigma}}+\boldsymbol{a}+\frac{1}{F}\left\{F'h-\mathrm{tr}\left(\frac{\partial f(\hat{p}_s,\ \hat{\chi}_s)}{\partial \boldsymbol{\beta}}\boldsymbol{b}\right)\right\}\hat{\boldsymbol{s}}, \tag{64}$$

$$\bar{\boldsymbol{N}}^* \equiv \bar{\boldsymbol{N}}-\frac{1}{3}(\mathrm{tr}\bar{\boldsymbol{N}})\boldsymbol{I}. \tag{65}$$

The bracket $\langle\ \rangle$ is Macauley bracket, i. e. $\langle u\rangle=0$ for $u=0$ and $\langle u\rangle=u$ for $u\geq 0$ resulting in $\langle u\rangle=u\bar{H}(u)$ with the Heviside unit-step function $\bar{H}$ for an arbitrary scalar variable u.

As was described in the previous article (Hashiguchi, 1993b), this model fulfills the *condition of continuity in the large* that a stress rate changes continuously for continuous change of a stress state, i. e.

$$\mathring{\boldsymbol{\sigma}}(\boldsymbol{\sigma}+\delta\boldsymbol{\sigma},\ \boldsymbol{H}_i,\ \boldsymbol{D})-\mathring{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \boldsymbol{H}_i,\ \boldsymbol{D})\to \boldsymbol{O}\ \ \textit{for}\ \ \delta\boldsymbol{\sigma}\to\boldsymbol{O} \tag{66}$$

so that the smooth elastic-plastic transition is described, where $\boldsymbol{H}_i(i=1, 2, \cdots, n)$ denote collectively scalar- or tensor-valued plastic internal state variables. Therefore, it is not required to incorporate the algorithm for judging the fulfillment of the yield condition and for selecting a stress increment for which a stress does not go out from the yield surface, e. g. the *Euler method* (Yamada *et al.*, 1968), *the mean normal method* (Pillinger *et al.*, 1986) and the *radial return method* (Krieg and Krieg, 1977), in the computer program. Also, the *Masing coefficient* (Hashiguchi, 1993b) is controllable by the selection of the value of the material function c so that the hysteresis loop would be depicted realistically. This model obeys the associated flow rule, while almost of all constitutive models for soils adopt the non-associated flow rule which violates the (second-order) *work rate-stiffness relaxation* as the fundamental requirement for the elastoplastic constitutive equations (Hashiguchi, 1993a) as was revealed by the author (Hashiguchi, 1991).

MATERIAL FUNCTIONS

Concrete forms of the material functions in the present model for soils are given in this section.

Let the following functions for the subloading surface and the limit surface for rotation of yield surface, etc. be adopted.

$$g(\bar{\chi})=1+\bar{\chi}^2,\ g(\bar{\chi}_s)=1+\bar{\chi}_s^2, \tag{67}$$

$$\bar{m}=\frac{2\sqrt{6}\sin\phi}{3-\sin^n\phi\sin 3\bar{\theta}},\quad \hat{m}_s=\frac{2\sqrt{6}\sin\phi}{3-\sin^n\phi\sin 3\hat{\theta}_s},\quad m_b=\frac{2\sqrt{6}\sin\phi_b}{3-\sin^n\phi_b\sin 3\bar{\theta}}. \tag{68}$$

n is the material parameter, while the equation $\|\bar{\boldsymbol{Q}}\|=\bar{m}$ with eqn $(68)_1$ for $n=1$ and $\bar{\boldsymbol{\alpha}}=0$ coincides with the Coulomb-Mohr failure criterion in the axisymmetric tension/compression states, i. e. $\|\boldsymbol{\sigma}^*\|/p=2\sqrt{6}/(3\pm\sin\phi)$($+$: extension (8=-n/6), $-$: compression ($\bar{\theta}=\pi/6$)).

For the formulation of the hardening function F(H) assume the linear relation of $\ln p-\ln v$ (Hashiguchi, 1974, 1994b; Hashiguchi and Ueno, 1977) ($p(\equiv-(\mathrm{tr}\boldsymbol{\sigma})/3)$: pressure, v: volume) as shown in Fig. 1 :

$$\begin{aligned}\varepsilon_v&=\varepsilon_v^p+\varepsilon_v^e\\&=\ln\frac{v^p}{v_0}+\ln\frac{v}{v^p}\\&=-(\mu-\gamma)\ln\frac{p_y}{p_{y0}}-\gamma\ln\frac{p}{p_0}\end{aligned} \tag{69}$$

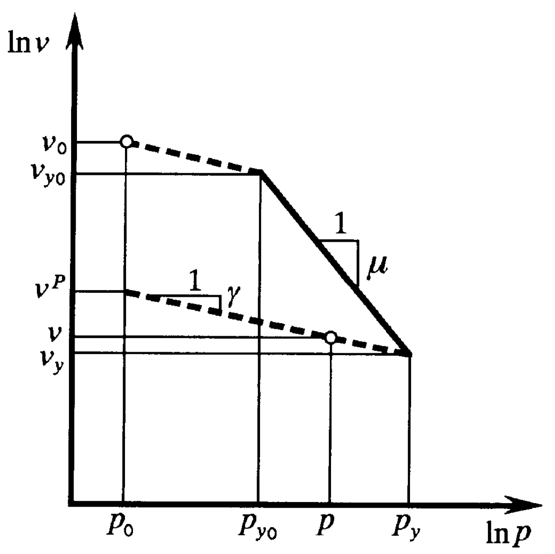

Fig. 1. The $\ln v$-$\ln p$ linear relation for isotropic consolidation of soils.

the time-derivative of which is given as

$$D_v = D_v^p + D_v^e = -(\mu-\gamma)\frac{\dot{p}_y}{p_y} - \gamma\frac{\dot{p}}{p}. \qquad (70)$$

where

$$\left.\begin{aligned} D_v &\equiv \mathrm{tr}\boldsymbol{D}, & \varepsilon_v &\equiv \int D_v dt, \\ D_v^e &\equiv \mathrm{tr}\boldsymbol{D}^e, & \varepsilon_v^e &\equiv \int D_v^e dt, \\ D_v^p &\equiv \mathrm{tr}\boldsymbol{D}^p, & \varepsilon_v^p &\equiv \int D_v^p dt. \end{aligned}\right\} \qquad (71)$$

$\varepsilon_v, \varepsilon_v^e$ and ε_v^p are volumetric, elastic volumetric and plastic volumetric logarithmic strain, respectively. v_0 is the initial volume and v^p is the volume in the unloaded state to the initial pressure p_0, and p_y and p_{y0} are yield pressure and its initial value, respectively. μ and γ are the slopes of normal consolidation (elastoplastic) curve and the swelling (elastic) curve, respectively, in the (lnp, lnv) plane.

By selecting F to be equal to p_y in eqn (69), one has

$$F = F_0 \exp\left(-\frac{\varepsilon_v^p}{\mu-\gamma}\right) \qquad (72)$$

for the isotropic consolidation process, where F_0 is the initial value of F. The hardening/softening of soils is substantially induced by the decrease/increase of plastic volumetric strain. In addition to this fact, the plastic deviatoric stretching induces a hardening for dense sands but a softening for loose ones. Taking this fact into account, assume the following evolution equation of isotropic hardening function F for the general loading process.

$$F = F_0 \exp\left(-\frac{H}{\mu-\gamma}\right), \qquad (73)$$

$$\dot{H} \equiv D_v^p - \xi(V^p - V_c)\|\boldsymbol{D}^{p*}\|, \qquad (74)$$

where $V^p(=V_0\exp(\varepsilon_v^p), V_0$: initial specific volume) is the specific volume in the unloaded state to the initial stress and V_c is the specific volume in the boundary between the states inducing a hardening and a softening due to a deviatoric plastic stretching and ξ is a material constant. From eqns (73) and (74) it holds for eqn (61) that

$$F' = -\frac{F}{P-Y}, \quad h = \mathrm{tr}\bar{\boldsymbol{N}} - \xi(V^p - V_c)\|\bar{\boldsymbol{N}}^*\|. \qquad (75)$$

Let the material function c in the evolution equation (42) of the similarity-center $\boldsymbol{s}$ be given as

$$c = c_1/R^{c_2}, \qquad (76)$$

where c_1 and c_2 are material constants.

Let the elastic bulk modulus K be given from eqn (70) as

$$K=\frac{p}{\gamma}. \tag{77}$$

The elastic stretching is given from eqn (2) with eqns (3) and (77) as

$$\boldsymbol{D}^e\equiv\frac{1}{3}\frac{\gamma}{p}\dot{\sigma}_m\boldsymbol{I}+\frac{1}{2G}\tilde{\boldsymbol{\sigma}}^*, \tag{78}$$

where

$$\sigma_m\equiv\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma},\ \boldsymbol{\sigma}^*\equiv\boldsymbol{\sigma}-\sigma_m\boldsymbol{I}. \tag{79}$$

COROTATIONAL RATE

The corotational rate of the second-order tensor $\boldsymbol{T}$ is given as

$$\mathring{\boldsymbol{T}}=\dot{\boldsymbol{T}}-\boldsymbol{W}_m\boldsymbol{T}+\boldsymbol{T}\boldsymbol{W}_m, \tag{80}$$

where $\boldsymbol{W}_m$ is the *corotational spin*, i. e. spin of a material substructure (Kratochvil, 1971 ; Mandel, 1971) and was given by Dafalias (1983, 1985) and Loret (1983) as follows :

$$\boldsymbol{W}_m=\boldsymbol{W}-\boldsymbol{W}^p, \tag{81}$$

where $\boldsymbol{W}$ is the *continuum spin*, i. e. the antiisymmetric part of the velocity gradient tensor and $\boldsymbol{W}^p$ is called the *plastic spin*. For the kinematic hardening $\boldsymbol{W}^p$ was given as

$$\boldsymbol{W}^p=\|\boldsymbol{D}^p\|\boldsymbol{\rho}(\boldsymbol{\sigma},\ \boldsymbol{\alpha}),\ \ \boldsymbol{\rho}=\eta(\boldsymbol{\alpha}\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{\alpha}), \tag{82}$$

where η is a material parameter (Dafalias, 1983, 1985 ; Loret, 1983).

The three tensor-valued plastic internal state variables, i. e. the kinematic hardening variable $\boldsymbol{\alpha}$, the rotational hardening variable $\boldsymbol{\beta}$ and the similarity-center $\boldsymbol{s}$ are incorporated in the present model. It is assumed that the kinematic hardening occurs only isotropically so that the yield surface expands and translates to the direction of the space diagonal in the principal stress space, keeping $a^*(\equiv \boldsymbol{a}-(\mathrm{tr}\boldsymbol{\alpha})/3)=0$ and involving the origin of stress space. Besides, note that the similarity-center approaches to the current stress, i. e. s- $\boldsymbol{\sigma}$ for a continuous plastic loading. Thus, it may be assumed that

$$\boldsymbol{W}^p=\|\boldsymbol{D}^{p*}\|\boldsymbol{\zeta}(\boldsymbol{\sigma},\ \boldsymbol{\beta}),\ \ \boldsymbol{\zeta}=\varphi(\boldsymbol{\beta}\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{\beta}), \tag{83}$$

where φ is a material parameter.

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APPENDIX

1. Equation of R

The substitution of eqn (30) into eqn (19) with eqns (20) and (67) leads to

$$\bar{\sigma}_m^2(\bar{m}^2+\beta_{ij}\beta_{ij})+2\,\bar{\sigma}^*_{ij}\beta_{ij}\bar{\sigma}_m+\bar{\sigma}^*_{ij}\bar{\sigma}^*_{ij}=-\bar{\sigma}_m\bar{m}^2RF, \tag{A1}$$

where

$$\left.\begin{aligned}\bar{\sigma}_m&\equiv\tilde{\sigma}_m+Rs_m,\\ \bar{\sigma}^*_{ij}&\equiv\tilde{\sigma}^*_{ij}+Rs^*_{ij}.\end{aligned}\right\} \tag{A2}$$

From eqn (Al) with eqn **(A2)** one has

$$R=(-B\pm\sqrt{B^2-AC})/A, \tag{A3}$$

where

$$\left.\begin{aligned}A&\equiv s_m^2\bar{m}^2+s_m^2\beta_{ij}\beta_{ij}+2s_ms^*_{ij}\beta_{ij}+s^*_{ij}s^*_{ij}+s_m\bar{m}^2F,\\ B&\equiv s_m\tilde{\sigma}_m\bar{m}^2+s_m\tilde{\sigma}_m\beta_{ij}\beta_{ij}+\tilde{\sigma}_ms^*_{ij}\beta_{ij}+s_m\tilde{\sigma}^*_{ij}\beta_{ij}+s^*_{ij}\tilde{\sigma}^*_{ij}+\frac{1}{2}\bar{m}^2F\tilde{\sigma}_m,\\ C&\equiv\tilde{\sigma}_m^2\bar{m}^2+\tilde{\sigma}_m^2\beta_{ij}\beta_{ij}+2\,\tilde{\sigma}_m\tilde{\sigma}^*_{ij}\beta_{ij}+\tilde{\sigma}^*_{ij}\tilde{\sigma}^*_{ij}.\end{aligned}\right\} \tag{A4}$$

2. Partial derivatives of material functions

The partial derivatives of material functions included in the present model are given in the following.

Because of

$$\frac{\partial f(\bar{p},\bar{\chi})}{\partial\sigma_{ij}}=\frac{\partial\bar{p}}{\partial\sigma_{ij}}g(\bar{\chi})+\bar{p}\frac{\partial g(\bar{\chi})}{\partial\bar{\chi}}\left(\frac{\partial\bar{\chi}}{\partial\bar{r}}\frac{\partial\bar{r}}{\bar{\eta}_{rs}}+\frac{\partial\bar{\chi}}{\partial\bar{m}}\frac{\partial\bar{m}}{\partial\sin3\bar{\theta}}\frac{\partial\sin3\bar{\theta}}{\partial\bar{\eta}_{rs}}\right)\frac{\partial\bar{\eta}_{rs}}{\partial\sigma_{ij}}, \tag{A5}$$

$$\left.\begin{aligned}&\frac{\partial\bar{p}}{\partial\sigma_{ij}}=-\frac{1}{3}\delta_{ij},\\ &\frac{\partial\bar{\chi}}{\partial\bar{r}}=\frac{1}{\bar{m}},\quad\frac{\partial\bar{\chi}}{\partial\bar{m}}=-\frac{\bar{r}}{\bar{m}^2},\\ &\frac{\partial\bar{r}}{\partial\bar{\eta}_{rs}}=\bar{t}_{rs},\quad\frac{\partial\sin3\bar{\theta}}{\partial\bar{\eta}_{rs}}=-\frac{3}{\bar{r}}(\sqrt{6}\,\bar{t}_{rp}\bar{t}_{ps}+\sin3\bar{\theta}\,\bar{t}_{rs}),\end{aligned}\right.$$

$$\left.\begin{aligned}\frac{\partial \bar{\eta}_{rs}}{\partial \sigma_{ij}} &= \frac{1}{3\bar{p}}(3\delta_{ir}\delta_{js} - \delta_{ij}\delta_{rs} + \bar{q}_{rs}\delta_{ij}),\\ \frac{\partial \bar{r}}{\partial \sigma_{ij}} &= \frac{1}{3\bar{p}}(3\bar{t}_{ij} + \bar{t}_{rs}\bar{q}_{rs}\delta_{ij}),\\ \frac{\partial \sin 3\bar{\theta}}{\partial \sigma_{ij}} &= -\frac{1}{\bar{p}\bar{r}}\{(\sin 3\bar{\theta}\,\bar{t}_{rs}\bar{q}_{rs} - \sqrt{6}\,\bar{t}_{rs}\bar{t}_{rs} + \sqrt{6}\,\bar{t}_{rs}\bar{t}_{st}\bar{q}_{tr})\delta_{ij} + 3\sin 3\bar{\theta}\,\bar{t}_{ij}\\ &\quad + 3\sqrt{6}\,\bar{t}_{ir}\bar{t}_{rj}\}.\end{aligned}\right| \tag{A6}$$

it holds that

$$\frac{\partial f(\bar{p},\bar{\chi})}{\partial \boldsymbol{\sigma}} = -\frac{1}{3}g(\bar{\chi})\boldsymbol{I} + \frac{1}{\bar{m}^2}\frac{\partial g(\bar{\chi})}{\partial \bar{\chi}}\Big[\bar{m}\{\bar{\boldsymbol{t}} + \frac{1}{3}\mathrm{tr}(\bar{\boldsymbol{t}}\bar{\boldsymbol{q}})\boldsymbol{I}\}$$
$$+\{(\sin 3\bar{\theta}\,\mathrm{tr}(\bar{\boldsymbol{t}}\bar{\boldsymbol{q}}) - \sqrt{6}\mathrm{tr}\,\bar{\boldsymbol{t}}^2 + \sqrt{6}\mathrm{tr}(\bar{\boldsymbol{t}}^2\bar{\boldsymbol{q}}))\boldsymbol{I} + 3\sin 3\bar{\theta}\,\bar{\boldsymbol{t}} + 3\sqrt{6}\,\bar{\boldsymbol{t}}^2\}\frac{\partial \bar{m}}{\partial \sin 3\bar{\theta}}\Big]. \tag{A7}$$

Because of

$$\frac{\partial f(\bar{p},\bar{\chi})}{\partial \beta_{ij}} = \bar{p}\frac{\partial g(\bar{\chi})}{\partial \bar{\chi}}\Big(\frac{\partial \bar{\chi}}{\partial \bar{r}}\frac{\partial \bar{r}}{\bar{\eta}_{rs}} + \frac{\partial \bar{\chi}}{\partial \bar{m}}\frac{\partial \bar{m}}{\partial \sin 3\bar{\theta}}\frac{\partial \sin 3\bar{\theta}}{\partial \bar{\eta}_{rs}}\Big)\frac{\partial \bar{\eta}_{rs}}{\partial \beta_{ij}}, \tag{A8}$$

$$\left.\begin{aligned}&\frac{\partial \bar{\eta}_{rs}}{\partial \beta_{ij}} = -\delta_{ir}\delta_{js},\quad \frac{\partial \bar{r}}{\partial \beta_{ij}} = -\bar{t}_{ij},\\ &\frac{\partial \sin 3\bar{\theta}}{\partial \beta_{ij}} = \frac{3}{\bar{r}}(\sqrt{6}\,\bar{t}_{ip}\bar{t}_{pj} + \sin 3\bar{\theta}\,\bar{t}_{ij})\end{aligned}\right\} \tag{A9}$$

it holds that

$$\frac{\partial f(\bar{p},\bar{\chi})}{\partial \boldsymbol{\beta}} = \frac{\bar{p}}{\bar{m}^2}\frac{\partial g(\bar{\chi})}{\partial \bar{\chi}}\Big\{\bar{m}\bar{\boldsymbol{t}} + 3(\sin 3\bar{\theta}\,\bar{\boldsymbol{t}} + \sqrt{6}\,\bar{\boldsymbol{t}}^2)\frac{\partial \bar{m}}{\mathrm{d}\sin 3\bar{\theta}}\Big\}. \tag{A10}$$

In the same manner it is also derived that

$$\frac{\partial f(\hat{p}_s,\hat{\chi}_s)}{\partial \boldsymbol{\beta}} = -\frac{\hat{p}_s}{\hat{m}_s^2}\frac{\partial g(\hat{\chi}_s)}{\partial \hat{\chi}_s}\Big\{\hat{m}_s\hat{\boldsymbol{t}}_s + 3(\sin 3\hat{\theta}_s\hat{\boldsymbol{t}}_s + \sqrt{6}\,\hat{\boldsymbol{t}}_s^2)\frac{\partial \hat{m}_s}{\partial \sin 3\hat{\theta}_s}\Big\}, \tag{A11}$$

where

$$\hat{\boldsymbol{t}}_s \equiv \frac{\hat{\boldsymbol{\eta}}_s}{\|\hat{\boldsymbol{\eta}}_s\|}, \tag{A12}$$

For eqns (67) and (68) it holds that

$$\frac{\partial g(\bar{\chi})}{\partial \bar{\chi}} = 2\bar{\chi},\quad \frac{\partial \bar{m}}{\partial \sin 3\bar{\theta}} = \frac{\sin^{n-1}\phi}{2\sqrt{6}}\bar{m}^2, \tag{A13}$$

$$\frac{\partial g(\hat{\chi}_s)}{\partial \hat{\chi}_s} = 2\hat{\chi}_s,\quad \frac{\partial \hat{m}_s}{\partial \sin 3\hat{\theta}_s} = \frac{\sin^{n-1}\phi}{2\sqrt{6}}\hat{m}_s^2. \tag{A14}$$