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Derivation of a Loading Criterion and the Associated Flow Rule

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A loading criterion in the elastoplasticity is derived, which is applicable to materials with not only hardening but also softening behavior. Further, the associated flow rule is derived from Ilyushin's postulate of the non-negative work done during a strain'cycle without the assumption of the existence of a purely elastic domain. These derivations are performed in the framework of the stress space formulation. Besides, irrationalities of the non-associated flow rule incorporating a plastic potential surface different from a yield or a loading surface are described.

INTRODUCTION

Physical interpretations of the associated flow rule were given by Drucker (1951) and by Ilyushin (1961). The former belongs to the stress space formulation and premises that the interior of the yield surface is purely elastic. Therefore, his interpretation is not applicable to constitutive equations which do not assume a purely elastic domain as in the subloading surface model (Hashiguchi, 1980) and the infinite surface model (Mroz *et al.*, 1981). The latter does not premise the existence of the purely elastic domain but belongs to the strain space formulation. In this paper, a loading criterion is derived, which is applicable to not only hardening but also softening states, and the associated flow rule is deduced from the Ilyushin's postulate of the non-negative work done during a strain cycle. They are done within the framework of stress space formulation,

A PLASTIC CONSTITUTIVE EQUATION

Now, we introduce a loading surface :

$$f(\boldsymbol{\sigma}, H_j) = 0 \quad (j = 1, 2, \cdots\cdots, n), \tag{1}$$

where $\boldsymbol{\sigma}$ is a stress and scalars or tensors H_j are plastic internal state variables, provided that $f < 0$ for the interior of the loading surface.

A differentiation of Eq. (1) leads to the consistency condition

$$tr\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}}\right) + \sum_{j=1}^{n} \frac{\partial f}{\partial H_j} \dot{H}_j = 0. \tag{2}$$

Here, let it be assumed that a plastic strain tate $\dot{\boldsymbol{\varepsilon}}^p$ is expressed as

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\boldsymbol{m} \;(\|\boldsymbol{m}\| = 1), \tag{3}$$

where $\dot{\lambda}$ is positive proportionality factor determined below and $\boldsymbol{m}$ is a normalized second-order tensor, which is function of stress and some plastic internal state variables. The notation $\| \|$ designates a norm (magnitude) .

It can be expressed by Eq. (3) that

$$\dot{H}_j = \dot{\lambda} h_j, \tag{4}$$

where scalars or tensors h_j are functions of $\boldsymbol{m}$ (degree one), stress and some plastic internal state variables.

Substituting Eq. (4) into Eq. (2), we have

$$\dot{\lambda} = \frac{tr\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}\right)}{-\sum_{j=1}^{n} \dfrac{\partial f}{\partial H_j} h_j} \tag{5}$$

or

$$\dot{\lambda} = \frac{tr(\boldsymbol{n}\dot{\boldsymbol{\sigma}})}{L}, \tag{6}$$

where

$$L = -\sum_{j=1}^{n} \frac{\partial f}{\partial H_j} h_j \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|, \tag{7}$$

$$\boldsymbol{n} \equiv \frac{\partial f}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|. \tag{8}$$

Substituting Eq. (3) and the relations

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p, \tag{9}$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{E}\dot{\boldsymbol{\varepsilon}}^e \tag{10}$$

into Eq. (6) we have

$$\dot{\lambda} = \frac{tr\{\boldsymbol{n}\boldsymbol{E}(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{m})\}}{\boldsymbol{L}}, \tag{11}$$

where $\dot{\boldsymbol{\varepsilon}}$ is a strain rate, $\dot{\boldsymbol{\varepsilon}}^e$ is its elastic component and the fourth-order tensor $\boldsymbol{E}$ is an elastic modulus. From Eq. (11) we obtain the expression of $\dot{\lambda}$ by the strain rate instead of the stress rate. Let it be denoted by $\dot{\Lambda}$:

$$\dot{\Lambda} = \frac{tr(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}})}{\boldsymbol{L} + tr(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n})} \tag{12}$$

by which Eq. (3) is rewritten as

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\Lambda}\boldsymbol{m} \tag{13}$$

A LOADING CRITERION

Let it be assumed that a plastic deformation occurs inevitably when a stress rate has an outward direction of a loading surface. Accordingly, $tr(\boldsymbol{n}\dot{\boldsymbol{\sigma}})>0$ is a sufficient condition for a loading state ($\dot{\boldsymbol{\varepsilon}}^p \neq 0$).

Then, it must hold that

$$tr(\boldsymbol{n}\dot{\boldsymbol{\sigma}}) \leqq 0 \tag{14}$$

in an unloading state ($\dot{\boldsymbol{\varepsilon}}^p = 0$), though Eq. *(14)* holds also in the loading state with a softening. It holds that $\dot{\boldsymbol{\sigma}} = \boldsymbol{E}\dot{\boldsymbol{\varepsilon}}$ in an unloading state ($\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e$). Then, the necessary condition (14) for an unloading state is rewritten as

$$tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) \leqq 0. \tag{15}$$

If $L + tr(\boldsymbol{nEn}) < 0$, a plastic deformation cannot occur for $tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}})>0$ by the collateral condition $\dot{\Lambda} > 0$ for $\dot{\boldsymbol{\varepsilon}}^p$ =t= 0. Further, if $L + tr(\boldsymbol{nEn}) = 0$, a plastic deformation can occur only for $tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) = 0$. Remind that an elastic state does not exist for $tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) > 0$ by the necessary condition (15) for this state. Thus, if $L + tr(\boldsymbol{nEn})$ IO, a strain rate bringing about $tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) > 0$ cannot occur, whereas real materials can undergo a strain rate of any direction (This point is fundamentally different from a stress rate). On the other hand. if $L + tr(\boldsymbol{nEn}) > 0$, an elastoplastic deformation proceeds for $tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) > 0$ by the condition $\dot{\Lambda} > 0$. Eventually, in order to describe a strain rate with any direction, it must hold that

$$L + tr(\boldsymbol{nEn}) > 0, \tag{16}$$

while L can take both positive and negative values but

$$tr(\boldsymbol{nEn}) > 0, \tag{17}$$

since $tr(\boldsymbol{nEn})$ is of the quadratic form and $\boldsymbol{E}$ is a positive definite as known from $tr(\dot{\boldsymbol{\varepsilon}}^e \boldsymbol{E} \dot{\boldsymbol{\varepsilon}}^e) = tr(\dot{\boldsymbol{\sigma}}\dot{\boldsymbol{\varepsilon}}^e) > 0 (\dot{\boldsymbol{\varepsilon}}^e \neq 0)$. Then, in a loading state it must hold that

$$tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) > 0 \tag{18}$$

which is a necessary condition for this state.

The fact that Eqs. (15) and (18) are necessary conditions for unloading and loading states, respectively, leads to the result that they are not only necessary but also sufficient conditions for each state. Thus, a loading criterion is given as

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &\neq 0 \quad \text{for'} \quad tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) > 0, \\ \dot{\boldsymbol{\varepsilon}}^p &= 0 \quad \text{for} \quad tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}}) \leqq 0, \end{aligned} \tag{19}$$

while constitutive equations assuming a purely elastic domain require further a judgement whether a yield condition is satisfied or not. Eq. (19) was shown by Mroz and

Zienkiewicz (1984) in a diffarent approach, *i.* e., a strain space formulation of plastic constitutive equations.

On the other hand, consider $\dot{\lambda}$ in Eq. (6) expressed by a stress rate. The loading in the state $L<0$ brings about a softening in which $tr(\boldsymbol{n}\dot{\boldsymbol{\sigma}})<0$ and $L<0$ leads to $\dot{\lambda}>0$. However, the unloading $tr(\boldsymbol{n}\dot{\boldsymbol{\sigma}})<0$ in the state $L<0$ also leads to $\dot{\lambda}>0$. Thus, $\dot{\lambda}>0$ is a necessary condition but not a sufficient condition for a loading state. Eventually, $\dot{\lambda}$ cannot be adopted as a quantity to define a loading criterion for materials exhibiting not only hardening but also softening behavior.

THE ASSOCIATED FLOW RULE

Ilyushin (1961) postulated that the work done during a strain cycle is non-negative, *i.* e.,

$$\oint_{\varepsilon} tr(\boldsymbol{\sigma} d\boldsymbol{\varepsilon}) \geqq 0. \tag{20}$$

For an infinitesimally small strain cycle, Eq. (20) is written as

$$tr(d\boldsymbol{\sigma}^p d\boldsymbol{\varepsilon}) \geqq 0 \quad \text{or} \quad tr(\dot{\boldsymbol{\sigma}}^p \dot{\boldsymbol{\varepsilon}}) \geqq 0, \tag{21}$$

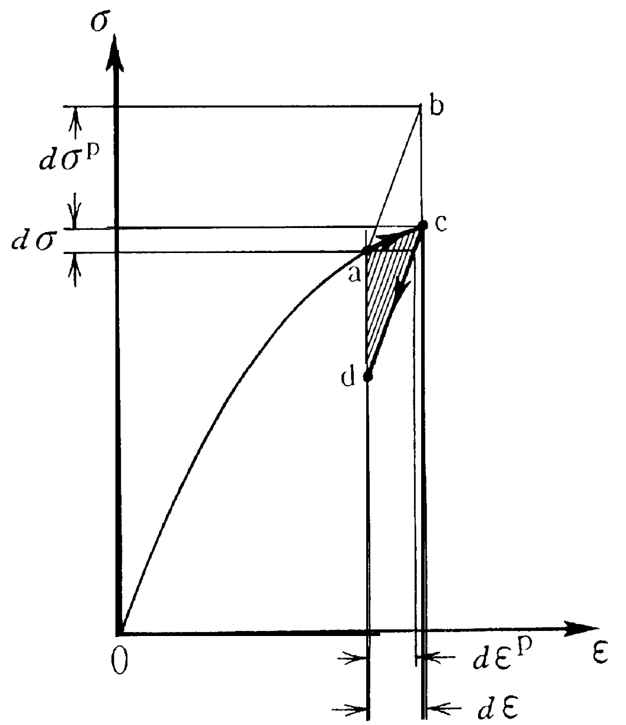

Fig.1 A non-negative work done during a strain cycle.

where $\dot{\boldsymbol{\sigma}}^p$ is a plastic relaxation stress rate, *i.* e.,

$$\dot{\boldsymbol{\sigma}}^p = \boldsymbol{E}\dot{\boldsymbol{\varepsilon}}^p, \tag{22}$$

referring to Fig. 1 in which the elastic lines a b and c d are regarded to be parallel since we consider an infinitesimally small strain cycle.

Substituting Eq. (13) with (12) into Eq. (21), we have

$$\frac{tr(\boldsymbol{nE}\dot{\boldsymbol{\varepsilon}})}{L+tr(\boldsymbol{nEn})}\, tr(\boldsymbol{mE}\dot{\boldsymbol{\varepsilon}}) > 0 \tag{23}$$

for a loading state.

In order that Eq. (23) is fulfilled for an arbitrary strain rate satisfying the loading condition $(19)_1$, it must hold that

$$\boldsymbol{m} = \boldsymbol{n} \tag{24}$$

Eq. (3) or (13) with Eq. (24) is the associated flow rule.

IRRATIONALITIES OF THE NON-ASSOCIATED FLOW RULE

Most of elastoplastic constitutive models for soils assume the non-associated flow rule which incorporates a plastic potential surface different from a yield or a loading surface. They encounter irrationalities described below.

1. There occurs an elastoplastic stiffness greater than an elastic stiffness as was indicated by Molenkamp and Ommen (1987).

2. A hardening, *i.* e., an expansion of normal-yield (or bounding) surface is predicted for loose sands subject to the stress path with $d\|\boldsymbol{\sigma}\|<0$ and $d(\|\boldsymbol{\sigma}'\|/P)>0$ (e. g., Lade et *al.*, 1987), where $P=-tr\boldsymbol{\sigma}, \boldsymbol{\sigma}'=\boldsymbol{\sigma}+P\boldsymbol{I}$. Such a stress path is observed also in an undrained state of loose sands undergoing a liquefaction. In reality, however, a liquefaction would bring about a softening.

Besides, needless to say, Drucker's and Ilyushin's postulates on the work done during a stress or a strain cycle are violated in the non-associated flow rule with a plastic potential surface, while these postulates are not thermodynamical requirements.

An adoption of the non-associated flow rule leads not only to a complexity of constitutive equations assuming a plastic potential surface in addition to a yield or a loading surface but also to the irrationalities described above. At present it would be advantageous to formulate constitutive equations by using the associated flow rule adequately (e. g. the extended subloading surface model (Hashiguchi, 1986)). Further, concurrently we should look for a more general flow rule which can describe the mutual dependency between directions of stress rate and of plastic strain rate, whlie one of trials was shown by the author (Hashiguchi, 1987).

REFERENCES

Drucker, D. C. 1951 A more fundamental approach to plastic stress-strain relations. *1st U.S.*

Congr. Appl. Mech. (ASME), **1** : 487-491

Hashiguchi, K. 1980 Constitutive equations of elastoplastic materials with elastic-plastic transition. *J. Appl. Mech. (ASME)*, **47** : ***266-272***

Hashiguchi, K. 1986 Elastoplastic constitutive equations with a subloading surface. *Proc. Int. Conf. Comput. Mech.*, IV65-70

Hashiguchi, K. 1987 Elastoplastic constitutive equations based on subloading surface concept. *Proc.* ***2nd Int.*** *Conf.* ***Constitutive Laws for Eng. Materials, 549-556.***

Ilyushin, A. A. 1961 0 postulate plastichnosti (On the postulate of plasticity). *Prikladnaya Mathematika* ***i Mekhanika. 25: 503-507***

Lade, P. V., R. B. Nelson and Y. M. Ito 1987 Nonassociated flow and stability of granular materials. ***J. Eng.*** *Mech.* ***(AXE), 113*** :1302-1318

Molenkamp, F. and A. Van Ommen 1987 Peculiarity of non-associativity in plasicity of soll mechanics. ***Int. J.*** *Numer.* ***Anal. Methods Geomech.,*** 11: 659-661

Mroz, Z., V. A. Norris and 0. C. Zienkiewicz 1981 An anisotropic, critical state model for soils subject to cyclic loading, ***Geotechnique, 31:*** 451-469

Mroz, Z., and 0. C. Zienkiewicz 1984 Uniform formalization of constitutive equations for clays and sands. ***Mechanics of Enginearing Materials.*** John Wiley & Sons, 415-450