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## Elastoplastic Constitutive Equations with Plural Yield Surfaces

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Various constitutive models for the description of the elastoplastic deformation with an anisotropic hardening and also a transition from the elastic to the distinct-yield (fully-plastic) state have been proposed in the past. Among them the two- or multi-surface theory with plural stratified yield surfaces which has been extended from the kinematic hardening model would be one of the most available models, and many constitutive equations have been presented using that theory. None of them have been formulated, however, in mathematically rational forms applicable to the generalized materials with not only hardening but also softening behaviours. In this paper, a reasonable formulation of the two- and the multi-surface theories will be given by deriving the mathematical condition which must be satisfied in order that the surfaces do not intersect each other at their relative translation and which will be called a "non-intersection condition" and by assuming a reasonable measure to represent the degree of distance from the distinct-yield state in the two-surface theory. Among them the two-surface theory may be simple enough to be adopted in numerical analyses of practical problems in engineering.

### INTRODUCTION

An extension of the kinematic hardening model advocated by Ishlinski (1954) and Prager (1956) so as to be able to describe even an elastoplastic deformation proceeding in the transition from the elastic to the distinct-yield (fully-plastic) state, which would obey Masing rule (1926), has been attempted by Iwan (1967) and Mroz (1967, 1969). The extended models by them are, however, of complex form assuming multiple subyield (nesting yield) surfaces encircled by a distinct-yield (bounding or limiting) surface, which have been called a multi-surface theory. Thereafter, based on them, simplified models employing a distinct-yield surface and only one subyield (inner-yield) surface have been formulated by many workers (Dafalias and Popov (1975, 1976), Krieger (1975), Mroz et al. (1979), Hashiguchi (1981)), which have been called a two-surface theory.

None of the models in the framework of the multi- or the two-surface theory have been formulated rationally on the basis of mathematical verifications. Especially, little consideration has been given to formulating the mathematical condition which regulates an inner surface so as not to protrude from an outer surface, i.e., which keeps the surfaces from their intersection

at the relative translation. Therefore, even though the foregoing models could analyze the deformation of specialized materials exhibiting only a hardening behavior, they would be confronted with the mathematical contradiction on the translation of assumed surfaces in the deformation analyses of generalized materials with not only hardening but also softening behaviours.

In this paper, the two- and the multi-surface theories are to be formulated in mathematically rational forms, especially deriving the nonintersection condition of the surfaces and assuming the reasonable measure of distance from the distinct-yield state in the two-surface theory.

#### BASIC CONSTITUTIVE EQUATIONS IN THE DISTINCT-YIELD STATE

A typical stress/strain curve of elastoplastic materials is schematically illustrated in Figure 1. First, we assume that the distinct-yield surface, which represents stresses in the distinct-yield state shown by the envelope curve of reloading curves, is described by the following equation.

$$f(\hat{\sigma}) - F(K) = 0, \quad (1)$$

where we set

$$\hat{\sigma} \equiv \mathbf{u} - \mathbf{a}. \quad (2)$$

The second-order tensor  $\sigma$  is a stress, and the scalar  $K$  and the second-order tensor  $\hat{\alpha}$  are parameters to describe, respectively, the expansion/contraction and the translation of the surface. For simplicity, we assume that the distinct-yield surface retains a similarity in a stress space. Therefore, the function  $f$  is to be a homogeneous function of its arguments. Then, let the

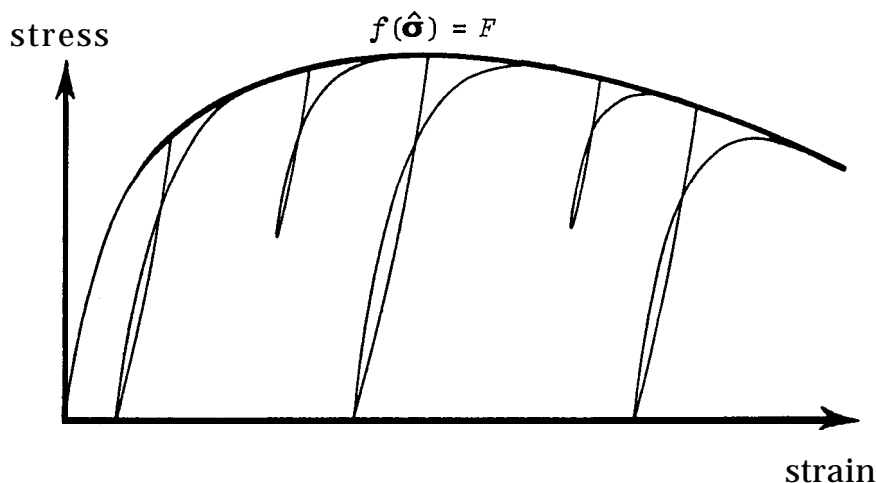


Fig 1. The distinct-yield state illustrated as an envelope curve of reloading curves.

degree of  $f$  be denoted by  $n$ .

Let  $\dot{K}$  where a superposed dot designates a material time derivative be a function of stress, plastic strain and plastic strain rate  $\dot{\epsilon}^p$  in degree one, which satisfies the condition  $\dot{K}=0$  when  $\dot{\epsilon}^p=0$ .

Further, let  $\hat{\alpha}$  be given as

$$\hat{\alpha} = A \dot{\epsilon}^p \mathbf{I} + B \operatorname{tr} \left( \dot{\epsilon}^p \frac{\hat{\sigma}}{|\hat{\sigma}|} \right) \frac{\hat{\sigma}}{|\hat{\sigma}|} \quad (3)$$

in accordance with Hashiguchi (1981), where

$$\dot{\epsilon}^p \equiv \operatorname{tr} \dot{\epsilon}^p \quad (4)$$

$A$  and  $B$  are scalar functions of  $K$  and  $\hat{\alpha}$  and the notation  $||$  is used to represent the magnitude.

By differentiating equation (1) and substituting equation (3) and the relation

$$\frac{\partial f}{\partial \hat{\sigma}} = \frac{nF}{\operatorname{tr}(\hat{n}\hat{\sigma})} \hat{n} \quad (5)$$

$$\hat{n} \equiv \frac{\frac{\partial f}{\partial \hat{\sigma}}}{\left| \frac{\partial f}{\partial \hat{\sigma}} \right|}, \quad (6)$$

we have the consistency condition

$$\operatorname{tr} \left[ \hat{n} \left\{ \dot{\sigma} - A \dot{\epsilon}^p \mathbf{I} - B \operatorname{tr} \left( \dot{\epsilon}^p \frac{\hat{\sigma}}{|\hat{\sigma}|} \right) \frac{\hat{\sigma}}{|\hat{\sigma}|} - \frac{1}{n} \frac{\dot{F}}{F} \hat{\sigma} \right\} \right] = 0. \quad (7)$$

Here, we assume the associated flow rule

$$\dot{\epsilon}^p = \langle \dot{\lambda} \rangle \hat{n}, \quad (8)$$

where  $\dot{\lambda}$  is a proportionality factor, and the symbol  $\langle \dot{\lambda} \rangle$  is MacCauley's bracket, that is, the operation  $\langle \dot{\lambda} \rangle = \dot{\lambda}$  when  $f(\hat{\sigma}) = F$  and  $\dot{\lambda} > 0$ , otherwise  $\langle \dot{\lambda} \rangle = 0$ . By substituting equation (8) into equation (7) the plastic strain rate is given as follows :

$$\dot{\epsilon}^p = \left\langle \frac{\operatorname{tr}(\hat{n}\dot{\sigma})}{\frac{1}{n} \operatorname{tr}(\hat{n}\hat{\sigma}) \frac{F'}{F} \hat{\kappa} + A \operatorname{tr}^2 \hat{n} + B \operatorname{tr}^2 \left( \hat{n} \frac{\hat{\sigma}}{|\hat{\sigma}|} \right)} \right\rangle \hat{n}, \quad (9)$$

where

$$F' \frac{dF}{dK}, \quad (10)$$

and  $\hat{\kappa}$  is a scalar function of stress, plastic strain and  $\hat{n}$  in degree one given

by

$$\hat{\kappa} \equiv \dot{K}/\dot{\lambda}, \quad (11)$$

### TWO-SURFACE MODEL

We introduce the subyield surface (see Figure 2) which is similar to the distinct-yield surface and translates within the distinct-yield surface. Here, assume that the current stress exists on or within the subyield surface and that the elastoplastic deformation can proceed when it exists on the subyield surface, but only the elastic deformation can proceed when it exists within the surface. Then, let the subyield surface be described by

$$f(\bar{\sigma}) - r^n F(K) = 0, \quad (12)$$

where we set

$$\bar{\sigma} \equiv \sigma - \bar{\mathbf{a}}, \quad (13)$$

$r(0 \leq r \leq 1)$  is a material constant and the second-order tensor  $\bar{\mathbf{a}}$  is a parameter to describe a translation of the surface.

Now, consider the subyield state  $f(\hat{\sigma}) < F$ . Let the conjugate point on the distinct-yield surface having the same outer-normal direction as that of the subyield surface at the current stress  $\sigma$  be denoted by  $\sigma_y$ . Namely, it holds that

$$\hat{\sigma}_y \equiv \sigma_y - \hat{\mathbf{a}}, \quad (14)$$

$$\hat{\mathbf{n}}_y = \bar{\mathbf{n}}, \quad (15)$$

where

$$\hat{\sigma}_y = \frac{1}{r} \bar{\sigma}, \quad (16)$$

$$\hat{\mathbf{n}}_y \equiv \frac{\frac{\partial f}{\partial \hat{\sigma}_y}}{\left| \frac{\partial f}{\partial \hat{\sigma}_y} \right|}, \quad (17)$$

$$\bar{\mathbf{n}} \equiv \frac{\frac{\partial f}{\partial \bar{\sigma}}}{\left| \frac{\partial f}{\partial \bar{\sigma}} \right|}. \quad (18)$$

Here, assume that the translation rule (3) of the distinct-yield surface holds in the subyield state, provided that the stress is replaced by the conjugate stress  $\sigma_y$ . Hence, noting the relation (16), we have

$$\hat{\mathbf{a}} = A \dot{\epsilon}_v^p \mathbf{I} + B \operatorname{tr} \left( \dot{\epsilon}^p \frac{\bar{\sigma}}{|\bar{\sigma}|} \right) \frac{\bar{\sigma}}{|\bar{\sigma}|}. \quad (19)$$

Now, we consider the formulation of  $\dot{\mathbf{a}}$ .

Since the subyield surface must exist within the distinct-yield surface, it must hold that

$$f(\hat{\sigma}_c) \leq F, \tag{20}$$

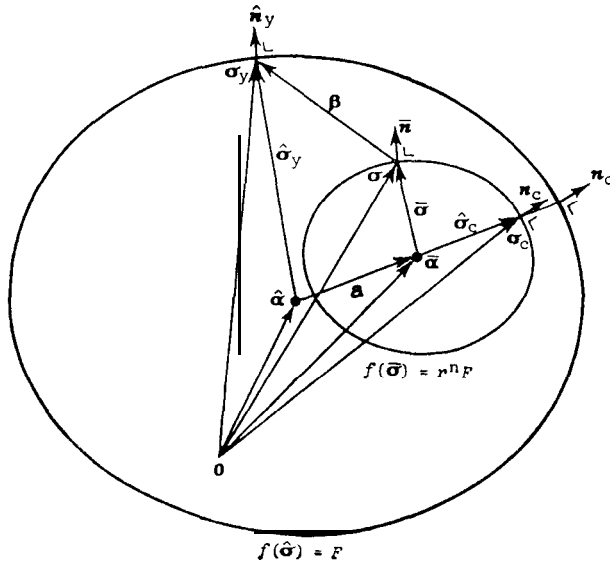


Fig. 2. The distinct-yield and the subyield surfaces;

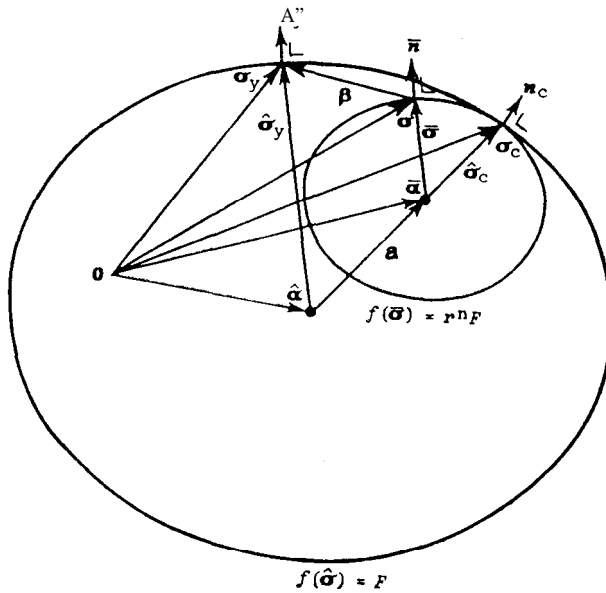


Fig. 3. The distinct-yield and the subyield surfaces in contact.

where

$$\hat{\sigma}_c \equiv \mathbf{a}, -\hat{\mathbf{a}}, \quad (21)$$

letting  $\sigma_c$  denote the intersecting point of the subyield surface and the half line starting from the point  $\hat{\mathbf{a}}$  and passing through the point  $\bar{\mathbf{a}}$  in the stress space (Figure 2).

Equation (20) can be written as

$$\text{tr} \left( \frac{\partial f}{\partial \sigma_c} \dot{\hat{\sigma}}_c \right) \leq \dot{F} \text{ when } f(\hat{\sigma}_c) = F \quad (22)$$

in a differential form. Let the condition which must be satisfied in order that the surfaces do not intersect but may contact with each other, such as equations (20) and (22), be called a "non-intersection condition".

In the state that the subyield surface contacts with the distinct-yield surface (Figure 3) it holds that

$$\hat{\sigma}_c = \frac{1}{1-r} \mathbf{a} = \frac{1}{r} \bar{\sigma}_c \text{ when } f(\hat{\sigma}_c) = F, \quad (23)$$

where

$$\mathbf{a} \equiv \bar{\mathbf{a}} - \hat{\mathbf{a}}, \quad (24)$$

$$\sigma_c \equiv \mathbf{a}, -\mathbf{a}. \quad (25)$$

By the relation (23), equation (22) can be expressed as

$$\text{tr} \left( \frac{\partial f}{\partial \mathbf{a}} \dot{\mathbf{a}} \right) \leq (1-r)^n \dot{F} \text{ when } f(\mathbf{a}) = (1-r)^n F. \quad (26)$$

Further, noting the relation

$$\frac{\partial f}{\partial \mathbf{a}} = \frac{n(1-r)^n F}{\text{tr}(\mathbf{n}_c \mathbf{a})} \mathbf{n}_c, \quad (27)$$

where

$$\mathbf{n}_c \equiv \frac{\frac{\partial f}{\partial \mathbf{a}}}{\left| \frac{\partial f}{\partial \mathbf{a}} \right|} = \frac{\frac{\partial f}{\partial \bar{\sigma}_c}}{\left| \frac{\partial f}{\partial \bar{\sigma}_c} \right|} = \frac{\frac{\partial f}{\partial \hat{\sigma}_c}}{\left| \frac{\partial f}{\partial \hat{\sigma}_c} \right|}, \quad (28)$$

the non-intersection condition (26) is written as

$$\text{tr} \left\{ \mathbf{n}_c \left( \dot{\mathbf{a}} - \frac{1}{n} \frac{\dot{F}}{F} \mathbf{a} \right) \right\} \leq 0 \text{ when } f(\mathbf{a}) = (1-r)^n F. \quad (29)$$

For equation (29) to be satisfied, we assume the following relation (Figure 3).

$$\dot{\mathbf{a}} - \frac{1}{n} \frac{\dot{F}}{F} \mathbf{a} = \dot{\mu} \boldsymbol{\beta}, \quad (30)$$

where  $\dot{\mu}$  ( $>0$ ) is a proportionality factor, and

$$\boldsymbol{\beta} \equiv \boldsymbol{\sigma}_y - \boldsymbol{\sigma} \quad (31)$$

which is equated as

$$\boldsymbol{\beta} = \frac{1}{r} \bar{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}} \quad (32)$$

by equations (13), (14) and (16).

From equation (30) we have

$$\dot{\mathbf{a}} = \hat{\mathbf{a}} + \frac{1}{n} \frac{\dot{F}}{F} \mathbf{a} + \dot{\mu} \boldsymbol{\beta}, \quad (33)$$

where  $\dot{\mu}$  is given as

$$\dot{\mu} = \frac{\text{tr}(\bar{\mathbf{n}}(\hat{\boldsymbol{\sigma}} - \frac{1}{n} \frac{\dot{F}}{F} \hat{\boldsymbol{\sigma}}))}{\text{tr}(\bar{\mathbf{n}} \boldsymbol{\beta})} \quad (34)$$

by substituting equation (33) into the consistency condition

$$\text{tr}\left\{\bar{\mathbf{n}}\left(\dot{\boldsymbol{\sigma}} - \frac{1}{n} \frac{\dot{F}}{F} \bar{\boldsymbol{\sigma}}\right)\right\} = 0 \quad (35)$$

which is derived by differentiating equation (12) and substituting the relation

$$\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} = \frac{n r^n F}{\text{tr}(\bar{\mathbf{n}} \bar{\boldsymbol{\sigma}})} \bar{\mathbf{n}} \quad (36)$$

In equation (34)  $\hat{\boldsymbol{\sigma}}$  is given by equation (2), letting a mean a current stress.

Finally, extending equation (9) to the subyield state  $\boldsymbol{\sigma} = \boldsymbol{\sigma}_y$ , we assume for the plastic strain rate to be given as follows:

$$\dot{\boldsymbol{\varepsilon}}^p = \left\langle \frac{\text{tr}(\hat{\mathbf{n}}_y \dot{\boldsymbol{\sigma}})}{\frac{1}{n} \text{tr}(\hat{\mathbf{n}}_y \hat{\boldsymbol{\sigma}}_y) \frac{F'}{F} \hat{\kappa}_y + A \text{tr}^2 \hat{\mathbf{n}}_y + B \text{tr}^2(\hat{\mathbf{n}}_y \frac{\hat{\boldsymbol{\sigma}}_y}{|\hat{\boldsymbol{\sigma}}_y|}) + H} \right\rangle \hat{\mathbf{n}}_y, \quad (37)$$

where  $H$  is monotonically increasing function of the scalar parameter

$$b \equiv \text{tr}\left(\hat{\mathbf{n}}_y \frac{\boldsymbol{\beta}}{F^{1/n}}\right), \quad (38)$$

satisfying the condition

$$b=0: H=0. \quad (39)$$

$\hat{\kappa}_y$  stands for a function given by replacing the argument  $\hat{\mathbf{n}}$  by  $\hat{\mathbf{n}}_y$  in the function  $\hat{\kappa}$ .



Equation (37) means that the magnitude of a plastic strain rate produced in the subyield state depends on the parameter  $b$ , that is, the projection of the vector  $\beta/F^{1/n}$  to the outer-normal direction of the subyield surface at  $a$ .

By substituting equations (15) and (16) into equations (37) and (38) we obtain

$$\dot{\epsilon}^p = \left\langle \frac{\text{tr}(\bar{n}\dot{\sigma})}{\frac{1}{nr} \text{tr}(\bar{n}\dot{\sigma}) \frac{F'}{F} \bar{\kappa} + A \text{tr}^2 \bar{n} + B \text{tr}^2 \left( \bar{n} \frac{\bar{\sigma}}{|\bar{\sigma}|} \right) + H} \right\rangle \bar{n}, \quad (40)$$

$$b = \frac{1}{F^{1/n}} \text{tr}(\bar{n}\beta), \quad (41)$$

where  $\bar{\kappa}$  stands for a function given by replacing the argument  $\hat{n}$  by  $\bar{n}$  in the function  $\hat{\kappa}$ . It is matter of course that equation (40) coincides with equation (9) when  $\sigma = \sigma_y$ .

Equation (40) has the form

$$\dot{\epsilon}^p = \langle \bar{\lambda} \rangle \bar{n}. \quad (42)$$

As was assumed earlier, only an elastic deformation occurs in the state of stress within the subyield surface. Further, assume that an elastic property of materials is not affected by the plastic deformation. On these assumptions the so-called postulate of maximum plastic work and thereby the associated flow rule and the convexity condition hold also for the subyield surface as was verified by Drucker (1951) for the conventional yield surface. Thus, the loading criterion for the subyield state is given as follows:

$$\langle \bar{\lambda} \rangle = \begin{cases} \dot{\bar{\lambda}} & \text{when } \mathbf{f}(\bar{\sigma}) = r^n F \text{ and } \bar{\lambda} > 0, \\ 0 & \text{when } \mathbf{f}(\bar{\sigma}) < r^n F \text{ or } \bar{\lambda} \leq 0. \end{cases} \quad (43)$$

#### MULTI-SURFACE MODEL

First, assume a set of nesting surfaces which are similar to each other (Figure 4), i.e.,

$$f_i(\bar{\sigma}_i) - r_i^n F(K) = 0: i=0, 1, \dots, m. \quad (44)$$

The surfaces for  $i=0$  and  $m$  correspond to the innermost surface enclosing the elastic domain and the distinct-yield surface, respectively. The constant  $r_i$  is a ratio of the size of the  $i$ -th surface to the distinct-yield surface, provided that

$$0 < r_i < 1 \text{ for } i=0, \dots, m-1; r_m = 1. \quad (45)$$

Now, let the current stress  $\sigma$  exist on the  $i$ -th surface. In the way

similar to that described in the formulation of two-surface theory, we obtain the non-intersection condition

$$\text{tr} \left\{ \bar{n}_i \left( \dot{\mathbf{a}}_i - \frac{1}{n} \frac{\dot{F}}{F} \mathbf{a}_i \right) \right\} \leq 0 \quad \text{when } f(\mathbf{a}_i) = \left( 1 - \frac{r_i}{r_{i+1}} \right)^n F, \quad (46)$$

where

$$\bar{n}_i^c = \frac{\frac{\partial f}{\partial \bar{\sigma}_i^c}}{\left| \frac{\partial f}{\partial \bar{\sigma}_i^c} \right|}, \quad (47)$$

$$\bar{\sigma}_i^c \equiv \sigma_i^c - \mathbf{a}_i, \quad (48)$$

$$\mathbf{a}_i \equiv \mathbf{a}_i - \mathbf{a}_{i+1}, \quad (49)$$

letting  $a$ : denote the intersecting point of the  $l$ -th surface and the half line stemming from point  $\mathbf{a}_{l+1}$  and passing through the point  $a$ .

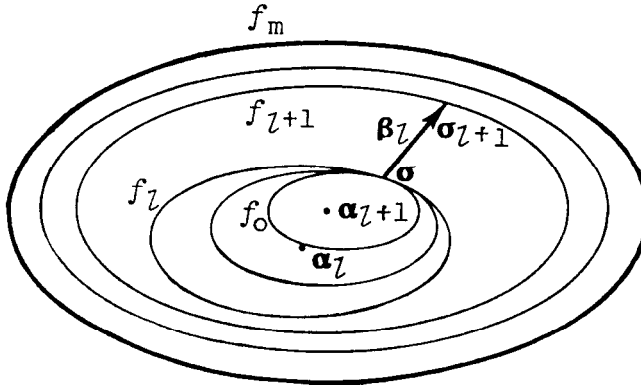


Fig. 4. Multi-surface model.

From equation (46)  $\dot{\mathbf{a}}_i$  is assumed to be given by

$$\dot{\mathbf{a}}_i = \dot{\mathbf{a}}_{i+1} + \frac{1}{n} \frac{\dot{F}}{F} \mathbf{a}_i + \dot{\mu} \beta_i, \quad (50)$$

$$\dot{\mu} = \frac{\text{tr} \left\{ \bar{n}_i \left( \dot{\sigma}_i - \frac{1}{n} \frac{\dot{F}}{F} \hat{\sigma}_i \right) \right\}}{\text{tr}(\bar{n}_i \beta_i)}, \quad (51)$$

where

$$\beta_i \equiv \sigma_{i+1} - \sigma = \frac{r_{i+1}}{r_i} \bar{\sigma}_i - \bar{\sigma}_{i+1}, \quad (52)$$

$$\bar{\mathbf{n}}_i \equiv \frac{\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}_i}}{\left| \frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}_i} \right|}, \quad (53)$$

$$\bar{\boldsymbol{\sigma}}_i \equiv \boldsymbol{\sigma} - \mathbf{a}_i, \quad (54)$$

letting  $\boldsymbol{\sigma}_{i+1}$  denote the conjugate point with the same outer-normal vector  $\bar{\mathbf{n}}_i$  on the  $(i+1)$ -th surface.

By the similarity of the surfaces,  $\mathbf{a}_i$  for the surfaces within the  $i$ -th surface are given by

$$\mathbf{a}_i = \left( -\frac{r_{i+1}}{r_i} \right) + \frac{r_{i+1}}{r_i} \mathbf{a}_i \text{ for } i=0, \dots, Z-1. \quad (55)$$

On the other hand, the outer surfaces ( $i=l+1, \dots, m$ ) should be assumed to retain their relative configurations, i.e.,

$$\dot{\mathbf{a}}_i = \dot{\hat{\boldsymbol{\alpha}}} + \frac{1}{n} \frac{F}{F'} (\mathbf{a}_i - \hat{\boldsymbol{\alpha}}) \text{ for } i=l+1, \dots, m, \quad (56)$$

while  $\hat{\boldsymbol{\alpha}}$  is given by (19) with the replacement of  $\bar{\boldsymbol{\sigma}}/|\bar{\boldsymbol{\sigma}}|$  to  $\bar{\boldsymbol{\sigma}}_i/|\bar{\boldsymbol{\sigma}}_i|$ , specifying  $\mathbf{a}_i$  by  $\hat{\boldsymbol{\alpha}}$ .

The plastic strain rate is given by defining piecewise hardening moduli in the interpolation rule, that is,

$$\dot{\boldsymbol{\epsilon}}^p = \left\langle \frac{\text{tr}(\bar{\mathbf{n}}_i \dot{\boldsymbol{\sigma}})}{H_y + (H_0 - H_y) \left( \frac{m-l}{m} \right)^\gamma} \right\rangle \bar{\mathbf{n}}_i, \quad (57)$$

$$H_y = \frac{1}{nr_i} \text{tr}(\bar{\mathbf{n}}_i \dot{\boldsymbol{\sigma}}_i) \frac{F'}{F} \bar{\kappa}_i + A \text{tr}^2 \bar{\mathbf{n}}_i + B \text{tr}^2 \left( \bar{\mathbf{n}}_i \frac{\bar{\boldsymbol{\sigma}}_i}{|\bar{\boldsymbol{\sigma}}_i|} \right), \quad (58)$$

where  $H_0$  and  $\gamma$  are material parameters, and  $\bar{\kappa}_i$  stands for a function given by replacing the argument  $\hat{\mathbf{n}}$  to  $\bar{\mathbf{n}}_i$  in the function  $\hat{\kappa}$ .

Further, assume the field of an infinite number of nesting surfaces the sizes of which range from the point (vanishing yield domain) to that of the distinct-yield surface. It was named an "infinite-surface model" by Mroz *et al.* (1979). In this model, hardening modulus could be given by

$$H = H_y + (H'_0 - H_y) \left\{ 1 - \left( \frac{f}{F} \right)^\nu \right\}, \quad (59)$$

where  $H'_0$  and  $\nu$  are material parameters. A salient feature of this model is the assumption of the "stress-reversal surface". New active loading surface expands contacting with the stress-reversal surface at the stress reversal point. Thereby, the translation rule is simplified considerably. Further, it needs to memorize only the active loading, the distinct-yield and the stress-reversal surfaces. In case of cyclic loading with decreasing amplitudes, many

stress-reversal surfaces need, however, to be memorized.

### COMMENTS ON PAST FORMULATIONS

Main differences of the formulations of this paper from the past ones are as follows :

1) Translation rule of surfaces

In the multi-surface model by Mroz (1967), it was assumed a *priori* that

$$\dot{\bar{\alpha}} = \dot{\mu}_1 \beta (\dot{\mu}_1 > 0), \quad (60)$$

However, the other equation

$$\dot{\beta} = \dot{\mu}_2 \beta (\dot{\mu}_2 > 0) \quad (61)$$

has been used after Mroz *et al.* (1978). Further, in the two-surface model, Dafalias and Popov (1975) proposed the equation

$$\dot{\bar{\alpha}} = \dot{\hat{\alpha}} + \dot{\mu}_3 \beta (\dot{\mu}_3 > 0). \quad (62)$$

These equations do not satisfy generally the non-intersection condition (29). According to them, the inner surface would protrude from the outer one in a softening state  $\dot{F} < 0$ .

2) Measure for a distance from the distinct-yield state in the two-surface model

Dafalias and Popov (1975) and Mroz *et al.* (1979) adopted  $|\beta|$  as the measure to represent the degree of distance from the distinct-yield state (Figure 5). On the other hand, the measure  $b$  prescribed in equation (41) is not the

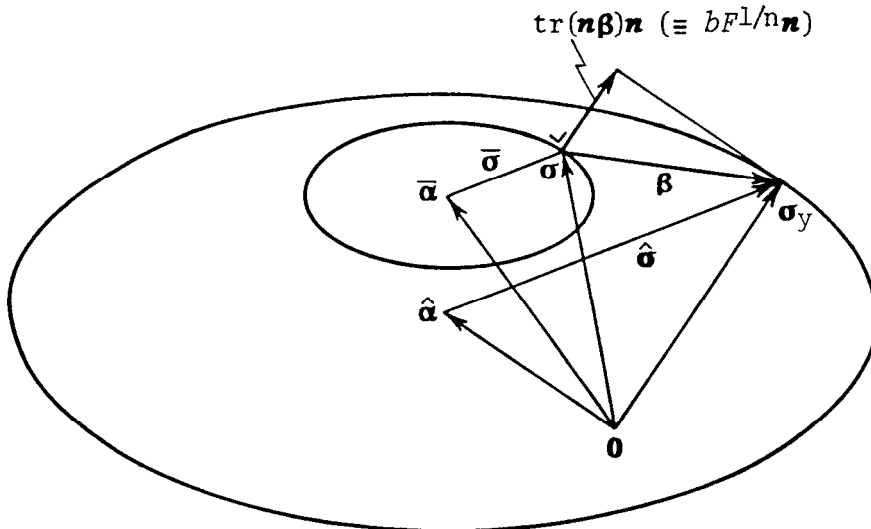


Fig. 5. Measures of the degree of distance from the distinct-yield state.

magnitude of  $\beta$  itself but its projection to the outer-normal direction of the subyield surface at the current stress (Figure 5). The latter would be more reasonable as that measure.

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