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## Exact Analytical Solutions to the Extension of Schumann's Theory on the Heat Transfer in Packed Bed

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An exact analytical solution is added to the author's previous work on the transient heat transfer in the packed bed of the particles of low thermal diffusivity. The solution has a simple integral form compared with the previous complicated series expression. The proofs of the convergence of the solution to the Schumann's at the zero Biot number are given. and a unique solution at the infinite Biot number **is** introduced.

### INTRODUCTION

The author (Murata, 1971) proposed an extension of Schumann's theory (Schumann, 1929) on the transient conduction of heat between gas and particles in packed bed. In that work the approximate analytical solutions similar to Furnas' solutions (Furnas, 1930), and the computing procedure of the rigorous series solution were given.

In this paper the author adds an exact solution of integral form and some supplemental proofs.

### THE LAPLACE TRANSFORM OF EXACT SOLUTION

The basic equations have been given by the author as follows:

$$\frac{\partial t_G}{\partial \xi} = -\frac{1}{\varepsilon}(t_G - T_s(a)), \tag{1}$$

$$t_s(a) = t_G - (2ah') \sum_{n=1}^{\infty} \frac{\sin^2 \beta_n}{\beta_n^2} \frac{\beta_n^2 + (ah' - 1)^2}{\beta_n^2 + ah'(ah' - 1)} \left( t_G - \frac{\beta_n^2}{\beta_1^2} \exp(-(\beta_n^2/\beta_1^2)\tau) \int_0^\tau t_G \exp((\beta_n^2/\beta_1^2)\lambda) d\lambda \right). \tag{2}$$

Eq. (2) is simply expressed by Duhamel's theorem as follows:

$$t_s(a) = \int_0^\tau t_G(\lambda) \frac{\partial}{\partial \tau} TS(\tau - \lambda) d\lambda, \tag{3}$$

where  $TS(\tau)$  is a surface temperature at the time  $\varepsilon$  in which radiation takes

place into a medium at temperature unity.

This  $TS(\tau)$  is known, and hence its Laplace transform can be derived directly from

$$\frac{\beta_1^2}{a^2} \frac{\partial TS}{\partial \tau} = \frac{\partial^2 TS}{\partial r^2} + \frac{1}{r} \frac{\partial TS}{\partial r}, \tag{4}$$

$$TS=0, \quad \tau=0, \quad 0 \leq r \leq a, \tag{5}$$

$$\frac{\partial TS}{\partial r} + h'(TS-1)=0, \quad r=a, \quad t \geq 0. \tag{6}$$

That is

$$FS = \frac{1}{s} \beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} + (ah' - 1). \tag{7}$$

Then, by convolution theorem, we have

$$F_s(a) = F_G(sFS) = F_G \frac{(ah')}{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} + (ah' - 1)}. \tag{8}$$

The Laplace transform of Eq.(1) is

$$\frac{\partial F_G}{\partial \xi} = -\frac{1}{\varepsilon} (F_G - F_s(a)). \tag{9}$$

Thus, by eliminating  $F_s$  from Eqs. (8) and (9), we have

$$\frac{\partial F_G}{\partial \xi} = \frac{1}{\varepsilon} \frac{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} - 1}{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} + (ah' - 1)} F_G, \tag{10}$$

and the solution at the inlet gas temperature unity as follows:

$$F_G = \frac{1}{s} \exp\left(-\frac{\xi}{\varepsilon} \frac{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} - 1}{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} + (ah' - 1)}\right) \tag{11}$$

### ANALYTICAL SOLUTION

The singularities except zero of Eq. (11) all lie to the left of the imaginary axis, so that the gas temperature  $t_G$  is determined by using the inversion theorem for the Laplace transformation as follows:

$$t_G = \frac{1}{2\pi i} \int_{Br} e^{sr} F_G ds = \frac{1}{2\pi i} \int_{r_1 r_2 r_3} e^{sr} F_G ds = I_1 + I_2 + I_3, \tag{12}$$

where  $r_1, r_2, r_3$  are the locuses of integration illustrated in Fig. 1 and  $I_1, I_2, I_3$  are the integrations along those locuses.

The limit value of the integrand at  $s=0$  is

$$\lim_{s \rightarrow 0} s(e^{sr} F_G) = 1. \tag{13}$$

Then we have

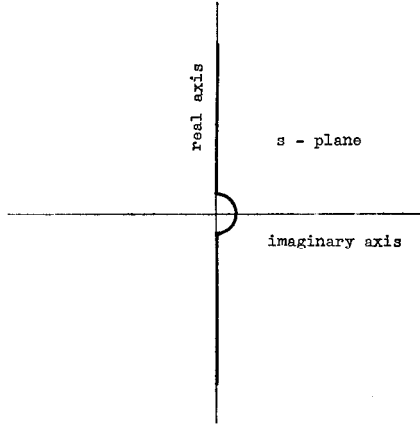


Fig. 1. The locus of integration.

$$I_2 = \frac{1}{2} . \tag{14}$$

The integrand is an analytic function and has a real value at origin as Eq. (13). Then, using the Image by inversion principle, we have

$$t_G = I_1 + I_2 + I_3 = 2I_3 + \frac{1}{2} = \frac{1}{2} + \frac{1}{\pi i} \int_{r_3} \mathcal{R}(e^{s^* F_G}) ds . \tag{15}$$

Substituting Eq. (11) into Eq. (15) with putting

$$s = i\lambda , \tag{16}$$

the following equation is obtained.

$$t_G = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathcal{R} \left( \frac{1}{i\lambda} \exp \left( -\frac{\xi}{\epsilon} \frac{\beta_1 \sqrt{i\lambda} \coth \beta_1 \sqrt{i\lambda} - 1}{\beta_1 \sqrt{i\lambda} \coth \beta_1 \sqrt{i\lambda} + (ah' - 1)} + i\tau\lambda \right) \right) d\lambda . \tag{17}$$

We know

$$\beta_1 \sqrt{i\lambda} = \beta_1 \sqrt{\lambda/2} (1+i) , \tag{18}$$

$$\coth \beta_1 \sqrt{i\lambda} = \frac{\sinh \beta_1 \sqrt{2\lambda} - i \sin \beta_1 \sqrt{2\lambda}}{\cosh \beta_1 \sqrt{2\lambda} - \cos \beta_1 \sqrt{2\lambda}} . \tag{19}$$

Then, Eq. (17) becomes

$$t_G = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty M d\lambda , \tag{20}$$

where

$$M = \frac{1}{\lambda} \sin(M_1 + \tau\lambda) \exp M_2 , \tag{21}$$

$$M_1 = -\frac{\xi}{\varepsilon} \frac{ah'(\beta_1\sqrt{\lambda/2})(\cosh\beta_1\sqrt{2\lambda} - \cos\beta_1\sqrt{2\lambda})}{((ah' - 1)(\cosh\beta_1\sqrt{2\lambda} - \cos\beta_1\sqrt{2\lambda}) + \beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda} + \sin\beta_1\sqrt{2\lambda} - \sin\beta_1\sqrt{2\lambda} - \sin\beta_1\sqrt{2\lambda}))^2 + \beta_1^2\lambda(\sinh\beta_1\sqrt{2\lambda} - \sin\beta_1\sqrt{2\lambda})^2/2} \tag{22}$$

$$M_2 = \frac{\xi}{\varepsilon} \frac{(\cosh\beta_1\sqrt{2\lambda} - \cos\beta_1\sqrt{2\lambda} - \beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda} + \sin\beta_1\sqrt{2\lambda}))}{((ah' - 1)(\cosh\beta_1\sqrt{2\lambda} - \cos\beta_1\sqrt{2\lambda}) + ((ah' - 1)(\csc\cos\beta_1\sqrt{2\lambda}) + \beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda} + \beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda} + \sin\beta_1\sqrt{2\lambda}))^2 + \beta_1^2\lambda(\sinh\beta_1\sqrt{2\lambda} - \sin\beta_1\sqrt{2\lambda}) - \beta_1^2\lambda(\sinh\beta_1\sqrt{2\lambda} - \sin\beta_1\sqrt{2\lambda})^2/2) \tag{23}$$

NUMERICAL CALCULATION OF EQ. (20)

Changing the right side of Eq.(20), we have

$$I_4 = \int_0^{\infty} Md\lambda = \int_{c \sin}^{\infty} S(\lambda)d\lambda + \int_0^{c \sin} S(\lambda)d\lambda + \int_{c \cos}^{\infty} C(\lambda)d\lambda + \int_0^{c \cos} C(\lambda)d\lambda = I_{\sin} + I_{c \sin} + I_{\cos} + I_{c \cos} \tag{24}$$

where

$$S(\lambda) = \frac{1}{\lambda} (\cos M_1 \exp(M_2)) \sin\tau\lambda, \\ C(\lambda) = \frac{1}{\lambda} (\sin M_1 \exp(M_2)) \cos\tau\lambda.$$

For numerical integration of the first and third terms, the following formula is introduced:

$$Z = \int_c^{\infty} f(\lambda) \sin(\tau\lambda + \sigma) d\lambda = h(\gamma f(c) \cos(\tau c + \sigma) + \delta (\sum_{j=0}^N f(\lambda_j) \sin(\tau\lambda_j + \sigma) - 0.5f(c) \sin(\tau c + \sigma))) + R_n + R_N \tag{25}$$

where

$$\gamma = \frac{\theta - \sin\theta}{\theta^2}, \quad \delta = \frac{2(1 - \cos\theta)}{\theta^2},$$

$\theta = \tau h$ ,  $h =$ divided interval of  $\lambda$  for numerical integration.

This formula is derived by fitting linear equation on  $f(\lambda)$ , and has simpler form and better convergency than Filon's method (Filon, 1929).

In this case  $f(\lambda)$  is given by

$$f_{\sin} = \frac{1}{\lambda} \cos M_1 \exp(M_2), \tag{26}$$

$$f_{\cos} = \frac{1}{\lambda} \sin M_1 \exp(M_2), \tag{27}$$

and, for sufficiently large  $\beta_1\sqrt{2\lambda_N}$ ,  $R_N$  is given by

$$R_{N\sin} \sim e^{-\frac{\xi}{\varepsilon}} / (\tau\lambda_N), \quad R_{N\cos} \sim e^{-\frac{\xi}{\varepsilon}} \frac{ah'}{\sqrt{2\varepsilon\beta_1\tau}} / (\lambda_N^{\frac{3}{2}}), \tag{28}$$

and, for  $ah' = \infty$

$$R_{N\sin} \sim R_{N\cos} \sim 2e^{\frac{\xi}{2}} E_i \left( -\frac{\xi}{2\sqrt{2}} \beta_1\sqrt{\lambda_N} \right). \tag{29}$$

$R_h$  may be estimated in the process of repeating computation by decreasing the value of  $h$ .

The second and fourth terms of Eq. (24) are computed by Simpson's method. The value of  $c_{\sin}$  and  $c_{\cos}$  are  $1.92645/r$  and  $0.617072/r$  which are determined as the smallest root of

$$\text{si}(c_{\sin})=0, \quad \text{ci}(c_{\cos})=0. \tag{30}$$

Then whole computing error of Eq. (20) becomes

$$R_{\text{whole}} = (R_{N\sin} + R_{N\cos} + R_{h\sin} + R_{h\cos} + R_{c\sin} + R_{c\cos})/\pi. \tag{31}$$

#### DIRECT NUMERICAL INVERSION OF EQ. (11)

Bellman et al. (1966) have shown the numerical inversion of the Laplace transform as follows :

$$t_G(\tau_i = -\phi \ln w_i) = \left( \sum_{j=1}^N a_j F(j/\phi) \right) / \phi, \tag{32}$$

where  $-\ln w_i = 0.693147$ ,  $a_{i1} = 1.857$ ,  $a_{i2} = -5.2.5$ ,  $a_{i3} = 288.75$ ,  $a_{i5} = 236.25$  in the case of  $N=5$  and  $i=3$ .

Then, taking  $N=5$  and  $i=3$ , we have

$$t_G(\tau) = (0.693147/r) \sum_{j=1}^5 a_{3j} F(0.693147j/\tau). \tag{33}$$

#### THE VALUES OF THE VARIABLES AND FUNCTION AT $ah' = 0$

The minimum root of

$$\beta \cot \beta - 1 = 0 \tag{34}$$

is given as

$$\beta_1 = 0. \tag{35}$$

Then, in neighbourhood of  $ah' = 0$ , we have

$$\beta_1 \left( \frac{1 - \beta_1^2/3}{\beta_1^2} \right) + ah' - 1 = 0. \tag{36}$$

From this, we have

$$\lim_{ah' \rightarrow 0} \beta_1^2/ah' = 3. \tag{37}$$

Then, we have

$$\begin{aligned} \lim_{ah' \rightarrow 0} \tau &= \lim_{ah' \rightarrow 0} (\kappa \beta_1^2/a^2) \theta' \\ &= \lim_{ah' \rightarrow 0} ((h/c_s \rho_s) (\beta_1^2/ah')/a) \theta' \\ &= ((h/c_s \rho_s) (3/a)) \theta' \\ &= ((h/c_s \rho_s) (A/V)) \theta' \\ &= ((h/c_s \rho_s) a_v / (1 - F_e)) \theta', \end{aligned} \tag{38}$$

$$\begin{aligned} \lim_{ah' \rightarrow 0} \varepsilon &= \lim_{ah' \rightarrow 0} (2ah') \frac{\sin^2 \beta_1}{\beta_1^2} \frac{\beta_1^2 + (ah' - 1)^2}{\beta_1^2 + ah'(ah' - 1)} \\ &= \lim_{ah' \rightarrow 0} 2 \left( \frac{\sin \beta_1}{\beta_1} \right)^2 \frac{\beta_1^2 + (ah' - 1)^2}{\beta_1^2/(ah') + (ah' - 1)} \\ &= 2 \times 1 \times \frac{0 + 1}{3 - 1} \\ &= 1, \end{aligned} \tag{39}$$

$$\begin{aligned} \lim_{ah' \rightarrow 0} F_c &= \lim_{ah' \rightarrow 0} \frac{1}{s} \exp \left( - \frac{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} - 1}{\beta_1 \sqrt{s} \coth \beta_1 \sqrt{s} + (ah' - 1)} \right) \\ &= \lim_{ah' \rightarrow 0} \frac{1}{s} \exp \left( - \frac{(\beta_1 \sqrt{s})^2/3}{(\beta_1 \sqrt{s})^2/3 + ah'} \right) \\ &= \lim_{ah' \rightarrow 0} \frac{1}{s} \exp \left( - \frac{s(\beta_1^2/(ah'))/3}{s(\beta_1^2/(ah'))/3 + 1} \right) \\ &= \frac{1}{s} \exp \left[ - \frac{s}{s+1} \right]. \end{aligned} \tag{40}$$

The last result is identical with the Laplace transform of the Furnas' solution.

### THE VALUES OF THE VARIABLES AND FUNCTION AS $ah'$ APPROACHES INFINITY

In a similar way as in the above mentioned procedure, we get the limits as  $ah'$  approaches infinity in finite thermal diffusivity or finite heat transfer coefficient.

The conditions for  $ah'$  to approach infinity are

$$\lim_{ah' \rightarrow \infty} \beta_1 = \pi, \tag{41}$$

$$\lim_{ah' \rightarrow \infty} (ah') \left( \frac{\sin \beta_1}{\beta_1} \right) = 1, \tag{42}$$

$$\lim_{ah' \rightarrow \infty} (ah') \varepsilon = 2, \tag{43}$$

$$\lim_{ah' \rightarrow \infty} \tau = \kappa \left( \frac{\pi}{a} \right)^2 \theta', \tag{44}$$

$$\lim_{ah' \rightarrow \infty} \xi = (2Ka_V) Z / (aC_G \rho_G F_s u), \tag{45}$$

$$\lim_{ah' \rightarrow \infty} F_G = e^{\frac{\xi}{2}} \frac{1}{s} \exp \left( -\frac{\xi}{2} (\pi \sqrt{s}) \coth(\pi \sqrt{\xi}) \right), \tag{46}$$

$$\lim_{ah' \rightarrow \infty} M_1 = -\frac{\xi}{4} \pi \sqrt{2\lambda} (\sinh \pi \sqrt{2\lambda} - \sin \pi \sqrt{2\lambda}) / (\cosh \pi \sqrt{2\lambda} - \cos \pi \sqrt{2\lambda}), \tag{47}$$

$$\lim_{ah' \rightarrow \infty} M_2 = -\frac{\xi}{4} (\pi \sqrt{2\lambda} (\sinh \pi \sqrt{2\lambda} + \sin \pi \sqrt{2\lambda}) / (\cosh \pi \sqrt{2\lambda} - \cos \pi \sqrt{2\lambda}) - 2). \tag{48}$$

In this case the surface temperature coincides with the gas temperature of particles.

We get also similar results by means of convolution theorem as follows:

$$F_s(r) = F_G \frac{(ah') \sinh(\beta_1 \sqrt{s}(r/a))}{(\beta_1 \sqrt{s}) \cosh \beta_1 \sqrt{s} + (ah' - 1) \sinh \beta_1 \sqrt{s}} \frac{a}{r}. \tag{49}$$

### NOMENCLATURE

- Br* : Bromwich integral
- E<sub>i</sub>* : Exponential integral
- F<sub>G</sub>* : Laplace transform of *t<sub>G</sub>*
- F<sub>s</sub>(r)* : Laplace transform of *t<sub>s</sub>(r)*
- FS* : Laplace transform of *TS*
- I<sub>1</sub>, J<sub>2</sub>, J<sub>3</sub>* : Integral defined by Eq. (12)
- M* : Defined by Eq. (21)
- M<sub>1</sub>* : Defined by Eq. (22)
- M<sub>2</sub>* : Defined by Eq. (23)
- s* : Parameter of Laplace transformation
- si* : Sine integral
- TS* : Defined by Eq. (4), Eq. (5) and Eq. (6) [°C]
- Γ<sub>1</sub>, Γ<sub>2</sub>, Γ<sub>3</sub>* : Loci of integration in Eq. (12)
- λ* : Variable of integration

! For the others, refer to the author's previous paper (Murata, 1971).



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