Exact Analytical Solutions to the Extension of Schumann's Theory on the Heat Transfer in Packed Bed

Murata, Satoshi
Laboratory of Agricultural Process Engineering, Faculty of Agriculture, Kyushu University

https://doi.org/10.5109/23768

出版情報：九州大学大学院農学研究院紀要. 27 (3/4), pp.151-158, 1983-02. Kyushu University

J. Fac. Agr., Kyushu Univ., 27 (3 • 4), 151-158 (1983)

Exact Analytical Solutions to the Extension of Schumann's Theory on the Heat Transfer in Packed Bed

Satoshi Murata

Laboratory of Agricultural Process Engineering, Faculty of Agriculture, Kyushu University 46-05, Fukuoka 812

(Received October 6, 1982)

An exact analytical solution is added to the author's previous work on the transient heat transfer in the packed bed of the particles of low thermal diffusivity. The solution has a simple integral form compared with the previous complicated series expression. The proofs of the convergence of the solution to the Schumann's at the zero Biot number are given. and a unique solution at the infinite Biot number is introduced.

INTRODUCTION

The author (Murata, 1971) proposed an extension of Schumann's theory (Schumann, 1929) on the transient conduction of heat between gas and particles in packed bed. In that work the approximate analytical solutions similar to Furnas' solutions (Furnas, 1930), and the computing procedure of the rigorous series solution were given.

In this paper the author adds an exact solution of integral form and some supplemental proofs.

THE LAPLACE TRANSFORM OF EXACT SOLUTION

The basic equations have been given by the author as follows:

$$\frac{\partial t_G}{\partial \xi} = -\frac{1}{\varepsilon}(t_G - T_S(a)), \tag{1}$$

$$t_S(a) = t_G - (2ah')\sum_{n=1}^{\infty}\frac{\sin^2\beta_n}{\beta_n^2}\,\frac{\beta_n^2+(ah'-1)^2}{\beta_n^2+ah'(ah'-1)}\Big(t_G - \frac{\beta_n^2}{\beta_1^2}\exp(-(\beta_n^2/\beta_1^2)\tau)\int_0^\tau t_G\exp((\beta_n^2/\beta_1^2)\lambda)\mathrm{d}\lambda\Big)\ . \tag{2}$$

Eq. (2) is simply expressed by Duhamel's theorem as follows:

$$t_S(a) = \int_0^\tau t_G(\lambda)\frac{\partial}{\partial\tau}TS(\tau-\lambda)\mathrm{d}\lambda\,, \tag{3}$$

where $TS(\tau)$ is a surface temperature at the time e in which radiation takes place into a medium at temperature unity.

This $TS(\tau)$ is known, and hence its Laplace transform can be derived directly from

$$\frac{\beta_1^2}{a^2}\frac{\partial TS}{\partial \tau}=\frac{\partial^2 TS}{\partial r^2}+\frac{1}{r}\frac{\partial TS}{\partial r}, \tag{4}$$

$$TS=0, \quad \tau=0,\ 0\leqq r\leqq a, \tag{5}$$

$$\frac{\partial TS}{\partial r}+h'(TS-1)=0, \quad r=a, \quad t\geqq 0. \tag{6}$$

That is

$$FS=\frac{1}{s}\frac{ah'}{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}+(ah'-1)}. \tag{7}$$

Then, by convolution theorem, we have

$$F_S(a)=F_G(sFS)=F_G\frac{(ah')}{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}+(ah'-1)}. \tag{8}$$

The Laplace transform of Eq.(1) is

$$\frac{\partial F_G}{\partial \xi}=-\frac{1}{\varepsilon}(F_G-F_S(a)). \tag{9}$$

Thus, by eliminating F_S from Eqs. (8) and (9), we have

$$\frac{\partial F_G}{\partial \xi}=-\frac{1}{\varepsilon}\frac{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}-1}{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}+(ah'-1)}F_G, \tag{10}$$

and the solution at the inlet gas temperature unity as follows:

$$F_G=\frac{1}{s}\exp\left(-\frac{\xi}{\varepsilon}\frac{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}-1}{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}+(ah'-1)}\right) \tag{11}$$

ANALYTICAL SOLUTION

The singularities except zero of Eq. (11) all lie to the left of the imaginary axis, so that the gas temperature t_G is determined by using the inversion theorem for the Laplace transformation as follows:

$$t_G=\frac{1}{2\pi i}\int_{Br} e^{s\tau}F_G ds=\frac{1}{2\pi i}\int_{\Gamma_1\Gamma_2\Gamma_3} e^{s\tau}F_G ds=I_1+I_2+I_3, \tag{12}$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ are the locuses of integration illustrated in Fig. 1 and I_1, I_2, I_3 are the integrations along those locuses.

The limit value of the integrand at $s=0$ is

$$\lim_{s\to 0} s(e^{s\tau}F_G) = 1. \tag{13}$$

Then we have

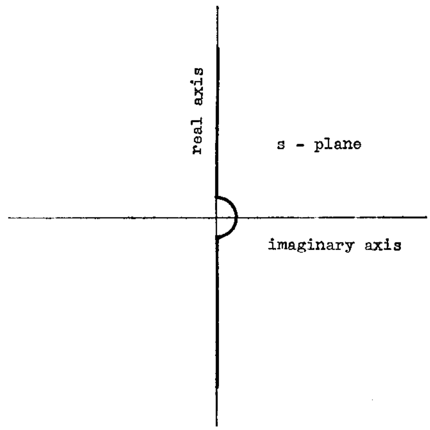

Fig. 1. The locus of integration.

$$I_2 = \frac{1}{2} . \tag{14}$$

The integrand is an analytic function and has a real value at origin as Eq. (13). Then, using the Image by inversion principle, we have

$$t_G = I_1 + I_2 + I_3 = 2I_3 + \frac{1}{2} = \frac{1}{2} + \frac{1}{\pi i}\int_{\Gamma_3} \mathscr{R}(e^{s\tau}F_G)\mathrm{d}s . \tag{15}$$

Substituting Eq. (11) into Eq. (15) with putting

$$s = i\lambda , \tag{16}$$

the following equation is obtained.

$$t_G = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathscr{R}\left(\frac{1}{i\lambda}\exp\left(-\frac{\xi}{\varepsilon}\frac{\beta_1\sqrt{i\lambda}\coth\beta_1\sqrt{i\lambda}-1}{\beta_1\sqrt{i\lambda}\coth\beta_1\sqrt{i\lambda}+(ah'-1)} + i\tau\lambda\right)\right)\mathrm{d}\lambda. \tag{17}$$

We know

$$\beta_1\sqrt{i\lambda} = \beta_1\sqrt{\lambda/2}(1+i), \tag{18}$$

$$\coth\beta_1\sqrt{i\lambda} = \frac{\sinh\beta_1\sqrt{2\lambda} - i\sin\beta_1\sqrt{2\lambda}}{\cosh\beta_1\sqrt{2\lambda} - \cos\beta_1\sqrt{2\lambda}} . \tag{19}$$

Then, Eq. (17) becomes

$$t_G = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty M\mathrm{d}\lambda , \tag{20}$$

where

$$M = \frac{1}{\lambda}\sin(M_1 + \tau\lambda)\exp M_2, \tag{21}$$

$$M_1 = -\frac{\xi}{\varepsilon}\,\frac{ah'(\beta_1\sqrt{\lambda/2})(\cosh\beta_1\sqrt{2\lambda}-\cos\beta_1\sqrt{2\lambda})(\sinh\beta_1\sqrt{2\lambda}-\sin\beta_1\sqrt{2\lambda})}{((ah'-1)(\cosh\beta_1\sqrt{2\lambda}-\cos\beta_1\sqrt{2\lambda})+\beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda}+\sin\beta_1\sqrt{2\lambda}))^2+\beta_1^2\lambda(\sinh\beta_1\sqrt{2\lambda}-\sin\beta_1\sqrt{2\lambda})^2/2}, \tag{22}$$

$$M_2 = \frac{\xi}{\varepsilon}\,\frac{(\cosh\beta_1\sqrt{2\lambda}-\cos\beta_1\sqrt{2\lambda}-\beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda}+\sin\beta_1\sqrt{2\lambda}))}{((ah'-1)(\cosh\beta_1\sqrt{2\lambda}-\cos\beta_1\sqrt{2\lambda})+}$$

$$\frac{((ah'-1)\quad(c\cos\beta_1\sqrt{2\lambda})+\beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda}+}{\beta_1\sqrt{\lambda/2}(\sinh\beta_1\sqrt{2\lambda}+\sin\beta_1\sqrt{2\lambda}))^2+\beta_1^2\lambda(\sinh\beta_1\sqrt{2\lambda}-}$$

$$\frac{\sin\beta_1\sqrt{2\lambda}))-\beta_1^2\lambda(\sinh\beta_1\sqrt{2\lambda}-\sin\beta_1\sqrt{2\lambda})^2/2}{\sin\beta_1\sqrt{2\lambda})^2/2} \tag{23}$$

NUMERICAL CALCULATION OF EQ. (20)

Changing the right side of Eq.(20), we have

$$I_4 = \int_0^\infty M\,\mathrm{d}\lambda = \int_{c\sin}^\infty S(\lambda)\mathrm{d}\lambda + \int_0^{c\sin} S(\lambda)\mathrm{d}\lambda + \int_{c\cos}^\infty C(\lambda)\mathrm{d}\lambda + \int_0^{c\cos} C(\lambda)\mathrm{d}\lambda$$
$$= I_{\sin} + I_{c\sin} + I_{\cos} + I_{c\cos}, \tag{24}$$

where

$$S(\lambda) = \frac{1}{\lambda}(\cos M_1 \exp(M_2))\sin\tau\lambda,$$

$$C(\lambda) = \frac{1}{\lambda}(\sin M_1 \exp(M_2))\cos\tau\lambda.$$

For numerical integration of the first and third terms, the following formula is introduced:

$$Z = \int_c^\infty f(\lambda)\sin(\tau\lambda+\sigma)\mathrm{d}\lambda$$
$$= h(\gamma f(c)\cos(\tau c+\sigma) + \delta(\sum_{j=0}^{N} f(\lambda_j)\sin(\tau\lambda_j+\sigma)$$
$$-0.5f(c)\sin(\tau c+\sigma))) + R_h + R_N, \tag{25}$$

where

$$\gamma = \frac{\theta-\sin\theta}{\theta^2}, \qquad \delta = \frac{2(1-\cos\theta)}{\theta^2},$$

$\theta = \tau h$, h=divided interval of λ for numerical integration.

This formula is derived by fitting linear equation on $f(\lambda)$, and has simpler form and better convergency than Filon's method (Filon, 1929).

In this case $f(\lambda)$ is given by

$$f_{\sin}=\frac{1}{\lambda}\cos M_1\exp(M_2), \tag{26}$$

$$f_{\cos}=\frac{1}{\lambda}\sin M_1\exp(M_2), \tag{27}$$

and, for sufficiently large $\beta_1\sqrt{2\lambda_N}$, R_N is given by

$$R_{N\sin}\sim e^{-\frac{\xi}{\varepsilon}}/(\tau\lambda_N),\quad R_{N\cos}\sim e^{-\frac{\xi}{\varepsilon}}\frac{ah'}{\sqrt{2\varepsilon\beta_1\tau}}/(\lambda_N^{\frac{3}{2}}), \tag{28}$$

and, for $ah'=\infty$

$$R_{N\sin}\sim R_{N\cos}\sim 2e^{\frac{\xi}{2}}\mathrm{E}_i\left(-\frac{\xi}{2\sqrt{2}}\beta_1\sqrt{\lambda_N}\right). \tag{29}$$

R_h may be estimated in the process of repeating computation by decreasing the value of *h.*

The second and fourth terms of Eq. (24) are computed by Simpson's method. The value of $c\sin$ and $c\cos$ are 1.92645/r and 0.617072/r which are determined as the smallest root of

$$\mathrm{si}(c\sin)=0,\quad \mathrm{ci}(c\cos)=0. \tag{30}$$

Then whole computing error of Eq. (20) becomes

$$R_{\text{whole}}=(R_{N\sin}+R_{N,,,}+R_{h\sin}+R_{h\cos}+R_{c\sin}+R_{c\cos})/\pi. \tag{31}$$

DIRECT NUMERICAL INVERSION OF EQ. (11)

Bellman et *al.* (1966) have shown the numerical inversion of the Laplace transform as follows :

$$t_G(\tau_i=-\psi\ln w_i)=(\sum_{j=1}^{N}a_{ij}\mathrm{F}(j/\psi))/\psi, \tag{32}$$

where $-\ln w_i=0.693147$, $a_{i1}=1.857$, $a_{,,}=-52.5$, $a_{i3}=288.75$, $a_{i5}=236.25$ in the case of $N=5$ and $i=3$.

Then, taking $N=5$ and $i=3$, *we* have

$$t_G(\tau)=(0.693147/\mathrm{r})\sum_{j=1}^{5}a_{3j}\mathrm{F}(0.693147j/\tau). \tag{33}$$

THE VALUES OF THE VARIABLES AND FUNCTION AT $ah'=0$

The minimum root of

$$\beta\cot\beta-1=0 \tag{34}$$

is given as

$$\beta_1=0. \tag{35}$$

Then, in neighbourhood of $ah'=0$, we have

$$\beta_1\left(\frac{1-\beta_1^2/3}{\beta_1^2}\right)+ah'-1=0. \tag{36}$$

From this, we have

$$\lim_{ah'\to 0}\beta_1^2/ah'=3. \tag{37}$$

Then, we have

$$\begin{aligned}\lim_{ah'\to 0}\tau&=\lim_{ah'\to 0}(\kappa\beta_1^2/a^2)\theta'\\&=\lim_{ah'\to 0}((h/c_S\rho_S)(\beta_1^2/ah')/a)\theta'\\&=((h/c_S\rho_S)(3/a))\theta'\\&=((h/c_S\rho_S)(A/V))\theta'\\&=((h/c_S\rho_S)a_V/(1-F_e))\theta',\end{aligned} \tag{36}$$

$$\begin{aligned}\lim_{ah'\to 0}\varepsilon&=\lim_{ah'\to 0}(2ah')\frac{\sin^2\beta_1^2}{\beta_1^2}\,\frac{\beta_1^2+(ah'-1)^2}{\beta_1^2+ah'(ah'-1)}\\&=\lim_{ah'\to 0}2\left(\frac{\sin\beta_1}{\beta_1}\right)^2\frac{\beta_1^2+(ah'-1)^2}{\beta_1^2/(ah')+(ah'-1)}\\&=2\times 1\times\frac{0+1}{3-1}\\&=1,\end{aligned} \tag{39}$$

$$\begin{aligned}\lim_{ah'\to 0}F_G&=\lim_{ah'\to 0}\frac{1}{s}\exp\left(-\frac{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}-1}{\beta_1\sqrt{s}\coth\beta_1\sqrt{s}+(ah'-1)}\right)\\&=\lim_{ah'\to 0}\frac{1}{s}\exp\left(-\frac{(\beta_1\sqrt{s})^2/3}{(\beta_1\sqrt{s})^2/3+ah'}\right)\\&=\lim_{ah'\to 0}\frac{1}{s}\exp\left(-\frac{s(\beta_1^2/(ah'))/3}{s(\beta_1^2/(ah'))/3+1}\right)\\&=\frac{1}{s}\exp\left(-\frac{s}{s+1}\right).\end{aligned} \tag{40}$$

The last result is identical with the Laplace transform of the Furnas' solution.

THE VALUES OF THE VARIABLES AND FUNCTION AS ah' APPROACHES INFINITY

In a similar way as in the above mentioned procedure, we get the limits as ah' approaches infinity in finite thermal diffusivity or finite heat transfer coefficient.

The conditions for ah' to approach infinity are

$$\lim_{ah' \to \infty} \beta_1 = \pi, \tag{41}$$

$$\lim_{ah' \to \infty} (ah') \left(\frac{\sin \beta_1}{\beta_1} \right) \text{ -1}, \tag{42}$$

$$\lim_{ah' \to \infty} (ah')\varepsilon = 2, \tag{43}$$

$$\lim_{ah' \to \infty} \tau = \kappa \left(\frac{\pi}{a} \right)^2 \theta', \tag{44}$$

$$\lim_{ah' \to \infty} \xi = (2Ka_V)Z/(aC_G\rho_G F_e u), \tag{45}$$

$$\lim_{ah' \to \infty} F_G = e^{\frac{\xi}{2}} \frac{1}{s} \exp\left(-\frac{\xi}{2} (\pi\sqrt{s}) \coth(\pi\sqrt{\xi}) \right), \tag{46}$$

$$\lim_{ah' \to \infty} M_1 = -\frac{\xi}{4} \pi\sqrt{2\lambda} (\sinh \pi\sqrt{2\lambda} - \sin \pi\sqrt{2\lambda}) / (\cosh \pi\sqrt{2\lambda} - \cos \pi\sqrt{2\lambda}), \tag{47}$$

$$\lim_{ah' \to \infty} M_2 = -\frac{\xi}{4} (\pi\sqrt{2\lambda} (\sinh \pi\sqrt{2\lambda} + \sin \pi\sqrt{2\lambda}) / (\cosh \pi\sqrt{2\lambda} - \cos \pi\sqrt{2\lambda}) \text{ -2}). \tag{48}$$

In this case the surface temperature coincides with the gas temperature of particles.

We get also similar results by means of convolution theorem as follows:

$$F_S(r) = F_G \frac{(ah')\sinh(\beta_1\sqrt{s}(r/a))}{(\beta_1\sqrt{s})\cosh\beta_1\sqrt{s} + (ah'-1)\sinh\beta_1\sqrt{s}} \frac{a}{r}. \tag{49}$$

NOMENCLATURE

Br : Bromwich integral
E_i : Exponential integral
F_G : Laplace transform of t_G
$F_s(r)$: Laplace transform of $t_S(r)$
FS : Laplace transform of *TS*
I_1, I_2, I_3 : Integral defined by Eq. (12)
M : Defined by Eq. (21)
M_1 : Defined by Eq. (22)
M_2 : Defined by Eq. (23)
s : Parameter of Laplace transformation
si : Sine integral
TS : Defined by Eq. (4), Eq. (5) and Eq. (6) [°C]
$\Gamma_1, \Gamma_2, \Gamma_3$: Locuses of integration in Eq. (12)
λ : Variable of integrvtion

For the others, refer to the author's previous paper (Murata, 1971).

REFERENCES

Bellmann, R., R. K. Kalaba and J. A. Locket 1966 *Inversion of Laplace Transform.* Elsevier, New York

Filon, L. N. G. 1929 On a quadrature for trigonometric integrals. *Proc. Soc. Edinburgh, 49: 38-47*

Furnas, C. C. 1930 Heat transfer from gas stream to a bed of broken solid. *Ind. Eng. Chem., 22: 721-731*

Murata, S. 1971 Extension of Schumann's theory to the case of low thermal diffusivity of solid particles. *J. Chem. Eng. Japan, 4: 140-146*

Schumann, T. E. W. 1929 Heat transfer--A liquid flowing through a porous prism-. *J. Franklin Inst.. 208* :405-416