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Hashiguchi, Koichi
Laboratory of Agricultural Machinery, Faculty of Agriculture, Kyushu University

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Koichi Hashiguchi

Laboratory of Agricultural Machinery, Faculty of Agriculture,
Kyushu University 46–05, Fukuoka 812

The associated flow rule in the traditional theory of plasticity has been recently adopted even in the plastic constitutive equations with the elastic-plastic transition. The proofs of this rule in the past, however, are confined to the classical idealization of elastoplastic continuum to ignore the elastic-plastic transition and the alteration in an elastic response. In this paper, based on two physically plausible assumptions, the associated flow rule is analytically derived for the generalized elastoplastic continuum with the elastic-plastic transition, the hardening, the perfectly-plastic and the softening behaviors and the alteration in an elastic response.

INTRODUCTION

Various constitutive equations describing a gradual transition from the elastic to the fully-plastic (yield) state (abbreviated as an **elastic-plastic transi-**tion) have been proposed along the line of the traditional theory of plasticity whose constitutive equation involves inherently a yield function (e.g. Mróz, 1967; Phillips and Sierakowski, 1965; Eisenberg and Phillips, 1971; Krieg, 1975 ; Dafalias and Popov, 1975; Hashiguchi, 1978, 1979 a, b). All these equations are premised on the so-called **associated** *flow* **rule.** Nothing, however, was mentioned as to whether or not this rule is applicable to such a material behavior without a physical contradiction. Formerly, Drucker (1951) advocated the applicability of this rule to hardening materials on the basis of the hypothesis concerning the net work done by an external agency during a closed stress cycle. And later Palmer, et **al.** (1967) persisted that this rule holds for wide classes of materials exhibiting not only the hardening but the perfectly-plastic and the softening behaviors, provided that the postulate of stability in the large (cf. Drucker, 1964) can reduce to the principle of maximum plastic work. This principle, however, holds only in the classical idealization of elastoplastic continuum to ignore the elastic-plastic transition and the alteration in an elastic response. On the other hand, based on the hypothesis concerning the work done during a closed strain cycle which is less restrictive than the Drucker's hypothesis, Il'ushin (1961) has concluded that this rule holds but for the alteration in an elastic response, while a linear relation between the stress and the elastic strain is assumed in his analysis. Recently, Naghdi and Trapp(1975 a, b), adopting the finite deformation theory by Green and Naghdi (1965), have elaborated the Il'ushin's approach and have obtained the

similar conclusion for the case of a linear relation between the stress rate
and the elastic strain rate. The elastic-plastic transition, however, is ignored
in these analyses. Against the above approaches, referring to the earlier
alternate approach by Prager (1949) in which any closed cycle of stress or
strain is not explicitly introduced, Phillips and Sierakowski (1965) have as-
serted the applicability of the associated flow rule even to an intermediate
loading surface assumed by themselves. But the Prager's approach is confined
to hardening materials satisfying the uniqueness condition (cf. Prager, 1949)
which may correspond to the postulate of stability in the small by Drucker
(1964). In this paper, based on physically plausible assumptions which can be
regarded as extensions of the Prager's conditions of uniqueness and continuity,
the associated flow rule is derived for the generalized elastoplastic continuum
with the elastic-plastic transition and the hardening, the perfectly-plastic and
the softening behaviors.

A DERIVATION OF THE ASSOCIATED FLOW RULE

Preliminarily consider the uniaxial loading of a bar. Observing the stress-
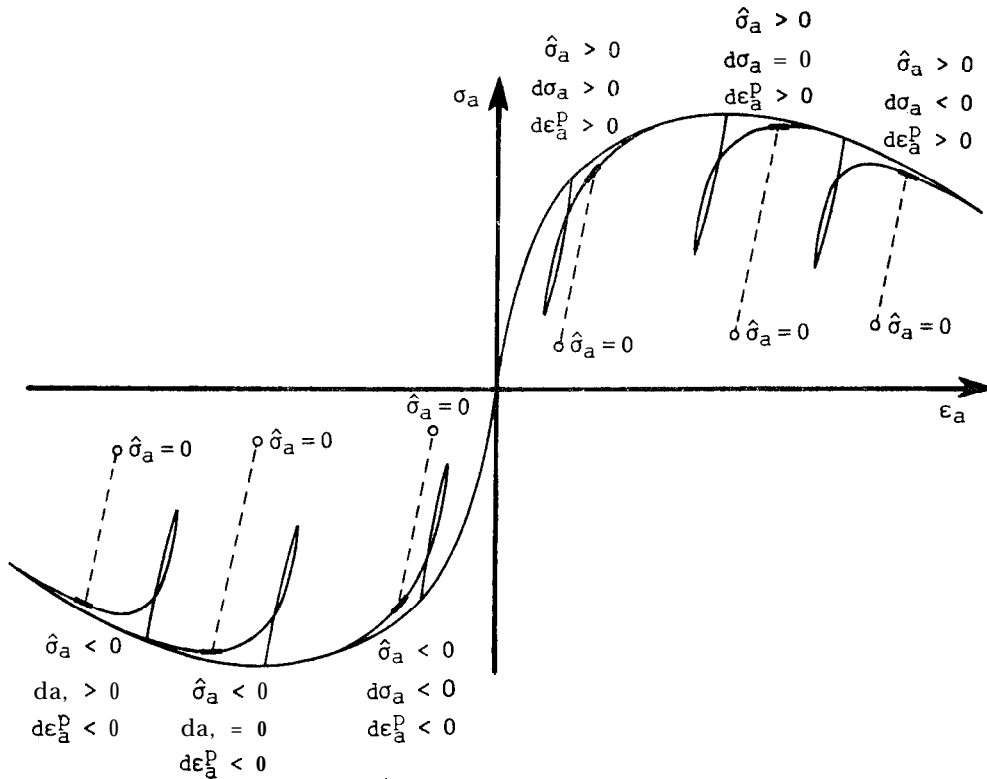strain curves schematically illustrated in Fig. 1, we could assume that in a

Fig. 1. Signs of $\hat{\sigma}_a, d\sigma_a$ and $d\varepsilon_a^p$ in the uniaxial loading,

loading state

$$\hat{\sigma}_a d\sigma_a > 0: \quad d\sigma_a d\varepsilon_a^p > 0,$$
$$\hat{\sigma}_a d\sigma_a = 0: \quad d\sigma_a d\varepsilon_a^p = 0, \tag{1}$$
$$\hat{\sigma}_a d\sigma_a < 0: \quad d\sigma_a d\varepsilon_a^p < 0,$$

and

$$d\sigma_a \to 0 \text{ as } d\varepsilon_a^p \to 0, \tag{2}$$

where

$$\hat{\sigma}_a \equiv \sigma_a - \alpha_a . \tag{3}$$

σ_a, α_a and ε_a^p are the normal components of the stress, the translation of load-ing surface and the strain along the axis respectively.

By extending the above notions to the general loading state, we introduce the following assumptions 1 and 2 from (1) and (2) respectively, in which the nine-dimensional stress space with superposed coordinates of stress components σ_{ij} and plastic strain increment components $d\varepsilon_{ij}^p$ is introduced.

Assumption 1. *When,* accompanying a plastic deformation (loading state), the state *of stress moves to the exterior, the tangency and the interior of the existing loading sur-face in a stress space, the plastic work done by the stress increment is positive, zero and negative respectively.*

Assumption 2. *As the plastic strain increment vanishes, the normal component of stress increment to the loading surface also vanishes.*

Assumption 1 can be written in analytical forms as follows.

$$tr\left(\frac{\partial f}{\partial \hat{\sigma}} d\sigma\right) > 0: \quad tr(d\sigma d\varepsilon^p) > 0,$$

$$tr\left(\frac{\partial f}{\partial \hat{\sigma}} d\sigma\right) = 0: \quad tr(d\sigma d\varepsilon^p) = 0, \tag{4}$$

$$tr\left(\frac{\partial f}{\partial \hat{\sigma}} d\sigma\right) < 0: \quad tr(d\sigma d\varepsilon^p) < 0$$

in the loading state, from which we obtain

$$tr(d\sigma d\varepsilon^p) = S tr\left(\frac{\partial f}{\partial \hat{\sigma}} d\sigma\right) \quad (S > 0), \tag{5}$$

where the second-order tensors $\sigma, d\varepsilon^p$ and $\mathbf{a}$ are the stress, the plastic strain increment and the parameter describing the translation of loading sur-face respectively. And we set

$$\hat{\sigma} \equiv \sigma - \alpha. \tag{6}$$

f is a yield function to describe the yield condition, which is a function of $\hat{\sigma}$ and some parameters representing the effect of a history of plastic deforma-

tion to the yield condition. S is a scalar function of stress and some internal variables. Assumption 1 can be regarded as the generalization of Prager's uniqueness condition for hardening materials ($tr(d\sigma d\varepsilon^p) > O$).

Assumption 2 can be expressed as

$$tr\left(\frac{\partial f}{\overset{\wedge}{\partial \sigma}}d\sigma\right) \to 0 \text{ as } d\varepsilon^p \to \mathbf{0}. \tag{7}$$

which is the inverse relation of Prager's continuity condition ($d\varepsilon^p \to \mathbf{0}$ as $tr(\partial f/\partial\sigma d\sigma) \to 0$) that does not hold in the loading state satisfying $tr(\partial f/\partial\sigma d\sigma) = 0$. In addition to assumption 2, we assume that there exists a linear relation between $d\varepsilon^p$ and da. Thus it can be written that

$$d\varepsilon^p = tr\left(\frac{\partial f}{\overset{\wedge}{\partial \sigma}}d\sigma\right)Q \quad (Q \rightleftharpoons \mathbf{0}), \tag{8}$$

where the second-order tensor Q is a function of stress and some internal variables. The relation (8) means that the direction of plastic strain increment is independent of the stress increment.

Substitution of (8) into (5) leads to

$$tr(Qd\sigma) = S(>0). \tag{9}$$

Then, noting that the stress increment can have any direction, it must hold that

$$Q = G\frac{\partial f}{\partial \sigma} , \tag{10}$$

with the condition

$$Gtr\left(\frac{\partial f}{\partial \sigma}d\sigma\right) > 0. \tag{11}$$

where G is a scalar function of stress and some internal variables. Thus, the plastic strain increment is given from (8)-(11) as follows:

$$d\varepsilon^p = d\lambda\frac{\partial f}{\partial \sigma} , \tag{12}$$

where

$$d\lambda \equiv Gtr\left(\frac{\partial f}{\partial \sigma}d\sigma\right)(>0) . \tag{13}$$

The equation (12) is to be the associated flow rule, while the restriction for $d\lambda$ to take the form (13) is imposed by assumption 2 and the linearity of the constitutive equation. Here, note that the convexity of the loading surface is not imposed in this approach.

DISCUSSION

In this paper the associated flow rule which is the most fundamental rule in the traditional theory of plasticity was analytically derived on the basis of assumptions 1 and 2. The above approach leads to the fact that this rule is applicable to the elastoplastic continuum with the elastic-plastic transition and the hardening, the perfectly-plastic and the softening behaviors independently of the elastic response. It refers to a more general elastoplastic continuum than those assumed in the past approaches. Here, remind that it is based on two assumptions which are generalization of distinct characteristics of the plastic deformation in the uniaxial loading. While these assumptions seem physically plausible, still it would be desirable to verify rationalities of them by any other fundamental principles on the plastic deformation.

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