Hirota's method and the search for integrable partial difference equations. 2. Equations on a 2×N stencil

Hietarinta, Jarmo
Department of Physics and Astronomy, University of Turku

Zhang, Da-jun
Department of Mathematics, Shanghai University

https://doi.org/10.15017/23389

出版情報：応用力学研究所研究集会報告. 22AO-S8 (5), pp.30-36, 2011-03. 九州大学応用力学研究所
バージョン：
権利関係：

応用力学研究所研究集会報告 No.22AO-S8
「非線形波動研究の新たな展開 — 現象とモデル化 —」（研究代表者　筧　三郎）
共催　九州大学グローバル COE プログラム
「マス・フォア・インダストリ教育研究拠点」

Reports of RIAM Symposium No.22AO-S8

Development in Nonlinear Wave: Phenomena and Modeling

Proceedings of a symposium held at Chikushi Campus, Kyushu Universiy,
Kasuga, Fukuoka, Japan, October 28 - 30, 2010

Co-organized by
Kyushu University Global COE Program
Education and Research Hub for Mathematics - for - Industry

Article No. 5 (pp. 30 - 36)

Hirota's method and the search for integrable partial difference equations. 2. Equations on a $2 \times N$ stencil

HIETARINTA　Jarmo（Hietarinta Jarmo），ZHANG Da-jun（Zhang Da-jun）

（Received 12 December 2011）

Research Institute for Applied Mechanics
Kyushu University
March, 2011

Hirota's method and the search for integrable partial difference equations. 2. Equations on a $2 \times N$ stencil

Jarmo Hietarinta[1*] and Da-jun Zhang[2†]

[1] *Department of Physics and Astronomy, University of Turku, FIN-20014 Turku, Finland*

[2] *Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China*

January 31, 2011

Abstract

The "direct method" proposed by R. Hirota in 1971 [1] is a very effective tool for constructing soliton solutions and has been applied to almost all integrable equations. In applying this method the condition of integrability appears when one tries to construct three-soliton solutions, whereas two-soliton solutions can be constructed even for non-integrable equations. Here we apply this method to fully discrete lattice equations defined on a $2 \times N$ stencil. It turns out that all the results obtained can also be obtained by reductions from the Hirota-Miwa equation. Thus the three-soliton condition is again found to give same results as other integrability criteria.

1 Introduction

It is well known that integrability of Partial Differential Equations (PDE) is associated with several different properties of the equation, such as the existence of conserved quantities, symmetries, Lax pair and multi-soliton solutions. These properties can also be used to search for integrable PDE's and it turns out that all methods give essentially the same set of integrable equations. Recently some of these methods have been applied to Partial Difference Equations (PΔE), although in that context there are also integrability criteria that do not have obvious continuous counterparts.

Here we consider the existence of multi-soliton solutions for PΔE's as an indicator of integrability. We use Hirota's bilinear method, which is well suited for constructing soliton solutions.

*E-mail: jarmo.hietarinta@utu.fi

†E-mail: djzhang@staff.shu.edu.cn

2 Hirota's direct method for PDE's

In Hirota's direct method [1, 2] a key ingredient is the transformation to new dependent variables, and in terms of which the soliton solution is given by a finite sum of exponentials. For example, in the case of the KdV equation $u_t + u_{xxx} + 6uu_x = 0$, the transformation $u \to F$ by $u = 2\partial_x^2 \log F$, leads to an equation, which after one integration can be written as

$$P(\vec{D})F \cdot F = 0, \tag{1}$$

where $P(D_x, D_t) = D_x^4 + D_x D_t$. Here the D's are *Hirota derivative operators* defined by

$$D_x^n f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1) g(x_2)\big|_{x_2 = x_1 = x}. \tag{2}$$

The D-derivative differs from the usual derivative by a crucial sign change. It is important to observe that equations written in Hirota bilinear form are gauge invariant:

$$P(\vec{D})(e^{\vec{x}\cdot\vec{p}} f) \cdot (e^{\vec{x}\cdot\vec{p}} g) = e^{2\vec{x}\cdot\vec{p}} P(\vec{D}) f \cdot g. \tag{3}$$

Gauge invariance is the property that allows generalizations of the Hirota bilinear form to other circumstances, such as higher multi-linearity [3] and discrete equations.

The existence of a bilinear form does not guarantee the existence of N-soliton solutions, but one- and two-soliton solutions are often easy to construct. For the KdV-class (1), where P is some *even* polynomial, the construction proceeds as follows:

One starts with the vacuum or background solution, which in this case is $F \equiv 1$ (which in the KdV case corresponds to $u \equiv 0$). This implies the condition $P(0) = 0$. Next starting with the vacuum $F = 1$ one builds a one-soliton solution (1SS) "perturbatively"

$$F = 1 + e^{\eta_i}, \text{ where } \eta_i = \vec{x} \cdot \vec{p}_i + \eta_i^0, \text{ with constant } \eta_i^0, \tag{4}$$

and upon substituting this to (1) one obtains the dispersion relation (DR)

$$P(\vec{p}_i) = 0, \tag{5}$$

which restricts the parameters of the soliton.

Next one finds that (1) also has two-soliton solutions of the form

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \tag{6}$$

where the phase factor A_{ij} is given by

$$A_{ij} = -\frac{P(\vec{p}_i - \vec{p}_j)}{P(\vec{p}_i + \vec{p}_j)}. \tag{7}$$

Here each $\vec{p}_i$ is restricted by the DR (5). Note that $A_{ij} = A_{ji}$ due to P being even.

In the above constructions crucial role is played by the minus sign in the definition (2) of the Hirota derivative, which implies

$$P(\vec{D}) e^{\vec{x}\cdot\vec{p}} \cdot e^{\vec{x}\cdot\vec{p}'} = P(\vec{p} - \vec{p}') e^{\vec{x}\cdot(\vec{p}+\vec{p}')}. \tag{8}$$

In practice this means that the first *and* the last terms of expressions like the 1SS (4) and 2SS (6) automatically satisfy the equation (if $P(0) = 0$).

The above construction shows a level of *partial integrability*: we can have elastic scattering of two solitons, for any dispersion relation, if the nonlinearity is suitable (namely if it arises from a bilinear equation as above). However, when one tries to follow this method and derive a 3SS, one finds an obstacle which only integrable equations pass. One can then say that a bilinear equation is *Hirota integrable* if the 1SS (4) can be extended to a NSS of the form

$$F = 1 + \epsilon \sum_{j=1}^{N} e^{\eta_j} + (\text{finite number of h.o. terms in } \epsilon)$$

without any further conditions on the parameters $\vec{p}_j$ of the individual solitons. The assumption that there are no further restrictions is important, because any equation has multi-soliton solutions for some restricted set of parameters.

If we now apply this principle to the 3SS we start with the ansatz

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \\ &\quad + A_{12} e^{\eta_1+\eta_2} + A_{23} e^{\eta_2+\eta_3} + A_{31} e^{\eta_3+\eta_1} + A_{12}A_{23}A_{13} e^{\eta_1+\eta_2+\eta_3}, \end{aligned} \tag{9}$$

which is fixed by the requirement that if in a NSS any soliton goes far away, the rest should look like the (N-1)SS. (In practice “going away” means either $e^{\eta_k} \to 0$ or $e^{\eta_k} \to \infty$.) This means that there is no freedom left: parameters are restricted only by the dispersion relation (5) and the phase factors A_{ij} were already given in (7). Thus the existence of three-soliton solutions is not automatic, indeed, when the ansatz (9) is substituted into (1) one obtains the three-soliton condition (3SC)

$$\begin{aligned} \sum_{\sigma_i=\pm1} & P(\sigma_1\vec{p}_1 + \sigma_2\vec{p}_2 + \sigma_3\vec{p}_3) P(\sigma_1\vec{p}_1 - \sigma_2\vec{p}_2) \\ & \times P(\sigma_2\vec{p}_2 - \sigma_3\vec{p}_3) P(\sigma_1\vec{p}_1 - \sigma_3\vec{p}_3) = 0, \end{aligned} \tag{10}$$

on the manifold $P(\vec{p}_i) = 0, \forall i$. This should be seen as a condition on the polynomial P. *If* the 3SC is satisfied then the equation most likely has NSS of the form

$$F = \sum_{\mu_i \in \{0,1\}} e^{\left[\sum_{i>j}^{(N)} a_{ij}\,\mu_i\,\mu_j + \sum_{i=1}^{N} \mu_i\,\eta_i\right]}, \quad \text{where} \quad \exp(a_{ij}) = A_{ij}. \tag{11}$$

The above method has been used for searches for various types of equations for which conditions similar to (10) can be derived [4, 5].

3 Hirota's bilinear formalism for lattice equations

We now turn to discrete lattice equations defined on the 2-dimensional Cartesian lattice. The discrete Hirota bilinear form is obtained by requiring gauge invariance under

$$f_j(n,m) \to f'_j(n,m) = A^n B^m\, f_j(n,m). \tag{12}$$

This leads us to the discrete Hirota bilinear (HB) form of the form

$$\sum_j c_j\, f_j(n+\nu_j^+, m+\mu_j^+)\, g_j(n+\nu_j^-, m+\mu_j^-) = 0$$

where the index sums $\mu_j^+ + \mu_j^- = \mu^s$, $\nu_j^+ + \nu_j^- = \nu^s$ do not depend on j.

Note that

$$e^{aD_x}\, f(x)\cdot g(x) = e^{a(\partial_x - \partial_{x'})}\, f(x)g(x')\big|_{x'=x} = f(x+a)g(x-a)\ .$$

Therefore the discrete version of $P(D)f\cdot f = 0$ should be an even function built up from exponentials of the Hirota derivative.

The first results on discrete bilinear soliton equations were obtained by Hirota in a series of papers in 1977 [6]. One major result was the "Discrete Analogue of a Generalized Toda Equation" (DAGTE) or "Hirota-equation" [7]

$$[Z_1\exp(D_1) + Z_1\exp(D_2) + Z_1\exp(D_3)]f\cdot f = 0, \quad Z_1+Z_2+Z_3 = 0. \tag{13}$$

This was later generalized [8] to a four term equation

$$\begin{aligned}&(a+b)(a+c)(b-c)\, f_{n+1,m,k}f_{n,m+1,k+1} + (b+c)(b+a)(c-a)\, f_{n,m+1,k}f_{n+1,m,k+1}\\ &+(c+a)(c+b)(a-b)\, f_{n,m,k+1}f_{n+1,m+1,k} + (a-b)(b-c)(c-a)\, f_{n+1,m+1,k+1}f_{n,m,k} = 0,\end{aligned} \tag{14}$$

which is often called the Hirota-Miwa equation.

Hirota's direct method has also been applied to construction of soliton solutions to nonlinear lattice equations [10, 11].

3.1 Searching for integrable bilinear lattice equations

One can also apply the three-soliton condition on lattice equations. Since we will here only consider 1-component equations, we still can take (1) as the basic class of equations, but now P would have to be a sum of exponentials.

One extra problem in the discrete case is that there are so many ways to discretize a derivative and thus there will be many more cases, see Figure 1.

For sub-figure 1a) the equation must have the form $a\, f_{n,m}f_{n+1,m+1} + b\, f_{n+1,m}f_{n,m+1} = 0$, and the existence of the vacuum soliton $f\equiv 1$ implies further that $b=-a$. In [12] we study equations defined on the arrangement given in sub-figure 1b).

Here we consider types c) and d) and in general the $2\times N$ stencil which corresponds to the difference equation

$$\sum_{s=1}^{N} c_s\, f_{n+\nu_s,m}\, f_{n-\nu_s,m+1} = 0. \tag{15}$$

In terms of Hirota's D-derivatives this can be written as (1), with the function

$$P(X,Y) := e^{-\frac{1}{2}Y}\, P_1(X), \quad P_1(X) := \sum_{s=1}^{N} c_s\, e^{\nu_s X}, \quad (\nu_s > \nu_t,\ \forall s<t),$$

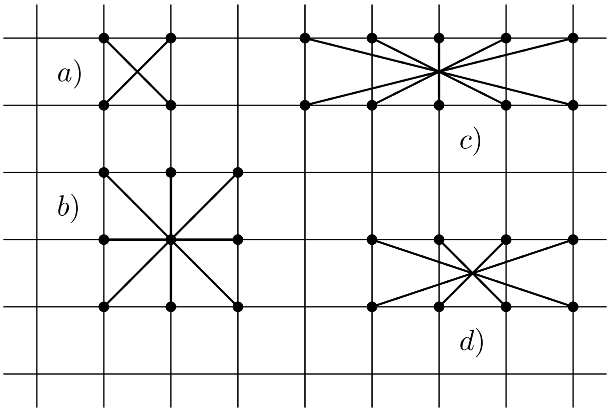

Figure 1: Configurations of points ("stencils") that can be used to define a PΔE of Hirota bilinear form.

followed by an overall shift of $\frac{1}{2}$ in the m (i.e., Y) direction. However, since the D-operator is antisymmetric and P operates on a symmetric target $f \cdot f$, only the symmetric part is relevant. Thus a more proper form is

$$P(X,Y) := e^{-\frac{1}{2}Y}\, P_1(X) + e^{\frac{1}{2}Y}\, P_1(-X). \tag{16}$$

After the transformation $Y \mapsto Y + (\nu_1 + \nu_N)X$ we may assume that $\nu'_N = -\nu'_1$.

Since equation (8) still holds the construction of 1SS and 2SS goes as usual, resulting with formulae (4-7), where now $\eta_j = p_j n + q_j m + \eta_j^0$. In particular, from the 0SS $f_{n,m} = 1,\ \forall n, m$ we get the condition

$$\sum_{s=1}^{N} c_s = 0, \tag{17}$$

and from the 1SS $f_{n,m} = 1 + e^{\eta_j}$ we get the DR $P(p_j, q_j) = 0$, from which we can solve for e^q:

$$e^{q_j} = -\frac{P_1(p_j)}{P_1(-p_j)}. \tag{18}$$

Using the DR we can also write

$$\begin{aligned} P(\sigma_1 p_1 + \sigma_2 p_2, \sigma_1 q_1 + \sigma_2 q_2) = \sigma_1\sigma_2 e^{-\frac{1}{2}(q_1+q_2)}[P_1(-p_1)P_1(-p_2)]^{-1}\times \\ [P_1(\sigma_1 p_1 + \sigma_2 p_2)P_1(-\sigma_1 p_1)P_1(-\sigma_2 p_2) \\ + P_1(-\sigma_1 p_1 - \sigma_2 p_2)P_1(\sigma_1 p_1)P_1(\sigma_2 p_2)], \end{aligned} \tag{19a}$$

$$\begin{aligned} P(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3, \sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3) = \sigma_1\sigma_2\sigma_3 e^{-\frac{1}{2}(q_1+q_2+q_3)}\times \\ [P_1(-p_1)P_1(-p_2)P_1(-p_3)]^{-1}[P_1(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)P_1(-\sigma_1 p_1)P_1(-\sigma_2 p_2)P_1(-\sigma_3 p_3) \\ - P_1(-\sigma_1 p_1 - \sigma_2 p_2 - \sigma_3 p_3)P_1(\sigma_1 p_1)P_1(\sigma_2 p_2)P_1(\sigma_3 p_3)]. \end{aligned} \tag{19b}$$

Note that these expressions are invariant under the simultaneous change of all signs σ_i. Using them we can in particular write the phase factor for the 2SS (7) in the form

$$A_{ij} = \frac{P_1(-p_i)P_1(p_i - p_j)P_1(p_j) + P_1(p_i)P_1(-p_i + p_j)P_1(-p_j)}{P_1(-p_i)P_1(p_i + p_j)P_1(-p_j) + P_1(p_i)P_1(-p_i - p_j)P_1(p_j)}.$$

Furthermore, using (19) we obtain the 3SC from (10), it will contain 64 terms when written in terms of $P_1(X)$. Since polynomial computations are easier it is useful to change variables

$$e^{p_j} = p'_j, \quad e^{q_j} = q'_j, \quad P_1(p) = \mathcal{P}(p')$$

so that the DR has the form $q'_j = -\frac{\mathcal{P}(p'_j)}{\mathcal{P}(1/p'_j)}$ and the phase factor becomes

$$A_{ij} = \frac{\mathcal{P}(1/p'_i)\mathcal{P}(p'_i/p'_j)\mathcal{P}(p'_j) + \mathcal{P}(p'_i)\mathcal{P}(p'_j/p'_i)\mathcal{P}(1/p_j)}{\mathcal{P}(1/p'_i)\mathcal{P}(p'_i p'_j)\mathcal{P}(1/p'_j) + \mathcal{P}(p'_i)\mathcal{P}(1/(p'_i p'_j))\mathcal{P}(p'_j)}.$$

It is now easy to verify (we used REDUCE[13] for this) that the 3SC has the solutions

$$\mathcal{P}(x) = ax^\nu + bx^\mu + cx^\kappa, \quad a + b + c = 0 \tag{20}$$

which corresponds to a sub-case of (13), and

$$\mathcal{P}(x) = ax^\nu + bx^\mu + cx^{-\mu} + dx^{-\nu}, \quad a + b + c + d = 0 \tag{21}$$

which corresponds to a reduction of (14) with $(n, m, k) \mapsto (n(\nu-\mu)-m(\nu+\mu)+\mu, k-\frac{1}{2})$. We have also scanned other four term equations of the type

$$\mathcal{P}(x) = ax^{n_1} + bx^{n_2} + cx^{n_3} + dx^{-n_1}, \quad n_1 = 2, 3, 4, \quad n_1 > n_2 > n_3 > -n_1.$$

but all equations that passed the 3SC condition turned out to be sub-cases of (21).

4 Conclusions

Hirota's direct method has turned out to be very efficient in deriving soliton solutions for a given equation, but it can also be used as a method for searching for integrable equations. The key requirement is the existence of three-soliton solutions without extra conditions on the soliton parameters. Here we have considered bilinear equations defined on a $2 \times N$ stencil of the Cartesian lattice, while the 3×3 case has been treated in [12]. In both cases the integrable equations found by the three-soliton condition turned out to be obtainable by a reduction from the Hirota-Miwa equation.

Acknowledgments

One of authors (DJZ) is supported by the NSF of China (11071157) and Shanghai Leading Academic Discipline Project (No.J50101).

References

[1] R. Hirota, *Exact Solution of the Korteweg–de Vries Equation for Multiple Collision of Solitons*, Phys. Rev. Lett. **27**, 1192-1194 (1971).

[2] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge Tracts in Mathematics 155, (Cambridge Univ. Press, 2004).

[3] B. Grammaticos, A. Ramani and J.Hietarinta, *Multilinear operators: the natural extension of Hirota's bilinear formalism*, Phys. Lett. A **190**, 65-70 (1994).

[4] J. Hietarinta, *A search of bilinear equations passing Hirota's three-soliton condition: I-IV*, J. Math. Phys. **28**, 1732-1742, 2094-2101, 2586-2592 (1987); ibid **29**, 628-635 (1988).

[5] J. Hietarinta, *Recent results from the search for bilinear equations having three-soliton solutions* in "Nonlinear Evolution Equations: Integrability and Spectral Methods", A. Degasperis, A.P. Fordy and M. Lakshmanan (eds.), (Manchester UP, 1990), pp. 307–317.

[6] R. Hirota, *Nonlinear partial difference equations I-III*, J. Phys. Soc. Japan **43**, 1424-1433, 2074-2089 (1977).

[7] R. Hirota, *Discrete analogue of generalized Toda equation*, J. Phys. Soc. Japan **50**, 3785-3791 (1981) .

[8] T. Miwa, *On Hirota's difference equations*, Proc. Japan Acad. **58**, 9-12 (1982).

[9] E. Date, M. Jimbo and T. Miwa, *Method for generating discrete soliton equations I-V*, J. Phys. Soc. Japan **51**, 4116-4124, 4125-4131 (1982), **52**, 388-393, 761-771 (1983).

[10] J. Atkinson, J. Hietarinta and F. Nijhoff, *Soliton solutions for Q3*, J. Phys. A: Math. Theor. **41**, 142001 (2008).

[11] J. Hietarinta and D.J. Zhang, *Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization*, J. Phys. A: Math. Theor. **42**, 404006 (30pp.) (2009).

[12] J. Hietarinta and D.J. Zhang, *Hirota's method and the search for integrable partial difference equations. 1. Equations on a* 3×3 *stencil*, work in progress.

[13] A.C. Hearn, "REDUCE User's Manual", Version 3.8 (2004).