ディジタル信号処理手法による連続系の同定に関する研究

楊, 子江
九州大学工学研究科電子工学専攻

https://doi.org/10.11501/3088155

出版情報: 九州大学, 1991, 博士(工学), 課程博士
バージョン:
権利関係:
IDENTIFICATION OF CONTINUOUS-TIME SYSTEMS USING DIGITAL SIGNAL PROCESSING TECHNIQUES

ZI-JIANG YANG
IDENTIFICATION OF CONTINUOUS-TIME SYSTEMS USING DIGITAL SIGNAL PROCESSING TECHNIQUES

ZI-JIANG YANG

Department of Electrical Engineering
Faculty of Engineering, Kyushu University
Hakozaki, Fukuoka, Japan

November, 1991
Contents

Preface and Acknowledgements v

Abbreviations vii

1 Introduction 1

2 Reexamination of the Integral-Equation Approach to Identification of Continuous-Time Systems 10
  2.1 Introduction 10
  2.2 Briefview of the integral-equation approach 11
  2.3 New integral-equation approach 14
  2.4 Effects of the measurement noise 17
  2.5 Illustrative examples 18
  2.6 Conclusion 22

3 A Unified Approach to Identification of Continuous-Time Systems Using Digital Low-Pass Filters 23
  3.1 Introduction 23
  3.2 Statement of the problem 25
  3.3 Approximated discrete-time estimation models 26
    3.3.1 FIR filtering approach 26
8.2 Brief review of bilinear transformation, delta operator and BPFs........... 147
8.3 Relation between the BPF model and the ZOH sampled model................. 151
8.4 Basic design of the indirect MRACS.............................................. 154
8.5 Digital implementation of the algorithm............................................. 156
8.6 Numerical examples.............................................................................. 160
8.7 Conclusion............................................................................................ 171

9 Conclusions............................................................................................. 172

References.................................................................................................... 176
Preface and Acknowledgements

Identification of dynamic-models for a physical system is usually conducted in discrete-time due to the rapid development and the wide uses of digital computers. Therefore discrete-time models have received more attention than continuous-time models. However, since the real world outside the digital computers is essentially continuous-time and hence continuous-time models play an important role in the design and conceptual analyses of systems, the relevance and importance of continuous-time model identification (CMI) purely using digital computers have received great attention in the last decade. Attempts to continuous system identification have been made by a lot of researchers, and various techniques of CMI have been reported widely in the literature (Unbehauen and Rao 1987, 1990). However, it is observed that coherence and unification in the field of CMI is not immediate, since these methods and algorithms have been developed by their researchers in their own styles instead of making the CMI procedure more general, flexible and systematic. The author, therefore, believes that the time has now come to develop a unified approach using the modern digital signal processing techniques to direct identification of continuous-time systems, from the viewpoint of using digital computers. This has been motivation for this work.

This dissertation is on the problem of identification and adaptive control of continuous-time system models purely using digital computers, based on discrete-time measurements. It has grown out of my research for the degree of doctor of engineering, at Department of Electrical Engineering, Faculty of Engineering, Kyushu University, Japan, under the direction of Professor Setsuo Sagara since April, 1987.

I wish to express my sincere thanks to all the persons who helped make this dissertation possible.

Especially, I would like to express my sincere appreciations to my supervisor, Professor Setsuo Sagara, whole provided inspiring suggestions, useful comments, and constant encouragement throughout the whole work. His serious attitude to research has always been a
reference model for me.

Special thanks are due to Professor T. Nagata and Professor T. Nishi. They provided valuable comments and advices on this dissertation, which are helpful in improving the quality of the dissertation.

I am grateful to Associate Professor Kiyoshi Wada. He helped me with critical discussions and extensive advices in every stage of the whole work.

I am also indebted to Mr. J. Imai for his helpful suggestions in computer programming and preparation of the dissertation.

Finally, but not least, the author would like to thank Associate Professor J. Murata, Mr. M. Ohbayashi and all the other members in System Control Laboratory of Department of Electrical Engineering in Kyushu University, who provided the inspiring working conditions and encouragement throughout my research life at Kyushu University.
### Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR</td>
<td>autoregressive</td>
</tr>
<tr>
<td>ARMA</td>
<td>autoregressive moving average</td>
</tr>
<tr>
<td>BCIV</td>
<td>bias compensated instrumental variable</td>
</tr>
<tr>
<td>BCLS</td>
<td>bias compensated least-squares</td>
</tr>
<tr>
<td>BPF</td>
<td>block-pulse function</td>
</tr>
<tr>
<td>CMI</td>
<td>continuous-time model identification</td>
</tr>
<tr>
<td>DMI</td>
<td>discrete-time model identification</td>
</tr>
<tr>
<td>EIV</td>
<td>extended instrumental variable</td>
</tr>
<tr>
<td>ELS</td>
<td>extended least squares</td>
</tr>
<tr>
<td>EV</td>
<td>errors-in-variables</td>
</tr>
<tr>
<td>DPS</td>
<td>distributed parameter system</td>
</tr>
<tr>
<td>FIF</td>
<td>finite integral filter</td>
</tr>
<tr>
<td>FIR</td>
<td>finite impulse response</td>
</tr>
<tr>
<td>GIV</td>
<td>generalized instrumental variable</td>
</tr>
<tr>
<td>GLS</td>
<td>generalized least squares</td>
</tr>
<tr>
<td>IIF</td>
<td>infinite integral filter</td>
</tr>
<tr>
<td>IIR</td>
<td>infinite impulse response</td>
</tr>
<tr>
<td>IV</td>
<td>instrumental variable</td>
</tr>
<tr>
<td>LS</td>
<td>least-squares</td>
</tr>
<tr>
<td>MA</td>
<td>moving average</td>
</tr>
<tr>
<td>MEIV</td>
<td>modified extended instrumental variable</td>
</tr>
<tr>
<td>MIMO</td>
<td>multiple-input-multiple-output</td>
</tr>
<tr>
<td>MRACS</td>
<td>model reference adaptive control system</td>
</tr>
<tr>
<td>NIIF</td>
<td>new infinite integral filter</td>
</tr>
<tr>
<td>NSR</td>
<td>noise/signal ratio</td>
</tr>
<tr>
<td>OF</td>
<td>orthogonal function</td>
</tr>
<tr>
<td>PEM</td>
<td>prediction error method</td>
</tr>
<tr>
<td>PFC</td>
<td>Poisson filter chains</td>
</tr>
<tr>
<td>PMF</td>
<td>Poisson moment functional</td>
</tr>
<tr>
<td>RML</td>
<td>recursive maximum likelihood</td>
</tr>
<tr>
<td>SISO</td>
<td>single-input-single-output</td>
</tr>
<tr>
<td>SVF</td>
<td>state variable filter</td>
</tr>
<tr>
<td>ZOH</td>
<td>zero-order hold</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

_System identification_ deals with the problem of building mathematical models of dynamic systems based on observed data from the systems for various purposes such as system analysis, control, prediction, fault detection and diagnosis etc. The subject is thus part of basic scientific methodology, and since dynamic systems are abundant in our environment, the techniques of system identification have a wide application area. Applications of system identification, in particular of _parameter estimation in dynamic models_, can be found in many fields, such as control engineering, biology, environmental sciences, econometrics and signal processing.

Techniques to infer a model from measurements typically contain two steps. First a family of candidate models is decided upon (Stoica, Eykhoff, Janssen, and Söderström 1986). Then we find the particular member of this family that satisfactorily (in some sense) describes the measured data (Ljung and Söderström 1983). In this dissertation, discussions are concentrated mostly on the second step, which in fact is the problem of _parameter estimation in dynamic models_. This is not to say that the first step is easy or obvious; it is, however, quite application-dependent, so that it is difficult to give a general discussion of this step.

The term 'model' is used in general to mean a handy entity representing the actual system (Ljung 1987). In the time-domain, all dynamic system models may be divided into two types based on whether they characterize continuous-time or discrete-time processes. Continuous-time models are usually described by _differential_ equations, whereas discrete-time models are described in _difference_ equations.

Originally, at the time when the digital computers were not widely used, most models of dynamic systems required for automatic control and system analysis were obtained
Introduction

by reference to either frequency response data or transient response data obtained during
planned experiments using simple stimuli, such as the unit step or impulse excitation. Since
most models encountered in the physical world are continuous and most classical control
systems theory at that time was based on transfer function models using either frequency
response methods or block diagram analysis, the models of dynamic systems were usually
formulated as continuous-time differential equations or their equivalent. A historical view of
these methods was given by Young (1981).

In the age of cheap computing power and digital electronics, it is quite natural, to not only
compute digitally but also model in discrete-time terms so that the mathematical characteri-
sation of the systems match the serial processing nature of the digital computer. Therefore a
'go-completely-digital' trend has been set up in the recent decades, even in situations wherein
the related systems and situations outside the digital computers are inherently continuous
in time. Owing to rapid developments and popular uses of digital computer technology,
the identification of dynamic models is usually conducted with sampled data in discrete-
time (Åström and Eykhoff 1971, Eykhoff 1974, Ljung and Söderström 1983, Söderström
and Stoica 1989). To enhance the use of discrete-time data for computer applications, the
discrete-time models have received much attention in developing identification theories and
techniques during the past two decades. And the rapid development of parameter estima-
tion procedures for discrete-time models has tended to obscure parallel developments in CMI.
However, since physical processes are usually continuous in time, many of us in the field of
systems and control owe a great deal to the continuous-time domain treatment for the basis
of our understanding of the subject. To most of us, the coefficients in discrete-time models
do not offer the same ease and appeal of physical interpretation as do the parameters in
continuous-time models. Therefore, continuous-time models still play an important role in
the design and conceptual analyses of systems.

The so-called indirect approach of CMI has also been implemented by many researchers
(Sinha 1972, Sinha and Lastman 1982, Huang Chen and Chao 1988). There are two main
steps for indirect identification methods. First, a discrete-time model is found by using some
standard identification methods previously mentioned; then the resulting discrete-time model
is converted into its equivalent continuous-time form. However, the task of going back to
the continuous-time equivalent is not without difficulties (Sinha 1972, Sarkar, Radharishna

With the above mentioned background, the relevance and importance of continuous-time
models have been increasingly recognized in recent years in areas such as identification and adaptive control (Young 1981, Unbehauen and Rao 1987, 1990, Gawthrop 1980, 1987).

Some advantages of the continuous-time models are summarized as follows (Unbehauen and Rao 1987, 1990, Gawthrop 1987).

1: In discrete-time model identification, the choice of sampling period is not a trivial matter. When the continuous-time model is identified, discretization of it is not difficult. However, in the initial setting of the task of discrete-time model identification (DMI), wherein \textit{a priori} knowledge of the range of the various time constants is insufficient, sampling and discretization of an unknown model will give rise to uncertainties in the resulting approximation. Therefore, selecting an appropriate sampling period is an important problem in DMI (Åström 1969, Ng and Goodwin 1976, Mulholland and Weidner 1980, Crittenden, Mulholland, Hill and Martinez 1983, Sinha and Puthenpula 1985, Sagara, Eguchi and Wada 1985a, 1985b).

2: In the presence of a possible and often unknown time delay which may not happen to be an integral multiple of the sampling time, the resulting discrete-time model may acquire the undesirable non-minimum phase property (Gawthrop 1980).

3: In the control problem, the properties of a controller often depend upon the sampling interval. At fast sampling rates, poles and some zeros cluster near the 1 point in the \(z\)-domain and consequently the computation of the control signal becomes sensitive to errors in the coefficients. Excessive sensitivity makes the computation of the control law from system coefficients numerically ill-conditioned (Gawthrop 1982).

4: A problem in the task of control system design using the discrete models is the loss of relative order information in the sampling process which reappears in the form of undesirable zero locations. However, for the continuous-time systems, the design method is matched to the actual system to be controlled. Thus system characteristics such as relative degree and zero location can be directly addressed (Gawthrop 1987).

5: Using the continuous-time models, artefacts of sampling such as sampled minimum phase systems having zeros outside the unit disc (Åström, Hagander and Sternby 1984) are avoided.

6: In the problem of controller design, the sampling interval is chosen \textit{after} the design stage, not before. Therefore the design procedure based on the continuous-time model is
more flexible.

7: Because of the simplicity and standardization in the model representation for the single-input-single-output (SISO) systems, indirect identification methods can provide results with reasonable accuracy. However, the situation becomes much more complicated in the multiple-input-multiple-output (MIMO) systems, due to the varieties and non-uniqueness of model representations (Huang Chen and Chao 1987, 1988).

With these considerations, a direct attack on the problem of continuous-time systems is clearly preferred if a continuous-time model is desired. Frequently identification methods are based on the system transfer function and the associated system ordinary differential equation. A major difficulty of identification of continuous-time models is that the derivatives of the system input-output signals are not measured directly and the differentiations may accentuate the noise effects. Therefore an important problem is how to handle the time derivatives. A historical view of the various methods reported widely in the literature is given here in brief.

- **Direct approximation of differentiation.**

  Since the continuous-time systems under study are described by differential equations and usually the signal derivatives are not measured directly, it is straightforward to use the direct approximations of the differentiations from sampled data, to identify the parameters. The approximation techniques for direct numerical differentiations were studied by Wang, Yang and Chang (1987) with a differentiation operational matrix using generalized orthogonal polynomials, and also by Kraus and Schaufelberger (1990) with differential operators based on block-pulse functions (BPFs). The method of backforward difference was investigated by (Ishimaru, Hanazaki and Akizuki 1988). Some other methods, such as the digital differentiators (Pintelon and Schoukens 1990) can also be applied to this approach. This approach, while being simple and straightforward is not robust to the noise since there is no low-pass pre-processing operation performed to clean-up the noisy measurements. Therefore, this approach is feasible only in deterministic cases.

- **Integral-equation.**

  Since the signal derivatives may accentuate the noise effects, many researchers have used the integral equation approach for CMI. Early work of the integral equation ap-
proach was done by Mathew and Fairman (1974). The multiple integrations are intro-
duced to convert the associated system ordinary differential equation to an equivalent
integral-equation. With the equivalent integral-equation of the continuous system, the
unknown model parameters can then be solved by a least-squares (LS) method. The
integral-equation approach to parameter estimation of continuous systems has been ex-
tensively reported in the literature during the last decade (Unbehauen and Rao 1987,
1990). Various methods employing this approach have been proposed, which are all
considered as approximation or implementation techniques of the integral operations.
Two classes of orthogonal functions (OFs), namely the piecewise constant orthogonal
functions such as Walsh functions (Rao and Palanisamy 1983), BPFs (Cheng and Hsu
1982, Sagara, Yuan and Wada 1988a), and the orthogonal polynomials such as La-
guerre polynomials (Hwang and Shih 1982), Jacobi polynomials (Liu and Shih 1985),
Legendre polynomials (Paraskevopoulos 1985), shifted Legendre polynomials (Hwang
and Guo 1984) and Fourier series (Mohan and Datta 1989) etc., have been reported
by many researchers. Among the OF methods, the BPF method which is very similar
to the trapezoidal integrating rule is the simplest one and is used most often since
the algorithms can be recursified in time, while for the other OFs, the algorithms
are not suitable for real-time estimation. Recently, the integral-equation approach to
continuous systems using the numerical integrating rules such as the Simpson’s and
the trapezoidal rules, have been proposed by Chao, Chang and Huang (1987), Whit-
field and Messali (1987). And the identification algorithms based on the numerical
integrating rules can be implemented in recursive form. A major disadvantage of the
methods mentioned above is that the non-zero initial conditions should be estimated
as unknown parameters (Eitelberg 1988). Another disadvantage is that the multiple
integrations of the observed system signals will grow rapidly with the time, especially
for the high-order systems. In such cases, the estimation algorithm may be reset after
a suitable period of time to prevent the blow up of the integrated signals, which may
cause numerical problems (Unbehauen and Rao 1990).

- **State variable filters (SVFs) and Poisson filter chains (PFCs).**

The continuous-time SVF approach is very popular in the literature (Kaya and Ya-
mamura 1962, Young 1970, Young and Jakeman 1980, 1981). In this approach, the
measured signals obtained from an unknown continuous process are passed through
analog SVFs and then sampled to provide the data to a simple recursive estimation
algorithm. And it was pointed by Young (1970) that if the SVFs have fast damped
transient response characteristics, the initial conditions die away quickly and can be neglected. However, such filters may have a broad pass-band and hence pass considerable noise. And in this case one has to start the identification procedure after the initial values damp out, to obtain satisfactory results. This may require a long time record of the data. To achieve the asymptotic statistical efficiency, it is was suggested that the pass-band of the SVFs should be chosen such that it matches that of the system under study, and in this case the adaptive SVF using the denominator of the system may be a good candidate (Young 1970, 1981 Young and Jakeman 1980).

In view of the desirability for a regular pattern of the impulse response functions, a chain of filters, each elements of which has a transfer function of the general form $1/(p + \lambda)$, $\lambda > 0$, where $p$ is a differential operator, is also applied. Owing to the resemblance of the related impulse response function of such a filter chain to the Poisson distribution function, the integrals, i.e. the outputs of the various stages of the chain excited by a signal, are termed Poisson moment functionals (PMFs) of the signal.

Extensive works have been reported on the PMF technique in recent years (Saha and Rao 1982, Saha and Mandal 1990). The PFCs may in particular conditions be seen to give rise to the SVFs. The Poisson filters have the low-pass filtering property as the SVFs, which reduces the noise effects. The PMFs of the derivatives of a given signal can be computed directly from those of the signal itself. With sufficiently large $\lambda$, the identification algorithms based on the PMFs become simpler, since the effects of initial conditions on PMFs become negligible. However, such broad-band filters may admit more noise.

- **Integral operations over finite time interval.**

The finite time integral operation approach has been studied by some researchers (Eidelberg 1988, Sagara and Zhao 1990, Schoukens 1990). The basic idea in this method is to replace the system signal derivatives by multiple integral operations over selected finite time intervals, and thus the initial condition problem and the blow up of the multiple integrated signals are avoided.

- **Modulating functions.**

Another classical method that can be applied to linear differential systems is Shinbrot's method of moment functionals, also called the modulating function approach, which facilitates converting a differential equation on a finite time interval into an algebraic equation in the continuous-time system parameters (Shinbrot 1957, Pearson
and LEE 1985a, 1985b, Jordan, Jalali-Naini and Mackie 1990). As introduced by Shinbrat (1957), the modulating function involves the use of a set of well-behaved known functions \( \{\phi_i(t)\} \) sufficiently differentiable on \([0, T]\). If the derivatives at the ends of the interval vanish, all the initial condition terms related to the signals also vanish reducing the burden of the identification problem. Although this approach does not involve the initial conditions and is suitable for identification based on input-output data observed over a finite time interval, it, however, has remained relatively obscure due in large measure to the rather severe computational burden associated with the linear functionals. Additionally, the modulating function approach usually gives rise to off-line algorithms (Unbehauen and Rao 1987, 1990).

The publications of Unbehauen and Rao (1987, 1990) undoubtedly contain a unified compendium of these methods in terms of 'linear dynamic operations'. However, it is observed that the above mentioned methods have been developed by their researchers in their own styles instead of making the continuous system identification procedure more general, flexible and systematic. Moreover, it is found by the author that the various methods reported widely in the literature so far, have a somewhat archaic flavour. They have been developed based more on classical integral operations which may have certain filtering effects, rather than on the modern digital filtering techniques. Motivated by this fact, and from the point of view of using digital computers, a unified approach using the modern digital filtering techniques to direct recursive identification of continuous systems is strongly required. And it should be pointed out that in spite of the fact that a great deal of efforts has been paid to the problem via integral equation approach, more detailed discussions are necessary to clarify the troubling initial condition problem which has been puzzling many researchers working at this subject. Early works of this direction were summarized in my master's thesis (Yang 1989) and my first publication in Japanese (Sagara, Yang and Wada 1990).

This dissertation is organized as follows.

- Chapter 2 reexamines the conventional integral-equation approach. The attention is focused on the initial condition problem which has been puzzling many researchers. A new calculation procedure of the multiple integrations of the system signal derivatives is proposed and a new estimation model is derived for which the initial conditions need not be identified as unknown parameters. Therefore the burden of the identification algorithms can be greatly reduced compared to the conventional methods (Sagara, Yang and Wada 1991a).
In chapter 3, a unified approach to direct recursive identification techniques of continuous systems from sampled input-output data using digital low-pass filters is discussed. Using a pre-designed digital low-pass filter, a discrete-time estimation model in continuous-time system parameters is constructed easily. Thus the system parameters can be identified directly by recursive identification algorithms. Numerical results show that if the filter is designed so that its pass-band matches that of the system under study closely and thus the noise effects are sufficiently reduced, accurate estimates can be obtained by recursive identification algorithms such as the LS method and the instrumental variable (IV) method. Two classes of filters (finite impulse response (FIR) digital filter and infinite impulse response (IIR) digital filter) are applied. And some well-known distinct methods mentioned previously are unified as either the IIR or the FIR filtering approach (Yang 1989, Sagara, Yang and Wada 1990, 1991a, 1991b).

In chapter 4 some recursive identification algorithms for continuous systems from sampled input and output data using an adaptive procedure are discussed. An approximated discrete-time model of the continuous system under study is first obtained by the bilinear transformation. Using the estimated denominator of the transfer function of the discrete-time model to construct the adaptive IIR filters which are introduced to avoid direct approximations of differentiations from sampled data, an approximated discrete-time estimation model with continuous system parameters is derived. With filtered inputs and delayed filtered outputs as instrumental variables, some kinds of recursive IV identification algorithms are proposed to obtain consistent estimates in the presence of noise. The proposed identification algorithms have close relations to the standard recursive identification algorithms for common discrete-time systems (Sagara, Yang and Wada 1991c).

In chapter 5, the problem of identification of continuous systems is considered when both the discrete input and output measurements are contaminated by white noises. It will be found that in the presence of input measurement noise, it is not appropriate to let the pass-band of the filters match that of the continuous system under study as suggested in some previous works. The simulation results will show that in this case the pass-band of the digital low-pass filters should be chosen such that it includes the main frequencies of both the system input and output signals in some range. When the noise effects cannot be neglected, the bias compensated LS (BCLS) method is applied to obtain a consistent estimate, which compensates the bias of the LS estimate with
the estimates of the noise variances (Sagara, Yang and Wada 1991d).

- Chapter 6 proposes the method for identification of continuous systems in the case where the discrete input measurement is corrupted by a white noise and the discrete output measurement is corrupted by a noise which may be coloured. The continuous system is identified through the discrete-time estimation model derived in chapter 4 using an adaptive procedure. The effects of the output noise is avoided by the IV method with filtered inputs and delayed filtered outputs as instrumental variables. Then the bias of the IV estimate due to the input noise is compensated by the proposed bias compensated IV (BCIV) method (Yang, Sagara and Wada 1991).

- Chapter 7 proposes a new approach to recursive parameter identification of second-order distributed parameter systems in the presence of measurement noise under unknown initial condition and boundary condition. A two-dimensional low-pass filter which is designed in continuous time-space domain and discretized by the bilinear transformation, is introduced to pre-filter the observed data corrupted by measurement noise. Thus a discrete estimation model of the system under study is easily constructed with filtered input-output data for recursive identification algorithms. The LS method is still efficient in the presence of low measurement noise if the filter parameters are designed so that the noise effects are reduced sufficiently. Using filtered input data as instrumental variables, an IV method is also presented to obtain consistent estimates when the digital low-pass filters are not designed successfully or when the output data is corrupted by high measurement noise (Sagara, Yang and Wada 1991e).

- In chapter 8, The implementation techniques of multi-rate indirect model reference adaptive control for continuous systems purely using digital computers are described. The scheme is composed of three components: a general recursive least squares type parameter estimator, a continuous plant model and a controller designed in continuous-time domain. To reduce the computational burden, the algorithm is implemented in a multi-rate manner with a small sampling interval of the system signals and a relatively large parameter estimation interval. Comparison of the discretization methods for the adaptive system using the BPFs, the trapezoidal integrating rule and the well-known delta operator are discussed through theoretical analysis and simulation study. It is shown that the block-pulse function method is the most effective one (Sagara, Yang and Wada 1991f).

- Chapter 9 summarizes the concluding results of this dissertation.
Chapter 2

Reexamination of the Integral-Equation Approach to Identification of Continuous-Time Systems

2.1 Introduction

Attempts to continuous system identification using multiple integral operations have been made by a lot of researchers. Early work of the integral-equation approach was done by Mathew and Fairman (1974). The multiple integrations are introduced to convert the associated system ordinary differential equation to an equivalent integral-equation. With the equivalent integral-equation of the continuous system, the unknown model parameters can then be solved by the LS method. This approach has been extensively reported in the literature during the last decade (Unbehauen and Rao 1987, 1990). Applications of the OFs, such as BPFs (Cheng and Hsu 1982, Sagara, Yuan and Wada 1988a), Walsh functions (Rao and Palanisamy 1983), Laguerre polynomials (Huang and Shih 1982), shifted Legendre polynomials (Hwang and Guo 1984), Fourier series (Mohan and Datta 1989) etc., have been reported by many researchers. These approaches first derive an operational matrix for integration from the OFs, then the differential equation which characterizes the dynamics of the system under study is converted into a set of over-determined linear algebraic equations by the operational matrices. Therefore, if the input-output data can be observed, the unknown parameters can be estimated directly by the LS algorithm without using direct differentiations which may accentuate measurement noise. Among the OF methods, the BPF...
method which is very similar to the trapezoidal integrating rule is the simplest one and is used most often, since the algorithms can be recursified in time. Recently, the method using the numerical integrating rules, which can be implemented in a recursive manner, has been proposed by Chao, Chen and Hwang (1987), Whitfield and Messali (1987).

A major disadvantage of the methods mentioned above is that the non-zero initial conditions should be estimated as unknown parameters (Eitelberg 1988). Especially for the high-order systems and MIMO systems, much more parameters should be estimated. This may increase the computational burden greatly. And since much more parameters including those concerning the initial conditions are estimated by the LS type algorithms, it should be careful to choose the input signals to obtain unique solution to the parameter estimation problem (Whitfield and Messali 1987).

Another problem of the integral-equation approach is that the multiple integrated system signals grow rapidly with the time. The problem is obviously more serious with high-order systems. In such cases, the estimation algorithm may be reset after a suitable period of time to prevent the blow up of the multiple integrations (Unbehauen and Rao 1987, 1990).

In this chapter, the integral-equation approach to identification of continuous systems described by differential equations is reexamined, focusing on the initial conditions (Sagara, Yang and Wada 1991a). It is pointed out that the terms concerning the initial values in the integral-equation arise due to the cancellation of the differential operators, then a new direct recursive computational procedure of the repeated integrations of signal derivatives is described without considering the unknown initial values. Thus the unknown initial values need not be estimated as unknown parameters. The numerical phenomenon due to the blow up of the multiple integrations is also investigated, taking into account the effects of the measurement noise (Sagara, Yang and Wada 1991a).

2.2 Briefview of the integral-equation approach

Consider the following SISO continuous system

\[
A(p)x(t) = B(p)u(t)
\]

\[
A(p) = \sum_{i=0}^{n} a_i p^{n-i} \quad (a_0 = 1)
\]

\[
B(p) = \sum_{i=1}^{n} b_i p^{n-i}
\]
Reexamination of the integral-equation approach

where \( p \) is a differential operator, \( u(t) \) and \( x(t) \) are the real input and the real output. And it is assumed that the system order \( n \) is known, \( A(p), B(p) \) are relatively prime.

Since differential operations may accentuate the measurement noise effects, it is inappropriate to identify the parameters using direct approximations of differentiations. A straightforward approach is to put the differential equation into an integral-equation.

We define the integral operator \( p^{-1} \) as

\[
p^{-1}f(t) = \int_{t_0}^{t} f(t_1) \, dt_1
\]

Integrating both sides of equation (2.1) \( n \) times leads to

\[
\sum_{i=0}^{n} a_i p^{-n-i} x(t) = \sum_{i=1}^{n} b_i p^{-n-i} u(t)
\]

(2.3)

Canceling the differential operator \( p \) by the integral operator \( p^{-1} \), we have the following integral-equation

\[
\sum_{i=0}^{n-1} a_i \left[ p^{-i} x(t) - \sum_{j=i}^{n-1} p^{j-i} x(t_0)(t - t_0)^j/j! \right] + a_n p^{-n} x(t) =
\]

\[
\sum_{i=1}^{n-1} b_i \left[ p^{-i} u(t) - \sum_{j=i}^{n-1} p^{j-i} u(t_0)(t - t_0)^j/j! \right] + b_n p^{-n} u(t)
\]

(2.4)

The integral-equation can be written into the following form by collecting the terms concerning the initial conditions (Whitfield and Messali 1987):

\[
x(t) = -\sum_{i=1}^{n} a_i p^{-i} x(t) + \sum_{i=1}^{n} b_i p^{-i} u(t) + \sum_{i=1}^{n} c_i (t - t_0)^{i-1}
\]

(2.5)

where

\[
c_1 = x(t_0)
\]

\[
c_i = \left[ \sum_{j=0}^{i-1} a_j p^{i-j-1} x(t_0) - \sum_{j=1}^{i-1} b_j p^{i-j-1} u(t_0) \right] / (i-1)! \quad (i = 2, 3, \ldots, n)
\]

(2.6)

The effects of the initial conditions are contained entirely in the terms \( \sum_{i=1}^{n} c_i (t - t_0)^{i-1} \), where \( c_i (i = 1, 2, \ldots, n) \) are unknown parameters to be identified as well as \( a_i, b_i \). It should be noted that the terms concerning the initial values in the integral-equation arise due to the cancellation of the differential operators.
Reexamination of the integral-equation approach

The implementation techniques of the multiple integrations have been studied by many researchers using OFs or numerical integrating rules. It is clear that

\[
p^{-i} f(t) = \int_{t_0}^{t} p^{-(i-1)} f(t_1) \, dt_1
\]

\[
= \int_{t_0}^{t-ST} p^{-(i-1)} f(t_1) \, dt_1 + \int_{t-ST}^{t} p^{-(i-1)} f(t_1) \, dt_1
\]

\[
= z^{-S} \{ p^{-i} f(t) \} + \int_{t-ST}^{t} p^{-(i-1)} f(t_1) \, dt_1
\]

\[
= \frac{1}{1 - z^{-S}} \int_{t-ST}^{t} p^{-(i-1)} f(t_1) \, dt_1
\]

\[
= \frac{1}{(1 - z^{-S})^i} \int_{t-ST}^{t} \int_{t_2-ST}^{t_2} \cdots \int_{t_i-ST}^{t_i} f(t_1) \, dt_1 \, dt_2 \cdots dt_i
\]

where \( z^{-1} \) is the shift operator which lets \( z^{-1} f(t) = f(t - iT) \), \( T \) is the sampling interval and \( S \) is a natural number which denotes the shortest integral interval of the applied numerical integrating techniques.

Since usually we can obtain only the discrete sampled data of the continuous system signals, for a sufficiently small sampling interval, we write the multiple integrations of continuous signal \( f(t) \) in the following discrete form:

\[
p^{-i} f(kT) = \frac{1}{1 - z^{-S}} \int_{kT-ST}^{kT} p^{-(i-1)} f(t_1) \, dt_1
\]

\[
\approx \frac{1}{1 - z^{-S}} (h_0 + h_1 z^{-1} + \cdots + h_S z^{-S}) p^{-(i-1)} f(kT)
\]

\[
= \frac{1}{(1 - z^{-S})^i} (h_0 + h_1 z^{-1} + \cdots + h_S z^{-S})^i f(kT)
\]

where the coefficients \( h_0, h_1, \cdots, h_S \) are determined by the applied numerical integrating rules or OFs.

Comment 2.1: Notice equation (2.8) is a general form of the calculation procedure of the multiple integral operations, where the OFs and the numerical integrating rules are most widely used. This is denoted as infinite integral filter (IIF), since the multiple integral operations are performed over the whole 'running' time interval \([t_0, t]\), in contrast to the finite time integral operation approach (Eitelberg 1988, Sagara and Zhao 1990, Schoukens 1990).
Reexamination of the integral-equation approach

For example:

\[ p^{-i} f(k) = \left( \frac{T}{2} \right) \frac{(1 + z^{-1})}{(1 - z^{-1})} p^{-(i-1)} f(k) \]  
(trapezoidal rule)

\[ = \left( \frac{T}{3} \right) \frac{(1 + \frac{1}{4} z^{-1} + z^{-2})}{(1 - z^{-2})} p^{-(i-1)} f(k) \]  
(Simpson’s 1/3 rule)

\[ = \left( \frac{T}{2} \right) \frac{(1 + z^{-1})}{(1 - z^{-1})} p^{-(i-1)} \bar{f}(k) \]  
(BPF)

where \( f(k) \) denotes the observed value of \( f(t) \) at \( t = kT \) and \( \bar{f}(k) \) denotes the block-pulse value over interval \([kT - T, kT)\).

As mentioned previously, since the BPF method which is very similar to the trapezoidal integrating rule is suitable for recursive calculation, only the BPFs among the OFs are applied here. A major disadvantage of the other OFs is the requirement that all the data acquired over the ‘running’ time interval \([t_0, t)\) is needed to find the coefficients in the OF expansion.

The integral-equation (2.5) can be written into the vector form as follows.

\[ x(k) = z^T_C(k) \theta_C \]

\[ z^T_C(k) = [-p^{-1}x(k), \ldots, -p^{-n}x(k), p^{-1}u(k), \ldots, p^{-n}u(k), 1,(t-t_0), \ldots, (t-t_0)^{n-1}] \]

\[ \theta^T_C = [a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n] \]  
(2.10)

Then the parameters can be estimated by the LS method:

\[ \hat{\theta}_C = \left[ \sum_{k=k_0+S}^{N} z_C(k)z^T_C(k) \right]^{-1} \left[ \sum_{k=k_0+S}^{N} z_C(k)x(k) \right] \]  
(2.11)

Although the integral-equation approach is very straightforward, a disadvantage is that more parameters should be estimated (see equation (2.10)). This may increase the complexity of the parameter estimation problem, especially for high-order systems and MIMO systems.

2.3 New integral-equation approach

A new calculation procedure of the multiple integrations of the system signal derivatives is proposed and the estimation model derived from the new calculation procedure of the multiple integrations need not estimate the parameters concerning the initial conditions.
Reexamination of the integral-equation approach

Define $\xi_{Rix}(t)$ and $\xi_{Riu}(t)$ as

$$\begin{align*}
\xi_{Rix}(t) &= p^{-n}p^{n-i}x(t) \\
\xi_{Riu}(t) &= p^{-n}p^{n-i}u(t)
\end{align*}$$

Then the integral-equation (2.3) can be written as

$$\sum_{i=0}^{n} a_i \xi_{Rix}(t) = \sum_{i=1}^{n} b_i \xi_{Riu}(t)$$

(2.13)

It is obvious that if $\xi_{Rix}(t) = p^{-n}p^{n-i}x(t)$ and $\xi_{Riu}(t) = p^{-n}p^{n-i}u(t)$ can be calculated directly from the observed input-output data, only the system parameters $a_i, b_i (i = 1, \ldots, n)$ need be estimated from the integral-equation (2.13). Thus the identification procedure based on equation (2.13) is more convenient than that based on equation (2.10). Notice that the integral operator $p^{-1}$ and the differential operator $p$ cannot be cancelled directly without the assumption of zero initial conditions, i.e. $p^{-1} \cdot p \neq 1$ (although $p \cdot p^{-1} = 1$). Our objective here is to find a numerical method to calculate $p^{-n}p^{n-i}f(t)$.

Various techniques can be applied to calculate $p^{-n}p^{n-i}f(t)$. Using the direct differential mapping method (the delta operator), we have

$$p^{-n}p^{n-i}f(kT) = \left(\frac{T}{1-z^{-1}}\right)^n \left(\frac{1-z^{-1}}{T}\right)^{n-i} f(kT)$$

(2.14)

It should be noted that generally, the multiple unstable zeros and poles on the unit circle cannot be cancelled without any consideration of the initial conditions.

In this chapter, we restrict our discussions on the methods using the numerical integrating techniques. Similarly to equation (2.7), it can be shown that

$$p^{-n}\{p^{n-i}f(t)\} = \frac{1}{(1-z^{-s})^n} \int_{t_{n-ST}}^{t_{n}} \cdots \int_{t_{1-ST}}^{t_{1}} \{p^{n-i}f(t_1)\} \ dt_1 \ dt_2 \cdots \ dt_n$$

(2.15)

Through straight calculations, a new type of integral filter named as new IIF (NIIF) is obtained as the following theorem:

**Theorem 2.1** Let $p^{n-i}f(t) = d^{n-i}f(t)/dt^{n-i}$ be the $(n-i)$th derivative of a continuous function $f(t)$. Then the $n$ times multiple integral operations of $p^{n-i}f(t)$ defined by $p^{-n}p^{n-i}f(t)$ are calculated as

$$p^{-n}p^{n-i}f(kT)$$

$$= \frac{1}{(1-z^{-s})^n} \int_{kT-ST}^{kT} \int_{t_{n-ST}}^{t_{n}} \cdots \int_{t_{1-ST}}^{t_{1}} (1-z^{-s})^{n-i}f(t_{n-i+1}) \ dt_{n-i+1} \cdots \ dt_n$$

$$\approx \frac{1}{(1-z^{-s})^n} (1-z^{-s})^{n-i}(h_0 + h_1 z^{-1} + \cdots + h_s z^{-s})f(kT)$$

(2.16)
Reexamination of the integral-equation approach

if the sampled values of \( f(t) \) are available with a sufficiently small sampling period \( T \).

For example, we have

\[
\frac{p^{-n}p^{n-i}f(kT)}{1 - z^{-1}n} = \frac{(1 - z^{-1})^{n-i} \left( \frac{T}{2} \right)^i (1 + z^{-1})^i}{(1 - z^{-1})^n} f(k) \quad \text{(trapezoidal rule)}
\]

\[
= \frac{(1 - z^{-2})^{n-i} \left( \frac{T}{3} \right)^i (1 + 1/4 z^{-1} + z^{-2})^i}{(1 - z^{-2})^n} f(k) \quad \text{(Simpson's 1/3 rule)}
\]

\[
= \frac{(1 - z^{-1})^{n-i} \left( \frac{T}{2} \right)^i (1 + z^{-1})^i}{(1 - z^{-1})^n} \tilde{f}(k) \quad \text{(BPF)}
\]

Equation (2.13) can be written as

\[
\sum_{i=0}^{n} a_i \xi_{Rix}(k) = \sum_{i=1}^{n} b_i \xi_{Riu}(k)
\]

\[
\xi_{Rix}(k) = \frac{(1 - z^{-S})^{n-i}(h_0 + h_1 z^{-1} + \cdots + h_S z^{-S})^i}{(1 - z^{-S})^n} x(k)
\]

\[
\xi_{Riu}(k) = \frac{(1 - z^{-S})^{n-i}(h_0 + h_1 z^{-1} + \cdots + h_S z^{-S})^i}{(1 - z^{-S})^n} u(k)
\]

The vector form is

\[
\xi_{R0e}(k) = x_R^T(k) \theta
\]

\[
x_R^T(k) = [-\xi_{R1x}(k), \cdots, -\xi_{Rnx}(k), \xi_{R1u}(k), \cdots, \xi_{Rnu}(k)]
\]

and the parameter vector \( \theta \) can be estimated by the LS method:

\[
\hat{\theta} = \left[ \sum_{k=k_0+S}^{N} x_R(k)x_R^T(k) \right]^{-1} \left[ \sum_{k=k_0+S}^{N} x_R(k)\xi_{R0e}(k) \right]
\]

The key point in the NIIF is that the integrations of the signal derivatives are calculated directly without any cancellation of the differential operators in contrast to the conventional method (see equations (2.4),(2.5)). It may be noted that our new integral-equation (2.18) is quite different in form from the conventional one (2.10), since the initial values do not appear explicitly in equation (2.18). However they are very closely related, since both can be derived directly from equation (2.3) and hence are viewed as the variations of equation (2.3). In fact, the effects of the initial values are implicitly included in the proposed NIIF (2.16).
Remark 2.1: The common polynomials \((1 - z^{-S})^n\) in the denominator and \((1 - z^{-S})^{n-i}\) in the numerator of the NIIF (2.16) cannot be cancelled in general case. Only in the case where all the initial values are zero, the term \((1 - z^{-S})^{n-i}\) can be cancelled, and in this case our NIIF (2.16) is equivalent to the conventional one (2.8).

Remark 2.2: Although our NIIF requires slightly more computational burden than the conventional IIF, no initial conditions need be considered and therefore the overall identification algorithms are simpler than the conventional ones.

Remark 2.3: Notice that in order to evaluate \(p^{-n}p^{n-i}f(kT)\) over interval \([k_0T, kT)\), considering the causality of the integral filters, one needs samples of \(f(t)\) at

\[
[k_0-(n-1)S]T, [k_0-((n-1)S-1)]T, \ldots, k_0T, [k_0+1]T, \ldots, [k_0+S]T, [k_0+S+1]T, \ldots, kT
\]

The computation for the output of \(p^{-n}p^{n-i}f(kT)\) should start at \([k_0 + S]T\). And the initial value of the output of \(p^{-n}p^{n-i}f(kT)\) for \(k < [k_0 + S]\) should be initialized strictly to be zero. Otherwise, we may have erroneous results.

Remark 2.4: It should be mentioned that although we treat a linear SISO system here, the basic idea can be extended to linear MIMO systems and linear-in-parameters non-linear systems, and to the case of the existence of deterministic disturbances, following the works of Whitfield and Messali (1987).

2.4 Effects of the measurement noise

For the integral-equation approach, only few works have been made on consistency problem in the presence of stochastic noise. Suppose the discrete measurement of the output is corrupted by a measurement noise \(\eta(k)\), we have the observation \(y(k)\) of the output \(x(k)\) as

\[
y(k) = x(k) + \eta(k) \tag{2.21}
\]

In this case, the estimation model (2.10) for the IIF becomes

\[
y(k) = z_{C\eta}^T(k)\theta_C + r_C(k)
\]

\[
z_{C\eta}^T(k) = [-p^{-1}y(k), \ldots, -p^{-n}y(k), p^{-1}u(k), \ldots, p^{-n}u(k), 1, (t-t_0), \ldots, (t-t_0)^{n-1}]
\]

\[
\theta_C^T = [a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n] \tag{2.22}
\]
Reexamination of the integral-equation approach

where

\[ r_C(k) = \sum_{i=0}^{n} p^{-i} \eta(k) \]  

(2.23)

and the NIIF model (2.19) becomes

\[
\begin{align*}
\xi_{R0y}(k) &= z^T_{R0}(k) \theta + r_R(k) \\
z^T_{R}(k) &= [-\xi_{R1y}(k), -\xi_{R2y}(k), \cdots, -\xi_{Rny}(k), \xi_{R1u}(k), \xi_{R2u}(k), \cdots, \xi_{Rnu}(k)] \\
\theta^T &= [a_1, \cdots, a_n, b_1, \cdots, b_n]
\end{align*}
\]

(2.24)

where

\[ r_R(k) = \sum_{i=0}^{n} p^{-i} \xi_{R0i}(k) \]  

(2.25)

and

\[
\begin{align*}
\xi_{Riy}(k) &= \frac{(1-z^{-S})^{n-i}(h_0 + h_1z^{-1} + \cdots + h_sz^{-S})^i y(k)}{(1-z^{-S})^n} \\
\xi_{Rii}(k) &= \frac{(1-z^{-S})^{n-i}(h_0 + h_1z^{-1} + \cdots + h_sz^{-S})^i \eta(k)}{(1-z^{-S})^n}
\end{align*}
\]

(2.26)

It should be noted that although our NIIF (2.16) is different from the conventional IIF (2.8) in time domain, the frequency response characteristics are equivalent. Hence the noise reducing effects of the two methods are expected to be similar. However, since the IIF and the NIIF can be viewed as unstable IIR filters which have multiple poles on the unit circle, the outputs of the integral filters will grow rapidly with the time, especially for the high-order systems. Therefore the equation errors \( r_C(k), r_R(k) \) increase very fast with the time ‘running’ time \( kT \), especially for the high-order systems. If there exists a considerable high measurement noise, when \( k \) is small and hence the equation errors are not large enough, the measurement noise does not influence the LS estimates so significantly, however, when \( k \) becomes large enough and hence the equation errors increase drastically, the LS estimates are not expected to converge. This fact will be verified through an example.

As a possible method to solve this problem, the algorithm may be reset after a suitable period of time to prevent the blow up of the integral filter outputs, which may cause numerical problems (Unbehauen and Rao 1990). However, it is not convenient for on-line identification.

2.5 Illustrative examples

Example 2.1: Comparison of the IIF and the NIIF in the deterministic case.
Consider a second-order SISO system tested with \( u(t) = 2t/(1 + 2t) \) (Whitfield and Messali
Reexamination of the integral-equation approach

Table 2.1: Results of Example 2.1.

<table>
<thead>
<tr>
<th></th>
<th>true value</th>
<th>NIIF</th>
<th>IIF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BPF</td>
<td>TIR</td>
<td>SIR</td>
</tr>
<tr>
<td>$a_1$</td>
<td>10.0</td>
<td>9.997</td>
<td>10.00</td>
</tr>
<tr>
<td>$a_2$</td>
<td>21.0</td>
<td>20.99</td>
<td>21.00</td>
</tr>
<tr>
<td>$b_1$</td>
<td>1.0</td>
<td>1.001</td>
<td>1.000</td>
</tr>
<tr>
<td>$b_2$</td>
<td>15.0</td>
<td>14.99</td>
<td>15.00</td>
</tr>
<tr>
<td>$c_1$</td>
<td>2.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$c_2$</td>
<td>1.0</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

(1987)

\[
\ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t) = b_1 \dot{u}(t) + b_2 u(t)
\]

(2.27)

with initial conditions $x(0) = 2.0$ and $\dot{x}(0) = -19.0$. The input-output signals are sampled over time interval \([0, 2.0]\), with sampling interval $T = 0.01$. The LS method is implemented in a recursive form. The LS estimates of the proposed method with trapezoidal integrating rule (TIR), Simpson’s integrating rule (SIR) and BPFs are shown in Table 2.1 together with the results of the IIF with TIR. The results show that our method gives very accurate estimates as the conventional one.

The noise rejection performances of the proposed NIIF and the IIF using the trapezoidal rule are compared by Example 2.2, when the discrete output measurement is corrupted by a low white measurement noise.

Example 2.2: Comparison of the IIF and the NIIF in the presence of the measurement noise.

Consider the following system

\[
\ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t) = b_1 \dot{u}(t) + b_2 u(t)
\]

(2.28)

with initial conditions $x(0) = 6.0$ and $\dot{x}(0) = -13.0$ (which let $c_1 = 6.0, c_2 = 5.0$). The input signal is

\[
u(t) = \sin(t) + \sin(1.5t) + 0.5\sin(3t) + 1.5\sin(4.5t) + 0.3\sin(5t) + 0.2\sin(7t) + 2.5\sin(7.5t) + 5.0\sin(10.5t)
\]

The input-output signals are sampled with sampling interval $T = 0.01$ and the noise/signal
Reexamination of the integral-equation approach

Figure 2.1: Results of the IIF (Example 2.2).

Figure 2.2: Results of the NIIF (Example 2.2).
Reexamination of the integral-equation approach

ratio (NSR)= 10%. The LS estimates are shown in Figures 2.1 and 2.2 respectively. Considering the estimates of the system parameters $a_1, a_2, b_1, b_2$, it may be noted that the noise rejection performances of the two methods are very similar, as mentioned previously. It should be mentioned that the NIIF method is more convenient, since with the new method, only four parameters need be estimated. However, for both methods, although the LS estimates converge to their true values for a while for small time $t$, it is observed that when $t$ increases large enough and hence the equation errors increase drastically, the estimates begin to diverge. This fact indicates that it is important to reset the algorithm after a suitable period when using the integral-equation method, to avoid the blow up of the outputs of the integral filters. However, to choose a suitable reset period is still a problem.
2.6 Conclusion

In chapter, the integral-equation approach to identification of continuous systems has been elaborated. It is the contribution that the troubling initial condition problem has been clarified. A general form of the IIF employed in the conventional integral-equation method concerning the OFs and the numerical integrating rules is formulated and it is found although this method has been treated widely by a lot of researchers, the initial condition problem remains still unclear.

It is pointed out that the terms concerning the initial values in the conventional integral-equation method arise due to the cancellation of the differential operators. Motivated by this, a new calculation procedure of the multiple integrations of the signal derivatives termed NIIF is proposed and thus the initial conditions need not be identified as unknown parameters. Therefore complexity of the identification algorithms can be greatly reduced compared with the conventional methods. This fact is specially significant for the MIMO systems and the high-order systems.

Effects of the measurement noise in integral-equation approach is also investigated. It is found that the noise reducing effects of the IIF and the proposed NIIF are similar, since the frequency responses are same. It is pointed out that since the IIF and the NIIF can be viewed as a kind of unstable IIR filters which have multiple poles on the unit circle, the equation error in the integral-equation due to the noise increases with the time and this can make the LS estimates diverge.

With these conclusions, it is found that although this chapter has clarified the unclear problem of the initial conditions, it is still strongly required to develop a more general, flexible and systematic pre-processing procedure for the task of CMI.
Chapter 3

A Unified Approach to Identification of Continuous-Time Systems Using Digital Low-Pass Filters

3.1 Introduction

Parameter identification of continuous-time models has a long history as shown by the surveys given by Young (1981), Unbehauen and Rao (1987, 1990). Frequently identification methods are based on the system transfer function and the associated system ordinary differential equation. A major difficulty of identification of continuous-time models is that the derivatives of the system input-output signals are not measured directly and the differentiations may accentuate the noise effects. Therefore an important problem is how to handle the time derivatives.

Subsequent to the appearance of Young's survey, two special monographs, one on the use of the integral-equation employing OFs or numerical integrating rules and another on the use of the PMFs, gave a comprehensive account of more recent developments in the CMI. Unbehauen and Rao (1987, 1990) attempted to provide a unified view of methods of handling the signal derivatives in terms of 'linear dynamic operations', however, it is observed that these methods have been developed by their researchers in their own styles instead of making the continuous system identification procedure more general, flexible and systematic. Moreover, it is found that some methods reported in the literature so far, have been developed based more on classical integral operations which may have certain filtering effects, rather than on the modern digital filtering techniques. Motivated by this fact, and
A unified approach to identification using digital filters

from the viewpoint of using digital computers, a unified approach using the digital filtering

Since identification techniques of common discrete-time systems have been discussed and
applied widely, it is a good idea to obtain an approximated discrete-time estimation model
with the continuous system parameters. Then we can estimate the continuous system parameters applying the existing recursive identification techniques such as the LS and IV methods
for common discrete-time systems. In this chapter, a unified approach to direct recursive
identification for linear SISO continuous systems using digital low-pass filtering techniques
is proposed. The digital low-pass filters are introduced to avoid direct approximations of
system signal derivatives from sampled system input-output data. Using a pre-designed
low-pass filter, an approximated discrete-time estimation model with the continuous system parameters is constructed easily. Thus the system parameters can be identified by the re-
cursive LS method or IV method. Numerical results show that the parameter estimates are
not so sensitive to the cut-off frequency of the filter, and that if the filter is designed so
that its pass-band matches that of the system closely and thus the noise effects are suffi-
ciently reduced, the LS method is still efficient for the case of low measurement noise. In
some practical situations, we may fail to design the digital filters appropriately, since little a priori knowledge of the unknown systems can be obtained. And some times, the output
measurement may be corrupted by a high measurement noise. In these cases, we can apply
the bootstrap method with the filtered input-output data of the estimated system model as
instrumental variables (Young 1970).

The direct methods for identification of continuous systems using digital low-pass filters
include the following steps.

1 Find a low-pass digital filter which is employed to pre-filter the sampled system data for
   the purpose of reducing the measurement noise effects.

2 Construct an approximated discrete-time estimation model with continuous system pa-
   rameters.

3 Use a recursive identification algorithm to estimate the system parameters from filtered
   input-output sampled data.

Clearly, the digital low-pass filters employed in continuous system identification can be
A unified approach to identification using digital filters

obtained using the modern digital filter design techniques (Oppenheim and Schafer 1975, Roberts and Mullis 1987) to have excellent filtering effects. There are two primary classes of digital filters. If the impulse response never decays exactly to zero, no matter how long a period of time elapses, the filter is classified as an IIR filter. If, however, the response does fall exactly to zero after a finite period of time, we classify the filter as an FIR filter. Both classes of filters are applied to identification of continuous systems. For the FIR filter, we consider an ideal low-pass FIR filter which is designed by window function techniques, and for the IIR filter, we use a Butterworth filter which is designed in continuous-time domain and discretized by the bilinear transformation. Simulation results show that both classes of the filters are effective in continuous system identification. A unified view of some well-known methods reported in the literature is also provided. It will be shown these methods can be unified as either the IIR or the FIR filtering approach, from the viewpoint of digital filtering.

3.2 Statement of the problem

Consider the following SISO continuous system

\[
\begin{align*}
A(p)x(t) &= B(p)u(t) \\
A(p) &= \sum_{i=0}^{n} a_i p^{n-i} \quad (a_0 = 1) \\
B(p) &= \sum_{i=1}^{n} b_i p^{n-i}
\end{align*}
\]  

Our goal is to identify the system parameters from the sampled input-output data. Practically the measurement of the output variable is corrupted by a measurement noise. In order to overcome the practical difficulties associated with the parameter estimation of a stochastic continuous-time noise model, the noise model may be assumed to be in a discrete-time form such as white noise, autoregressive (AR) noise, moving average (MA) noise and autoregressive moving average (ARMA) noise (Young and Jakeman 1980, Huang, Chen and Chao 1987). The sampled measurement of the output is described as

\[
y(k) = x(k) + \eta(k)
\]

where \(\eta(k)\) denotes the measurement noise. The measurement noise \(\eta(k)\) is assumed to be a stationary time series with zero-mean.
As mentioned previously, since differential operations may accentuate the measurement
noise, it is inappropriate to identify the parameters using direct approximations of differ­
entiations. Our objective here is to introduce a digital low-pass filter which would reduce
the noise effects sufficiently. Then we can obtain an approximated discrete-time estimation
model with continuous system parameters which is composed of filtered input-output data.
The derived model does not involve any initial conditions and is thus suitable for on-line
identification.

3.3 Approximated discrete-time estimation models

In this section, we describe the design techniques of the two classes of digital filters and
the approximated discrete-time estimation models derived by the pre-designed filters. It will
be shown that some other methods can be unified as either the IIR or the FIR filtering
approach.

3.3.1 FIR filtering approach

Replacing the differential operator in equation (3.1) by the bilinear transformation which
is closely related to the trapezoidal integration rule

\[ p = \frac{2(1 - z^{-1})}{T(1 + z^{-1})} \]  

we have the following approximated discrete-time model (Krishna 1988):

\[ A'(z^{-1})x(k) = B'(z^{-1})u(k) \]

\[ A'(z^{-1}) = \sum_{i=0}^{n} a_i \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} \]  

\[ B'(z^{-1}) = \sum_{i=1}^{n} b_i \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} \]  

The bilinear transformation (Tustin’s method) has been widely used in simulation, digital
filter and control system design (Haykin 1972, Haberland and Rao 1973, Sinha and Lastman
if the sampling period \( T \) is sufficiently small, the truncation error between the continuous-
time system (3.1) and the discrete-time model (3.4) can be neglected, and in this case the
discrete-time model and the continuous-time model are equivalent.
A unified approach to identification using digital filters

Introduce a low-pass FIR digital $Q_F(z^{-1})$

$$Q_F(z^{-1}) = \sum_{m=0}^{M_F} q_m z^{-m} \quad (3.5)$$

Multiplying both sides of equation (3.4) by a pre-designed filter $Q_F(z^{-1})$ we have

$$\sum_{i=0}^{n} a_i \xi_{Fiw}(k) = \sum_{i=1}^{n} b_i \xi_{Fiu}(k) \quad (3.6)$$

where

$$\xi_{Fiw}(k) = Q_F(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} x(k) \quad (3.7)$$

$$\xi_{Fiu}(k) = Q_F(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} u(k) \quad (3.8)$$

Notice that in the frequency domain, we have

$$Q_F(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} = F(z^{-1}) \left( \frac{2,1 - z^{-1}}{T,1 + z^{-1}} \right)^n \quad (3.9)$$

This fact implies that from the point of view of pre-filtering, the derivation of the estimation model (3.6) is equivalent in the frequency domain to multiplying the original system differential equation (3.1) by a pre-filter $F(z^{-1})$.

Substituting equation (3.2) into equation (3.6), We have

$$\sum_{i=0}^{n} a_i \xi_{Fiw}(k) = \sum_{i=1}^{n} b_i \xi_{Fiu}(k) + \sum_{i=0}^{n} a_i \xi_{Fii}(k) \quad (3.10)$$

where

$$\xi_{Fiw}(k) = Q_F(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} y(k) \quad (3.11)$$

$$\xi_{Fiu}(k) = Q_F(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} u(k) \quad (3.11)$$

Many types of FIR digital filters can be applied. For simplicity, we consider a desired ideal low-pass filter which has the specification:

$$H_a(\omega) = \begin{cases} 
1 & |\omega| \leq \omega_{dc} \\
0 & \text{otherwise}
\end{cases} \quad (3.12)$$

Various design techniques such as the window method, the frequency sampling method, minimax design method (Roberts and Mullis 1987) are used to design FIR filters. The window method for the ideal low-pass filter is outlined here.
A unified approach to identification using digital filters

Assume that the sampling period is $T$ ($\omega_{dc} \leq \pi/T$), since the frequency response of a digital FIR filter is periodic of period $2\pi/T$, we can represent $H_d(\omega)$ in a form of Fourier expansion:

$$H_d(\omega) = \sum_{m=-\infty}^{\infty} h_d(m)e^{-j\omega T}$$

(3.13)

where

$$h_d(m) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} H_d(\omega)e^{j\omega T} d\omega$$

(3.14)

Our goal is to choose an FIR filter $H_1(e^{j\omega T})$ which is close (in some sense) to $H_d(\omega)$. One criterion is to choose the actual response

$$H_1(e^{j\omega T}) = \sum_{m=-M_F}^{M_F} h_1(m)e^{-j\omega T}$$

(3.15)

so that the mean-square error $\epsilon^2$ between $H_d(\omega)$ and $H_1(e^{j\omega T})$ on $[-\pi/T, \pi/T]$ is minimized, where $M_F$ is an even natural number. This is

$$\epsilon^2 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} |H_d(\omega) - H_1(e^{j\omega T})|^2 d\omega$$

(3.16)

From the theory of Fourier series, we can minimize $\epsilon^2$ by choosing $h_1(m)$ to be the Fourier coefficients of $H_d(\omega)$ for $-M_F/2 \leq m \leq M_F/2$:

$$h_1(m) = h_d(m) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} H_d(\omega)e^{j\omega T} d\omega = \frac{\sin(mT\omega_{dc})}{\pi m}$$

(3.17)

We may rewrite $H_1(e^{j\omega T})$ in a more general form utilizing $z$-transformation as follows:

$$H_1(z^{-1}) = \sum_{m=-M_F/2}^{M_F/2} c_m z^{-m} \quad (c_m = h_1(m))$$

(3.18)

The transfer function will not give us exactly the desired frequency response because of the missing higher order terms. To reduce the effect of truncation, we apply the window function directly to the infinite series obtained by the Fourier-series expansion (Roberts and Mullis 1987). The resulting truncated windowed series can then be inserted directly, (as the coefficients) into our transfer function in the $z$-domain. It may be noted that our window function automatically becomes truncated by multiplying the undesired higher-order terms by zero and adjusting the amplitudes of the non-vanishing portion of the series by means of appropriate multiplying $w_m$. For example, we can use the Hamming window:

$$w_m = \begin{cases} 
0.54 + 0.46 \cos(\pi m/M_F/2) & |m| \leq M_F/2 \\
0 & \text{otherwise}
\end{cases}$$

(3.19)
A unified approach to identification using digital filters

The term-by-term modification of the coefficients has the simple form:

\[ c'_m = c_m \cdot w_m \]  

(3.20)

Then we have the windowed transfer function:

\[ H'_1(z^{-1}) = \sum_{m=-M_F/2}^{M_F/2} c'_m z^{-m} \]  

(3.21)

Finally, a causal filter having the same low-pass characteristics can be obtained by delaying the entire sequence by \( M_F/2 \) sampling intervals:

\[ Q_F(z^{-1}) = \sum_{m=0}^{M_F} q_m z^{-m} \quad (q_m = c'_{m-M_F/2}) \]  

(3.22)

The design procedure of \( Q_F(z^{-1}) \) can be summarized as follows.

1: Select an appropriate \( M_F \), sampling interval \( T \), and desired cut-off frequency \( \omega_{dc} \).

2: Perform the inverse Fourier transform to give \( c_m \) in equation (3.18).

3: Multiply \( c_m \) by a suitable window function to give the filter coefficients \( c'_m \), and finally give \( Q_F(z^{-1}) \).

4: Investigate whether the actual frequency response of filter \( F(z^{-1}) \) described in equation (3.9) is acceptable, if not, return to step 1.

The design procedure can be carried out by a program in FORTRAN77 language.

Remark 3.1: It should be noted that usually with a finite length \( M_F \), the frequency response of \( Q_F(z^{-1}) \) does not give the desired ideal low-pass characteristics as described in equation (3.12). Therefore, for high-order systems, it is still necessary to investigate if the frequency response of

\[ Q_F(z^{-1})(1 - z^{-1})^n = F(z^{-1}) \left( \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^n = F(z^{-1}) \cdot p^n \]

has low-pass properties in the high frequency-domain. If not, we should increase the filter length \( M_F \) or construct the pre-filter \( F(z^{-1}) \) by the series connection of several pre-designed FIR filters \( Q_F(z^{-1}) \).
A unified approach to identification using digital filters

3.3.2 IIR filtering approach

There are many design methods for IIR digital filters. One of the most popular formulations is to use the large body of knowledge of the continuous-time or analog filters such as Butterworth filter, Chebyshev filter, inverse Chebyshev filter, Elliptic or Cauer filters etc. When a continuous filter is designed, we can transform it in some manner such as the bilinear transformation to obtain the IIR digital filter (Haykin 1972, Roberts and Mullis 1987).

In this chapter, we choose an $m$th ($m \geq n$) order Butterworth filter $F_I(p)$ which has the basic form for the 'amplitude-squared function' as

$$|F_I(\omega)|^2 = F_I(p)F_I(-p) |_{p=j\omega} = \frac{1}{1 + (\omega/\omega_c)^{2m}} \quad (3.23)$$

where $\omega_c$ is the cut-off frequency for which

$$|F_I(\omega)|^2 \leq \frac{1}{2}, \quad |\omega| \geq \omega_c \quad (3.24)$$

The Butterworth filter has a magnitude response which is maximally flat for $\omega \leq \omega_c$ and declines monotonically for $\omega > \omega_c$. The poles of the filter must satisfy the equation

$$1 + (-jp/\omega_c)^{2m} = 0 \quad (3.25)$$

obtained by setting $\omega = p/j = -jp$. The roots of this equation lie on the unit circle and are given by

$$\lambda_i = e^{j\theta_i}, \quad \theta_i = \frac{\pi}{2} \left( 1 + \frac{2i - 1}{m} \right) \quad (i = 1, 2, \ldots, m) \quad (3.26)$$

Only the left half plane roots are kept. Therefore the Butterworth filter is described to be

$$F_I(p) = \prod_{i=1}^{m} \frac{1}{(p/\omega_c - \lambda_i)} \quad (3.27)$$

$$= \frac{1}{(p/\omega_c)^m + c_1(p/\omega_c)^{m-1} + c_2(p/\omega_c)^{m-2} + \cdots + c_m}$$

For instance, a second-order Butterworth filter is given by

$$F_{I2}(p) = \frac{1}{(p/\omega_c)^2 + \sqrt{2}(p/\omega_c) + 1} \quad (3.28)$$

Multiplying both sides of the system equation (3.1) by the pre-designed $F_I(p)$, we have

$$F_I(p)x(t) + \sum_{i=1}^{n} a_i F_I(p)p^{n-i}x(t) = \sum_{i=1}^{n} b_i F_I(p)p^{n-i}u(t) \quad (3.29)$$
A unified approach to identification using digital filters

Discretizing it by the bilinear transformation, we obtain

\[ \sum_{i=0}^{n} a_i \xi_{iuy}(k) = \sum_{i=1}^{n} b_i \xi_{iuv}(k) \]  

(3.30)

where

\[ \xi_{iuv}(k) = Q_I(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^n \eta_i x(k) \]  

(3.31)

\[ \xi_{iuy}(k) = Q_I(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^n \eta_i u(k) \]

and

\[ Q_I(z^{-1}) = \left( \frac{1 - z^{-1}}{\omega_c} \right)^m + \sum_{i=1}^{m} c_i \left( \frac{1 - z^{-1}}{\omega_c} \right)^{m-i} \left( \frac{T}{2} \right)^i (1 + z^{-1})^i \]  

(3.32)

Substituting equation (3.2) into equation (3.30), We have

\[ \sum_{i=0}^{n} a_i \xi_{iuy}(k) = \sum_{i=1}^{n} b_i \xi_{iuv}(k) + \sum_{i=0}^{n} a_i \xi_{iin}(k) \]  

(3.33)

where

\[ \xi_{iuy}(k) = Q_I(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^n \eta_i y(k) \]  

(3.34)

\[ \xi_{iin}(k) = Q_I(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^n \eta_i \eta(k) \]

Although we only treat a Butterworth filter here, similar results can be obtained using Chebyshev filter, inverse Chebyshev filter, Elliptic or Cauer filters etc.

Remark 3.2: It is clear that when the filter order \( m = n \), \( F_I(p)p^n \) does not give low-pass properties in the high frequency-domain. Therefore, a pre-filter with order \( m = n + 1 \) may yield more excellent filtering effects although it requires more computational burden, since in this case \( F_I(p)p^n \) has low-pass properties in the high frequency-domain. As shown in chapter 5, it is found that in the presence of input and output noises, for the second-order system under simulation study, a third-order filter gives much more accurate results than a second-order filter, when the BCLS method is used. However, experience tells that in most cases \( m = n \) is sufficient for the standard LS and IV methods.

Comment 3.1: It should be noted that the above approximated discrete-time estimation model is similar to the one described in equation (3.10). The difference is that \( Q_I(z^{-1}) \) in equation (3.34) is an IIR low-pass filter while \( Q_F(z^{-1}) \) in equation (3.11) is an FIR low-pass filter. Both types of digital filters are useful. The FIR filters are realized with nonrecursive
A unified approach to identification using digital filters

structures and are therefore always stable. Usually, for the FIR filters realized with nonre­
cursive structures, errors resulting from quantization or round-off are more easily predictable
and less critical than the recursive IIR filters. However, the IIR filters have simple, hand cal­
culation design methods using the bilinear transformation. And the IIR filters can produce
desired amplitude response with significantly few coefficients than nonrecursive FIR filters.
Thus the IIR filtering approach requires less computational burden and memory of digital
computers than the FIR filtering approach. Therefore, in practice, the IIR filters are more
convenient. Detailed discussions about the digital low-pass filters are given for example, in
the text books of Oppenheim and Schafer (1975) and Roberts and Mullis (1987).

3.4 Recursive identification algorithms

When the digital low-pass filter is designed, we have the approximated discrete-time
estimation model of equation (3.10) for the FIR filtering approach, or the model of equation
(3.33) for the IIR filtering approach. Both can be written in vector form:

\[
\begin{align*}
\xi_{ip}(k) &= z^T(k)\theta + r(k) \\
z^T(k) &= [-\xi_{1p}(k), \cdots, -\xi_{np}(k), \xi_{1u}(k), \cdots, \xi_{nu}(k)] \\
\theta^T &= [a_1, \cdots, a_n, b_1, \cdots, b_n] \\
r(k) &= \sum_{i=0}^{n} a_i \xi_{ip}(k)
\end{align*}
\]

(3.35)

where

\[
\begin{align*}
\xi_{ip}(k) &= \xi_{Fip}(k), \quad \xi_{iu}(k) = \xi_{Fiu}(k), \quad \xi_{in}(k) = \xi_{Fin}(k) \quad \text{(FIR filter)}
\end{align*}
\]

(3.36)

or

\[
\begin{align*}
\xi_{ip}(k) &= \xi_{Iip}(k), \quad \xi_{iu}(k) = \xi_{Iiu}(k), \quad \xi_{in}(k) = \xi_{Iin}(k) \quad \text{(IIR filter)}
\end{align*}
\]

(3.37)

It is clear that the discrete-time model (3.35) is suitable for some recursive identifi­
cation algorithms for common discrete-time system models (Ljung and Söderström 1983,
Söderström and Stoica 1989). Among these algorithms, the LS method is perhaps the most
commonly used method owing to its simplicity. This method is first considered. The pa­
rameter \(\theta\) is to be estimated from filtered discrete measurements of the system signals. The
LS estimate of \(\theta\) is defined as the vector \(\hat{\theta}\) that minimizes the loss function

\[
V_{LS}(\theta) = \frac{1}{2} \sum_{k=k_s}^{k=N} r^2(k)
\]

(3.38)
A unified approach to identification using digital filters

If the input signals are sufficiently rich so that

\[
\begin{bmatrix}
\sum_{k=k_S}^N z(k)z^T(k)
\end{bmatrix}
\]

is nonsingular and positive determined, the solution of the minimization problem is given as

\[
\hat{\theta} = \left[ \sum_{k=k_S}^N z(k)z^T(k) \right]^{-1} \cdot \left[ \sum_{k=k_S}^N z(k)x(k) \right]
\]

(3.39)

If the frequency response of the pre-designed digital low-pass filter is designed so that the effects of the measurement noise are reduced sufficiently, the LS method is still efficient. However, in some practical situations, we may fail to determine the filters appropriately, since little a priori knowledge of the unknown systems can be obtained. And sometimes, the output measurement may be corrupted by a high measurement noise. In these cases, the output measurement noise may influence the LS estimates. It might be noted that even when the measurement noise \( \eta(k) \) is a white noise, the equation error \( r(k) \) becomes an MA process noise of a long length, or an ARMA process noise, with respect to the applied digital filters. Therefore, in the presence of high noise, the LS estimates are usually biased.

In the presence of noise, the IV method has by now been used by many researchers. Usually, the instrumental variables are formed as different combinations of inputs, delayed outputs, delayed inputs, filtered inputs, and external setpoint variations. Detailed investigations of different variants of IVs are given by Söderström and Stoica (1981, 1983). Among the various variants of the IVs, the most common candidate may be the one in a bootstrap manner (Wong and Polak 1967, Young 1970, 1976, Rowe 1970, Söderström and Stoica 1981, 1983) with the filtered input-output data of the estimated system model as instrumental variables. The advantage of this candidate is that the knowledge of the noise model is not required to know. Here only the bootstrap type IV method is discussed.

From the approximated discrete-time model of equation (3.1), we have

\[
\tilde{x}(k) = \frac{\sum_{i=1}^n \tilde{b}_i \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} u(k)}{\sum_{i=0}^n \tilde{a}_i \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i}}
\]

(3.40)

where \( \tilde{a}_i, \tilde{b}_i \) are the parameter estimates.

For on-line identification, it is more convenient in practice to generate \( \tilde{x}(k) \) recursively by the above approximated discrete-time model than to generate the estimated real output by
A unified approach to identification using digital filters

constructing the continuous-time system model in a differential equation with the parameter estimates.

The instrumental variable vector is constructed to be

$$\hat{m}(k) = \begin{bmatrix} -\hat{e}_{1z}(k), \ldots, -\hat{e}_{nx}(k), \hat{e}_{1u}(k), \ldots, \hat{e}_{nu}(k) \end{bmatrix}$$  \hspace{1cm} (3.41)

where

$$\hat{e}_{iz}(k) = \hat{e}_{iz}(k) = Q_F(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} \hat{x}(k) \hspace{1cm} (FIR \ filter)$$  \hspace{1cm} (3.42)

or

$$\hat{e}_{iz}(k) = \hat{e}_{iz}(k) = Q_I(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} \hat{x}(k) \hspace{1cm} (IIR \ filter)$$  \hspace{1cm} (3.43)

If

$$\lim_{N \to \infty} \frac{1}{N - k_s + 1} \sum_{k=k_s}^N \hat{m}(k)z^T(k)$$

exists, and is nonsingular, we have the following IV estimate:

$$\hat{\theta} = \left[ \sum_{k=k_s}^N \hat{m}(k)z^T(k) \right]^{-1} \left[ \sum_{k=k_s}^N \hat{m}(k)\xi_{dy}(k) \right]$$  \hspace{1cm} (3.44)

Using equation (3.35), we have

$$\lim_{N \to \infty} \hat{\theta}$$

$$= \left[ \lim_{N \to \infty} \frac{1}{N - k_s + 1} \sum_{k=k_s}^N \hat{m}(k)z^T(k) \right]^{-1} \left[ \lim_{N \to \infty} \frac{1}{N - k_s + 1} \sum_{k=k_s}^N \hat{m}(k)\{z^T(k)\theta + r(k)\} \right]$$

$$= \theta + \left[ \lim_{N \to \infty} \frac{1}{N - k_s + 1} \sum_{k=k_s}^N \hat{m}(k)z^T(k) \right]^{-1} \left[ \lim_{N \to \infty} \frac{1}{N - k_s + 1} \sum_{k=k_s}^N \hat{m}(k)r(k) \right]$$  \hspace{1cm} (3.45)

Since the input $u(k)$ and the estimated real output $\hat{x}(k)$ are not correlated with the measurement noise, we have

$$\lim_{N \to \infty} \frac{1}{N - k_s + 1} \sum_{k=k_s}^N \hat{m}(k)r(k) = 0$$  \hspace{1cm} (3.46)

Therefore the IV method gives consistent estimates.

The estimate (3.44) can be interpreted as the minimizing element of

$$V_{IV}(\theta) = \left\| \sum_{k=k_s}^N \hat{m}(k)r(k) \right\|^p$$  \hspace{1cm} (3.47)
where the norm \( \| \cdot \| \) can be chosen freely and \( p \) arbitrary \( > 0 \). This can be seen from the following simple calculation (Söderström and Stoica 1981):

\[
V_{IV}(\theta) = \left\| \sum_{k=k_s}^{n} \tilde{m}(k) [\xi_{0\theta}(k) - z^T(k)\theta] \right\|^p
\]

\[
= \left\| \left[ \sum_{k=k_s}^{n} \tilde{m}(k)\xi_{0\theta}(k) \right] - \left[ \sum_{k=k_s}^{n} \tilde{m}(k)z^T(k) \right] \theta \right\|^p
\]

\[= 0\]  

(3.48)

In the last equality, the IV estimate (3.44) has been used. Since the loss function always has a minimum value equal to zero, it is in some sense degenerated.

The recursive estimation algorithms of the LS and IV methods can all be described by an algorithm of the following form (Söderström et al. 1978):

\[
\hat{\theta}(k) = \hat{\theta}(k-1) + L(k)e(k)
\]

\[
e(k) = \xi_{0\theta}(k) - \phi^T(k)\hat{\theta}(k-1)
\]

\[
L(k) = \frac{P(k-1)e(k)}{\rho(k) + \phi^T(k)P(k-1)e(k)}
\]

\[
P(k) = \frac{1}{\rho(k)} \left[ P(k-1) - \frac{P(k-1)e(k)\phi^T(k)P(k-1)}{\rho(k) + \phi^T(k)P(k-1)e(k)} \right]
\]

where \( \rho(k) \) is the forgetting factor and a typical choice is given as (Ljung and Söderström 1983, Söderström and Stoica 1989)

\[
\rho(k) = (1 - 0.01)\rho(k - 1) + 0.01, \quad \rho(k_{s}) = 0.95
\]

(3.50)

Now the methods under discussion will be obtained as special cases of the recursive algorithm. The LS method is obtained with

\[
\phi(k) = z(k), \quad \psi(k) = z(k)
\]

(3.51)

and the IV method is obtained with

\[
\phi(k) = z(k), \quad \psi(k) = \tilde{m}(k)
\]

(3.52)
Table 3.1: LS estimates using FIR filter ($M_F = 50$).

| $\omega_{dc}$ ($\omega_{ac}$) | $\hat{a}_1$ (3.0) | $\hat{a}_2$ (4.0) | $\hat{b}_1$ (0.0) | $\hat{b}_1$ (4.0) | $\Delta||\theta||$ $\pm\sigma_\theta$ |
|-----------------------------|-----------------|-----------------|-----------------|-----------------|-------------------------------|
| 10.0 (8.62)                | 2.8482 ±0.0733 | 4.1525 ±0.0427 | 0.0043 ±0.0088 | 4.0303 ±0.0795 | 0.2173 ±0.0510               |
| 8.0 (6.69)                 | 2.9161 ±0.0435 | 4.0747 ±0.0453 | 0.0042 ±0.0145 | 4.0215 ±0.0599 | 0.1044 ±0.0408               |
| 7.0 (5.69)                 | 2.9315 ±0.0405 | 4.0414 ±0.0471 | 0.0051 ±0.0170 | 4.0059 ±0.0450 | 0.0804 ±0.0374               |
| 5.0 (3.78)                 | 2.9357 ±0.0396 | 3.9821 ±0.0457 | 0.0093 ±0.0151 | 3.9612 ±0.0561 | 0.0776 ±0.0391               |
| 4.0 (3.01)                 | 2.9370 ±0.0344 | 4.0107 ±0.0436 | 0.0068 ±0.0162 | 3.9850 ±0.0495 | 0.0660 ±0.0359               |
| 3.0 (2.49)                 | 2.9153 ±0.0572 | 3.9254 ±0.0589 | 0.0058 ±0.0161 | 3.9032 ±0.0756 | 0.1494 ±0.0519               |
| 1.0 (2.05)                 | 2.8559 ±0.0657 | 3.8407 ±0.0596 | 0.0323 ±0.0145 | 3.8032 ±0.0833 | 0.2930 ±0.0558               |

Figure 3.1: Frequency responses of the FIR filters for Table 3.1.
A unified approach to identification using digital filters

3.5 Illustrative examples

To illustrate the effectiveness of the proposed estimation algorithms, we consider a second-order system described by

\[ \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t) = b_1 \dot{u}(t) + b_2 u(t) \]

\[ a_1 = 3.0, \ a_2 = 4.0, \ b_1 = 0.0, \ b_2 = 4.0 \] (3.53)

Simulation experiments are carried out under the following conditions:

\textit{Input signal:}

\[ u(x) = \sin(t) + \sin(1.5t) + 0.5 \sin(3t) + 1.5 \sin(4.5t) + 0.3 \sin(5t) + 0.2 \sin(7t) + 2.5 \sin(7.5t) + \sin(10.5t) \]

\textit{Measurement noise:} white noise.

Example 3.1: Effects of the filter characteristics.

The effects of the filter characteristics on the results of the LS method are investigated with sampling period \( T = 0.04 \) and NSR = 20%; 2500 samples are taken, and the LS estimates are shown in Table 3.1 for the FIR filters, and Table 3.2 for the IIR filters.

In Table 1, \( \omega_{ac} \) denotes the actual cut-off frequency of the FIR digital low-pass filter \( F(z^{-1}) \) defined in equation (3.9) for which

\[ |F(\omega)|^2 \leq \frac{1}{2}, \quad |\omega| \geq \omega_{ac} \] (3.54)
Table 3.2: LS estimates using IIR filter (m=2).

<table>
<thead>
<tr>
<th>$\omega_c$</th>
<th>$a_1$ (3.0)</th>
<th>$a_2$ (4.0)</th>
<th>$b_1$ (0.0)</th>
<th>$b_1$ (4.0)</th>
<th>$\Delta|\theta|$ $\pm\sigma_\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0</td>
<td>2.6765</td>
<td>4.2443</td>
<td>0.0114</td>
<td>3.9769</td>
<td>0.4062</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0786$</td>
<td>$\pm0.0470$</td>
<td>$\pm0.0086$</td>
<td>$\pm0.0939$</td>
<td>$\pm0.0570$</td>
</tr>
<tr>
<td>6.0</td>
<td>2.9842</td>
<td>4.1128</td>
<td>0.0053</td>
<td>4.0004</td>
<td>0.1937</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0620$</td>
<td>$\pm0.0441$</td>
<td>$\pm0.0081$</td>
<td>$\pm0.0795$</td>
<td>$\pm0.0484$</td>
</tr>
<tr>
<td>5.0</td>
<td>2.9011</td>
<td>4.0683</td>
<td>0.0030</td>
<td>4.0050</td>
<td>0.1203</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0526$</td>
<td>$\pm0.0424$</td>
<td>$\pm0.0080$</td>
<td>$\pm0.0712$</td>
<td>$\pm0.0436$</td>
</tr>
<tr>
<td>4.0</td>
<td>2.9423</td>
<td>4.0350</td>
<td>0.0014</td>
<td>4.0037</td>
<td>0.0675</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0445$</td>
<td>$\pm0.0406$</td>
<td>$\pm0.0080$</td>
<td>$\pm0.0639$</td>
<td>$\pm0.0393$</td>
</tr>
<tr>
<td>2.0</td>
<td>2.9571</td>
<td>3.9691</td>
<td>0.0029</td>
<td>3.9560</td>
<td>0.0687</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0401$</td>
<td>$\pm0.0370$</td>
<td>$\pm0.0082$</td>
<td>$\pm0.0549$</td>
<td>$\pm0.0350$</td>
</tr>
<tr>
<td>1.5</td>
<td>2.9298</td>
<td>3.9260</td>
<td>0.0088</td>
<td>3.9089</td>
<td>0.1369</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0475$</td>
<td>$\pm0.0395$</td>
<td>$\pm0.0088$</td>
<td>$\pm0.0615$</td>
<td>$\pm0.0393$</td>
</tr>
<tr>
<td>1.0</td>
<td>2.8350</td>
<td>3.7897</td>
<td>0.0319</td>
<td>3.7600</td>
<td>0.3605</td>
</tr>
<tr>
<td></td>
<td>$\pm0.0811$</td>
<td>$\pm0.0568$</td>
<td>$\pm0.0175$</td>
<td>$\pm0.1016$</td>
<td>$\pm0.0643$</td>
</tr>
</tbody>
</table>

Each of the tables includes the mean and standard deviation of the estimates obtained from Monte-Carlo simulation of 20 experiments. And in each table, the mean error $\Delta\|\theta\|$ and the average standard deviation $\sigma_\theta$ are defined as

$$\Delta\|\theta\| = \|\theta - \bar{\theta}_{mean}\|$$

$$\sigma_\theta = \frac{\sigma_{a_1} + \ldots + \sigma_{a_n} + \sigma_{b_1} + \ldots + \sigma_{b_n}}{2n}$$

They are included in the tables for quick comparison.

The frequency responses of the FIR filters and IIR filters used in Table 3.1 and Table 3.2 are shown in Figures 3.1 and 3.2 respectively. Considering the estimates in Table 3.1 and Table 3.2 and the filter characteristics in Figures 3.1 and 3.2, it is clear that in the case of low measurement noise, if the pass-band of the digital low-pass filters is chosen to match that of the system closely and thus the noise effects are sufficiently reduced, the LS estimates are still acceptable. It should be noted that the LS estimates are not so sensitive to design of the pass-band of the filters. However if the pass-band of the filters is not selected appropriately, the LS method would fail to give unbiased estimates due to the noise effects.

Example 3.2: Parameter estimates when the filters are designed unsuccessfully.
Table 3.3: Parameter estimates for the filters with large cut-off frequency.

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>LS Estimate</th>
<th>IV Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIR filter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_P = 24$</td>
<td>2.7505 ± 0.0540</td>
<td>3.0137 ± 0.0586</td>
</tr>
<tr>
<td>$\omega_{dc} = 10.0$</td>
<td>4.1275 ± 0.0381</td>
<td>4.0008 ± 0.0379</td>
</tr>
<tr>
<td>$\omega_{ac} = 9.32$</td>
<td>0.0121</td>
<td>0.0016</td>
</tr>
<tr>
<td>$\Delta | \theta | = 0.2857 ± 0.0611</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIR filter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 2$</td>
<td>2.6190 ± 0.0515</td>
<td>3.0156 ± 0.0597</td>
</tr>
<tr>
<td>$\omega_c = 10.0$</td>
<td>4.2187 ± 0.0402</td>
<td>4.0004 ± 0.0381</td>
</tr>
<tr>
<td>$\Delta | \theta | = 0.4487 ± 0.0614</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4: Parameter estimates in the presence of high measurement noise.

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>LS Estimate</th>
<th>IV Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIR filter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_P = 40$</td>
<td>2.5557 ± 0.0849</td>
<td>2.9968 ± 0.1047</td>
</tr>
<tr>
<td>$\omega_{dc} = 6.0$</td>
<td>4.0770 ± 0.0755</td>
<td>3.9859 ± 0.0824</td>
</tr>
<tr>
<td>$\omega_{ac} = 5.70$</td>
<td>0.0246</td>
<td>0.0039</td>
</tr>
<tr>
<td>$\Delta | \theta | = 0.4938 ± 0.1081</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIR filter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 2$</td>
<td>2.3659 ± 0.0785</td>
<td>2.9950 ± 0.1067</td>
</tr>
<tr>
<td>$\omega_c = 6.0$</td>
<td>4.1760 ± 0.0777</td>
<td>3.9824 ± 0.0813</td>
</tr>
<tr>
<td>$\Delta | \theta | = 0.7143 ± 0.1191</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.3: LS estimates using FIR filters (Example 3.3).
Figure 3.4: IV estimates using FIR filters (Example 3.3).

Figure 3.5: LS estimates using IIR filters (Example 3.3).

Figure 3.6: IV estimates using IIR filters (Example 3.3).
A unified approach to identification using digital filters

Comparison of the LS method and the IV method is taken when the pass-band of the filters is not designed appropriately with sampling period $T = 0.02$ and NSR = 20%; 5000 samples are taken, and the results are shown in Table 3.3. It is clear that both classes of filters have too wide a pass-band compared to that of the system. Thus the noise effects cannot be reduced sufficiently, therefore the LS method does not give acceptable results. In this case, the IV method is very satisfactory. Therefore if we use the IV method, design of the filters becomes to be less critical.

Example 3.3: Parameter estimates in the presence of high measurement noise.

Comparison of the LS method and the IV method is taken when the output is corrupted by a high measurement noise with sampling period $T = 0.02$ and NSR = 50%; 5000 samples are taken, and the results are shown in Table 3.4. In the presence of high measurement noise, it is difficult to obtain accurate estimates with the LS method. However, the IV method is very efficient for both the FIR digital low-pass filters and the IIR digital low-pass filters. To show the properties of the LS and IV estimators, the examples of the estimates are given in Figures 3.3~3.6.

3.6 Unification of the other methods

Unification of some other methods studied widely by a lot of researchers is provided here, in terms of ‘digital low-pass filters’. It is found that some direct parameter identification methods for continuous systems (Unbehauen and Rao 1990) would include the following.

1 Find a pre-processing method to avoid the direct derivatives.

2 Find the approximating or discretizing techniques so that the pre-processing procedure can be performed purely digitally.

3 Use an identification algorithm to estimate the system parameters from sampled input-output data.

For convenience of exposition, we consider that these methods can be unified in the sense of a pre-processing operation $H(p)$ in continuous-time domain. The pre-processing operation $H(p)$ is such that it facilitates generation of the appropriate pre-processed signals for parameter estimation, while retaining the parameters of the continuous system model.
in their actual original form. Multiplying both sides of the system equation (3.1) by the pre-processing operation $H(p)$ leads to

$$H(p)p^n x(t) + \sum_{i=1}^{n} a_i H(p)p^{n-i}x(t) = \sum_{i=1}^{n} b_i H(p)p^{n-i}u(t)$$ (3.56)

It will be shown that some direct methods for identification of continuous systems are obtained by selecting a particular form of $H(p)$.

Direct approximation of differentiation

A simple way is to choose $H(p)$ to be an all-pass filter:

$$H(p) = 1$$ (3.57)

Then equation (3.56) becomes to be the original system differential equation:

$$p^n x(t) + a_1 p^{n-1}x(t) + \cdots + a_n x(t) = b_1 p^{n-1}u(t) + b_2 p^{n-2}u(t) + \cdots + b_n u(t)$$ (3.58)

The problem is how to use direct approximations of the differentiations to identify the parameters. The approximation techniques for direct numerical differentiations were studied for example, by Wang, Yang and Chang (1987) with a differentiation operational matrix using generalized orthogonal polynomials, and also by Kraus and Schaufelberger (1990) with differential operators. This method while being simple and straightforward is not robust to the noise since there is no low-pass pre-processing operation performed to clean-up the noisy measurements.

Integral equation

As discussed detailedy in chapter 2, instead of the approximations of the numerical differentiations, the system differential equation is integrated $n$ times from time $t_0$ to $t$.

In this case we have

$$H(p) = p^{-n}$$ (3.59)

Integrating both sides of equation (3.1) $n$ times leads to

$$\sum_{i=0}^{n} a_i p^{-n}p^{n-i}x(t) = \sum_{i=1}^{n} b_i p^{-n}p^{n-i}u(t)$$ (3.60)
A unified approach to identification using digital filters

Especially when the trapezoidal rule is applied, the conventional integral-equation approach termed as the IIF method is given as

\[ \sum_{i=0}^{n} a_i p^{-i} x(k) = \sum_{i=1}^{n} b_i p^{-i} u(t) + \sum_{i=1}^{n} c_i (t - t_0)^{i-1} \]

\[ p^{-i} x(kT) = \left( \frac{T 1 + z^{-1}}{2 1 - z^{-1}} \right)^i x(kt) \]  \hspace{1cm} (3.61)

\[ p^{-i} u(kT) = \left( \frac{T 1 + z^{-1}}{2 1 - z^{-1}} \right)^i u(kT) \]

and the new integral-equation approach termed as the NIIF method is given by

\[ \sum_{i=0}^{n} a_i \xi_{ix}(k) = \sum_{i=1}^{n} b_i \xi_{iu}(k) \]

\[ \xi_{ix}(k) = \left( \frac{T}{2} \right)^i \frac{(1 - z^{-1})^{n-i}(1 + z^{-1})^i}{(1 - z^{-1})^n} x(k) \]  \hspace{1cm} (3.62)

\[ \xi_{iu}(k) = \left( \frac{T}{2} \right)^i \frac{(1 - z^{-1})^{n-i}(1 + z^{-1})^i}{(1 - z^{-1})^n} u(k) \]

Some comments to the results obtained in chapter 2 are given as follows.

Comment 3.2: The IIF requires to estimate the initial conditions whereas the NIIF can avoid the initial condition problem.

Comment 3.3: The low-pass characteristics of both the IIF and NIIF are similar and hence the noise reducing properties are similar.

Comment 3.4: Both the IIF and the NIIF can be viewed as a kind of unstable IIR filters which have multiple poles on the unit circle. And hence the blow up of the outputs of the unstable filters may give rise to some numerical difficulties in some cases, especially in the situation where the output signal is corrupted by a considerable high noise. In these cases, the algorithm to obtain consistent estimates becomes to be very sophisticated (Sagara, Yuan and Wada 1988a).

Integral operations over finite time interval

The finite time integral operation approach has been studied by some researchers (Eitzenberg 1988, Sagara and Zhao 1990, Schoukens 1990). The basic idea in this method is to replace the system signal derivatives by multiple integral operations over selected finite time intervals, and thus the initial condition problem is avoided.
A unified approach to identification using digital filters

Assume that the differential operator $p$ is viewed as a Laplace operator and then define an integral operation over a selected interval $[t - LT, t]$ as

$$
(1 - e^{-LTp})p^{-1}x(t) = \int_0^t x(t) \, dt - \int_0^{t-LT} x(t) \, dt = \int_{t-LT}^t x(t) \, dt
$$

(3.63)

where $L$ is a natural number.

In this case, we take $H(p)$ as

$$
H(p) = (1 - e^{-LTp})^n p^{-n}
$$

(3.64)

with the frequency response

$$
H(j\omega) = \left( \frac{1 - e^{-j\omega LT}}{j\omega} \right)^n
$$

(3.65)

$$
= \left[ \frac{2\pi \sin \pi(\omega/\omega_s)}{\omega_s \pi(\omega/\omega_s)} e^{-j\pi \omega_s / \omega_s} \right]^n, \quad \omega_s = \frac{2\pi}{LT}
$$

Clearly, $H(j\omega)$ depends on the selection of $LT$. It is clear that for a determined $LT$, a small $T$ requires a large $L$, which may increase the computational burden and the memory of the computer. Therefore a trade-off should be taken between the truncation accuracy and the computer load.

In this case, the system differential equation becomes to be

$$
\sum_{i=0}^n a_i H_i(p)x(t) = \sum_{i=1}^n b_i H_i(p)u(t)
$$

$$
H_i(p) = (1 - e^{-LTp})^n p^{-i} = (1 - e^{-LTp})^{n-i} \left[ (1 - e^{-LTp})^i p^{-i} \right]
$$

(3.66)

$H_i(p)$ can be discretized by Simpson’s or trapezoidal rules:

$$
H_i'(z^{-1}) = (1 - z^{-L})^{n-i} (h_0 + h_1 z^{-1} + \ldots + h_L z^{-L})^i
$$

(3.67)

which are a kind of FIR digital filters. Therefore, this approach is one kind of the FIR filtering approach. We term this method as finite integral filter (FIF) method in contrast to the IIF and NIF. Especially, when $H_i(p)$ is discretized by the trapezoidal rule, we have

$$
H_i'(z^{-1}) = (1 - z^{-L})^{n-i} \left( \frac{T}{2} \right)^i (1 + 2z^{-1} + \ldots + 2z^{-(L-1)} + z^{-L})^i
$$

$$
= T_i(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i}
$$

(3.68)
A unified approach to identification using digital filters

where

\[ T_r(z^{-1}) = (1 + z^{-1} + \cdots + z^{-(m-1)})^n \] (3.69)

Hence we have the following approximated model:

\[ \sum_{i=0}^{n} a_i T_{ix}(k) = \sum_{i=1}^{n} b_i T_{iu}(k) \]

\[ \xi_{T_{ix}}(k) = T_r(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i(1 - z^{-1})^n-i x(k) \] (3.70)

\[ \xi_{T_{iu}}(k) = T_r(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i(1 - z^{-1})^n-i u(k) \]

It is interesting to find that in this case, \( T_r(z^{-1}) \) is a special one of the FIR filter \( Q_F(z^{-1}) \) used in equation (3.10). However, inspection of equations (3.12) and (3.65) indicates that the modern digital filtering techniques may yield more excellent filtering effects than the classical integral operations, since the frequency response of the integral operations described in equation (3.65) has too many zeros, which may filter off the real system signals in particular situations.

State variable filters

The continuous-time SVF approach is very popular in the literature. Applications of the continuous-time SVFs have a long history in identification of continuous-time models (Young 1981). Levadi (1964) used to propose a purely analog IV method combined with the analog SVF. Young (1970) developed it in a hybrid approach with the analog SVF and the discrete IV method. And it was pointed out by Young and Jakeman (1980) that the SVF should be designed so that its pass-band matches that of the system under study as possible. To the knowledge of the authors, discretization techniques of the SVFs have not been published so much as those of the multiple integrations. Therefore it is still worthwhile to discuss about the discrete-time SVFs (Sagara, Yang and Wada 1990) which seems to be more satisfactory than the multiple integrations which are approximated or discretized by orthogonal functions or numerical integrating rules.

Consider an \( m \)th order continuous SVF

\[ H(p) = \frac{1}{(p + \lambda)^m} \quad (m \geq n) \] (3.71)

where \( \lambda \) is a constant which determines the pass-band of \( H(p) \). Usually, it is determined that \( \lambda > 0 \).
A unified approach to identification using digital filters

The system equation (3.1) becomes to be

$$\sum_{i=0}^{n} a_i \frac{1}{(p + \lambda)^m} p^{-i} x(t) = \sum_{i=1}^{n} b_i \frac{1}{(p + \lambda)^m} p^{-i} u(t)$$

(3.72)

It is a straightforward development to discretize the continuous SVF by the bilinear transformation which is closely related to the trapezoidal rule so that the whole identification algorithm can be performed purely in a digital manner (Sagara, Yang and Wada 1990):

$$\sum_{i=0}^{n} a_i \xi_{Sx}(k) = \sum_{i=1}^{n} b_i \xi_{Su}(k)$$

$$\xi_{Sx}(k) = Q_S(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})(1 - z^{-1})^{n-i} x(k)$$

(3.73)

$$\xi_{Su}(k) = Q_S(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})(1 - z^{-1})^{n-i} u(k)$$

where

$$Q_S(z^{-1}) = -\left( \frac{T}{2} \right)^{m-n} \frac{(1 + z^{-1})^{m-n}}{[(1 + \frac{T}{2} \lambda) + (\frac{T}{2} \lambda - 1) z^{-1}]^m}$$

(3.74)

Considering the model (3.30), it is obvious that the discrete-time SVF approach is one kind of the IIR digital filtering approach similar to the method using Butterworth filter, Chebyshev filter, inverse Chebyshev filter, Elliptic or Cauer filters etc.

It is clear that if $\lambda = 0$ and $m = n$, the discrete SVFs are equivalent to our new integral filters derived by the trapezoidal rule (see equation (3.62)). Therefore, the proposed NIIF approach using the trapezoidal rule can be viewed as a special case of the discrete SVF approach obtained by the bilinear transformation. It should be noted that the conventional IIF approach and the SVF approach have been viewed as two different approaches in the literature. The essential difference is that the conventional IIF approach requires to estimate the initial conditions whereas the SVF approach eliminates the initial condition problem.

Some comments are given here for the NIIF and the SVF.

**Comment 3.5:** To compute out the outputs of the discrete SVFs, the initial values of the SVFs should be set strictly to be zero following the suggestion of **Remark 2.3** in chapter 2.

**Comment 3.6:** The NIIFs (or $\lambda = 0$ for the discrete SVFs) can be viewed as a kind of unstable IIR filters which have multiple poles on the unit circle, therefore the outputs of the integral filters will grow rapidly with the time, especially for the high-order systems. In such
cases, the estimation algorithm may be reset after a suitable period of time to prevent the blow up of the integral filter outputs, which may cause numerical problems as discussed in chapter 2. In contrast, the discrete SVFs with $\lambda > 0$ are viewed as stable IIR filters, and thus the filter outputs are always bounded if the filter inputs are bounded. We emphasize that with the aid of the bilinear transformation, the NIIF and the SVF approach can be unified as the IIR digital filtering approach with $\lambda \geq 0$.

Comment 3.7: When the effects of the measurement noise cannot be neglected, an appropriate choice of $\lambda$ should be taken to reduce the noise effects, whereas in some previous works such as analog SVFs (Young 1970) or the PFCs (Saha and Rao 1982, Saha and Mandal 1990), it was suggested that $\lambda$ should be chosen large enough so that the initial conditions decay as soon as possible, although a filter with large $\lambda$ may pass considerable noise. Therefore a trade-off between these two purposes should be taken. However, considering the discussions in chapter 2 and the results obtained in Example 2.2, even when $\lambda = 0$, the IIR filtering approach works quite well in the presence of large non-zero initial values. On the analogy of this, it is clear that under the non-zero initial conditions, $\lambda$ need not be chosen large enough to eliminate the initial conditions.

Comment 3.8: When the initial conditions need be estimated, the PMF techniques may be preferable to the discrete SVFs obtained by the bilinear transformation, since for the estimation model derived by the Poisson filters, the initial conditions and the systems parameters are estimated simultaneously by the LS techniques (Saha and Mandal 1990). However, in the case where only the system parameters require estimation, the IIR digital filters derived by the bilinear transformation may be simpler.

### 3.7 Discussions on the problem of the initial conditions

The derivations of the two types of estimation models (3.6) and (3.30) can also be divided into the following two steps.

1 Obtain the approximated discrete-time model (3.4) which does not contain the terms concerning the initial conditions by the bilinear transformation.

2 Multiply both sides of equation (3.4) by the pre-designed $Q_i(z^{-1})$ or $Q_f(z^{-1})$ and then obtain (3.6) or (3.30).
Therefore, in contrast to the conventional IIF approach, the identification algorithms based on the IIR filtering and FIR filtering approaches can start at any time without any restrictions of the initial conditions, since the initial conditions do not appear in the both types of estimation models. For the estimation model (3.6) of the FIR filtering approach, it is not difficult to understand this feature, since equation (3.6) contains only the moving average processes of the sampled system signals. However, for the model (3.30) of the IIR filtering approach, the effects of the initial conditions have not been clarified in the literature, because of some preconceptions.

In spite of some preconceptions, we will give clarified discussions on the initial conditions.

One may construct the following estimation model in stead of equation (3.72) without any resistance:

\[
\sum_{i=0}^{n} a_i p^{n-i} x^*(t) = \sum_{i=1}^{n} b_i p^{n-i} u x^*(t) \tag{3.75}
\]

where the filtered signals \( u^*(t) \) and \( x^*(t) \) are given by

\[
x^*(t) = \frac{1}{(p + \lambda)^m} x(t) \]

\[
u^*(t) = \frac{1}{(p + \lambda)^m} u(t) \tag{3.76}
\]

The filtered input-output signals and their derivatives are computed simultaneously in some manner such as a state space realization method.

However, one may find to his experience, that for the above model the non-zero initial conditions can affect the performance of the estimators (Homsi, Titli and Despujols 1991). It should be pointed out that this conclusion is not appropriate, since usually \( H(p)p^{n-i} \neq p^{n-i}H(p) \).

This is first shown through a simple example before theoretical analysis.

Example 3.4: Comparison of the estimation models (3.72) and (3.75).

Consider a first order system:

\[
px(t) + a_1 x(t) = u(t) \quad (a_1 = 1) \tag{3.77}
\]

with input-output signals

\[
u(t) = \sin(t) + \cos(t)\]

\[
x(t) = \sin(t) + x(0)e^{-t} \tag{3.78}
\]
A unified approach to identification using digital filters

where \( x(0) \) is a non-zero initial value. The pre-filter \( H(p) \) is given as

\[
H(p) = \frac{1}{p + \lambda} \tag{3.79}
\]

For the model (3.72), if \( x(t) \) and \( u(t) \) are available, we have the filtered signals as

\[
\xi_{0u}(t) = \frac{1}{p + \lambda} px(t)
\]
\[
\xi_{1u}(t) = \frac{1}{p + \lambda} x(t)
\]
\[
\xi_{1u}(t) = \frac{1}{p + \lambda} u(t) \tag{3.80}
\]

When the filters are computed on a digital computer, the initial values of the outputs of the filters can be assumed to be zero. Then when \( \lambda \neq 1 \), some straightforward but tedious calculations show

\[
\xi_{0u}(t) = \frac{\lambda \cos t + \sin t}{1 + \lambda^2} - \frac{x(0)e^{-t}}{\lambda - 1} - \left( \frac{\lambda}{1 + \lambda^2} - \frac{x(0)}{\lambda - 1} \right)e^{-\lambda t}
\]
\[
\xi_{1u}(t) = \frac{\lambda \sin t - \cos t}{1 + \lambda^2} + \frac{x(0)e^{-t}}{\lambda - 1} - \left( -\frac{1}{1 + \lambda^2} + \frac{x(0)}{\lambda - 1} \right)e^{-\lambda t} \tag{3.81}
\]
\[
\xi_{1u}(t) = \frac{\lambda \sin t - \cos t + \lambda \cos t + \sin t}{1 + \lambda^2} - \left( \frac{\lambda - 1}{1 + \lambda^2} \right)e^{-\lambda t}
\]

It is easy to verify the fact that

\[
\xi_{0u}(t) + a_1 \xi_{1u}(t) = \xi_{1u}(t) \tag{3.82}
\]

Then parameter \( a_1 \) can be estimated by the LS method from the signals \( \xi_{0u}(t) \), \( \xi_{1u}(t) \) and \( \xi_{1u}(t) \).

For the model (3.75), we have

\[
x^*(t) = \frac{1}{p + \lambda} x(t)
\]
\[
= \frac{\lambda \sin t - \cos t}{1 + \lambda^2} + \frac{x(0)e^{-t}}{\lambda - 1} - \left( -\frac{1}{1 + \lambda^2} + \frac{x(0)}{\lambda - 1} \right)e^{-\lambda t}
\]
\[
u^*(t) = \frac{1}{p + \lambda} u(t)
\]
\[
= \frac{\lambda \sin t - \cos t + \lambda \cos t + \sin t}{1 + \lambda^2} - \left( \frac{\lambda - 1}{1 + \lambda^2} \right)e^{-\lambda t} \tag{3.83}
\]

and

\[
px^*(t) = \frac{\lambda \cos t + \sin t}{1 + \lambda^2} - \frac{x(0)e^{-t}}{\lambda - 1} + \left( -\frac{\lambda}{1 + \lambda^2} + \frac{\lambda x(0)}{\lambda - 1} \right)e^{-\lambda t} \tag{3.84}
\]
A unified approach to identification using digital filters

However, unfortunately, we find

\[ px^*(t) + a_1 x^*(t) = u^*(t) + x(0) e^{-\lambda t} \] (3.85)

This implies that the equality of the estimation model (3.75) holds only when \( x(0) = 0 \) or \( \lambda \) is sufficiently large and hence \( x(0) e^{-\lambda t} \) dies away quickly. It is clear if \( \lambda = 0 \), \( x(0) e^{-\lambda t} \) never decays, and in this case the above estimation model is one case of the IIF model of (3.61).

When the term \( x(0) e^{-\lambda t} \) has decayed sufficiently, the parameter \( a_1 \) can be estimated by the LS method from

\[ px^*(t) + a_1 x^*(t) = u^*(t) \] (3.86)

This should be the reason that one may suggest that \( \lambda \) should be chosen large enough so that the initial conditions decay as soon as possible, although a filter with large \( \lambda \) may pass considerable noise.

This example implies that under large non-zero conditions, the estimation model (3.75) is inappropriate. Only when all the initial values of the system under study are zero, (3.72) and (3.75) are equivalent. Motivated by this we have the following theorem:

**Theorem 3.1** Consider a general pre-filter

\[ H(p) = \frac{1}{c_0 p^m + c_1 p^{m-1} + \cdots + c_m} \quad (m \geq n) \] (3.87)

The commutative law between the linear operations \( H(p) \) and \( p^{n-i} \) does not hold under non-zero initial conditions, i.e.

\[ H(p)p^{n-i}x(t) \neq p^{n-i}H(p)x(t) \quad (i = 1, \cdots, n-1) \] (3.88)

Only when \( x^{(n-i)}(0)|_{t=0} = 0 \) for \( i = 1, \cdots, n \), the commutative law holds:

\[ H(p)p^{n-i}x(t) = p^{n-i}H(p)x(t) \] (3.89)

**Proof**

Define

\[ \xi_{ix}(t) = H(p)p^{n-i}x(t) \]

\[ = \frac{1}{c_0 p^m + c_1 p^{m-1} + \cdots + c_m \{ p^{n-i}x(t) \}} \] (3.90)
and
\[
\xi_i^*(t) = p^{n-i}H(p)x(t)
\]
\[
= p^{n-i}\left[\frac{1}{c_0p^m + c_1p^{m-1} + \cdots + c_m}x(t)\right]
\] (3.91)

When the filters are implemented on a digital filter, it is reasonable to let \(\xi_i(0) = \xi_i^*(0) = 0\), for \(i = 0, 1, \cdots, m\).

Perform the Laplace transformation, we have
\[
\xi_i(s) = \frac{s^{n-i}x(s)}{c_0s^m + c_1s^{m-1} + \cdots + c_m} - \sum_{k=1}^{n-i} s^{n-i-k}x^{(k-1)}(0)
\] (3.92)

and
\[
\xi_i^*(s) = \frac{s^{n-i}x(s)}{c_0s^m + c_1s^{m-1} + \cdots + c_m}
\] (3.93)

Then \(\xi_i(t)\) and \(\xi_i^*(t)\) can be solved by performing the inverse Laplace transformation to the above two equations. And it is clear that the results of the theorem hold, i.e. only when \(x^{(n-0)}(0) = 0\) for \(i = 1, \cdots, n\), the relation
\[
H(p)p^{n-i}x(t) = p^{n-i}H(p)x(t)
\] (3.94)

holds.

It is not difficult to understand that when we use the IIR filtering approach, if the effects of the initial values are not significant, one may encounter the initial condition problem if he neglects the results of Theorem 3.1.
3.8 Conclusion

In this chapter, a unified approach to recursive identification of continuous systems from sampled input-output data using digital low-pass filters has been discussed.

The digital low-pass filters are introduced to avoid direct approximations of system signal derivatives from sampled system input-output data. Using a pre-designed digital low-pass filter, an approximated discrete-time estimation model with continuous system parameters is constructed easily. Thus the system parameters can be identified directly by recursive identification algorithms.

Two classes of filters (FIR filter and IIR filter) have been applied to identification of continuous systems. And a unified view of some well-known direct methods for identification of continuous systems has been carried out. As mentioned in section 3.6, all the methods can be unified in three steps. The essential difference of the methods is in the first step, i.e. to choose an effective pre-processing procedure $H(p)$. It is pointed out that some well-known methods reported in the literature are thought to be either the FIR or the IIR filtering approach.

Some new comments to the initial condition problem which is still unclear in the literature have been given. The parameter estimation procedure based on the approximated discrete-time model derived here can be made starting at any time of interest, without any restriction on the initial conditions.

Simulation results show that both classes of filters are effective in continuous system identification. Numerical results show that the parameter estimates are not so sensitive to the pass-band of the filter. It is often a point of practical policy to attempt to filter off unwanted frequency components not thought useful to the analysis by the introduction of band-pass filters that match as closely as possible the frequency band of importance to the analysis. If the filter is designed so that its pass-band matches that of the system closely and thus the noise effects are sufficiently reduced, acceptable estimates can be obtained by the recursive LS identification algorithms for the case of low measurement noise. When there is not any a priori knowledge of the unknown systems to be obtained and thus we may fail to determine the filters appropriately, or when the output measurement is corrupted by a high measurement noise, the LS method can not give excellent estimates. In these cases, the IV method in a bootstrap manner is shown to be a quite efficient method.
Chapter 4

Recursive Identification Algorithms for Continuous-Time Systems Using an Adaptive Procedure

4.1 Introduction

In chapter 3, a unified approach to identification of continuous-time systems has been discussed in terms of 'digital filtering'. And it was emphasized that using the bilinear transformation, the discrete versions of some well-known pre-processing methods which are introduced to avoid the direct signal derivatives can be unified as either the digital FIR or the IIR filtering approach.

In the literature of discrete-time system identification, the well-known standard recursive methods would include LS, generalized least squares (GLS), extended least squares (ELS), recursive maximum likelihood (RML) and bootstrap techniques (Söderström and Stoica 1989). These methods, while promising to give consistent estimates in the presence of measurement noise, are computationally demanding and therefore they are difficult to apply to the continuous-time systems directly.

However, as shown in chapter 3, for the discrete-time estimation models derived by the digital filters, the equation error in the estimation model including the filtered measurement noise is in a quite different form from the measurement noise added to the sampled data of system output. For instance, even when the measurement noise is white noise, the equation error in the discrete-time estimation model becomes to be an MA process of large length for the FIR filtering approach, or an ARMA process for the IIR filtering approach. Therefore
for the discrete-time estimation model derived by those digital filters, it is more difficult to obtain consistent estimates applying the existing standard recursive identification algorithms for common discrete-time systems directly. It is also difficult to estimate the associated measurement noise model.

It was confirmed by simulation study that the pass-band of the pre-filter should be chosen such that it matches that of the system under study. However, in some practical situations, we may fail to determine the filters appropriately, since little a priori knowledge of the unknown systems can be obtained. In continuous-time domain, it is reasonable to choose \( H(p) = 1/A(p) \) in equation (3.56), since, usually the pass-band of a physical system depends mainly on its poles (Young 1976, Young and Jakeman 1980).

In this chapter we present some recursive identification techniques for linear SISO continuous systems using an adaptive IIR filtering procedure. An approximated discrete-time model of the continuous system is first obtained by the bilinear transformation (Krishna 1988) as equation (3.4). Then using the estimated denominator of the transfer function of the discrete-time model as an adaptive IIR digital pre-filter \( 1/A'(z^{-1}) \), an approximated discrete-time estimation model with continuous system parameters is derived. The low-pass pre-filter \( 1/A'(z^{-1}) \) is introduced to attempt to filter off unwanted high frequency components of the noises (Young 1976). Another advantage of using such a pre-filter is that the output noise remains in its original form in the equation error (Young and Jakeman 1980), and this fact makes it possible to apply the existing recursive identification algorithms to the derived approximated discrete-time estimation model. The adaptive IIR digital filters can be thought to be discrete-time versions of the adaptive continuous-time SVFs suggested by Young and Jakeman (1980) to avoid direct approximations of the system signal derivatives from sampled data. Usually, the consistency property is discussed in discrete-time. It seems that the approximated discrete-time estimation model is much more suitable for recursive identification algorithms, since our method is less computationally demanding and easy to implement on digital computers.

It will be found that the conventional LS method is still efficient due to the excellent noise reducing effects of the filters in the presence of low measurement noise. However, when the output is corrupted by a high measurement noise, it will be found through a theoretical analysis that the conventional LS method cannot give consistent estimates even when the measurement noise is white noise. This fact is quite different from the case of the Steiglitz and McBride algorithm (Steiglitz and McBride 1965) for common discrete-time systems. To
obtain consistent estimates in the presence of white noise of high level, we can consider an IV method which modifies the LS method by introducing an instrumental variable vector with filtered inputs and delayed filtered outputs. Some variations of the IV method are also proposed for various types of coloured measurement noises. The well-known bootstrap method will also be applied if the noise model need not be treated. The proposed identification algorithms have close similarity to the standard recursive identification algorithms for discrete-time systems. It is our major purpose of this chapter to investigate how the standard recursive identification algorithms can be applied to identification of continuous systems. Simulation results show that the proposed algorithms are quite satisfactory.

4.2 Estimation model

Consider the following SISO continuous system

\[
A(p)x(t) = B(p)u(t)
\]

\[
A(p) = \sum_{i=0}^{n} a_ip^{n-i} \quad (a_0 = 1)
\]

\[
B(p) = \sum_{i=1}^{n} b_ip^{n-i}
\]  

(4.1)

An approximated discrete-time model can be obtained by the bilinear transformation as mentioned in chapter 3:

\[
A'(z^{-1})x(k) = B'(z^{-1})u(k)
\]

\[
A'(z^{-1}) = \sum_{i=0}^{n} a_i \left( \frac{T}{2} \right)^i (1 + z^{-1})(1 - z^{-1})^{n-i}
\]

\[
B'(z^{-1}) = \sum_{i=1}^{n} b_i \left( \frac{T}{2} \right)^i (1 + z^{-1})(1 - z^{-1})^{n-i}
\]

(4.2)

As mentioned previously, the measurement of the output variable is corrupted by a measurement noise. The sampled measurement of the output is described as

\[
y(k) = x(k) + \eta(k)
\]

(4.3)

where \(\eta(k)\) denotes the measurement noise.
Recursive identification algorithms using an adaptive procedure

Then the approximated discrete-time model becomes

\[(1 - z^{-1})^n y(k) + \sum_{i=1}^{n} a_i \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} y(k) = \]

\[\sum_{i=1}^{n} b_i \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} u(k) + A'(z^{-1}) \eta(k)\]

(4.4)

Since \((1 - z^{-1})^n y(k)\) may accentuate the effects of the measurement noise, we can rewrite the approximated discrete-time model into the following estimation model which is composed of filtered sampled input-output data:

\[\xi_{A0p}(k) + \sum_{i=1}^{n} a_i \xi_{Aip}(k) = \sum_{i=1}^{n} b_i \xi_{Aiu}(k) + \eta(k)\]

\[\xi_{Aiu}(k) = Q_{IA}(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} u(k)\]

\[\xi_{Aip}(k) = Q_{IA}(z^{-1}) \left( \frac{T}{2} \right)^i (1 + z^{-1})^i (1 - z^{-1})^{n-i} y(k)\]

(4.5)

where

\[Q_{IA}(z^{-1}) = \frac{1}{A'(z^{-1})}\]

(4.6)

can be considered as a digital filter of IIR kind constructed in an adaptive manner with the consistent estimate of \(A'(z^{-1})\).

Remark 4.1: The IIR filters can be constructed in an adaptive manner with the estimate of \(A'(z^{-1})\). The adaptive IIR digital filters used here can be thought to be discrete-time versions of the adaptive continuous-time SVFs suggested by (Young and Jakeman 1980) to avoid direct approximations of the system signal derivatives from sampled data and also to achieve the asymptotic statistical efficiency. The estimation model (4.5) can also be obtained through choosing \(H(p) = 1/A(p)\) in equation (3.56) and then discretizing equation (3.56) by the bilinear transformation. In continuous-time domain, the low-pass pre-filter \(H(p) = 1/A(p)\) is introduced to attempt to filter off unwanted high frequency (higher than the cut-off frequency determined by \(1/A(p)\)) components of the noises.

Remark 4.2: It is possible to apply the existing standard identification algorithms such as the LS, GLS, ELS, ML and the bootstrap techniques for the common discrete-time system to identify the approximated discrete-time estimation model of the continuous system described in equation (4.5).

Remark 4.3: It should be noted that for the approximated model of equation (4.5), even if \(\eta(k)\) is a white noise, the conventional LS method cannot give consistent estimates since
Recursive identification algorithms using an adaptive procedure

\( \xi_{A_{0\theta}}(k)(i = 1, \cdots, n) \) are correlated with \( \eta(k) \). Therefore the identification algorithms for the approximated discrete-time estimation model may be different from those for the common discrete-time systems.

4.3 Estimation methods

Now we will consider the identification algorithms which are closely similar to the existing discrete-time recursive identification techniques for various types of the measurement noises.

Case 1: The white noise case

The first case is that the measurement noise is assumed to be a white noise, i.e.

\[ \eta(k) = e(k) \quad (4.7) \]

where \( e(k) \) is white noise with zero-mean and variance \( \sigma_e^2 \).

The discrete-time estimation model of equation (4.5) can be written in a vector form as

\[z^T(k)\theta + e(k) \]

where

\[
\begin{bmatrix}
-\xi_{A_{1\theta}}(k), \cdots, -\xi_{A_{n\theta}}(k), \\
\xi_{A_{1\theta}}(k), \cdots, \xi_{A_{n\theta}}(k)
\end{bmatrix}
\]

\( \theta^T = [a_1, \cdots, a_n, b_1, \cdots, b_n] \)

If the input signals are sufficiently rich so that

\[
\left[ \sum_{k=k_g}^{N} z(k)z^T(k) \right]^{-1}
\]

exists, then using the conventional LS method, we have the estimate of \( \theta \):

\[
\hat{\theta} = \left[ \sum_{k=k_g}^{N} z(k)z^T(k) \right]^{-1} \left[ \sum_{k=k_g}^{N} z(k)\xi_{A_{0\theta}}(k) \right] \quad (4.9)
\]

It is necessary to analyze the consistency of the LS estimate, i.e. the limiting behavior of the LS estimates when the number of data tends to infinity. It will be found that usually the LS estimates are asymptotically biased even when the sampled output is corrupted by a
Recursive identification algorithms using an adaptive procedure

white noise. Using equation (4.8) we have

$$\text{plim} \hat{\theta}_{N \to \infty} = \left[ \text{plim} \frac{1}{N - k_S + 1} \sum_{k=k_S}^{N} z(k)z^T(k) \right]^{-1} \cdot \left[ \text{plim} \frac{1}{N - k_S + 1} \sum_{k=k_S}^{N} z(k)\{z^T(k)\theta + e(k)\} \right]$$

$$= \theta + \left[ \text{plim} \frac{1}{N - k_S + 1} \sum_{k=k_S}^{N} z(k)z^T(k) \right]^{-1} \cdot \left[ \text{plim} \frac{1}{N - k_S + 1} \sum_{k=k_S}^{N} z(k)e(k) \right]$$

(4.10)

Furthermore, we obtain the following result due to the fact that $\xi_{Aly}(k)(i = 1, \cdots, n)$ are correlated with $e(k)$:

$$\text{plim} \frac{1}{N - k_S + 1} \sum_{k=k_S}^{N} z(k)e(k) = E \begin{bmatrix} -\xi_{Aly}(k) \\ \vdots \\ -\xi_{Any}(k) \\ \xi_{Aly}(k) \\ \vdots \\ \xi_{Any}(k) \end{bmatrix} e(k) \neq 0$$

(4.11)

which means that using the conventional LS method we cannot obtain consistent estimates even if the measurement noise is white noise. This fact is different from the case of the Steiglitz and McBride algorithm (Steiglitz and McBride 1965) for common discrete-time systems which may give consistent estimates in the presence of white noise (Stoica and Söderström 1981).

An approach to this problem is to choose an IV vector $m^T(k)$ whose elements are highly correlated with the sampled real system output $x(k)$, but totally uncorrelated with the noise $e(k)$ (Young 1970, Söderström and Stoica 1981). For example, we can introduce an IV vector with filtered inputs and delayed filtered outputs to avoid the correlations between $\xi_{Aly}(k)(i = 1, \cdots, n)$ and $e(k)$:

$$m^T(k) = [-\xi_{Aly}(k - l), \cdots, -\xi_{Any}(k - l), \xi_{Aly}(k), \cdots, \xi_{Any}(k)], \quad l \geq 1$$

(4.12)

Remark 4.4: Although the delay parameter $l = 1$ is sufficient, during the estimation process, the pre-filter $1/A'(z^{-1})$ is constructed adaptively by the parameter estimates, and any error of the estimates may make the IV vector $m(k)$ be slightly correlated with the noise $e(k)$. These possible correlations can be reduced by taking a considerably large $l > 1$, and hence the estimates may be more accurate. However, an unnecessarily large $l$ should be avoided, since too large a $l$ may destroy the existence of $[\sum_{k=k_S}^{N} m(k)z^T(k)]^{-1}$ (Söderström
and Stoica (1981) and hence may make the algorithms very sensitive or even unstable. In this chapter, \( l \) is chosen to be 1, although some other choices may give better results.

When the input signals are sufficiently rich so that

\[
\left[ \sum_{k=k_S}^{N} m(k)z^T(k) \right]^{-1}
\]

exists, we have the following IV method:

\[
\hat{\theta} = \left[ \sum_{k=k_S}^{N} m(k)z^T(k) \right]^{-1} \cdot \left[ \sum_{k=k_S}^{N} m(k)\xi_{A0\nu}(k) \right]
\]

which modifies the LS method given in equation (4.9) by avoiding the correlations between \( \xi_{A0\nu}(k) \) \((i = 1, \ldots, n)\) and \( e(k) \).

Also using equation (4.8), we have

\[
\lim_{N \to \infty} \hat{\theta} = \left[ \lim_{N \to \infty} \frac{1}{N-k_S+1} \sum_{k=k_S}^{N} m(k)z^T(k) \right]^{-1} \cdot \left[ \lim_{N \to \infty} \frac{1}{N-k_S+1} \sum_{k=k_S}^{N} m(k)\{z^T(k)\theta + e(k)\} \right]
\]

\[
= \theta + \left[ \lim_{N \to \infty} \frac{1}{N-k_S+1} \sum_{k=k_S}^{N} m(k)z^T(k) \right]^{-1} \cdot \left[ \lim_{N \to \infty} \frac{1}{N-k_S+1} \sum_{k=k_S}^{N} m(k)e(k) \right]
\]

(4.14)

It is clear that

\[
\lim_{N \to \infty} \frac{1}{N-k_S+1} \sum_{k=k_S}^{N} m(k)e(k) = E \begin{bmatrix} -\xi_{A1\nu}(k-l) \\ \vdots \\ -\xi_{A\nu}(k-l) \\ \xi_{A1u}(k) \\ \vdots \\ \xi_{A\nu u}(k) \end{bmatrix} e(k) = 0
\]

(4.15)

Therefore, the IV method gives the consistent estimate of \( \theta \) asymptotically, since the correlations between \( \xi_{A1\nu}(k) \)(\(i = 1, \ldots, n\)) and \( e(k) \) are avoided by introducing the instrumental variables.

With the same idea to modify the LS method by introducing an IV vector, we can treat the problems of the coloured measurement noise.

Case 2: The AR noise case
Recursive identification algorithms using an adaptive procedure

When the measurement noise is described as

\[ \eta(k) = \frac{1}{C(z^{-1})}e(k) \]

\[ C(z^{-1}) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_r z^{-r} \]  \hspace{1cm} (4.16)

the discrete-time estimation model becomes to be

\[ \xi_{A0y}(k) + \sum_{i=1}^{n} a_i \xi_{Aiy}(k) = \sum_{i=1}^{n} b_i \xi_{Au}(k) + \frac{1}{C(z^{-1})}e(k) \]  \hspace{1cm} (4.17)

Whitening the noise term leads to

\[ \tilde{\xi}_{A0y}(k) + \sum_{i=1}^{n} a_i \tilde{\xi}_{Aiy}(k) = \sum_{i=1}^{n} b_i \tilde{\xi}_{Au}(k) + e(k) \]  \hspace{1cm} (4.18)

where

\[ \tilde{\xi}_{Aiy}(k) = C(z^{-1})\xi_{Aiy}(k) \]

\[ \tilde{\xi}_{Au}(k) = C(z^{-1})\xi_{Au}(k) \]  \hspace{1cm} (4.19)

The vector form is

\[ \tilde{\xi}_{A0y}(k) = \tilde{z}^T(k)\theta + e(k) \]

\[ \tilde{z}^T(k) = [-\tilde{\xi}_{y}(k), \ldots, -\tilde{\xi}_{y}(k), \tilde{\xi}_{u}(k), \ldots, \tilde{\xi}_{u}(k)] \]  \hspace{1cm} (4.20)

\[ \theta^T = [a_1, \ldots, a_n, b_1, \ldots, b_n] \]

and

\[ \eta(k) = r^T(k)c + e(k) \]

\[ r^T(k) = [-\eta(k-1), -\eta(k-2), \ldots, -\eta(k-r)] \]  \hspace{1cm} (4.21)

\[ c^T = [c_1, c_2, \ldots, c_r] \]

Introduce a filtered IV vector:

\[ \tilde{m}(k) = C(z^{-1})m(k) \]  \hspace{1cm} (4.22)

where \( m(k) \) is defined in equation (4.12).

Then we can use the following algorithm to obtain the estimates \( \hat{\theta} \) and \( \hat{c} \):

\[ \hat{\theta} = \left[ \sum_{k=k_S}^{N} \tilde{m}(k)\tilde{z}^T(k) \right]^{-1} \cdot \left[ \sum_{k=k_S}^{N} \tilde{m}(k)\tilde{\xi}_{A0y}(k) \right] \]

\[ \hat{\eta}(k) = \xi_{A0y}(k) - z^T(k)\hat{\theta} \]  \hspace{1cm} (4.23)

\[ \hat{c}(k) = \left[ \sum_{k=k_S}^{N} \tilde{f}(k)\tilde{z}^T(k) \right]^{-1} \cdot \left[ \sum_{k=k_S}^{N} \tilde{f}(k)\tilde{\eta}(k) \right] \]
Recursive identification algorithms using an adaptive procedure

This algorithm is named to be the generalized instrumental variable (GIV) method since it is similar to the GLS method.

**Case 3: The MA noise case**

When the measurement noise is described to be

\[ \eta(k) = C(z^{-1})e(k) \]

\[ C(z^{-1}) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_r z^{-r} \]  (4.24)

The approximated discrete-time estimation model becomes to be

\[ \xi_{A_{0y}}(k) + \sum_{i=1}^{n} a_i \xi_{A_{iy}}(k) = \sum_{i=1}^{n} b_i \xi_{A_{iu}}(k) + C(z^{-1})e(k) \]  (4.25)

The approximated discrete-time estimation model can be written in a vector form as

\[ \xi_{A_{0y}}(k) = z^T_E(k) \theta + e(k) \]

\[ z^T_E(k) = [-\xi_{A_{1y}}(k), \cdots, -\xi_{A_{ny}}(k), \xi_{A_{1u}}(k), \cdots, \xi_{A_{nu}}(k), e(k-1), \cdots, e(k-r)] \]  (4.26)

\[ \theta^T = [a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_r] \]

Introduce the IV vector as

\[ m^T_E(k) = [-\xi_{A_{1y}}(k-l), \cdots, -\xi_{A_{ny}}(k-l), \xi_{A_{1u}}(k), \cdots, \xi_{A_{nu}}(k), e(k-1), \cdots, e(k-r)] \]  (4.27)

Then we have the following algorithm to estimate the parameters:

\[ \hat{\theta} = \left( \sum_{k=kg}^{N} m^T_E(k) \hat{z}^T_E(k) \right)^{-1} \left( \sum_{k=kg}^{N} m^T_E(k) \xi_{A_{0y}}(k) \right) \]  (4.28)

where

\[ \hat{z}^T_E(k) = [-\xi_{A_{1y}}(k), \cdots, -\xi_{A_{ny}}(k), \xi_{A_{1u}}(k), \cdots, \xi_{A_{nu}}(k), e(k-1), \cdots, e(k-r)] \]

\[ \hat{m}^T_E(k) = [-\xi_{A_{1y}}(k-l), \cdots, -\xi_{A_{ny}}(k-l), \xi_{A_{1u}}(k), \cdots, \xi_{A_{nu}}(k), e(k-1), \cdots, e(k-r)] \]

\[ e(k) = \xi_{A_{0y}}(k) - \hat{z}^T_E(k) \hat{\theta} \]  (4.29)

This method is named to be the extended instrumental variable (EIV) method since it is similar to the extended least squares (ELS) method.

As studied by Ljung, Söderström and Gustavsson (1975), Söderström, Ljung and Gustavsson (1978) and Frieland (1982) for the recursive version of the ELS method, the convergence of the recursive algorithm described later in the next section is related to the positive realness of \( 1/C(z^{-1}) - 1/2 \). When the pre-filter \( Q_{IA}(z^{-1}) \) is constructed by the true value
of \(1/A'(z^{-1})\), it can be shown through ordinary differential equation (ODE) analysis, the
algorithm converge globally, if \(1/C(z^{-1}) - 1/2\) is strictly positive real (Ljung 1977a, 1977b,

Influenced by the recursive maximum likelihood (RML) method (Ljung, Söderström and
Gustavsson 1975, Söderström, Ljung and Gustavsson 1978, Frielander 1982), to improve the
convergence property and the asymptotic statistical efficiency, we can modified the recursive
EIV algorithm described by equations (4.37) and (4.45) by replacing \(m(k)\) and \(z(k)\) as follows
if \(1/C(z^{-1})\) is stable:

\[
\begin{align*}
\tilde{m}_E(k) &= \frac{m_E(k)}{C(z^{-1})} \\
\tilde{z}_E(k) &= \frac{z_E(k)}{C(z^{-1})}
\end{align*}
\]  

(4.30)

This recursive modified algorithm is named to be the modified EIV (MEIV) method. This
method is similar to the RML method. And in this case, it can also be shown through ODE
analysis, that when the pre-filter \(QIA(z^{-1})\) is constructed by the true value of \(1/A'(z^{-1})\),
if \(\hat{C}(z^{-1})/C(z^{-1}) - 1/2\) is strictly positive real, the algorithm converge globally (Ljung,
1987, Söderström and Stoica 1989). Therefore the convergency condition is weakened.

Note that it is an open question whether the EIV and MEIV methods with adaptive pre­
filter \(QIA(z^{-1}) = 1/A'(z^{-1})\) will give convergence or not. However, simulation experiments
show that the EIV method and the MEIV method with the adaptive procedure will converge
in most cases, requiring long data records.

**Case 4: The ARMA noise case**

Consider the following ARMA noise model:

\[
\begin{align*}
\eta(k) &= \frac{D(z^{-1})}{C(z^{-1})}e(k) \\
C(z^{-1}) &= 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_r z^{-r} \\
D(z^{-1}) &= d_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_r z^{-r}
\end{align*}
\]  

(4.31)

The approximated discrete-time model becomes to be

\[
\xi_{\text{AIV}}(k) + \sum_{i=1}^{n} a_i \xi_{\text{AIV}}(k) = \sum_{i=1}^{n} b_i \xi_{\text{AIV}}(k) + \frac{D(z^{-1})}{C(z^{-1})}e(k)
\]  

(4.32)
And the vector form is
\[ \mathbf{z}^T(k) = g(z^{-1})^T \mathbf{e}(k) \]
\[ \mathbf{z}^T(k) = [-\xi A_1 y(k), \cdots, -\xi A_n y(k), \xi A_1 u(k), \cdots, \xi A_n u(k)] \]  \hspace{1cm} (4.33)
\[ \mathbf{\theta}^T = [a_1, \cdots, a_n, b_1, \cdots, b_n] \]

We can apply the bootstrap method (Young 1970, Söderström and Stoica 1981) with the filtered estimated system model output \( \hat{x}(k) \) as instrumental variables, as discussed in chapter 3.

The instrumental variable vector is constructed to be
\[ \mathbf{\bar{m}}_B^T(k) = [-\xi A_1 x(k), \cdots, -\xi A_n x(k), \xi A_1 u(k), \cdots, \xi A_n u(k)] \]  \hspace{1cm} (4.34)
where
\[ \xi A_i = Q_A (z^{-1}) \left( \frac{T_i}{2} \right) (1 + z^{-1})(1 - z^{-1})^{n - i} \hat{x}(k) \]  \hspace{1cm} (4.35)

Hence we have the following bootstrap method:
\[ \mathbf{\bar{\theta}} = \left[ \sum_{k=k_S}^{N} \mathbf{\bar{m}}_B(k) \mathbf{z}^T(k) \right]^{-1} \left[ \sum_{k=k_S}^{N} \mathbf{\bar{m}}_B(k) \xi A_0 y(k) \right] \]  \hspace{1cm} (4.36)

The bootstrap method is an useful tool when we have not any \textit{a priori} knowledge about the stochastic properties of measurement noise.

4.4 Recursive estimation algorithms

The recursive estimation algorithms can all be described by an algorithm of the following form (Söderström et al. 1978):
\[ \mathbf{\hat{\theta}}(k) = \mathbf{\hat{\theta}}(k-1) + L(k) \mathbf{e}(k) \]
\[ L(k) = \frac{P(k-1) \mathbf{\psi}(k)}{\rho(k) + \mathbf{\phi}^T(k) P(k-1) \mathbf{\psi}(k)} \]  \hspace{1cm} (4.37)
\[ P(k) = \frac{1}{\rho(k)} \left[ P(k-1) - \frac{P(k-1) \mathbf{\psi}(k) \mathbf{\phi}^T(k) P(k-1)}{\rho(k) + \mathbf{\phi}^T(k) P(k-1) \mathbf{\psi}(k)} \right] \]

Now the methods under discussion will be obtained as special cases of the recursive algorithm described in the above equation.
Recursive identification algorithms using an adaptive procedure

The LS method

The conventional LS method is obtained with

$$\hat{\theta}^T = [\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n]$$

$$\phi(k) = \mathbf{z}(k)$$

$$\psi(k) = \mathbf{z}(k)$$

$$\varepsilon(k) = \xi_{A_{0y}}(k) - \mathbf{z}^T(k)\hat{\theta}(k - 1)$$

where \(\mathbf{z}(k)\) is defined in equation (4.8)

The IV method

The proposed IV method is obtained with

$$\hat{\theta}^T = [\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n]$$

$$\phi(k) = \mathbf{z}(k)$$

$$\psi(k) = \mathbf{m}(k)$$

$$\varepsilon(k) = \xi_{A_{0y}}(k) - \mathbf{z}^T(k)\hat{\theta}(k - 1)$$

where \(\mathbf{z}(k), \mathbf{m}(k)\) are defined in equation (4.8), equation (4.12) respectively.

The GIV method

The proposed GIV method can be described as follows.

Step 1

Filter \(\xi_{A_{iu}}(k)\) and \(\xi_{A_{iy}}(k)\):

$$\tilde{\xi}_{A_{iy}}(k) = \tilde{C}(z^{-1})\xi_{A_{iy}}(k)$$

$$\tilde{\xi}_{A_{iu}}(k) = \tilde{C}(z^{-1})\xi_{A_{iu}}(k)$$

Step 2

Estimate \(\hat{\theta}\):

$$\hat{\theta}(k) = \hat{\theta}(k - 1) + L(k)\tilde{\varepsilon}(k)$$

$$\tilde{\varepsilon}(k) = \tilde{\xi}_{A_{0y}}(k) - \tilde{z}^T(k)\hat{\theta}(k - 1)$$

$$L(k) = \frac{P(k - 1)\tilde{m}(k)}{\rho(k) + \tilde{z}^T(k)P(k - 1)\tilde{m}(k)}$$

$$P(k) = \frac{1}{\rho(k)} \left[ P(k - 1) - \frac{P(k - 1)\tilde{m}(k)\tilde{z}^T(k)P(k - 1)}{\rho(k) + \tilde{z}^T(k)P(k - 1)\tilde{m}(k)} \right]$$
Recursive identification algorithms using an adaptive procedure

\[ \hat{\theta}^T = [\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n] \]  

(4.42)

and \( \bar{z}(k), \bar{m}(k) \) are defined in equation (4.20), equation (4.22) respectively.

**Step 3**

Estimate noise \( \eta(k) \):

\[ \hat{\eta}(k) = \xi_{Ao}(k) - z^T(k)\hat{\theta}(k) \]  

(4.43)

**Step 4**

Estimate \( c \):

\[
\begin{align*}
\dot{c}(k) &= \dot{c}(k-1) + L_c(k)e_c(k) \\
e_c(k) &= \hat{\eta}(k) - \bar{f}^T(k)\dot{c}(k-1) \\
L_c(k) &= \frac{P_c(k-1)\bar{f}(k)}{\rho(k) + \bar{f}^T(k)P_c(k-1)\bar{f}(k)} \\
P_c(k) &= \frac{1}{\rho(k)} \left[ P_c(k-1) - \frac{P_c(k-1)\bar{f}(k)\bar{f}^T(k)P_c(k-1)}{\rho(k) + \bar{f}^T(k)P_c(k-1)\bar{f}(k)} \right]
\end{align*}
\]  

(4.44)

**Step 5**

Return to step 1 until finished.

The EIV method

The proposed EIV method is obtained with

\[ \hat{\theta}^T = [\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n, \tilde{c}_1, \ldots, \tilde{c}_r] \]

\[ \phi(k) = \bar{z}_E(k) \]

\[ \psi(k) = \bar{m}_E(k) \]

(4.45)

\[ \epsilon(k) = \xi_{Ao}(k) - z_E^T(k)\hat{\theta}(k-1) \]

where \( \bar{z}_E(k), \bar{m}_E(k) \) are defined in equation (4.29).

The MEIV method

For this case we take

\[ \hat{\theta}^T = [\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n, \tilde{c}_1, \ldots, \tilde{c}_r] \]

\[ \phi(k) = \bar{z}_E(k) \]

\[ \psi(k) = \bar{m}_E(k) \]

(4.46)

\[ \epsilon(k) = \xi_{Ao}(k) - z_E^T(k)\hat{\theta}(k-1) \]
where $\tilde{E}(k)$, $\tilde{E}_E(k)$ are defined in equation (4.30).

**The bootstrap method**

The bootstrap method is obtained with

$$
\begin{align*}
\widetilde{\theta}^T &= [\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n] \\
\phi(k) &= \mathbf{z}(k) \\
\psi(k) &= \tilde{m}_B(k) \\
\varepsilon(k) &= \xi_{Ao_0}(k) - \mathbf{z}^T(k)\tilde{\theta}(k-1)
\end{align*}
$$

where $\mathbf{z}(k), \tilde{m}_B(k)$ are defined in equation (4.33), equation (4.34) respectively.

### 4.5 Implementation of the algorithms

All the recursive identification algorithms described in section 4.4 can be implemented in an on-line manner on a digital computer. Considering the IIR digital filters described in equation (4.6), we can construct $Q_{IA}(z^{-1})$ adaptively with the estimates of $a_i (i = 1, \ldots, n)$, since the parameters $a_1, a_2, \ldots, a_n$ are unknown.

It is clear that our adaptive IIR filters are time-variant, especially for the beginning recursions of the identification algorithms. Therefore it is critical to choose the forgetting factor $\rho(k)$ in the recursive identification algorithm given in equation (4.37) to improve convergence properties. When $\rho(k) < 1$, the parameter estimates approach the true values more rapidly, i.e. the adaptive IIR filters are adapted more rapidly, however the algorithms are more sensitive to the noise, and when $\rho(k) < 1$ the parameter estimates do not converge, but oscillate around the true values. In practice, $\rho(k)$ should be small (that is slightly less than 1) for small $k$ to make the transient phase short. After some time, $\rho(k)$ should get close to 1 to enable convergence and to decrease the oscillations around the true values.

In our study, the forgetting factor $\rho(k)$ is chosen to be

$$
\begin{align*}
\rho(k) &= \rho(k_N), \quad \rho(k_N) < 1 \quad k \leq N_0 \\
\rho(k) &= (1 - \lambda)\rho(k - 1) + \lambda, \quad \lambda \ll 1 \quad k > N_0
\end{align*}
$$

Clearly, $\rho(k)$ tends exponentially to 1 when $k$ increases greatly than $N_0$. For larger $N_0$, $\rho(k)$ grow more slowly to 1. Therefore a tradeoff between the convergence rate of the identification algorithms and insensitivity to noise effects should be taken by selecting an appropriate $N_0$. 
Recursive identification algorithms using an adaptive procedure

For detailed discussions about the forgetting factor, the reader is referred to the works by Ljung and Söderström (1983), Söderström and Stoica (1989).

Denote $a_1, a_2, \cdots, a_n$ in a vector form as

$$a^T = [a_1, a_2, \cdots, a_n]$$

(4.49)

Since the estimate vector $\bar{a}$ changes roughly during the beginning recursions of the estimation algorithms, we can smooth $\bar{a}$ by a low-pass filter prior to the adaptations of $Q_{IA}(z^{-1})$ as

$$\begin{cases} 
\bar{a}(k) = \bar{a}(k_d) & k \leq N_0 \\
\bar{a}(k) = (1 - \mu)\bar{a}(k - 1) + \mu\bar{a}(k - d) & k > N_0 
\end{cases}$$

(4.50)

where $\mu$ and $d$ can be chosen to be

$$\begin{align*}
\mu &= 0.05 \\
d &= 10 
\end{align*}$$

(4.51)

This means that during the first $N_0$ times recursions of the estimation algorithms, the parameters of $Q_{IA}(z^{-1})$ can be chosen to be constant ones by an appropriate guess. When $k$ increases greater than $N_0$, $Q_{IA}(z^{-1})$ can be constructed adaptively with the low-pass filter's output $\bar{a}$.

Finally, we note that the adaptive procedure apparently requires monitoring the stability of $Q_{IA}(z^{-1})$ and keeping $\bar{a}$ within a stable region if instability of $Q_{IA}(z^{-1})$ is detected.

Remark 4.5:

For the GIV method, when $k \leq N_0$, it is not necessary to estimate the noise $\eta(k)$ and the noise model parameter $c$. Only the system parameter $\theta$ is estimated, assuming $C(z^{-1}) = 1$.

Remark 4.6:

For the MEIV method, when $k \leq N_0$, to improve the numerical performance of the algorithm, the algorithm is implemented as the EIV method, assuming $\bar{C}(z^{-1}) = 1$. And when $k > N_0$, the MEIV method starts.
Recursive identification algorithms using an adaptive procedure

Table 4.1: Simulation results of the LS method (NSR=20%).

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{a}_1$</td>
<td>3.0338</td>
<td>2.9887</td>
<td>2.9763</td>
<td>2.9765</td>
</tr>
<tr>
<td>(3.0)</td>
<td>± 0.6369</td>
<td>± 0.0562</td>
<td>± 0.0446</td>
<td>± 0.0238</td>
</tr>
<tr>
<td>$\hat{a}_2$</td>
<td>4.0615</td>
<td>3.9399</td>
<td>3.9787</td>
<td>3.9790</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.3400</td>
<td>± 0.0468</td>
<td>± 0.0987</td>
<td>± 0.0223</td>
</tr>
<tr>
<td>$\hat{b}_1$</td>
<td>0.0048</td>
<td>0.0005</td>
<td>0.0015</td>
<td>0.0020</td>
</tr>
<tr>
<td>(0.0)</td>
<td>± 0.0041</td>
<td>± 0.0101</td>
<td>± 0.0059</td>
<td>± 0.0046</td>
</tr>
<tr>
<td>$\hat{b}_2$</td>
<td>4.0819</td>
<td>3.9947</td>
<td>3.9677</td>
<td>3.9705</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.4172</td>
<td>± 0.0684</td>
<td>± 0.0488</td>
<td>± 0.0300</td>
</tr>
</tbody>
</table>

4.6 Illustrative examples

To illustrate the effectiveness of the proposed estimation algorithms, we consider a second-order system described by

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t) = b_1 \dot{u}(t) + b_2 u(t)$$

$$a_1 = 3.0, \quad a_2 = 4.0, \quad b_1 = 0.0, \quad b_2 = 4.0$$

Simulation experiments are carried out under the following conditions:

**Sampling period:** $T = 0.01$

**Input signal:**

$$u(t) = \sin(t) + \sin(1.5t) + 0.5 \sin(3t) + 1.5 \sin(4.5t) + 0.3 \sin(5t) + 0.2 \sin(7t) + 2.5 \sin(7.5t) + 5.0 \sin(10.5t)$$

**Forgetting factor:**

$$\rho(k) =
\begin{cases}
0.95 & \text{if } k \leq N_0 (= 500) \\
(1 - 0.01) \rho(k - 1) + 0.01 & \text{if } k > N_0
\end{cases}$$

**Initialization of $Q_{IA}(z^{-1})(i = 0, 1, 2)$ for $k \leq N_0 (= 500)$:**

$$Q_{IA}(z^{-1}) = \frac{1}{\sum_{i=0}^{2} a_i (\frac{T}{2})^i (1 + z^{-1})^i (1 - z^{-1})^{2-i}}$$

$a_0 = 1.0, \quad a_1 = 11.0, \quad a_2 = 10.0$
Recursive identification algorithms using an adaptive procedure

Figure 4.1: LS estimates (Table 4.1).

Table 4.2: Simulation results of the LS method (NSR=50%).

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{a}_1 ) (3.0)</td>
<td>3.1075 ± 2.2710</td>
<td>2.8984 ± 0.1265</td>
<td>2.8592 ± 0.0817</td>
<td>2.8584 ± 0.0514</td>
</tr>
<tr>
<td>( \hat{a}_2 ) (4.0)</td>
<td>4.1257 ± 1.3661</td>
<td>3.9226 ± 0.1131</td>
<td>3.8931 ± 0.0861</td>
<td>3.8816 ± 0.0550</td>
</tr>
<tr>
<td>( \hat{b}_1 ) (0.0)</td>
<td>0.0336 ± 0.1428</td>
<td>0.0125 ± 0.0260</td>
<td>0.0099 ± 0.0137</td>
<td>0.0114 ± 0.0128</td>
</tr>
<tr>
<td>( \hat{b}_2 ) (4.0)</td>
<td>4.2603 ± 1.2136</td>
<td>3.9010 ± 0.1578</td>
<td>3.8230 ± 0.0897</td>
<td>3.8279 ± 0.0704</td>
</tr>
</tbody>
</table>

Figure 4.2: LS estimates (Table 4.2).
Recursive identification algorithms using an adaptive procedure

Table 4.3: Simulation results of the IV method.

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{a}_1$</td>
<td>3.2018 ± 2.3829</td>
<td>3.0495 ± 0.1416</td>
<td>2.9967 ± 0.1029</td>
<td>2.9999 ± 0.0595</td>
</tr>
<tr>
<td>(3.0)</td>
<td>4.1067 ± 1.2987</td>
<td>4.0464 ± 0.1258</td>
<td>3.9891 ± 0.0949</td>
<td>3.9996 ± 0.0595</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0341 ± 0.1346</td>
<td>0.0005 ± 0.0216</td>
<td>0.0017 ± 0.0148</td>
<td>0.0022 ± 0.0113</td>
</tr>
<tr>
<td>(0.0)</td>
<td>4.2341 ± 1.2401</td>
<td>4.0854 ± 0.1680</td>
<td>3.9928 ± 0.1125</td>
<td>4.0004 ± 0.0759</td>
</tr>
</tbody>
</table>

Figure 4.3: IV estimates (Table 4.3).

Table 4.4: Simulation results of the GIV method.

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{a}_1$</td>
<td>3.0248 ± 0.0591</td>
<td>3.0020 ± 0.0139</td>
<td>3.0015 ± 0.0144</td>
<td>3.0023 ± 0.0121</td>
</tr>
<tr>
<td>(3.0)</td>
<td>4.0159 ± 0.0499</td>
<td>4.0016 ± 0.0149</td>
<td>3.9999 ± 0.0127</td>
<td>3.9999 ± 0.0122</td>
</tr>
<tr>
<td>$b_1$</td>
<td>-0.0013 ± 0.0079</td>
<td>-0.0001 ± 0.0030</td>
<td>0.0001 ± 0.0021</td>
<td>0.0001 ± 0.0014</td>
</tr>
<tr>
<td>(0.0)</td>
<td>4.0106 ± 0.0595</td>
<td>4.0025 ± 0.0188</td>
<td>4.0023 ± 0.0180</td>
<td>4.0019 ± 0.0122</td>
</tr>
<tr>
<td>$c_1$</td>
<td>1.4650 ± 0.0527</td>
<td>1.4884 ± 0.0216</td>
<td>1.4959 ± 0.0144</td>
<td>1.4995 ± 0.0122</td>
</tr>
<tr>
<td>(1.5)</td>
<td>0.7288 ± 0.0448</td>
<td>0.7422 ± 0.0190</td>
<td>0.7479 ± 0.0133</td>
<td>0.7481 ± 0.0101</td>
</tr>
<tr>
<td>$c_2$</td>
<td>0.7288 ± 0.0448</td>
<td>0.7422 ± 0.0190</td>
<td>0.7479 ± 0.0133</td>
<td>0.7481 ± 0.0101</td>
</tr>
</tbody>
</table>
Recursive identification algorithms using an adaptive procedure

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>( \hat{a}_2 )</th>
<th>( \hat{a}_1 )</th>
<th>( \hat{b}_2/2 )</th>
<th>( \hat{b}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>2.0</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>3.0</td>
<td>3.5</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>4.5</td>
<td>5.0</td>
<td>2.5</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Figure 4.4: GIV estimates (Table 4.4).
Example 4.1: The conventional LS method.

The conventional LS method is investigated with a white measurement noise of \( \eta(k) = e(k) \) and \( \text{NSR}= 20\%, 50\% \).

The results are shown in Table 4.1 and Table 4.2. Each of the tables includes the mean and standard deviation of the estimates obtained from Monte-Carlo simulation of 20 experiments for different sample sizes \( N \). And Figures 4.1~4.2 plot the estimates of one realization of the results of Tables 4.1~4.2 respectively. It is shown in Table 4.1 that in the presence of low measurement noise, the conventional LS method is still efficient due to the excellent noise reducing effects of the adaptive IIR filters. However, in the presence of white noise of high level, it is known from Table 4.2 that using the LS method we cannot obtain consistent estimates.

Example 4.2: The IV method.

The proposed IV method is illustrated with a white measurement noise of \( \eta(k) = e(k) \) and \( \text{NSR}= 50\% \).

The results are shown in Table 4.3. Compared with the results in Table 4.2, it is shown that the proposed IV method is quite superior to the conventional LS method in the presence of white noise of high level. The results agree with the consistency discussions in section 4.3.
Recursive identification algorithms using an adaptive procedure

Figure 4.5: EIV estimates (Table 4.5).
Recursive identification algorithms using an adaptive procedure

Table 4.6: Simulation results of the EIV method (Example 4.5).

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>2.9781</td>
<td>2.9983</td>
<td>2.9918</td>
<td>3.0022</td>
</tr>
<tr>
<td>(3.0)</td>
<td>± 0.1946</td>
<td>± 0.0460</td>
<td>± 0.0409</td>
<td>± 0.0224</td>
</tr>
<tr>
<td>$a_2$</td>
<td>4.0291</td>
<td>4.0154</td>
<td>3.9998</td>
<td>4.0049</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.1648</td>
<td>± 0.0340</td>
<td>± 0.0240</td>
<td>± 0.0196</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0041</td>
<td>0.0003</td>
<td>0.0006</td>
<td>-0.0008</td>
</tr>
<tr>
<td>(0.0)</td>
<td>± 0.0191</td>
<td>± 0.0073</td>
<td>± 0.0053</td>
<td>± 0.0038</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4.0012</td>
<td>4.0087</td>
<td>3.9868</td>
<td>3.9969</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.1419</td>
<td>± 0.0607</td>
<td>± 0.0600</td>
<td>± 0.0358</td>
</tr>
<tr>
<td>$\delta_1$</td>
<td>-1.2755</td>
<td>-1.3733</td>
<td>-1.3987</td>
<td>-1.4277</td>
</tr>
<tr>
<td>(-1.50)</td>
<td>± 0.0760</td>
<td>± 0.0422</td>
<td>± 0.0377</td>
<td>± 0.0312</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>0.4484</td>
<td>0.5303</td>
<td>0.5707</td>
<td>0.6053</td>
</tr>
<tr>
<td>(0.75)</td>
<td>± 0.0697</td>
<td>± 0.0455</td>
<td>± 0.0466</td>
<td>± 0.0455</td>
</tr>
</tbody>
</table>

Example 4.3 The GIV method.

The proposed GIV method is illustrated with a measurement noise of

$$\eta(k) = \frac{e(k)}{1 + c_1 z^{-1} + c_2 z^{-2}}$$

$c_1 = 1.50, \ c_2 = 0.75$

and is chosen that NSR= 50%.

The results are shown in Table 4.4. It is shown that the parameter estimates of both the system and the noise model converge to their true values. Thus the GIV method is quite efficient when the measurement noise is modeled by an AR process. This method is similar to the well-known GLS method.

Example 4.4 The EIV method.

The proposed EIV method is illustrated with a measurement noise of

$$\eta(k) = (1 + c_1 z^{-1} + c_2 z^{-2}) e(k)$$

$c_1 = 0.40, \ c_2 = 0.30$

and NSR= 50%.

The results are shown in Table 4.5. It should be noted that $1/C(z^{-1}) - 1/2$ is strictly positive real. Therefore it can be concluded by the simulation results that the EIV method
Figure 4.6: EIV estimates (Table 4.6).
Recursive identification algorithms using an adaptive procedure

Figure 4.7: MEIV estimates (Table 4.7).
Recursive identification algorithms using an adaptive procedure

Table 4.7: Simulation results of the MEIV method.

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{a}_1$</td>
<td>2.9759</td>
<td>3.0033</td>
<td>2.9932</td>
<td>3.0030</td>
</tr>
<tr>
<td>(3.0)</td>
<td>± 0.2285</td>
<td>± 0.0617</td>
<td>± 0.0252</td>
<td>± 0.0206</td>
</tr>
<tr>
<td>$\hat{a}_2$</td>
<td>4.0302</td>
<td>4.0134</td>
<td>4.0018</td>
<td>4.0026</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.1653</td>
<td>± 0.0254</td>
<td>± 0.0190</td>
<td>± 0.0159</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0048</td>
<td>-0.0002</td>
<td>-0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>(0.0)</td>
<td>± 0.0297</td>
<td>± 0.0051</td>
<td>± 0.0046</td>
<td>± 0.0030</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4.0016</td>
<td>4.0190</td>
<td>3.9995</td>
<td>4.0035</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.1639</td>
<td>± 0.0312</td>
<td>± 0.0277</td>
<td>± 0.0192</td>
</tr>
<tr>
<td>$\hat{c}_1$</td>
<td>-1.2784</td>
<td>-1.4600</td>
<td>-1.4855</td>
<td>-1.5005</td>
</tr>
<tr>
<td>(-1.50)</td>
<td>± 0.0512</td>
<td>± 0.0237</td>
<td>± 0.0276</td>
<td>± 0.0154</td>
</tr>
<tr>
<td>$\hat{c}_2$</td>
<td>0.4935</td>
<td>0.6953</td>
<td>0.7309</td>
<td>0.7505</td>
</tr>
<tr>
<td>(0.75)</td>
<td>± 0.0727</td>
<td>± 0.0238</td>
<td>± 0.0274</td>
<td>± 0.0181</td>
</tr>
</tbody>
</table>

can give excellent results when $1/C(z^{-1}) - 1/2$ is strictly positive real. This is analogous to the discussions of Ljung et al. (1975), Söderström et al. (1978) for the ELS method. Further discussions about the EIV method will be taken in Example 4.5 through comparison of the EIV method and the MEIV method.

Example 4.5 Comparison of the EIV method and the MEIV method.

The proposed EIV method and the MEIV method are compared with a measurement

Table 4.8: Simulation results of the bootstrap method (white noise).

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{a}_1$</td>
<td>2.8630</td>
<td>3.0690</td>
<td>2.9789</td>
<td>2.9912</td>
</tr>
<tr>
<td>(3.0)</td>
<td>± 2.2377</td>
<td>± 0.1695</td>
<td>± 0.1107</td>
<td>± 0.0581</td>
</tr>
<tr>
<td>$\hat{a}_2$</td>
<td>3.9585</td>
<td>4.0340</td>
<td>3.9731</td>
<td>3.9853</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 1.2905</td>
<td>± 0.1454</td>
<td>± 0.1069</td>
<td>± 0.1008</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0115</td>
<td>0.0035</td>
<td>0.0071</td>
<td>0.0087</td>
</tr>
<tr>
<td>(0.0)</td>
<td>± 0.1050</td>
<td>± 0.0385</td>
<td>± 0.0286</td>
<td>± 0.0258</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4.1742</td>
<td>4.1060</td>
<td>3.9961</td>
<td>3.9932</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 1.1635</td>
<td>± 0.2068</td>
<td>± 0.0852</td>
<td>± 0.0913</td>
</tr>
</tbody>
</table>
Recursive identification algorithms using an adaptive procedure

Figure 4.8: Bootstrap estimates in the presence of white measurement noise (Table 4.8).

Table 4.9: Simulation results of the bootstrap method (AR noise).

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{a}_1$</td>
<td>2.9545</td>
<td>2.9820</td>
<td>2.9969</td>
<td>2.9974</td>
</tr>
<tr>
<td>(3.0)</td>
<td>± 0.1848</td>
<td>± 0.0401</td>
<td>± 0.0246</td>
<td>± 0.0171</td>
</tr>
<tr>
<td>$\hat{a}_2$</td>
<td>3.9797</td>
<td>3.9973</td>
<td>3.9984</td>
<td>3.9933</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.5298</td>
<td>± 0.0691</td>
<td>± 0.0352</td>
<td>± 0.0240</td>
</tr>
<tr>
<td>$\hat{b}_1$</td>
<td>0.0039</td>
<td>0.0016</td>
<td>0.0004</td>
<td>0.0002</td>
</tr>
<tr>
<td>(0.0)</td>
<td>± 0.0090</td>
<td>± 0.0053</td>
<td>± 0.0028</td>
<td>± 0.0029</td>
</tr>
<tr>
<td>$\hat{b}_2$</td>
<td>3.9769</td>
<td>3.9875</td>
<td>3.9983</td>
<td>3.9984</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.1041</td>
<td>± 0.0523</td>
<td>± 0.0326</td>
<td>± 0.0226</td>
</tr>
</tbody>
</table>

Figure 4.9: Bootstrap estimates in the presence of AR measurement noise (Table 4.9).
Recursive identification algorithms using an adaptive procedure

The noise of

\[ \eta(k) = (1 + c_1 z^{-1} + c_2 z^{-2})e(k) \]

and NSR = 50%

The results are shown in Table 4.6 and Table 4.7 respectively. It is obvious that \( 1/C(z^{-1}) - 1/2 \) is not strictly positive real. It is shown in Table 4.6 that when \( 1/C(z^{-1}) - 1/2 \) is not strictly positive real, with the EIV method, the noise model parameter estimates do not converge to their true values although the system parameter estimates are still acceptable. Additionally, from Table 4.7, we know that with the MEIV method, the parameter estimates of both the system and the noise model converge accurately to the true values and therefore the convergence properties are improved. The results are similar to those discussed by Ljung et al. (1975), Söderström et al. (1978) and Frielander (1982) for the ELS method and the RML method, although a theoretical analysis of convergence is difficult to be taken in our case, because the identification algorithms based on the adaptive procedure are strongly nonlinear. For some numerical examples studied by the authors, the improvement of the convergence properties of the MEIV method is remarkable.

Example 4.6 The bootstrap method.

The bootstrap method is illustrated for various types of measurement noises:

**Measurement noise:**

- **White noise:**
  \[ \eta(k) = e(k) \]  
- **AR noise:**
  \[ \eta(k) = \frac{e(k)}{1 + 1.50z^{-1} + 0.75z^{-2}} \]  
- **MA noise:**
  \[ \eta(k) = (1 - 1.50z^{-1} + 0.75z^{-2})e(k) \]  
- **ARMA noise:**
  \[ \eta(k) = \frac{1.0 - 1.0z^{-1} + 0.2z^{-2}}{1.0 - 1.5z^{-1} + 0.7z^{-2}}e(k) \]

and NSR = 50%

The results are shown in Tables 4.8~4.11. It is shown that the convergence rate of the estimates may depend on the type of the measurement noise. For large sample sizes, the estimates obtained by the bootstrap method for various types of noises are quite accurate. Therefore the bootstrap method is a quite useful method when we do not treat the noise model.
Recursive identification algorithms using an adaptive procedure

Table 4.10: Simulation results of the bootstrap method (MA noise).

<table>
<thead>
<tr>
<th>N</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>2.9525±0.1862</td>
<td>2.9890±0.0387</td>
<td>2.9983±0.0244</td>
<td>2.9994±0.0180</td>
</tr>
<tr>
<td>a₂</td>
<td>4.0912±0.4087</td>
<td>3.9950±0.0518</td>
<td>4.0008±0.0329</td>
<td>3.9990±0.0220</td>
</tr>
<tr>
<td>b₁</td>
<td>0.0039±0.0107</td>
<td>0.0005±0.0036</td>
<td>-0.0012±0.0029</td>
<td>0.0004±0.0026</td>
</tr>
<tr>
<td>b₂</td>
<td>3.9850±0.1271</td>
<td>3.9916±0.0514</td>
<td>3.9994±0.0317</td>
<td>4.0000±0.0232</td>
</tr>
</tbody>
</table>

Figure 4.10: Bootstrap estimates in the presence of MA measurement noise (Table 4.10).
Recursive identification algorithms using an adaptive procedure

Figure 4.11: Bootstrap estimates in the presence of ARMA measurement noise (Table 4.11).

Table 4.11: Simulation results of the bootstrap method (ARMA noise).

<table>
<thead>
<tr>
<th></th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>2.9266</td>
<td>3.0383</td>
<td>2.9974</td>
<td>2.9980</td>
</tr>
<tr>
<td>(3.0)</td>
<td>± 0.2468</td>
<td>± 0.0211</td>
<td>± 0.0255</td>
<td>± 0.0180</td>
</tr>
<tr>
<td>$a_2$</td>
<td>3.9961</td>
<td>3.9925</td>
<td>3.9988</td>
<td>3.9970</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.5331</td>
<td>± 0.0629</td>
<td>± 0.0361</td>
<td>± 0.0238</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.0003</td>
<td>-0.0001</td>
<td>-0.0001</td>
<td>-0.0001</td>
</tr>
<tr>
<td>(0.0)</td>
<td>± 0.0102</td>
<td>± 0.0033</td>
<td>± 0.0030</td>
<td>± 0.0032</td>
</tr>
<tr>
<td>$b_2$</td>
<td>3.9892</td>
<td>3.9900</td>
<td>3.9994</td>
<td>3.9990</td>
</tr>
<tr>
<td>(4.0)</td>
<td>± 0.1096</td>
<td>± 0.0236</td>
<td>± 0.0377</td>
<td>± 0.0236</td>
</tr>
</tbody>
</table>
4.7 Conclusion

In this chapter, the recursive identification algorithms for continuous systems using an adaptive procedure have been discussed. The continuous system is identified through an approximated discrete-time estimation model with continuous system parameters. With the aid of the bilinear transformation, the approximated discrete-time estimation model of the continuous system involving adaptive IIR filters is easily derived. The approximated discrete-time estimation model used in this chapter is very suitable for recursive identification algorithms. From the viewpoint of pre-filtering, it is reasonable to choose the adaptive pre-filter, since, usually the pass-band of a physical system depends mainly on its poles.

A class of recursive identification algorithms which are closely similar to the existing standard recursive identification algorithms for common discrete-time systems have been presented to obtain consistent estimates in the presence of various types of measurement noises. Through the study of this chapter, we know that the continuous-time system parameters can be identified in very similar ways as those for common discrete-time systems. The continuous-time system identification requires a digital filtering procedure to avoid direct approximation of differentiations from sampled data while the discrete-time system identification is usually based directly on a linear regression model composed of delayed sampled input-output data.

The proposed identification algorithms for the continuous systems are easy to implement on digital computers and numerical examples show that the proposed recursive identification algorithms are quite efficient.