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Introduction

The problem of giving the reciprocity law of the power residue symbols in algebraic number fields can be reduced to giving the reciprocity law of the Hilbert norm residue symbols in p-adic number fields [Ha1, Teil II, §12].

Let p be a rational prime number and $k/\mathbb{Q}_p$ a finite extension, where $\mathbb{Q}_p$ denotes the rational p-adic number field. Let K/k be a finite abelian extension and let

$$(\ \ , K/k) : k^\times \longrightarrow \mathrm{Gal}(K/k)$$

denote the *reciprocity map* (the *norm residue map*) in local class field theory. The symbol $(\alpha, K/k)$ (called the *norm residue symbol*) for $\alpha \in k^\times$ has the *norm property*:

$$\alpha \in N_{K/k}(K^\times) \text{ if and only if } (\alpha, K/k) = 1,$$

where $N_{K/k} : K \longrightarrow k$ means the norm map. If $(\alpha, K/k)$ has an *explicit* expression in terms of α and K/k, then we call it an *explicit reciprocity formula* (or an *explicit reciprocity law*). In particular, if k contains ζ_m, a primitive m-th root of unity, and $K = k(\sqrt[m]{\beta}\,)$, $\beta \in k^\times$, then the m-th *Hilbert(-Hasse) norm residue symbol*

$$(\alpha,\beta)_m = \sqrt[m]{\beta}^{\,(\alpha,\, k(\sqrt[m]{\beta})/k) - 1} \in \langle \zeta_m \rangle$$

can be defined. Various explicit formulas that express $(\alpha,\beta)_m$ in terms of α and β have been known [AH, Br, Co3, Ha2, Ha3, Ha4, Hay, Hel, Hen, Iw1, Kn, Ku, Sa, Sen, Sh1, Sh2, SI, Ta1, Vo1, Vo2, Ya1, Ya2]. Recently, these formulas have been generalized and refined by using formal groups [CW, Co2, dS2, Ko, Sh4, Vo3, Vo4, Vo5, Vo6, Wi]. In this article, we shall give some explicit formulas for the generalized Hilbert norm residue symbol $(\alpha,\beta)_n^F$ defined on fields L $(\supset k)$ generated by torsion points of some formal groups F [Su1, Su2, Su3].

Let k'/k be the unramified extension of degree d $(d \geq 1)$. We denote by φ the Frobenius automorphism of k'/k. Let $\mathfrak{o}$, $\mathfrak{o}'$ denote the integer rings of k, k', respectively, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. For a prime element π of k' we take a power series $f \in \mathfrak{o}'[[X]]$ satisfying

$$f(X) \equiv \pi X \mod \deg 2, \quad f(X) \equiv X^q \mod \pi,$$

where $q = p^f$ denotes the number of elements in the residue field $\mathfrak{o}/\mathfrak{p}$. There exists a unique one-dimensioal commutative formal group law $F_f \in \mathfrak{o}'[[X,Y]]$ such that $F_f^{\varphi}(f(X),f(Y)) = f(F_f(X,Y))$ [Sh3, dS1]. We call it a *relative Lubin-Tate formal group* (relative to the unramified extension k'/k). We write $X \underset{f}{+} Y = F_f(X,Y)$. For any $c \in \mathfrak{o}$ there exists a unique endomorphism $[c]_f \in \mathfrak{o}'[[X]]$ of F_f such that $[c]_f(X) \equiv cX$ (mod deg 2) and $[c]_f^{\varphi} \circ f = f \circ [c]_f$.

Let Ω denote the completion of the algebraic closure of k, $\mathfrak{p}_\Omega$ the maximal ideal of the integer ring of Ω, and $F_f(\mathfrak{p}_\Omega)$ the associated formal $\mathfrak{o}$-module. Let W_f^n $(\subset F_f(\mathfrak{p}_\Omega))$, $n \geq 1$ be the set of all $\mathfrak{p}^n$-torsion points of F_f. Put $\xi = N_{k'/k}\,\pi$. Then the field $k_{\xi,n} = k'(W_f^n)$, $n \geq 1$, does not depend on the choice of π and f, and is the class field over k corresponding to the subgroup $\langle\xi\rangle \times (1 + \mathfrak{p}^n)$ $(\subset k^\times)$. Let $\mathfrak{p}_{\xi,n}$ denote the maximal ideal of the integer ring of $k_{\xi,n}$. For $\alpha \in k_{\xi,n}^\times$ and $\beta \in F_f(\mathfrak{p}_{\xi,n})$ we define a "Kummer pairing" [Fr, Wi, dS2]

$$(\alpha,\beta)_{f,n} = (\alpha,\ k_{\xi,n}(\rho)/k_{\xi,n})(\rho) \underset{\varphi^{-n}(f)}{-} \rho, \quad \rho \in W^n_{\varphi^{-n}(f)},$$

$$\rho \in F_{\varphi^{-n}(f)}(\mathfrak{p}_\Omega), \quad \beta = f^{\varphi^{-1}} \circ \cdots \circ f^{\varphi^{-n}}(\rho).$$

We call $(\alpha,\beta)_{f,n}$ the *generalized Hilbert norm residue symbol*. In particular, if $k' = k = \mathbf{Q}_p$, $\xi = \pi = p$ and $f(X) = (1 + X)^p - 1$, then $(\alpha,\beta)_{f,n} + 1$ $(= (\alpha,\ 1 + \beta)_{p^n})$ is the p^n-th Hilbert symbol defined on the cyclotomic field $\mathbf{Q}_p(\zeta_{p^n})$.

In Chapter 1, we shall give a *complete* formula for $(\alpha,\beta)_{f,n}$. Let $\omega = (\omega_i)_i$ be an $\mathfrak{o}$-generator of the Tate module $W_f = \varprojlim_i W^i_{\varphi^{-i}(f)}$. Take *any* power series $s \in X\mathfrak{o}'[[X]]$ satisfying $\beta = s(\omega_n)$ and define

$$\theta_f^n s = \lambda_f \circ s - \frac{1}{\pi}\lambda_f^\varphi \circ s^\varphi \circ f^{\varphi^{-n}} \in X\mathfrak{o}'[[X]],$$

where $\lambda_f : F_f \xrightarrow{\sim} \mathbf{G}_a$ denotes the logarithm map satisfying $(d\lambda_f/dX)(0) = 1$. On the other hand, by a generalization of Coleman's *interpolation theorem* [Co1, dS1], there exists a power series $t \in \mathfrak{o}'((X))^\times$ such that $t^{\varphi^{-i}}(\omega_i) = N_{k_{\xi,n}/k_{\xi,i}}\alpha$ for all i $(1 \leq i \leq n)$. Put $\delta_f t = \dfrac{1}{d\lambda_f/dX}\cdot\dfrac{dt/dX}{t} \in X^{-1}\mathfrak{o}'[[X]]$ and define

$$\langle t,s\rangle_{f,n} = T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\{\sum_{\gamma\in W^n_{\varphi^{-n}(f)}} ((\theta^n_f s)(\delta_f t)^{\varphi^{-n}})(\gamma)\right.$$

$$\left. + \frac{ds}{dX}(0)(1 - \left((\mathcal{N}_f t)^{\varphi^{-1}}/t\right)^{\varphi^{-n}}(0))\}\right) \in k,$$

where $T_{k'/k} : k' \longrightarrow k$ means the trace map and $\mathcal{N}_f : \mathfrak{o}'((X)) \longrightarrow \mathfrak{o}'((X))$ denotes Coleman's *norm operator* [Co1, dS1]. The main result of Chapter 1 is the following

Theorem A (Theorem 1.20). We have

$$\langle t,s\rangle_{f,n} \in \mathfrak{o}; \quad (\alpha,\beta)_{f,n} = [\langle t,s\rangle_{f,n}]_{\varphi^{-n}(f)}(\omega_n).$$

If $d = 1$, then F_f is an ordinary Lubin-Tate group [LT] and Theorem A becomes the Coleman-de Shalit formula [Co2, Co3, dS2]. Next, let $m \geq n$ and suppose that $\alpha = N_{k_{\xi,m}/k_{\xi,n}}\alpha'$, $\alpha' \in k^{\times}_{\xi,m}$. Take *any* power series $t' \in \mathfrak{o}'((X))^{\times}$ such that $t'^{\varphi^{-m}}(\omega_m) = \alpha'$, and define

$$\langle t',\beta\rangle_{f,m} = T_{k_{\xi,m}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-m}}}\lambda_f(\beta)(\delta_f t')^{\varphi^{-m}}(\omega_m)\right) \in k.$$

Evaluating the "correction term" of the formula in Theorem A, we shall prove the following

Theorem B (Theorem 1.22). If (a) $m \geq 2n$, or if (b) $m \geq n + \ell$ and $\beta \in F_f(\mathfrak{p}^{2q^{n-1-\ell}}_{\xi,n})$ for some ℓ $(0 \leq \ell \leq n - 1)$, then

$$\langle t',\beta\rangle_{f,m} \in \mathfrak{o}; \quad (\alpha,\beta)_{f,n} = [\langle t',\beta\rangle_{f,m}]_{\varphi^{-n}(f)}(\omega_n).$$

The conditions in Theorem B refines those in the formulas of Iwasawa, Kudo and Wiles [Iw1, Ku, Wi].

In Chapter 2, we deal with ordinary Lubin-Tate groups. Fix $\pi \in k$, $f \in \mathfrak{o}[[X]]$, and write $F = F_f$, $X \underset{F}{+} Y = X \underset{f}{+} Y$, $[c]_F = [c]_f$, $\lambda_F = \lambda_f$, $k_n = k_{\pi,n}$, $\mathfrak{p}_n = \mathfrak{p}_{\pi,n}$ and $(\ ,\ )_n^F = (\ ,\ )_{f,n}$. We define two power series [Vo3, Sh4, Sa]:

$$\begin{cases} E_F(X) = \lambda_F^{-1}\left(\sum_{\ell=0}^{\infty} \dfrac{X^{q^\ell}}{\pi^\ell}\right) \in X\mathfrak{o}[[X]], \\ E(X) = 1 + E_{\mathbf{G}_m}(X) = \exp\left(\sum_{m=0}^{\infty} \dfrac{X^{p^m}}{p^m}\right) \in 1 + X\mathbf{Z}_p[[X]]. \end{cases}$$

Let F_0 denote the basic Lubin-Tate group associated with the polynomial $[\pi]_{F_0}(X) = X^q + \pi X$ and put $u_n = (\lambda_{F_0}^{-1} \circ \lambda_F)(\omega_n)$. Let $\mathfrak{R} = \{\theta \in \mathfrak{o}^\times \mid \theta^{q-1} = 1\}$ be the multiplicative representative set of $\mathfrak{o}/\mathfrak{p}$ and e the ramification index of $k/\mathbf{Q}_p$. We define two sets:

$$R_1 = \{E(\theta u_n^j) \mid \theta \in \mathfrak{R},\ 1 \le j < \frac{pe(q-1)q^{n-1}}{p-1}\ (p \nmid j) \text{ or } j = \frac{pe(q-1)q^{n-1}}{p-1}\},$$

$$R_2 = \{E_F(u_n^i) \mid 1 \le i < q^n\ (q \nmid i)\} \cup \{\kappa_F\},$$

where $\kappa_F = \lambda_F^{-1}\left(\sum_{\ell=0}^{\infty} \frac{1}{\pi^\ell}[\pi^n]_{F_0}(u_n^{q^\ell})\right)$. Then, R_1 is a set of $\mathbf{Z}_p$-generators of the principal units $1 + \mathfrak{p}_n$ of $k_n^\times$ [Sa], and R_2 represents an $\mathfrak{o}/(\pi^n)$-basis of the formal module $F(\mathfrak{p}_n)/[\pi^n]_F(F(\mathfrak{p}_n))$ [Vo3]. These sets generalize the *Takagi basis* [Ta1], as explained below. For $\eta \in \mathfrak{R}$ let σ_η be an element of $\mathrm{Gal}(k_n/k)$ such that $\sigma_\eta(\omega_n) = [\eta]_F(\omega_n)$. Put $H = \{\sigma_\eta \mid \eta \in \mathfrak{R}\}$ ($\subset \mathrm{Gal}(k_n/k)) \simeq \mathrm{Gal}(k_1/k)$. Then, as an $\mathfrak{o}[H]$-module, $F(\mathfrak{p}_n)$ has a direct sum decomposition

$$F(\mathfrak{p}_n) = A^{(1)} \oplus \cdots \oplus A^{(q-1)},$$

$$A^{(i)} = \{\beta \in F(\mathfrak{p}_n) \mid \sigma_\eta(\beta) = [\eta^i]_F(\beta) \quad \text{for all} \quad \eta \in \mathfrak{R}\}.$$

We shall generalize the characterization of the Takagi basis [Ta1, SI] as follows.

Theorem C (Theorem 2.1). For $1 \le i \le q-1$ we have

$$A^{(i)} = \begin{cases} \langle E_F(u_n^j) \in R_2 \mid j \equiv i \bmod (q-1)\rangle & \text{if} \quad i \ne 1, \\ \langle E_F(u_n^j) \in R_2 \mid j \equiv 1 \bmod (q-1)\rangle \oplus \langle \kappa_F \rangle & \text{if } i = 1. \end{cases}$$

Using the Coleman-de Shalit formula, Shiratani [Sh4] generalized Takagi's formulas [Ta1] as follows.

A complementary law: If p is *odd* and $q > 2n$, then

$$(u_n, E_F(u_n^i))_n^F = \begin{cases} 0 & (1 \le i < q^n), \\ \omega_n & (i = q^n). \end{cases}$$

A general law for $n = 1$: If p is *odd* and $j, i \ge 1$, then

$$(E(u_1^j), E_F(u_1^i))_1^F = \begin{cases} [j]_F(\omega_1) & (i + p^m j = q \text{ for some } m \ge 0), \\ 0 & (\text{otherwise}). \end{cases}$$

If $F = \mathbb{G}_m$ and $n = 1$, these formulas become Takagi's formulas. Using Vostokov's formula [Vo3, Vo4], we shall obtain

Theorem D (Theorem 2.7) (General laws for $n \geq 1$). If p is *odd*, $E(\theta u_n^j) \in R_1$ and $E_F(u_n^i) \in R_2$, then

$$(E(\theta u_n^j),\ E_F(u_n^i))_n^F = \left[\mathrm{res}_X\ \Phi/[\pi^n]_{F_0}\right]_F(\omega_n),$$

where

$$\Phi(X) = -\sum_{m=0}^{f-1}\sum_{\ell=0}^{\infty}\frac{q^{\ell+1} i\theta^{p^m}}{p^m\pi^{\ell+1}}X^{p^m j+q^{\ell+1} i-1} + \sum_{m=0}^{\infty} j\theta^{p^m}X^{i+p^m j-1},$$

$$\Phi/[\pi^n]_{F_0} \in \mathfrak{o}\{X\} = \{\sum_{i=-\infty}^{\infty} a_i X^i \mid a_i \in \mathfrak{o},\ a_i \longrightarrow 0(i \longrightarrow -\infty)\}$$

and $\mathrm{res}_X\left(\sum_{i=-\infty}^{\infty} a_i X^i\right) = a_{-1}$.

Using Theorem D, we can obtain formulas for $(E(\theta u_n^j),\ E_F(u_n^i))_n^F$, $n = 1, 2, 3, \cdots$. As n grows, these formulas become complicated rapidly. For general n, we shall prove the following

Theorem E (Theorem 2.10). If p is *odd*, $E(\theta u_n^j) \in R_1$ and $E_F(u_n^i) \in R_2$, then,

$$\begin{cases} (E(\theta u_n^j),\ E_F(u_n^i))_n^F = [j\theta^{p^m}]_F(\omega_n) & (i + p^m j = q^n,\quad m \in \mathbb{Z}), \\ [\pi^{n-1}]_F\left((E(\theta u_n^j),\ E_F(u_n^i))_n^F\right) = 0 & (\text{otherwise}). \end{cases}$$

Furthermore we have

$$(E(\theta u_n^j),\ E_F(u_n^i))_n^F = 0,$$

if one of the following conditions holds:

(a) $p \nmid j$ and $q \nmid (i + p^m j)$ for all m $(0 \leq m \leq f - 1)$,

(b) $i + p^{m} j \not\equiv 1 \mod (q - 1)$ for all m $(0 \leq m \leq f - 1)$,

(c) $i + p^{f-1} j < q^{n}$.

Finally, in Chapter 3, we deal with the case where $p = 2$. We shall obtain some formulas for $(\alpha, \beta)_n^F$ $(\alpha \in \{-u_n\} \cup R_1,\ \beta \in R_2)$, by using the Coleman-de Shalit formula.

Theorem F (Theorem 3.1) (A complementary law). If $p = 2$ and $q \geq 2n$, then

$$(-u_n, E_F(u_n^i))_n^F = \begin{cases} 0 & (1 \leq i < q^n), \\ \omega_n & (i = q^n). \end{cases}$$

Theorem G (Theorem 3.2) (A general law for $n = 1$). If $p = 2$, $\theta \in \mathfrak{R}$ and $j, i \geq 1$, then

$$(E(\theta u_1^j), E_F(u_1^i))_1^F = \begin{cases} [j\theta^{2^m}]_F(\omega_1) & (i + 2^m j = q,\quad m \in \mathbb{Z}), \\ 0 & (\text{otherwise}). \end{cases}$$

These formulas are 2-adic versions of Shiratani's formulas mentioned above.

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Chapter 1

Explicit reciprocity laws by relative Lubin-Tate groups

In [dS2], de Shalit proved an explicit reciprocity formula that had been conjectured [Co2] and proved for the multiplicative formal group [Co3] by Coleman. It is a *complete* formula for the generalized Hilbert norm residue symbol on fields generated by torsion points of Lubin-Tate formal groups, and generalizes the formulas of Artin-Hasse, Iwasawa, Kudo and Wiles [AH, Iw1, Ku, Wi].

In this chapter, we shall extend it to *relative* Lubin-Tate formal groups and give a refinement of the explicit formulas of Iwasawa, Kudo and Wiles. Relative Lubin-Tate groups were first introduced by Shiratani [Sh3]. Their basic properties and relations with local class field theory were studied by Menéndez [Me] and by de Shalit [dS1, dS3]. Recently, Iwasawa [Iw3] used them to construct local class field theory by means of the theory of formal groups.

§1.1. Generalized Hilbert norm residue symbol

Let p be a prime number and $k/\mathbf{Q}_p$ a finite extension. Let Ω denote the completion of the algebraic closure of k, and K $(\subset \Omega)$ the closure of the maximal unramified extension of k. Let $\nu : \Omega^\times \longrightarrow \mathbf{Q}$ denote the additive valuation of Ω normalized so that $\nu(k^\times) = \mathbf{Z}$. Fix a positive integer d and let $k'(\subset K)$

be the unramified extension of k of degree d. In this chapter, we fix an element $\xi \in k$ such that $\nu(\xi) = d$.

Let φ denote the Frobenius automorphism of K/k and let $\varphi' = \varphi^d$ be that of K/k'. Let $\mathfrak{o}$, $\mathfrak{o}'$ and $\mathfrak{o}_K$ denote the integer rings of k, k' and K, respectively, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. We denote by q the number of elements in the residue field $\mathfrak{o}/\mathfrak{p}$. For a finite extension L/M in Ω we denote by $N_{L/M} : L \longrightarrow M$ and $T_{L/M} : L \longrightarrow M$ the norm map and the trace map, respectively. Furthermore, if $M/\mathbf{Q}_p$ is finite and L/M is abelian, then we denote by

$$(\ \ , L/M) : M^{\times} \longrightarrow \mathrm{Gal}(L/M)$$

the *reciprocity map* (the *norm residue map*) in local class field theory. Basic references for (local and global) class field theory are Artin [Ar], Artin-Tate [AT], Cassels-Fröhlich [CF], Hasse [Ha1], Iwasawa [Iw2, Iw3], Lang [La], Serre [Ser], Takagi [Ta2] and Weil [We].

Let π be a prime element of K and take a power series $f \in \mathfrak{o}_K[[X]]$ (called a *Frobenius power series* belonging to π) satisfying

$$f(X) \equiv \pi X \mod \deg 2, \quad f(X) \equiv X^q \mod \pi.$$

Then there exists a unique one-dimensioal commutative formal group law $F_f(X,Y) \in \mathfrak{o}_K[[X,Y]]$ such that $F_f^{\varphi}(f(X),f(Y)) = f(F_f(X,Y))$ [Sh3, Theorem 1; dS1, Theorem 1; Iw3, Proposition 4.3]. Since $F_f^{\varphi} = F_{\varphi(f)}$ by definition, we have $f \in \mathrm{Hom}_{\mathfrak{o}_K}(F_f, F_{\varphi(f)})$ (= the set of all homomorphisms $g : F_f \longrightarrow F_{\varphi(f)}$ with coefficients in $\mathfrak{o}_K$). The formal group F_f is called a *relative Lubin-Tate formal group* (relative to the unramified extension K/k) [dS1]. We write

$X \underset{f}{+} Y = F_f(X,Y)$. For $c \in \mathfrak{o}$ there exists a unique endomorphism $[c]_f \in \mathfrak{o}_K[[X]]$ of F_f such that

$$[c]_f(X) \equiv cX \mod \deg 2, \quad [c]_f^{\varphi} \circ f = f \circ [c]_f.$$

The map $[\ ]_f : \mathfrak{o} \longrightarrow \mathrm{End}_{\mathfrak{o}_K}(F_f)$ (= the endomorphism ring of F_f) is a ring isomorphism [Sh3, Theorem 1]. If $f \in \mathfrak{o}'[[X]]$, then $F_f \in \mathfrak{o}'[[X,Y]]$ and $[c]_f \in \mathfrak{o}'[[X]]$, $c \in \mathfrak{o}$. In this case, F_f is a relative Lubin-Tate group (relative to the extension k'/k). If $f \in \mathfrak{o}[[X]]$, then F_f ($\in \mathfrak{o}[[X,Y]]$) is an ordinary Lubin-Tate group introduced by Lubin and Tate [LT] and $f = [\pi]_f$ is an endomorphism of F_f.

Let $\mathfrak{p}_\Omega$ denote the maximal ideal of the integer ring of Ω and $F_f(\mathfrak{p}_\Omega)$ the associated formal $\mathfrak{o}$-module with addition and $\mathfrak{o}$-action by

$$\alpha \underset{f}{+} \beta = F_f(\alpha,\beta), \quad c \underset{f}{\cdot} \alpha = [c]_f(\alpha)$$

for α, $\beta \in \Omega$ and $c \in \Omega$. Let

$$W_f^i = \{\gamma \in F_f(\mathfrak{p}_\Omega) \mid [c]_f(\gamma) = 0 \text{ for all } c \in \mathfrak{p}^i\} \quad (i \geq 1)$$

denote the set of all $\mathfrak{p}^i$-torsion points of F_f. Put $W_f^0 = \{0\}$ and $\tilde{W}_f^i = W_f^i - W_f^{i-1}$ $(i \geq 1)$.

The field $K_i = K(W_f^i)$, $i \geq 1$ does not depend on the choice of π and f, and is a totally ramified abelian extension of K of degree $(q-1)q^{i-1}$ [Iw3, Proposition 4.11]. Any element of $\tilde{W}_f^i$ is a prime element of K_i. Let U denote the unit group of $\mathfrak{o}$. There exists a surjective homomorphism $\delta_i : U \longrightarrow \mathrm{Gal}(K_i/K)$ such that, for any π and f as above, we have

$$\delta_i(u)(\gamma) = [u]_f(\gamma) \quad (u \in U,\ \gamma \in W_f^i),$$

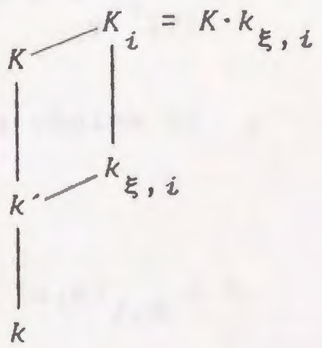
which induces an isomorphism $U/(1+\mathfrak{p}^i) \xrightarrow{\simeq} \mathrm{Gal}(K_i/K)$.

The *Tate module* $W_f = \varprojlim_i W^i_{\varphi^{-i}(f)}$ (the projective limit is taken with respect to the maps $\varphi^{-i}(f) : W^i_{\varphi^{-i}(f)} \longrightarrow W^{i-1}_{\varphi^{-i+1}(f)}$) of F_f is a free $\mathfrak{o}$-module of rank 1 [dS1, Proposition 1] with $\mathfrak{o}$-action

$$c \underset{f}{\cdot} \gamma = [c]_f(\gamma) = ([c]_{\varphi^{-i}(f)}(\gamma_i))_i \quad (c \in \mathfrak{o},\ \gamma = (\gamma_i)_i \in W_f).$$

Further, $\tilde{W}_f = \varprojlim_i \tilde{W}^i_{\varphi^{-i}(f)}$ is the set of all $\mathfrak{o}$-generators of W_f.

If $f \in \mathfrak{o}'[[X]]$ and $N_{k'/k}\pi = \xi$, then the field $k_{\xi,i} = k'(W^i_f)$, $i \geq 1$ does not depend on the choice of such π and f, and is an abelian extension over k of degree $d(q-1)q^{i-1}$, which corresponds to the subgroup $\langle\xi\rangle \times (1+\mathfrak{p}^i)$ $(= N(k_{\xi,i}/k)) \subset k^\times$ by local class field theory [dS1; Iw3, p.79 Remark].



Further, we have a surjective homomorphism $\delta_{\xi,i} : U \longrightarrow \mathrm{Gal}(k_{\xi,i}/k')$, such that, for any π and f as above, we have

$$\delta_{\xi,i}(u)(\gamma) = [u]_f(\gamma) \quad (u \in U,\ \gamma \in W^i_f).$$

Furthermore, by local class field theory [LT; Iw3, (6.2) and Lemma 6.1], we have

$$(1.1) \quad (u, k_{\xi,i}/k) = \delta_{\xi,i}(u^{-1}) \ (u \in U), \quad (\xi, k_{\xi,i}/k) = \textit{identity}.$$

In particular, if $k = \mathbb{Q}_p$, $\pi = p$ and $f(X) = (1+X)^p - 1$, then $F_f = \mathbb{G}_m$, $k_{p,n} = \mathbb{Q}_p(\zeta_{p^n})$ and (1.1) is the reciprocity law in cyclotomic fields [Dw].

Let $\mathfrak{o}_{\xi,i}$ denote the integer ring of $k_{\xi,i}$, and $\mathfrak{p}_{\xi,i}$ the maximal ideal of $\mathfrak{o}_{\xi,i}$. Let n be a positive integer and $\alpha \in k_{\xi,n}^{\times}$, $\beta \in F_f(\mathfrak{p}_{\xi,n})$. Let ρ be an element of $F_{\varphi^{-n}(f)}(\mathfrak{p}_\Omega)$ such that $f^{\varphi^{-1}} \circ \cdots \circ f^{\varphi^{-n}}(\rho) = \beta$, then $k_{\xi,n}(\rho)/k_{\xi,n}$ is a finite abelian extension with Galois group $\subset W^n_{\varphi^{-n}(f)}$. We define the *generalized Hilbert norm residue symbol* (a *Kummer pairing*) $(\alpha,\beta)_{f,n}$ [Fr, p.125; Wi; dS2] by

$$(\alpha,\beta)_{f,n} = (\alpha,\ k_{\xi,n}(\rho)/k_{\xi,n})(\rho) \underset{\varphi^{-n}(f)}{-} \rho \in W^n_{\varphi^{-n}(f)} .$$

Then, the symbol $(\alpha,\beta)_{f,n}$ does not depend on the choice of ρ and has the *norm property*:

$$\alpha \in N_{k_{\xi,n}(\rho)/k_{\xi,n}}(k_{\xi,n}(\rho)^{\times}) \text{ if and only if } (\alpha,\beta)_{f,n} = 0.$$

Furthermore, $(\alpha,\beta)_{f,n}$ is linear in α and $\mathfrak{o}$-linear in β. In later sections, we shall give a complete formula for $(\alpha,\beta)_{f,n}$, and a simpler formula under some restrictions on α and β.

§1.2. Preliminary lemmas

In this section, we shall state basic results concerning relative Lubin-Tate groups, which will be used in later sections.

Let π and π_1 be prime elements of K, and take Frobenius power series $f,\ f_1 \in \mathfrak{o}_K[[X]]$ belonging to π and π_1, respectively. Further, let U' and U_K denote the unit groups of $\mathfrak{o}'$ and $\mathfrak{o}_K$, respectively. Put

$$\begin{cases} \mathfrak{o}_K(\pi,\pi_1) = \{\eta \in \mathfrak{o}_K \mid \eta^{\varphi}\pi = \eta\pi_1\}, \\ U_K(\pi,\pi_1) = U_K \cap \mathfrak{o}_K(\pi,\pi_1) = \{\eta \in U_K \mid \eta^{\varphi-1} = \pi_1/\pi\}. \end{cases}$$

Lemma 1.1 ([Iw3, Lemma 3.11]). Let $i : \mathfrak{o} \longrightarrow \mathfrak{o}_K$ and $i : U \longrightarrow U_K$ be inclusion maps, and let

$$\begin{cases} \varphi - 1 : \mathfrak{o}_K \ni \eta \longmapsto \varphi(\eta) - \eta \in \mathfrak{o}_K, \\ \varphi - 1 : U_K \ni \eta \longmapsto \eta^{\varphi-1} = \varphi(\eta)/\eta \in U_K. \end{cases}$$

Then the following sequences are exact:

$$\begin{cases} 0 \longrightarrow \mathfrak{o} \xrightarrow{\ i\ } \mathfrak{o}_K \xrightarrow{\varphi-1} \mathfrak{o}_K \longrightarrow 0, \\ 1 \longrightarrow U \xrightarrow{\ i\ } U_K \xrightarrow{\varphi-1} U_K \longrightarrow 1. \end{cases}$$

Corollary 1.2. We have

$$U_K(\pi,\pi_1) \neq \phi, \quad \mathfrak{o}_K(\pi,\pi_1) \neq \phi, \quad \mathfrak{o}_K(\pi,\pi) = \mathfrak{o}.$$

If $\pi, \pi_1 \in k'$ and $N_{k'/k}\pi = N_{k'/k}\pi_1$, then $\mathfrak{o}_K(\pi,\pi_1) \subset \mathfrak{o}'$.

Proof. The first assertion is an easy consequence of the previous lemma. On the other hand, the exactness of the sequence

$$0 \longrightarrow \mathfrak{o}' \xrightarrow{\ i\ } \mathfrak{o}_K \xrightarrow{\varphi'-1} \mathfrak{o}_K \longrightarrow 0$$

implies the second assertion:

$$\mathfrak{o}_K(\pi,\pi_1) = \{\eta \in \mathfrak{o}_K \mid \eta^{\varphi}\pi = \eta\pi_1\} \subset \{\eta \in \mathfrak{o}_K \mid \eta^{\varphi'} = \eta\} = \mathfrak{o}'.$$

Lemma 1.3 ([Iw3, Proposition 3.12]). Let

$$L(X_1,\cdots,X_m) = \alpha_1 X_1 + \cdots + \alpha_m X_m, \quad \alpha_i \in \mathfrak{o}_K$$

be a linear form satisfying

$$\pi \cdot L^{\varphi}(X_1,\cdots,X_m) = \pi_1 \cdot L(X_1,\cdots,X_m).$$

Then there exists a unique power series $F \in \mathfrak{o}_K[[X_1,\cdots,X_m]]$ such that

$$F \equiv L \mod \deg 2, \quad F^{\varphi}(f(X_1),\cdots,f(X_m)) = f_1(F(X_1,\cdots,X_m)).$$

Furthermore, if $f, f_1 \in \mathfrak{o}'[[X]]$ and $N_{k'/k}\,\pi = N_{k'/k}\,\pi_1$, then $L, F \in \mathfrak{o}'[[X_1,\cdots,X_m]]$.

Corollary 1.4 ([LT, Lemma 2; dS1, Theorems 2 and 3]).
(1) For any $\eta \in \mathfrak{o}_K(\pi,\pi_1)$ there exists a unique power series $\theta = [\eta]_{f,f_1} \in \mathrm{Hom}_{\mathfrak{o}_K}(F_f, F_{f_1})$ such that

$$\theta(X) \equiv \eta X \mod \deg 2, \quad \theta^{\varphi} \circ f = f_1 \circ \theta.$$

If $\eta \in U_K(\pi,\pi_1)$, then $\theta \in \mathrm{Iso}_{\mathfrak{o}_K}(F_f, F_{f_1})$ (= the set of all isomorphisms $g : F_f \xrightarrow{\simeq} F_{f_1}$). We have

$$f = [\pi]_{f,\varphi(f)}, \quad [c]_f = [c]_{f,f} \quad \text{for } c \in \mathfrak{o}.$$

If $f, f_1 \in \mathfrak{o}'[[X]]$ and $N_{k'/k}\,\pi = N_{k'/k}\,\pi_1$, then $\theta \in \mathfrak{o}'[[X]]$. In particular, if $f \in \mathfrak{o}'[[X]]$ and $N_{k'/k}\,\pi = \xi$, then we have $f^{\varphi^{d-1}} \circ \cdots \circ f = [\xi]_f$.
(2) The map $[\ \ ]_{f,f_1} : \mathfrak{o}_K(\pi,\pi_1) \longrightarrow \mathrm{Hom}_{\mathfrak{o}_K}(F_f, F_{f_1})$ is an

injective additive homomorphism. Let π_2 be another prime element of K and let $f_2 \in \mathfrak{o}_K[[X]]$ be a Frobenius power series belonging to π_2. Then

$$[\eta_1\eta]_{f,f_2} = [\eta_1]_{f_1,f_2} \circ [\eta]_{f,f_1}, \quad [c]_{f_1} \circ [\eta]_{f,f_1} = [\eta]_{f,f_1} \circ [c]_f$$

for $\eta_1 \in \mathfrak{o}_K(\pi_1,\pi_2)$, $\eta \in \mathfrak{o}_K(\pi,\pi_1)$ and $c \in \mathfrak{o}$.

If $\eta \in U_K(\pi,\pi_1)$, then the map $\theta = [\eta]_{f,f_1}$ defines an $\mathfrak{o}$-module isomorphism $\theta : W_f \ni (\gamma_i)_i \longmapsto (\theta^{\varphi^{-i}}(\gamma_i))_i \in W_{f_1}$. Let $\omega = (\omega_i)_i \in \tilde{W}_f$ be an $\mathfrak{o}$-generator of W_f.

Lemma 1.5 ([Iw3, Lemma 8.1 and Proposition 8.2]).
(1) Let $g \in \mathfrak{o}_K[[X]]$. If $g^{\varphi^{-i}}(\omega_i) = 0$ for all i $(1 \le i \le n)$, then g can be divided (in $\mathfrak{o}_K[[X]]$) by $f^{\varphi^{n-1}} \circ \cdots \circ f(X)/X$. Furthermore, if $g^{\varphi^{-i}}(\omega_i) = 0$ for all $i \ge 1$, then $g = 0$.
(2) For $\omega_1 \in \tilde{W}_{f_1}$ there exists a unique element $\eta \in U_K(\pi,\pi_1)$ such that $[\eta]_{f,f_1}(\omega) = \omega_1$. In this case, if $\omega = \omega_1$, then $\eta = 1$, $f = f_1$ and $\pi = \pi_1$. In particular, $\tilde{W}_f \cap \tilde{W}_{f_1} = \phi$ if $f \ne f_1$.

We recall basic properties of Coleman's *norm operator* $\mathcal{N}_f : \mathfrak{o}_K((X)) \longrightarrow \mathfrak{o}_K((X))$ [Col; dS1; Iw3, §5.2], which is a unique multiplicative operator satisfying

(𝒩0) $$(\mathcal{N}_f t)\circ f(X) = \prod_{\gamma\in W_f^1} t(X \underset{f}{+} \gamma) \quad (\text{in } K((X)))$$

for all $t \in \mathfrak{o}_K((X))$. If $t \in \mathfrak{o}_K[[X]]$, then $\mathcal{N}_f t \in \mathfrak{o}_K[[X]]$. If $f \in \mathfrak{o}'[[X]]$ and $t \in \mathfrak{o}'((X))$, then $\mathcal{N}_f t \in \mathfrak{o}'((X))$.

(𝒩1) $$\mathcal{N}_f t \equiv t^{\varphi} \mod \pi \quad \text{for } t \in \mathfrak{o}_K((X)).$$

(𝒩2) Put $\mathcal{N}_f^i = \mathcal{N}_{\varphi^{i-1}(f)} \circ\cdots\circ \mathcal{N}_f$ $(i \geq 1)$, then

$$(\mathcal{N}_f^i t)\circ f^{\varphi^{i-1}} \circ\cdots\circ f(X) = \prod_{\gamma\in W_f^i} t(X \underset{f}{+} \gamma), \quad t \in \mathfrak{o}_K((X)).$$

(𝒩3) If $t \in X^j \mathfrak{o}_K[[X]]^{\times}$, $j \in \mathbb{Z}$, then $\mathcal{N}_f^i t \in X^j \mathfrak{o}_K[[X]]^{\times}$ and

$$\mathcal{N}_f^{i+1} t/(\mathcal{N}_f^i t)^{\varphi} \equiv 1 \mod \pi^{i+1}.$$

(𝒩4) For $t \in \mathfrak{o}_K((X))$ we have

$$\begin{cases} (\mathcal{N}_f t)^{\varphi^{-i}}(\omega_{i-1}) = N_{K_i/K_{i-1}}(t^{\varphi^{-i}}(\omega_i)), & i \geq 2, \\ (\mathcal{N}_f^{n-i} t)^{\varphi^{-n}}(\omega_i) = N_{K_n/K_i}(t^{\varphi^{-n}}(\omega_n)), & 1 \leq i \leq n. \end{cases}$$

Remark 1.6. Let $t \in \mathfrak{o}_K((X))^{\times}$. It follows from (𝒩4), (𝒩1) and Lemma 1.5(1) that

$$N_{K_i/K_{i-1}}(t^{\varphi^{-i}}(\omega_i)) = t^{\varphi^{-i+1}}(\omega_{i-1}) \quad \text{for all } i\ (2 \leq i \leq n)$$

$$\iff (\mathcal{N}_f t)^{\varphi^{-i}}(\omega_{i-1}) = t^{\varphi^{-i+1}}(\omega_{i-1}) \quad \text{for all } i\ (2 \leq i \leq n)$$

$$\iff (\mathcal{N}_f t)^{\varphi^{-1}}/t \in 1 + \frac{\pi}{X} f^{\varphi^{n-2}} \circ\cdots\circ f(X)\mathfrak{o}_K[[X]].$$

Therefore, we define [Co2, p.459; dS2, §2] subgroups

$$M_f^n = \{t \in \mathfrak{o}_K((X))^\times \mid (N_f t)^{\varphi^{-1}}/t \in 1 + \frac{\pi}{X} f^{\varphi^{n-2}} \circ \cdots \circ f(X)\mathfrak{o}_K[[X]]\}$$

of $\mathfrak{o}_K((X))^\times$ for $n \geq 1$ (in particular, $M_f^1 = \mathfrak{o}_K((X))^\times$), and put

$$M_f^\infty = \bigcap_{n=1}^\infty M_f^n = \{t \in \mathfrak{o}_K((X))^\times \mid N_f t = t^\varphi\}.$$

Let $\mathfrak{O}_i$ denote the integer ring of K_i, and $\mathfrak{P}_i$ the maximal ideal of $\mathfrak{O}_i$.

Lemma 1.7 (A generalization of [Iw3, Lemma 5.10]). Let $\alpha_i \in \pi^{n-i}\mathfrak{P}_1\mathfrak{O}_i$ for $1 \leq i \leq n$. Then there exists a power series $h \in X\mathfrak{o}_K[[X]]$ such that

$$h^{\varphi^{-i}}(\omega_i) = \alpha_i \quad \text{for all } i \ (1 \leq i \leq n).$$

If $f \in \mathfrak{o}'[[X]]$, $N_{K'/K}\pi = \xi$ and $\alpha_i \in \pi^{n-i}\mathfrak{p}_{\xi,1}\mathfrak{o}_{\xi,i}$ $(1 \leq i \leq n)$, then h can be taken from $X\mathfrak{o}'[[X]]$.

Proof. For $1 \leq i \leq n$, we put

$$g_i = (f^{\varphi^{n-1}} \circ \cdots \circ f)(f^{\varphi^{i-2}} \circ \cdots \circ f)/(f^{\varphi^{i-1}} \circ \cdots \circ f) \in X\mathfrak{o}_K[[X]].$$

Then, for $1 \leq i, j \leq n$, we have

$$g_i^{\varphi^{-j}}(\omega_j) = \begin{cases} \pi^{\varphi^{n-1-i}+\cdots+1}\omega_1 & (j = i), \\ 0 & (\text{otherwise}). \end{cases}$$

Since we can write $\alpha_i = \pi^{\varphi^{n-1-i}+\cdots+1}\omega_1 h_i^{\varphi^{-i}}(\omega_i)$ with some

$h_i \in \mathfrak{o}_K[[X]]$, the power series $h = \sum_{i=1}^{n} g_i h_i \in X\mathfrak{o}_K[[X]]$ satisfies that $h^{\varphi^{-i}}(\omega_i) = \alpha_i$ for all i $(1 \le i \le n)$. The second assertion can be proved in the same way.

Let $\nu_i : K_i^\times \longrightarrow \mathbf{Z}$ denote the normalized valuation of K_i. The next proposition is a slight generalization of Coleman's *interpolation theorem* [Col, Theorems 15 and 16; dS1, Theorem 4; dS3, 2.2. Theorem].

Proposition 1.8. (1) Let $\alpha \in K_n^\times$, $\nu_n(\alpha) = j \in \mathbf{Z}$ and put $\alpha_i = N_{K_n/K_i}\alpha$ $(1 \le i \le n)$. Then there exists a power series $t \in X^j\mathfrak{o}_K[[X]]^\times$ (called *a Coleman power series* for α) such that

$$t^{\varphi^{-i}}(\omega_i) = \alpha_i \quad \text{for all} \quad i \ (1 \le i \le n), \quad \text{i.e.,} \quad t \in M_f^n.$$

If $f \in \mathfrak{o}'[[X]]$, $N_{k'/k}\pi = \xi$ and $\alpha \in k_{\xi,n}^\times$, then t can be taken from $X^j\mathfrak{o}'[[X]]^\times$.

(2) Let $j \in \mathbf{Z}$. For $a = (\alpha_i)_i \in A = \varprojlim_i K_i^\times$ (the project limit is taken with respect to the norm maps) such that $\nu_i(\alpha_i) = j$, there exists a unique power series $t = t_{\omega,a} \in X^j\mathfrak{o}_K[[X]]^\times$ (called *the Coleman power series* for a) such that

$$t^{\varphi^{-i}}(\omega_i) = \alpha_i \quad \text{for all} \quad i \ge 1, \quad \text{i.e.,} \quad t \in M_f^\infty.$$

If $f \in \mathfrak{o}'[[X]]$, $N_{k'/k}\pi = \xi$ and $a \in A_\xi = \varprojlim_i k_{\xi,i}^\times$, then $t \in X^j\mathfrak{o}'[[X]]^\times$.

Proof. (1) Write $\alpha = t'^{\varphi^{-n}}(\omega_n)$ with $t' \in X^j \mathfrak{o}_K[[X]]^\times$. Then, by ($\mathcal{N}$4), we have

$$\alpha_i = N_{K_n/K_i}(t'^{\varphi^{-n}}(\omega_n)) = (\mathcal{N}_f^{n-i} t')^{\varphi^{-n}}(\omega_i), \quad 1 \le i \le n.$$

Since $(\mathcal{N}_f^n t')^{\varphi^{-i}} / \mathcal{N}_f^{n-i} t' \equiv 1 \mod \pi^{n-i+1}$ by ($\mathcal{N}$3), we put

$$\alpha_i' = \alpha_i / (\mathcal{N}_f^n t')^{\varphi^{-n-i}}(\omega_i) - 1 \in \pi^{n-i+1}\mathfrak{O}_i \subset \pi^{n-i}\mathfrak{P}_1\mathfrak{O}_i.$$

By Lemma 1.7, there exists a power series $h \in X\mathfrak{o}_K[[X]]$ such that $h^{\varphi^{-i}}(\omega_i) = \alpha_i'$ for all i $(1 \le i \le n)$. Therefore, the power series

$$t = (\mathcal{N}_f^n t')^{\varphi^{-n}}(1 + h) \in X^j \mathfrak{o}_K[[X]]^\times$$

satisfies $t^{\varphi^{-i}}(\omega_i) = \alpha_i$ for all i $(1 \le i \le n)$. The second assertion can be proved in the same way.
(2) By (1), there exists $t_n \in X^j \mathfrak{o}_K[[X]]^\times$ for each $n \ge 1$ such that $t_n^{\varphi^{-i}}(\omega_i) = \alpha_i$ for all i $(1 \le i \le n)$. Therefore, by Lemma 1.5(1), $f^{\varphi^{n-1}} \circ \cdots \circ f(X)/X$ divides $X^{-j}(t_{n+1} - t_n)$ in $\mathfrak{o}_K[[X]]$, which implies that the sequence $\{t_n\}_{n\ge1}$ converges to a power series $t \in X^j \mathfrak{o}_K[[X]]^\times$. Hence, for all $i \ge 1$, we have $t^{\varphi^{-i}}(\omega_i) = t_n^{\varphi^{-i}}(\omega_i) = \alpha_i$ $(n \ge i)$. The uniqueness of t also follows from Lemma 1.5(1). The second assertion can be proved in the same way.

Corollary 1.9. For $a = (\alpha_i)_i \in A$ satisfying $\nu_i(\alpha_i) = 1$, there exists a unique Frobenius power series $f_a \in \mathfrak{o}_K[[X]]$

(belonging to some prime element π_a of K) such that $a \in \widetilde{W}_{f_a}$. If $a \in A_\xi$, then $f_a \in \mathfrak{o}'[[X]]$ and $N_{k'/k}\pi_a = \xi$.

Proof. Let $t = t_{\omega,a} = \eta X + \cdots \in X\mathfrak{o}_K[[X]]^\times$ be the Coleman power series for a. Put $f_a = t^\varphi \circ f \circ t^{-1} = \eta^{\varphi - 1}\pi X + \cdots \in X\mathfrak{o}_K[[X]]$, where t^{-1} denotes the inverse of t in the multiplicative group $(X\mathfrak{o}_K[[X]]^\times, \circ)$. Then f_a is a Frobenius power series belonging to $\pi_a = \eta^{\varphi-1}\pi \in K$ and $a \in \widetilde{W}_{f_a}$. The uniqueness of f_a follows from Lemma 1.5(2). If $a \in A_\xi$, we take $f \in \mathfrak{o}'[[X]]$ and $\pi \in k'$ so that $N_{k'/k}\pi = \xi$. Then it follows that $t \in X\mathfrak{o}'[[X]]^\times$, $f_a \in \mathfrak{o}'[[X]]$ and $N_{k'/k}\pi_a = \xi$.

Remark 1.10. If ω $(\in \widetilde{W}_f)$ belongs to A, we call ω *normed*. Since $t_{\omega,\omega}(X) = X$, we see that ω is normed if and only if $X \in M_f^\infty$ (i.e., $\mathcal{N}_f X = X$). So, we call f *normed* if $\mathcal{N}_f X = X$. Assume that f is normed, then $\mathcal{N}^n_{\varphi^{-n}(f)} X = X$. Therefore, by ($\mathcal{N}$2),

$$f^{\varphi^{-1}} \circ \cdots \circ f^{\varphi^{-n}}(X) = \prod_{\gamma \in W^n_{\varphi^{-n}(f)}} (X \underset{\varphi^{-n}(f)}{+} \gamma).$$

Hence, if $\beta \in F_f(\mathfrak{p}_{\xi,n}) - \{0\}$ and $\beta = f^{\varphi^{-1}} \circ \cdots \circ f^{\varphi^{-n}}(\rho)$, $\rho \in F_{\varphi^{-n}(f)}(\mathfrak{p}_\Omega)$, then we have

$$\beta = \prod_{\gamma \in W^n_{\varphi^{-n}(f)}} (\xi \underset{\varphi^{-n}(f)}{+} \gamma) \in N_{k_{\xi,n}(\rho)/k_{\xi,n}}(k_{\xi,n}(\rho)^\times).$$

This implies that $(\beta,\beta)_{f,f} = 0$ if f is normed [Iw3, Lemma 8.6].

Let $\lambda_f : F_f \xrightarrow{\simeq} \mathbf{G}_a$ be the logarithm map of F_f satisfying $(d\lambda_f/dX)(0) = 1$.

Lemma 1.11 ([Co1, Lemmas 20 and 21; Wi, Lemmas 3, 4, 5 and 11]). Let s, $h \in X\mathfrak{o}_K[[X]]$ and suppose that $h(X) \equiv X^q \mod \pi$.

(1) $\lambda_f = \lim_{n\to\infty} f^{\varphi^{n-1}} \circ \cdots \circ f/\pi^{\varphi^{n-1}+\cdots+1}$.

(2) $f^{\varphi^{n-1}} \circ \cdots \circ f \circ s \equiv f^{\varphi^{n-1}} \circ \cdots \circ f^{\varphi} \circ s^{\varphi} \circ h \mod \pi^n$.

(3) $\lambda_f \circ s - \frac{1}{\pi}\lambda_f^{\varphi} \circ s^{\varphi} \circ h \in X\mathfrak{o}_K[[X]]$.

(4) Let $\mathcal{L}$ be the $\mathfrak{o}_K[[X]]$-submodule of $K[[X]]$ generated by $\{X^{q^i}/\pi^i\}_{i=0}^{\infty}$. Then $\lambda_f \in \mathcal{L}$. In particular, $\lambda_f(\beta)$ converges for any $\beta \in F_f(\mathfrak{p}_\Omega)$.

(5) Put $D = \{\beta \in \mathfrak{p}_\Omega \mid \nu(\beta) > \frac{1}{q-1}\}$ and let $e_f = \lambda_f^{-1} : \mathbf{G}_a \longrightarrow F_f$ be the exponential series for F_f. Then $e_f(\beta)$ converges for $\beta \in D$. Furthermore, $\lambda_f : F_f(D) \longrightarrow D$ and $e_f : D \longrightarrow F_f(D)$ are $\mathfrak{o}$-isomorphisms, and the equalities $\nu(\lambda_f(\beta)) = \nu(e_f(\beta)) = \nu(\beta)$ $(\beta \in D)$ hold.

(6) Let $\beta \in \mathfrak{p}_\Omega - D$ and take an integer $i \geq 1$ such that $\frac{1}{q-1} < q^i \cdot \nu(\beta) \leq \frac{q}{q-1}$, then

$$\nu(\lambda_f(\beta)) \geq q^i \cdot \nu(\beta) - i.$$

(7) Let $\mu \in \mathbf{Q}$ be such that $0 < \mu \leq \frac{1}{q-1}$. For such μ, we put $D(\mu) = \{\beta \in \mathfrak{p}_\Omega \mid \nu(\beta) \geq \mu\}$. Take an integer $i = i(\mu) \geq 1$ such that $\frac{1}{q-1} < q^i \cdot \mu \leq \frac{q}{q-1}$, then

$$\nu(\lambda_f(\beta)) \geq q^i \cdot \mu - i \quad \text{for all } \beta \in D_\mu.$$

(8) If $f \in \mathfrak{o}'[[X]]$, $N_{k'/k}\,\pi = \xi$ and $\beta \in F_f(\mathfrak{p}_{\xi,n})$, then

$$T_{k_{\xi,n}/k}(\frac{1}{\pi}\lambda_f(\beta)\mathfrak{o}_{\xi,n}) \subset \mathfrak{o}.$$

Proof. (1) Put $g_n = f^{\varphi^{n-1}}\circ\cdots\circ f/\pi^{\varphi^{n-1}+\cdots+1}$. Let (π, X) denote the ideal of $\mathfrak{o}_K[[X]]$ generated by π and X. Since $f^{\varphi^{n-1}}\circ\cdots\circ f \in (\pi, X)^{n+1}$ and $f^{\varphi^n}(X) - \pi^{\varphi^n}X \equiv 0 \pmod{\deg 2}$, we have

$$f^{\varphi^n}\circ f^{\varphi^{n-1}}\circ\cdots\circ f - \pi^{\varphi^n}f^{\varphi^{n-1}}\circ\cdots\circ f \in (\pi, X)^{2n+2}.$$

Therefore the sequence $\{g_n\}_{n\geq 1}$ converges to some $g \in XK[[X]]$. On the other hand,

$$f^{\varphi^{n-1}}\circ\cdots\circ f(X \underset{f}{+} Y) - f^{\varphi^{n-1}}\circ\cdots\circ f(X) - f^{\varphi^{n-1}}\circ\cdots\circ f(Y) \in (\pi, X)^{2n+2}$$

implies that

$$g(X \underset{f}{+} Y) = \lim_{n\to\infty} g_n(X \underset{f}{+} Y) = \lim_{n\to\infty}(g_n(X) + g_n(Y)) = g(X) + g(Y).$$

Since $\frac{dg}{dX}(0) = \lim_{n\to\infty}\frac{dg_n}{dX}(0) = 1$, we conclude that $g = \lambda_f$.

(2) If $n = 1$, then

$$f\circ s(X) \equiv s(X)^q \equiv s^{\varphi}(X^q) \equiv s^{\varphi}\circ h \quad \text{mod } \pi.$$

Next, let $n \geq 1$ and suppose that

$$f^{\varphi^{n-1}}\circ\cdots\circ f\circ s = f^{\varphi^{n-1}}\circ\cdots\circ f^{\varphi}\circ s^{\varphi}\circ h + \pi^n v, \quad f^{\varphi^n}(X) = X^q + \pi w(X)$$

with some $v, w \in X\mathfrak{o}_K[[X]]$. Then

$$f^{\varphi^n}\circ f^{\varphi^{n-1}}\circ\cdots\circ f\circ s \equiv (f^{\varphi^{n-1}}\circ\cdots\circ f^{\varphi}\circ s^{\varphi}\circ h)^q + \pi w(f^{\varphi^{n-1}}\circ\cdots\circ f^{\varphi}\circ s^{\varphi}\circ h) \pmod{\pi^{n+1}}$$

$$= f^{\varphi^n} \circ f^{\varphi^{n-1}} \circ \cdots \circ f^{\varphi} \circ s^{\varphi} \circ h.$$

By induction, this completes the proof.

(3) By (1) and (2), we see that

$$\lambda_f \circ s - \frac{1}{\pi}\lambda_f^{\varphi} \circ s^{\varphi} \circ h = \lim_{n\to\infty} \frac{f^{\varphi^{n-1}} \circ \cdots \circ f \circ s - f^{\varphi^{n-1}} \circ \cdots \circ f^{\varphi} \circ s^{\varphi} \circ h}{\pi^{n-1} + \cdots + 1} \in X\mathfrak{o}_K[[X]].$$

(4) Since $\mathcal{L} = \{ \sum_{j=1}^{\infty} a_j X^j \in K[[X]] \mid \log_q j + \nu(a_j) \geq 0\}$ is a closed $\mathfrak{o}_K[[X]]$-submodule of $K[[X]]$, it is enough to show that

$$f^{\varphi^{n-1}} \circ \cdots \circ f \in \mathcal{L}_n = \{ \sum_{j=1}^{\infty} a_j X^j \in K[[X]] \mid \log_q j + \nu(a_j) \geq n\}$$

for all $n \geq 1$. If $n = 1$, this follows from $f(X) \equiv X^q \mod \pi$. Let $n \geq 1$ and suppose that $f^{\varphi^{n-1}} \circ \cdots \circ f(X) = \sum_{j=1}^{\infty} a_j X^j$, $\log_q j + \nu(a_j) \geq n$. Since $\mathcal{L}_n$ is an $\mathfrak{o}_K[[X]]$-module, we see that

$$f^{\varphi^n} \circ f^{\varphi^{n-1}} \circ \cdots \circ f = (\sum_{j=1}^{\infty} a_j X^j)^q + \pi w(\sum_{j=1}^{\infty} a_j X^j) \in \mathcal{L}_{n+1}.$$

By induction, this completes the proof.

(5) Let $z \in \mathfrak{p}_{\Omega}$ be such that $\nu(z) \geq \frac{1}{q-1}$, and define $\Lambda_z(X) = \lambda_f(zX)/z$. Then, by (4), we see that $\Lambda_z \in X\mathfrak{o}_K[[X]]^{\times}$. Put $E_z = \Lambda_z^{-1} \in X\mathfrak{o}_K[[X]]^{\times}$. Solving $Y = \lambda_f(X) = z\Lambda_z(X/z)$, we obtain $X = e_f(Y) = zE_z(Y/z)$, i.e., $e_f(X) = zE_z(X/z)$. Therefore, $e_f(\beta)$ converges for $\beta \in D$ and, $e_f : D \longrightarrow F_f(D)$ is an $\mathfrak{o}$-isomorphism. On the other hand, it follows from $\lambda_f(X) = z\Lambda_z(X/z)$ that $\lambda_f : F_f(D) \longrightarrow D$ is also an $\mathfrak{o}$-isomorphism. Finally, for $\beta \in D$, we see that

$$\nu(\lambda_f(\beta)) = \nu(z\Lambda_z(\beta/z)) = \nu(\beta) = \nu(zE_z(\beta/z)) = \nu(e_f(\beta)).$$

(6) For $y \in F_f(\mathfrak{p}_\Omega)$, we have

$$\begin{cases} \nu(f(y)) = \begin{cases} q\cdot\nu(\beta) & \text{if } \nu(y) < \dfrac{1}{q-1}, \\ \nu(\beta)+1 & \text{if } \nu(y) > \dfrac{1}{q-1}, \end{cases} \\ \nu(f(y)) \geq \dfrac{q}{q-1} \quad \text{if } \nu(y) = \dfrac{1}{q-1}. \end{cases}$$

Hence, by (5), we see that

$$\nu(\lambda_f(\beta)) = \nu\left(\frac{1}{\pi^{\varphi^{i-1}+\cdots+1}}\lambda_{\varphi^i(f)} \circ f^{\varphi^{i-1}} \circ\cdots\circ f(\beta)\right) \geq q^i\cdot\nu(\beta) - i.$$

(7) Define $P(\mu) = q^{i(\mu)}\mu - i(\mu)$ for μ $(0 < \mu \leq \frac{1}{q-1})$. We extend this function to $\{\mu \in \mathbf{Q} \mid \mu > 0\}$ by $P(\mu) = \mu$ for $\mu > \frac{1}{q-1}$. Then P is continuous and monotone increasing. Since we have $\nu(\lambda_f(\beta)) \geq P(\nu(\beta))$ for all $\beta \in F_f(\mathfrak{p}_\Omega)$ by (5) and (6), the assertion of (7) follows.

(8) Since $\beta \in D(1/(q-1)q^{n-1})$, we see from (7) that

$$\nu(\tfrac{1}{\pi}\lambda_f(\beta)) \geq \frac{q}{q-1} - n - 1 = -(n - \frac{1}{q-1}).$$

Since the different of $k_{\xi,n}/k$ is

$$\mathfrak{p}_{\xi,n}^{(q-1)q^{n-1}\{n-1/(q-1)\}} \sim \mathfrak{p}^{n-1/(q-1)},$$

the assertion of (8) follows.

Finally, we recall basic properties of Coleman's *trace operator* $\mathcal{G}_f : \mathfrak{o}_K((X)) \longrightarrow \mathfrak{o}_K((X))$ [Col], which is a unique additive operator satisfying

$$(\mathcal{G}0)\qquad (\mathcal{G}_f t)\circ f(X) = \sum_{\gamma\in W_f^1} t(X \underset{f}{+} \gamma) \quad (\text{in } K((X)))$$

for all $t \in \mathfrak{o}_K((X))$. If $t \in \mathfrak{o}_K[[X]]$, then $\mathcal{G}_f t \in \mathfrak{o}_K[[X]]$. If $f \in \mathfrak{o}'[[X]]$ and $t \in \mathfrak{o}'((X))$, then $\mathcal{G}_f t \in \mathfrak{o}'((X))$. Put $\mathcal{G}_f^i = \mathcal{G}_{\varphi^{i-1}(f)} \circ\cdots\circ \mathcal{G}_f$ $(i \geq 1)$, then, for $t \in \mathfrak{o}_K((X))$, we have

$$(\mathcal{G}1)\qquad (\mathcal{G}_f^i t)\circ f^{\varphi^{i-1}} \circ\cdots\circ f(X) = \sum_{\gamma\in W_f^i} t(X \underset{f}{+} \gamma),$$

$$(\mathcal{G}2)\qquad \mathcal{G}_f^i t \equiv 0 \mod \pi^i,$$

$$(\mathcal{G}3)\qquad \begin{cases} (\mathcal{G}_f t)^{\varphi^{-i}}(\omega_{i-1}) = T_{K_i/K_{i-1}}(t^{\varphi^{-i}}(\omega_i)), & i \geq 2, \\ (\mathcal{G}_f^{n-i} t)^{\varphi^{-n}}(\omega_i) = T_{K_n/K_i}(t^{\varphi^{-n}}(\omega_n)), & 1 \leq i \leq n. \end{cases}$$

§1.3. Analytical pairing $\langle t,s\rangle_{f,n}$

In this section, we define an analytical pairing $\langle t,s\rangle_{f,n}$ for $t \in M_f^n$, $s \in F_f(X\mathfrak{o}_K[[X]])$ and state its basic properties.

As in the previous section, let π and π_1 be prime elements of K, and let f, $f_1 \in \mathfrak{o}_K[[X]]$ be Frobenius power series belonging to π and π_1, respectively. Let $t \in \mathfrak{o}_K((X))^\times$ and $s \in F_f(X\mathfrak{o}_K[[X]])$. We put

$$\begin{cases} \delta_f t = \dfrac{1}{d\lambda_f/dX}\cdot\dfrac{dt/dX}{t} \in X^{-1}\mathfrak{o}_K[[X]], \\ \theta_f^n s = \lambda_f\circ s - \dfrac{1}{\pi}\lambda_f^{\varphi}\circ s^{\varphi}\circ f^{\varphi^{-n}} \in X\mathfrak{o}_K[[X]] \quad (\text{Lemma } 1.11(3)). \end{cases}$$

If $t \in M_f^n$, then we define

$$\langle t,s\rangle_{f,n}^1 = \sum_{i=0}^{d-1}\left\{\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\sum_{\gamma\in W^n_{\varphi^{-n}(f)}}\left((\theta_f^n\, s)(\delta_f t)^{\varphi^{-n}}\right)(\gamma)\right\}^{\varphi^i}$$

$$= \sum_{i=0}^{d-1}\left\{\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\left(\mathcal{Y}_{\varphi^{-n}(f)}((\theta_f^n\, s)(\delta_f t)^{\varphi^{-n}})\right)(0)\right\}^{\varphi^i} \in \mathfrak{o}_K$$

(by ($\mathcal{Y}$2)),

$$\langle t,s\rangle_{f,n}^2 = \sum_{i=0}^{d-1}\left\{\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\frac{ds}{dX}(0)\left(1-((N_f t)^{\varphi^{-1}}/t)^{\varphi^{-n}}(0)\right)\right\}^{\varphi^i} \in \mathfrak{o}_K$$

(by the definition of M_f^n),

$$\langle t,s\rangle_{f,n} = \langle t,s\rangle_{f,n}^1 + \langle t,s\rangle_{f,n}^2 \in \mathfrak{o}_K.$$

In particular, if f, t and s have coefficients in $\mathfrak{o}'$, then

$$\langle t,s\rangle_{f,n}^1 = T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\sum_{\gamma\in W^n_{\varphi^{-n}(f)}}\left((\theta_f^n\, s)(\delta_f t)^{\varphi^{-n}}\right)(\gamma)\right) \in \mathfrak{o},$$

$$\langle t,s\rangle_{f,n}^2 = T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\frac{ds}{dX}(0)\left(1-((N_f t)^{\varphi^{-1}}/t)^{\varphi^{-n}}(0)\right)\right) \in \mathfrak{o}.$$

Obviously, $\langle t,s\rangle_{f,n}$ is $\mathfrak{o}$-linear in s, and $\langle t,s\rangle_{f,n}^1$ is linear in t. On the other hand, for t_1, $t_2 \in M_f^n$, we see that

$$1-((N_f(t_1t_2))^{\varphi^{-1}}/t_1t_2)^{\varphi^{-n}}(0) = 1-((N_f t_1)^{\varphi^{-1}}/t_1)^{\varphi^{-n}}(0)$$

$$+ \left((\mathcal{N}_f t_1)^{\varphi^{-1}}/t_1\right)^{\varphi^{-n}}(0)\left(1-\left((\mathcal{N}_f t_2)^{\varphi^{-1}}/t_2\right)^{\varphi^{-n}}(0)\right)$$

$$\equiv 1 - \left((\mathcal{N}_f t_1)^{\varphi^{-1}}/t_1\right)^{\varphi^{-n}}(0) + 1 - \left((\mathcal{N}_f t_2)^{\varphi^{-1}}/t_2\right)^{\varphi^{-n}}(0) \mod \pi^{2n}.$$

Therefore, $\langle t,s\rangle^2_{f,n} \pmod{\pi^n}$ is linear in t.

Lemma 1.12. Let $\eta \in U_K(\pi,\pi_1)$ and $\theta = [\eta]_{f,f_1} \in X\mathfrak{o}_K[[X]]^\times$. Put

$$t_1 = t\circ\theta^{-1} \in \mathfrak{o}_K((X))^\times, \quad s_1 = \theta\circ s\circ(\theta^{\varphi^{-n}})^{-1} \in F_{f_1}(X\mathfrak{o}_K[[X]]).$$

Then,

$$\begin{cases} \delta_f t = \eta(\delta_{f_1} t_1)\circ\theta, \quad \mathcal{N}_f t = (\mathcal{N}_{f_1} t_1)\circ\theta^{\varphi}, \\ \theta^n_f\, s = \eta^{-1}(\theta^n_{f_1} s_1)\circ\theta^{\varphi^{-n}}. \end{cases}$$

In particular, if $t \in M^n_f$, then $t_1 \in M^n_{f_1}$ and

$$\langle t,s\rangle_{f,n} = \langle t_1,s_1\rangle_{f_1,n}.$$

Proof. The assertion of the lemma follows from

$$\theta^{\varphi}\circ f = f_1\circ\theta, \quad \lambda_{f_1}\circ\theta = \eta\lambda_f, \quad \eta^{\varphi-1} = \pi_1/\pi,$$

$$(\mathcal{N}_f t)\circ f(X) = \prod_{\gamma\in W^1_f} t(X \underset{f}{+} \gamma) = \prod_{\gamma'\in W^1_{f_1}} t_1(\theta(X) \underset{f_1}{+} \gamma')$$

$$= (\mathcal{N}_{f_1} t_1)\circ f_1\circ\theta(X) = (\mathcal{N}_{f_1} t_1)\circ\theta^{\varphi}\circ f(X).$$

Lemma 1.13 ([Co1, p.115, Remark]). We have

$$\pi\delta_{\varphi(f)}(\mathscr{N}_f t) = \mathscr{G}_f(\delta_f t), \quad t \in \mathfrak{o}_K((X))^{\times}.$$

Proof. From $\lambda_f(X \underset{f}{+} \gamma) = \lambda_f(X) + \lambda_f(\gamma)$ and $\lambda_{\varphi(f)} \circ f = \pi\lambda_f$, we have

$$\frac{d(X \underset{f}{+} \gamma)}{dX} = \frac{d\lambda_f/dX}{(d\lambda_f/dX)\circ(X \underset{f}{+} \gamma)}, \qquad \frac{d\lambda_f}{dX} = \frac{1}{\pi}\left(\frac{d\lambda_{\varphi(f)}}{dX}\circ f\right)\frac{df}{dX}.$$

Therefore, logarithmically differentiating both sides of ($\mathscr{N}$0), we see that

$$\frac{df}{dX}\left(\frac{d(\mathscr{N}_f t)/dX}{\mathscr{N}_f t}\right)\circ f = \sum_{\gamma\in W_f^1} \frac{d(X \underset{f}{+} \gamma)}{dX}\left(\frac{dt/dX}{t}\right)\circ(X \underset{f}{+} \gamma)$$

$$= \frac{1}{\pi}\cdot\frac{df}{dX}\left(\frac{d\lambda_{\varphi(f)}}{dX}\circ f\right) \sum_{\gamma\in W_f^1} (\delta_f t)\circ(X \underset{f}{+} \gamma).$$

This implies that

$$\pi\delta_{\varphi(f)}(\mathscr{N}_f t) = \mathscr{G}_f(\delta_f t).$$

Lemma 1.14 ([Co3, Lemma 13]). Suppose that $f \in \mathfrak{o}'[[X]]$, $N_{k'/k}\,\pi = \xi$, $t \in M_f^n \cap \mathfrak{o}'((X))^{\times}$, and that $s \in F_f(X\mathfrak{o}'[[X]])$. Let $\gamma_i \in \widetilde{W}^i_{\varphi^{-n}(f)}$ $(1 \le i \le n)$, and $\gamma_0 = 0$. Write

$$v = (\mathscr{N}_f t)^{\varphi^{-1}}/t = 1 + \pi f^{\varphi^{n-1}} f^{\varphi^{n-2}} \circ\cdots\circ f(X)w(X)/X, \quad w \in \mathfrak{o}'[[X]].$$

Then

$$\langle t,s\rangle_{f,n} = T_{k'/k}\Big(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \sum_{\gamma\in \tilde{W}^{n}_{\varphi^{-n}(f)}} \lambda_f\circ s(\gamma)(\delta_f t)^{\varphi^{-n}}(\gamma)$$

$$- \sum_{\gamma\in W^{n-1}_{\varphi^{-n}(f)}} \big(\frac{\lambda_f\circ s}{X} w^{\varphi^{-n}}\big)(\gamma)\Big)$$

$$= T_{k_{\xi,n}/k}\Big(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \lambda_f\circ s(\gamma_n)(\delta_f t)^{\varphi^{-n}}(\gamma_n)\Big)$$

$$- \sum_{i=0}^{n-1} T_{k_{\xi,i}/k}\Big(\big(\frac{\lambda_f\circ s}{X} w^{\varphi^{-n}}\big)(\gamma_i)\Big),$$

where $k_{\xi,0} = k'$.

Proof. Since f, t and s have coefficients in $\mathfrak{o}'$, the term corresponding to $\gamma = 0$ in $\langle t,s\rangle^1_{f,n}$ is equal to

$$T_{k'/k}\Big(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\big(\frac{ds}{dX}(0) - \frac{\pi^{\varphi^{-n}}}{\pi}\frac{ds}{dX}(0)^{\varphi}\big)j\Big) = 0,$$

where $j \in \mathbf{Z}$ is such that $t \in X^j\mathfrak{o}'[[X]]^{\times}$. By Lemma 1.13, we have $\mathcal{G}_f(\delta_f t) = \pi\delta_{\varphi(f)}(\mathcal{N}_f t) = \pi(\delta_f t + \delta_f v)^{\varphi}$. Therefore, we see that

$$T_{k'/k}\Big(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \sum_{\gamma\in W^{n}_{\varphi^{-n}(f)}-\{0\}} \frac{1}{\pi}\lambda_f^{\varphi}\circ s^{\varphi}\circ f^{\varphi^{-n}}(\gamma)(\delta_f t)^{\varphi^{-n}}(\gamma)\Big)$$

$$= T_{k'/k}\Big(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \sum_{\gamma'\in W^{n-1}_{\varphi^{-n+1}(f)}-\{0\}} \frac{1}{\pi}\lambda_f^{\varphi}\circ s^{\varphi}(\gamma')(\mathcal{G}_f(\delta_f t))^{\varphi^{-n}}(\gamma')\Big)$$

$$= T_{k'/k}\left(\frac{1}{\pi^{1+\cdots+\varphi^{-n+1}}} \sum_{\gamma' \in W^{n-1}_{\varphi^{-n+1}(f)} - \{0\}} \lambda_f^{\varphi} \circ s^{\varphi}(\gamma')(\delta_f t + \delta_f v)^{\varphi^{-n+1}}(\gamma')\right)$$

$$= T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \sum_{\gamma \in W^{n-1}_{\varphi^{-n}(f)} - \{0\}} \lambda_f \circ s(\gamma)(\delta_f t + \delta_f v)^{\varphi^{-n}}(\gamma)\right).$$

This implies that

$$\langle t,s\rangle^1_{f,n} = T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\left\{ \sum_{\gamma \in \tilde{W}^n_{\varphi^{-n}(f)}} \lambda_f \circ s(\gamma)(\delta_f t)^{\varphi^{-n}}(\gamma) - \sum_{\gamma \in W^{n-1}_{\varphi^{-n}(f)} - \{0\}} \lambda_f \circ s(\gamma)(\delta_f v)^{\varphi^{-n}}(\gamma)\right\}\right).$$

On the other hand, since $(\delta_f v)^{\varphi^{-n}}(\gamma) = \pi^{\varphi^{-1}+\cdots+\varphi^{-n}} w^{\varphi^{-n}}(\gamma)/\gamma$ for $\gamma \in W^{n-1}_{\varphi^{-n}(f)} - \{0\}$, and since

$$\langle t,s\rangle^2_{f,n} = -T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \frac{ds}{dX}(0)\pi^{\varphi^{-1}+\cdots+\varphi^{-n}} w^{\varphi^{-n}}(0)\right)$$

$$= -T_{k'/k}\left(\left(\frac{\lambda_f \circ s}{X} w^{\varphi^{-n}}\right)(0)\right),$$

we finally obtain

$$\langle t,s\rangle_{f,n} = T_{k'/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}} \sum_{\gamma \in \tilde{W}^n_{\varphi^{-n}(f)}} \lambda_f \circ s(\gamma)(\delta_f t)^{\varphi^{-n}}(\gamma) - \sum_{\gamma \in W^{n-1}_{\varphi^{-n}(f)}} \left(\frac{\lambda_f \circ s}{X} w^{\varphi^{-n}}\right)(\gamma)\right)$$

$$= T_{k_{\xi,n}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\lambda_f\circ s(\gamma_n)(\delta_f t)^{\varphi^{-n}}(\gamma_n)\right)$$

$$- \sum_{i=0}^{n-1} T_{k_{\xi,i}/k}\left(\left(\frac{\lambda_f\circ s}{X}w^{\varphi^{-n}}\right)(\gamma_i)\right).$$

Remark 1.15. Let the notation and the assumption be as in Lemma 1.14. Since

$$\sum_{i=0}^{n-1} T_{k_{x,i}/k}\left(\left(\frac{\lambda_f\circ s}{X}w^{\varphi^{-n}}\right)(\gamma_i)\right) \in \mathfrak{o},$$

by Lemma 1.11(8), we put $\beta = s(\gamma_n) \in F_f(\mathfrak{p}_{\xi,n})$ and define

$$\langle t,\beta\rangle_{f,n} = T_{k_{\xi,n}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\lambda_f(\beta)(\delta_f t)^{\varphi^{-n}}(\gamma_n)\right) \in \mathfrak{o}.$$

In particular, if $t \in M_f^{\infty}$, then $w = 0$ and $\langle t,\beta\rangle_{f,n} = \langle t,s\rangle_{f,n}$. In this case, using $\mathcal{G}_f(\delta_f t) = \pi\delta_{\varphi(f)}(\mathcal{N}_f t) = \pi(\delta_f t)^{\varphi}$, we obtain

$$\langle t,\beta\rangle_{f,i} = \langle t,\beta\rangle_{f,n} \quad \text{for all} \quad i \geq n.$$

§1.4. A complete formula for $(\alpha,\beta)_{f,n}$

In this section, following the method of de Shalit [dS2], we shall prove a *complete* formula for the generalized Hilbert symbol $(\alpha,\beta)_{f,n}$ $(\alpha \in k_{\xi,n}^{\times},\ \beta \in F_f(\mathfrak{p}_{\xi,n}))$. Here, as in the previous sections, ξ is a fixed element of k such that $\nu(\xi) = d$, and $f \in \mathfrak{o}'[[X]]$ is a Frobenius power series belonging to a prime element π of k' satisfying $N_{k'/k}\,\pi = \xi$.

Lemma 1.16. (1) If $\alpha = N_{k_{\xi,n+1}/k_{\xi,n}}\alpha'$, $\alpha' \in k_{\xi,n+1}^{\times}$, then

$$f^{\varphi^{-n-1}}((\alpha',\beta)_{f,n+1}) = (\alpha,\beta)_{f,n}.$$

(2) Let π_1 be another prime element of k' such that $N_{k'/k}\pi_1 = \xi$, and let $f_1 \in \mathfrak{o}'[[X]]$ be a Frobenius power series belonging to π_1. Let $\eta \in U_K(\pi,\pi_1) \subset U'$ and put $\theta = [\eta]_{f,f_1} \in \mathrm{Iso}_{\mathfrak{o}'}(F_f, F_{f_1})$. Then

$$\theta^{\varphi^{-n}}((\alpha,\beta)_{f,n}) = (\alpha,\theta(\beta))_{f_1,n}.$$

Proof. The assertions of the lemma easily follow from the definition of the symbol $(\alpha,\beta)_{f,n}$.

First, we consider the case where α is a *universal norm* from the tower $\{k_{\xi,m}\}_{m\geq n}$, i.e., $\alpha \in \bigcap_{m\geq n} N_{k_{\xi,m}/k_{\xi,n}}(k_{\xi,m}^{\times})$. Take an element $a = (\alpha_i)_i \in A_\xi$ such that $\alpha_n = \alpha$. By Lemma 1.16(1), we can define an element $(a,\beta)_f$ $(= ((a,\beta)_{f,i})_i) \in W_f$ by

$$(a,\beta)_{f,i} = (\alpha_i,\beta)_{f,i} \quad \text{for all} \quad i \geq n.$$

On the other hand, take an element $\omega = (\omega_i)_i \in \widetilde{W}_f$ and let $t_{\omega,a} \in M_f^{\infty} \cap \mathfrak{o}'((x))^{\times}$ be the Coleman power series for a. In view of the fact that $\langle t_{\omega,a},\beta\rangle_{f,i} = \langle t_{\omega,a},\beta\rangle_{f,n}$ for all $i \geq n$ (Remark 1.15), we define an element $[a,\beta]_\omega$ $(= ([a,\beta]_{\omega,i})_i) \in W_f$ by

$$[a,\beta]_\omega = [\langle t_{\omega,a},\beta\rangle_{f,n}]_f(\omega),$$

namely,

$$[a,\beta]_{\omega,i} = \left[T_{k_{\xi,i}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-i}}}\lambda_f(\beta)(\delta_f t_{\omega,a})^{\varphi^{-i}}(\omega_i)\right)\right]_{\varphi^{-i}(f)}(\omega_i)$$

for all $i \geq n$. By definition, both symbols $(a,\beta)_f$, $[a,\beta]_\omega$ are linear in a and $\mathfrak{o}$-linear in β.

Lemma 1.17. Let π_1, f_1, η and θ be as in Lemma 1.16(2). Then

$$\begin{cases} \theta((a,\beta)_f) = (a,\theta(\beta))_{f_1}, \\ \theta([a,\beta]_\omega) = [a,\theta(\beta)]_{\theta(\omega)}, \quad \theta(\omega) \in \widetilde{W}_{f_1}. \end{cases}$$

Proof. The first formula follows from Lemma 1.16(2). The second formula follows from Lemma 1.12:

$$\theta([a,\beta]_\omega) = [\langle t_{\omega,a},\beta\rangle_{f,n}]_{f_1}(\theta(\omega))$$

$$=[\langle t_{\theta(\omega),a},\theta(\beta)\rangle_{f_1,n}]_{f_1}(\theta(\omega)) = [a,\theta(\beta)]_{\theta(\omega)}.$$

The following theorem is a generalization of [Iw3, Theorems 8.16 and 8.18].

Theorem 1.18 (Iwasawa-Wiles formula for relative Lubin-Tate groups). Let the notation be as above. Then

$$(a,\beta)_f = [a,\beta]_\omega.$$

In particular,

$$(\alpha,\beta)_{f,n} = \left[T_{k_{\xi,n}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-n}}}\lambda_f(\beta)(\delta_f t_{\omega,a})^{\varphi^{-n}}(\omega_n)\right)\right]_{\varphi^{-n}(f)}(\omega_n).$$

Proof. Since both symbols $(a,\beta)_f$ and $[a,\beta]_\omega$ are linear in a, we may assume that $\nu_i(\alpha_i) = 1$, i.e., α_i is a prime element of $k_{\xi,i}$. Furthermore, by Corollary 1.9, Lemma 1.5(2) and Lemma 1.17, we may assume that $a = \omega \in A_\xi \cap \widetilde{W}_f$, i.e., ω is normed (and f is normed). Since $(\omega,\beta)_f$ and $[\omega,\beta]_\omega$ are elements of W_f, it suffices to show that, for all $i\ (\geq n) \in d\mathbf{Z}$, their i-th components coincide, namely, $(\omega_i,\beta)_{f,i} = [\omega,\beta]_{\omega,i}$. Fix such an i and put $\beta_j = [\xi^{(j-i)/d}]_f(\beta)$ for $j\ (\geq i) \in d\mathbf{Z}$. Then, by Lemma 1.16(1) and by the $\mathfrak{o}$-linearity of $(\ ,\ )_f$, we see that

$$\begin{aligned}(\omega_i,\beta)_{f,i} &= f^{\varphi^{-i-1}}\circ\cdots\circ f^{\varphi^{-j}}((\omega_j,\beta)_{f,j})\\ &= [\xi^{(j-i)/d}]_f((\omega_j,\beta)_{f,j}) = (\omega_j,\beta_j)_{f,j}.\end{aligned}$$

On the other hand, by the definition of $[\ ,\ ]_\omega$, we see that

$$[\omega,\beta]_{\omega,i} = [\xi^{(j-i)/d}]([\omega,\beta]_{\omega,j}) = [\omega,\beta_j]_{\omega,j}.$$

Hence, it suffices to show that

$$(\omega_j,\beta_j)_{f,j} = [\omega,\beta_j]_{\omega,j}\ \left(= \left[\frac{1}{\xi^{j/d}}T_{k_{\xi,j}/k}\left(\frac{\lambda_f(\beta_j)}{((d\lambda_f/dX)(\omega_j)\cdot\omega_j)}\right)\right]_f(\omega_j)\right)$$

for sufficiently large $j(\geq i) \in d\mathbf{Z}$. Let m be a positive integer. In the following, we write $\equiv \pmod{\pi^{mj-M}}$ if there exists an integer M, independent of j, such that the congruence holds for all $j\ (\geq i) \in d\mathbf{Z}$. The constant M may differ in different places, but will be denoted by the same letter. Since

$\beta_j = [\xi^{(j-i)/d}]_f(\beta)$, and since $f(X) \equiv \pi X$ (mod deg 2) and $f(X) \equiv X^q$ (mod π), we have

$$\beta_j \equiv 0 \mod \pi^{j-M}.$$

If we write $\omega_j \underset{f}{+} \beta_j = \omega_j(1 + \varepsilon_j)$, $\varepsilon_j \in k_{\xi,j}$, then it follows that

$$\varepsilon_j \equiv 0 \mod \pi^{j-M}.$$

Since f is normed, we have

$$(\omega_j, \omega_j)_{f,j} = (\omega_j(1 + \varepsilon_j), \omega_j \underset{f}{+} \beta_j)_{f,j} = 0.$$

Hence, by the linearity of the symbol $(\ ,\ )_{f,j}$, we obtain

$$(\omega_j, \beta_j)_{f,j} = \underset{f}{-} (1 + \varepsilon_j, \beta_j)_{f,j} \underset{f}{-} (1 + \varepsilon_j, \omega_j)_{f,j}.$$

Here,

$$(1 + \varepsilon_j, \beta_j)_{f,j} = [\xi^{(j-i)/d}]_f((1 + \varepsilon_j, \beta)_{f,j})$$

$$= (N_{k_{\xi,j}/k_{\xi,i}}(1 + \varepsilon_j), \beta)_{f,i} = 0$$

for sufficiently large $j \in d\mathbf{Z}$, because $\varepsilon_j \equiv 0$ (mod π^{j-M}) implies that, for such j,

$$N_{k_{\xi,j}/k_{\xi,i}}(1 + \varepsilon_j) \in N_{k_{\xi,i}(\rho')/k_{\xi,i}}(k_{\xi,i}(\rho')^{\times}),$$

where $\beta = [\xi^{i/d}]_f(\rho')$, $\rho' \in F_f(\mathfrak{p}_\Omega)$. On the other hand, we see from (1.1) that

$$(1 + \varepsilon_j, \omega_j)_{f,j} = (1 + \varepsilon_j, k_{\xi,2j}/k_{\xi,j})(\omega_{2j}) \underset{f}{-} \omega_{2j}$$

$$= (N_{k_{\xi,j}/k}(1+\varepsilon_j),\ k_{\xi,2j}/k)(\omega_{2j}) -_f \omega_{2j}$$

$$= \left[N_{k_{\xi,j}/k}(1+\varepsilon_j)^{-1} - 1\right]_f(\omega_{2j}).$$

Since the different of $k_{\xi,j}/k$ is $p^{j-1/(q-1)}$, we see that

$$T_{k_{\xi,j}/k}(\varepsilon_j) \equiv 0 \mod \pi^{2j-M},$$

$$T_{k_{\xi,j}/k}(\varepsilon_j^2) \equiv 0 \mod \pi^{3j-M},$$

$$N_{k_{\xi,j}/k}(1+\varepsilon_j) = \prod_{\sigma\in \mathrm{Gal}(k_{\xi,j}/k)} (1+\varepsilon_j)^\sigma$$

$$\equiv 1 + T_{k_{\xi,j}/k}(\varepsilon_j) + \sum_{\sigma\neq\tau} \varepsilon_j^\sigma\cdot\varepsilon_j^\tau \mod \pi^{3j-M},$$

$$= 1 + T_{k_{\xi,j}/k}(\varepsilon_j) + \frac{1}{2}\{T_{k_{\xi,j}/k}(\varepsilon_j)^2 - T_{k_{\xi,j}/k}(\varepsilon_j^2)\}$$

$$\equiv 1 + T_{k_{\xi,j}/k}(\varepsilon_j) \mod \pi^{3j-M}.$$

Therefore, we have

$$N_{k_{\xi,j}/k}(1+\varepsilon_j)^{-1} - 1 \equiv -T_{k_{\xi,j}/k}(\varepsilon_j) \mod \pi^{3j-M}.$$

By Taylor expansion, we see from $\omega_j +_f \beta_j = \omega_j(1+\varepsilon_j)$ that

$$\lambda_f(\omega_j) + \lambda_f(\beta_j) \equiv \lambda_f(\omega_j) + \frac{d\lambda_f}{dX}(\omega_j)\cdot\omega_j\varepsilon_j \mod \pi^{2j-M},$$

namely, we have

$$\varepsilon_j \equiv \frac{\lambda_f(\beta_j)}{(d\lambda_f/dX)(\omega_j)\cdot\omega_j} \mod \pi^{2j-M}.$$

Hence, we obtain the desired formula:

$$(\omega_i, \beta)_{f,i} = (\omega_j, \beta_j)_{f,j} = \bar{f}\, (1 + \varepsilon_j,\ \omega_j)_{\omega,j} = \left[T_{k_{\xi,j}/k}(\varepsilon_j)\right]_f(\omega_{2j})$$

$$= \left[\frac{1}{\xi^{j/d}}\, T_{k_{\xi,j}/k}\left(\frac{\lambda_f(\beta_j)}{(d\lambda_f/dX)(\omega_j)\cdot\omega_j}\right)\right]_f(\omega_j) = [\omega, \beta_j]_{\omega,j} = [\omega, \beta]_{\omega,i}$$

for sufficiently large $j \in d\mathbb{Z}$. This concludes the proof of Theorem 1.18.

Remark 1.19. In the case where p is *odd*, another proof of Theorem 1.18 was obtained by Imada [Im], by using Wiles' method [Wi], i.e., by computing the symbol $(\alpha, \beta)_f$ for the basic Lubin-Tate group associated with the polynomial $f(X) = X^q + \pi X$.

Theorem 1.20. Let $\alpha \in k_{\xi,n}^{\times}$ and $\beta \in F_f(\mathfrak{p}_{\xi,n})$. Take a Coleman power series $t \in M_f^n \cap \mathfrak{o}'((X))^{\times}$ for α, i.e., $t^{\varphi^{-i}}(\omega_i) = N_{k_{\xi,n}/k_{\xi,i}}\alpha$ for all i $(1 \le i \le n)$, and any power series $s \in F_f(X\mathfrak{o}'[[X]])$ such that $\beta = s(\omega_n)$. Then

$$(\alpha, \beta)_{f,n} = [\langle t, s\rangle_{f,n}]_{\varphi^{-n}(f)}(\omega_n).$$

Proof. Since both sides of the equality are linear in α, it suffices to prove the formula in the case where α is a prime element of $k_{\xi,n}$. Put $\pi_1 = N_{k_{\xi,n}/k'}\alpha$, then π_1 is a prime element of k'. Put $\xi_1 = N_{k'/k}\pi_1 \in k$ and $u = \xi_1/\xi \in U$. Then $u \equiv 1 \pmod{\pi^n}$, $k_{\xi_1,i} = k_{\xi,i}$ for all i $(1 \le i \le n)$, and α is a universal norm from the tower $\{k_{\xi_1,m}\}_{m \ge n}$, by local class field theory. Take an element $a = (\alpha_i)_i \in A_{\xi_1}$ such that $\alpha_n = \alpha$. Take a Frobenius power series $f_1 \in \mathfrak{o}'[[X]]$ belonging to π_1. Let $\eta \in U_K(\pi, \pi_1)$, and put $\theta = [\eta]_{f,f_1} \in \mathrm{Iso}_{\mathfrak{o}_K}(F_f, F_{f_1})$ and $\omega' (= (\omega'_i)_i) = \theta(\omega) \in \tilde{W}_{f_1}$. Since $\eta^{\varphi'-1} = N_{k'/k}(\pi_1/\pi) = u$, we have

$$\theta^{\varphi'} = \theta \circ [u]_f = [u]_{f_1} \circ \theta.$$

As in Lemma 1.12, put

$$\begin{cases} t_1 = t \circ \theta^{-1} \in M^n_{f_1}, \\ s_1 = \theta \circ s \circ (\theta^{\varphi^{-n}})^{-1} \in F_{f_1}(X\mathfrak{o}_K[[X]]). \end{cases}$$

Then, t_1 and s_1 do not necessarily have coefficients in $\mathfrak{o}'$. Let $t' \in M^\infty_{f_1} \cap X\mathfrak{o}'[[X]]^\times$ be the Coleman power series for $a = (\alpha_i)_i \in A_{\xi_1}$. On the other hand, by Lemma 1.1, we see that there exists a power series $h \in F_{f_1}(X\mathfrak{o}_K[[X]])$ such that $s_1 = h^{\varphi'} \underset{f_1}{-} h$. Then, from the above, we see that

$$(h^{\varphi'} \underset{f_1}{-} h)^{\varphi'} = s_1^{\varphi'} = [u]_{f_1} \circ s_1 \circ [u]^{-1}_{\varphi^{-n}(f_1)}$$

$$= ([u]_{f_1} \circ h \circ [u]^{-1}_{\varphi^{-n}(f_1)})^{\varphi'} \underset{f_1}{-} [u]_{f_1} \circ h \circ [u]^{-1}_{\varphi^{-n}(f_1)}.$$

This implies that the power series

$$s' = h^{\varphi'} \underset{f_1}{-} [u]_{f_1} \circ h \circ [u]^{-1}_{\varphi^{-n}(f_1)} \quad (\in F_{f_1}(X\mathfrak{o}_K[[X]]))$$

has coefficients in $\mathfrak{o}'$. So, we put

$$\beta' = s'(\omega'_n) \in F_{f_1}(\mathfrak{p}_{\xi,n}).$$

First, we prove the following equality:

$$(1.2) \qquad \theta^{\varphi^{-n}}((\alpha,\beta)_{f,n}) = (\alpha,\beta')_{f_1,n}.$$

Let $\rho \in F_{\varphi^{-n}(f)}(\mathfrak{p}_\Omega)$ be such that $\beta = f^{\varphi^{-1}} \circ \cdots \circ f^{\varphi^{-n}}(\rho)$. Since $u \equiv 1 \pmod{\pi^n}$, we see that

$$\beta' = s'(\omega'_n) = s_1(\omega'_n) \underset{f_1}{+} (h \underset{f_1}{-} [u]_{f_1} \circ h \circ [u]^{-1}_{\varphi^{-n}(f_1)})(\omega'_n)$$

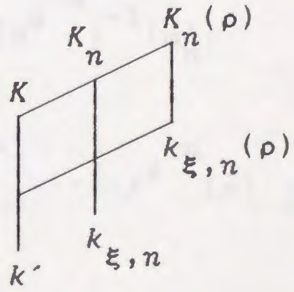
$$= \theta(\beta) \underset{f_1}{+} [1-u]_{f_1} \circ h(\omega'_n)$$

$$= f_1^{\varphi^{-1}} \circ \cdots \circ f_1^{\varphi^{-n}} \circ \theta^{\varphi^{-n}}(\rho)$$

$$\underset{f_1}{+} f_1^{\varphi^{-1}} \circ \cdots \circ f_1^{\varphi^{-n}} \circ \left[\frac{1-u}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}}\right]_{f_1,\varphi^{-n}(f_1)} \circ h(\omega'_n).$$

$$= f_1^{\varphi^{-1}} \circ \cdots \circ f_1^{\varphi^{-n}}(\theta^{\varphi^{-n}}(\rho) \underset{\varphi^{-n}(f_1)}{+} \bar{\rho}),$$

where $\bar{\rho} = \left[(1-u)/\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}\right]_{f_1,\varphi^{-n}(f_1)} \circ h(\omega'_n)$ $\in F_{\varphi^{-n}(f_1)}(\mathfrak{P}_n)$. Put $\sigma = (\alpha, k_{\xi,n}(\rho)/k_{\xi,n})$. Since α is a prime element of $k_{\xi,n}$, we have $\sigma = \varphi'$ on $k_{\xi,n}(\rho) \cap K$. Therefore, we can



extend σ to $K_n(\rho) = K\cdot k_{\xi,n}(\rho)$ so that $\sigma|_K = \varphi'$. Then, the equality (1.2) is equivalent to

$$\theta^{\varphi^{-n}}(\sigma(\rho) \underset{\varphi^{-n}(f)}{-} \rho) = \sigma(\theta^{\varphi^{-n}}(\rho) \underset{\varphi^{-n}(f_1)}{+} \bar{\rho}) \underset{\varphi^{-n}(f_1)}{-} (\theta^{\varphi^{-n}}(\rho) \underset{\varphi^{-n}(f_1)}{+} \bar{\rho}),$$

namely,

$$\theta^{\varphi^{-n}}(\sigma(\rho)) \underset{\varphi^{-n}(f_1)}{-} \sigma(\theta^{\varphi^{-n}}(\rho)) = \sigma(\bar{\rho}) \underset{\varphi^{-n}(f_1)}{-} \bar{\rho}.$$

The left-hand side of this equality is equal to

$$\sigma(\theta^{\varphi^{-n}} \circ [u]^{-1}_{\varphi^{-n}(f)}(\rho) \underset{\varphi^{-n}(f_1)}{-} \theta^{\varphi^{-n}}(\rho))$$

$$= \sigma(\theta^{\varphi^{-n}} \circ [u^{-1}-1]_{\varphi^{-n}(f)}(\rho)) = \theta^{\varphi^{-n}} \circ [1-u]_{\varphi^{-n}(f)}(\rho).$$

Here, we used the fact that

$$[u^{-1}-1]_{\varphi^{-n}(f)}(\rho) = \left[(u^{-1}-1)/\pi^{\varphi^{-1}+\cdots\varphi^{-n}}\right]_{f,\varphi^{-n}(f)}(\beta) \in k_{\xi,n}.$$

On the other hand, since $\sigma|_K = \varphi'$, we see that

$$\sigma(\bar{\rho}) \underset{\varphi^{-n}(f_1)}{-} \bar{\rho} = \left[(1-u)/\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}\right]_{f_1,\varphi^{-n}(f_1)} \circ (h^{\varphi'} \underset{f_1}{-} h)(\omega_n')$$

$$= \left[(1-u)/\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}\right]_{f_1,\varphi^{-n}(f_1)} \circ \theta \circ s \circ (\theta^{\varphi^{-n}})^{-1}(\omega_n')$$

$$= \left[(1-u)/\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}\right]_{f_1,\varphi^{-n}(f_1)} \circ \theta \circ f^{\varphi^{-1}} \circ \cdots \circ f^{\varphi^{-n}}(\rho)$$

$$= \theta^{\varphi^{-n}} \circ [1-u]_{\varphi^{-n}(f)}(\rho).$$

This completes the proof of the equality (1.2). Then, it follows from (1.2) and Theorem 1.18 that

$$(\alpha,\beta)_{f,n} = (\theta^{\varphi^{-n}})^{-1}((\alpha,\beta')_{f_1,n}) = [\langle t',s'\rangle_{f_1,n}]_{\varphi^{-n}(f)}(\omega_n).$$

On the other hand, by Lemma 1.12, we have

$$[\langle t,s\rangle_{f,n}]_{\varphi^{-n}(f)}(\omega_n) = [\langle t_1,s_1\rangle_{f_1,n}]_{\varphi^{-n}(f)}(\omega_n).$$

Hence, it is sufficient to show that

$$\langle t_1,s_1\rangle_{f_1,n} \equiv \langle t',s'\rangle_{f_1,n} \quad \text{mod } \pi^n,$$

or equivalently, that

$$(1.3) \qquad \langle t_1,\ s_1 \bar{-}_{f_1} s'\rangle_{f_1,n} \equiv \langle t'/t_1,\ s'\rangle_{f_1,n} \quad \text{mod } \pi^n.$$

Since

$$s_1 \bar{-}_{f_1} s' = [u]_{f_1} \circ h \circ [u]^{-1}_{\varphi^{-n}(f_1)} \bar{-}_{f_1} h \in X^2 \mathfrak{o}_K[[X]],$$

we have

$$\langle t_1,\ s_1 \bar{-}_{f_1} s'\rangle^2_{f_1,n} = 0.$$

For the same reason, the term corresponding to $\gamma = 0$ in $\langle t_1,\ s_1 \bar{-}_{f_1} s'\rangle^1_{f_1,n}$ is equal to 0. Since $u \equiv 1 \pmod{\pi^n}$, we see that

$$\theta^n_{f_1}(s_1 -_{f_1} s')(\gamma) = (u-1)(\theta^n_{f_1} h)(\gamma), \quad \gamma \in W^n_{\varphi^{-n}(f_1)} - \{0\}.$$

Hence, using $\delta_{f_1} t_1 = X^{-1} + \cdots$ and $(\mathcal{G}2)$, we see that

$$\langle t_1, s_1 -_{f_1} s'\rangle_{f_1,n} = \langle t_1, s_1 -_{f_1} s'\rangle^1_{f_1,n}$$

$$= \sum_{i=0}^{d-1}\left\{\frac{u-1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}} \sum_{\gamma\in W^n_{\varphi^{-n}(f_1)}-\{0\}} \left((\theta^n_{f_1} h)(\delta_{f_1} t_1)^{\varphi^{-n}}\right)(\gamma)\right\}^{\varphi^i}$$

$$\equiv \sum_{i=0}^{d-1}\left\{\frac{u-1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}}\left((\theta^n_{f_1} h)(\delta_{f_1} t_1)^{\varphi^{-n}}\right)(0)\right\}^{\varphi^i} \quad \bmod \pi^n$$

$$= -\sum_{i=0}^{d-1}\left\{\frac{u-1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}}\left(\frac{dh}{dX}(0) - \frac{\varphi^{-n}(\pi_1)}{\pi_1}\cdot\frac{dh}{dX}(0)^{\varphi}\right)\right\}^{\varphi^i}$$

$$= -\frac{u-1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}}\left(\frac{dh}{dX}(0) - \frac{dh}{dX}(0)^{\varphi'}\right)$$

$$= \frac{u-1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}}\cdot\frac{ds_1}{dX}(0).$$

Next, we compute the right-hand side of (1.3). Put $r = t'/t_1 \in \mathfrak{o}_K[[X]]^{\times}$. Then, $\delta_f r \in \mathfrak{o}_K[[X]]$ implies that the term corresponding to $\gamma = 0$ in $\langle r, s'\rangle^1_{f_1,n}$ is equal to 0. Further, since $r^{\varphi^{-i}}(\omega'_i) = 1$ for all i $(1 \le i \le n)$, we can write

$$r(X) = 1 + f_1^{\varphi^{n-1}} \circ\cdots\circ f_1(X) y(X)/X, \quad y \in \mathfrak{o}_K[[X]].$$

Therefore, using $\lambda_{\varphi^n(f)} \circ f_1^{\varphi^{n-1}} \circ \cdots \circ f_1 = \pi_1^{\varphi^{n-1}+\cdots+1} \lambda_{f_1}$, we see that, for $\gamma \in W^n_{\varphi^{-n}(f_1)} - \{0\}$,

$$(\delta_{f_1} r)^{\varphi^{-n}}(\gamma) = \pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}} \cdot \frac{y^{\varphi^{-n}}(\gamma)/\gamma}{(d\lambda_{f_1}/dX)(0)\cdot 1}$$

$$= \pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}} \cdot y^{\varphi^{-n}}(\gamma)/\gamma.$$

Then, using $(\mathcal{G}2)$, we see that

$$\langle r, s' \rangle^1_{f_1,n} = \sum_{i=0}^{d-1} \left\{ \sum_{\gamma \in W^n_{\varphi^{-n}(f_1)} - \{0\}} \left((\theta^n_{f_1} s') y^{\varphi^{-n}} / X \right)(\gamma) \right\}^{\varphi^i}$$

$$\equiv - \sum_{i=0}^{d-1} \left\{ \left((\theta^n_{f_1} s') y^{\varphi^{-n}} / X \right)(0) \right\}^{\varphi^i} \mod \pi^n$$

$$= - \sum_{i=0}^{d-1} \left\{ \left(\frac{ds'}{dX}(0) - \frac{\varphi^{-n}(\pi_1)}{\pi_1} \cdot \frac{ds'}{dX}(0)^{\varphi} \right) y^{\varphi^{-n}}(0) \right\}^{\varphi^i}.$$

On the other hand, by $(\mathcal{N}0)$, we have

$$(\mathcal{N}_{f_1} r)(0) = r(0) = 1 + \pi_1^{\varphi^{n-1}+\cdots+1} \cdot y(0).$$

Therefore, we see that

$$\langle r, s' \rangle^2_{f_1,n} = \sum_{i=0}^{d-1} \left\{ \frac{1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}} \cdot \frac{ds'}{dX}(0) \left(1 - \frac{1 + \pi_1^{\varphi^{-2}+\cdots+\varphi^{-n-1}} y^{\varphi^{-n-1}}(0)}{1 + \pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}} y^{\varphi^{-n}}(0)} \right) \right\}^{\varphi^i}$$

$$\equiv \sum_{i=0}^{d-1}\left\{\frac{ds'}{dX}(0)\left(y^{\varphi^{-n}}(0) - \frac{\varphi^{-n-1}(\pi_1)}{\varphi^{-1}(\pi_1)}y^{\varphi^{-n-1}}(0)\right)\right\}^{\varphi^i} \mod \pi^n.$$

Hence, adding $\langle r,s'\rangle^1_{f_1,n}$ and $\langle r,s'\rangle^2_{f_1,n}$, we obtain

$$\langle r,s'\rangle_{f_1,n} \equiv \frac{\varphi^{-n-1}(\pi_1)}{\varphi^{-1}(\pi_1)}\cdot\frac{ds'}{dX}(0)\left(y^{\varphi^{-n-1+d}}(0) - y^{\varphi^{-n-1}}(0)\right) \mod \pi^n.$$

Here, since $r(0) = (t'/t_1)(0) = \eta\cdot(t'/t)(0)$, $(t'/t)(0) \in k'^{\times}$, we see that

$$y^{\varphi^{-n-1+d}}(0) - y^{\varphi^{-n-1}}(0) = \frac{1}{\pi_1^{\varphi^{-2}+\cdots+\varphi^{-n-1}}}\left(r^{\varphi^{-n-1+d}}(0) - r^{\varphi^{-n-1}}(0)\right)$$

$$= \frac{1}{\pi_1^{\varphi^{-2}+\cdots+\varphi^{-n-1}}}(\eta^{\varphi^d} - 1)\cdot r^{\varphi^{-n-1}}(0)$$

$$\equiv \frac{u-1}{\pi_1^{\varphi^{-2}+\cdots+\varphi^{-n-1}}} \mod \pi^n.$$

Hence, we finally obtain

$$\langle r,s'\rangle_{f_1,n} \equiv \frac{u-1}{\pi_1^{\varphi^{-1}+\cdots+\varphi^{-n}}}\cdot\frac{ds'}{dX}(0) \mod \pi^n.$$

Since

$$\frac{ds'}{dX}(0) = \frac{dh}{dX}(0)^{\varphi'} - \frac{dh}{dX}(0) = \frac{ds_1}{dX}(0),$$

we obtain the desired congruence (1.3):

$$\langle t_1, s_1 \underset{f_1}{-} s' \rangle_{f_1,n} \equiv \langle r, s' \rangle_{f_1,n} \qquad \bmod \pi^n.$$

This concludes the proof of Theorem 1.20.

§1.5. A simpler formula for $(\alpha,\beta)_{f,n}$

In this section, we evaluate the "correction term" of $\langle t,s \rangle_{f,n}$ in the form as in Lemma 1.14 (§1.3), and obtain a simpler formula for $(\alpha,\beta)_{f,n}$ under some restrictions on α and β.

Let ξ, π, f, α, β and ω be as in §1.4. Let $m \geq n$ and suppose that $\alpha = N_{k_{\xi,m}/k_{\xi,n}} \alpha'$, $\alpha' \in k_{\xi,m}^{\times}$. Let $t \in M_f^m \cap \mathfrak{o}'((X))^{\times}$ be a Coleman power series for α'. By Remark 1.15, we have

$$\langle t,\beta \rangle_{f,m} \in \mathfrak{o}.$$

Theorem 1.21. If (a) $m \geq 2n$, or if (b) $m \geq n + \ell$ and $\alpha \in F_f(\mathfrak{p}_{\xi,n}^{q^{n-1-\ell}+1})$ for some ℓ $(0 \leq \ell \leq n - 1)$, then we have

$$(\alpha,\beta)_{f,n} = [\langle t,\beta \rangle_{f,m}]_{\varphi^{-n}(f)}(\omega_n).$$

Proof. Take a power series $s \in F_f(X\mathfrak{o}'[[X]])$ satisfying $\alpha = s(\omega_m)$. By assumption, we may assume that $s \in X^{q^{m-n}}\mathfrak{o}'[[X]]$ (in the case (a)), or that $s \in X^{q^{m-n}(q^{n-1-\ell}+1)}\mathfrak{o}'[[X]]$ (in the case (b)), respectively. By Lemma 1.16(1) and Theorem 1.20, we see that

$$(\alpha,\beta)_{f,n} = f^{\varphi^{-n-1}}\circ\cdots\circ f^{\varphi^{-m}}((\alpha',\beta)_{f,m}) = [\langle t,s\rangle_{f,m}]_{\varphi^{-n}(f)}(\omega_n).$$

Write

$$(N_f t)^{\varphi^{-1}}/t = 1 + \pi^{\varphi^{m-1}} f^{\varphi^{m-2}}\circ\cdots\circ f(X)w(X)/X, \quad w \in \mathfrak{o}'[[X]].$$

Let $\gamma_i \in \tilde{W}^i_{\varphi^{-m}(f)}$, $1 \le i < m$ and $\gamma_0 = 0$. Then, by Lemma 1.14 and Remark 1.15, we have

$$\langle t,s\rangle_{f,m} = \langle t,\beta\rangle_{f,m} - \sum_{i=0}^{m-1} T_{k_{\xi,i}/k}\left((\frac{\lambda_f\circ s}{X}w^{\varphi^{-m}})(\gamma_i)\right).$$

We evaluate the second term of the right hand side of this equality. Since $s \in X^2\mathfrak{o}'[[X]]$ in both cases, we have

$$T_{k'/k}\left((\frac{\lambda_f\circ s}{X}w^{\varphi^{-m}})(0)\right) = 0.$$

Using Lemma 1.11(4) and the inequality $(q^j - 1)/(q - 1) \ge j$ for all $j \in \mathbb{Z}$, we evaluate the other terms. Let $i \in \{1,\cdots,m - 1\}$. In the case (a), we see that

$$\begin{aligned}\nu\left((\frac{\lambda_f\circ s}{X}w^{\varphi^{-m}})(\gamma_i)\right) &\ge \min_{j\ge0} \nu\left(\gamma_i^{q^{m-n}\cdot q^j}/\pi^j\right) - \nu(\gamma_i)\\ &= \min_{j\ge0} \{q^{m-n+j}/q^{i-1}(q-1) - j\} - 1/q^{i-1}(q-1)\\ &\ge n - (i - \frac{1}{q-1}).\end{aligned}$$

Similarly, in the case (b), we see that

$$\nu\left((\frac{\lambda_f\circ s}{X}w^{\varphi^{-m}})(\gamma_i)\right) \ge \min_{j\ge0} \nu\left(\gamma_i^{q^{m-n}(q^{n-1-\ell}+1)\cdot q^j}/\pi^j\right) - \nu(\gamma_i)$$

$$= \min_{j\geq 0} \{q^{m-n+j}(q^{n-1-\ell}+1)/q^{i-1}(q-1) - j\} - 1/q^{i-1}(q-1)$$

$$\geq n - (i - \frac{1}{q-1}).$$

Since the different of $k_{\xi,i}/k$ is $\mathfrak{p}^{i-1/(q-1)}$, we obtain

$$T_{k_{\xi,i}/k}\left((\frac{\lambda_f \circ S}{X} w^{\varphi^{-m}})(\gamma_i)\right) \equiv 0 \mod \pi^n$$

in both cases. This concludes the proof of Theorem 1.21.

Next, take *any* power series $t' \in \mathfrak{o}'((X))^\times$ such that $t'^{\varphi^{-m}}(\omega_m) = \alpha'$, and put

$$\langle t', \beta\rangle_{f,m} = T_{k_{\xi,m}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-m}}} \lambda_f(\beta)(\delta_f t')^{\varphi^{-m}}(\omega_m)\right) \in k.$$

Theorem 1.22. If (a) $m \geq 2n$, or if (b) $m \geq n + \ell$ and $\beta \in F_f(\mathfrak{p}_{\xi,n}^{2q^{n-1-\ell}})$ for some ℓ $(0 \leq \ell \leq n-1)$, then

$$\langle t', \beta\rangle_{f,m} \in \mathfrak{o}; \quad (\alpha, \beta)_{f,n} = [\langle t', \beta\rangle_{f,m}]_{\varphi^{-n}(f)}(\omega_n).$$

Proof. Write $t' = tz$ with $z \in \mathfrak{o}'[[X]]^\times$. Then, by Theorem 1.21, we only need to show that

$$\langle z, \beta\rangle_{f,m} = T_{k_{\xi,m}/k}\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-m}}} \lambda_f(\beta)(\delta_f z)^{\varphi^{-m}}(\omega_m)\right) \equiv 0 \mod \pi^n.$$

Since $z^{\varphi^{-m}}(\omega_m) = 1$, we can write

$$z = 1 + (f^{\varphi^{m-1}} \circ \cdots \circ f / f^{\varphi^{m-2}} \circ \cdots \circ f) z', \quad z' \in \mathfrak{o}'[[X]].$$

Therefore, using $\lambda_{\varphi^m(f)} \circ f^{\varphi^{m-1}} \circ \cdots \circ f = \pi^{\varphi^{m-1}+\cdots+1} \lambda_f$, we see that

$$(\delta_f z)^{\varphi^{-m}}(\omega_m) = \frac{1}{(d\lambda_f/dX)^{\varphi^{-m}}(\omega_m)} \cdot \left(\frac{dz/dX}{z}\right)^{\varphi^{-m}}(\omega_m)$$

$$= \pi^{\varphi^{-1}+\cdots+\varphi^{-m}} \cdot \frac{z'^{\varphi^{-m}}(\omega_m)/\omega_1}{(d\lambda_f/dX)(0)\cdot 1}.$$

In the case (a), since $\nu(\beta) \geq 1/(q-1)q^{n-1}$, we see from Lemma 1.11(7) that

$$\nu\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-m}}} \lambda_f(\beta)(\delta_f z)^{\varphi^{-m}}(\omega_m)\right) \geq \nu(\lambda_f(\beta)/\omega_1)$$

$$\geq \frac{q}{q-1} - n - \frac{1}{q-1} \geq n - \left(m - \frac{1}{q-1}\right).$$

Similarly, in the case (b), since $\nu(\beta) \geq 2q^{n-1-\ell}/(q-1)q^{n-1} = 2/(q-1)q^{\ell}$, we see that

$$\nu\left(\frac{1}{\pi^{\varphi^{-1}+\cdots+\varphi^{-m}}} \lambda_f(\beta)(\delta_f z)^{\varphi^{-m}}(\omega_m)\right) \geq \nu(\lambda_f(\beta)/\omega_1)$$

$$\geq \frac{2}{q-1} - \ell - \frac{1}{q-1} \geq n - \left(m - \frac{1}{q-1}\right).$$

Since the different of $k_{\xi,m}/k$ is $\mathfrak{p}^{m-1/(q-1)}$, we obtain the desired congruence in both cases and conclude the proof of Theorem 1.22.

Remark 1.23. The conditions in Theorem 1.22 are refinements of those of [Iw1, Theorems 1 and 2], [Ku, Theorems 1′ and 2′] and [W, Theorems 1 and 23], where (a′) $\mathfrak{m} \geq 2n$, or (b′) $\mathfrak{m} = n$ and $\beta \in F_f(\mathfrak{p}_{\xi,n}^{2q^{n-1}})$ (and $n = 1$ if $F_f \neq \mathbb{G}_m$) were assumed.

Chapter 2

Takagi's formulas by Lubin-Tate groups

Let p be an odd prime number and $Q_p(\xi_p)$ the prime cyclotomic field. Let $\sqrt[p-1]{-p}$ be a prime element of $Q_p(\xi_p)$ such that $\sqrt[p-1]{-p} \equiv \xi_p - 1 \pmod{(\xi_p - 1)^2}$. In [Ta1], Takagi constructed principal units $\kappa_1, \cdots, \kappa_p \in Q_p(\xi_p)^\times$ (called a *Takagi basis*) for which $\{\sqrt[p-1]{-p}, \kappa_1, \cdots, \kappa_p\}$ represents a basis of $Q_p(\xi_p)^\times / Q_p(\xi_p)^{\times p}$ and the following explicit formulas

a complementary law: $$(\sqrt[p-1]{-p}, \kappa_i)_p = \begin{cases} \xi_p & (i = p), \\ 1 & (\text{otherwise}), \end{cases}$$

a general law: $$(\kappa_j, \kappa_i)_p = \begin{cases} \xi_p^{-j} & (j + i = p), \\ 1 & (\text{otherwise}) \end{cases}$$

hold.

Using the Iwasawa-Wiles formula [Iw1, Iw3, Wi] and the Coleman-de Shalit formula [Co2, Co3, dS2], Shiratani [Sh4] generalized these formulas to the field generated by the prime division points of a Lubin-Tate group F. In this chapter, we shall extend Shiratani's formula to the field generated by the π^n-division points of F by using Vostokov's formula [Vo3, Vo4]. We shall also obtain a generalization of the characrterization of the Takagi basis.

§2.1. Generalized Takagi basis

In this chapter, we deal with ordinary Lubin-Tate groups. So, let the notation be as in §1.1, and fix a prime element π of k and a Frobenius power series $f \in \mathfrak{o}[[X]]$ belonging to π. In this section, we do not assume that p is odd. We write $F = F_f$, $X \underset{F}{+} Y = X \underset{f}{+} Y$, $[c]_F = [c]_f$, $\lambda_F = \lambda_f$, $k_n = k_{\pi,n}$, $\mathfrak{p}_n = \mathfrak{p}_{\pi,n}$, $W_F = W_f$ and $(\ ,\ )_n^F = (\ ,\ )_{f,n}$. Let e be the ramification index of $k/\mathbf{Q}_p$ and let $\mathfrak{f}$ be such that $q = p^{\mathfrak{f}}$. Let $\mathfrak{R} = \{\theta \in \mathfrak{o}^{\times} \mid \theta^{q-1} = 1\}$ be the multiplicative representative set of $\mathfrak{o}/\mathfrak{p}$ and put $V = \{\eta \in \mathbf{Z}_p^{\times} \mid \eta^{p-1} = 1\} \subset \mathfrak{R}$.

By the generalization [Sh4, Lemma1] of the Dieudonné-Dwork lemma [Di], the power series

$$\lambda_A = \sum_{\ell=0}^{\infty} \frac{X^{q^{\ell}}}{\pi^{\ell}} \in Xk[[X]]$$

is the logarithm map of a Lubin-Tate group F_A belonging to π. We call λ_A the generalized Artin-Hasse logarithm associated with π. Similarly,

$$\lambda_a = \sum_{m=0}^{\infty} \frac{X^{p^m}}{p^m} \in X\mathbf{Q}_p[[X]]$$

is the logarithm map of a Lubin-Tate group F_a belonging to p, which is known as the (ordinary) Artin-Hasse logarithm. Using these power series, we define two power series [Vo3, §2; Sh4, §2; Sa, §1]:

$$\begin{cases} E_F(X) = \lambda_F^{-1}\circ\lambda_A \in X\mathfrak{o}[[X]], \\ E(X) = 1 + E_{\mathbb{G}_m}(X) = \exp\circ\lambda_a \in 1 + X\mathbb{Z}_p[[X]]. \end{cases}$$

Let F_0 denote the basic Lubin-Tate group associated with the polynomial $[\pi]_{F_0}(X) = X^q + \pi X$. Let $\omega = (\omega_i)_i \in \tilde{W}_F$ and put $(u_i)_i = \left((\lambda_{F_0}^{-1}\circ\lambda_F)(\omega_i)\right)_i \in \tilde{W}_{F_0}$. We define two sets:

$$R_1 = \{E(\theta u_n^j) \mid \theta \in \mathfrak{R},\ 1 \le j < \frac{pe(q-1)q^{n-1}}{p-1}\ (p \nmid j) \text{ or } j = \frac{pe(q-1)q^{n-1}}{p-1}\},$$

$$R_2 = \{E_F(u_n^i) \mid 1 \le i < q^n\ (q \nmid i)\} \cup \{\kappa_F\},$$

where $\kappa_F = \lambda_F^{-1}\left(\sum_{\ell=0}^{\infty} \frac{1}{\pi^\ell}[\pi^n]_{F_0}(u_n^{q^\ell})\right)$ is a π^n-primary element of $F(\mathfrak{p}_n)$ defined in [Vo3, Proposition 1] (i.e., the $[\pi^n]_F$-division points of κ_F generate an unramified extension over k_n). Then R_1 is a set of $\mathbb{Z}_p$-generators of the principal units $1 + \mathfrak{p}_n$ of $k_n^\times$ [Sa, §1] and the set R_2 represents an $\mathfrak{o}/(\pi^n)$-basis of the formal module $F(\mathfrak{p}_n)/[\pi^n]_F(F(\mathfrak{p}_n))$ [Vo3, Proposition 2 and its Remark].

For $\eta \in \mathfrak{R}$ let σ_η be an element of $\mathrm{Gal}(k_n/k)$ such that $\sigma_\eta(\omega_n) = [\eta]_F(\omega_n)$. Put $H = \{\sigma_\eta \mid \eta \in \mathfrak{R}\}$ $(\subset \mathrm{Gal}(k_n/k)) \simeq \mathrm{Gal}(k_1/k)$ and define

$$1_i = \frac{1}{q-1}\sum_{\eta\in\mathfrak{R}} \eta^{-i}\sigma_\eta \in \mathfrak{o}[H] \quad (1 \le i \le q-1),$$

then

$$1 = \sum_{i=1}^{q-1} 1_i, \quad 1_i \cdot 1_j = \delta_{ij}\cdot 1_i \quad (\delta_{ij}\text{: Kronecker's delta}),$$

and 1_i $(1 \le i \le q-1)$ are the primitive orthogonal idempotents of the group ring $\mathfrak{o}[H]$. As an $\mathfrak{o}[H]$-module, $F(\mathfrak{p}_n)$ has a direct sum decomposition

$$F(\mathfrak{p}_n) = A^{(1)} \oplus \cdots \oplus A^{(q-1)},$$

where

$$A^{(i)} = [1_i]_F(\mathfrak{p}_n) = \{\beta \in F(\mathfrak{p}_n) \mid \sigma_\eta(\beta) = [\eta^i]_F(\beta) \quad (\eta \in \mathfrak{R})\}.$$

Theorem 2.1. For $1 \le i \le q-1$ we have

$$A^{(i)} = \begin{cases} \langle E_F(u_n^j) \in R_2 \mid j \equiv i \bmod (q-1)\rangle & \text{if } i \ne 1, \\ \langle E_F(u_n^j) \in R_2 \mid j \equiv 1 \bmod (q-1)\rangle \oplus \langle \kappa_F \rangle & \text{if } i = 1. \end{cases}$$

Proof. Let $\eta \in \mathfrak{R}$. It is well-known [Wi, Lemma 20] that $[\eta]_{F_0}(X) = \eta X$. On the other hand, since $\lambda_a \circ [\eta]_{F_a}(X) = \eta\lambda_a(X) = \lambda_a(\eta X)$, we have $[\eta]_{F_a}(X) = \eta X$. Therefore we see that

$$\sigma_\eta(u_n) = (\lambda_{F_0}^{-1} \circ \lambda_F)([\eta]_F(\omega_n)) = [\eta]_{F_0}(u_n) = \eta u_n,$$

$$\sigma_\eta(E_F(u_n^j)) = E_F(\eta^j u_n^j) = [\eta^j]_F(E_F(u_n^j)),$$

$$\sigma_\eta(\kappa_F) = \lambda_F^{-1}\left(\sum_{\ell=0}^{\infty} \frac{1}{\pi^\ell}[\pi^n]_{F_0}(\eta^{q^\ell} u_n^{q^\ell})\right)$$

$$= \lambda_F^{-1}\left(\eta \sum_{\ell=0}^{\infty} \frac{1}{\pi^\ell}[\pi^n]_{F_0}(u_n^{q^\ell})\right) = [\eta]_F(\kappa_F).$$

Hence we obtain the assertion of the theorem.

Remark 2.2. Let $E(\theta u_n^j) \in R_1$. Since E is a $\mathbf{Z}_p$-isomorphism, we see that, for any $\eta \in V$,

$$\sigma_\eta(E(\theta u_n^j)) = E(\theta \eta^j u_n^j) = E(\theta u_n^j)^{\eta^j}.$$

Corollary 2.3. Let $E(\theta u_n^j) \in R_1$ and $E_F(u_n^i) \in R_2$. Then

$$\begin{cases} (E(\theta u_n^j),\ E_F(u_n^i))_n^F = 0 & \text{if} \quad j + i \not\equiv 1 \mod (p-1), \\ (E(\theta u_n^j),\ \kappa_F)_n^F = 0 & \text{if} \quad j \not\equiv 0 \mod (p-1). \end{cases}$$

Proof. For any $\eta \in V$ we see that

$$\begin{aligned} [\eta]_F\Big((E(\theta u_n^j),\ E_F(u_n^i))_n^F\Big) &= \sigma_\eta\Big((E(\theta u_n^j),\ E_F(u_n^i))_n^F\Big) \\ &= (E(\theta \eta^j u_n^j),\ E_F(\eta^i u_n^i))_n^F \\ &= (E(\theta u_n^j)^{\eta^j},\ [\eta^i]_F(E_F(u_n^i)))_n^F \\ &= [\eta^{j+i}]_F\Big((E(\theta u_n^j),\ E_F(u_n^i))_n^F\Big). \end{aligned}$$

From this, we obtain the first formula. The second formula can be proved in the same way.

In particular, let $k = \mathbf{Q}_p$, $\pi = p$ and $f(X) = (1 + X)^p - 1$, then $F = \mathbf{G}_m = X + Y + XY$ and $[c]_{\mathbf{G}_m}(X) = (1 + X)^c - 1$ $(c \in \mathbf{Z}_p)$. Put $\zeta_{(i)} = \omega_i - 1$, then $\zeta_{(i)}$ is a primitive p^i-th root of unity satisfying $\zeta_{(i+1)}^p = \zeta_{(i)}$. Furthermore, $[p]_{F_0}(X) = X^p + pX$

and $u_1 = \sqrt[p-1]{-p} \equiv \zeta_{(1)} - 1 \pmod{(\zeta_{(1)} - 1)^2}$. The p^n-th Hilbert symbol $(\alpha,\beta)_{(n)}$ ($\alpha, \beta \in k_n^{\times} = \mathbf{Q}_p(\zeta_{p^n})^{\times}$) defined by

$$(\alpha,\beta)_{(n)} = \sqrt[p^n]{\beta}^{(\alpha,\, \mathbf{Q}_p(\zeta_{(n)}, \sqrt[p^n]{\beta})/\mathbf{Q}_p(\zeta_{(n)})) - 1} \in \langle \zeta_{(n)} \rangle$$

satisfies

$$(\alpha,\beta)_{(n)} = 1 + (\alpha, \beta - 1)_n^{\mathbf{G}_{\mathfrak{m}}} \quad (\alpha \in \mathbf{Q}_p(\zeta_{(n)})^{\times},\ \beta \in U_n),$$

where $U_n = 1 + \mathfrak{p}_n$ denotes the set of principal units of $\mathbf{Q}_p(\zeta_{p^n})$. We put

$$\varepsilon_i = \begin{cases} E(u_n^i) & (1 \le i < p^n,\ p \nmid i), \\ \exp\left(\sum_{\ell=0}^{\infty} \frac{1}{p^{\ell}} [p^n]_{F_0} (u_n^{p^{\ell}}) \right) & (i = p^n), \end{cases}$$

Let $R = \{\varepsilon_1, \cdots, \varepsilon_{p^n}\}$. Then the set $\{u_n\} \cup R$ represents a $\mathbf{Z}_p/(p^n)$-basis of $\mathbf{Q}_p(\zeta_{(n)})^{\times}/\mathbf{Q}_p(\zeta_{(n)})^{\times p^n}$. Put $H' = \{\sigma_\eta \mid \eta \in V\}$ ($\subset \mathrm{Gal}(\mathbf{Q}_p(\zeta_{(n)})/\mathbf{Q}_p)) \simeq \mathrm{Gal}(\mathbf{Q}_p(\zeta_{(1)})/\mathbf{Q}_p)$. As a $\mathbf{Z}_p[H']$-module, U_n has a direct product decomposition

$$U_n = A'^{(1)} \times \cdots \times A'^{(p-1)},$$

where

$$A'^{(i)} = U_n^{1'_i} = \{\varepsilon \in U_n \mid \varepsilon^{\sigma_\eta} = \varepsilon^{\eta^i} \text{ for all } \eta \in V\},$$

$$1'_i = \frac{1}{p-1} \sum_{\eta \in V} \eta^{-i} \sigma_\eta \in \mathbf{Z}_p[H'] \quad (1 \le i \le p-1).$$

Corollary 2.4. For $1 \leq i \leq p-1$ we have

$$A'^{(i)} = \langle \varepsilon_j \in R \mid j \equiv i \mod (p-1) \rangle.$$

Let $\varepsilon_j, \varepsilon_i \in R$, then we have

$$(\varepsilon_j, \varepsilon_i)_{(n)} = 1 \quad \text{if} \quad j + i \not\equiv 1 \mod (p-1).$$

Remark 2.5. In the case where p is *odd* and $n = 1$, the ordinary Takagi basis $\{\kappa_1, \cdots, \kappa_p\}$ is uniquely determined modulo U_1^p by the following conditions [Ta, §1]:

$$\begin{cases} \kappa_i^{\sigma_\eta} \equiv \kappa_i^{\eta^i} & \mod U_1^p \quad (i = 1, \cdots, p; \quad \eta \in V), \\ \kappa_i \equiv 1 - (1 - \zeta_{(1)})^i & \mod (1 - \zeta_{(1)})^{i+1} \quad (i = 1, \cdots, p), \\ \kappa_1 \equiv \zeta_{(1)} & \mod U_1^p. \end{cases}$$

The following characterization was obtained in [SI, Theorem 3.1].

$$\kappa_i \equiv \varepsilon_i^{(-1)^{i-1}} \mod U_1^p \quad (i = 1, \cdots, p).$$

Therefore, Theorem 2.1, Remark 2.2 and Corollary 2.4 imply that the sets R_2, R_1 and R generalize the Takagi basis $\{\kappa_1, \cdots, \kappa_p\}$.

§2.2. Known formulas

In the following, we assume that p is *odd*. In this section, we describe known formulas for $(\alpha, \beta)_n^F$ $(\alpha \in \{u_n\} \cup R_1,\ \beta \in R_2)$. By [Vo3, Proposition 1], we have

$$(2.1)\qquad \begin{cases} (u_n, \kappa_F)_n^F = (\lambda_F^{-1}\circ\lambda_{F_0})\left((u_n, \kappa_{F_0})_n^{F_0}\right) = \omega_n, \\ (E(\theta u_n^j), \kappa_F)_n^F = 0 \qquad \text{for} \quad E(\theta u_n^j) \in R_1. \end{cases}$$

Shiratani's formulas [Sh4, Theorems 1 and 2] read as follows.

A complementary law: If $q > 2n$, then

$$(2.2)\qquad (u_n, E_F(u_n^i))_n^F = \begin{cases} 0 & (1 \le i < q^n), \\ \omega_n & (i = q^n). \end{cases}$$

A general law for $n = 1$: If $j \ge 1$ and $i \ge 1$, then

$$(2.3)\qquad (E(u_1^j), E_F(u_1^i))_1^F = \begin{cases} [j]_F(\omega_1) & (i + p^m j = q,\ m \in \mathbb{Z}), \\ 0 & (\text{otherwise}). \end{cases}$$

If $F = \mathbb{G}_m$ and $n = 1$, Shiratani's formulas give Takagi's formulas. In §2.3, we shall describe general laws for $n \ge 2$.

§2.3. General laws for $n \ge 2$

We still assume that p is *odd*. To compute $(E(\theta u_n^j), E_F(u_n^i))_n^F$ for $n \ge 2$, we use Vostokov's formula. Since

$$(E(\theta u_n^j), E_F(u_n^i))_n^F = (\lambda_F^{-1}\circ\lambda_{F_0})\left((E(\theta u_n^j), E_{F_0}(u_n^i))_n^{F_0}\right),$$

it is sufficient to compute the symbol in the case where $F = F_0$.

For $\alpha \in k_n^\times$ and $\beta \in F_0(\mathfrak{p}_n)$ we write

$$\begin{cases} \alpha = u_n^a \cdot \eta \cdot \varepsilon(u_n), & a \in \mathbb{Z},\ \eta \in \Re,\ \varepsilon(X) \in 1 + X\mathfrak{o}_T[[X]], \\ \beta = B(u_n), & B(X) \in X\mathfrak{o}[[X]], \end{cases}$$

where $\mathfrak{o}_T$ is the integer ring of the inertia subfield T of $k/\mathbb{Q}_p$. Put $A(X) = X^a \cdot \eta \cdot \varepsilon(X)$. Then *Vostokov's formula* [Vo4, Theorem4] is:

$$(2.4) \qquad (\alpha, \beta)_n^{F_0} = \left[\operatorname{res}_X \Phi/[\pi^n]_{F_0}\right]_{F_0}(u_n),$$

where

$$\Phi(X) = -\frac{1}{\pi}\left(\log \varepsilon(X) - \frac{1}{q} \log \varepsilon(X^q)\right)\frac{d}{dX}\left(\lambda_{F_0} \circ B(X^q)\right)$$

$$+ \left(\lambda_{F_0} \circ B(X) - \frac{1}{\pi} \lambda_{F_0} \circ B(X^q)\right) A^{-1} \frac{dA}{dX} \in \mathfrak{o}[[X]],$$

$$\Phi/[\pi^n]_{F_0} \in \mathfrak{o}\{X\} = \{ \sum_{i=-\infty}^{\infty} a_i X^i \mid a_i \in \mathfrak{o},\ a_i \longrightarrow 0 (i \longrightarrow -\infty)\}$$

and $\operatorname{res}_X \left(\sum_{i=-\infty}^{\infty} a_i X^i\right) = a_{-1}$. We know from [Vo3, §1] that $\Psi(X) = \sum_{i=-\infty}^{\infty} a_i X^i \in \mathfrak{o}\{X\}$ is invertible in $\mathfrak{o}\{X\}$ if and only if there exists an i for which a_i is a unit in $\mathfrak{o}$. Furthermore, if i' is the least integer such that $a_{i'}$ is a unit in $\mathfrak{o}$ and $\Psi(X) = a_{i'} X^{i'} (1 + \psi(X))$, then $\frac{1}{\Psi} = a_{i'}^{-1} X^{-i'} (1 - \psi + \psi^2 - \cdots)$. In particular, $[\pi^n]_{F_0}(X) = X^{q^n} + \cdots + \pi^n X$ is invertible in $\mathfrak{o}\{X\}$.

Using (2.4), we can remove the assumption $q \geq 2n$ in Shiratani's formula (2.2).

Proposition 2.6. We have

$$(u_n, E_F(u_n^i))_n^F = \begin{cases} 0 & (1 \leq i < q^n), \\ \omega_n & (i = q^n). \end{cases}$$

Proof. Let $\alpha = u_n$ and $\beta = E_{F_0}(u_n^i)$, then we can take

$$\begin{cases} A(X) = X, \quad \eta = 1, \quad \varepsilon(X) = 1, \\ B(X) = E_{F_0}(X^i) = \lambda_{F_0}^{-1}\left(\sum_{\ell=0}^{\infty} \dfrac{X^{q^\ell i}}{\pi^\ell}\right) \in X\mathfrak{o}[[X]]. \end{cases}$$

Therefore, by (2.4), we have

$$\begin{aligned} (u_n, E_F(u_n^i))_n^F &= (\lambda_F^{-1} \circ \lambda_{F_0})\left((u_n, E_{F_0}(u_n^i))_n^{F_0}\right) \\ &= \left[\mathrm{res}_X\, \Phi/[\pi^n]_{F_0}\right]_F(\omega_n), \end{aligned}$$

where

$$\Phi(X) = \frac{1}{X}\left(\sum_{\ell=0}^{\infty} \frac{X^{q^\ell i}}{\pi^\ell} - \frac{1}{\pi}\sum_{\ell=0}^{\infty} \frac{X^{q^{\ell+1} i}}{\pi^\ell}\right) = X^{i-1}.$$

If we write

$$[\pi^n]_{F_0}(X) = X^{q^n} + \cdots + \pi^n X = X^{q^n}(1 + \psi), \quad \psi \in \mathfrak{o}\{X\},$$

then ψ has only the terms of negative degree. Hence, we see that

$$\Phi/[\pi^n]_{F_0}(X) = X^{i-1-q^n}(1 - \psi + \psi^2 - \cdots),$$

$$\mathrm{res}_X\, \Phi/[\pi^n]_{F_0} = \begin{cases} 0 & (1 \le i < q^n), \\ 1 & (i = q^n). \end{cases}$$

This concludes the proof.

Next, for $\alpha = E(\theta u_n^j)$, $\beta = E_{F_0}(u_n^i)$ we can take

$$\begin{cases} A(X) = \varepsilon(X) = E(\theta X^j) = \exp\left(\sum_{m=0}^{\infty} \theta^{p^m} \cdot \frac{X^{p^m j}}{p^m}\right) \in 1 + X\mathfrak{o}_T[[X]], \\ B(X) = E_{F_0}(X^i) = \lambda_{F_0}^{-1}\left(\sum_{\ell=0}^{\infty} \frac{X^{q^\ell i}}{\pi^\ell}\right) \in X\mathfrak{o}[[X]]. \end{cases}$$

Therefore it follows that

$$\Phi(X) = -\frac{1}{\pi}\left(\sum_{m=0}^{\infty} \theta^{p^m} \cdot \frac{X^{p^m j}}{p^m} - \frac{1}{q}\sum_{m=0}^{\infty} \theta^{p^m} \cdot \frac{X^{qp^m j}}{p^m}\right) \cdot \frac{d}{dX}\left(\sum_{\ell=0}^{\infty} \frac{X^{q^{\ell+1} i}}{\pi^\ell}\right)$$

$$= \left(\sum_{\ell=0}^{\infty} \frac{X^{q^\ell i}}{\pi^\ell} - \sum_{\ell=0}^{\infty} \frac{X^{q^{\ell+1} i}}{\pi^{\ell+1}}\right) \cdot \sum_{m=0}^{\infty} j\theta^{p^m} X^{p^m j}$$

$$= -\sum_{m=0}^{f-1} \sum_{\ell=0}^{\infty} \frac{q^{\ell+1} i\theta^{p^m}}{p^m \pi^{\ell+1}} X^{p^m j + q^{\ell+1} i - 1} + \sum_{m=0}^{\infty} j\theta^{p^m} X^{i+p^m j-1}.$$

Hence we obtain the following

Theorem 2.7. For $E(\theta u_n^j) \in R_1$ and $\beta = E_{F_0}(u_n^i) \in R_2$ we have

$$(E(\theta u_n^j),\ E_F(u_n^i))_n^F = \left[\operatorname{res}_X \Phi/[\pi^n]_{F_0}\right]_F(\omega_n),$$

where

$$\Phi(X) = -\sum_{m=0}^{f-1} \sum_{\ell=0}^{\infty} \frac{q^{\ell+1} i\theta^{p^m}}{p^m \pi^{\ell+1}} X^{p^m j + q^{\ell+1} i - 1} + \sum_{m=0}^{\infty} j\theta^{p^m} X^{i+p^m j-1}.$$

Computing $1/[\pi^n]_{F_0}$ for $n = 2, 3$, we see that

$$[\pi^2]_{F_0}(X) = (X^q + \pi X)^q + \pi(X^q + \pi X) \equiv X^{q^2} + \pi X^q \mod \pi^2,$$

$$1/[\pi^2]_{F_0} \equiv X^{-q^2}(1 - \pi X^{-q^2+q}) \mod \pi^2,$$

$$[\pi^3]_{F_0}(X) \equiv (X^{q^2} + \pi X^q)^q + \pi(X^{q^2} + \pi X^q)$$

$$\equiv X^{q^3} + q\pi X^{q^3-q^2+q} + \pi X^{q^2} + \pi^2 X^q \mod \pi^3,$$

$$1/[\pi^3]_{F_0} \equiv X^{-q^3}(1 - q\pi X^{-q^2+q} - \pi X^{-q^3+q^2} - \pi^2 X^{-q^3+q} + \pi^2 X^{-2q^3+2q^2}) \pmod{\pi^3}.$$

Therefore, by Theorem 2.7, we obtain the following

Proposition 2.8. Let $E(\theta u_n^j) \in R_1$ and $E_F(u_n^i) \in R_2$. Then we have

$$(E(\theta u_2^j), E_F(u_2^i))_2^F = \begin{cases} [-p\theta]_F(\omega_2) & (e = f = 1,\ j = p^2,\ i = p-1), \\ [j\theta^{p^m}]_F(\omega_2) & (i + p^m j = q^2,\ m \in \mathbb{Z}), \\ [-j\theta^{p^m}]_F(\omega_1) & (i + p^m j = 2q^2 - q,\ m \in \mathbb{Z}), \\ 0 & (\text{otherwise}). \end{cases}$$

$$(E(\theta u_3^j), E_F(u_3^i))_3^F$$

$$= \begin{cases} [-p^2\theta + \frac{p^2(p-1)}{\pi}\theta]_F(\omega_3) & (e = f = 1,\ j = p^3,\ i = p-1), \\ [-p\theta]_F(\omega_2) & (e = f = 1,\ j = p^3,\ i = p^2 - 1), \\ [2p^2\theta]_F(\omega_3) & (e = f = 1,\ j = p^3,\ i = 2p-2), \\ [j\theta^{p^m}]_F(\omega_3) & (i + p^m j = q^3,\ m \in \mathbb{Z}), \end{cases}$$

$$
\begin{cases}
[-jp\theta]_F(\omega_2) & (e = f = 1,\ i + j = p^3 + p^2 - p), \\
[-j\theta^{p^m}]_F(\omega_2) & (i + p^m j = 2q^3 - q^2,\ \ m \in \mathbb{Z}), \\
[-j\theta^{p^m}]_F(\omega_1) & (i + p^m j = 2q^3 - q,\ \ m \in \mathbb{Z}), \\
[j\theta^{p^m}]_F(\omega_1) & (i + p^m j = 3q^3 - 2q^2,\ \ m \in \mathbb{Z}), \\
0 & (\text{otherwise}).
\end{cases}
$$

Corollary 2.9. Let $1 \le j,\ i < p^n,\ \ p \nmid j,\ \ p \nmid i$. Then we have

$$
(\varepsilon_j, \varepsilon_i)_{(2)} = \begin{cases}
\xi_{(2)}^{j} & (j + i = p^2), \\
\xi_{(1)}^{-j} & (j + i = 2p^2 - p), \\
1 & (\text{otherwise}),
\end{cases}
$$

$$
(\varepsilon_j, \varepsilon_i)_{(3)} = \begin{cases}
\xi_{(3)}^{j} & (j + i = p^3), \\
\xi_{(1)}^{-j} & (j + i = p^3 + p^2 - p), \\
\xi_{(2)}^{-j} & (j + i = 2p^3 - p^2), \\
\xi_{(1)}^{-j} & (j + i = 2p^3 - p), \\
1 & (\text{otherwise}).
\end{cases}
$$

$$
(\varepsilon_j, \varepsilon_i)_{(4)} = \begin{cases}
\xi_{(4)}^{j} & (j + i = p^4), \\
\xi_{(1)}^{-j} & (j + i = p^4 + p^2 - p), \\
\xi_{(2)}^{-j} & (j + i = p^4 + p^3 - p^2), \\
\xi_{(1)}^{-j} & (j + i = p^4 + p^3 - p), \\
\xi_{(1)}^{-\frac{p-1}{2}j} & (j + i = p^4 + 2p^3 - 2p^2),
\end{cases}
$$

$$
\begin{cases}
\xi_{(3)}^{-j} & (j+i=2p^4-p^3), \\
\xi_{(1)}^{-j} & (j+i=2p^4-p^3+p^2-p), \\
\xi_{(2)}^{-j}\cdot\xi_{(1)}^{2j} & (j+i=2p^4-p^2), \\
\xi_{(1)}^{-j} & (j+i=2p^4-p), \\
1 & (\text{otherwise}).
\end{cases}
$$

Proof. It follows from Theorem 2.7 that

$$(\varepsilon_j, \varepsilon_i)_n = \xi_{(n)}^{\mathrm{res}_X \varphi}, \qquad \varphi = \Phi/[p^n]_{F_0},$$

$$\Phi(X) = -\sum_{\ell=0}^{\infty} iX^{j+p^{\ell+1}i-1} + \sum_{m=0}^{\infty} jX^{i+p^m j-1}.$$

The formulas for $n = 2, 3$ follow from Proposition 2.8. For $n = 4$, we see that

$$
\begin{aligned}
[p^4]_{F_0} &\equiv (X^{p^3}+p^2X^{p^3-p^2+p}+pX^{p^2}+p^2X^p)^p+p(X^{p^3}+p^2X^{p^3-p^2+p}+pX^{p^2}+p^2X^p) \\
&\equiv X^{p^4}(1+p^3X^{-p^2+p}+p^2X^{-p^3+p^2}+p^3X^{-p^3+p}+\frac{p(p-1)}{2}p^2X^{-2p^3+2p^2} \\
&\qquad +pX^{-p^4+p^3}+p^3X^{-p^4+p^3-p^2+p}+p^2X^{-p^4+p^2}+p^3X^{-p^4+p}) \mod p^4, \\
1/[p^4]_{F_0} &\equiv X^{-p^4}(1-p^3X^{-p^2+p}-p^2X^{-p^3+p^2}-p^3X^{-p^3+p}-\frac{p(p-1)}{2}p^2X^{-2p^3+2p^2} \\
&\qquad -pX^{-p^4+p^3}-p^3X^{-p^4+p^3-p^2+p}-p^2X^{-p^4+p^2}-p^3X^{-p^4+p}+2p^3X^{-p^4+p^2} \\
&\qquad +p^2X^{-2p^4+2p^3}+2p^3X^{-2p^4+p^3+p^2}-p^3X^{-3p^4+3p^3}) \mod p^4.
\end{aligned}
$$

From this we obtain the formula for $n = 4$.

For general n, we have the following

Theorem 2.10. Let $E(\theta u_n^j) \in R_1$ and $E_F(u_n^i) \in R_2$, then

$$\begin{cases} (E(\theta u_n^j),\ E_F(u_n^i))_n^F = [j\theta^{p^m}]_F(\omega_n) & (i + p^m j = q^n,\ m \in \mathbb{Z}), \\ [\pi^{n-1}]_F\left((E(\theta u_n^j),\ E_F(u_n^i))_n^F\right) = 0 & (\text{otherwise}). \end{cases}$$

Furthermore we have

$$(E(\theta u_n^j),\ E_F(u_n^i))_n^F = 0,$$

if one of the following conditions holds:

(a) $p \nmid j$ and $q \nmid (i + p^m j)$ for all m $(0 \le m \le f - 1)$,

(b) $i + p^m j \not\equiv 1 \mod (q - 1)$ for all m $(0 \le m \le f - 1)$,

(c) $i + p^{f-1} j < q^n$.

Proof. Let $[\pi^n]_{F_0}(X) = \sum_{s=1}^{q^n} a_s^{(n)} X^s$ $(a_{q^n}^{(n)} = 1)$. By induction on n, we prove $a_s^{(n)} \equiv 0 \pmod{\pi^n}$ for $s \not\equiv q \mod q(q-1)$. If $n = 1$, this is obvious. Let $n \ge 1$ and assume that $a_s^{(n)} \equiv 0 \pmod{\pi^n}$ for $s \not\equiv q \mod q(q-1)$. Then

$$[\pi^n]_{F_0}(X) \equiv \sum_{\substack{1 \le s \le q^n \\ s \equiv q \bmod q(q-1)}} a_s^{(n)} X^s \mod \pi^n,$$

and we have

$$[\pi^{n+1}]_{F_0}(X) \equiv \Big(\sum_{\substack{1 \le s \le q^n \\ s \equiv q \bmod q(q-1)}} a_s^{(n)} X^s\Big)^q + \pi\Big(\sum_{\substack{1 \le s \le q^n \\ s \equiv q \bmod q(q-1)}} a_s^{(n)} X^s\Big) \mod \pi^{n+1}.$$

This implies that $a_s^{(n+1)} \equiv 0 \pmod{\pi^{n+1}}$ for $s \not\equiv q \mod q(q-1)$. This completes the induction. Therefore, if we write

$$1/[\pi^n]_{F_0} = X^{-q^n} \sum_{s=0}^{\infty} b_s X^{-s} \qquad (b_0 = 1),$$

then $b_s \equiv 0 \pmod{\pi^n}$ for $s \not\equiv 0 \mod q(q-1)$. Hence, following the notation of Theorem 2.7, we obtain

$$(2.5)\ \operatorname{res}_X \Phi/[\pi^n]_{F_0} \equiv \sum_{\substack{s\geq 0\\ q(q-1)|s}} \left(- \sum_{\substack{p^{m}j+q^{\ell+1}i=q^n+s\\ 0\leq m\leq f-1\\ \ell\geq 0}} \frac{q^{\ell+1} i\theta^{p^{m}}}{p^{m}\pi^{\ell+1}} + \sum_{\substack{i+p^{m}j=q^n+s\\ m\geq 0}} j\theta^{p^{m}}\right) b_s \pmod{\pi^n}.$$

First, we assume that $i + p^{m'}j = q^n$ for some $m' \geq 0$. If $p|j$, then $j = \dfrac{pe(q-1)q^{n-1}}{p-1} = eq^n \dfrac{1-1/q}{1-1/p} \geq q^n$ and therefore $q^n = i + p^{m'}j > q^n$, which is a contradiction. Hence $p \nmid j$. Then we have $q \nmid (p^{m}j + q^{\ell+1}i)$ for $0 \leq m \leq f-1$ and $\ell \geq 0$, so there is no term in the first sum of the right-hand side of (2.5). On the other hand, if $i + p^{m}j = q^n + s$ for some $m \geq 0$ and $s \geq 1$ such that $q(q-1)|s$, then we have $i + p^{m}j = i + p^{m'}j + s$, $m > m'$ and $p^{m'}(p^{m-m'} - 1)j = s$. Since $p\nmid j$ and $q|s$, this implies that $m' \geq f$ and $q|i$, which contradicts the assumption. Hence, there remains only the the term $j\theta^{p^{m'}}$ (corresponding to $s = 0$) in the second sum of the right-hand side of (2.5). Therefore, by Theorem 2.7, we obtain

$$(E(\theta u_n^j),\ E_F(u_n^i))_n^F = [j\theta^{p^{m'}}]_F(\omega_n).$$

Next, we assume that $i + p^{\mathfrak{m}} j \neq q^n$ for all $\mathfrak{m} \geq 0$. Since $[\pi^n]_{F_0}(X) \equiv X^{q^n} \pmod{\pi}$, we have $1/[\pi^n]_{F_0} \equiv X^{-q^n} \pmod{\pi}$. Hence, by Theorem 2.7, we have

$$(2.6) \quad \mathrm{res}_X\ \Phi/[\pi^n]_{F_0} \equiv - \sum_{\substack{p^{\mathfrak{m}} j + q^{\ell+1} i = q^n \\ 0 \leq \mathfrak{m} \leq \mathfrak{f}-1 \\ \ell \geq 0}} \frac{q^{\ell+1} i \theta^{p^{\mathfrak{m}}}}{p^{\mathfrak{m}} \pi^{\ell+1}} + \sum_{\substack{i + p^{\mathfrak{m}} j = q^n \\ \mathfrak{m} \geq 0}} j \theta^{p^{\mathfrak{m}}} \mod \pi.$$

By assumption, there is no term in the second sum of the right-hand side of (2.6). On the other hand, if $p^{\mathfrak{m}} j + q^{\ell+1} i = q^n$ for some $\mathfrak{m}$ $(0 \leq \mathfrak{m} \leq \mathfrak{f} - 1)$ and $\ell \geq 0$, then $p | j$ and $j = \dfrac{pe(q-1)q^{n-1}}{p-1} \geq q^n$, which is a contradiction. Therefore there is no term in the first sum of the right-hand side of (2.6). Hence, we obtain

$$[\pi^{n-1}]_F\left((E(\theta u_n^j),\ E_F(u_n^i))_n^F\right) = 0.$$

Finally, if we assume one of the conditions (a), (b), or (c), then it is easy to see that there is no term in the right-hand side of (2.5). This completes the proof of Theorem 2.10.

Corollary 2.11. Let $\varepsilon_j, \varepsilon_i \in R$, then we have

$$(\varepsilon_j, \varepsilon_i)_{(n)} = \begin{cases} \zeta_{(n)}^j & \text{if } j + i = p^n, \\ 1 & \begin{cases} \text{if } j + i \not\equiv p \mod p(p-1), \\ \text{or if } j + i < p^n. \end{cases} \end{cases}$$

This corollary is a generalization of Takagi's general law.

On the other hand, a generalization of Takagi's complementary law can be obtained from (2.1) and Proposition 2.6:

$$(u_n, \varepsilon_i)_{(n)} = \begin{cases} \zeta_{(n)} & (i = p^n), \\ 1 & (\text{otherwise}). \end{cases}$$

Chapter 3

Takagi's formulas in 2-adic number fields

Let $k/\mathbf{Q}_2$ be a finite extension. In this chapter, we shall obtain some explicit formulas for the generalized Hilbert norm residue symbol on the field generated by π^n-division points of a Lubin-Tate formal group defined over k.

Our formulas are 2-adic versions of Shiratani's formulas [Sh4] mentioned in §2.2. These formulas give explicit reciprocity laws for certain generators of the multiplicative group and of the formal module, and consist of *complementary laws* and *general laws*.

In §3.1, we shall obtain a complementary law which is slightly different from the odd p case. In §3.2, we give a general law which has the same shape as the odd p case.

§3.1. A complementary law

Let the notation be as in §2.1 and assume that $p = 2$. In this section, we prove the following

Theorem 3.1. Under the assumption $q \geq 2n$ we have

$$(-u_n, E_F(u_n^i))_n^F = \begin{cases} 0 & (1 \leq i < q^n), \\ \omega_n & (i = q^n). \end{cases}$$

Proof. We compute $(-u_n, E_F(u_n^i))_n^F$ in the same way as in the proof of [Sh4, Theorem 1]. Since $(-u_i)_i$ $(\in \widetilde{W}_f)$ is normed (§1.2, Remark 1.10), we see from the Iwasawa-Wiles formula [Iw1, Iw3, Wi; §1.4, Theorem 1.18] that

$$(3.1)\qquad (-u_n, E_F(u_n^i))_n^F = (\lambda_F^{-1}\circ\lambda_{F_0})\Big((-u_n, E_{F_0}(u_n^i))_n^{F_0}\Big)$$

$$= \left[\frac{1}{\pi^n}\sum_{\ell=0}^{\infty}\frac{1}{\pi^\ell}T_{k_n/k}\left(\frac{u_n^{iq^\ell-1}}{\lambda_{F_0}'(u_n)}\right)\right]_F(\omega_n).$$

From [Sh4, Lemma 4] and the discussion after that lemma, we have

$$(3.2)\qquad \frac{1}{\pi^\ell}T_{k_n/k}\left(\frac{u_n^{iq^\ell-1}}{\lambda_{F_0}'(u_n)}\right)$$

$$= \begin{cases} -\displaystyle\sum_{\substack{s_m\ge r_m\ge 0\ (1\le m\le n-1)\\ (q-1)(s_m+1)+r_m=r_{m-1}}} (-\pi)^{s_1+\cdots+s_{n-1}-(r_1+\cdots+r_{n-1})+\frac{r_{n-1}}{q-1}}\binom{s_1}{r_1}\cdots\binom{s_{n-1}}{r_{n-1}}\pi^{n-1-\ell} & (iq^\ell \ge q^{n-1},\ i\equiv 1 \mod (q-1)),\\ 0 & (\text{otherwise})\end{cases}$$

$$= \begin{cases} \displaystyle\sum_{\substack{0\le j_m\le \frac{1}{q}(j_{m-1}-1)\\ (1\le m\le n-1)}} (-1)^{j_0-(q-1)(j_1+\cdots+j_{n-1})-n}\binom{s_1}{r_1}\cdots\binom{s_{n-1}}{r_{n-1}}\pi^{j_0-(q-1)(j_1+\cdots+j_{n-1})-\ell} & (iq^\ell \ge q^{n-1},\ i\equiv 1 \mod (q-1)),\\ 0 & (\text{otherwise}),\end{cases}$$

where $r_0 = iq^\ell - 1$, $j_m = \dfrac{r_m}{q-1}$ $(\ge 0) \in \mathbf{Z}$ $(0 \le m \le n-1)$ and $s_m = j_{m-1} - j_m - 1$ $(1 \le m \le n-1)$. In particular, from (3.1) and

(3.2) we obtain

$$(-u_n\ ,\ E_F(u_n^i))_n^F = 0 \qquad \text{if} \quad i \not\equiv 1 \mod (q-1).$$

In the sequel, we assume that $iq^{\ell} \geq q^{n-1}$ and $i \equiv 1 \mod (q-1)$. For simplicity, we write

$$A = A(\ell; j_1, \cdots, j_{n-1}) = (-1)^{j_0-(q-1)(j_1+\cdots+j_{n-1})-n}\binom{s_1}{r_1}\cdots\binom{s_{n-1}}{r_{n-1}},$$

$$B = B(\ell; j_1, \cdots, j_{n-1}) = j_0 - (q-1)(j_1 + \cdots + j_{n-1}) - \ell,$$

in the sum of the right-hand side of (3.2). We shall compute $A\pi^B \pmod{\pi^{2n}}$ under the assumption $q \geq 2n$. Let $t\ (0 \leq t \leq n) \in \mathbb{Z}$ be such that $q^t \leq i < q^{t+1}$. It follows from $iq^{\ell} \geq q^{n-1}$ that $q^{t+1+\ell} > q^{n-1}$, so that $\ell \geq n-1-t$. Since

$$j_0 = \frac{iq^{\ell}-1}{q-1},\ j_1 \leq \frac{1}{q}(j_0 - 1) = \frac{iq^{\ell-1}-1}{q-1}, \cdots,$$

$$j_{n-1} \leq \frac{1}{q}(j_{n-2} - 1) \leq \frac{iq^{\ell-n+1}-1}{q-1},$$

we have

$$B \geq \frac{iq^{\ell}-1}{q-1} - (q-1)\left(\frac{iq^{\ell-1}-1}{q-1} + \cdots + \frac{iq^{\ell-n+1}-1}{q-1}\right) - \ell$$

$$= \frac{iq^{\ell-n+1}-1}{q-1} + (n-1) - \ell$$

$$\geq \frac{q^{t+\ell-n+1}-1}{q-1} + (n-1-\ell) \geq t \geq 0.$$

Here, the equalities hold if and only if $t = 0$, $\ell = n$ or $n-1$, and $j_1 = \dfrac{q^{\ell-1}-1}{q-1}, \cdots, j_{n-1} = \dfrac{q^{\ell-n+1}-1}{q-1}$. In these cases, we have

$s_1 = r_1, \cdots, s_{n-1} = r_{n-1}$ and therefore

$$A = (-1)^{B+\ell-n} = (-1)^{\ell-n} = \begin{cases} 1 & (\ell = n), \\ -1 & (\ell = n-1). \end{cases}$$

Hence the corresponding terms cancel out. If $t = 0$, the next minimal value of B is $q - 1$ ($\geq 2n - 1$) and it occurs when

$$\ell = n+1,\ j_1 = \frac{q^n - 1}{q - 1}, \cdots, j_{n-1} = \frac{q^2 - 1}{q - 1}$$
$$(s_1 = r_1, \cdots, s_{n-1} = r_{n-1}),$$

or when

$$\ell = n,\ j_1 = \frac{q^{n-1} - 1}{q - 1}, \cdots, j_{n-2} = \frac{q^2 - 1}{q - 1},\ j_{n-1} = 0$$
$$(s_1 = r_1, \cdots, s_{n-2} = r_{n-2}, r_{n-1} = 0).$$

In these cases, we have

$$A = (-1)^{B+\ell-n} = (-1)^{(q-1)+(\ell-n)} = \begin{cases} 1 & (\ell = n+1), \\ -1 & (\ell = n). \end{cases}$$

So, the corresponding terms cancel out again. If $t = 0$, there is no other term (mod π^{2n}) in the right-hand side of (3.2), and this settles the case where $t = 0$. Next, we treat the case where $t \geq 1$. In this case, we only need to consider the terms corresponding to the minimal value of B where ℓ and $j_1, \cdots, j_{n-1}$ run over the possible values under the above conditions, because otherwise, we have

$$B \geq \begin{cases} (q-1) + t \geq q \geq 2n & \left(\begin{array}{l}\text{if at least one of } j_1, \cdots, j_{n-1} \\ \text{does not take the maximal value}\end{array}\right), \\ \dfrac{q^2-1}{q-1} + t - 2 \geq q \geq 2n & (\text{if } \ell \geq n - t + 1). \end{cases}$$

The minimal value of B occurs when

$$\ell = n - t,\ j_1 = \frac{iq^{n-t-1} - 1}{q-1}, \cdots,\ j_{n-t} = \frac{i-1}{q-1},$$

$$j_{n-t+1} = \left[\frac{1}{q}\left(\frac{i-1}{q-1} - 1\right)\right] = \left[\frac{i-q^t}{q(q-1)}\right] + \frac{q^{t-1}-1}{q-1}, \cdots,$$

$$j_{n-1} = \left[\frac{i-q^t}{q^{t-1}(q-1)}\right] + \frac{q-1}{q-1}$$

$$\left(\begin{array}{l} s_1 = r_1, \cdots, s_{n-t} = r_{n-t},\ s_{n-t+1} = j_{n-t} - j_{n-t+1} - 1, \\ r_{n-t+1} = (q-1)j_{n-t+1}, \cdots \end{array}\right),$$

or when

$$\ell = n - t - 1\ (\geq 0),\ j_1 = \frac{iq^{n-t-2} - 1}{q-1}, \cdots,\ j_{n-t-1} = \frac{i-1}{q-1},$$

$$j_{n-t} = \left[\frac{i-q^t}{q(q-1)}\right] + \frac{q^{t-1}-1}{q-1}, \cdots,$$

$$j_{n-2} = \left[\frac{i-q^t}{q^{t-1}(q-1)}\right] + \frac{q-1}{q-1},\ j_{n-1} = 0$$

$$\left(\begin{array}{l} s_1 = r_1, \cdots, s_{n-t-1} = r_{n-t-1},\ s_{n-t} = j_{n-t-1} - j_{n-t} - 1, \\ r_{n-t} = (q-1)j_{n-t}, \cdots \end{array}\right).$$

Here, $[x]$ means the integral part of a rational number $x \in \mathbf{Q}$. In the above cases, we have

$$B = t + s_q\left(\frac{i-q^t}{q-1}\right),$$

$$A = \begin{cases} (-1)^{s_q\left(\frac{i - q^t}{q - 1}\right)} \begin{pmatrix} \frac{i - q^{t-1}}{q - 1} - \left[\frac{i - q^t}{q(q - 1)}\right] - 1 \\ (q - 1)\left(\left[\frac{i - q^t}{q(q - 1)}\right] + \frac{q^{t-1} - 1}{q - 1}\right) \end{pmatrix} \cdots & (\ell = n - t), \\ (-1)^{s_q\left(\frac{i - q^t}{q - 1}\right)+1} \begin{pmatrix} \frac{i - q^{t-1}}{q - 1} - \left[\frac{i - q^t}{q(q - 1)}\right] - 1 \\ (q - 1)\left(\left[\frac{i - q^t}{q(q - 1)}\right] + \frac{q^{t-1} - 1}{q - 1}\right) \end{pmatrix} \cdots & (\ell = n - t - 1 \geq 0), \end{cases}$$

where $s_q(x) = a_0 + a_1 + \cdots + a_m$ for $x = a_0 + a_1 q + \cdots + a_m q^m \in \mathbb{N} \cup \{0\}$, $0 \leq a_i \leq q - 1$ ($a_i \in \mathbb{Z}$). Therefore, if $1 \leq t \leq n - 1$, the corresponding terms cancel out. If $t = n$, then $i = q^n$, $B = n$, and there remains only the term corresponding to $\ell = 0$, $j_1 = \frac{q^{n-1} - 1}{q - 1}, \ldots, j_{n-1} = \frac{q - 1}{q - 1}$ ($s_1 = r_1, \ldots, s_{n-1} = r_{n-1}$). In this case, we have $A = 1$. Hence, from (3.1), (3.2) and the computation above, we obtain

$$(-u_n, E_F(u_n^i))_n^F = \begin{cases} 0 & (1 \leq i < q^n), \\ \omega_n & (i = q^n). \end{cases}$$

This concludes the proof of Theorem 3.1.

§3.2. A general law

In this section, we prove the following

Theorem 3.2. For $\theta \in \mathfrak{R}$ and $j,\ i \geq 1$ we have

$$(E(\theta u_1^j),\ E_F(u_1^i))_1^F = \begin{cases} [j\theta^{2^m}]_F(\omega_1) & (i + 2^m j = q,\quad m \in \mathbb{Z}), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Since the proof is almost the same as the proof of [Sh4, Theorem 2], we omit the details. First, we compute $(1 - \theta u_1^j\ ,\ E_F(u_1^i))_1^F$. Put $h(X) = E_{F_0}(X^i)$. Then, by the Coleman-de Shalit formula [Co2, Co3, dS2; §1.4, Theorem 1.20], we have

$$(1 - \theta u_1^j\ ,\ E_F(u_1^i))_1^F = (\lambda_F^{-1} \circ \lambda_{F_0})\left((1 - \theta u_1^j\ ,\ E_{F_0}(u_1^i))_1^{F_0}\right)$$

$$= \left[\frac{1}{\pi}\left\{T_{k_1/k}\left(\sum_{\ell=0}^{\infty} \frac{u_1^{q^\ell i}}{\pi^\ell} \cdot \frac{1}{\lambda'_{F_0}(u_1)} \cdot \frac{-j\theta u_1^{j-1}}{1 - \theta u_1^j}\right) + h'(0)\left(1 - N_{k_1/k}(1 - \theta u_1^j)\right)\right\}\right]_F(\omega_1).$$

We have

$$\frac{1}{\pi}\ h'(0)\left(1 - N_{k_1/k}(1 - \theta u_1^j)\right) \equiv \begin{cases} -j\theta^{\frac{q-1}{j}} \bmod \pi & (i = 1,\ j \mid (q - 1)), \\ 0 \bmod \pi & (\text{otherwise}), \end{cases}$$

$$\frac{1}{\pi}\ T_{k_1/k}\left(\sum_{\ell=0}^{\infty} \frac{u_1^{q^\ell i}}{\pi^\ell} \cdot \frac{1}{\lambda'_{F_0}(u_1)} \cdot \frac{-j\theta u_1^{j-1}}{1 - \theta u_1^j}\right)$$

$$= \frac{j\theta}{\pi(q - 1)}\ T_{k_1/k}\left(\sum_{\ell=0}^{\infty} \sum_{a=0}^{\infty} \frac{\theta^a}{\pi^\ell}\ u_1^{q^\ell i + (a+1)j - 1}\right),$$

where

$$\frac{j\theta}{\pi(q-1)} T_{k_1/k}\left(\frac{\theta^a}{\pi^{\ell}} u_1^{q^{\ell} i+(a+1)j-1}\right) = \begin{cases} j\theta^{\frac{q-1}{j}} & (\ell=1,\ i=1,\ (a+1)j = q-1), \\ -j\theta^{\frac{q-i}{j}} & (\ell=0,\ i + (a+1)j = q), \\ c\pi \text{ for some } c \in \mathfrak{o} & (\text{otherwise}). \end{cases}$$

Hence, we obtain

$$(1 - \theta u_1^j,\ E_F(u_1^i))_1^F = \begin{cases} [-j\theta^{\frac{q-i}{j}}]_F(\omega_1) & \\ & (i+(a+1)j = q \text{ for some } a \geq 0), \\ 0 & (\text{otherwise}). \end{cases}$$

Using

$$E(\theta u_1^j) = \prod_{(b,2)=1} (1 - \theta^b u_1^{bj})^{-\frac{\mu(b)}{b}},$$

we see that

$$(E(\theta u_1^j),\ E_F(u_1^i))_1^F = \Sigma_{(F)\,(b,2)=1} \left[-\frac{\mu(b)}{b}\right]_F \left((1 - \theta^b u_1^{bj},\ E_F(u_1^i))_1^F\right)$$

$$= \begin{cases} \displaystyle\Sigma_{(F)\,(b,2)=1,\ bj\mid(q-i)} \left[-\frac{\mu(b)}{b}(-bj)\theta^{b\cdot\frac{q-i}{bj}}\right]_F(\omega_1) & (i + (a+1)j = q \text{ for some } a \geq 0), \\ 0 & (\text{otherwise}) \end{cases}$$

$$= \begin{cases} \left[j\theta^{\frac{q-i}{j}} \sum_{\substack{(b,2)=1 \\ bj|(q-i)}} \mu(b) \right]_F (\omega_1) & (i + (a+1)j = q \text{ for some } a \ge 0), \\ 0 & (\text{otherwise}) \end{cases}$$

$$= \begin{cases} [j\theta^{2^m}]_F(\omega_1) & (i + 2^m j = q \text{ for some } m \ge 0), \\ 0 & (\text{otherwise}), \end{cases}$$

where $\sum_{(F)}$ means the sum in the formal module $F(p_1)$. This concludes the proof of Theorem 3.2.

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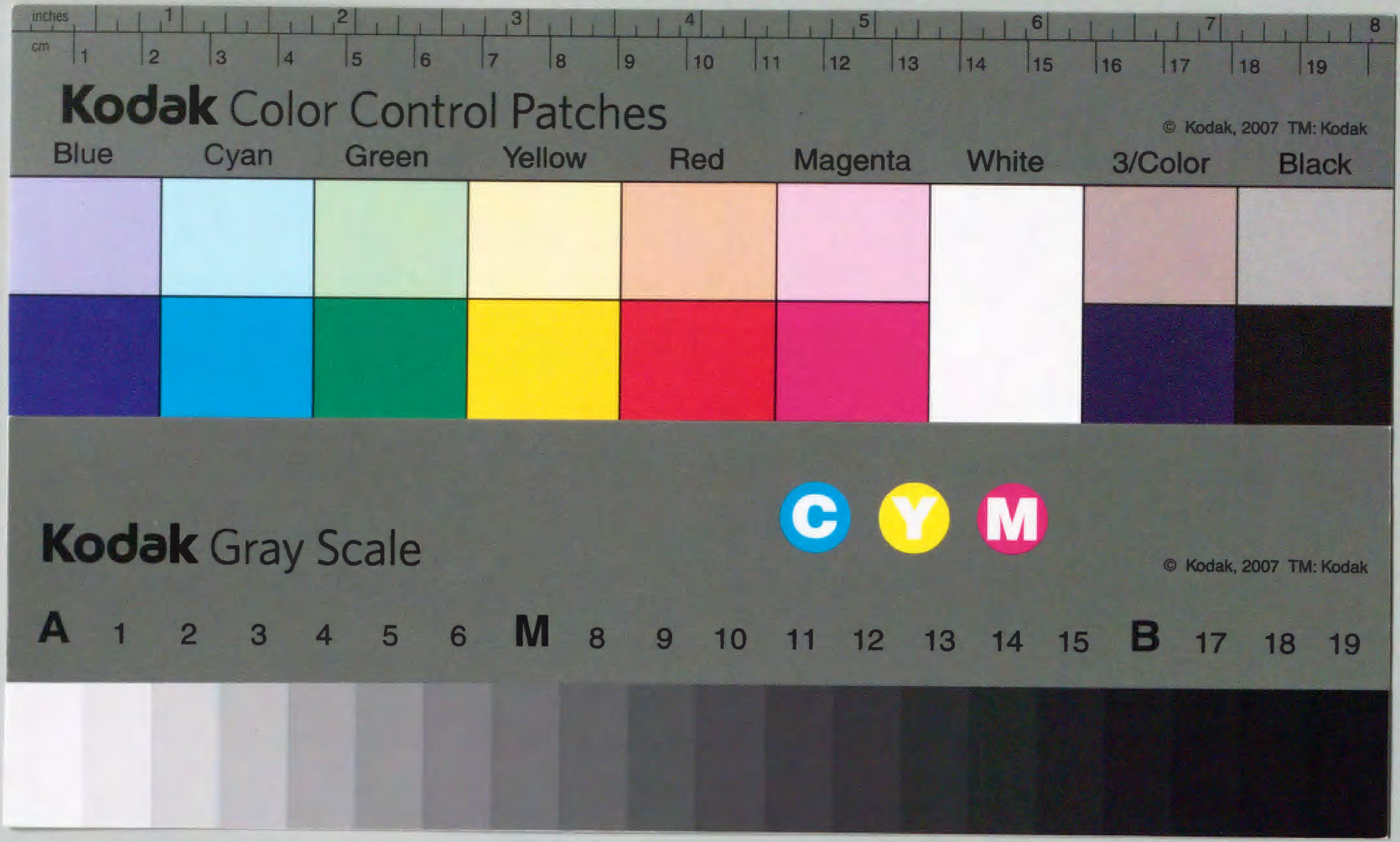
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