Studies on the Levi Problem and the Indicator of Entire Functions in Infinite Dimensional Spaces

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https://doi.org/10.11501/3070071

出版情報：九州大学, 1993, 博士（理学）, 論文博士
バージョン：
権利関係：

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Studies on the Levi Problem and the Indicator of Entire Functions in Infinite Dimesional Spaces

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Preface

Let E be a complex locally convex Hausdorrf space and f be an entire function of exponential type. Then the *indicator* I_f of the entire function f is the function on E with values in $[-\infty, \infty)$ defined by

$$I_f(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |f(tz')|$$

for every $z \in E$. The indicator has the following properties.

(1) I_f is plurisubharmonic.

(2) I_f is positively homogeneous of order 1, that is, $I_f(tz) = tI_f(z)$ for every $t > 0$ and $z \in E$

Conversely when given a plurisubharmonic function p on E which is positively homogeneous of order 1, we consider the problem to ask whether there exists an entire function f of exponential type on E with $I_f = p$. Kiselman([18] in Part II), Lelong([19] in Part II) and Martineau([22] in Part II) solved affirmatively this problem in the case of $E = \boldsymbol{C}^n$. Their results are called the *indicator theorem* of entire functions of exponential type on $\boldsymbol{C}^n$.

In the present thesis we investigate first the Levi problem in infinite dimensional projective spaces, and then apply this result to the research of the indicator of entire functions of of exponential type in infinite dimensional spaces. The main theorems are the following two theorems.

Theorem 1. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space and (ω, φ) be a Riemann domain over the complex projective space $\boldsymbol{P}(E)$ induced from E. Assume that ω is not homeomorphic to $\boldsymbol{P}(E)$ through φ. Then the following statements* (1), (2), (3), (4) *and* (5) *are equivalent. Moreover if ω is an open subset of $\boldsymbol{P}(E)$, the statements* (1), (2), (3), (4), (5) *and* (6) *are equivalent.*

(1) *ω is pseudoconvex.*

(2) *For any finite dimensional subspace F of E, $\varphi^{-1}(\boldsymbol{P}(E))$ is a Stein manifold.*

(3) *ω is a domain of holomorphy.*

(4) *ω is a domain of holomorphy and holomorphically separated.*

(5) *ω is a domain of existence.*

(6) *There exists a non-constant holomorphic function f on ω such that, for every connected open neighborhood V of an arbitrary point on the boundary of ω, each component of $\omega \cap V$ contains zero of f of arbitrarily high order.*

Theorem 2. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space, and p be a plurisubharmonic function in E which is positively homogeneous of order 1. Then there exists an entire function f of exponential type on E such that*

$$P(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |f(tz')|$$

for every $z \in E$.

Corollary 3. *If E is a nuclear Fréchet space with the bounded approximation property or a DFN-space, ther exists an analytic functional μ on the strong dual space E' of the space E such that*

$$p(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |\mu(\exp(tz'))|$$

for every $z \in E$.

In order to prove Theorem 1 we first prove by using the method of Ueda([43] in Part II) that a pseudoconvex Riemann domain (ω, φ) over the projective space $\boldsymbol{P}(E)$ is the quotient space, by $\boldsymbol{C}^*$, of a pseudoconvex Riemann domain (Ω, Φ) with $\boldsymbol{C}^*$-action over E. And then by researching invariant properties, with $\boldsymbol{C}^*$-action,

of the Riemann domain (Ω, Φ) over E, we do those of the Riemann domain (ω, φ) over $\boldsymbol{P}(E)$. Last, we prove the theorem of Cartan-Thullen's type in a Riemann domain over a projective space induced from a separable metrizable locally convex space, and then prove that a pseudoconvex Riemann domain in Theorem 1 satisfies the assumption of the Theorem of Cartan-Thullen's type. Thus we complete the proof of Theorem 1. The proof of Theorem 2 is based on the characterization of pseudoconvex domains of the projective space $\boldsymbol{P}(E)$ in Theorem 1. We use, for the proof of Corollary 3, the bijection between the space of all entire functions of exponential type on E and the space of all analytic functionals on the dual space E' by the Fourier-Borel transformation.

The author would like to express his sincere gratitude to Professor Joji Kajiwara for his continuous encouragement and valuable suggestions.

Part I

On a pseudoconvex domain spread over a complex projective space induced from a complex Banach space with a Schauder basis

Introduction.

Oka[18] solved the Levi problem, which is the problem to ask if a pseudoconvex domain is a domain of holomorphy, in a domain spread over $\boldsymbol{C}^n$. At the same time, Bremermann[1] and Norguet[16] solved this problem in $\boldsymbol{C}^n$. Their results were extended to a domain spread over the complex projective space $\boldsymbol{P}_n(\boldsymbol{C})$ of dimension n by Fujita[4], Kiselman[9] and Takeuchi[22].

In the last fifteen years, the Levi problem has been discussed in various infinite dimensional spaces. Gruman[5] and Gruman and Kiselman[6] solved this problem in a complex Banach space E with a Schauder basis, and Hervier[7] extended this result to a domain spread over E. Dineen[2] and Gruman[5] solved this problem in an infinite dimensional vector space E with the finite open topology, and Kajiwara[8] extended this result to a domain of the complex projective space induced from E.

The aim of this paper is to prove the following two theorems having their sources

in the Levi problem and in the imbedding theorem of a Stein manifold.

Theorem 1. *Let E be a complex Banach space with a Schauder basis, and $\boldsymbol{P}(E)$ the complex projective space induced from E. Let (Ω, ϕ) be a domain spread over the complex projective space $\boldsymbol{P}(E)$. Suppose that Ω is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ. Then the following conditions are equivalent:*

(1) *Ω is pseudoconvex.*

(2) *For every finite dimensional linear subspace F of E and the projective space $\boldsymbol{P}(F)$ induced from F, the inverse image $\phi^{-1}(\boldsymbol{P}(F))$ of $\boldsymbol{P}(F)$ by ϕ is a Stein manifold.*

(3) *Ω is a domain of holomorphy.*

(4) *Ω is a domain of existence.*

Theorem 2. *Let H be a separable complex Hilbert space, $\{e_j\}_{j=1}^{\infty}$ an orthonormal basis of H, and $\boldsymbol{P}(H)$ the complex projective space induced from H. Let (Ω, ϕ) be a pseudoconvex domain spread over $\boldsymbol{P}(H)$. Suppose that Ω is not homeomorphic to $\boldsymbol{P}(H)$ through ϕ. We denote by H_n the linear span of the set $\{e_1, e_2, \cdots, e_n\}$ and denote by $\boldsymbol{P}(H_n)$ the complex projective space induced from H_n. Then there exists an injective holomorphic mapping f of Ω into H such that for every positive integer n the restriction mapping $f|\phi^{-1}(\boldsymbol{P}(H_n))$ of f on $\phi^{-1}(\boldsymbol{P}(H_n))$ is a regular and proper holomorphic mapping of $\phi^{-1}(\boldsymbol{P}(H_n))$ into H.*

The author would like to thank the referees for their kindly advice, valuable suggestion and encouragement.

1 Banach complex manifolds and domains spread over Banach complex manifolds.

Let E and F be complex Banach spaces, and U an open subset of E. A mapping

$f : U \to F$ is said to be *holomorphic* in U if f is continuous in U and if, for any $(a, b) \in U \times (E - \{0\})$ and for any continuous linear functional $\alpha \in F'$, the composite mapping $\lambda \to \alpha \circ f(a + \lambda b)$ $(\lambda \in \boldsymbol{C})$ is holomorphic where it is defined. A function $p : U \to [-\infty, +\infty)$ is said to be *plurisubharmonic* if p is upper semicontinuous in U and if, for any point (a, b) of $U \times (E - \{0\})$, the function $\lambda \to p(a + \lambda b)$ $(\lambda \in \boldsymbol{C})$ is subharmonic where it is defined.

A Hausdorff space M is called *a complex manifold modeled on a complex Banach space* E if there exists a family $\mathcal{F} = \{(U_i, \phi_i); i \in I\}$ of pairs (U_i, ϕ_i) of open sets U_i of M and homeomorphisms ϕ_i of open sets U_i onto open sets of E satisfying the following conditions:

(1) For any elements i, j of I with $U_i \cap U_j \neq \emptyset$, the mapping $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ between open sets in E is holomorphic.

(2) $\cup_{i \in I} U_i = M$.

$\mathcal{F}$ is called the *atlas* of M. An element of F is called a *chart* of M.

Let M and N be complex manifolds with atlases $\{(U_i, \phi_i); i \in I\}$ and $\{(U'_\alpha, \phi'_\alpha); \alpha \in A\}$ respectively. Then a mapping $f : M \to N$ is said to be *holomorphic* if, for any $i \in I$ and $\alpha \in A$ with $f(U_i) \cap U'_\alpha \neq \emptyset$, the mapping $\phi'_\alpha \circ f \circ \phi_i^{-1}$ is holomorphic. Particularly, if $N = \boldsymbol{C}$, f is called a *holomorphic function*. We denote by $H(M)$ the family of all holomorphic functions in M. A function $p : M \to [-\infty, +\infty)$ is said to be *plurisubharmonic* if, for any $i \in I$, the function $f \circ \phi_i^{-1}$ is plurisubharmonic.

We consider subsets Δ_1 and Δ_2 in $\boldsymbol{C}^2$ defined by

$$\Delta_1 = \{(z_1, z_2) \in \boldsymbol{C}^2;\ |z_1| = 1,\ z_2 \in [0, 1]\} \cup \{|z_1| \leq 1,\ z_2 = 0\}, \tag{1.1}$$

$$\Delta_2 = \{|z_1| \leq 1,\ z_2 \in [0, 1]\}. \tag{1.2}$$

A complex manifold M is said to be *satisfy the Kontinuitätssatz* if any holomorphic mapping of a neighborhood of Δ_1 into M is extended holomorphically to Δ_2.

Let M be a complex manifold. If there exists a local biholomorphic mapping ϕ of a complex manifold Ω into M, (Ω, ϕ) is called a *region sprea over* M. Moreover, if Ω is connected, (Ω, ϕ) is called a *domain spread over* M.

Let (Ω, ϕ) and (Ω', ϕ') be regions spread over M. If a holomorphic mapping λ of Ω into Ω' satisfies $\phi = \phi' \circ \lambda$, λ is called a *mapping of* $(\Omega.\phi)$ *into* (Ω', ϕ'). If (Ω', ϕ') is a region spread over M, then a mapping λ of (Ω, ϕ) into (Ω', ϕ') is said to be an $\mathcal{F}$-*extension* of Ω if for each $f \in \mathcal{F}$ there exists a unique $f' \in H(\Omega')$ such that $f' \circ \lambda = f$. A mapping λ of (Ω, ϕ) into (ω', ϕ') is said to be a *holomorphic extension* of Ω if λ is an $H(\Omega)$-extension of Ω. Ω is said to be an $\mathcal{F}$-*domain of holomorphy* if each $\mathcal{F}$-extension of Ω is an isomorphism. Ω is said to be a *domain of holomorphy* if Ω is an $H(\Omega)$-domain of holomorphy. Ω is said to be a *domain of existence* if there exists $f \in H(\Omega)$ such that Ω is an $\{f\}$-domain of holomorphy.

Let E be a complex Banach space with a norm $\| \cdot \|$ and let (Ω, ϕ) be a region spread over E. For a point z of E and for a positive number ϵ, we define the open ball $B(z, \epsilon)$ by

$$B(z, \epsilon) = \{w \in E \ ; \ \| w - z \| < \epsilon\}. \tag{1.3}$$

For any point x of Ω, there exists a positive number $\epsilon(x)$ such that, for any positive number ϵ with $\epsilon < \epsilon(x)$, there exists uniquely an open neighborhood $\Delta(x, \epsilon)$ of x which is mapped by ϕ homeomorphically onto the open ball $B(\phi(x), \epsilon)$. The open neighborhood $\Delta(x, \epsilon)$ is called the *open ball in* Ω *with center* x *and with radius* ϵ. We define the *boundary distance function* $d_\Omega(x)$ on Ω by

$$d_\Omega(x) = \sup\{x; \text{ the open ball } \Delta(x, \epsilon) \text{ exists}\}. \tag{1.4}$$

Let a and b be points of Ω. By a *line segment* $[a, b]$ in Ω we mean a set in Ω containing the points a and b and homeomorphic under ϕ to the line segment $[\phi(a), \phi(b)]$ in E. By a *polygonal line* $[x_0, x_1, \cdots, x_n]$ in Ω we mean a finite union of line segments of the form $[x_{j-1}, x_j]$ with $j = 1, \cdots, n$.

Remark 1.1 *Let x and y be two points which belong to a connected component of Ω. Since there exists a polygonal line $[x_0, x_1, \cdots, x_n]$ with $x_0 = x$ and with $x_n = y$, there exists a finite dimensional linear subspace F of E such that the set $\{x, y\}$ is contained in a connected component of the inverse image $\phi^{-1}(F)$ of F by ϕ .*

2 Complex projective spaces induced from complex Banach spaces.

In this section we first give some properties of a complex projective space induced from a complex Banach space. Then we give the definition of pseudoconvexity of a domain spread over the complex projective, and prove some lemmas with respect to pseudoconvexity.

Let E be a complex Banach space with the norm $\|\cdot\|$. Let z and z' be points in $E - \{0\}$. z and z' are said to be *equivalent* if there exists a complex number $\lambda \in \boldsymbol{C} - \{0\}$ such that $z' = \lambda z$. The quotient space $\boldsymbol{P}(E)$ of $E - \{0\}$ by this equivalence relation is called the *complex projective space induced from* E. We denote by Q the quotient map of $E - \{0\}$ onto $\boldsymbol{P}(E)$. For any $\xi \in E - \{0\}$, we denote by $[\xi]$ the equivalence class of ξ. Then we have $Q(\xi) = [\xi]$.

Let E' be the complex Banach space of continuous linear functionals on E. We set

$$S = \{(f, a) \in E' \times E;\ f(a) \neq 0\}. \tag{2.1}$$

For each $f \in E' - \{0\}$, we consider a hyperplane $E(f)$ of E and an open subset $U(f)$ of $\boldsymbol{P}(E)$ defined by

$$E(f) = \{\xi \in E;\ f(\xi) = 0\}, \tag{2.2}$$

$$U(f) = \{[\xi] \in \boldsymbol{P}(E);\ f(\xi) \neq 0\} \tag{2.3}$$

respectively. For every $(f, a) \in S$, we define a homeomorphism $\phi_{(f,a)}$ of $U(f)$ onto $E(f)$ by

$$\phi_{(f,a)}([\xi]) = (1/f(\xi))\xi - (1/f(a))a$$

for every $[\xi] \in U(f)$. The family $\{U(f), \phi_{(f,a)}\}_{(f,a)\in S}$ defines the complex structure of the projective space $\boldsymbol{P}(E)$.

Let $S(E)$ be the unit sphere in E. Then the topological space $\boldsymbol{P}(E)$ is a quotient space of $S(E)$. The topology of $S(E)$ as a subspace of E induces the topology on the quotient space $\boldsymbol{P}(E)$. $S(E)$ is a principal fibre bundle over $\boldsymbol{P}(E)$ with circle group. Since $S(E)$ is a subspace of the metric space E, the metric on $S(E)$ induces a metric $d(,)$ on $\boldsymbol{P}(E)$ by

$$d(p, p') = \inf\{\| z - z' \|;\ z \in Q^{-1}(p) \cap S(E),\ z' \in Q^{-1}(p') \cap S(E)\} \tag{2.4}$$

for any points p and p' of $\boldsymbol{P}(E)$. Since E is complete and $S(E)$ is closed, $S(E)$ is a complete metric space. From the compactness of the fibre of $S(E)$, it follows that $\boldsymbol{P}(E)$ is also complete.

Let (Ω, ϕ) be a domain spread over the complex projective space $\boldsymbol{P}(E)$ induced from E. $E - \{0\}$ is the total space of the holomorphic principal bundle over $\boldsymbol{P}(E)$ with the complex multiplicative group $\boldsymbol{C}^*$. We consider the fibre product X of Ω and $E - \{0\}$ given by

$$X = \{(z, w) \in \Omega \times (E - \{0\});\ \phi(z) = Q(w)\}. \tag{2.5}$$

We denote by $\tilde{\phi}$ and $\tilde{Q}$ projections of the fibre product X into $E - \{0\}$ and into Ω respectively. Then $(X, \tilde{\phi})$ is a domain spread over E.

For any $(z, w) \in X$ and for any $\lambda \in \boldsymbol{C}^*$, we set

$$\lambda \cdot (z, w) = (z, \lambda w) \tag{2.6}$$

Then points $\lambda \cdot (z, w)$ of $\Omega \times (E - \{0\})$ belong to X for all $\lambda \in \boldsymbol{C}^*$. The mapping $(\lambda, x) \to \lambda \cdot x$ is a holomorphic mapping of $\boldsymbol{C}^* \times X$ onto X. Then Ω is the quotient space of X by this $\boldsymbol{C}^*$-action and $\tilde{Q}$ is the quotient map of X onto Ω. X is the total space of a holomorphic principal bundle over Ω with the complex multiplicative group $\boldsymbol{C}^*$. We have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\tilde{Q}"] \arrow[d, "\tilde{\phi}"'] & \Omega \arrow[d, "\phi"] \\ E - \{0\} \arrow[r, "Q"'] & \boldsymbol{P}(\mathrm{E}) \end{tikzcd} \tag{2,7}$$

Let f be a holomorphic function in X. We set

$$\tilde{f}(x) = (1/2\pi) \int_o^{2\pi} f(e^{i\theta} \cdot x) d\theta \tag{2.8}$$

for every $x \in X$. Then $\tilde{f}$ is a holomorphic function in X and we have

$$\tilde{f}(e^{i\eta} \cdot x) = \tilde{f}(x)$$

for every $\eta \in [0, 2\pi)$ and for every $x \in X$. By the identity theorem of a complex variable holomorphic function theory, we have

$$\tilde{f}(\lambda \cdot x) = \tilde{f}(x) \tag{2.10}$$

for every $\lambda \in \boldsymbol{C}^*$. Therefore $\tilde{f}$ is constant on $\tilde{Q}^{-1}(z)$ for every $z \in \Omega$. We define a holomorphic function f^* in Ω by

$$f^*(z) = \tilde{f}(\tilde{Q}^{-1}(z)) \tag{2.11}$$

for every $z \in \Omega$. We have

$$(g \circ \tilde{Q})^* = g \tag{2.12}$$

for every $g \in H(\Omega)$. Hence we obtain the following lemma.

Lemma 2.1. *For any $f \in H(X)$, a holomorphic function $\tilde{f}$ in X defined by* (2.8) *is constant on $\tilde{Q}^{-1}(z)$ for every $z \in \Omega$. Thus we can define a holomorphic function f^* in Ω by* (2.11).

Let F be a closed linear subspace of E. We denote by X_F and by Ω_F regions spread over F and spread over the complex projective space $\boldsymbol{P}(F)$ induced from F, respectively, defined by

$$X_F = \tilde{\phi}^{-1}(F - \{0\}), \tag{2.13}$$

$$\Omega_F = \phi^{-1}(\boldsymbol{P}(F)). \tag{2.14}$$

X_F is a holomorphic principal bundle over Ω_F with the complex multiplicative group $\boldsymbol{C}^*$. We have the following commutative diagram induced from the commutative diagram (2.7):

$$\begin{tikzcd} X_F \arrow[r, "\tilde{Q}|X_F"] \arrow[d, "\tilde{\phi}|X_F"'] & \Omega_F \arrow[d, "\phi|\Omega_F"] \\ F - \{0\} \arrow[r, "Q|(F - \{0\})"'] & \boldsymbol{P}(F) \end{tikzcd} \tag{2.15}$$

Let (Ω, ϕ) be a region spread over a complex projective space $\boldsymbol{P}(E)$ induced from a complex Banach space E. Then the region (Ω, ϕ) is said to be *pseudoconvex* if, for every $f \in E' - \{0\}$ and for the open set $U(f)$, defined by (2.3), of $\boldsymbol{P}(E)$, the open set $\phi^{-1}(U(f))$ of Ω satisfies the Kontinuitätssatz.

Lemma 2.2. *Let E be a complex Banach space and (Ω, ϕ) be a domain spread over the complex projective space $\boldsymbol{P}(E)$. Suppose that Ω is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ. Then for any finite dimensional linear subspace F of E and for any connected component V_F of Ω_F, there exist a finite dimensional linear subspace G of E and a connected component V_G of Ω_G satisfying the following conditions:*

(1) *V_F is a closed complex submanifold of V_G.*

(2) *V_G is not homeomorphic to $\boldsymbol{P}(G)$ through $\phi|V_G$.*

Proof. By Remark 1.1 and by the commutative diagram (2.15), there exist a finite dimensional linear subspace F_0 of E and a connected component V_{F_0} of Ω_{F_0} such that V_{F_0} is not homeomorphic to $\boldsymbol{P}(F_0)$ through $\phi|V_{F_0}$. We take a point z of V_F and a point w of V_{F_0}. By Remark 1.1 and by the commutative diagram (2.15), there exists a finite dimensional subspace F_1 such that a connected component V_{F_1} of Ω_{F_1} contains the set $\{z, w\}$. Let G be the complex vector space spanned by all elements of the union $F \cup F_0 \cup F_1$. Then $\boldsymbol{P}(F)$ and $\boldsymbol{P}(F_0)$ are closed complex submanifolds of $\boldsymbol{P}(G)$. We denote by V_G the connected component of Ω_G containing the set $\{z, w\}$. Since $(V_G, \phi|V_G)$ is a domain spread over $\boldsymbol{P}(G)$, both V_F and V_{F_0} are closed complex submanifolds of V_G. Then V_G satisfies the required conditions (1) and (2). This completes the proof.

Lemma 2.3. *Suppose that Ω is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ and that Ω is pseudoconvex. Then, for any finite dimensional linear subspace F of E, Ω_F is a Stein manifold. X satisfies the Kontinuitätssatz.*

Proof. Let F be a finite dimensional linear subspace of E. Let V_F be any component of Ω_F. By Lemma 2.2 there exists a finite dimensional subspace G of E and a component V_G of Ω_G satisfying the conditions (1) and (2) in Lemma 2.2. Since Ω is pseudoconvex, V_G is also pseudoconvex. By Fujita[4], Kiselman[9] and Takeuchi[22], the pseudoconvex domain V_G spread over $\boldsymbol{P}(G)$ is a Stein manifold. Since V_F is a closed complex submanifold of the Stein manifold V_G, V_F is a Stein manifold. Thus Ω_F is a Stein manifold. X_F is the total space of a holomorphic principal bundle over the Stein manifold Ω_F with the complex multiplicative group $\boldsymbol{C}^*$. Therefore X_F is a Stein manifold by Matsushima and Morimoto[12]. Since $(X, \tilde{\phi})$ is a domain spread over E, X satisfies the Kontinuitätssatz by Noverraz[17]. This completes the proof.

Lemma 2.4 *With the assumption of Lemma 2.2 the following conditions are equivalent:*

(1) *Ω is pseudoconvex.*

(2) *Ω_F is a Stein manifold for every finite dimensional linear subspace F of E .*

Proof. It follows from Lemma 2.3 that (1) implies (2).

We will show that (2) implies (1). Let f be an element of $E' - \{0\}$. BY the assumption, for every finite dimensional linear subspace F of E with $\dim_{\boldsymbol{C}} F \geq 2$ and $F \not\subset \{f = 0\}$, Ω_F is a Stein manifold. We set $H = \phi^{-1}(\{[\xi] \in \boldsymbol{P}(F);\ f(\xi) = 0\})$. Since H is a hypersurface of Ω_F and $\Omega_F \cap \phi^{-1}(U(f)) = \Omega_F \backslash H$, $\Omega_F \cap \phi^{-1}(U(f))$ is a Stein manifold. $\phi^{-1}(U(f))$ and $\Omega_F \cap \phi^{-1}(U(f))$ are identified with regions spread over the Banach space $\{f = 0\}$ and spread over the finite dimensional subspace $\{f = 0\} \cap F$ of $\{f = 0\}$ respectively. Therefore by Noverraz[17] the domain $\phi^{-1}(U(f))$ satisfies the Kontinuitätssatz. Thus Ω is pseudoconvex. This completes the proof.

3 Some properties of the fibre product X.

In this section we will research some properties of the fibre product X, defined in the preceding section, of Ω and $E-\{0\}$ for complex Banach space E with a Schauder basis and for a pseudoconvex domain (Ω, ϕ) spread over the complex projective space $\boldsymbol{P}(E)$.

Let E be a complex Banach space with the norm $\|\cdot\|$ and a Schauder basis $\{e_j\}_{j=1}^{\infty}$. Let (Ω, ϕ) be a pseudoconvex domain, which is not homeomorphic to $\boldsymbol{P}(E)$ through ϕ, spread over the complex projective space $\boldsymbol{P}(E)$.

Since Ω is pseudoconvex, by Lemma 2.3 X satisfies the Kontinuitätssatz. By Noverraz[17], we have the following Lemma 3.1.

Lemma 3.1. *$-\log d_X$ is a continuous plurisubharmonic function in X where d_X is the boundary distance function on X. For any finite dimensional linear subspace F of E, $\tilde{\phi}^{-1}(F)$ is a Stein manifold.*

We can choose a Schauder basis $\{e_j\}_{j=1}^{\infty}$ of E such that the intersection of the image of $\tilde{\phi}$ and the linear space $\{\lambda e_1;\ \lambda \in \boldsymbol{C}\}$ is nonempty. For every $\xi \in E$, ξ can be represented in a unique way

$$\xi = \sum_{n=1}^{\infty} \xi_n e_n. \tag{3.1}$$

We denote by E_n the linea span of the set $\{e_1, e_2, \cdots, e_n\}$, and by u_n the mapping of E onto E_n defined by

$$u_n(\xi) = \sum_{j=1}^{n} \xi_j e_j. \tag{3.2}$$

We denote by μ_n a continuous linear functional of E defined by

$$\mu_n(\xi) = \xi_n \tag{3.3}$$

for every $\xi = \sum_{j=1}^{\infty} \xi_j e_j$.

Lemma 3.2. *There exist a norm $|||\cdot|||$ of E and positive constants c_1 and c_2 satisfying the following conditions:*

(1) $c_1 \| \xi \| \leq |||\xi||| \leq c_2 \| \xi \|$ *for every $\xi \in E$.*

(2) $|||u_n(\xi)||| \leq |||\xi|||$ *for every positive integer n.*

The proof of Lemma 3.2 is in Singer[21]. The condition (1) of Lemma 3.2 implies that the Banach space $(E, |||\cdot|||)$ with the norm $|||\cdot|||$ is equivalent to the Banach space E with the original norm $\|\cdot\|$. Therefore we may assume that the norm E satisfies the condition

$$\| u_n(\xi) \| \leq \| \xi \| \tag{3.4}$$

for every positive integer n.

Let x_0 be a point of X with $\tilde{\phi}(x_0) \in E_1$. We may assume that the norm $\|\cdot\|$ of E is chosen such that $d_X(x_0) \geq 1$. For every n we set

$$X_n = \tilde{\phi}^{-1}(E_n), \tag{3.5}$$

$$A_n = \{x \in X;\ \sup_{m \geq n} \| u_m \circ \tilde{\phi}(x) - \tilde{\phi}(x) \| < d_X(x)\}, \tag{3.6}$$

$$\nu_n(x) = (\tilde{\phi}|\Delta(x, d_X(x)))^{-1} \circ u_n \circ \tilde{\phi}(x) \tag{3.7}$$

for every $x \in A_n$. Then $\sup_{m \geq n} \| u_m \circ \tilde{\phi}(x) - \tilde{\phi}(x) \|$ is continuous on X, and A_n is an open subset of X. ν_n is a holomorphic mapping of A_n into X_n for every n.

Let (Y, ψ) be a region spread over a complex Banach space F. Then we use the notation $d_Y(A) = \inf\{d_Y(x);\ x \in A\}$ where A is a subset of Y.

The proof of the following lemma is in Lemma 54.5 of Mujica[13].

Lemma 3.3. *There exist two increasing sequences* $\{B_n\}_{n=1}^{\infty}$ *and* $\{C_n\}_{n=1}^{\infty}$ *of open sets* B_n *and* C_n *of* X *such that*

(a) $x_0 \subset C_n \subset B_n \subset A_n$ *for every* $n \geq 1$, $X = \cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} C_n$.

(b) $d_{A_n}(B_n) \geq 2^{-n}$ *and* $B_m \cap X_n$ *is relatively compact in* $A_m \cap X_n$ *for every* m, $n \geq 1$. (c) $d_{C_{m+1}}(C_m) \geq 2^{-m-1}$ *and* $\nu_n(C_m) \subset B_m \cap X_n$ *for every* $m \geq 1$ *and every* $n \geq m$.

For every $x \in X$, we define the sets $V(x)$ and $S(x)$ by

$$V(x) = \{\lambda \cdot x;\ \lambda \in \boldsymbol{C}^*\}, \tag{3.8}$$

$$S(x) = \{e^{i\theta} \cdot x;\ 0 \leq \theta \leq 2\pi\}. \tag{3.9}$$

Let K be a compact subset of a Stein manifold S. We use the notation

$$K(S) = \{x \in S;\ |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in H(S)\}. \tag{3.10}$$

The set $K(S)$ is the *holomorphically convex hull* of K in the Stein manifold S. If $K(S) = K$, K is said to be *Runge* in S. Let S_1 be a Stein manifold and S_2 be a Stein open subset of S_1. S_2 is said to be *Runge relative to* S_1 if, for any compact subset K of S_2, $K(S_1)$ is a compact subset in S_2.

We denote by K_n the holomorphically convex hull of the topological closure of the set $B_n \cap X_{n+1}$ in the Stein manifold X_{n+1}. Since X_{n+1} is a Stein manifold, K_n is a compact subset of X_{n+1} and Runge in X_{n+1}. On the other hand $\sup_{m \geq n} \| u_m \circ \tilde{\phi}(x) - \tilde{\phi}(x) \|$ is continuous in X, and $\sup_{m \geq n} \log \| u_m \circ \tilde{\phi} - \tilde{\phi}(x) \| - \log d_X(x)$ is a continuous plurisubharmonic function of X into $[-\infty, \infty)$. Therefore by Narasimhan[15], $A_n \cap X_{n+1}$ is Runge relative to X_{n+1} and K_n is compact in $A_n \cap X_{n+1}$.

Lemma 3.4. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of points of X such that $c_n \in X_n$, $c_n \notin X_{n-1}$ and $V(c_n) \subset X \backslash K_n$. Then, for any sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions f_n in X_n such that*

$$f_{n+1}|X_n = f_n, \tag{3.11}$$

$$|f_{n+1}(x) - f_n \circ \nu_n(x)| < 1/2^n \tag{3.12}$$

for any $x \in K_n$, and

$$Ref_n(x) \geq \lambda_n \tag{3.13}$$

for any $x \in S(c_n)$ where Ref_n represents the real part of f_n.

Proof. We will show this lemma by induction with respect to n. We set $f_1(x) = \lambda_1$ for every $x \in X$. Then f_1 satisfies (3.13). We assume that there exist holomorphic functions f_k in X_k $(1 \leq k \leq n)$ with (3.11), (3.12) and (3.13). We set

$$g(x) = f_n \circ \nu_n(x) \tag{3.14}$$

for every $x \in X_{n+1} \cap A_n$. Closed subsets $K_n \cup X_n$ and $(X_{n+1} \backslash A_n)$ are mutually disjoint because K_n is a compact subset of $X_{n+1} \cap A_n$. Therefore there exists a C^∞-function η in X_{n+1} such that $\eta = 1$ on a neighborhood of $K_n \cup X_n$, and that $\eta = 0$ on a neighborhood of $(X_{n+1} \backslash A_n)$.

We consider a $\bar{\partial}$-equation on X_{n+1}:

$$\bar{\partial}\nu = (\mu_{n+1} \circ \tilde{\phi}(x))^{-1} g \bar{\partial}\eta \tag{3.15}$$

where μ_j are defined in (3.3). Since X_{n+1} is a Stein manifold, and since the right hand side of (3.15) is $\bar{\partial}$-closed, there exists a C^∞-function ν on X_{n+1} satisfying (3.15). We set

$$h(x) = \eta(x)g(x) - (\mu_{n+1} \circ \tilde{\phi}(x))\nu(x) \tag{3.16}$$

for every $x \in X_{n+1}$. Then h is holomorphic in X_{n+1} and satisfies $h|X_n = f_n$. Since ν is holomorphic in a neighborhood of a Runge compact subset K_n of X_{n+1}, by Oka-Weil theorem there exists a holomorphic function w in X_{n+1} such that

$$|\nu(x) - w(x)| < 1/(2^{n+1}M) \tag{3.17}$$

for every $x \in K_n$ where $M = \sup\{|\mu_{n+1} \circ \tilde{\phi}(x)|;\ x \in K_n\}$. We set

$$F(x) = h(x) + (\mu_{n+1} \circ \tilde{\phi}(x))w(x) \tag{3.18}$$

for every $x \in X_{n+1}$. Then we have

$$|F(x) = f_n \circ \nu_n(x)| < 1/2^{n+1} \tag{3.19}$$

for every $x \in K_n$.

We set

$$T = S(c_{n+1}) \cup K_n, \tag{3.20}$$

$$V_{n+1} = V(c_{n+1}). \tag{3.21}$$

We denote by $\hat{T}$ the holomorphically convex hull of T in X_{n+1}. Since X_{n+1} is Stein, $\hat{T}$ is compact in X_{n+1}.

We will show that $\hat{T} \subset V_{n+1} \cup K_n$. Let x be a point of $X_{n+1} \backslash (V_{n+1} \cup K_n)$. Since X_{n+1} is a Stein manifold, by Oka-Cartan theorem there exists a holomorphic function s in X_{n+1} with $s = 0$ on V_{n+1} and with $s(x) = 1$. Since K_n is a Runge compact subset of X_{n+1}, there exists a holomorphic function t in X_{n+1}, such that $|t(x)| > 1$ and $\| t \|_{K_n} < 1/(\| s \|_{K_n} + 1)$ where $\| s \|_{K_n}$ and $\| t \|_{K_n}$ represent supremums of functions $|s(\cdot)|$ and $|t(\cdot)|$, respectively, on the compact set K_n. Then we have $|s(x)t(x)| > 1$ and $\sup\{|s(y)t(y)|;\ y \in T\} < 1$. Therefore x cannot belong to $\hat{T}$. Thus we have $\hat{T} \subset V_{n+1} \cup K_n$.

Since $\hat{T}$ is a Runge compact subset of X_{n+1}, there exist Stein neighborhoods Δ_1 and Δ_2 of $(\hat{T} \cap V_{n+1})$ and of K_n, respectively, in X_{n+1} with $\Delta_1 \cap \Delta_2 = \emptyset$. We set $L = \sup\{|F(x)|;\ x \in S(c_{n+1})\}$. We define a holomorphic function α in a Stein manifold $\Delta_1 \cap V_{n+1}$ by

$$\alpha(\lambda \cdot c_{n+1}) = (L + \lambda_{n+1} + 1)/\lambda\mu_{n+1} \circ \tilde{\phi}(c_{n+1}) \tag{3.22}$$

for every $\lambda \cdot c_{n+1} \in \Delta_1 \cap V_{n+1}$ $(\lambda \in \boldsymbol{C} - \{0\})$. Since $\Delta_1 \cap V_{n+1}$ is a closed complex submanifold of Δ_1, by Oka-Cartan theorem there exists a holomorphic function A in Δ_1 such that $A|V_{n+1} \cap \Delta_1 = \alpha$. We define a holomorphic function B on $\Delta_1 \cup \Delta_2$ by $B|\Delta_1 = A$ and $B|\Delta_2 = 0$. Since $\Delta_1 \cup \Delta_2$ is a neighborhood of the Runge compact subset $\tilde{T}$ in X_{n+1}, there exists a holomorphic function G on X_{n+1} such that

$$|G(x) - B(x)| < 1/\{2^{n+1}(L' + 1)\} \tag{3.23}$$

for every $x \in \hat{T}$ where $L' = \sup\{|\mu_{n+1} \circ \tilde{\phi}(x)|;\ x \in S(c_{n+1}) \cup K_n)\}$. We set $f_{n+1}(x) = F(x) + (\mu_{n+1} \circ \tilde{\phi}(x))G(x)$ for every $x \in X_{n+1}$. By (3.19) and (3.23) we have

$$|f_{n+1}(x) - f_n \circ \nu_n(x)| < 1/2^n \tag{3.24}$$

for every $x \in K_n$. By (3.22) and (3.23) we have

$$\mathrm{Re} f_{n+1}(e^{i\theta} \cdot c_{n+1}) \geq \lambda_{n+1} \tag{3.25}$$

for every $\theta \in \boldsymbol{R}$. Since $f_{n+1}|X_n = f_n$, this completes the proof.

Lemma 3.5. *Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \epsilon_n < \infty$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions f_n in X_n such that $f_{n+1}|X_n = f_n$ and $|f_{n+1}(x) - f_n \circ \nu_n(x)| < \epsilon_n$ for every $x \in K_n$. Then there exists a holomorphic function f in X such that $f|X_n = f_n$.*

Proof. Since, by Lemma 3.3, $\nu_{n+j}(C_{n+j-1}) \subset B_{n+j-1} \cap X_{n+j} \subset K_{n+j-1}$ and $C_n \subset C_{n+j-1}$, we have $|f_{n+j} \circ \nu_{n+j}(x) - f_{n+j-1} \circ \nu_{n+j-1}(x)| = |f_{n+j}(\nu_{n+j}(x)) - f_{n+j-1} \circ \nu_{n+j-1}(\nu_{n+j}(x))| < \epsilon_{n+j-1}$ for any positive integers n and j and for any $x \in C_n$. Thus for any m, n we have

$$\begin{aligned} |f_{n+m} \circ \nu_{n+m}(x) - f_n \circ \nu_n(x)| &\leq \sum_{j=1}^{m} |f_{n+j} \circ \nu_{n+j}(x) - f_{n+j-1} \circ \nu_{n+j=1}(x)| \\ &\leq \sum_{j=1}^{m} \epsilon_{n+j-1} \leq \sum_{j=1}^{\infty} \epsilon_j \end{aligned}$$

for every $x \in C_n$. Therefore the sequence $\{f_n \circ \nu_n\}_{n=1}^{\infty}$ converges uniformly on each C_n to a function $f \in H(X)$. Then f satisfies $f|X_n = f_n$. This completes the proof.

We can obtain the following two lemmas by the application of Lemma 3.4 and Lemma 3.5.

Lemma 3.6. *With the conditions of Lemma* 3.4, *there exists a holomorphic function f in X such that* $\operatorname{Re} f(x) \geq \lambda_n$ *for every n and for every $x \in S(c_n)$.*

Lemma 3.7. *Let F be any finite dimensional complex linear subspace of E. Then the restriction mapping of $H(X)$ into $H(\tilde{\phi}^{-1}(F))$ is surjective.*

4 Proof of Theorem 1 and Theorem 2.

In order to prove Theorem 1 and Theorem 2, we will prepare some lemmas. Throughout this section E means a complex Banach space with a Schauder basis $\{e_n\}_{n=1}^{\infty}$ and (Ω, ϕ) means a domain, which is not homeomorphic to the projective space $\boldsymbol{P}(E)$ through ϕ, spread over $\boldsymbol{P}(E)$.

Lemma 4.1. *If Ω is a domain of holomorphy, Ω is pseudoconvex.*

Proof. For any continuous linear functional f of E and the open set $U(f) = \{[\xi] \in \boldsymbol{P}(E);\ f(\xi) \neq 0\}$, we have only to show that the domain $\phi^{-1}(U(f))$ satisfies

the Kontinuitätssatz. Since there exists a biholomorphic mapping p of $U(f)$ onto the complex Banach space $L = \{\xi \in E;\ f(\xi) = 0\}$, the domain $(\phi^{-1}(U(f)), p \circ (\phi|^{-1}(U(f))))$ is a domain spread over L. Since Ω is a domain of holomorphy and since, for any sequence $\{x_n\}_{n=1}^{\infty}$ of $\phi^{-1}(U(f))$ converging to a point of $\Omega \backslash \phi^{-1}(U(f))$, the set $\{p \circ \phi(x_n)\}$ is an unbounded subset of L, $\phi^{-1}(U(f))$ is also a domain of holomorphy. By Noverraz[17], $\phi^{-1}(U(f))$ satisfies the Kontinuitätssatz. This completes the proof.

With the conditions and notations in Section 3, we set

$$S(K_n) = \{e^{i\theta} \cdot x\ ;\ \theta \in [0, 2\pi],\ x \in K_n\} \tag{4.1}$$

for each n. $S(K_n)$ is compact in $X_{n+1} \cap A_n$. We denote by $\widehat{S(K_n)}$ the holomorphically convex hull of $S(K_n)$ in X_{n+1}. Since $A_n \cap X_{n+1}$ is Runge relative to X_{n+1}, $\widehat{S(K_n)}$ is a compact subset of $A_n \cap X_{n+1}$. We set $e^{i\theta} \cdot C_n = \{e^{i\theta} \cdot x\ ;\ x \in C_n\}$. For any $\theta \in \boldsymbol{R}$, we have

$$e^{i\theta} \cdot C_n \cap X_{n+1} \subset S(K_n) \subset \widehat{S(K_n)}, \tag{4.2}$$

$$\nu_n(e^{i\theta} \cdot C_n) \subset S(K_n) \subset \widehat{S(K_n)}. \tag{4.3}$$

Hereafter we assume that Ω is pseudoconvex in a series of lemmas.

Lemma 4.2. *Then for any holomorphic function f in X_n there exists a sequence $\{f_{n+k}\}_{k=0}^{\infty}$ of holomorphic functions in X_{n+k} satisfying the following conditions:*

(1) $f_n = f$,

(2) $f_{n+k}|X_{n+k-1} = f_{n+k-1}$,

(3) $|f_{n+k}(x) - f_{n+k-1} \circ \nu_{n+k-1}(x)| < 1/2^{n+k}$ *for every* $x \in \widehat{S(K_n)}$.

Proof. We can prove this lemma by the same way as the proof of Lemma 3.4.

Remark 4.3. *By the same way as the proof of Lemma 3.5, we can prove that there exists a holomorphic function F in X such that $F|X_{n+k} = f_{n+k}$ and $F(x) = \lim_{k\to\infty} f_{n+k} \circ \nu_{n+k}(x)$ for every $x \in X$. By (4.2) and (4.3), we have*

$$\begin{aligned} |F(x)| &= \lim_{m\to\infty} |f_m \circ \nu_m(x)| \\ &\leq \limsup_{m\to\infty} \left\{ \sum_{k=N}^{m} |f_k \circ \nu_k(x) - f_{k-1} \circ \nu_{k-1}(x)| + |f_N \circ \nu_N(x)| \right\} \\ &\leq 2^{-N} + \sup\{|f_N \circ \nu_N(y)|; y \in S(C_N)\} < \infty \end{aligned}$$

for every $N \geq n$ and for every $x \in S(C_N)$ where $S(C_N)$ is the set $\{e^{i\theta} \cdot z;\ (\theta, z) \in \boldsymbol{R} \times C_N\}$. Thus we have $\sup\{|F(x)|; x \in S(C_N)\} < \infty$ for every $N \geq 1$.

We denote by D_m an open subset of Ω defined by $D_m = \tilde{Q}(C_m)$ for every $m \geq 1$.

Lemma 4.4. *For any holomorphic function f in $\phi^{-1}(\boldsymbol{P}(E_n))$ there exists a holomorphic function F in Ω such that $F|\phi^{-1}(\boldsymbol{P}(E_n)) = f$ and $\sup\{|F(x)|\ ;\ x \in D_m\} < \infty$ for every $m \geq 1$.*

Proof. We consider a holomorphic function g in X_n defined by $g = f \circ (\tilde{Q}|X_n)$. By Lemma 4.2 and by Remark 4.3, there exists a holomorphic function G in X such that $G|X_n = g$ and $\sup\{|G(x)|\ ;\ x \in S(C_m)\} < \infty$ for every $m \geq 1$. We set

$$\tilde{G}(x) = (1/2\pi) \int_0^{2\pi} G(e^{i\theta} \cdot x) d\theta$$

for every $x \in X$. Then $\tilde{G}$ is a holomorphic function in X and constant on $\tilde{Q}^{-1}(z)$ for every $z \in \Omega$. We define a holomorphic function F by $F(x) = \tilde{G} \circ \tilde{Q}^{-1}(z)$ for every $z \in \Omega$. Then we have $F|\phi^{-1}(\boldsymbol{P}(E_n)) = f$ and $\sup\{|F(x)|\ ;\ z \in D_m\} \leq \sup\{|G(x)|\ ;\ z \in S(C_m)\} < \infty$ for every $m \geq 1$. This completes the proof.

Lemma 4.5. *For any different points z and w in Ω, there exists a holomorphic function f in Ω such that $f(z) \neq f(w)$ and that* $\sup\{|f(p)| \ ; \ p \in D_m\} < \infty$ *for every $m \geq 1$.*

Proof. There exist two diffent points x and y in X such that $\tilde{Q}(x) = z$ and $\tilde{Q}(y) = w$. There exists a positive integer N such that the set $\{x, y, \nu_N(x), \nu_N(y)\}$ is contained in C_N and that $\tilde{Q}(\nu_N(x)) \neq \tilde{Q}(\nu_N(y))$. Then the compact sets $S(x)$, $S(y)$, $S(\nu_N(x))$ and $S(\nu_N(y))$, defined in (3.9), are contained in $S(C_N)$. We consider closed submanifolds $V(\nu_N(x))$ and $V(\nu_N(y))$, defined in (3.8), of the Stein manifold X_N. By Oka-Cartan theorem, there exist a holomophic function g in X_N satistying $g|V(\nu_N(x)) = 2$ and $g|V(\nu_N(y)) = 0$. By Lemma 4.2, there exists a sequence $\{g_m\}_{m=N}^{\infty}$ of holomorphic functions g_m in X_{N+m} such that $g_m|X_{m-1} = g_{m-1}$, $g_N = g$ and $|g_m \circ \nu_m(t) - g_{m-1} \circ \nu_{m-1}(t))| < 1/2^m$ for every $m > N$ and every $t \in S(C_{m-1})$. Let G be a holomorphic function defined by $G(t) = \lim_{m\to\infty} g_m \circ \nu_m(t)$ for every $t \in X$. Then we have $|G(t) - g \circ \nu_N(t)| \leq 1/2^N$ for every $t \in S(C_N)$. Thus we have $\mathrm{Re}\ G(e^{i\theta} \cdot x) \geq \mathrm{Re}\ g \circ \nu_N(e^{i\theta} \cdot x) - 1/2^n \geq 3/2$ and $\mathrm{Re}\ G(e^{i\theta} \cdot y) \leq \mathrm{Re}\ g \circ \nu_N(e^{i\theta} \cdot y) + 1/2^N \leq 1/2$. By Remark 4.3, the holomorphic function G in X satisfies $\sup\{|G(t)| \ ; \ t \in S(C_m)\} < \infty$ for every $m \geq 1$. We set

$$\tilde{G}(t) = (1/2\pi) \int_0^{2\pi} G(e^{i\theta} \cdot t) d\theta$$

for every $t \in X$. Then $\tilde{G}$ is a holomorphic function in X and constant on $\tilde{Q}^{-1}(\zeta)$ for every $\zeta \in \Omega$. We set $f(\zeta) = \tilde{G} \circ \tilde{Q}^{-1}(\zeta)$ for every $\zeta \in \Omega$. Then f is a holomorphic function and satisfies $\mathrm{Re}\ f(w) \leq 1/2 < 3/2 \leq \mathrm{Re}\ f(z)$. Moreover we have $\sup\{|f(\zeta)| \ ; \ \zeta \in D_m\} \leq \sup\{|G(t)| \ ; \ t \in S(C_m)\} < \infty$. f satisfies the requirement of this lemma. This completes the proof.

We set $\mathcal{D} = \{D_n\}_{n=1}^{\infty}$ and set $|f|_n = \sup\{|f(x)| \ ; \ x \in D_n\}$ for every $f \in H(\Omega)$

and every $n \geq 1$. We denote by $A(\mathcal{D})$ the Fréchet space defined by

$$A(\mathcal{D}) = \{f \in H(\Omega) \ ; \ |f|_n < \infty \quad \text{for every } n\}.$$

Lemma 4.6. *For each countable set P of Ω there exists a function $g \in A(\mathcal{D})$ such that $g(x) \neq g(y)$ for all $(x, y) \in P \times P \backslash \Delta$ where Δ is the diagonal set of the product $P \times P$.*

Proof. By Lemma 4.5, the set $S_{xy} = \{g \in A(\mathcal{D}) \ ; \ g(x) \neq g(y)\}$ is nonempty for each $(x, y) \in P \times P \backslash \Delta$. The set S_{xy} is open in $A(\mathcal{D})$. We claim that S_{xy} is dense in $A(\mathcal{D})$. Let f be an element of $A(\mathcal{D})$ with $f \notin S_{xy}$. We choose $g \in S_{xy}$ and set $g_n = f + (1/n)g$. Then we have $g_n \in S_{xy}$ for every n and $g_n \to f$ in $A(\mathcal{D})$. Since $A(\mathcal{D})$ is a Baire space, the set $S = \cap\{S_{xy} \ ; \ (x, y) \in P \times P \backslash \Delta\}$ is dense in $A(\mathcal{D})$, and in particular nonempty. This completes the proof.

Proof of Theorem 1. It follows from Lemma 2.4 that (1) and (2) are equivalent. It follows from Lemma 4.1 that (3) implies (1). It is clear that (4) implies (3).

Now we will show that (1) implies (4). Let E_n be the linear span of the set $\{e_1, \cdots, e_n\}$. We may assume that $Q(e_1) \in \phi(\Omega)$. Since $\boldsymbol{P}(E)$ is separable, there exists a countable dense subset D of $\boldsymbol{P}(E)$. We set $P = \phi^{-1}(D)$. Then P is a countable dense subset of Ω. By Lemma 4.6, there exists a holomorphic function $g \in A(\mathcal{D})$ such that $g(x) \neq g(y)$ for every $(x, y) \in P \times P \backslash \Delta$. Let d be the distance of $\boldsymbol{P}(E)$ defined by (2.4). We denote by Ω_n the region, defined by $\Omega_n = \phi^{-1}(\boldsymbol{P}(E_n))$, spread over $\boldsymbol{P}(E_n)$ for every n. We denote by d_n the boundary distance function of the region $(\Omega_n, \phi|\Omega_n)$ with respect to $d|\boldsymbol{P}(E_n)$. For each $x \in \Omega_n$ we denote by $B_n(x)$ the open neighborhood, which is homeomorphically mapped by $\phi|\Omega_n$ onto the set $\{\zeta \in \boldsymbol{P}(E_n) \ ; \ d(\phi(x), \zeta) \leq d_n(x)\}$, of x in Ω_n. We set $L_n = \tilde{Q}(K_n)$ for

each n where K_n is defined in Section 3. Each L_n is a compact subset of Ω_n and $\cup_{n=1}^{\infty} L_n = \cup_{n=1}^{\infty} \Omega_n$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points a_n in Ω_n such that $\{a_n\}_{n=1}^{\infty}$ is dense in Ω. We can find a sequence $\{b_n\}_{n=1}^{\infty}$ in Ω such that $b_n \in B_n(a_n) \backslash L_n$ and $b_n \in \Omega_n \backslash \Omega_{n-1}$. Ther exists a sequence $\{c_n\}_{n=1}^{\infty}$ in X such that $\tilde{Q}(c_n) = b_n$. Then we have $V(c_n) \cap K_n = \emptyset$. By Lemma 3.6, there exists a holomorphic function f in X such that $\mathrm{Re} f(x) \geq n + |g(b_n)|$ for every n and for every $x \in S(c_n)$. We set

$$\tilde{f}(x) = (1/2\pi) \int_0^{2\pi} f(e^{i\theta} \cdot x) d\theta$$

for every $x \in X$. Then $\tilde{f}$ is holomorphic function in X and constant on the fibre $\tilde{Q}^{-1}(z)$ for every $z \in \Omega$. We set $f^*(z) = \tilde{f}(\tilde{Q}^{-1}(z))$ for every $z \in \Omega$. Then f^* is holomorphic in Ω and satisfies $\mathrm{Re}\, f^*(b_n) \geq n + |g(b_n)|$. Since the set of quotient $(f^*(x) - f^*(y))/(g(x) - g(y))$ with $(x, y) \in P \times P \backslash \Delta$ is countable, there exists $\theta \in (0, 1)$ such that $f^*(x) - f^*(y) \neq \theta(g(x) - g(y))$ for every $(x, y) \in P \times P \backslash \Delta$. If we set $h = f^* - \theta g$, then $h \in H(\Omega)$, $h(x) \neq h(y)$ for every $(x, y) \in P \times P \backslash \Delta$ and

$$\mathrm{Re}\, h(b_n) \geq n \tag{4.4}$$

for every $n \geq 1$. We will show that Ω is the domain of existence of h. Let $\lambda : \Omega \to \tilde{\Omega}$ be an $\{h\}$-extension of Ω, and let $\tilde{h} \in H(\tilde{\Omega})$ with $\tilde{h} \circ \lambda = h$. To prove that λ is injective, let a, $b \in \Omega$ with $\lambda(a) = \lambda(b)$. There exist an open neighborhood $U(a)$ of a and an open neighborhood $U(b)$ of b such that $\lambda(U(a)) = \lambda(U(b))$ and that $\lambda|U(a)$, $\lambda|U(b)$, $\phi|U(a)$ and $\phi|U(b)$ are isomorphisms. Then we have $\lambda(x) = \lambda(y)$, if $(x, y) \in U(a) \times U(b)$ and $\phi(x) = \phi(y)$. Thus we have $h(x) = \tilde{h} \circ \lambda(x) = \tilde{h} \circ \lambda(y) = h(y)$, if $(x, y) \in U(a) \times U(b)$ and $\phi(x) = \phi(y)$. We set $W = \phi(U(a)$. Then we have $W = \phi(U(a)) = \phi(U(b))$ and W is an open subset of $\boldsymbol{P}(E)$. $W \cap D$ is nonempty. Thus there exist $x_0 \in U(a)$ and $y_0 \in U(b)$ such that $\phi(x_0) = \phi(y_0) \in W \cap D$. Then $h(x_0) = h(y_0)$. Since $(x_0, y_0) \in P \times P \backslash \Delta$, this is a contradiction. Therefore λ is

injective. To prove that λ is surjective, we assume that $\tilde{\Omega} \neq \lambda(\Omega)$. Then there exists a point b_0 of $(\tilde{\Omega}\backslash\lambda(\Omega)) \cap \overline{\lambda(\Omega)}$ where $\overline{\lambda(\Omega)}$ is the topological closure of $\lambda(\Omega)$ in $\tilde{\Omega}$. Then there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}_{n=1}^{\infty}$ such that $\{\lambda(b_{n_k})\}$ converges to b_0. Then we have

$$|\tilde{h}(\lambda(b_{n_k}))| \geq \mathrm{Re}\, \tilde{h} \circ \lambda(b_{n_k}) = \mathrm{Re}\, h(b_{n_k}) \geq n_k.$$

This implies that $\tilde{h}$ is unbounded in a neighborhood of b_0. This is a contradiction. Thus λ is surjective. Therefore λ is an isomorphism. This implies that Ω is a domain of existence of h. This completes the proof.

Proof of Theorem 2. Let Δ be the diagonal set of the product space $\Omega \times \Omega$. Let (z, w) be any point of $\Omega \times \Omega\backslash\Delta$. By Lemma 4.5, there exists a holomorphic function $g_{(z,w)} \in A(\mathcal{D})$ such that $g_{(z,w)}(z) \neq g_{(z,w)}(w)$. There exists an open neighborhood $U((z, w))$ of (z, w) in $\Omega \times \Omega\backslash\Delta$ such that $g_{(z,w)}(\zeta_1) \neq g_{(z,w)}(\zeta_2)$ for every $(\zeta_1, \zeta_2) \in U((z, w))$. Since $\cup\{U((z, w))\ ;\ (z, w) \in \Omega\times\Omega\backslash\Delta\} = \Omega\times\Omega\backslash\Delta$ and the open set $\Omega \times \Omega\backslash\Delta$ satisfies the Lindelöf property, there exists a sequence $\{(z_j, w_j)\}_{j=1}^{\infty}$ of elements of $\Omega \times \Omega\backslash\Delta$ such that $\cup_{j=1}^{\infty} U((z_j, w_j)) = \Omega \times \Omega\backslash\Delta$. We set $g_n = g_{(z_n,w_n)}$ and $M_n = \sup\{|g_n(\zeta)|\ ;\ \zeta \in D_n\}$ for every positive integer n. Each M_n is a finite positive number. We define an injective holomorphic mapping g of Ω into l^2 by

$$g = \left(\frac{1}{M_1}g_1, \frac{1}{2M_2}g_2, \cdots, \frac{1}{nM_n}g_n, \cdots\right).$$

Since $\phi^{-1}(\boldsymbol{P}(H_{n+1}))$ is a Stein manifold of dimension n for every n, by Narasimhan[14] and by Remmert[20] there exists (2n+1)-holomorphic function $h_{n,j}$ $(1 \leq j \leq 2n+1)$ such that $h_n = (h_{n,1}, h_{n,2}, \cdots, h_{n,2n+1})$ is a regular, injective and proper holomorphic mapping of $\phi^{-1}(\boldsymbol{P}(H_{n+1}))$ into $\boldsymbol{C}^{2n+1}$. By Lemma 4.4, there exists a holomorphic mapping $\tilde{h_n}$ of Ω into $\boldsymbol{C}^{2n+1}$ such that $\tilde{h_n}|\phi^{-1}(\boldsymbol{P}(H_{n+1})) = h_n$ and $\sup\{\|\tilde{h}_n(x)\|_{2n+1};\ x \in D_m\} < \infty$ for every $m \geq 1$ where $\|\cdot\|_{2n+1}$ is the Euclidean

norm of $\boldsymbol{C}^{2n+1}$. We set $k_n = \sup\{\| \tilde{h}_n(x) \|_{2n+1};\ x \in D_n\}$ for every n. We define a holomorphic mapping h of Ω into l^2 by

$$h = (\frac{1}{k_1}\tilde{h}_1, \frac{1}{2k_2}\tilde{h}_2, \cdots, \frac{1}{nk_n}\tilde{h}_n, \cdots).$$

Then $h|\phi^{-1}(\boldsymbol{P}(H_n))$ is a regular, injective, proper holomorphic mapping of $\phi^{-1}(\boldsymbol{P}(H_n))$ into l^2. There exists an isomorphism α of of $l^2 \times l^2$ onto H. We define a holomorphic mapping f of Ω into H by $f(z) = \alpha(g(z), h(z))$ for every z. Then f satisfies the requirement of this theorem. This completes the proof.

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Part II

On the indicator of growth of entire functions of exponential type in infinite dimensional spaces and the Levi problem in infinite dimensional projective spaces

Abstract

Let E be a separable complex Fréchet space with the bounded approximation property, or a complex DFN-space and $P(E)$ be the complex projective space induced from E. Then we solve affirmatively the Levi problem in a Riemann domain over the projective space $P(E)$. By using this result, we give the infinite dimensional version of the indicator theorem of entire functions of exponential type on C^n.

1 Introduction.

Let E be a locally convex space, here always assumed to be complex and Hausdorff. Let f be an entire function of exponential type on E. Then the *indicator* I_f of the entire function f is the function on E with values in $[-\infty, \infty)$ defined by

$$I_f(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |f(tz')|$$

for every $z \in E$. The indicator has the following properties.

(1) I_f is plurisubharmonic.

(2) I_f is positively homogeneous of order 1, that is, $I_f(tz) = t\ I_f(z)$ for every positive number t and every $z \in E$.

Conversely when given a plurisubharmonic function p on E which is positively homogeneous of order 1, we consider the problem to ask whether or not there exists an entire function f of exponential type on E with $I_f = p$. Kiselman[18], Lelong[19] and Martineau[22] solved affirmatively this problem in case the dimension of E is finite. Their results are called the *indicator theorem* of entire functions of exponential type on $\boldsymbol{C}^n$.

This paper is concerned with the Levi problem in infinite dimensional projective spaces and with the indicator theorem of entire functions of exponential type in infinite dimensional spaces. The main theorems of this paper are the following two theorems.

Theorem 1. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space and (ω, φ) be a Riemann domain over the complex projective space $\boldsymbol{P}(E)$ induced from E. Assume that ω is not homeomorphic to $\boldsymbol{P}(E)$ through φ. Then the following statements* (1), (2), (3), (4), *and* (5) *are equivalent. Moreover if ω is an open subset of $\boldsymbol{P}(E)$, the statements* (1), (2), (3), (4), (5), *and* (6) *are equivalent.*

(1) *ω is pseudoconvex.*

(2) *For any finite dimensional subspace F of E, $\varphi^{-1}(\boldsymbol{P}(E))$ is a Stein manifold.*

(3) *ω is a domain of holomorphy.*

(4) *ω is a domain of holomorphy and holomorphically separated.*

(5) *ω is a domin of existence.*

(6) *There exists a non-constant holomorphic function f on ω such that, for every connected open neighborhood V of an arbitrary point on the boundary of ω,*

each component of $\omega \cap V$ contains zero of f of arbitrarily high order.

Theorem 2. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space, and p be a plurisubharmonic function in E which is positively homogeneous of order 1. Then there exists an entire function f of exponential type on E such that*

$$p(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |f(tz')|$$

for every $z \in E$.

Corollary 3. *If E is a nuclear Fréchet space with the bounded approximation property or a DFN-space, there exists an analytic functional μ on the strong dual space E' of the space E such that*

$$p(z) = \limsup_{z' \to z} \limsup_{t \to \infty} \frac{1}{t} \log |\mu(\exp(tz'))|$$

for every $z \in E$.

The proof of Theorem 2 is based on the characterization of pseudoconvex domains of the projective space $\boldsymbol{P}(E)$ in Theorem 1. This method of the proof was first given by Kiselman[18] in case $E = \boldsymbol{C}^n$.

The Levi problem was first solved by Oka[35] in $\boldsymbol{C}^2$. Moreover Oka[36] extended his result to Riemann domains over $\boldsymbol{C}^n$. At the same time, Bremermann[3] and Norguet[32] solved this problem in $\boldsymbol{C}^n$. The Levi problem in infinite dimensional spaces is also the important object of study in infinite dimensional complex analysis, and has been solved affirmatively in various infinite dimensional spaces(cf.Aurich[1], Colomeau and Mujica[5], Dineen[6], [8, Appendix 1], Dineen, Noverraz and Schottenloher[9], Gruman[11], Gruman and Kiselman[12], Hervier[14], Hirchowitz[15],

Matos[23], Mujica[25], [27], Noverraz[33], [34], Pomes[39], Popa[40], Schottenloher[41]). Josefson[16] gave an example of a non-separable Banach space in which the Levi problem is negative. Fujita[10], Kiselman[18] and Takeuchi[42] extended the result of the Levi problem in Riemann domains over $\boldsymbol{C}^n$ to those over the complex projective space $\boldsymbol{P}(\boldsymbol{C}^{n+1})$ of dimension n. Kajiwara[17] and Nishihara[31] investigated the Levi problem in Riemann domains over infinite dimensional projective spaces. In case E is a topological vector space with the finite open topology, Kajiwara[17] solved affirmatively the Levi problem in the projective space $\boldsymbol{P}$ (E). In case E is a Banach space with a Schauder basis, Nishihara[31] solved affirmatively this problem in Riemann domains over the projective spaces $\boldsymbol{P}(E)$. Therefore Theorem 1 is the extension of Nishihara[31].

2 Notations and preliminaries.

Let E be a locally convex spaces and $cs(E)$ be the set of all nontrivial continuous seminorms on E.

A Hausdorff space M is called a *complex manifold* modeled on the space E if there exists a family $\mathcal{F} = \{(U_i, \varphi_i)\ ;\ i \in I\}$ of pairs (U_i, φ_i) of open sets U_i of M and homeomorphisms φ_i of open sets U_i onto open sets of E satisfying the following conditions.

(1) For any $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the mappings $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ between open sets in E are holomorphic.

(2) $\bigcup_{i \in I} U_i = M$.

$\mathcal{F}$ is called the *atlas* of M, and an element of $\mathcal{F}$ is called a *chart* of M.

Let M and N be complex manifolds with atlases $\{(U_i, \varphi_i);\ i \in I\}$ and $\{(U'_\alpha, \varphi'_\alpha);\ \alpha \in A\}$ respectively. Then a mapping $f : M \to N$ is said to be *holomorphic* if, for any

$i \in I$ and any $\alpha \in A$ with $f(U_i) \cap U'_\alpha \neq \emptyset$, the mapping $\varphi'_\alpha \circ f \circ \varphi_i^{-1}$ is holomorphic when it is defined. Particularly, if $N = \boldsymbol{C}$, f is called a *holomorphic function* on M. We denote by $H(M)$ the vector space of all holomorphic functions in M. A function $p : M \to [-\infty, \infty)$ is said to be plurisubharmonic if for each $i \in I$, the function $f \circ \varphi_i^{-1}$ is plurisubharmonic. We denote by $ps(M)$ the set of all plurisubharmonic functions on M. We can define a *submanifold* of the complex manifold M, a *product manifold* of M and another complex manifold, a *holomorphic fibre bundle* over M, a *holomorphic principal bundle* over M and a *holomorphic vector bundle* over M by the same way as in case the dimension of M is finite. If there exists a local biholomorphic mapping φ of a complex manifold ω into the complex manifold M, (ω, φ) is called a *Riemann domain* over M. A *section* of ω is a continuous mapping $\sigma : A \to \omega$, with $A \subset M$, such that $\varphi \circ \sigma = \mathrm{id}$ on A.

Let (ω, φ) and (ω', φ') be a Riemann domain over a complex manifold M. If a holomorphic mapping λ of ω into ω' satisfies $\varphi = \varphi' \circ \lambda$, the mapping λ is called a *morphism* of (ω, φ) into (ω', φ'). Let (ω, φ) be a Riemann domain over M, and let $\mathcal{F} \subset H(\omega)$. If (ω', φ') is a Riemann domain over M, then a morphism λ of (ω, φ) into (ω', φ') is said to be an *$\mathcal{F}$-extension* of ω if for each $f \in \mathcal{F}$ there exists a unique $f' \in H(\omega')$ such that $f' \circ \lambda = f$. A morphism λ of (ω, φ) into (ω', φ') is said to be a *holomorphic extension* of ω if λ is an *$H(\omega)$-extension* of ω. ω is said to be an *$\mathcal{F}$-domain of holomorphy* if each $\mathcal{F}$-extension of ω is an isomorphism. ω is said to be a *domain of holomorphy* if ω is an $H(\omega)$-domain of holomorphy. ω is said to be a *domain of existence* if there exists $f \in H(\omega)$ such that ω is an $\{f\}$-domain of holomorphy. Let (ω, φ) be a Riemann domain over the complex manifold M and let $\mathcal{F} \subset H(\omega)$. A morphism $\lambda : \omega \to \omega'$ is called an *$\mathcal{F}$-envelope of holomorphy* of ω if

:

(a) λ is $\mathcal{F}$-extension of ω.

(b) If $\mu : \omega \to \omega''$ is an $\mathcal{F}$-extension of ω, then there exists a morphism $\nu : \omega'' \to \omega'$ such that $\nu \circ \mu = \lambda$.

By the same way as Mujica[27, Theorem 56.4] we can prove the following theorem.

Theorem 2.1. *Let (ω, φ) be a Riemann domain over a complex manifold M and let $\mathcal{F} \subset H(\omega)$. Then there exists the $\mathcal{F}$-envelope of holomorphy of ω and the existence of it is unique up to isomorphism.*

For $\mathcal{F} \subset H(\omega)$, we denote by $E_{\mathcal{F}}(\omega)$ the $\mathcal{F}$-envelope of holomorphy of a Riemann domain ω. Then we can prove the following proposition.

Proposition 2.2. *Let (ω, φ) be a Riemann domain over a complex manifold and $\mathcal{F} \subset H(\omega)$.*

(1) *Let $\lambda : \omega \to \omega'$ be an $\mathcal{F}$-extension of ω. Then $\omega' = E_{\mathcal{F}}(\omega)$ if and only if ω' is an $\mathcal{F}$-domain of holomorphy.*

(2) *$\omega = E_{\mathcal{F}}(\omega)$ if and only if ω is an $\mathcal{F}$-domain of holomorphy.*

Let M be a complex manifold and S be a subset of M. For a complex valued function f and for a real valued function R on M, we write

$$|f|_S = \sup\{\, |f(x)| \; ; \; x \in S \,\}, R(S) = \inf\{\, R(x) \; ; \; x \in S \,\}.$$

For $\mathcal{F} \subset H(M)$ we write

$$\hat{S}_{\mathcal{F}} = \{\, y \in M \; ; \; |f(y)| \leq |f|_S \text{ for all } f \in \mathcal{F}\}.$$

Likewise, for $S \subset M$ and $\mathcal{F} \subset Ps(M)$ we set

$$\hat{S}_{\mathcal{F}} = \{\, y \in M \; ; \; f(y) \leq \sup_{x \in S} f(x), \text{ for all } f \in \mathcal{F}\}.$$

Let (ω, φ) be a Riemann domain over M. Let $\mathcal{F} \subset H(\omega)$, ω is said to be *$\mathcal{F}$-separated* if for each pair (x, y) of points of ω satisfying $x \neq y$ there exists a holomorphic function $h \in \mathcal{F}$ such that $h(x) \neq h(y)$. ω is said to be *holomorphically separated* if ω is $H(\omega)$-separated. ω is said to be *$\mathcal{F}$-fibre separated* if for each pair (x, y) of points of ω, satisfying $x \neq y$ and $\varphi(x) = \varphi(y)$, there exists a holomorphic function $h \in \mathcal{F}$ such that $h(x) \neq h(y)$.

We shall collected some properties of Riemann domains over a locally convex space, for which we have use afterwards.

Let E be a locally convex space and (Ω, Φ) be a Riemann domain over E. For $S \subset \Omega$ and for a convex balanced neighborhood V of 0 in E we write $S + V \subset \Omega$ if for each $x \in S$ there exists a section $\sigma : \Phi(x) + V \to \Omega$ such that $\sigma \circ \Phi(x) = x$.

We define the distance functions $d^{\alpha}_{\Omega} : \Omega \to [0, +\infty]$, for $\alpha \in cs(E)$, and $\delta_{\omega} : \Omega \times E \to (0, +\infty]$, as follows:

$$d^{\alpha}_{\Omega}(x) = \sup(\{\ r > 0\ ;\ \text{there is a section}\ \ \sigma : B^{\alpha}_{E}(\Phi(x), r) \to \Omega \\ \text{with}\ \sigma \circ \Phi(x) = x\} \cup \{0\})$$

and

$$\delta_{\Omega}(x, a) \ = \ \sup\{\ r > 0\ ;\ \text{there is a section}\ \sigma : D_{E}(\Phi(x), a, r) \to \Omega \\ \text{with}\ \sigma \circ \Phi(x) = x\}$$

where for $\xi, a \in E$ and $r > 0$ we write

$$B^{\alpha}_{E}(\xi, r) = \{\xi + b\ ;\ b \in E,\ \alpha(b) < r\},$$

$$D_{E}(\xi, a, r) = \{\xi + \lambda a\ ;\ \lambda \in \boldsymbol{C},\ |\lambda| < r\}.$$

If $d^{\alpha}_{\Omega}(x) > 0$ then for each $r \in (0, d^{\alpha}_{\Omega}(x)]$ there is a unique set $B^{\alpha}_{\Omega}(x, r) \subset \Omega$ containing x such that $\Phi : B^{\alpha}_{\Omega}(x, r) \to B^{\alpha}_{E}(\Phi(x), r)$ is a bijection. Likewise, for

each $x \in \Omega$, $a \in E$ and $r \in (0, \delta_\Omega(x,a)]$ there is a unique set $D_\Omega(x,a,r) \subset \Omega$ containing x such that $\varphi : D_\Omega(x,a,r) \to D_E(\varphi(x),a,r)$ is bijection. The function d^α_Ω is continuous, and the function δ_Ω is lower semicontinuous. The domain Ω is said to be *pseudoconvex* if the function $-\log \delta_\Omega$ is plurisubharmonic on $\Omega \times E$. The following proposition is on Noverraz[34].

Proposition 2.3. *For a Riemann domain (Ω, Φ) over a locally convex space E, the following conditions are equivalent.*

(a) *Ω is pseudoconvex.*

(b) *$d^\alpha_\Omega(\hat{X}_{Ps(\Omega)}) = d^\alpha_\Omega(X)$ for every $X \subset \Omega$ and $\alpha \in cs(E)$.*

(c) *For each compact set K of Ω there exists $\alpha \in cs(E)$ such that $d^\alpha_\Omega(\hat{K}_{Ps(\Omega)}) > 0$.*

(d) *$\Phi^{-1}(F)$ is a Stein Manifold for each finite dimensional linear subspace F of E.*

Let E be a Fréchet space. A sequence (e_n) in the Fréchet space E is said to be a *Schauder basis* if every $x \in E$ admits a unique representation as a series $x = \sum_{n=1}^\infty \xi_n(x)\, e_n$ where the series converges in the ordinary sense for the topology of E. Let E_n be the linear span of the set $\{e_1, e_2, \ldots, e_n\}$ and let $T_n : E \to E_n$ be the canonical projection. Then it follows from the open mapping theorem that the sequence (T_n) is equicontinuous and converges to the identity uniformly on compact sets, and that the space E has a fundamental sequence of continuous seminorms α_j which satisfy the conditions $\alpha_j = \sup_n \alpha_j \circ T_n$.

3 Riemann domains with C^*-action.

In this section we investigate properties of Riemann domains with $\boldsymbol{C}^*$-action over locally convex spaces. Results in this section are usefull to investigate some properties of Riemann domains over projective spaces.

A Riemann domain (Ω, Φ) over a locally convex space E is said to be *with* $\boldsymbol{C}^*$-*action* if (Ω, Φ) satisfies the following conditions.

(1) $\boldsymbol{C}^*$ acts freely on Ω on the left : $(\lambda, x) \in \boldsymbol{C}^* \times \Omega \to \lambda \cdot x \in \Omega$.

(2) The action $(\lambda, x) \in \boldsymbol{C}^* \times \Omega \to \lambda \cdot x \in \Omega$ is holomorphic.

(3) $\Phi(\lambda \cdot x) = \lambda \Phi(x)$ for every $(\lambda, x) \in \boldsymbol{C}^* \times \Omega$.

Let S be a subset of Ω or E . We set

$$\lambda \cdot S = \{ \lambda \cdot x \ ; \ x \in S \}. \tag{3.1}$$

Then we can prove the following lemma.

Lemma 3.1. *Let E be a locally convex space and (Ω, Φ) be a Riemann domain with $\boldsymbol{C}^*$-action over E. Then we have*

$$d_\Omega^\alpha(\lambda \cdot x) = |\lambda| d_\Omega^\alpha(x), \tag{3.2}$$

$$\delta_\Omega(\lambda \cdot x, a) = |\lambda| \delta_\Omega(x, a) = \delta_\Omega(x, \lambda^{-1} \cdot a) \tag{3.3}$$

for any $(\lambda, x) \in \boldsymbol{C}^* \times \Omega$, $a \in E$ and $\alpha \in cs(E)$.

Proof. We shall show first the equality (3.3). Since the second equality of (3.3) is trivial, we shall show only the first equality of (3.3). Let (x, a) be a point of $\Omega \times E$. Let r be a real number with $0 < r < \delta_\Omega(x, a)$. There exists a section $\sigma : D_E(\Phi(x), a, r) \to \Omega$ with $\sigma \circ \Phi(x) = x$. For each $\lambda \in \boldsymbol{C}^*$, a mapping $z \in \sigma(D_E(\Phi(x), a, r)) \to \lambda \cdot z \in \Omega$ is a biholomorphic mapping of $\sigma(D_E(\Phi(x), a, r))$ onto $\lambda \cdot \sigma(D_E(\Phi(x), a, r))$. Since $\lambda \cdot D_E(\Phi(x), a, r) = D_E(\Phi(\lambda \cdot x), a, |\lambda| \ r)$ and $\Phi(\lambda \cdot \sigma(D_E(\Phi(x), a, r)) = \lambda \cdot D_E(\Phi(x), a, r)$, a mapping $\xi \in D_E(\Phi(\lambda \cdot x), a, |\lambda| \ r) \to \lambda \cdot \sigma(\lambda^{-1} \xi)$ is a section of Ω and satisfies $\lambda \cdot \sigma(\lambda^{-1} \xi) = \lambda \cdot x$ if $\xi = \Phi(\lambda \cdot x)$. Therefore we have

$$(3.4) \qquad \delta_\Omega(\lambda \cdot x, a) \geq |\lambda| \, \delta_\Omega(x, a).$$

Since x, a and λ are given arbitrarily, by (3.4) we have

$$\begin{aligned} \delta_\Omega(\lambda \cdot x, a) &= |\lambda| \, |\lambda^{-1}| \, \delta_\Omega(\lambda \cdot x, a) \\ &\leq |\lambda| \, \delta_\Omega(\lambda^{-1} \, \lambda \cdot x, a) \\ &= |\lambda| \, \delta_\Omega(x, a). \end{aligned}$$

Thus we obtain the equality (3.3).

The equality (3.2) is obtained from (3.3) and from the equality $d^\alpha_\Omega(x) = \inf\{ \, \delta_\Omega(x, a) \, ; \, \alpha(a) = 1 \, \}$. This completes the proof.

Let E be a Fréchet space with a Schauder basis (e_n) and (Ω, Φ) be a pseudoconvex Riemann domain with $\boldsymbol{C}^*$-action over E.

If U is any open subset of Ω, then we consider the functions $\eta^n_U(x) : U \to [0, +\infty]$ definded by

$$(3.5) \qquad \eta^n_U(x) = \inf_{k \geq n} \, \delta_U(x, T_k \circ \Phi(x) - \Phi(x)).$$

These functions were introduced by Schottenloher[41], who proved that they are strictly positive and lower semicontinuous on U. Thus the functions $-\log \, \eta^n_U$ are plurisubharmonic on U whenever U is pseudoconvex. We set

$$A_n = \{ \, x \in \Omega \, ; \, \eta^n_\Omega(x) > 1 \},$$

$$\tau_n(x) = (\Phi | D_x)^{-1} \circ T_n \circ \Phi(x) \qquad (x \in A_n),$$

where $D_x = D_\Omega(x, T_n \circ \Phi(x) - \Phi(x), \eta^n_\Omega(x))$. Then the following lemma can be verified.

Lemma 3.2. *We set $\Omega_n = \Phi^{-1}(E_n)$ for every n. Then there exist a sequence of open sets $A_n \subset \Omega$ and a sequence of holomorphic mappings $\tau_n : A_n \to \Omega_n$ with the following properties :*

(a) $\Omega = \cup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$ *and* $\Omega_n \subset A_n$ *for every* n.

(b) $\tau_n = id$ *on* Ω_n, $\Phi \circ \tau_n = T_n \circ \Phi$ *on* A_n *and* $\tau_n \circ \tau_{n+1} = \tau_{n+1} \circ \tau_n = \tau_n$ *on* A_n *for every* n.

(c) *For each compact subset* K *of* Ω *and a balanced open neighborhood* V *of* 0 *in* E *with* $K + V \subset \Omega$ *there exists a positive integer* n *such that* $K \subset A_n$ *and* $\tau_k(x) \in x + V$ *for every* $x \in K$ *and* $k \geq n$.

(d) $\hat{K}_{Ps(\Omega)} \subset A_n$ *for every compact subset* K *of* A_n.

(e) $\lambda \cdot A_n = A_n$ *for every* $\lambda \in \boldsymbol{C}^*$ *with* $|\lambda| = 1$.

Proof. The proof of statement (a), (b), (c), and (d) is in Schottenloher[41]. The statement (e) follows from (3.3) and (3.5).

Lemma 3.3. *Assume that the Fréchet space* E *has a continuous norm and that* Ω *is connected. Let* (α_n) *be a fundamental sequence of continuous norms on* E *with* $\alpha_{n+1} \geq 2\alpha_n$ *and* $\alpha_n = \sup_k \alpha_n \circ T_k$ *for every* n. *Let* (A_n) *and* (τ_n) *be two sequences satisfying the conditions in Lemma* 3.2. *Then there are two sequences of open sets* $C_n \subset B_n \subset A_n$ *and a sequence* (V_n) *of balanced convex open neighborhoods of* 0 *in* E *with the following properties:*

(a) $\Omega = \cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} C_n$, $B_n \subset B_{n+1}$ *and* $C_n + V_n \subset C_{n+1}$ *for every* n.

(b) $B_n \cap \Omega_k \subset\subset A_n \cap \Omega_k$ *for every* n *and* k.

(c) $\tau_k(C_n) \subset B_n \cap \Omega_k$ *whenever* $k > n$.

(d) *The set* $(B_n \cap \Omega_k)^{\wedge}_{H(\Omega_k)}$ *is relatively compact in* $A_n \cap \Omega_k$ *for every* n *and* k.

(e) $d_{\Omega}^{\alpha_n}(B_n) \geq 1$ *for every* n.

(f) $\sup \{\alpha_1(\Phi(x)) \ ; \ x \in B_n\} \leq n$ *for every* n.

(g) $\lambda \cdot B_n = B_n$, $\lambda \cdot C_n = C_n$ *for every* $\lambda \in \boldsymbol{C}^*$ *with* $|\lambda| = 1$.

Proof. By Mujica[26, Lemma 2.6] and by an examination of the proof of Mujica[26, Lemma 2.6] there exist two sequences of open sets $C_n \subset B_n \subset A_n$

and a sequence V_n of balanced convex open neighborhoods of 0 in E satisfying the statements (a), (b), (c), (d), (e) and (f). We replace newly two sets $\cup_{|\lambda|=1}\lambda \cdot B_n$ and $\cup_{|\lambda|=1}\lambda \cdot C_n$ by B_n and C_n respectively for each n. Then the two sets B_n and C_n are open sets of Ω and satisfy the required conditions. This completes the proof.

A holomorphic function f in Ω is said to be $\boldsymbol{C}^*$ *-invariant* if

$$f(\lambda \cdot x) = f(x)$$

for every $(\lambda, x) \in \boldsymbol{C}^* \times \Omega$. For each $f \in H(\Omega)$, we set

$$\tilde{f}(x) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} \cdot x)\, d\theta \tag{3.6}$$

for every $x \in \Omega$. Then $\tilde{f}$ is holomorphic on Ω. For each x, a function $\lambda \to \tilde{f}(\lambda \cdot x)$ $(\lambda \in \boldsymbol{C}^*)$ is holomorphic and constant on $|\lambda| = 1$. Therefore by the identity theorem, the function $\lambda \to \tilde{f}(\lambda \cdot x)$ $(\lambda \in \boldsymbol{C}^*)$ is constant. Thus the function $\tilde{f}$ is $\boldsymbol{C}^*$-invariant.

Lemma 3.4. *With the conditions and the notation of Lemma* 3.3, *for each* $\boldsymbol{C}^*$*-invariant function* $f_n \in H(\Omega)$ *and for each* $\varepsilon > 0$ *there exists* $\boldsymbol{C}^*$*-invariant function* $f \in H(\Omega)$ *such that*

(a) $f = f_n$ *on* Ω_n,

(b) $|f - f_n \circ \tau_n|_{C_n} \leq \varepsilon$

(c) $|f|_{C_j} < +\infty$ *for every* j.

Proof. By Mujica[23, Lemma 2.7], there exists a holomorphic function f satisfying the conditions (a), (b), and (c). Then the function f may not be $\boldsymbol{C}^*$-invariant. Let $\tilde{f}$ be a $\boldsymbol{C}^*$-invariant holomorphic function on Ω_n defined by (3.6). Then it is easy to show that the function $\tilde{f}$ satisfies the conditions (a), (b), and (c). This completes the proof.

The following lemma is on Dineen[7]

Lemma 3.5. *Let E be a Fréchet space with a Schauder basis. Let α be a continuous seminorm on E satisfying the condition*

$$\alpha(x) = \sup_{m \geq 1} \alpha\left(\sum_{n=1}^{m} \xi_n(x)\, e_n\right) \tag{3.7}$$

for every $x \in E$. If we set

$$Z^\alpha = \{\, n \in \boldsymbol{N} \;;\; \alpha(e_n) = 0 \,\},$$

$$E^\alpha = \{\, x \in E \;;\; \xi_n(x) = 0 \text{ for every } n \in Z^\alpha \}, \tag{3.8}$$

then E^α has a Schauder basis and a continuous norm, and E is the topological direct sum of E^α and $\alpha^{-1}(0)$.

Let (Ω, Φ) be a Riemann domain with $\boldsymbol{C}^*$-action over E and A be a subset of Ω or of E. Then we set

$$\boldsymbol{C}^* \cdot A = \{\, \xi \cdot x \;;\; \xi \in \boldsymbol{C}^*,\; x \in A \}.$$

Lemma 3.6. *Let (Ω, Φ) be a connected pseudoconvex Riemann domain with $\boldsymbol{C}^*$-action over a Fréchet space E which has a Schauder basis. Let $x_0 \in \Omega$ and let α be a continuous seminorm on E satisfying the condition (3.7) in Lemma 3.5 and $d^\alpha_\Omega(x_0) > 0$. Let $\pi_\alpha : E \to E^\alpha$ be the canonical projection and $\Omega^\alpha = \Phi^{-1}(E^\alpha)$. Then we have the following:*

(a) *There is a holomorphic mapping $\sigma_\alpha : \Omega \to \Omega^\alpha$ such that $\sigma_\alpha = id$ on Ω^α, $\Phi \circ \sigma_\alpha = \pi_\alpha \circ \Phi$ on Ω and $\sigma_\alpha(\lambda \cdot x) = \lambda \cdot \sigma_\alpha(x)$ for every $(\lambda, x) \in \boldsymbol{C}^* \times \Omega$.*

(b) *Let U be any connected pseudoconvex open subset of Ω such that $d^\alpha_U(y_0) > 0$ for some $y_0 \in U$. Then $U = \sigma_\alpha^{-1}(U \cap \Omega^\alpha)$ and $f \circ \sigma_\alpha = f$ on U for every $f \in H(U)$ which is bounded on an α-neghborhood of y_0.*

(c) *For x, $y \in \Omega$ we have $x = y$ if and only if $\Phi(x) = \Phi(y)$ and $\sigma_\alpha(x) = \sigma_\alpha(y)$.*

(d) *For each $a \in E$ and $t \in \Omega^\alpha$ with $\pi_\alpha(a) = \Phi(t)$ there is a unique $x \in \Omega$ such that $\Phi(x) = a$ and $\sigma_\alpha(x) = t$.*

(e) *A net (x_i) in Ω converges in Ω if and only if $(\Phi(x_i))$ converges in E and $(\sigma_\alpha(x_i))$ converges in Ω^α.*

Proof. Any other things except for the equality $\sigma_\alpha(\lambda \cdot x) = \lambda \cdot \sigma_\alpha(x)$ for every $(\lambda, x) \in \mathrm{C}^* \times \Omega$ were proved in Mujica[26, Lemma 3.2]. Therefore we shall show only this equality. Let (λ, x) be any element of $\boldsymbol{C}^* \times \Omega$. We set $z = \sigma_\alpha(x)$ and $w = \sigma_\alpha(\lambda \cdot x)$, and then we have only to show the equality $\lambda \cdot z = w$. We remark $\Phi(\lambda \cdot z) = \Phi(w)$. Since $\Phi(\xi \cdot z) = \xi\Phi(z)$ for every $\xi \in \boldsymbol{C}^*$, a mapping σ of $\boldsymbol{C}^* \cdot \{\Phi(z)\}$ into Ω defined by $\sigma(\xi\Phi(z)) = \sigma_\alpha(\xi \cdot x)$ for every $\xi \in \boldsymbol{C}^*$ and a mapping $\sigma' : \xi \cdot \Phi(z) \in \boldsymbol{C}^* \cdot \{\Phi(z)\} \to \xi \cdot z$ are sections of Ω with $\sigma \circ \Phi(z) = z$ and $\sigma' \circ \Phi(z) = z$ respectively. Therefore it follows from the uniqueness of existence of a section of Ω that $\xi \cdot z = \sigma_\alpha(\xi \cdot x)$ for every $\xi \in \boldsymbol{C}^*$. Especially we have $\lambda \cdot z = \sigma_\alpha(\lambda \cdot x) = w$. This completes the proof.

We define a holomorphic mapping $\tilde{\Phi}$ of the product manifold $\Omega^\alpha \times \alpha^{-1}(0)$ of Ω^α and $\alpha^{-1}(0)$ into E by $\tilde{\Phi}(x, \xi) = \Phi(x) + \xi$ for every $(x, \xi) \in \Omega^\alpha \times \alpha^{-1}(0)$. Then $(\Omega^\alpha \times \alpha^{-1}(0), \tilde{\Phi})$ is a Riemann domain over E. Moreover $(\Omega^\alpha \times \alpha^{-1}(0), \tilde{\Phi})$ is with $\boldsymbol{C}^*$-action :

$$\boldsymbol{C}^* \times (\Omega^\alpha \times \alpha^{-1}(0)) \in (\lambda, (x, \xi)) \to (\lambda \cdot x, \lambda \cdot \xi) \in \Omega^\alpha \times \alpha^{-1}(0).$$

We define a morphism μ of Ω into $\Omega^\alpha \times \alpha^{-1}(0)$ by $\mu(x) = (\sigma_\alpha(x), \Phi(x) - \pi_\alpha \cdot \Phi(x))$ for every $x \in \Omega$. Then we have $\mu(\lambda \cdot x) = \lambda \cdot \mu(x)$ for every $(\lambda, x) \in \boldsymbol{C}^* \times \Omega$ and μ is an isomorphism. Thus we have the following Lemma 3.7.

Lemma 3.7. *Let (Ω, Φ) be a connected pseudoconvex Riemann domain with*

$\boldsymbol{C}^*$-action over a Fréchet space E which has a Schauder basis. With the conditions and the notations of Lemma 3.6, (Ω, Φ) is identified with the Riemann domain $(\Omega^\alpha \times \alpha^{-1}(0), \tilde{\Phi})$ with $\boldsymbol{C}^*$-action over $E = E^\alpha \bigoplus \alpha^{-1}(0)$.

4 Riemann domains over projective spaces.

Let E be a locally convex space. Let z and z' be points in $E - \{0\}$. z and z' are said to be *equivalent* if there exists $\lambda \in \boldsymbol{C}^*$ such that $z' = \lambda z$. We denote by $\boldsymbol{P}(E)$ the quotient space of $E - \{0\}$ by this equivalent relation. Then $\boldsymbol{P}(E)$ is a Hausdorff space. The Hausdorff space $\boldsymbol{P}(E)$ is called the *complex projective space induced from E*. We denote by q the quotient map of $E - \{0\}$ onto $\boldsymbol{P}(E)$. For any $\xi \in E - \{0\}$, we denote by $[\xi]$ the equivalent class of ξ (i.e. $q(\xi) = [\xi]$). Let E' be the complex vector space of all continuous linear functional on E. We set

$$\mathcal{S}_E = \{ (f, a) \in E' \times E \ ; \ f(a) \neq 0 \}. \tag{4.1}$$

For each $f \in E' - \{0\}$, we define a hyperplane $E(f)$ of E and an open subset $U(f)$ of $\boldsymbol{P}(E)$ by

$$E(f) = \{ \xi \in E \ ; \ f(\xi) = 0 \}, \tag{4.2}$$

$$U(f) = \{ [\xi] \in \boldsymbol{P}(E) \ ; \ f(\xi) \neq 0 \} \tag{4.3}$$

respectively. For every $(f, a) \in \mathcal{S}_E$, we define a homeomorphism $\varphi_{(f,a)}$ of $U(f)$ onto $E(f)$ by

$$\varphi_{(f,a)}([\xi]) = \frac{1}{f(\xi)}\xi - \frac{1}{f(a)}a \tag{4.4}$$

for every $[\xi] \in U(f)$. Then the family $\{U(f), \varphi_{(f,a)}\}_{(f,a)\in S_E}$ defines the complex structure of the projective space $\boldsymbol{P}(E)$.

Let (ω, φ) be a Riemann domain over the complex projective space $\boldsymbol{P}(E)$ induced from E. We consider the fibre product Ω of ω and $E - \{0\}$ defined by

$$\Omega = \{ (z, w) \in \omega \times (E - \{0\}) \ ; \ \varphi(z) = q(w) \}. \tag{4.5}$$

We denote by Φ and Q projections of the fibre product Ω into $E - \{0\}$ and ω respectively. Then (Ω, Φ) is a Riemann domain over E. For each $(z, w) \in \Omega$ and for each $\lambda \in \boldsymbol{C}^*$, we set

$$\lambda \cdot (z, w) = (z, \lambda w). \tag{4.6}$$

Then points $\lambda \cdot (z, w)$ of $\omega \times (E - \{0\})$ belongs to Ω for all $(z, w) \in \Omega$ and all $\lambda \in \boldsymbol{C}^*$. The mapping $(\lambda, x) \in \boldsymbol{C}^* \times \Omega \to \lambda \cdot x$ is holomorphic. By this action, (Ω, Φ) is the Riemann domain with $\boldsymbol{C}^*$-action over E. The Riemann domain (Ω, Φ) with $\boldsymbol{C}^*$-action over E is called the *Riemann domain associated with* the Riemann domain (ω, φ). The Riemann domain ω is the quotient space of Ω by this $\boldsymbol{C}^*$-action and Q is the quotient map of Ω onto ω. Ω is the total space of a holomorphic principal bundle over ω with the complex multiplicative group $\boldsymbol{C}^*$. We have the following commutative diagram:

$$\begin{tikzcd} \Omega \arrow[r, "Q"] \arrow[d, "\Phi"'] & \omega \arrow[d, "\varphi"] \\ E - \{0\} \arrow[r, "q"'] & \boldsymbol{P}(E). \end{tikzcd} \tag{4.7}$$

Let E be a locally convex space and (ω, φ) be a Riemann domain over the projective space $\boldsymbol{P}(E)$. Then the Riemann domain ω is said to be *pseudoconvex* if for each $(f, a) \in \boldsymbol{S}_E$ the Riemann domain $\left(\varphi^{-1}(U(f)), \varphi_{(f,a)} \circ \varphi|\varphi^{-1}(U(f))\right)$ over $E(f)$ is pseudoconvex.

Let F be a closed linear subspace of E. We set

$$\Omega_F = \Phi^{-1}(F), \tag{4.8}$$

$$\omega_F = \varphi^{-1}(\boldsymbol{P}(F)). \tag{4.9}$$

Ω_F is a holomorphic principal bundle over ω_F with the complex multiplicative group $\boldsymbol{C}^*$.

Let (Ω, Φ) be a Riemann domain over a locally convex space E. Let a and b be points of Ω. By a *line segment* $[a, b]$ in Ω we mean a set in Ω containing the points a and b and homeomorphic under Φ to the line segment $[\Phi(a), \Phi(b)]$ in E. By a *polygonal line* $[x_0, x_1, \ldots, x_n]$ in Ω we mean a finite union of line segments of the form $[x_{j-1}, x_j]$ with $j = 1, \ldots, n$.

Remark 4.1. *Let x and y be two points which belong to a connected component of Ω. Since there exists a polygonal line $[x_0, x_1, \ldots, x_n]$ with $x_0 = x$ and with $x_n = y$, there exists a finite dimensional linear subspace F of E such that the set $\{x, y\}$ is contained in a connected component of the set $\Phi^{-1}(F)$.*

Lemma 4.2. *Let E be a locally convex space and (ω, φ) be a Riemann domain over the complex projective space $\boldsymbol{P}(E)$. Assume that ω is not homeomorphic to $\boldsymbol{P}(E)$ through φ. Then for any finite dimensional linear subspace F of E and for any connected component V_F of ω_F, there exists a finite dimensional linear subspace G of E and a connected component V_G of $\omega_G = \varphi^{-1}(\boldsymbol{P}(G))$ satisfying the following conditions.*

(1) *V_F is a closed complex submanifold of V_G.*

(2) *V_G is not homeomorphic to $\boldsymbol{P}(G)$ through φ.*

Proof. By Remark 4.1 and by the commutative diagram (4.7), there exist a finite dimensional linear subspace F_0 of E and a connected component V_{F_0} of ω_{F_0}

such that V_{F_0} is not homeomorphic to $\boldsymbol{P}(F_0)$ through φ. We take a point z of V_F and a point w of V_{F_0}. By Remark 4.1 and by the commutative diagram (4.7), there exist a finite dimensional subspace F_1 and a connected component V_{F_1} of ω_{F_1} such that V_{F_1} contains the set $\{z, w\}$. Let G be the complex vector space spanned by all elements of the union $F \cup F_0 \cup F_1$. Then both $\boldsymbol{P}(F)$ and $\boldsymbol{P}(F_0)$ are closed complex submanifolds of $\boldsymbol{P}(G)$. We denote by V_G the connected component of ω_G containing the set $\{z, w\}$. Since $(V_G, \varphi|V_G)$ is a Riemann domain over $\boldsymbol{P}(G)$, both V_F and V_{F_0} are closed comlex submanifolds of V_G. Then V_G satisfies the required conditions (1) and (2). This completes the proof.

Lemma 4.3. *In adition to the assumption of Lemma* 4.2, *we assume that* ω *is pseudoconvex. Then, for any finite dimensional linear subspace* F *of* E, ω_F *is a Stein manifold. Moreover* Ω *is pseudoconvex.*

Proof. Let F be a finite dimensional linear subspace of E. Let V_F be any component of ω_F. By Lemma 4.2 there exists a finite dimensional linear subspace G of E and a component V_G of ω_G satisfying the conditions (1) and (2) in Lemma 4.2. Since ω is pseudoconvex, V_G is also pseudoconvex. By Fujita[10], Kiselman[18] and Takeuchi[42], the pseudoconvex Riemann domain V_G over the projective space $\boldsymbol{P}$(G) is a Stein manifold. Since V_F is a closed complex submanifold of the Stein manifold V_G, V_F is also a Stein manifold. Therefore ω_F is a Stein manifold.

For every finite dimensional linear subspace F of E, Ω_F is the total space of a holomorphic principal bundle over the Stein manifold ω_F with the complex multiplicative group $\boldsymbol{C}^*$. Therefore by Matsushima and Morimoto[24] Ω_F is a Stein manifold. Thus it follows from Proposition 2.3 that Ω is pseudoconvex. This completes the proof.

Proposition 4.4. *With the assumption of Lemma* 4.2 *the following statements*

are equivalent.

(1) *ω is pseudoconvex.*

(2) *ω_F is a Stein manifold for every finite dimensional linear subspace F of E.*

(3) *Ω is pseudoconvex.*

Proof. It follows from Lemma 4.3 that (1) implies (2). An examination of the proof of Lemma 4.3 shows that (2) implies (3)

We shall show that (3) implies (1). For any $(f, a) \in \mathcal{S}_E$ in (4.1), we have only to prove that the Riemann domain $(\varphi^{-1}(U(f)), \varphi_{(f,a)} \circ \varphi|\varphi^{-1}(U(f)))$ over the vector space $E(f)$ is pseudoconvex. Let L be a finite dimensional linear subspace of $E(f)$. Then by Proposition 2.3 we have only to show that $\varphi^{-1} \circ \varphi^{-1}_{(f,a)}(L)$ is a Stein manifold. We set $F = L \bigoplus \langle a \rangle$ where $\langle a \rangle$ is the linear span of the set $\{a\}$. Since $f = 0$ on L, $\varphi^{-1}_{(f,a)}(L) = q(L + a)$. By the assumption Ω_F is a Stein manifold. Since Ω_F is the total space of a holomorphic principal bundle over the complex manifold ω_F with the complex multiplicative group $\boldsymbol{C}^*$ and since $\boldsymbol{C}^*$ is the complexification of the compact group $\{e^{i\theta}; \theta \in \boldsymbol{R}\}$, it follows from Matsushima and Morimoto[24] that ω_F is also a Stein manifold. Since $L + a$ is an affine subspace of the finite dimensional space F, the set $\varphi^{-1} \circ \varphi^{-1}_{(f,a)}(L) = \varphi^{-1}(q(L + a))$ is a closed submanifold of the Stein manifold ω_F. Therefore the complex manifold $\varphi^{-1} \circ \varphi^{-1}_{(f,a)}(L)$ is a Stein manifold. This completes the proof.

After this we assume that Riemann domains (ω, φ) over the projective space $\boldsymbol{P}(E)$ are not homeomorphic to $\boldsymbol{P}(E)$ through φ.

Let E be a locally convex space, and let α and β be nontrivial continuous seminorms on E with $\alpha \leq \beta$. We set

$$\boldsymbol{P}(E)_\alpha = \{[x] \in \boldsymbol{P}(E) \ ; \ \alpha(x) \neq 0 \ \}.$$

We define a pseudodistance $\rho_E^{\alpha,\beta}$ on the open set $\boldsymbol{P}(E)_\alpha$ by

$$\rho_E^{\alpha,\beta}([x],[y]) \;=\; \inf\{\beta\left(e^{i\theta}\tfrac{x}{\alpha(x)} - e^{i\theta'}\tfrac{y}{\alpha(y)}\right) \;;\; \theta,\ \theta' \in \boldsymbol{R}\,\} \tag{4.10}$$

for every $[x],[y] \in \boldsymbol{P}(E)_\alpha$. Let (ω,φ) be a Riemann domain over $\boldsymbol{P}(E)$ with $\varphi(\omega) \subset \boldsymbol{P}(E)_\alpha$, that is, (ω,φ) be a Riemann domain over the complex manifold $\boldsymbol{P}(E)_\alpha$. Let a be a point of $\boldsymbol{P}(E)_\alpha$ and r be a positive number, we denote by $B_{P(E)}(a,r;\ \rho_E^{\alpha,\beta}\ )$ the open ball $\{\ b \in \boldsymbol{P}(E)_\alpha\ ;\ \rho_E^{\alpha,\beta}(a,b)\ <\ r\ \}$ with respect to the pseudodistance $\rho_E^{\alpha,\beta}$ with center a and with radius r. We define the *boundary distance function* $\Delta_\omega^{\alpha,\beta}\ :\ \omega \to [0,\infty)$ for any $\alpha,\ \beta \in cs(E)$ with $\beta \geq \alpha$ by

$$\Delta_\omega^{\alpha,\beta}(x) \;=\; \sup\{\ r\ ;\ \text{there is a section } \sigma : B_{P(E)}(\varphi(x),r\ ;\ \rho_E^{\alpha,\beta}\ ) \to \omega \\ \text{with } \sigma\circ\varphi(x) = x\ \}.$$

If $\Delta_\omega^{\alpha,\beta}(x) > 0$, then for each $r \in (0,\Delta_\omega^{\alpha,\beta}(x)]$ there exists a unique subset $B_\omega(x,r\ ;\ \rho_E^{\alpha,\beta})$ of ω containing x such that a mapping $\varphi : B_\omega(x,r\ ;\ \rho_E^{\alpha,\beta}) \to B_{P(E)}^{\alpha,\beta}(\varphi(x),r\ ;\ \rho_E^{\alpha,\beta})$ is bijective.

Lemma 4.5. *Let E be a locally convex space and (ω,φ) be a connected pseudoconvex Riemann domain. Then if a continuous seminorm α on E satisfies $d_\Omega^\alpha(a) > 0$ for some points a of Ω, we have $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha^{-1}(0)$ and $\varphi(\omega) \subset \boldsymbol{P}(E)_\alpha$.*

Proof. Since d_Ω^α is continuous, there exists an open neighborhood $N(a)$ such that $d_\Omega^\alpha > 0$ on $N(a)$. For any $v \in \alpha^{-1}(0)$ and any $x \in N(a)$, we have $\delta(x,v) = \infty$. Since ω is pseudoconvex, it follows from Proposition 4.4 that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha^{-1}(0)$. Thus for any $x \in \Omega$, there exists a section σ of Ω on $\Phi(x) + \alpha^{-1}(0)$. Therefore the set $\Phi(x) + \alpha^{-1}(0)$ is contained in $E - \{0\}$. Thus $\Phi(x)$ is not contained in $\alpha^{-1}(0)$. Thus $\alpha(\Phi(x)) \neq 0$ for every $x \in \Omega$. It follows from the commutative diagram (4.7) that $\varphi(\omega) \subset \boldsymbol{P}(E)_\alpha$. This completes the proof.

Lemma 4.6. *Let E be a locally convex space. Let (ω, φ) be a Riemann domain over the projective space $\boldsymbol{P}(E)$. Let S be a subset of ω. Then the mapping $\varphi|S :\to \boldsymbol{P}(E)$ is injective if and only if the mapping $\Phi|Q^{-1}(S) : Q^{-1}(S) \to E - \{0\}$ is injective.*

Proof. Assume that the mapping $\varphi|S$ is injective. Let a and b be any different points of $Q^{-1}(S)$. If $Q(a) \neq Q(b)$, it follows from the commutative diagram (4.7) and from the injectivity of $\varphi|S$ that $\Phi(a) \neq \Phi(b)$. If $Q(a) = Q(b)$, there exist different points w_1 and w_2 of $E - \{0\}$ such that $a = (Q(a), w_1), b = (Q(b), w_2)$. Since $w_1 = \Phi(a)$ and $w_2 = \Phi(b)$, $\Phi(a) \neq \Phi(b)$. Therefore $\Phi|Q^{-1}(S)$ is injective.

Assume that the mapping $\Phi|Q^{-1}(S)$ is injective. Let a and b be different points of S. Then there exist points w_1 and w_2 of $E - \{0\}$ such that $(a, w_1), (b, w_2) \in Q^{-1}(S)$. Then $(a, \lambda \cdot w_1)$, $(b, \lambda' \cdot w_2) \in Q^{-1}(S)$ for any $\lambda, \lambda' \in \boldsymbol{C}^*$. Since $(a, \lambda \cdot w_1) \neq (b, \lambda' \cdot w_2)$ for any λ, $\lambda' \in \boldsymbol{C}^*$ and $\Phi|Q^{-1}(S)$ is injective, $\lambda \cdot w_1 \neq \lambda' \cdot w_2$ for any λ, $\lambda' \in \boldsymbol{C}^*$. Therefore $q(w_1) \neq q(w_2)$. Thus it follows from $q \circ \Phi = \varphi \circ Q$ that $\varphi(a) \neq \varphi(b)$. Therefore $\varphi|S$ is injective. This completes the proof.

For a subset S of Ω, we set

$$V(S) = Q^{-1} \circ Q(S).$$

Lemma 4.7. *Let a be a point of Ω. Let α be a continuous seminorm on E with $d_\Omega^\alpha(a) > 0$. For every positive number r with $0 < r < d_\Omega^\alpha(a)$, the mapping $\Phi|V(B_\Omega^\alpha(a, r)) : V(B_\Omega^\alpha(a, r)) \to E - \{0\}$ is injective.*

Proof. Let (z_1, w_1) and (z_2, w_2) be different points of $V(B_\Omega^\alpha(a, r)) \subset \omega \times (E - \{0\})$. We have only to show that $w_1 \neq w_2$. We assume that $w_1 = w_2$. Since $(z_1, w_1) \neq (z_2, w_2)$, we have $z_1 \neq z_2$. Since $\varphi(z_1) = q(w_1) = q(w_2) = \varphi(z_2)$, both z_1 and z_2 belong to $\varphi^{-1}(\varphi(z_1))$. Since both (z_1, w_1) and (z_2, w_2) belong to $V(B_\Omega^\alpha(a, r))$,

there exists complex number $\lambda_1, \lambda_2 \in \boldsymbol{C}^*$ such that $(z_1, \lambda_1 \cdot w_1), (z_2, \lambda_2 \cdot w_2) \in B^\alpha_\Omega(a, r)$. Since $\Phi | B^\alpha_\Omega(a, r)$ is injective, $\lambda_1 \cdot w_1 \neq \lambda_2 \cdot w_2$. Since $B^\alpha_E(\Phi(a), r)$ is convex, the line segment $[\lambda_1 \cdot w_1, \lambda_2 \cdot w_2]$ is contained in $B^\alpha_E(\Phi(a), r)$. The set $\{ (z_1, (1-t)\lambda_1 \cdot w_1 + t\lambda_2 \cdot w_2) \ ; \ t \in [0, 1] \}$ is homeomorphically mapped by Φ onto $[\lambda_1 \cdot w_1, \lambda_2 \cdot w_2]$. Since $(z_1, \lambda_1 \cdot w_1) \in B^\alpha_\Omega(a, r)$ and $[\lambda_1 \cdot w_1, \lambda_2 \cdot w_2] \subset B^\alpha_E(\Phi(a), r)$, it is valid that $(z_1, \lambda_2 \cdot w_2) \in B^\alpha_\Omega(a, r)$. Then we have $\Phi((z_1, \lambda_2 \cdot w_2)) = \Phi((z_2, \lambda_2 \cdot w_2))$. Since $\Phi | B^\alpha_\Omega(a, r)$ is injective, it follows that $z_1 \neq z_2$. This is a contradiction. This completes the proof.

We obtain the following Lemma 4.8 from Lemma 4.6 and 4.7.

Lemma 4.8. *With the assumption of Lemma* 4.7. *The mapping* $\varphi | Q(B^\alpha_\Omega(a, r))$ *is injective.*

Lemma 4.9. *We assume that there exists a nontrivial continuous seminorm* α *on* E *such that* $\varphi(\omega) \subset \boldsymbol{P}(E)_\alpha$. *Let* a *a point of* Ω. *Let* β *be a continuous seminorm on* E *with* $\beta \geq \alpha$ *and with* $d^\beta_\Omega(a) > 0$. *For every positive number* r *with* $0 < r < d^\beta_\Omega(a)$, *the open set* $\varphi \circ Q(B^\beta_\Omega(a, r))$ *contains the open set* $B_{P(E)}(\varphi \circ Q(a), r/\alpha(\Phi(a)) \ ; \ \rho^{\alpha,\beta}_E)$.

Proof. Let u be a point of $B_{P(E)}(\varphi \circ Q(a), r/\alpha(\Phi(a)) \ ; \ \rho^{\alpha,\beta}_E)$. Then there exist a point w of $E - \{0\}$ with $\alpha(w) = 1$ and a real number θ such that $u = q(w)$ and

$$\beta\left(e^{i\theta} w - \frac{\Phi(a)}{\alpha(\Phi(a))}\right) < \frac{r}{\alpha(\Phi(a))}.$$

This implies that $\beta(e^{i\theta}\alpha(\Phi(a))w - \Phi(a)) < r$. Therefore $e^{i\theta}\alpha(\Phi(a))w$ belongs to $B^\beta_E(\Phi(a), r)$. Since the mapping $\Phi : B^\beta_\Omega(a, r) \to E - \{0\}$ is injective, there is a unique point z of ω such that $(z, e^{i\theta}\alpha(\Phi(a))w)$ belongs to $B^\beta_\Omega(a, r)$. Then we have $\varphi \circ Q(z, e^{i\theta}\alpha(\Phi(a))w) \in \varphi \circ Q(B^\beta_\Omega(a, r))$ and $\varphi \circ Q(z, e^{i\theta}\alpha(\Phi(a))w) = q \circ \Phi(z, e^{i\theta}\alpha(\Phi(a))w) = q(e^{i\theta}\alpha(\Phi(a))w) = q(w) = u$. Therefore we have $u \in \varphi \circ Q(B^\beta_\Omega(a, r))$. This completes the proof.

Therefore from Lemma 4.8 and 4.9 we obtain the following Lemma 4.10 which plays the important role in section 6

Lemma 4.10. *Let a be a point of Ω with $d_\Omega^\beta(a) \geq r > 0$. Then we have*

$$\Delta_\omega^{\alpha,\beta}(Q(a)) \geq \frac{r}{\alpha(\Phi(a))}.$$

We end this section by proving the following Proposition 4.11.

Proposition 4.11. *Let E be a locally convex space, (ω, φ) be the projective space $\boldsymbol{P}(E)$ and $\mathcal{F} \subset H(\omega)$. If ω is an $\mathcal{F}$-domain of holomorphy, ω is pseudoconvex.*

Proof. Let a be any point of $E - \{0\}$ and f be any continuous linear functional of E with $f(a) \neq 0$. For the open set $U(f)$ defined by (4.3), we set

$$\omega_f = \varphi^{-1}(U(f)). \tag{4.11}$$

We have only to show that the Riemann domain $(\omega_f, \varphi_{(f,a)} \circ \varphi|\omega_f)$ over $E(f)$ is pseudoconvex. We set

$$\mathcal{F}' = \{ h|\omega_f \ ; \ h \in \mathcal{F}\} \cup \{ g/f \ ; \ g \in E'\}.$$

Then ω_f is an $\mathcal{F}'$-domain of holomorphy. Thus by Noverraz[32], ω_f is pseudoconvex.

5 Cartan-Thullen type theorem.

Let E be a locally convex space and (ω, φ) be a Riemann domain over the projective space $\boldsymbol{P}(E)$.

An increasing sequence $\mathcal{U} = (U_j)_{j=1}^\infty$ of open subsets of ω is called a *regular cover* of ω if $\omega = \cup_{j=1}^\infty U_j$ and if there exists an increasing sequence $(\alpha_j)_{j=1}^\infty$ of continuous seminorms on E such that

$$\varphi(\omega) \subset \boldsymbol{P}(E)_{\alpha_1}, \ \Delta_{U_{j+1}}^{\alpha_1, \alpha_j}(U_j) > 0$$

for every j. We denote by $H^\infty(\mathcal{U})$ the Fréchet algebra

$$H^\infty(\mathcal{U}) \;=\; \{\, f \in H(\omega)\;;\; |f|_{U_j} \;<\; \infty \text{ for every } j \,\}$$

endowed with the topology generated by the norms $f \to |f|_{U_j}$.

Let E be a locally convex space, (Ω, Φ) be a Riemann domain and f be a holomorphic function on Ω. For any point a of Ω, there exist continuous n-homogeneous polynomials $P_n \;:\; E \to \boldsymbol{C}$ and a balanced convex open neighborhood V of 0 in E such that $a + V \subset \Omega$ and

$$f(a+x) \;=\; \sum_{n=0}^{\infty} P_n(x)$$

uniformly for $x \in V$. We denote by $o(f, a)$ the smallest integer n such that P_n are not identically zero in E. We write $o(f, a) = \infty$ if P_n are identically zero in E for all n. We call $o(f, a)$ the *order of zero* of f at a. If functions f and g are holomorphic in a neighborhood of a point a in E, $o(fg, a) = o(f, a) + o(g, a)$. f is identically zero in a neiborhood of a if and only if $o(f, a) = \infty$.

Let E be a metrizable locally convex space and $(\alpha_j)_{j=1}^\infty$ be a fundamental sequence of continuous seminorms on E. We set

$$\rho_E^{\{\alpha_j\}}(x, y) \;=\; \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_E^{\alpha_1,\, \alpha_j}(x, y)}{1 + \rho_E^{\alpha_1,\, \alpha_j}(x, y)}$$

for every $x,\; y \in \boldsymbol{P}(E)_{\alpha_1}$. Then $\rho_E^{\{\alpha_j\}}$ is a continuous distance of $\boldsymbol{P}(E)_{\alpha_1}$ which defines the same topology as the initial topology of $\boldsymbol{P}(E)_{\alpha_1}$. We denote by $\Delta_\omega^{\{\alpha_j\}}$ the *boundary distance function* of ω with respect to the distance $\rho_E^{\{\alpha_j\}}$ and by $B_\omega^{\{\alpha_j\}}(x,\; r)$ the open neighborhood of x in ω which is homeomorphic to the open set $\{\; z \in \boldsymbol{P}(E)_{\alpha_1}\;;\; \rho_E^{\{\alpha_j\}}(\varphi(x),\; z) < r \;\}$ through φ for r with $r \le \Delta_\omega^{\{\alpha_j\}}(x)$. We set

$$B_\omega^{\{\alpha_j\}}(x) \;=\; B_\omega^{\{\alpha_j\}}(x,\; \Delta_\omega^{\{\alpha_j\}}(x)\;).$$

Theorem 5.1. *Let E be a separable metrizable locally convex space and (ω, φ) be a connected Riemann domain over the projective space $\boldsymbol{P}(E)$. Assume that there*

exist a regular cover $\mathcal{U} = (U_i)_{i=1}^{\infty}$ of ω and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms on E such that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_1^{-1}(0)$, that ω is $H^\infty(\mathcal{U})$-fibre separated and that $\Delta_\omega^{\alpha_1,\,\alpha_j}(\hat{U}_{j\ H(\omega)}) > 0$ for every j. Then ω is a domain of existence.

Proof. We remark that it follows from an examination of the proof of Lemma 4.5 that $\delta(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_1^{-1}(0)$ implies $\varphi(\omega) \subset \boldsymbol{P}(E)_{\alpha_1}$. Since the projective space $\boldsymbol{P}(E)$ is separable, there exists a countable dense subset D of $\boldsymbol{P}(E)$. We set $A = \varphi^{-1}(D)$. Let (x_k) be a sequence in A with the property that each point of A appears in the sequence (x_k) infinitely many times . We set $V_k = \hat{U}_{k\ H(\omega)}$ for each $k \geq 1$. By the assumption, we have $\Delta_\omega^{\{\alpha_j\}}(\hat{U}_{k\ H(\omega)}) > 0$. Thus $B_\omega^{\{\alpha_j\}}(x)$ is not contained in V_k for each $x \in \omega$ and $k \geq 1$. After replacing a sequence (V_k) by subsequence, if necessary, we can find a sequence (y_k) in ω such that $y_k \in B_\omega^{\{\alpha_j\}}(x_k)$, $y_k \notin V_k$ and $y_k \in V_{k+1}$ for every $k \geq 1$. Hence we can inductively find a sequence (f_k) in $H(\omega)$ such that

$$|f_k|_{V_k} \ < \ 2^{-k} \text{ and } f_k(y_k) = 1$$

for every $k \geq 1$. Since $\sum_{k=1}^{\infty} \frac{k}{2^k}$ is convergent, the infinite product

$$\prod_{k=1}^{\infty} (1 - f_k)^k$$

converges uniformly on V_k for each k and there it defines a function $f \in H(\omega)$ which is not identically zero in ω. We set $N(f) = \{\ x \in \omega\ ;\ f(x) = 0\ \}$ and $A' = A \setminus N(f)$. Then A' is a countable dense subset of ω. We set $B = \{\ (x,y) \in A' \times A'\ ;\ \varphi(x) = \varphi(y) \text{ and } x \neq y\ \}$. B is a countable subset of $\omega \times \omega$. Since ω is $H^\infty(\mathcal{U})$-fibre separated, the set $S_{x,y} = \{\ g \in H^\infty(\mathcal{U})\ ;\ Re\ g(x) \neq Re\ g(y)\ \}$ is nonvoid for each $(x,y) \in B$. The set $S_{x,y}$ is open in $H^\infty(\mathcal{U})$. We claim the set $S_{x,y}$ is dense in $H^\infty(\mathcal{U})$. Indeed, given $f \in H^\infty(\mathcal{U})$ with $f \notin S_{x,y}$, choose $g \in S_{x,y}$ and set $g_n = f + (1/n)g$. Then $g_n \in S_{x,y}$ for every n and the sequence (g_n) converges to f in $H^\infty(\mathcal{U})$. Since

$H^\infty(\mathcal{U})$ is a Baire space, the set $S = \cap\{\ S_{x,y}\ ;\ (x,y) \in B\ \}$ is dense in $H^\infty(\mathcal{U})$. Thus there exists a function $g \in H^\infty(\mathcal{U})$ such that $Re\ g(x) \neq Re\ g(y)$ for every $(x,y) \in B$. Since the set of the quotient

$$\frac{(\log|f(x)| - \log|f(y)|)}{Re\ (g(x) - g(y))}$$

with $(x,y) \in B$ is countable, there exists $\theta \in (0,1)$ such that $\log|f(x)| - \log|f(y)| \neq \theta\ Re\ (g(x) - g(y))$ for every $(x,y) \in B$. We set

$$h(x) \ = \ f(x)\exp(-\theta g(x))$$

for every $x \in \omega$. Then we have $h(x) \neq h(y)$ for every $(x,y) \in B$.

We shall show that ω is the domain of existence of h. Let $\lambda : \omega \to \tilde{\omega}$ be an $\{h\}$-envelope of holomorphy of ω, and let $\tilde{h} \in H(\tilde{\omega})$ with $\tilde{h} \circ \lambda = h$. We denote by $\tilde{\varphi}$ the projection of the Riemann domain $\tilde{\omega}$ into $\boldsymbol{P}(E)$. We remark that by Proposition 2.2 and Lemma 4.11 $\tilde{\omega}$ is pseudoconvex. To prove that λ is injective, we assume that there exist distinct points a and b of ω such that $\lambda(a) = \lambda(b)$. Then there exist an open neighborhood $U(a)$ of a and an open neighborhood $U(b)$ of b with $U(a) \cap U(b) = \emptyset$ such that $\lambda|U(a)$, $\lambda|U(b)$, $\varphi|U(a)$ and $\varphi|U(b)$ are homeomorphisms and that $\lambda(U(a)) = \lambda(U(b))$. Then we have $\lambda(x) = \lambda(y)$ if $(x,y) \in U(a) \times U(b)$ and $\varphi(x) = \varphi(y)$. Thus we have $h(x) = \tilde{h} \circ \lambda(x) = \tilde{h} \circ \lambda(y) = h(y)$ if $(x,y) \in U(a) \times U(b)$ and $\varphi(x) = \varphi(y)$. We set $W = \varphi(U(a)) = \varphi(U(b))$, $S_1 = \varphi(U(a) \cap N(f))$ and $S_2 = \varphi(U(b) \cap N(f))$. The set $S_1 \cup S_2$ is an analytic subset of the open set W of $\boldsymbol{P}(E)$. Therefore $W \setminus (S_1 \cup S_2)$ is a dense open subset of W. Therefore we have $D \cap (W \setminus (S_1 \cup S_2)) \neq \emptyset$. Hence there exists a point p of W such that $p \notin S_1 \cup S_2$, $p \in D$. Then there exists a point $(x,y) \in U(a) \times U(b)$ with $\varphi(x) = \varphi(y) = p$. Since $p \notin S_1 \cup S_2$, $\{x,\ y\} \cap N(f) = \emptyset$. Therefore (x,y) belongs to B. Thus we have $h(x) \neq h(y)$. On the other hand since $\varphi(x) = \varphi(y)$ and $(x,y) \in U(a) \times U(b)$, $h(x) = h(y)$. This is a contradiction. Therefore λ is injective.

To prove that λ is surjective, we assume that $\tilde{\omega} \neq \lambda(\omega)$. Then there exists a point $z_0 \in (\tilde{\omega} \setminus \lambda(\omega)) \cap \overline{\lambda(\omega)} \neq \emptyset$ where $\overline{\lambda(\omega)}$ is the topological closure of $\lambda(\omega)$ in $\tilde{\omega}$. Since $\tilde{\omega}$ is pseudoconvex, it follows from an examination of the proof of Lemma 4.5 that $\tilde{\varphi}(\tilde{\omega}) \subset \boldsymbol{P}\,(E)_{\alpha_1}$. We set $a = \tilde{\varphi}(z_0)$. There exists a continuous linear functional μ on E such that $\mu(a) \neq 0$. Then $\tilde{\varphi}^{-1}(U(\mu))$ is an open subset of $\tilde{\omega}$ and contains the subset $\{z_0\}$ of $\tilde{\omega}$ where $U(\mu)$ is in (4.3). $(\tilde{\varphi}^{-1}(U(\mu)),\ \varphi_{(\mu,a)} \circ \tilde{\varphi})$ is a Riemann domain over the locally convex space $E(\mu)$ where $E(\mu)$ and $\varphi_{(\mu,a)}$ are in (4.3) and in (4.4) respectively. There exists an open neighborhood V of 0 in $E(\mu)$ such that there exists a section s of the Riemann domain $\tilde{\varphi}^{-1}(U(\mu))$ on V. Then $\tilde{h} \circ s$ is holomorphic in V. For any $x \in V$ there exists a sequence of n-homogeneous polynomials $P_x^n : E \to \boldsymbol{C}$ and a convex balanced open neighborhood U of 0 in E such that $x + U \subset V$ and

$$\tilde{h} \circ s(\xi) \;=\; \tilde{h} \circ s(x) \;+\; \sum_{n=1}^{\infty} P_x^n(\xi)$$

uniformly for $\xi \in U$. Then $P_x^n(\xi)$ is given by

$$P_x^n(\xi) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(x + e^{i\theta} \cdot \xi)d\theta$$

for any $\xi \in E$. Since $\tilde{h} \circ s$ is not idetically 0 in V, the order $o(\tilde{h} \circ s, 0)$ of zero of $\tilde{h} \circ s$ at 0 is finite. We set $n(0) = o(\tilde{h} \circ s, 0)$. Then there exists $\xi_0 \in E$ such that $P_0^{n(0)}(\xi_0) \neq 0$. Since $x \to P_x^{n(0)}(\xi_0)$ is continuous, there exists an open neighborhood $N(0)$ of 0 in V such that $P_x^{n(0)}(\xi_0) \neq 0$ for any $x \in N(0)$. Therefore we have $o(\tilde{h} \circ s, x) \leq n(0)$ for every $x \in N(0)$. There exists a positive number r such that $2r < \Delta_{\tilde{\omega}}^{\{\alpha_j\}}(z_0)$ and $\varphi_{(\mu,a)} \circ \tilde{\varphi}(B_{\tilde{\omega}}^{\{\alpha_j\}}(z_o, 2r)) \subset N(0)$. We can find $p \in A$ such that $\lambda(p) \in B_{\tilde{\omega}}^{\{\alpha_j\}}(z_0, r)$. Then we have $\Delta_{\omega}^{\{\alpha_j\}}(p) < r$ and it follows that

$$\lambda(B_{\omega}^{\{\alpha_j\}}(p)) \;=\; B_{\tilde{\omega}}^{\{\alpha_j\}}(\lambda(p), \Delta_{\omega}^{\{\alpha_j\}}(p)) \subset B_{\tilde{\omega}}^{\{\alpha_j\}}(\lambda(p), r) \subset B_{\tilde{\omega}}^{\{\alpha_j\}}(z_0, 2r).$$

By the definition of the sequence (x_k) there exists a strictly increasing sequence (k_n) of natural numbers such that $x_{k_n} = p$ for every n. Hence each y_{k_n} belongs to $B_\omega^{\{\alpha_j\}}(p)$ and therefore $\lambda(y_{k_n}) \in B_{\tilde{\omega}}^{\{\alpha_j\}}(z_0, 2r)$. We set $z_{k_n} = \varphi_{(\mu,a)} \circ \tilde{\varphi}(y_{k_n})$. Then z_{k_n} belong to $N(0)$. On the other hand we have $o(\tilde{h} \circ s, z_{k_n}) \geq k_n$. Since $o(\tilde{h} \circ s, x) \leq n(0)$ for every $x \in N(0)$, this is a contradiction. This completes the proof.

6 Levi problem in a Riemann domain over an infinite dimensional comlex projective space.

The aim of this section is to prove Theorem 1. Let E be a Fréchet space with a Schauder basis $(e_n)_{n=1}^\infty$. We shall first assume that E has a continuous norm. Let (ω, φ) be a connected pseudoconvex Riemann domain over the comlex projective space $\boldsymbol{P}(E)$. Let (Ω, Φ) be the Riemann domain with $\boldsymbol{C}^*$-action associated with th Riemann doman (ω, φ) over $\boldsymbol{P}(E)$. We choose a fundamental sequece $(\alpha_n)_{n=1}^\infty$ of continuous norms on E with $\alpha_{n+1} \geq 2\alpha_n$ and $\alpha_n = \sup_k \alpha_n \circ T_k$ for every n. With the notatins of Lemma 3.2 and Lemma 3.3, we set $\omega_n = \varphi^{-1}(\boldsymbol{P}(E_n))$, $A_{n,\omega} = Q(A_n)$, $B_{n,\omega} = Q(B_n)$, $C_{n,\omega} = Q(C_n)$ and

$$\tau_{n,\omega}(z) = Q \circ \tau_n \circ (Q|A_n)^{-1}(z) \tag{6.1}$$

for every $z \in A_{n,\omega}$. Then the mapping $\tau_{n,\omega}$ is a holomorphic mapping of $A_{n,\omega}$ into ω_n. By Lemma 3.3 (e), (f) and Lemma 4.10, we have

$$\Delta_\omega^{\alpha_1,\alpha_n}(B_{n,\omega}) \geq 1/n. \tag{6.2}$$

A sequence $\mathcal{C} = (C_{j,\omega})_{j=1}^\infty$ of open sets of ω is a regular cover of ω. In fact, by Lemma 3.3 (a) and Lemma 4.10, there exists an increasing sequence $(\beta_j)_{j=1}^\infty$ of continuous normes on E such that $\beta_1 \geq \alpha_1$ and

$$\Delta_{C_{j+1,\omega}}^{\alpha_1,\beta_j}(C_{j,\omega}) \geq 1/j \tag{6.3}$$

for every $j \geq 1$.

Lemma 6.1. *For each $f_n \in H(\omega_n)$ and for each $\epsilon > 0$ there exists $f \in H(\omega)$ such that*

(a) $$f = f_n \quad on\ \omega_n,$$

(b) $$|f - f_n \circ \tau_{n,\omega}|_{C_{n,\omega}} \leq \epsilon,$$

(c) $$|f|_{C_{j,\omega}} < \infty \qquad for\ every\ j.$$

Proof. We define a holomorphic function g_n on $\Omega_n = \Phi^{-1}(E_n)$ by $g_n = f_n \circ (Q|\Omega_n)$ for each n. Each g_n is a $\boldsymbol{C}^*$-invariant function on Ω_n. By Lemma 3.4, there exists a $\boldsymbol{C}^*$-invariant function $g \in H(\Omega)$ such that $g = g_n$ on Ω, $|g - g_n \circ \tau_n|_{C_n} \leq \epsilon$, *and* $|g|_{C_j} < \infty$ for every j. We define a holomorphic function f on ω by

$$f(z) = g \circ Q^{-1}(z) \qquad \text{for every } z \in \omega.$$

Then f satisfies the required conditions (a), (b) and (c). This completes the proof.

Lemma 6.2. *ω is $H^\infty(\mathcal{C})$-separated.*

Proof. Let a and b be any different points of ω. There exists a positive integer N such that the set $\{a, b, \tau_{N,\omega}(a), \tau_{N,\omega}(b)\}$ is contained in $C_{N,\omega}$ and that $\tau_{N,\omega}(a) \neq \tau_{N,\omega}(b)$. By proposition 4.4, ω_N is a Stein manifold. By Oka-Cartan theorem, there exists a holomorphic function f_N on ω_n such that $f_N(\tau_{N,\omega}(a)) = 2$ and $f_N(\tau_{N,\omega}(b)) = 0$. By Lemma 6.1, there exists a holomorphic function $f \in H(\omega)$ such that

(a) $$f = f_N \qquad \text{on } \omega_N$$

(b) $$|f - f_N \circ \tau_{N,\omega}|_{C_{N,\omega}} \leq 1/2,$$

(c) $\qquad |f|_{C_{j,\omega}} < \infty \qquad$ for every $j \geq 1$.

Since $\{a, b\} \subset C_{N,\omega}$,

$$\begin{aligned} |f(a) - f(b)| &\geq |f_N \circ \tau_{N,\omega}(a) - f_N \circ \tau_{N,\omega}(b)| - |f(a) - f_n \circ \tau_{N,\omega}(a)| - |f(b) - f_N \circ \tau_{N,\omega}(b)| \\ &\geq 2 - 1/2 - 1/2 = 1 \end{aligned}$$

Therefore we have $f(a) \neq f(b)$. Thus ω is $H^\infty(\mathcal{C})$-separated. This completes the proof.

Lemma 6.3. *It is valid that*

$$\Delta_\omega^{\alpha_1,\alpha_n}(\hat{C}_{n,\omega\ H^\infty(\mathcal{C})}) > 0$$

for every n.

Proof. Let n be any positive integer and x_0 be any point of $\hat{C}_{n,\omega\ H^\infty(\mathcal{C})}$. Then there exists a positive integer n_0 such that $n_0 \geq n$ and $x_0 \in C_{k,\omega}$ for every $k \geq n_0$. By Lemma 6.1, for each $k \geq n_0$, each $f_k \in H(\omega_k)$ and each $\epsilon > 0$, there exists a function $f \in H^\infty(\mathcal{C})$ such that $|f - f_k \circ \tau_{k,\omega}|_{C_{k,\omega}} \leq \epsilon$. Therefore

$$\begin{aligned} |f_k \circ \tau_{k,\omega}(x_0)| &\leq |f(x_0)| + \epsilon \\ &\leq |f|_{C_{n,\omega}} + \epsilon \\ &\leq |f_k \circ \tau_{k,\omega}|_{C_{n,\omega}} + 2\epsilon \\ &\leq |f_k|_{B_{n,\omega} \cap \omega_k} + 2\epsilon. \end{aligned}$$

Since ϵ is arbitrarily given, $\tau_{k,\omega}(x_0) \in \widehat{(B_{n,\omega} \cap \omega_k)}_{H(\omega_k)}$. We shall show that $\Delta_\omega^{\alpha_1,\alpha_n}(x_0) \geq 1/n$. We assume that $\Delta_\omega^{\alpha_1,\alpha_n}(x_0) < 1/n$. Then there exists an integer $N \geq n_0$ such that

$$(6.4) \qquad \Delta_{\omega}^{\alpha_1,\alpha_n}(\tau_{m,\omega}(x_0)) < 1/n$$

for every $m \geq N$. There exists a point y_0 of Ω such that $Q(y_0) = x_0$. We set

$$\begin{aligned} V(\tau_m(y_0)) &= \{\lambda \cdot \tau_m(y_0);\ \lambda \in \boldsymbol{C}^*\}, \\ S(\tau_m(y_0)) &= \{e^{i\theta} \cdot \tau_m(y_0);\ \theta \in \boldsymbol{R}\} \text{ and} \\ K_m &= (B_n \cap \Omega_m)^{\hat{}}_{H(\Omega_m)} \end{aligned}$$

for every $m \geq N$. Since Ω is pseudoconvex, since $d_{\Omega}^{\alpha_n}(B_n \cap \Omega_m) \geq 1$ and $\sup_{B_n} \alpha_1 \leq n$, we have $d_{\Omega}^{\alpha_n}(K_m) \geq 1$ and $\sup_{K_m} \alpha \leq n$. Therefore by Lemma 4.10 we have

$$(6.5) \qquad \Delta_{\omega}^{\alpha_1,\alpha_n}(Q(K_m)) \geq \tfrac{1}{n}.$$

Since $Q(\tau_m(y_0)) = \tau_{m,\omega}(x_0)$, it follows from (6.4) and (6.5) that $K_m \cap V(\tau_m(y_0)) = \emptyset$ for every $m \geq N$. We set

$$T_m = K_m \cup S(\tau_m(y_0))$$

for every $m \geq N$. Since Ω_m is a Stein manifold, $\hat{T}_{m\ H(\Omega_m)}$ is compact in Ω_m. We write $\hat{T}_m = \hat{T}_{m\ H(\Omega_m)}$. We remark that the set $\hat{T}_m$ is contained in the set $V(\tau_m(y_0)) \cup K_m$. In fact, let x be a point of $\Omega_m \setminus (V(\tau_m(y_0)) \cup K_m)$. Since Ω_m is a Stein manifold and $V(\tau_m(y_0))$ is a closed submanifold of Ω_m, by Oka-Cartan theorem there exists a holomorphic function s in Ω_m with $s = 0$ on $V(\tau_m(y_0))$ and with $s(x) = 1$. Since K_m is a Runge compact subset of Ω_m, there exists a holomorphic function t in Ω such that $|t(x)| > 1$, $|t|_{K_m} < 1/(|s|_{K_m} + 1)$. Then we have $|s(x)t(x)| > 1$ and $|st|_{T_m} < 1$. Therefore x cannot belong to $\hat{T}_m$. Thus the set $\hat{T}_m$ is contained in the set $V(\tau_m(y_0)) \cup K_m$. Since $V(\tau_m(y_0)) \cap K_m = \emptyset$, it follows that $\hat{T}_m \cap V(\tau_m(y_0)) \cap K_m = \emptyset$ and $\hat{T}_m = (\hat{T}_m \cap V(\tau_m(y_0))) \cup K_m$. Let V_1 and V_2 be open neighborhoods of $\hat{T}_m \cap V(\tau_m(y_0))$ and of K_m, respectively, with $V_1 \cap V_2 = \emptyset$. Let g be a holomorphic function on $V_1 \cup V_2$ defined by $g = 2$ on V_1 and $g = 0$ on V_2.

Since $\hat{T}_m$ is a Runge compact subset of Ω_m, there exists a function $h \in H(\Omega_m)$ such that $|h - g|_{\hat{T}_m} < 1/2$. Then we have $Re\ h \geq 3/2$ on $S(\tau_m(y_0))$ and $|h|_{B_n \cap \Omega_m} \leq 1/2$ where $Re\ h$ is the real part of h. Let $\tilde{h}$ be a holomorphic function on Ω_m defined by (3.6). The holomorphic function $\tilde{h}$ is constant on $Q^{-1}(z)$ for every $z \in \omega_m$. Thus we can define a holomorphic function h^* on ω_m by $h^*(z) = \tilde{h} \circ Q^{-1}(z)$ for every $z \in \omega_m$. Then we have $|h^*|_{B_{n,\omega} \cap \omega_m} \leq 1/2$ and $h^*(\tau_{m,\omega}(x_0)) \geq 3/2$. Therefore $\tau_{m,\omega}(x_0)$ does not belong to $(B_{n,\omega} \cap \omega_m)^{\wedge}_{H(\omega_m)}$. This is a contradiction. Therefore we have

$$\Delta^{\alpha_1,\alpha_n}(\hat{C}_{n,\omega\ H^\infty(\mathcal{C})}) \geq 1/n > 0.$$

This completes the proof.

From Lemma 6.2 and 6.3 we obtain the following Proposition 6.4.

Proposition 6.4. *Let E be a Fréchet space with a Schauder basis and with a continuous norm. Let (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\boldsymbol{P}(E)$. Then there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of ω and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous norms on E such that ω is $H^\infty(\mathcal{U})$-separated and $\Delta_\omega^{\alpha_1,\alpha_j}(\hat{U}_{j\ H^\infty(\mathcal{U})}) > 0$ for every j.*

Next we shall assume that the Fréchet space E has not a continuous norm. Let (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\boldsymbol{P}(E)$. Let (Ω, Φ) be the Riemann domain over E associated with the Riemann domain (ω, φ) over $\boldsymbol{P}(E)$. Then by Lemma 4.5 there exists a continuous seminorm α_0 on E such that α_0 satisfies the condition (3.7), that $\delta_\omega(\cdot, \cdot) = \infty$ on $\Omega \times \alpha_0^{-1}(0)$ and that $\varphi(\omega) \subset \boldsymbol{P}(E)_{\alpha_0}$. Let (e_n) be a Schauder basis of E and E^{α_0} be a Fréchet space, defined by (3.8), with a Schauder basis and with a continuous norm. E^{α_0} is a closed subspace of E and E is the topological direct sum of E^{α_0} and of $\alpha_0^{-1}(0)$. We set $\omega^{\alpha_0} = \varphi^{-1}(\boldsymbol{P}(E^{\alpha_0}))$. We denote by π_{α_0} the canonical projection of E onto

E^{α_0}. We set $\Omega^{\alpha_0} = \Phi^{-1}(E^{\alpha_0})$ and $\omega^{\alpha_0} = \varphi^{-1}(\boldsymbol{P}(E^{\alpha_0}))$. Then $(\Omega^{\alpha_0}, \Phi|\Omega^{\alpha_0})$ and $(\omega^{\alpha_0}, \varphi|\omega^{\alpha_0})$ are Riemann domains over E^{α_0} and $\boldsymbol{P}(E^{\alpha_0})$ respectively. We set

$$\tilde{\sigma}_{\alpha_0}(z) = (Q|\Omega^{\alpha_0}) \circ \sigma_{\alpha_0} \circ Q^{-1}(z)$$

for every $z \in \omega$ where σ_{α_0} is a holomorphic mapping of Ω onto Ω^{α_0} in Lemma 3.6 (a). Since by Lemma 3.6 (a) we have $\sigma_{\alpha_0}(\lambda \cdot x) = \lambda \cdot \sigma_{\alpha_0}(x)$ for every $(\lambda, x) \in \boldsymbol{C}^* \times \Omega$, the mapping $\tilde{\sigma}_{\alpha_0}$ is well-defined and a holomorphic mapping of ω onto ω^{α_0}. Moreover we have $\tilde{\sigma}_{\alpha_0} = id$ on ω^{α_0}. $(\omega^{\alpha_0}, \varphi|\omega^{\alpha_0})$ is the connected pseudoconvex Riemann domain over the projective space $\boldsymbol{P}(E^{\alpha_0})$ and E^{α_0} has a Schauder basis and a continuous norm. Thus, by Proposition 6.4, there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of ω^{α_0} and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ with $\alpha_1 \geq \alpha_0|E^{\alpha_0}$ of continuous norms on E^{α_0} such that ω^{α_0} is $H^{\infty}(\mathcal{U})$-separated and $\Delta_{\omega}^{\alpha_1,\alpha_j}(\hat{U}_{j\ H^{\infty}(\mathcal{U})}) > 0$ for every j. We set $V_j = \tilde{\sigma}_{\alpha_0}^{-1}(U_j)$ for every j and $\mathcal{V} = (V_j)_{j=1}^{\infty}$. For each continuous norm β on E^{α_0}, we define a continuous seminorm $\tilde{\beta}$ on E by $\tilde{\beta}(x) = \beta(\pi_{\alpha_0}(x))$ for every $x \in E$. If a continuous norm β on E^{α_0} with $\beta \geq \alpha_1$ satisfies $\Delta_{U_{j+1}}^{\alpha_1,\beta}(U_j) > 0$, it is valid that $\Delta_{V_{j+1}}^{\tilde{\alpha}_1,\tilde{\beta}}(V_j) = \Delta_{U_{j+1}}^{\alpha_1,\beta}(U_j) > 0$. Since $\hat{V}_{j\ H(\omega)} \subset \hat{V}_{j\ H^{\infty}(\mathcal{V})}$,

$$\begin{aligned}
\Delta_{\omega}^{\tilde{\alpha}_1,\tilde{\alpha}_j}(\hat{V}_{j\ H(\omega)}) &\geq \Delta_{\omega}^{\tilde{\alpha}_1,\tilde{\alpha}_j}(\hat{V}_{j\ H^{\infty}(\mathcal{V})}) \\
&\geq \Delta_{\omega}^{\tilde{\alpha}_1,\tilde{\alpha}_j}(\tilde{\sigma}_{\alpha_0}^{-1}(\hat{U}_{j\ H^{\infty}(\mathcal{U})})) \\
&= \Delta_{\omega^{\alpha_0}}^{\alpha_1,\alpha_j}(\hat{U}_{j\ H^{\infty}(\mathcal{U})}) > 0.
\end{aligned}$$

Let w be a point of $\varphi(\omega)$. Let a and b be different points in the set $\varphi^{-1}(w)$. Then there exists points x and y of Ω such that $Q(x) = a$ and $Q(y) = b$. Then we have $\Phi(x) = \Phi(y)$ and $x \neq y$. By Lemma 3.6 (c), $\sigma_{\alpha}(x) \neq \sigma_{\alpha}(y)$. Thus $\tilde{\sigma}_{\alpha_0}(a) \neq \tilde{\sigma}_{\alpha_0}(b)$. Since ω^{α_0} is $H^{\infty}(\mathcal{U})$-separated, there exists $f \in H^{\infty}(\mathcal{U})$ such that $f(\tilde{\sigma}_{\alpha_0}(a)) = f(\tilde{\sigma}_{\alpha_0}(b))$. Since $f \circ \tilde{\sigma}_0 \in H^{\infty}(\mathcal{V})$, ω is $H^{\infty}(\mathcal{V})$-fibre separated. Thus we can obtain the following Proposition 6.5.

Proposition 6.5. *Let E be a Fréchet space with a Schauder basis and (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\boldsymbol{P}(E)$. Then there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of ω and an increasing sequence $(\alpha_j)_{j=0}^{\infty}$ of continuous seminorms on E such that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_0^{-1}(0)$, that ω is $H^\infty(\mathcal{U})$-fibre separated and that $\Delta_\omega^{\alpha_1,\alpha_j}(\hat{U}_{j\ H(\omega)}) > 0$ for every j.*

A separable Fréchet space is said to have the *bounded approximation property* if there is a sequence of continuous linear operator of finite rank which converges pointwise to the identity. Pelczynski[38] has shown that every separable Fréchet space with the bounded approximation property is topologically isomorphic to a complement subspace of a Fréchet space with a Schauder basis.

Proposition 6.6. *Let E be a separable Fréchet space with the bounded approximation property or DFN-space, and (ω, φ) be a connected pseudoconvex Riemann domain over the complex projective space $\boldsymbol{P}(E)$. Then there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of ω and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms on E such that $\delta_\Omega(\cdot,\cdot) = \infty$ on $\Omega \times \alpha_1^{-1}(0)$, that ω is $H^\infty(\mathcal{U})$-fibre separated and that $\Delta_\omega^{\alpha_1,\alpha_j}(\hat{U}_{j\ H(\omega)}) > 0$ for every j.*

Proof. If E is a separable Fréchet space with the bounded approximation property, by Pelczynski[38] there exist complex Fréchet spaces F and G such that G is the topological direct sum of E and F and that G has a Schauder basis. Let $\tilde{\Omega}$ be the product space of the Riemann domain Ω associated with the Riemann domain (ω, φ) and the space F and $\tilde{\Phi}$ be a mapping of $\tilde{\Omega}$ into the space G defined by $\tilde{\Phi}(x, y) = \Phi(x) + y$ for every $(x, y) \in \tilde{\Omega} = \Omega \times F$. We set

$$\lambda \cdot (x, y) \;=\; (\lambda \cdot x, \lambda \cdot y).$$

Then $(\tilde{\Omega}, \tilde{\Phi})$ is a connected pseudoconvex Riemann domain with $\boldsymbol{C}^*$-action over

$G = E \oplus F$. Let $\tilde{\omega}$ be the quotient space by this $\boldsymbol{C}^*$-action. Let $\tilde{Q}$ be the quotient map of $\tilde{\Omega}$ onto $\tilde{\omega}$. We denote by $\tilde{q}$ the quotient map of $G - \{0\}$ the projective space $\boldsymbol{P}(G)$. Let $\tilde{\varphi}$ be the mapping of $\tilde{\omega}$ into $\boldsymbol{P}(G)$ defined by $\tilde{\varphi}(x) = \tilde{q} \circ \tilde{\Phi} \circ \tilde{Q}^{-1}(x)$ for every $x \in \tilde{\omega}$. Then $\tilde{\varphi}$ is well-defined and $(\tilde{\omega}, \tilde{\varphi})$ is a connected Riemann domain over $\boldsymbol{P}(G)$. Moreover the Riemann domain $(\tilde{\Omega}, \tilde{\Phi})$ is that associated with the Riemann domain $(\tilde{\omega}, \tilde{\varphi})$. Since $\tilde{\Omega}$ is pseudoconvex, it follows from Proposition 4.4 that $\tilde{\omega}$ is also pseudoconvex. By Proposition 6.5, there exist a regular cover $\mathcal{U} = (U_j)_{j=1}^{\infty}$ of $\tilde{\omega}$ and an increasing sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms on G such that $\delta_{\tilde{\Omega}}(\cdot, \cdot) = \infty$ on $\tilde{\Omega} \times \alpha_1^{-1}(0)$, that $\tilde{\omega}$ is $H^{\infty}(\mathcal{U})$-fibre separated and that $\Delta_{\tilde{\omega}}^{\alpha_1, \alpha_j}(\hat{U}_{j\ H(\tilde{\omega})}) > 0$ for every j. Riemann domains Ω and ω are identified with a closed submanifold $\Omega \times \{0\}$ of $\tilde{\Omega}$ and with a closed submanifold $\tilde{Q}(\Omega \times \{0\})$ of $\tilde{\omega}$ respectively. We set

$$V_j = U_j \cap \omega \qquad \text{and} \qquad \beta_j = \alpha_j | E.$$

Then a sequence $\mathcal{V} = (V_j)_{j=1}^{\infty}$ of open sets in ω and a sequence $(\alpha_j)_{j=1}^{\infty}$ of continuous seminorms of E satisfy the required condition.

If E is a DFN-space, by Colombeau and Mujica[5] and Nachbin[28] and Paques and Zaine[37] there exists a Hilbert norm α on E such that (Ω, Φ) is a Riemann domain over the separable pre-Hilbert space (E, α). Thus by the same way as the proof of Proposition 6.4, we can obtain a regular cover $(U_j)_{j=1}^{\infty}$ of ω satisfying the required conditions. This completes the proof.

Proposition 6.7. *With the condition of Proposition 6.5, ω is holomorphically separated.*

Proof. Let E be a separable Fréchet space with the bounded approximation property. By an examination of the proof of Proposition 6.6, there exist a Fréchet space G with a Schauder basis and a pseudoconvex Riemann domain $(\tilde{\omega}, \tilde{\varphi})$ over the

projective space $\boldsymbol{P}(G)$ such that the space E is a complement subspace of G and that ω is a closed submanifold of $\tilde{\omega}$. Therefore we have only to prove this proposition in case E has a Schauder basis. Let (Ω, Φ) be the Riemann domain over E associated with the Riemann domain (ω, φ) over $\boldsymbol{P}(E)$. Let a and b be different points of ω. There exist points x and y of Ω such that $Q(x) = a, Q(y) = b$. We choose a Schauder basis $(e_n)_{n=1}^{\infty}$ of E such that the linear span of the set $\{e_1, e_2\}$ contains the set $\{\Phi(x), \Phi(y)\}$. By an examination of the proof of Proposition 6.4, there exist a continuous seminorm α, a complement subspace E^{α} of E and a holomorphic mapping $\tilde{\sigma}_{\alpha} : \omega \to \omega^{\alpha} = \varphi^{-1}(\boldsymbol{P}(E^{\alpha}))$ such that E^{α} has a Schauder basis and a continuous norm, that $\{e_1, e_2\} \subset E^{\alpha}$ and that $\tilde{\sigma}_{\alpha} = id$ on ω^{α}. By Lemma 6.2, ω^{α} is $H(\omega^{\alpha})$-separated. Since $\{a, b\} \subset \omega^{\alpha}$, there exists a function $h \in H(\omega^{\alpha})$ such that $h(a) \neq h(b)$. We define a holomorphic function f on ω by $f = h \circ \tilde{\sigma}_{\alpha}$. Since $f|\omega^{\alpha} = h$, we have $f(a) \neq f(b)$. Thus ω is $H(\omega)$-separated.

Let E be a DFN-space. Then by Colombeau and Mujica[5], Nachbin[28] and Paque and Zaine[37], there exists a Hilbert norm α of E such that (Ω, Φ) is a Riemann domain over the separable pre-hilbert space (E, α). Thus by the same way as the proof of Lemma 6.2, we can show that ω is $H(\omega)$-separated. This completes the proof.

Proof of Theorem 1. Without loss of generality we may assume that ω is connected. We remark that (Ω, Φ) can be regarded as a Riemann domain over a separable pre-Hilbert space by Colombeau and Mujica[5], Nachbin[28] and Paque and Zaine[35] if E is a DFN-space. The proof of this theorem is completed by Proposition 6.6, Proposition 6.7, Theorem 5.1, an examination of the proof of Theorem 5.1 and Proposition 4.11.

7 The indicator of entire functions of exponential type.

The aim of this section is to prove Theorem 2 and Corollary 3. Let E be a locally convex space. An entire function $f \in H(E)$ is said to be *exponential type* if

$$\limsup_{\boldsymbol{C} \ni \zeta \to \infty} \frac{\log |f(\zeta z)|}{|\zeta|} < \infty. \tag{7.1}$$

for every $z \in E$. If f is an entire function of exponential type on E, by Hervé[13] there exists a continuous seminorm α on E such that

$$|f(z)| \leq |f(0)| + \exp \alpha(z) - 1 \tag{7.2}$$

for every $z \in E$. We denote by $EXP(E)$ the space of all entire functions of exponential type on the space E. A function p on E is said to be *positively homogeneous of order* σ if $p(\lambda z) = \lambda^{\sigma} p(z)$ for every $\lambda > 0$ and every $x \in E$. For an entire function f of exponential type on E, the indicator I_f of the function f on E is plurisubharmonic and positively homogeneous of order 1.

Lemma 7.1. *Let E be a seprable Fréchet space with the bounded approximation property or a DFN-space and Ω be a pseudoconvex domain of E. Let F be a finite dimensional linear subspace of E. Then the restriction mapping of $H(\Omega)$ into $H(\Omega \cap F)$ is surjective.*

Proof. If E is a DFN-space, it follows from Colombeau-Mujica[5, Lemma 4.2] that this restriction mapping is surjective. Therefore it is sufficient to show this lemma in case E is a separable Fréchet space with the bounded approximation property. Let E be a separable Fréchet space with the bounded approximation property. By Pelczynski[38] there exist a Fréchet space E_1 with a Schauder basis and a Fréchet space E_2 such that E_1 is the topological direct sum of E and E_2.

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of F where n is the dimension of the space F. We choose a sequence $(e_j)_{j=1}^{\infty}$ of E_1 so that $(e_j)_{j=1}^{\infty}$ is a Schauder basis of E_1. There exists uniquely a sequence $(\xi_n)_{n=1}^{\infty}$ of a continuous linear functionals on E such that $x = \sum_{n=1}^{\infty} \xi_n(x) e_n$ for each $x \in E$. We denote by π the canonical projection of E_1 onto E. Since the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\Omega)$ is surjective, we have only to show that the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\Omega \cap F)$ is surjective. If E_1 has a continuous norm, it follows from Mujica[26, Lemma 2.7] that this restriction mapping is surjective. We assume that E_1 has not a continuous norm. Let x_0 be a point of $\pi^{-1}(\Omega)$. Then there exists a continuous seminorm α on E_1 such that $d^{\alpha}_{\pi^{-1}(\Omega)}(x_0) > 0$, $\alpha(e_j) \neq 0$ for every j with $1 \leq j \leq n$ and $\alpha(x) = \sup\{\alpha(\sum_{n=1}^{m} \xi_n(x) e_n)\ ;\ m \geq 1\}$. Since $\pi^{-1}(\Omega)$ is pseudoconvex, by Lemma 4.5 the continuous seminorm α on E_1 satisfies that $\delta_{\pi^{-1}(\Omega)} = \infty$ on $\pi^{-1}(\Omega) \times \alpha^{-1}(0)$. By Lemma 3.5, there exists a Fréchet space E_1^{α} with a Schauder basis and with a continuous norm such that the space E_1 is the topological direct sum of E_1^{α} and $\alpha^{-1}(0)$. Then $\pi^{-1}(\Omega) = \{\ x + y \in E_1;\ x \in \pi^{-1}(\Omega) \cap E_1^{\alpha},\ y \in \alpha^{-1}(0)\}$. Since $\pi^{-1}(\Omega) \cap E_1^{\alpha}$ is a peudoconvex domain of E_1^{α} and since F is a subspace of E_1^{α}, the restriction mapping $H(\pi^{-1}(\Omega) \cap E_1^{\alpha}) \to H(\Omega \cap F)$ is surjective. Since the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\pi^{-1}(\Omega) \cap E_1^{\alpha})$ is surjective, the restriction mapping $H(\pi^{-1}(\Omega)) \to H(\Omega \cap F)$ is surjective. This completes the proof.

Lemma 7.2. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space. Let Ω be a pseudoconvex domain with $\boldsymbol{C}^*$-action of the product space $E \times \boldsymbol{C}$ with $0 \notin \Omega$ and $\Omega \cap (E \times \{0\}) \neq \emptyset$. We denote by q the quotient map of $(E \times \boldsymbol{C}) - \{0\}$ onto the projective space $\boldsymbol{P}(E \times \boldsymbol{C})$. We set $\omega = q(\Omega)$. Then if the domain ω of $\boldsymbol{P}(E \times \boldsymbol{C})$ is pseudoconvex, there exists a holomorphic function f on ω such that f is not identically zero and $f \circ q(x, 0) = 0$ for every $(x, 0) \in \Omega$.*

Proof. There exists a finite dimensional linear subspace F of E such that $(F \times \boldsymbol{C}) \cap \Omega \neq \emptyset$. We set $\omega_{F\times\boldsymbol{C}} = \omega \cap \boldsymbol{P}(F \times \boldsymbol{C})$. By Oka-Cartan theorem, there exists a holomorphic function h on $\omega_{F\times\boldsymbol{C}}$ such that h is not identically zero and $h \circ q(x, 0) = 0$ for every $x \in F$ with $(x, 0) \in \Omega$. Then a function

$$(x, \zeta) \to \frac{h \circ q(x, \zeta)}{\zeta} \qquad ((x, \zeta) \in (F \times \boldsymbol{C}) \cap \Omega)$$

is holomorphic in $(F \times \boldsymbol{C}) \cap \Omega$. By Lemma 7.1, there exists a holomorphic function $\tilde{h}$ on Ω such that $\tilde{h}(x, \zeta) = \frac{h \circ q(x,\zeta)}{\zeta}$ for every $(x, \zeta) \in \Omega \cap (F \times \boldsymbol{C})$. We define a holomorphic function g on Ω by $g(x, \zeta) = \zeta\tilde{h}(x, \zeta)$ for every $(x, \zeta) \in \Omega \subset E \times \boldsymbol{C}$. We define a $\boldsymbol{C}^*$-invariant holomorphic function $\tilde{g}$ on Ω by

$$\tilde{g}(x, \zeta) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta}x, e^{i\theta}\zeta)d\theta$$

for every $(x, \zeta) \in \Omega \subset E \times \boldsymbol{C}$. Then we have $\tilde{g}|\Omega \cap (F \times \boldsymbol{C}) = h \circ q$. We define a holomorphic function f on ω by $f(z) = \tilde{g} \circ q^{-1}(z)$ for every $z \in \omega$. Then we have $f \circ q(x, 0) = 0$ for every $(x, 0) \in \Omega \subset E \times \boldsymbol{C}$. Since $f|\omega_{F\times\boldsymbol{C}} = h$, f is not identically zero. This completes the proof.

We recall that the Borel transform of an entire function F of exponential type in one complex variable is given for large $|t|$ by

$$H(t) = \sum_{j=1}^{\infty} A_j t^{-j-1} \qquad \text{if } F(\tau) = \sum_{j=1}^{\infty} A_j \frac{\tau^j}{j!}$$

The corresponding integral representation is

$$H(t) = \int_0^{\infty} F(s\tau)e^{-st\tau}\tau ds,$$

where $\tau \in \boldsymbol{C}$ has to chosen suitably for every t. It follows from this formula that H can be holomorphically continued into the complement of the convex compact set

$$K = \{t \in \boldsymbol{C};\ \text{for all } \tau \in \boldsymbol{C},\ \limsup_{s\to\infty} \frac{1}{s}\log|F(st)| \geq Re\ t\tau\}.$$

Conversely we have

$$F(\tau) = \frac{1}{2\pi i}\int_{\Gamma} H(t)e^{t\tau}dt,$$

where Γ is some large circle. This integral representation of F shows immediately that for all $\epsilon > 0$ we have

$$|F(\tau)| \leq C_\epsilon \exp\left(\sup_{t\in L} Re\ t\tau + \epsilon|\tau|\right)$$

if H is holomorphic outside a compact convex set $L \subset \boldsymbol{C}$.

Let E be a locally convex space and p be a positively homogeneous plurisubharmonic function of order 1 on E with values in $[-\infty, \infty)$. Then we set

$$D_p = \cup_{t\in\boldsymbol{C}}\{z \in E;\ p(tz) < Re\ t\}, \tag{7.3}$$

$$\Omega_p = \cup_{t\in\boldsymbol{C}}\{(z,\zeta) \in E \times \boldsymbol{C};\ p(tz) < Re\ t\zeta\}, \tag{7.4}$$

$$\omega_p = q(\Omega_p). \tag{7.5}$$

Then by Kiselman[18, Theorem 3.1 and 3.3] and by Proposition 2.3, we have the following Lemma 7.3.

Lemma 7.3. *The open set D_p of E is connected and pseudoconvex. The open set ω_p of the projective space $\boldsymbol{P}(E \times \boldsymbol{C})$ is connected, proper, and pseudoconvex if and only if p is not identically $-\infty$. Moreover ω_p determines F uniquely : if plurisubharmonic functions p and r on E are positively homogeneous of order 1, we have $r \leq p$ if and only if $\omega_p \subset \omega_r$*

Proof of Theorem 2. If $p = -\infty$, we take $f = 0$. We assume that p is not identically $-\infty$. Then we consider the open sets Ω_p and ω_p, defined by (7.4) and (7.5), of $(E\times\boldsymbol{C})-\{0\}$ and of the projective space $\boldsymbol{P}(E\times\boldsymbol{C})$ respectively. By Lemma

7.3 the open set ω_p of the projective space $\boldsymbol{P}(E \times \boldsymbol{C})$ is pseudoconvex. Therefore by Theorem 1 there exists a non-constant holomorphic function f_1 of ω_p such that for every connected open neighborhood V of an arbitrary point on the boundary of ω_p, each component of $V \cap \omega_p$ contains zero of f_1 of arbitrarily high order. By Lemma 7.2 there exists a function $f_2 \in H(\omega_p)$ such that f_2 is not identically zero and $f_2 \circ q|\omega \cap (E \times \{0\}) = 0$. We set $f_3 = f_1 f_2$. Then the domain ω_p is the domain of existence of a holomorphic function f_3 on ω_p. We define a holomorphic function g on D_p by $g(z) = f_3 \circ q(z, 1)$ for every $z \in D_p$. The open set D_p of the space E is connected and contains the origin 0 in E. Thus there exist continuous n-homogeneous polynomials $p_n : E \to \boldsymbol{C}$ and a balanced open neighborhood W of 0 in E such that $W \subset D_p$ and that the expansion

$$g(z) = \sum_{n=0}^{\infty} p_n(z)$$

uniformly for $z \in W$. We define an entire function on E by

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} p_n(z)$$

for every $z \in E$. Then by Boas[2, Theorem 5.3.1] and by Hervé[13, Theorem 3.3.9] the function f is an entire function of exponential type on E. We consider a function r on E which is positively homogeneous of order 1 defined by

$$r(z) = \limsup_{s \to \infty} \frac{\log |f(sz)|}{s}$$

for every $z \in E$. We set $h_z(t) = g(z/t)/t = f_3 \circ q(z, t)/t$ for some fixed $z \in E$. Then h_z is the Borel transform of $\tau \to f(\tau z)$ so that

$$f(\tau z) = \frac{1}{2\pi i} \int_\Gamma h_z(t) e^{t\tau} dt.$$

In view of our choice of $f_3 \circ q$, h_z can be holomorphically continued to every point t such that $(z, t) \in \Omega_p$; in particular there is no singularity at the origin if $(z, 0) \in \Omega_p$.

We can therefore choose Γ in any neighborhood of the convex set

$$\{t \in \boldsymbol{C};\ \text{for all } \tau \in \boldsymbol{C},\ p(\tau z) \geq Re\ t\tau\}.$$

Thus for every $\epsilon > 0$ there exists an positive constant C_ϵ such that

$$|f(\tau z)| \leq C_\epsilon \exp\left(p(\tau z) + \epsilon|\tau|\right)$$

for every $\tau \in \boldsymbol{C}$ (z is fixed). Hence $r(z) \leq p(z)$ and since z is arbitrary, $r^* \leq p$ where we denote by r^* the upper regularized of the function r on E.

On the other hand, the integral

$$h_z(t) = \int_0^\infty f(s\tau z)e^{-st\tau}\tau ds$$

converges absolutely and uniformly for all (z,t) satisfying $r(\tau z) \leq Re\ t\tau - \epsilon$. It follows that $h_z(\zeta)$ is a holomorphic function of (z,ζ) in Ω_{r^*}, in particular $\zeta h_z(\zeta) = f_3 \circ q(z,\zeta)$ can be holomorphically continued to a function in $H(\Omega_{r^*})$. Since $\Omega_{r^*} = q^{-1}(\omega_{r^*})$ and the function $(z,\zeta) \to \zeta h_z(\zeta) = f_3 \circ q(z,\zeta)$ is $\boldsymbol{C}^*$-invariant, the function f_3 is continued holomorphically to ω_{r^*}. Since $\omega_{r^*} \cap \omega_p \neq \emptyset$, since ω_p is the domain of existence of f_3 and since by Lemma 7.3 ω_{r^*} is connected, we have $\omega_{r^*} \subset \omega_p$. Thus by Lemma 7.3, $p \leq r^*$. This completes the proof .

Let E be a locally convex space. We induce the compact open topology in the space $H(E)$ of all entire functions in E. A continuous linear functional μ on $H(E)$ or, in other words, an element of the dual space $H(E)'$ of the space $H(E)$, is called an *analytic functional* in E. Let $\mu \in H(E)'$. The (generalized) Laplace transform $\hat{\mu}$ of μ is defined by $\hat{\mu}(\varphi) = \mu(e^\varphi)$, $\varphi \in H(E)$. Then the restriction of $\hat{\mu}$ on the dual space E' of E is the Fourier-Borel transformation of μ, which is an entire function of exponential type on E'.

Proof of Corollary 3. If E is a Fréchet nuclear space or a DFN- space, the corespondence $H(E')' \ni \mu \to \hat{\mu} \in EXP(E)$ by the Fourier-Borel transformation is bijective(cf. Colombeau[4] and Dineen[8]). Thus by Theorem 2 the proof of Corollary 3 is completed.

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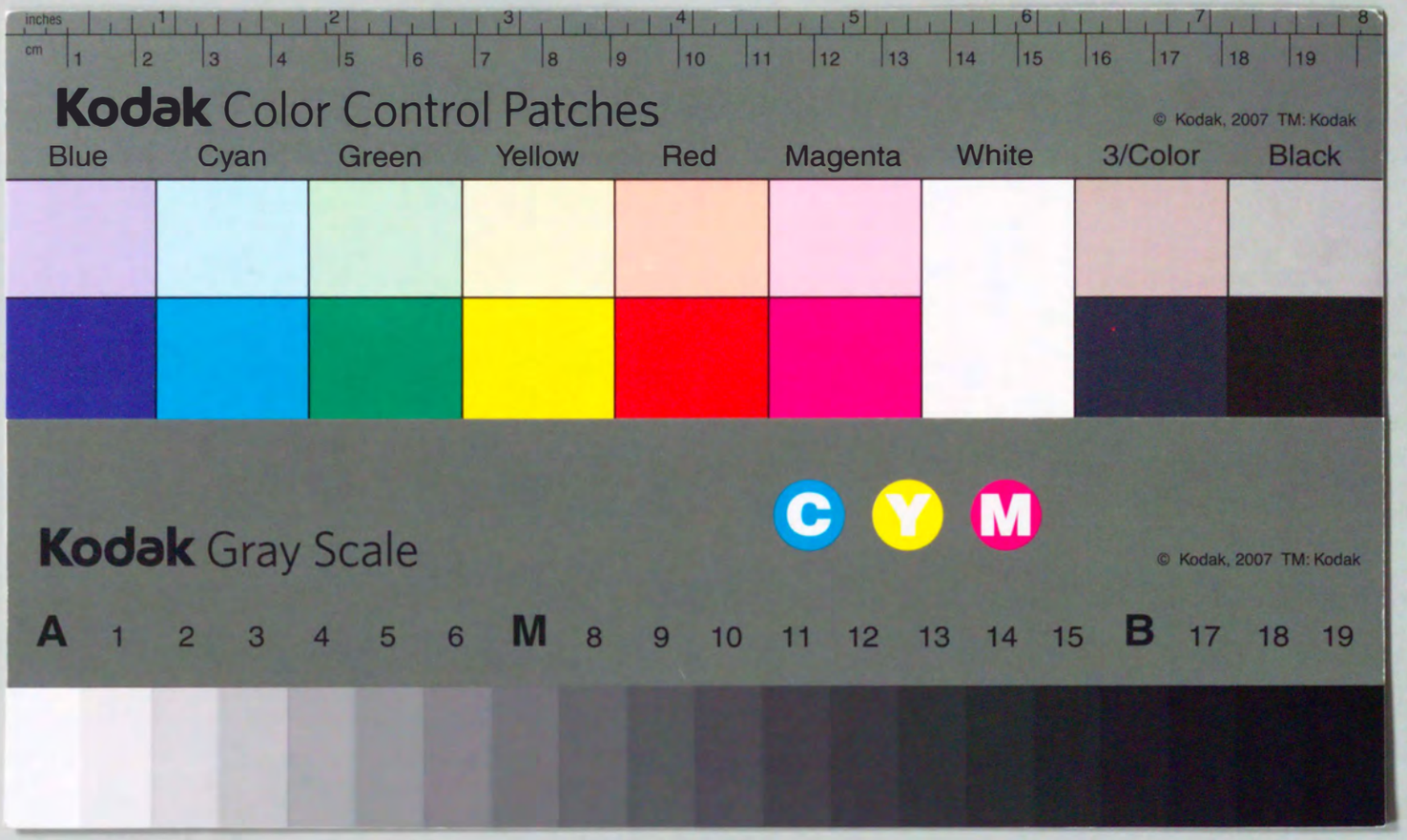
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