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Theory of Eddy Viscosity for Two-Dimensional Inviscid Barotropic Fluid

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Abstract

Although it is possible as a first approximation to regard the terrestrial atmosphere as an inviscid fluid in considering large scale atmospheric motions, there are some phenomena that can be explained only if the atmosphere has an enormous viscosity coefficient comparison to the molecular viscosity coefficient of the air. The enormous viscosity originates from turbulence and is called the eddy viscosity.

The eddy viscosity is a parameter to characterize a transport phenomenon of momentum, and the concept of it is independent of whether fluid is inviscid or not. Therefore, the eddy viscosity coefficient is treated as a transport coefficient for an inviscid fluid in this study. For simplicity, the eddy viscosity for two-dimensional inviscid barotropic fluid is investigated in this study.

According to Mori's theory, the spectral form of the vorticity equation for two-dimensional inviscid barotropic fluid is exactly reduced to the generalized Langevin equation for the vorticity:

$$\frac{d}{dt}\,\zeta_t(\mathbf{k}) = i\,\Omega_{\mathbf{k}}\,\zeta_t(\mathbf{k}) - \int_0^t d\tau\,\gamma_{\mathbf{k}}(t-\tau)\,\zeta_\tau(\mathbf{k}) + R_t(\mathbf{k}),$$

where $\zeta_t(\mathbf{k})$ is the Fourier component of the vorticity with wave number $\mathbf{k}$ at time t, $\Omega_{\mathbf{k}}$ is a frequency and $R_t(\mathbf{k})$ is a random force. The damping term in the Langevin equation is interpreted as the eddy viscosity damping term, because it originates from the nonlinear effect from which turbulence come out. The memory function $\gamma_{\mathbf{k}}(t)$ is expressed in terms of the correlation function of the random force $R_t(\mathbf{k})$, that is constructed by the vorticity equation. The relation between the memory function and the random force is called the fluctuation-dissipation theorem. A formula for the eddy viscosity coefficient is derived by using the fluctuation-dissipation theorem.

The eddy viscosity coefficient thus obtained depends on the length scale of a phenomenon of interest. Under Cheju Island scale, the theory derives that the eddy

viscosity coefficient is about eight orders of magnitude larger than that of the molecular viscosity coefficient of the air. This value is of the same order as the eddy viscosity coefficient obtained by applying the Reynolds' law of similarity to atmospheric Kármán vortices generated on the leeward of Cheju Island. It is theoretically derived that the reason for the enormousness of the atmospheric eddy viscosity coefficient in comparison to the molecular viscosity coefficient of the air comes from that of the length scale of interest.

§1. Introduction

It is widely known that the terrestrial atmosphere can be approximately considered as an inviscid fluid, because atmospheric waves, e.g., inertio-gravity waves, propagate up to the great height almost conserving their energy density (e.g., see Lindzen, 1990). On the other hand, there are some examples contradictory to what the atmosphere can be considered as inviscid. A famous example is one as follows. In laboratory experiments, it is known that a condition for being generated the Kármán vortices on the leeward of a cylinder is given by Reynolds' number $R_e = UD/\nu \sim 10^2$, where U is the mean velocity of the fluid, D is the diameter of the cylinder and ν is the kinematic viscosity coefficient of the fluid (Batchelor, 1967). If we apply the Reynolds' law of similarity to atmospheric Kármán vortices generated on the leeward of Cheju Island of South Korea, we have $\nu = UD / R_e \sim (10\ \mathrm{ms^{-1}}) \times (30\ \mathrm{km}) / 10^2 \sim 10^3\ \mathrm{m^2s^{-1}}$. This means that the atmosphere must have an effective viscosity coefficient of eight orders of magnitude larger than that of the molecular viscosity coefficient of the air, which is $\nu_{mol} \sim 10^{-5}\ \mathrm{m^2s^{-1}}$, in order to explain the generation of the atmospheric Kármán vortices (Tsuchiya, 1969). Since this anomalous viscosity originates from turbulence in the atmosphere, it is called the eddy viscosity. The eddy viscosity has been explained by the Reynolds' stress, qualitatively (e.g., see Holton, 1979). However no quantitative arguments on the eddy viscosity, especially elucidation of anomalousness of it, from the fundamental equation of fluid dynamics have been performed yet.

Kraichnan (1976) investigated the eddy viscosity by using both a model dynamical equation for turbulence and an energetic consideration. He and his colleagues (Kraichnan, 1970; Leith, 1971; Herring and Kraichnan, 1972) developed a model equation for the turbulent velocity field in order to study the concept of the eddy viscosity in the frame of the direct interaction approximation (ordinary, referred as DIA) (Kraichnan, 1959). The model equation is constructed by replacing the nonlinear term of the Navier-Stokes equation by a random variable which has same covariance as the velocity field and adding a dynamical damping term with memory. The damping term is chosen to hold the energy balance between the damping term and the random variable. The DIA also leads to a family of related approximations of similar structure. If the random variable is replaced by a white noise in time, then the memory of dynamical damping term reduces to zero. A model equation obtained by applying this approximation is called the test field model (ordinary, referred as TFM) equation (Kraichnan, 1971). Kraichnan (1976) defined an effective eddy viscosity acting on large scales due to sub-grid scales in terms of both an energy transfer function which is constructed by the TFM equation and the energy spectrum function and obtained the negative eddy viscosity coefficient for a two-dimensional flow in the energy inertial range, which originates from the upward or inverse energy cascade (Kraichnan, 1967).

In this paper, we derive the eddy viscosity coefficient by a statistical mechanical theory which derives a transport coefficient in macroscopic irreversible processes from a microscopic reversible physical law. The eddy viscosity is a parameter to characterize a transport phenomenon of momentum[1] and the concept of it is independent of whether fluid is inviscid or not. Therefore, we regard the eddy viscosity as a transport coefficient for an inviscid fluid.

Prandtl (1925) discussed the eddy viscosity according to the kinetic theory, which is one of the statistical mechanical theories to derive the transport coefficient from microscopic properties of matter. In the kinetic theory, the occurrences of transport phenomena are due to collisions of constructing particles of matter. Prandtl hypothesized that the eddy viscosity comes from collisions between lumps of fluid. Although the kinetic theory requires assumptions on the mean collision time or the mean free path of the constructing particles, it is possible to provide them reasonably from the

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[1] Since the viscosity, the thermal conduction and the diffusion relate to the transport of momentum, energy and mass, respectively, these phenomena and their similar ones, e.g., electric conduction, are called the transport phenomena and the viscosity coefficient, the thermal conductivity and the diffusion coefficient are called the transport coefficients.

size of the particles. However, it is impossible to provide a reasonable assumption on the mean collision time and the mean free path of the lumps of fluid, which is called the mixing length, because the size and density of the lumps are arbitrary. Therefore, the Prandtl's theory gives a physical picture to the eddy viscosity coefficient, but can not provide any quantitative explanation of the magnitude of the coefficient.

We discuss the eddy viscosity within the framework of the linear response theory, especially the fluctuation-dissipation theorem. The linear response theory derives the transport coefficient by calculating the relaxation rate to an equilibrium state of a system when it is disturbed by a small perturbation from the equilibrium. Our theory is proposed by the reinterpretation of Gambo's study (Gambo, 1982, hereafter, referred as G82). He showed by data analysis that the quasi-geostrophic vorticity equation for the ultra-long waves in the mid-latitude might be treated in the same way as the simple Langevin equation for Brownian motion. By the way, in the statistical mechanics, it is known that a fluctuating motion of a mode in a macroscopic dynamical system consisting of a very large number of particles or a large number of degrees of freedom is generally described by a Langevin equation. The vorticity equation can be written by the spectral form which is the system of the ordinary differential equations with infinite degrees of freedom for the Fourier components of the vorticity. Therefore, G82 may be reinterpreted as follows: a fluctuating motion of the modes for the ultra-long waves in the spectral form of the vorticity equations may be treated in the same way as the simple Langevin equation.

Mori (1965a) derived the generalized Langevin equation from the equation of motion for dynamical variables. The damping term in the generalized Langevin equation is related to the correlation function of the random force which is quite different from the random force in the simple Langevin equation and comes from a nonlinearity of the system, i.e., it is not an externally added force, but the random force is constructed by the equation of motion. The relation between the memory function and the random force is called the fluctuation-dissipation theorem.

We consider a two-dimensional (2-D) inviscid barotropic vorticity equation, for simplicity. As suggested by the reinterpretation of G82, according to Mori's theory, the spectral form of the vorticity equation is exactly reduced to the generalized Langevin equation for the vorticity. Since the damping term in the generalized Langevin equation comes from a nonlinearity of the system, it is interpreted as the eddy viscosity damping term. Hence we define the eddy viscosity coefficient as a damping coefficient of the damping term to be of diffusion type and derive it by the fluctuation-dissipation theorem.

In section 2, we give an explanation of the concept of the eddy viscosity. Here, it is clearly explained that the eddy viscosity coefficient is the transport coefficient of the inviscid fluid. Then Prandtl's mixing length theory and Kraichnan's theory are reviewed. In section 3, a derivation of the generalized Langevin equation from the equation of motion for dynamical variables is presented in detail and a comment on the random force is given. Section 4 presents our theory of the eddy viscosity coefficient. In section 5, a preliminary numerical simulation is performed by using Kells and Orszag's model with 20 independent variables (Kells and Orszag, 1978) in order to examine the validity of an approximation used in section 4. Moreover, numerical results of our theory are presented. Discussions are also described. We devote the last section to the summary and perspectives.

§2. Brief Review

In this section, at first, we give an explanation of the concept of the eddy viscosity. Next we review two conventional theories to determine the eddy viscosity coefficient, Prandtl's mixing length theory (Prandtl, 1925) and Kraichnan's theory (Kraichnan, 1976).

2.1 Eddy Viscosity

Now we consider an inviscid and incompressible fluid. The motion of the fluid is governed by the Euler's equation,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \tag{2.1}$$

and the continuity equation,

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2.2}$$

Here we use Einstein's rule of summation. If we express the velocity field u_i as the summation of the average over a time interval, $\langle u_i \rangle$, and its deviation from the average, u_i', which can be regarded as eddy, the equation of motion for $\langle u_i \rangle$ is given by

$$\frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i} - \frac{\partial}{\partial x_j}\langle u_i' u_j' \rangle. \tag{2.3}$$

Equation (2.3) can be rewritten as the equation for the momentum density by

$$\frac{\partial}{\partial t}\rho \langle u_i \rangle = -\frac{\partial}{\partial x_j}\langle \Pi_{ij} \rangle, \tag{2.4}$$

$$\langle \Pi_{ij} \rangle = \langle p \rangle \, \delta_{ij} + \rho \langle u_i \rangle \langle u_j \rangle + \rho \langle u_i' u_j' \rangle, \tag{2.5}$$

where Π_{ij} is called the momentum flux density tensor. A tensor $\tau_{ij} \equiv \rho \langle u_i' u_j' \rangle$ in the last term of the right hand side of (2.5) is called the Reynolds' stress tensor, that is the i-th component of the force acting on unit surface normal to the j-axis. On the other hand, the momentum flux density tensor for an incompressible viscous fluid is given by

$$\Pi_{ij} = p\ \delta_{ij} + \rho\ u_i\ u_j + \rho\ \nu_{mol} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{2.6}$$

where ν_{mol} is called the kinematic viscosity coefficient or the molecular viscosity coefficient (Landau and Lifshitz, 1987). On the analogy of (2.6), we suppose that the Reynolds' stress is expressed by

$$\tau_{ij} \equiv \rho\ \langle u_i' u_j' \rangle = \rho\ \nu_{eddy} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right), \tag{2.7}$$

i.e., the momentum transfer due to eddies can be written in terms of the mean flow and causes an effective viscosity on the mean flow. The coefficient ν_{eddy} in (2.7) is called the eddy viscosity coefficient. If the turbulence is isotropic and ν_{eddy} does not change noticeably throughout the fluid, equation (2.3) reduces to

$$\frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} = - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} + \nu_{eddy} \frac{\partial^2}{\partial {x_i}^2} \langle u_i \rangle, \tag{2.8}$$

which has an analogous form with the Navier-Stokes equation.

2.2 Prandtl's Mixing Length Theory

Although the value of the eddy viscosity coefficient ν_{eddy} should be determined by (2.7), the estimation of the coefficient ν_{eddy} have been practically performed by the Prandtl's mixing length theory (Prandtl, 1925), even in the present day since a long ago. This theory treats turbulent motion by the analogy with the kinetic theory of gases. That is, it hypothesizes that the eddy viscosity coefficient must be the product of a "mixing length" and a certain suitable velocity, because the molecular viscosity coefficient is expressed in terms of the product of the mean free path and the mean value of the fluctuating velocity (Lifshitz and Pitaevskii, 1981). We review the Prandtl's mixing length theory in the following after Stanišić (1988).

Now we consider an incompressible, steady flow with shear, $\langle \mathbf{U} \rangle = (\langle u(z) \rangle, 0, 0)$, as shown in Fig. 2.1. Prandtl assumed that lumps of fluid move in the x- and z-direction with conserving the x-component of their momentum over a certain characteristic length which is called the Prandtl's mixing length and is denoted by l. The velocity field can be expanded around a point z_0, because l is thought to be small,

$$\left\langle u(z_0 \pm l)\right\rangle = \langle u(z_0)\rangle \pm l \left.\frac{d\langle u(z)\rangle}{dz}\right|_{z = z_0}. \tag{2.9}$$

The lumps from $z_0 + l$ arrive at z_0 possessing a greater velocity in the x-direction. On the other hand, the lumps from $z_0 - l$ arrive at z_0 possessing a smaller velocity in the x-direction. Therefore the magnitude of the fluctuation of velocity in the x-direction is

$$|u'| = l \frac{d\langle u\rangle}{dz}. \tag{2.10}$$

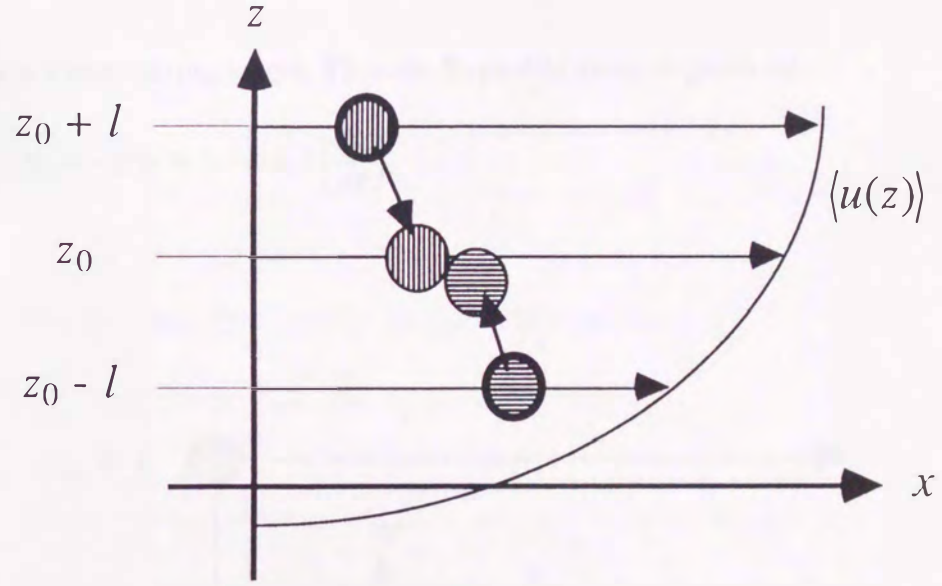

Fig. 2.1. Geometry of flow for Prandtl's mixing length theory.

We physically consider that the fluctuation of velocity in the z-direction, w', occurs as follows. If the lumps from $z_0 + l$ arrive at z_0 just left of those from $z_0 - l$, then they collide forcing fluid out in the z-direction (see Fig. 2.2). However, if the lumps from $z_0 + l$ arrive at z_0 just right of those from $z_0 - l$, then they move apart with a relative velocity $2\,u'$, causing a transverse flow to fill to the void left between them (see Fig. 2.3). Hence we can consider that w' and u' are same order of magnitude,

$$|w'| = c_1\,|u'|, \tag{2.11}$$

where c_1 is a positive constant of order of unity. Moreover, since we see that both of the lumps from $z_0 + l$ and $z_0 - l$ arrive at z_0 with negative momentum flux density $\rho \langle u'w' \rangle < 0$, we can assume that

$$\langle u'w' \rangle = - c_2 |u'| |w'|, \tag{2.12}$$

where c_2 is also a positive constant of order of unity. Introducing (2.10) and (2.11) into (2.12), we have

$$\langle u'w' \rangle = - c_1 c_2 l^2 \left(\frac{d\langle u \rangle}{dz} \right)^2 = - l_m{}^2 \left(\frac{d\langle u \rangle}{dz} \right)^2. \tag{2.13}$$

Here l_m is a new mixing length. Thus the Reynolds' stress is given by

$$\tau_{xz} = - \rho \langle u'w' \rangle = \rho\, l_m{}^2 \left(\frac{d\langle u \rangle}{dz} \right)^2. \tag{2.14}$$

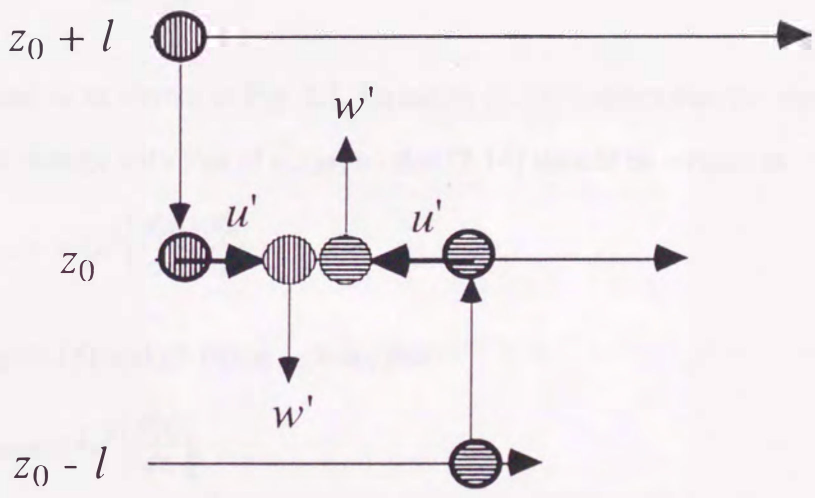

Fig. 2.2. Geometry of inter collision of lumps.

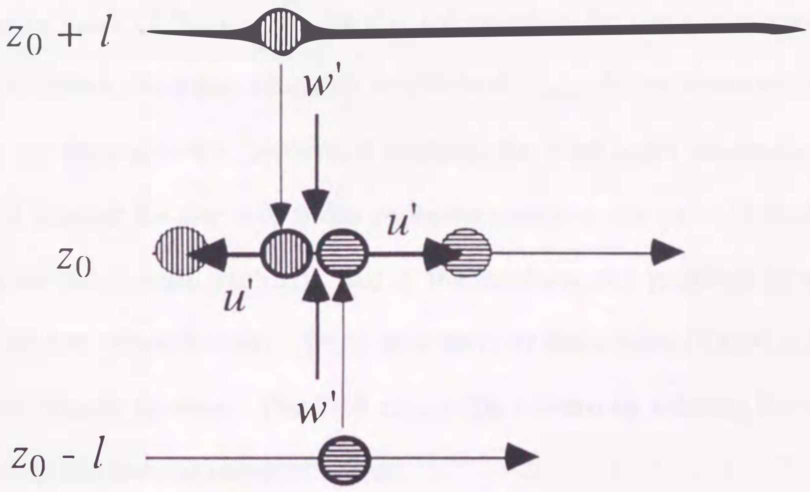

Fig. 2.3. Geometry of collision of lumps.

On the other hand, we see from (2.7) that the Reynolds' stress is

$$\tau_{xz} = \rho\, \nu_{eddy} \frac{d\langle u \rangle}{dz} \tag{2.15}$$

for the situation as shown in Fig. 2.1. Equation (2.15) implies that the sign of Reynolds' stress must change with that of shear, so that (2.14) should be written as

$$\tau_{xz} = \rho\, {l_m}^2 \left| \frac{d\langle u \rangle}{dz} \right| \frac{d\langle u \rangle}{dz}. \tag{2.16}$$

Comparing (2.15) and (2.16), it follows that

$$\nu_{eddy} = {l_m}^2 \left| \frac{d\langle u \rangle}{dz} \right|. \tag{2.17}$$

The eddy viscosity coefficient was replaced by the mixing length l_m. However, the mixing length is determined only by observations or experiments. Therefore, although the Prandtl's mixing length theory gives a physical picture to the eddy viscosity, it is impossible to determine the value of eddy viscosity coefficient and to explain anomalousness of it in the atmosphere from the first principle of physics, i.e., the equation of motion of the fluid dynamics.

2.3 Kraichnan's Theory

As seen from (2.7), we require the information for the second order moment in order to determine the eddy viscosity coefficient ν_{eddy}. If we construct the equation of motion for the second order moment, it contains the third order moments. In general, the equation of motion for the n-th order moment contains the $(n + 1)$-th order moments. This is called the closure problem, that is the fundamental problem in turbulence. The direct interaction approximation (DIA) proposed by Kraichnan (1959) is one of methods to avoid the closure problem. The DIA closes the system by relating the higher statistics to the terms of the second order moments.

Kraichnan and his colleagues (Kraichnan, 1970; Leith, 1971; Herring and Kraichnan, 1972) derived a model dynamical equation by the DIA to study the concept of the eddy viscosity. Kraichnan (1976) examined the dependence of the eddy viscosity coefficient on the wave number by using a test field model (TFM) that is one of the simplification of the DIA model.

In this sub-section, we present the DIA model equation, whose form is similar to that of the generalized Langevin equation which is one of the main subjects of this paper, and the Kraichnan's theory of the eddy viscosity on purpose to compare with our theory.

The equation of motion for an incompressible viscous fluid, Navier-Stokes equation, in a box of side L with cyclic boundary condition can be written in the spectral form as

$$\left(\frac{\partial}{\partial t} + \nu_{mol}\, k^2\right) u_i(\mathbf{k}, t) = - i\, k_m\, P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p}, t)\, u_m(\mathbf{q}, t), \tag{2.18}$$

where $u_i(\mathbf{k}, t)$ is the Fourier component with wave vector $\mathbf{k}$ of the i-th component of velocity, k_i is the i-th component of $\mathbf{k}$, k is the absolute value of $\mathbf{k}$ and

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i\, k_j \,/\, k^2. \tag{2.19}$$

The DIA may be regarded as a device for neglecting the statistical correlations among the terms contributing to the right hand side of (2.18), while systematic collection for the energy budget is made by adding an additional term which has the form of a generalized eddy damping. The model equation in the DIA for isotropic turbulence is given by

$$\left(\frac{\partial}{\partial t} + \nu_{mol}\, k^2\right) u_i(\mathbf{k}, t) + \int_0^t \eta(k, t, s)\, u_i(\mathbf{k}, s)\, ds$$

$$= - i\, k_m\, P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \xi_j(\mathbf{p}, t)\, \xi_m(\mathbf{q}, t). \quad (2.20)$$

Here $\xi_i(\mathbf{p}, t)$, which replaces $u_i(\mathbf{p}, t)$ in the nonlinear terms, is a multivariate-normal velocity field which is related to the field $u_i(\mathbf{p}, t)$ only in that they have the same second order moment:

$$\left\langle \xi_i(\mathbf{k}, t)\, \xi_j(\mathbf{k}, t')^* \right\rangle = \left\langle u_i(\mathbf{k}, t)\, u_j(\mathbf{k}, t')^* \right\rangle, \quad (2.21)$$

where the angular brackets denote average over an ensemble and * denotes the complex conjugate. The function $\eta(k, t, s)$ in the damping term which restores energy conservation is a functional of the second order moment and a Green's function. It is given by

$$\eta(k, t, s) = \pi\, k \int_\Delta b_3(k, p, q)\, G(p, t, s)\, U(q, t, s)\, p\, q\, dp\, dq\ , \quad (2.22)$$

where $G(k, t, s)$ is the Green's function associate with (2.18) and $U(k, t, s)$ is defined by

$$U(k, t, s) = \left(\frac{L}{2\,\pi}\right)^3 \left\langle u_i(k, t)\, u_i(k, s)^* \right\rangle. \quad (2.23)$$

The coefficient $b_3(k, p, q)$, which arises from dot products of P_{ij} operators, is given by

$$b_3(k, p, q) = \frac{1}{2}\, k^{-4} \sin^2\alpha \left[\left(k^2 - q^2\right)\left(p^2 - q^2\right) + k^2\, p^2\right], \quad (2.24)$$

where α is the internal angle opposite k in a triangle whose sides are k, p and q. The integration in (2.22) extends over all wave numbers p and q which can form a triangle together with k. The equations of motion for $G(k, t, s)$ and $U(k, t, s)$ are

$$\left.\begin{aligned}&\left(\frac{\partial}{\partial t} + \nu_{mol}\, k^2\right) G(k, t, t') + \int_0^t \eta(k, t, s)\, G(k, s, t')\, ds = 0,\\ &G(k, t', t') = 1,\end{aligned}\right\} \tag{2.25}$$

and

$$\left(\frac{\partial}{\partial t} + \nu_{mol}\, k^2\right) U(k, t, t')$$

$$= \pi k \int_\Delta b_3(k, p, q) \left[\int_0^t G(k, t', s)\, U(p, t, s)\, U(q, t, s)\, ds \right.$$

$$\left. - \int_0^{t'} G(p, t, s)\, U(k, t', s)\, U(q, t, s)\, ds \right] p\, q\, dp\, dq\,, \tag{2.26}$$

respectively. The equations (2.22)-(2.26) are the system of equations in the DIA. They can be solved only by numerical technique, because (2.25) and (2.26) are integro-differential equations.

The equation for the energy spectrum $E(k, t)$, which is determined by

$$E(k, t) = 2\, \pi\, k^2\, U(k, t, t), \tag{2.27}$$

is given by

$$\left(\frac{\partial}{\partial t} + \nu_{mol}\, k^2\right) E(k, t) = T(k, t). \tag{2.28}$$

Here $T(k, t)$ is an energy transfer function, which is constructed in terms of the right hand side of (2.26). The energy transfer function satisfies the overall conservation and the detailed conservation, because the DIA model equation (2.20) is the one for energy consistent model.

Several relatives of the DIA can be obtained by altering the form of the random nonlinear terms in the model equation or by modifying (2.25) and (2.26) (Leith, 1971;

Herring and Kraichnan, 1972). The TFM model equation is obtained by replacing the nonlinear term by a white noise in time (Kraichnan, 1971b). It is

$$\left(\frac{\partial}{\partial t} + \nu_{mol}\, k^2 + \eta(k, t)\right) u_i(\mathbf{k}, t)$$

$$= - i\, k_m\, P_{ij}(\mathbf{k})\, w(t) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \left[\theta_{kpq}(t)\right]^{1/2} \xi_j(\mathbf{p}, t)\, \xi_m(\mathbf{q}, t) \quad , (2.29)$$

where $\xi_i(\mathbf{p}, t)$ is again a random velocity field related to $u_i(\mathbf{p}, t)$ by (2.21). The new function $w(t)$ is a random, zero-mean, white noise process which satisfies

$$\langle w(t)\, w(t') \rangle = \delta(t - t'). \qquad (2.30)$$

The function θ_{kpq} is a characteristic interaction time for the wave number triad (k, p, q). It is determined by the equations which measure the self-distortion of the velocity field by advection and pressure force (Kraichnan, 1971a). The memory of the dynamical damping term in the model equation is lost, because the nonlinear term is replaced by the white noise in time. The damping coefficient $\eta(t)$ is given by

$$\eta(k, t) = \pi\, k \int_{\Delta} b_3(k, p, q)\, \theta_{kpq}\, U(q, t, t)\, p\, q\, dp\, dq \quad . \qquad (2.31)$$

Kraichnan (1976) constructed an energy transfer function $T(k \mid k_m)$ that is contribution to $T(k)$ from the triads (k, p, q) such that p and/or $q > k_m$ and $k < k_m$ in the TFM. Then he defined an effective viscosity acting on the mode of wave number k due to dynamical interaction with wave number $> k_m$, i.e., the eddy viscosity, by

$$\nu_{eddy}(k \mid k_m) = - T(k \mid k_m) \,/\, \left[2\, k^2\, E(k)\right]. \qquad (2.32)$$

In both two- and three-dimensions, the eddy viscosity coefficient (2.32) for $k \ll k_m$ was independent of k and of the local shape of the spectrum. In two-dimension, this asymptotic eddy viscosity coefficient was negative. It may be interpreted that this negative value means the existence of the upward (or inverse) cascade of energy in two-dimensional flow (Kraichnan, 1967).

§3. Generalized Langevin Equation

In this section, we give a detailed explanation of a derivation of the generalized Langevin equation from an equation of motion of dynamical variables after Kubo, et al. (1985) and Lovesey (1980, 1986).

The fundamental logical structure of the statistical physics is the derivation of the macroscopic laws through a various stage of coarse graining of microscopic law. As proceeding one step of coarse graining, information are contracted. This contraction is a projection of the object onto a certain cross section of our understandings. The projected process is recognized as a stochastic process. For example, the stochastic process called Brownian motion is the projection of microscopic motion of a colloid particle and all the molecules of the surrounding liquid onto the dimension of the motion of the colloid particle only. The motion of the colloid particle of mass m is governed by the simple Langevin equation,

$$m \frac{dv(t)}{dt} = - \gamma v(t) + R(t), \tag{3.1}$$

where $v(t)$ is the velocity of the colloid particle, γ is the frictional constant and $R(t)$ is a random force whose auto-correlation is proportional to the Dirac delta function, $\langle R(0)\, R(t)\rangle \propto \delta(t)$, where brackets denote the average over an ensemble. γ and $R(t)$ come from interactions between the colloid particle and the molecules.

Brownian motion is not restricted to the colloid particle. It is, generally speaking, a fluctuating motion of a mode in a macroscopic dynamical system with a very large number of particle or a large number of degrees of freedom. It is merely for a particle much heavier than the molecules in the medium, described well by the simple Langevin equation (3.1). However, various modifications are required for the Langevin equation to be applied to more general sorts of Brownian motion. One modification is to abandon the assumption of a white spectrum for the random force $R(t)$. This means that the retarded friction is account for, i.e., the stochastic evolution of a system depends on the history of the evolution.

In the simple Langevin equation (3.1), the friction is assumed to be determined by the instantaneous velocity of particle. Introducing the retarded friction in the simple Langevin equation (3.1), it can be generalized to

$$m \frac{dv(t)}{dt} = - \int_0^t \gamma(t - t')\, v(t')\, dt' + R(t), \tag{3.2}$$

where $\gamma(t)$ expresses the friction retardation. Equation (3.2) is called a generalized Langevin equation.

Mori (1965a) discussed a microscopic derivation of the generalized Langevin equation. He applied the damping formalism to an equation of motion of dynamical variables, transforming it to a form which corresponds to the Langevin equation and provided a stochastic interpretation to it. Thus Mori's generalized Langevin equation is the exact equation of motion for variables of interest. The validity of a Langevin equation is prescribed by the time scale of the random force. The simple Langevin equation (3.1) is valid for the description of processes which occur on a time scale much longer than the mean interval between atomic collision. On the other hand, the generalized Langevin equation is not restricted to the description of dynamical system in long-time. The useful properties of this equation is to be able to separate the force into components with different time scale.

The form of the exact generalized Langevin equation, which will be derived in the following, for a variable A is

$$\frac{d}{dt} A_t = i\, \Omega\, A_t - \int_0^t \gamma(t - t')\, A_{t'}\, dt' + R(t), \tag{3.3}$$

where Ω is the frequency and $\gamma(t)$ is called the memory function. Before the derivation of (3.3), we perform some preliminary discussions on a scalar product, projection operators and the equation of motion of dynamical variables.

3.1 Preliminary Discussions

3.1.1 Equation of Motion of Dynamical Variables

We denote a point in phase space by $(\mathbf{p}, \mathbf{q})$. By the motion of the system, a phase point $(\mathbf{p}, \mathbf{q})$ moves to $(\mathbf{p}_t, \mathbf{q}_t)$ in a time interval t. Now we consider the evolution of a dynamical variable A for a conservative system. The evolution of A is denoted by

$$A_t \equiv A(\mathbf{p}_t, \mathbf{q}_t) = A(\mathbf{p}, \mathbf{q}; t). \tag{3.4}$$

The last expression indicates that the value of A_t is determined by the initial phase point $(\mathbf{p}, \mathbf{q})$ and time t. The equation of motion of A_t is given by

$$\begin{aligned}\frac{d}{dt} A_t &= \left(\dot{\mathbf{q}}_t \cdot \frac{\partial A_t}{\partial \mathbf{q}_t} + \dot{\mathbf{p}}_t \cdot \frac{\partial A_t}{\partial \mathbf{p}_t}\right) = \left(\frac{\partial H_t}{\partial \mathbf{p}_t} \cdot \frac{\partial A_t}{\partial \mathbf{q}_t} - \frac{\partial H_t}{\partial \mathbf{q}_t} \cdot \frac{\partial A_t}{\partial \mathbf{p}_t}\right) \\ &= \{H_t, A_t\}_{\mathbf{p}_t, \mathbf{q}_t}. \end{aligned} \tag{3.5}$$

Here H_t is the Hamiltonian of the system at time t and the curly bracket denotes the Poisson bracket in the classical mechanics. The phase point at time t is function of the initial phase point and time, i.e.,

$$\left.\begin{aligned} \mathbf{p}_t &= \mathbf{p}(\mathbf{p}, \mathbf{q}; t), \\ \mathbf{q}_t &= \mathbf{q}(\mathbf{p}, \mathbf{q}; t). \end{aligned}\right\} \tag{3.6}$$

Therefore it is able to describe the initial phase point $(\mathbf{p}, \mathbf{q})$ by $(\mathbf{p}_t, \mathbf{q}_t)$. This is the canonical transformation from $(\mathbf{p}_t, \mathbf{q}_t)$ to $(\mathbf{p}, \mathbf{q})$, because both the phase points satisfy the canonical equations, respectively,

$$\left.\begin{aligned} \dot{\mathbf{p}}_t &= -\frac{\partial H_t}{\partial \mathbf{q}_t}, \qquad \dot{\mathbf{q}}_t = \frac{\partial H_t}{\partial \mathbf{p}_t}, \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}}, \qquad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}. \end{aligned}\right\} \tag{3.7}$$

Since the Poisson bracket is invariant under the canonical transformations (e.g., Landau and Lifshitz, 1976), equation (3.5) can be rewritten as follows[1] :

$$\frac{d}{dt}A_t = \{H_t, A_t\}_{\mathbf{p},\,\mathbf{q}} = \{H, A_t\}_{\mathbf{p},\,\mathbf{q}}$$

$$= \left(\frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial A_t}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial A_t}{\partial \mathbf{p}}\right) = \left(\dot{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}}\right) A_t$$

$$\equiv \Gamma A_t. \tag{3.8}$$

Here Γ is the Liouville operator which is constructed by the initial phase points. A formal solution of (3.8) is written as

$$A_t = e^{t\,\Gamma} A. \tag{3.9}$$

The operator $e^{t\,\Gamma}$ is called the propagator. Equation (3.9) implies that the value of A_t can be obtained by operating the propagator to the initial value A. The validity of this description (3.9) is readily examined for the harmonic oscillator, $H = \frac{p^2}{2\,m} + \frac{1}{2}\,k\,q^2$.

From the conservation of the number of representative points in the phase space, a distribution function ρ_t in the phase space satisfies the continuity equation,

$$\frac{\partial}{\partial t}\rho_t + \frac{\partial}{\partial \mathbf{p}_t} \cdot (\rho_t\, \dot{\mathbf{p}}_t) + \frac{\partial}{\partial \mathbf{q}_t} \cdot (\rho_t\, \dot{\mathbf{q}}_t) = 0. \tag{3.10}$$

Since the second and the third term in (3.10) reduce to

$$\frac{\partial}{\partial \mathbf{p}_t} \cdot (\rho_t\, \dot{\mathbf{p}}_t) + \frac{\partial}{\partial \mathbf{q}_t} \cdot (\rho_t\, \dot{\mathbf{q}}_t) = \left\{\dot{\mathbf{p}}_t \cdot \frac{\partial}{\partial \mathbf{p}_t} + \dot{\mathbf{q}}_t \cdot \frac{\partial}{\partial \mathbf{q}_t}\right\} \rho_t$$

with the aid of the canonical equations (3.7), equation (3.10) becomes

$$\frac{\partial}{\partial t}\rho_t = -\left\{\dot{\mathbf{p}}_t \cdot \frac{\partial}{\partial \mathbf{p}_t} + \dot{\mathbf{q}}_t \cdot \frac{\partial}{\partial \mathbf{q}_t}\right\} \rho_t \equiv \Gamma^{(d)} \rho_t\,. \tag{3.11}$$

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[1] It can be seen that (3.8) is valid without noticing the invariant of the Poisson bracket under the canonical transformations. The derivative dA_t/dt can depend on the properties of the motion of the system, and not on the particular choice of variables, (see Landau and Lifshitz, 1976, §45).

This is called the Liouville equation. The Liouville operator Γ for the equation of motion (3.8) has a negative sign in contrast to that of the Liouville equation (3.11) for the distribution function in the phase space and is constructed by the initial phase points.

3.1.2 Inner Product and Projection Operators

Now we define the inner product of the arbitrary phase functions f and g, which is denoted by (f, g), as

$$(f, g) \equiv \int d\mathbf{p}\, d\mathbf{q}\, \rho f g \ , \tag{3.12}$$

where ρ is the phase distribution function which satisfy the stationary Liouville equation,

$$\Gamma \rho = 0, \tag{3.13}$$

and the integration extends over whole phase space. It is readily seen that $(f, g) = (g, f)$. Moreover,

$$\begin{aligned}(f_t, g_t) &= \int d\mathbf{p}\, d\mathbf{q}\, \rho f_t g_t \\ &= \int d\mathbf{p}_t\, d\mathbf{q}_t\, \rho_t f_t g_t \\ &= (f, g).\end{aligned} \tag{3.14}$$

In the second modification, the Liouville's theorem $d\mathbf{p}_t\, d\mathbf{q}_t = d\mathbf{p}\, d\mathbf{q}$ and the stationary property of ρ were used. From (3.14), we see that

$$\frac{d}{dt}\left(f_t, g_t^*\right) = \left(\Gamma f_t, g_t^*\right) + \left(f_t, \left(\Gamma g_t\right)^*\right) = 0$$

or

$$\left(\Gamma f_t, g_t^*\right) = -\left(f_t, \left(\Gamma g_t\right)^*\right). \tag{3.15}$$

That is, the Liouville operator Γ is anti-Hermitian.

Next we define the projection operator $\widehat{P}$ that projects a function f onto the A axis by

$$\widehat{P} f \equiv \frac{(f,\, A^*)}{(A,\, A^*)} A. \tag{3.16}$$

Here * denotes the complex conjugate. It is easily seen that $\widehat{P}$ satisfies the following properties,

$$\left(\widehat{P} f,\, g^*\right) = \left(f,\, \left(\widehat{P} g\right)^*\right) \tag{3.17}$$

and

$$\widehat{P}^2 = \widehat{P}. \tag{3.18}$$

The other projection operator $\widehat{Q}$ is defined by

$$\widehat{Q} \equiv 1 - \widehat{P}. \tag{3.19}$$

$\widehat{Q}$ also satisfies the same properties of $\widehat{P}$, i.e.,

$$\left(\widehat{Q} f,\, g^*\right) = \left(f,\, \left(\widehat{Q} g\right)^*\right) \tag{3.20}$$

and

$$\widehat{Q}^2 = \widehat{Q}. \tag{3.21}$$

Equations (3.18) and (3.21) are why the operators $\widehat{P}$ and $\widehat{Q}$ are called the projection operators. The projection operators satisfy the Hermitian condition, (3.17) and (3.20).

3.2 Derivation of the Generalized Langevin Equation

In the derivation of the generalized Langevin equation (3.3), the projection operators, $\widehat{P}$ and $\widehat{Q}$, are used to separate off the time-dependent part of the variable of interest,

$$A_t = \widehat{P}\, A_t + \widehat{Q}\, A_t = \Xi(t)\, A + A_t', \tag{3.22}$$

where $\Xi(t)$ is the auto-correlation function of A, which is defined by

$$\Xi(t) \equiv \frac{\left(A_t,\, A^*\right)}{\left(A,\, A^*\right)}. \tag{3.23}$$

The second equality in (3.22) defines A_t'. Because we see that $A = \ \Xi(0)\, A + A'$ from (3.22) and $\Xi(0) = 1$ from (3.23), we have

$$A' = 0. \tag{3.24}$$

A_t' is orthogonal to A for all time $t > 0$,

$$\left(A_t',\, A^*\right) = \left(\widehat{Q}\, A_t,\, A^*\right) = \left(A_t,\, \left(\widehat{Q}\, A\right)^*\right) = 0, \tag{3.25}$$

where (3.20) was used in the second modification. The Laplace transform of a function f_t is denoted by $f[s]$,

$$f[s] \equiv \int_0^\infty e^{-s\,t} f_t\, dt\, . \tag{3.26}$$

We use the notation $\dot{A} = \Gamma A$.

The Laplace transform of the generalized Langevin equation (3.3) and of (3.22) are

$$\left(s - i\,\Omega + \gamma[s]\right) A[s] = A + R[s] \tag{3.27}$$

and

$$A[s] = \Xi[s]\, A + A'[s], \tag{3.28}$$

respectively. Equation (3.27) is consistent with (3.28), provided that

$$\Xi[s] = \frac{1}{\left(s - i\,\Omega + \gamma[s]\right)} \tag{3.29}$$

and

$$A'[s] = \frac{R[s]}{\left(s - i\,\Omega + \gamma[s]\right)} = \Xi[s]\; R[s], \tag{3.30}$$

or

$$A_t{}' = \int_0^t \Xi(t')\; R_{t-t'}\; dt'. \tag{3.31}$$

Substituting (3.31) into (3.25),

$$\left(A_t{}',\; A^*\right) = \int_0^t \Xi(t')\left(R_{t-t'},\; A^*\right) dt' = 0, \tag{3.32}$$

we see that R_t is orthogonal to A for all time $t > 0$.

Next we deduce the equation of motion of $A_t{}'$.

$$\begin{aligned} \frac{d}{dt} A_t' &= \frac{d}{dt} \widehat{Q}\, e^{t\,\Gamma} A = \widehat{Q}\,\Gamma e^{t\,\Gamma} A = \widehat{Q}\,\Gamma A_t \\ &= \widehat{Q}\,\Gamma\left(\Xi(t)\; A + A_t'\right) \\ &= \Xi(t)\; \widehat{Q}\,\dot{A} + \widehat{Q}\,\Gamma A_t'. \end{aligned} \tag{3.33}$$

Thus noting that (3.24) and $\Xi(0) = 1$,

$$\dot{A}' = \widehat{Q}\,\dot{A}. \tag{3.34}$$

Moreover from (3.31), we have

$$\frac{d}{dt} A_t' = \Xi(t)\; R \tag{3.35}$$

and

$$\dot{A}' = R\;. \tag{3.36}$$

Hence

$$R = \widehat{Q}\dot{A} = \widehat{Q}\,\Gamma A\,. \tag{3.37}$$

Substituting (3.37) into (3.33) and integrating it, we obtain

$$A_t' = \int_0^t dt'\; e^{(t-t')\widehat{Q}\,\Gamma}\, \Xi(t)\, R\,. \tag{3.38}$$

Comparing (3.31) with (3.38), we obtain

$$R_t = e^{t\widehat{Q}\,\Gamma} R\,. \tag{3.39}$$

We can show that the random force R_t is orthogonal to A by using (3.20) and (3.21),

$$\begin{aligned}\left(R_t,\, A^*\right) &= \left(e^{t\widehat{Q}\,\Gamma} R,\, A^*\right) = \left(e^{t\widehat{Q}^2\Gamma} R,\, A^*\right) = \left(e^{t\widehat{Q}\,\Gamma} R,\, \left(\widehat{Q}A\right)^*\right)\\ &= \left(R_t,\, \left(\widehat{Q}A\right)^*\right) = 0.\end{aligned} \tag{3.40}$$

An important feature of R_t is that the temporal evolution is given by the modified propagator $e^{t\widehat{Q}\,\Gamma}$. The role of the projection operator $\widehat{Q}$ in the propagator $e^{t\widehat{Q}\,\Gamma}$ will be considered in the next sub-section.

Next we derive the equation of motion for the auto-correlation function of A, $\Xi(t)$.

$$\begin{aligned}\frac{d}{dt}\,\Xi(t) &= \frac{\left(\Gamma A_t,\, A^*\right)}{\left(A,\, A^*\right)} = \frac{\left(\Gamma e^{t\Gamma} A,\, A^*\right)}{\left(A,\, A^*\right)} = \frac{\left(\dot{A},\, \left(e^{-t\Gamma} A\right)^*\right)}{\left(A,\, A^*\right)}\\ &= \frac{\left(\dot{A},\, A_{-t}^*\right)}{\left(A,\, A^*\right)} = \frac{\left(\dot{A},\, \left(\widehat{P}A_{-t} + \widehat{Q}A_{-t}\right)^*\right)}{\left(A,\, A^*\right)}\\ &= \frac{\left(\dot{A},\, \left\{\dfrac{\left(A_{-t},\, A^*\right)}{\left(A,\, A^*\right)}A\right\}^*\right)}{\left(A,\, A^*\right)} + \frac{\left(\dot{A},\, A_{-t}'^*\right)}{\left(A,\, A^*\right)}.\end{aligned} \tag{3.41}$$

$$\Xi(-t)^* = \left\{\frac{\left(A_{-t},\, A^*\right)}{\left(A,\, A^*\right)}\right\}^*$$

$$= \frac{\left(A_{-t}^{*}, A\right)}{\left(A^{*}, A\right)} = \frac{\left(\left(e^{-t\Gamma} A\right)^{*}, A\right)}{\left(A, A^{*}\right)} = \frac{\left(A^{*}, e^{t\Gamma} A\right)}{\left(A, A^{*}\right)} = \frac{\left(A^{*}, A_{t}\right)}{\left(A, A^{*}\right)}$$

$$= \Xi(t). \tag{3.42}$$

Substituting (3.42) and (3.31) into the first and second term on the right hand side of (3.41), respectively, we have

$$\frac{d}{dt}\Xi(t) = \frac{\left(\dot{A}, A^{*}\right)}{\left(A, A^{*}\right)}\Xi(t) + \int_{0}^{-t} dt' \frac{\left(\dot{A}, R_{-t-t'}^{*}\right)}{\left(A, A^{*}\right)}\Xi(t')^{*}$$

$$= \frac{\left(\dot{A}, A^{*}\right)}{\left(A, A^{*}\right)}\Xi(t) - \int_{0}^{t} dt' \frac{\left(\dot{A}, R_{-t+t'}^{*}\right)}{\left(A, A^{*}\right)}\Xi(t'). \tag{3.43}$$

By using (3.15), (3.20), (3.21) and $\widehat{Q}\, e^{t\Gamma\widehat{Q}} = e^{t\widehat{Q}\Gamma}\widehat{Q}$, we have

$$\left(\dot{A}, R_{-t}^{*}\right) = \left(e^{t\Gamma\widehat{Q}}\dot{A}, R^{*}\right) = \left(\widehat{Q}\, e^{t\Gamma\widehat{Q}}\dot{A}, R^{*}\right) = \left(e^{t\widehat{Q}\Gamma}R, R^{*}\right)$$

$$= \left(R_{t}, R^{*}\right). \tag{3.44}$$

Hence, we obtain the equation of motion for $\Xi(t)$,

$$\frac{d}{dt}\Xi(t) = \frac{\left(\dot{A}, A^{*}\right)}{\left(A, A^{*}\right)}\Xi(t) - \int_{0}^{t} dt' \frac{\left(R_{t-t'}, R^{*}\right)}{\left(A, A^{*}\right)}\Xi(t'). \tag{3.45}$$

The Laplace transform of (3.45) is given by

$$\Xi[s] = \left\{ s - \frac{\left(\dot{A}, A^{*}\right)}{\left(A, A^{*}\right)} + \int_{0}^{\infty} dt\, e^{-st} \frac{\left(R_{t}, R^{*}\right)}{\left(A, A^{*}\right)} \right\}^{-1}. \tag{3.46}$$

Comparing (3.29) with (3.46), we have the explicit expressions for the frequency Ω and the memory function $\gamma(t)$ as

$$i\,\Omega = \frac{\left(\dot{A},\, A^*\right)}{\left(A,\, A^*\right)} \tag{3.47}$$

and

$$\gamma(t) = \frac{\left(R_t,\, R^*\right)}{\left(A,\, A^*\right)}. \tag{3.48}$$

Equation (3.48) relates the fluctuation of the system to the dissipation of it so that it is called the fluctuation-dissipation theorem. After substitution of (3.47) and (3.48) into (3.45), we obtain the final form of the equation of motion for $\Xi(t)$,

$$\frac{d}{dt}\,\Xi(t) = i\,\Omega\,\Xi(t) - \int_0^t dt'\;\gamma(t - t')\;\Xi(t'). \tag{3.49}$$

On the course of the derivation of the generalized Langevin equation (3.3) from the equation of motion (3.8), we used the anti-Hermitian property for Liouville operator Γ, (3.15), and the Hermitian property for the projection operators $\widehat{P}$ and $\widehat{Q}$, (3.17) and (3.20), only. Therefore we can transform the equation of motion for any dynamical variables to the generalized Langevin equation, provided that the Liouville operator is anti-Hermitian and we can define the projection operators to be Hermitian.

3.3 Comments on the Random Force

The random force R_t is residue on $\dot{A}_t$ removed the systematic part and whose temporal evolution is controlled by the modified propagator $e^{t\widehat{Q}\Gamma}$. Now we consider the effects of the projection operator $\widehat{Q}$ in the propagator $e^{t\widehat{Q}\Gamma}$. To do so, we eliminate the $\widehat{Q}$ in $e^{t\widehat{Q}\Gamma}$ and see what occurs the elimination.

We introduce a variable

$$I_t = e^{t\Gamma} R\,. \tag{3.50}$$

The Laplace transform of the auto-correlation for I is given by

$$\int_0^\infty dt\, e^{-st}\left(I_t, I^*\right) = \int_0^\infty dt\left(e^{(-s+\Gamma)t} R, R^*\right)$$

$$= \left(\left(s - \Gamma\right)^{-1} R, R^*\right) = \left(\left(s - \Gamma\widehat{Q} - \Gamma\widehat{P}\right)^{-1} R, R^*\right). \tag{3.51}$$

From the operator identity for any operators X and Y,

$$(X + Y)^{-1} = X^{-1} - X^{-1}\, Y\, (X + Y)^{-1}, \tag{3.52}$$

we obtain

$$\left(\left(s - \Gamma\widehat{Q} - \Gamma\widehat{P}\right)^{-1} R, R^*\right) = \left(\left(s - \Gamma\widehat{Q}\right)^{-1} R, R^*\right)$$

$$+ \left(\left(s - \Gamma\widehat{Q}\right)^{-1}\left(\Gamma\widehat{P}\right)\left(s - \Gamma\right)^{-1} R, R^*\right)$$

$$= \left(\widehat{Q}\left(s - \Gamma\widehat{Q}\right)^{-1} R, R^*\right) + \left(\left(s - \Gamma\widehat{Q}\right)^{-1} \Gamma \frac{\left(\left(s - \Gamma\right)^{-1} R, A^*\right)}{\left(A, A^*\right)} A, R^*\right)$$

$$= \left(\left(s - \widehat{Q}\,\Gamma\right)^{-1} R, R^*\right) + \frac{\left(\left(s - \Gamma\right)^{-1} R, A^*\right)}{\left(A, A^*\right)}\left(\widehat{Q}\left(s - \Gamma\widehat{Q}\right)^{-1} \Gamma A, R^*\right)$$

$$= \gamma[s]\left(A, A^*\right) + \frac{\left(\left(s - \Gamma\right)^{-1} R, A^*\right)}{\left(A, A^*\right)}\left(\left(s - \widehat{Q}\,\Gamma\right)^{-1} \widehat{Q}\,\Gamma A, R^*\right)$$

$$= \gamma[s]\left(A, A^*\right) + \frac{\left(\left(s - \Gamma\right)^{-1} R, A^*\right)}{\left(A, A^*\right)}\left(\left(s - \widehat{Q}\,\Gamma\right)^{-1} R, R^*\right)$$

$$= \gamma[s]\left(A, A^*\right) + \left(\left(s - \Gamma\right)^{-1} R, A^*\right)\gamma[s]. \tag{3.53}$$

From (3.3),

$$R = \dot{A} - i\,\Omega A = \left(\Gamma - i\,\Omega\right) A, \tag{3.54}$$

thus we have

$$\left(\left(s - \Gamma\right)^{-1} R, A^*\right) = \left(\left(s - \Gamma\right)^{-1}\left(\Gamma - i\,\Omega\right) A, A^*\right). \tag{3.55}$$

We see that

$$\gamma[s]\ \Xi[s] = 1 - \left(s - i\,\Omega\right) \Xi[s], \tag{3.56}$$

from (3.29). Moreover,

$$\Xi[s] = \int_0^\infty dt\, e^{-s\,t} \frac{\left(A_t, A^*\right)}{\left(A, A^*\right)} = \frac{\left(\left(s - \Gamma\right)^{-1} A, A^*\right)}{\left(A, A^*\right)}. \tag{3.57}$$

Substituting (3.57) into the right hand side of (3.56), we obtain

$$\begin{aligned} \gamma[s]\ \Xi[s] &= 1 - \left(s - i\,\Omega\right) \frac{\left(\left(s - \Gamma\right)^{-1} A, A^*\right)}{\left(A, A^*\right)} \\ &= \frac{\left(\left\{\left(s - \Gamma\right) - \left(s - i\,\Omega\right)\right\} \left(s - \Gamma\right)^{-1} A, A^*\right)}{\left(A, A^*\right)} \\ &= - \frac{\left(\left(\Gamma - i\,\Omega\right) \left(s - \Gamma\right)^{-1} A, A^*\right)}{\left(A, A^*\right)}. \end{aligned} \tag{3.58}$$

Comparing (3.55) with (3.58), we see that

$$\frac{\left(\left(s - \Gamma\right)^{-1} R, A^*\right)}{\left(A, A^*\right)} = - \gamma[s]\ \Xi[s]. \tag{3.59}$$

Introducing (3.59) into (3.53) we have

$$\int_0^\infty dt\, e^{-s\,t} \frac{\left(I_t, I^*\right)}{\left(A, A^*\right)} = \gamma[s] \left(1 - \Xi[s]\ \gamma[s]\right), \tag{3.60}$$

or

$$\frac{\left(I_t, I^*\right)}{\left(A, A^*\right)} = \gamma(t) - \int_0^t dt' \int_0^{t'} dt''\ \gamma(t - t')\ \Xi(t' - t'')\ \gamma(t''). \tag{3.61}$$

If we assume that $\gamma(t) = 2\,L\,\delta(t)$, then (3.61) reduces to

$$\frac{\left(I_t,\, I^*\right)}{\left(A,\, A^*\right)} = 2\,L\,\delta(t) - L^2\,\Xi(t). \tag{3.62}$$

Equation (3.62) implies that the correlation function for I includes the slow process contained in $\Xi(t)$. Therefore the role of the projection operator $\widehat{Q}$ in the propagator $e^{t\,\widehat{Q}\,\Gamma}$ is to remove these process in the evolution of the random force R_t.

Here, if we put $s = i\,\Omega$ in (3.60), then (3.60) exactly reduces to 0. This means that the pseudo-memory function $(I_t, I^*)\,/\,(A, A^*)$, that propagates with $e^{t\,\Gamma}$, can not give any information to the damping effect of A_t. However, (3.60) can still give us a useful information if we set the upper limit of the integral not to be infinite but to be a certain finite value of τ_c. Usually, τ_c is taken to be the characteristic relaxation time of the auto-correlation function (R_t, R^*) of the random force R_t. When the integration range is cut off at $t = \tau_c$, we have the integrated value of (3.60) taking the maximum value, i.e., the plateau value. This is so-called the plateau value problem (Kubo, 1957). Accordingly we must not eliminate the projection operator $\widehat{Q}$ in the modified propagator $e^{t\,\widehat{Q}\,\Gamma}$ of the random force (Mori, 1965a).

§4. Theory

For the 2-D flow of an inviscid barotropic fluid, the Euler's equation becomes the vorticity equation

$$\frac{\partial \zeta(\mathbf{r}, t)}{\partial t} + \mathbf{v}(\mathbf{r}, t)\cdot\nabla\zeta(\mathbf{r}, t) = 0 \ , \tag{4.1}$$

where $\mathbf{v}(\mathbf{r}, t)$ is the velocity field and

$$\zeta(\mathbf{r}, t) = \mathbf{e}_z \cdot \left(\nabla \times \mathbf{v}(\mathbf{r}, t)\right) \tag{4.2}$$

is the vorticity. Here a vector $\mathbf{e}_z$ is the unit vector having the direction of the positive z-axis. The continuity equation is

$$\nabla\cdot\mathbf{v}(\mathbf{r}, t) = 0 \ . \tag{4.3}$$

We suppose that the fluid is confined to a square of side L with periodic boundary condition. We can expand $\mathbf{v}(\mathbf{r}, t)$ and $\zeta(\mathbf{r}, t)$ into Fourier series

$$\left.\begin{aligned} \mathbf{v}(\mathbf{r}, t) &= \sum_{\mathbf{k}} \mathbf{v}_t(\mathbf{k}) \ \exp\left(i\,\mathbf{k}\cdot\mathbf{r}\right), \\ \zeta(\mathbf{r}, t) &= \sum_{\mathbf{k}} \zeta_t(\mathbf{k}) \ \exp\left(i\,\mathbf{k}\cdot\mathbf{r}\right). \end{aligned}\right\} \tag{4.4}$$

Here $\mathbf{k} = \left(2\,\pi\, n_x / L,\ 2\,\pi\, n_y / L\right)$ with n_x and n_y being integers and summation runs over all wave vectors. From (4.2), (4.3) and (4.4), we have

$$\mathbf{v}_t(\mathbf{k}) = -\frac{i}{k^2}\left(\mathbf{e}_z \times \mathbf{k}\right)\zeta_t(\mathbf{k}), \tag{4.5}$$

where k is the absolute value of wave vector $\mathbf{k}$. By making use of (4.5), after some manipulation, (4.1) becomes the spectral form,

$$\frac{d}{dt}\,\zeta_t(\mathbf{k}) = \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\ \zeta_t(\mathbf{p})\ \zeta_t(\mathbf{k} - \mathbf{p}), \tag{4.6}$$

where $\mathbf{p}$ is desecrate 2-D wave vectors and

$$D(\mathbf{k}, \mathbf{p}) = \frac{1}{2}\,\mathbf{e}_z \cdot \left(\mathbf{k} \times \mathbf{p}\right)\left(\frac{1}{|\mathbf{p}|^2} - \frac{1}{|\mathbf{k} - \mathbf{p}|^2}\right). \tag{4.7}$$

We rewrite the spectral form of the vorticity equation (4.6) into the form of (3.8), i.e., the description by the Liouville operator. Now we consider a phase distribution function $f(t, \zeta_t)$ in a multi-dimensional phase space spanned by the Fourier components of the vorticity[1]. From the conservation of the number of representative points in the phase space, the continuity equation for $f(t, \zeta_t)$ becomes

$$\frac{\partial}{\partial t} f(t, \zeta_t) + \sum_{\mathbf{k}} \frac{\partial}{\partial \zeta_t(\mathbf{k})}\left\{f(t, \zeta_t)\ \dot{\zeta}_t(\mathbf{k})\right\}$$

$$= \frac{\partial}{\partial t} f(t, \zeta_t) + \sum_{\mathbf{k}} \left\{\dot{\zeta}_t(\mathbf{k})\ \frac{\partial}{\partial \zeta_t(\mathbf{k})} f(t, \zeta_t) + f(t, \zeta_t)\ \frac{\partial}{\partial \zeta_t(\mathbf{k})}\dot{\zeta}_t(\mathbf{k})\right\} = 0. \tag{4.8}$$

By using (4.6), the second term in the curly bracket of (4.8) is identically zero. Therefore the Liouville equation for $f(t, \zeta_t)$ is given by

$$\frac{\partial}{\partial t} f(t, \zeta_t) = \Gamma^{(d)} f(t, \zeta_t), \tag{4.9}$$

where

$$\Gamma^{(d)} = -\left\{\sum_{\mathbf{k}} \dot{\zeta}_t(\mathbf{k})\ \frac{\partial}{\partial \zeta_t(\mathbf{k})}\right\}. \tag{4.10}$$

As described in the sub-subsection 3.1.1, the Liouville operator Γ for an equation of motion has a negative sign in contrast to that of the Liouville equation for the phase distribution function and is constructed by the initial phase points. Hence the spectral form of the vorticity equation can be rewritten by

$$\frac{d}{dt}\,\zeta_t(\mathbf{k}) = \Gamma\,\zeta_t(\mathbf{k}), \tag{4.11}$$

where

---

[1] $\zeta_t \equiv \{\,\zeta_t(\mathbf{k});$ all wave vectors $\mathbf{k}.\}$

$$\Gamma = \sum_{\mathbf{k}} \left\{ \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \right\} \frac{\partial}{\partial \zeta(\mathbf{k})} \tag{4.12}$$

and $\zeta(\mathbf{k}) \equiv \zeta_{t=0}(\mathbf{k})$.

Next we define projection operators and an inner product. Let $\widehat{P}$ and $\widehat{Q}$ be projection operators that project a function F onto a phase space spanned by observed variables and projected-out variables, respectively. In this work, $\widehat{P}$ and $\widehat{Q}$ are taken as follows:

$$\left.\begin{aligned} \widehat{P}\, F &\equiv \frac{\left(F, \zeta(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)} \zeta(\mathbf{k}), \\ \widehat{Q} &\equiv 1 - \widehat{P}. \end{aligned}\right| \tag{4.13}$$

These projection operators mean that only one variable $\zeta(\mathbf{k})$ is regarded as the observed variable and all the other variables, $\left\{\zeta(\mathbf{k}');\ \text{all wave vectors } \mathbf{k}' \text{ except } \mathbf{k}\right\}$, are regarded as the projected-out variables. The parenthesis denotes the inner product of two functions, which is defined by

$$(F, G) \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}\zeta\, \rho\, F\, G\,, \tag{4.14}$$

where integration runs over all phase space, a function ρ is an ensemble which satisfy the stationary Liouville equation,

$$\Gamma \rho = 0, \tag{4.15}$$

and * denotes the complex conjugate. Note that $\zeta(\mathbf{k})^* = \zeta(-\mathbf{k})$. Equation (4.14) implies the average of $F\,G$ over the ensemble ρ. We easily see that the projection operators, which are defined by (4.13), satisfy the properties (3.17), (3.18), (3.20) and (3.21). Moreover, by (4.12) and partial integration, we have

$$\left(\Gamma F, G^*\right) = \int_{-\infty}^{\infty} d\zeta\, \rho\, G^* \sum_{\mathbf{k}} \left\{ \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \right\} \frac{\partial F}{\partial \zeta(\mathbf{k})}$$

$$= \sum_{\mathbf{k}} \int_{-\infty}^{\infty} d\zeta_{\text{except } d\zeta(\mathbf{k})} \left\{ \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \right\} \rho\, F\, G^* \Big|_{\zeta(\mathbf{k}) = -\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} d\zeta \left\{ \Gamma\left(\rho\, G^*\right) \right\} F. \tag{4.16}$$

We can reasonably suppose that functions F, G and ρ rapidly decrease to zero as $\zeta(\mathbf{k}) \to \pm\infty$, so that the first term in (4.16) vanishes. By (4.15) and $\Gamma^* = \Gamma$, which is seen by (4.12), we obtain

$$\left(\Gamma F, G^*\right) = -\left(F, \left(\Gamma G\right)^*\right). \tag{4.17}$$

Hence, the Liouville operator which is defined by (4.12) is the anti-Hermitian.

Since the projection operators and the Liouville operator are respectively the Hermitian and anti-Hermitian, according to Mori's theory (Mori, 1965a), the generalized Langevin equation for the vorticity of interest is derived from (4.11), as stated in the subsection 3.2:

$$\frac{d}{dt} \zeta_t(\mathbf{k}) = i\, \Omega_{\mathbf{k}}\, \zeta_t(\mathbf{k}) - \int_0^t d\tau\, \gamma_{\mathbf{k}}(t - \tau)\, \zeta_\tau(\mathbf{k}) + R_t(\mathbf{k}). \tag{4.18}$$

The first term on the right-hand side of (4.18) is an oscillating term. The frequency $\Omega_{\mathbf{k}}$ is described by

$$\Omega_{\mathbf{k}} = -i \frac{\left(\Gamma \zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)}$$

$$= - i \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p}) \frac{\left(\zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}), \zeta(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)}. \tag{4.19}$$

The real part of the second term represents an effective damping, i.e., the eddy viscosity damping, of the vorticity due to nonlinear effects from which turbulence come out. The third term is a randomly fluctuating force and is expressed as

$$R_t(\mathbf{k}) = e^{t \widehat{Q} \Gamma}\, \widehat{Q}\; \Gamma \zeta(\mathbf{k})$$

$$= e^{t \widehat{Q} \Gamma}\, \widehat{Q} \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p})\,. \tag{4.20}$$

Equation (4.20) represents an evolution of the initial value of the nonlinear term in the phase space spanned by the projected-out variables. The relation between the random force and the memory function is given by

$$\gamma_{\mathbf{k}}(t) = \frac{\left(R_t(\mathbf{k}), R(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)}. \tag{4.21}$$

This is the fluctuation-dissipation theorem.

The equation of motion for the auto-correlation function of $\zeta_t(\mathbf{k})$ is derived by (4.18):

$$\frac{d}{dt} \Xi_{\mathbf{k}}(t) = i\, \Omega_{\mathbf{k}}\, \Xi_{\mathbf{k}}(t) - \int_0^t d\tau\; \gamma_{\mathbf{k}}(t - \tau)\, \Xi_{\mathbf{k}}(\tau), \tag{4.22}$$

$$\Xi_{\mathbf{k}}(t) \equiv \frac{\left(\zeta_t(\mathbf{k}), \zeta(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)}. \tag{4.23}$$

As seen from (2.8), the eddy viscosity is an effective viscosity simultaneously acting on the motion of the variables of interest and does not accompany retardation. If we suppose that the spectrum of the random force is white or equivalently the memory function $\gamma_{\mathbf{k}}(t)$ behaves as the delta function,

$$\gamma_{\mathbf{k}}(t) = 2 \left\{ \int_0^\infty \gamma_{\mathbf{k}}(t)\, dt \right\} \delta(t), \tag{4.24}$$

where $\delta(t)$ is the Dirac delta function, the memory of the damping term in the generalized Langevin equation reduces to zero, i.e., the retardation is not account for. This assumption means that the time scale of the projected-out variables is short enough in comparison to that of the variables of interest. Since the time scale of phenomena in the meteorological systems is proportional to the length scale of those, (4.24) is satisfied, provided that we are interested in ultra-long waves. Gambo (1982) actually showed that the spectrum of the random force for the ultra-long waves in the mid-latitude in winter time is white, but not for long waves. If we are interested in the long waves, the projected-out variables include the variables which have time scale longer than the time scale of the variables of interest, i.e., the random force for the long waves includes the ultra-long waves, then (4.24) is consequently not satisfied. However, equation (4.24) may hold, provided that we take into account the contribution of only phenomena whose length scale are smaller than that of variables of interest. Then the generalized Langevin equation becomes the Langevin equation for Brownian motion,

$$\frac{d}{dt} \zeta_t(\mathbf{k}) = i\, \Omega_{\mathbf{k}}\, \zeta_t(\mathbf{k}) - \widetilde{\nu_{eddy}}\, k^2\, \zeta_t(\mathbf{k}) + R_t(\mathbf{k})\,, \tag{4.25}$$

where

$$\widetilde{\nu_{eddy}} = \left\{ \int_0^\infty \gamma_{\mathbf{k}}(t)\, dt \right\} / k^2. \tag{4.26}$$

The eddy viscosity coefficient $\widetilde{\nu_{eddy}}$ is defined by setting the damping term to be of the diffusion type. Equation (4.26) is the general expression for the eddy viscosity coefficient. Then the auto-correlation function is expressed as

$$\Xi_{\mathbf{k}}(t) = \exp\left[\left(i\, \Omega_{\mathbf{k}} - \widetilde{\nu_{eddy}} k^2\right) t\right]. \tag{4.27}$$

We will next determine the memory function definitely from the fluctuation-dissipation theorem (4.21).

In order to express the memory function definitely, we require a explicit form of the distribution function ρ of fluctuation of the vorticity. It has commonly been hypothesized that the statistical dynamics of classical 2-D turbulence is controlled by two quadratic invariants of energy and enstrophy (Kraichnan, 1975, Shepherd, 1987). Thus in this work, we use a canonical ensemble which has energy, $E = \sum_{\mathbf{k}} \left|\zeta(\mathbf{k})\right|^2 / \left(2\, k^2\right)$, and enstrophy, $Z = \sum_{\mathbf{k}} \left|\zeta(\mathbf{k})\right|^2 / 2$, per unit mass as the invariants,

$$\begin{aligned}\rho &= C \exp\left[- \left(\frac{\mu E}{B} + \frac{Z}{B}\right)\right] \\ &= C \prod_{\mathbf{k}} \exp\left[- \frac{k^2 + \mu}{B\, k^2} \times \frac{\left|\zeta(\mathbf{k})\right|^2}{2}\right],\end{aligned} \tag{4.28}$$

where parameters B and μ are determined by the energy and enstrophy per unit mass. B corresponds to the temperature parameter in the boson problem, while μ is the chemical potential (Kraichnan, 1975). C is the normalization constant. It easily follows that (4.28) obeys the stationary Liouville equation (4.15). In the limit $L \to \infty$, the power spectrum $e(k)$ of the energy density of the vortices with wave number k is given by

$$e(k) = \left(\frac{L}{2\, \pi}\right)^2 \frac{B\, k\, \pi}{k^2 + \mu}. \tag{4.29}$$

Then E and Z are

$$\left.\begin{aligned} E &= \int_{k_{min}}^{k_{max}} e(k)\, dk = \left(\frac{L}{2\, \pi}\right)^2 \frac{\pi B}{2} \log\left[\frac{k_{max}{}^2 + \mu}{k_{min}{}^2 + \mu}\right], \\ Z &= \int_{k_{min}}^{k_{max}} k^2 e(k)\, dk = \left(\frac{L}{2\, \pi}\right)^2 \frac{\pi B}{2} \left\{ \left(k_{max}{}^2 - k_{min}{}^2\right) - \mu \log\left[\frac{k_{max}{}^2 + \mu}{k_{min}{}^2 + \mu}\right]\right\}, \end{aligned}\right\} \tag{4.30}$$

respectively. Here, $k_{min} = (2\, \pi) / L$ and k_{max} is the maximum wave number or the truncation wave number. The power spectrum $e(k)$ depends on k as k^{-1} for larger wave numbers and has a maximum value at $\sqrt{\mu}$. The validity of (4.28) will be confirmed in the next section by comparing the ensemble-averaged second order moments calculated by (4.28) with the time-averaged ones obtained by a simulation.

Using this distribution function ρ, we calculate the frequency $\Omega_{\mathbf{k}}$ and the memory function $\gamma_{\mathbf{k}}(t)$. Then the frequency $\Omega_{\mathbf{k}}$ is given by

$$\Omega_{\mathbf{k}} = - i \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p}) \frac{\left(\zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}),\, \zeta(\mathbf{k})^{*}\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^{*}\right)} = 0. \tag{4.31}$$

Since the propagator $e^{t \widehat{Q} \Gamma}$ having appeared in the random force (4.20) is expressed by the expanded form about time,

$$e^{t \widehat{Q} \Gamma} = 1 + t \widehat{Q} \Gamma + \frac{1}{2}\left(t \widehat{Q} \Gamma\right)^{2} + \cdots, \tag{4.32}$$

the random force becomes

$$R_{t}(\mathbf{k}) = R^{(0)}(\mathbf{k}) + t\, R^{(1)}(\mathbf{k}) + \frac{1}{2} t^{2} R^{(2)}(\mathbf{k}) + \cdots, \tag{4.33}$$

where

$$\left.\begin{aligned} R^{(0)}(\mathbf{k}) &= R(\mathbf{k}) = \widehat{Q} \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}), \\ R^{(1)}(\mathbf{k}) &= \widehat{Q}\ \Gamma R(\mathbf{k}), \\ R^{(2)}(\mathbf{k}) &= \widehat{Q}\ \Gamma R^{(1)}(\mathbf{k}). \end{aligned}\right\} \tag{4.34}$$

The explicit form of $R(\mathbf{k})$ and $R^{(1)}(\mathbf{k})$ are described in Appendix A. The memory function is also described by the expanded form,

$$\gamma_{\mathbf{k}}(t) = \gamma_{\mathbf{k}}^{(0)} + t\, \gamma_{\mathbf{k}}^{(1)} - \frac{1}{2} t^{2} \gamma_{\mathbf{k}}^{(2)} + \cdots, \tag{4.35}$$

where

$$\left.\begin{aligned}
\gamma_{\mathbf{k}}^{(0)} &= \frac{\left(R(\mathbf{k}),\, R(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}, \\
\gamma_{\mathbf{k}}^{(1)} &= \frac{\left(R^{(1)}(\mathbf{k}),\, R(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}, \\
\gamma_{\mathbf{k}}^{(2)} &= -\frac{\left(R^{(2)}(\mathbf{k}),\, R(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)} = \frac{\left(R^{(1)}(\mathbf{k}),\, R^{(1)}(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}.
\end{aligned}\right\} \tag{4.36}$$

The explicit form of them are as follows: the first term on the right-hand side of (4.35) is given by

$$\gamma_{\mathbf{k}}^{(0)} = \sum_{\mathbf{p}} 2\, D(\mathbf{k}, \mathbf{p})^2 \frac{\left\langle|\zeta(\mathbf{p})|^2\right\rangle \left\langle|\zeta(\mathbf{k} - \mathbf{p})|^2\right\rangle}{\left\langle|\zeta(\mathbf{k})|^2\right\rangle} \equiv \sum_{\mathbf{p}} \gamma_{\mathbf{k}}^{\mathbf{p}}, \tag{4.37}$$

where $\gamma_{\mathbf{k}}^{\mathbf{p}}$ is a positive quantity. The second and the third term on the right-hand side of (4.35) are expressed by

$$\gamma_{\mathbf{k}}^{(1)} = 0 \tag{4.38}$$

and

$$\begin{aligned}
\gamma_{\mathbf{k}}^{(2)} = &\sum_{\mathbf{p},\, \mathbf{q}} 2\, \gamma_{\mathbf{k}}^{\mathbf{p}}\, \gamma_{\mathbf{p}}^{\mathbf{q}} \\
&+ \sum_{\mathbf{p},\, \mathbf{q}} \frac{\left\langle|\zeta(\mathbf{p})|^2\right\rangle \left\langle|\zeta(\mathbf{q})|^2\right\rangle \left\langle|\zeta(\mathbf{k} - \mathbf{p} - \mathbf{q})|^2\right\rangle}{\left\langle|\zeta(\mathbf{k})|^2\right\rangle} \\
&\qquad \times 16 D(\mathbf{k}, \mathbf{p}) D(\mathbf{k} - \mathbf{p}, \mathbf{q}) D(\mathbf{k}, \mathbf{q}) D(\mathbf{k} - \mathbf{q}, \mathbf{p}) \\
&+ \sum_{\mathbf{p}} 4\, \gamma_{\mathbf{k}}^{\mathbf{p}}\, \gamma_{\mathbf{p}}^{\mathbf{k} - \mathbf{p}} \frac{\left\langle|\zeta(\mathbf{k} - \mathbf{p})|^4\right\rangle}{\left(\left\langle|\zeta(\mathbf{k} - \mathbf{p})|^2\right\rangle\right)^2},
\end{aligned} \tag{4.39}$$

respectively. The second order moment $\left\langle|\zeta(\mathbf{k})|^2\right\rangle$ and the fourth order moment $\left\langle|\zeta(\mathbf{k})|^4\right\rangle$ are calculated according to (4.14) and (4.28):

$$\left\langle \left|\zeta(\mathbf{k})\right|^2 \right\rangle \equiv \left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^* \right) = \frac{B\,k^2}{k^2 + \mu}, \tag{4.40}$$

$$\left\langle \left|\zeta(\mathbf{k})\right|^4 \right\rangle \equiv \left(\zeta(\mathbf{k})\,\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\,\zeta(\mathbf{k})^* \right) = 3\left\{ \left\langle \left|\zeta(\mathbf{k})\right|^2 \right\rangle \right\}^2. \tag{4.41}$$

When equation (4.24) holds, development of the system at time t depends on simultaneous variables or history of variables from very near past t - ε, where suppose ε be infinitesimally small quantity. On the course of time ε, the memory function in (4.18) changes from $\gamma_{\mathbf{k}}(t - (t - \varepsilon)) = \gamma_{\mathbf{k}}(\varepsilon)$ to $\gamma_{\mathbf{k}}(t - t) = \gamma_{\mathbf{k}}(0)$. Therefore the following approximation is available: we terminate (4.35) at the third term and express it in a Gauss function at first, that is, $\gamma_{\mathbf{k}}(t) \approx \gamma_{\mathbf{k}}^{(0)} \exp\left[- \left(\gamma_{\mathbf{k}}^{(2)} / \gamma_{\mathbf{k}}^{(0)} \right) \cdot \left(t^2 / 2 \right) \right]$, moreover, approximate the Gauss function by an exponential function,

$$\gamma_{\mathbf{k}}(t) \approx \gamma_{\mathbf{k}}^{(0)} \exp\left[- \varepsilon_{\mathbf{k}}\, t \right]. \tag{4.42}$$

Here, by making equal the integrated values of both the approximated functions about time from 0 to ∞, we have

$$\varepsilon_{\mathbf{k}} = \sqrt{ \frac{2}{\pi} \frac{\gamma_{\mathbf{k}}^{(2)}}{\gamma_{\mathbf{k}}^{(0)}} }\,. \tag{4.43}$$

Then the Laplace transformed auto-correlation function is given by

$$\Xi_{\mathbf{k}}[s] = \cfrac{1}{s + \cfrac{\gamma_{\mathbf{k}}^{(0)}}{s + \varepsilon_{\mathbf{k}}}}, \tag{4.44}$$

where

$$\Xi_{\mathbf{k}}[s] \equiv \int_0^{\infty} \exp\left[- s\, t \right] \Xi_{\mathbf{k}}(t)\, dt\,. \tag{4.45}$$

Under this approximation, equation (4.26) becomes

$$\widetilde{\nu_{eddy}} = \sqrt{\pi \left(\gamma_{\mathbf{k}}^{(0)} \right)^3 / \left(2\, \gamma_{\mathbf{k}}^{(2)} \right)} \,/\, k^2. \tag{4.46}$$

This is a explicit expression for the eddy viscosity coefficient. The eddy viscosity coefficient $\widetilde{\nu_{eddy}}$ is numerically calculated, provided that the parameters B and μ are numerically given, i.e., the canonical distribution (4.28) is specified, because equation (4.46) includes only the parameters B and μ as unknown parameters (see (4.40) and (4.41)).

Next we would like to extend the discussion to the continuous wave number case. Then the eddy viscosity coefficient is readily calculated for given characteristic vales of the system. However, in the limit $L \to \infty$, it is difficult to express the summation in (4.39) by an integration and to integrate it. Thus we estimate $\widetilde{\nu_{eddy}}$ approximately by a self-consistent treatment. The validity of the self-consistent treatment in the following is examined in the next section.

By an analogy to $\gamma_{\mathbf{k}}^{(0)}$, we consider $\gamma_{\mathbf{k}}(t)$ as follows:

$$\gamma_{\mathbf{k}}(t) = \sum_{\mathbf{p}} 2\, D(\mathbf{k}, \mathbf{p})^2 \frac{\left(\zeta_t(\mathbf{p}), \zeta(\mathbf{p})^*\right)\left(\zeta_t(\mathbf{k} - \mathbf{p}), \zeta(\mathbf{k} - \mathbf{p})^*\right)}{\left(\zeta(\mathbf{k}), \zeta(\mathbf{k})^*\right)}. \tag{4.47}$$

Equation (4.47) is consistent with (4.37) at $t = 0$. We substitute (4.27) with (4.31) into (4.47), then we obtain

$$\gamma_{\mathbf{k}}(t) = \sum_{\mathbf{p}} 2\, D(\mathbf{k}, \mathbf{p})^2 \frac{\left\langle|\zeta(\mathbf{p})|^2\right\rangle\left\langle|\zeta(\mathbf{k} - \mathbf{p})|^2\right\rangle}{\left\langle|\zeta(\mathbf{k})|^2\right\rangle} \exp\left[-\left\{\widetilde{\nu_{eddy}}\, p^2 + \widetilde{\nu_{eddy}}\, |\mathbf{k} - \mathbf{p}|^2\right\} t\right] \tag{4.48}$$

and

$$\begin{aligned}\widetilde{\nu_{eddy}} &= \left\{\int_0^\infty \gamma_{\mathbf{k}}(t)\, dt\right\} / k^2 \\ &= \sum_{\mathbf{p}} 2\, D(\mathbf{k}, \mathbf{p})^2 \frac{\left\langle|\zeta(\mathbf{p})|^2\right\rangle\left\langle|\zeta(\mathbf{k} - \mathbf{p})|^2\right\rangle}{\left\langle|\zeta(\mathbf{k})|^2\right\rangle\left\{\widetilde{\nu_{eddy}}\, p^2 + \widetilde{\nu_{eddy}}\, |\mathbf{k} - \mathbf{p}|^2\right\} k^2}.\end{aligned} \tag{4.49}$$

Thus we obtain

$$\widetilde{\nu_{eddy}} \cong \sqrt{\sum_{\mathbf{p}} 2\, D(\mathbf{k}, \mathbf{p})^2 \frac{\left\langle|\zeta(\mathbf{p})|^2\right\rangle \left\langle|\zeta(\mathbf{k} - \mathbf{p})|^2\right\rangle}{\left\langle|\zeta(\mathbf{k})|^2\right\rangle \left\{p^2 + |\,\mathbf{k} - \mathbf{p}\,|^2\right\} k^2}} \,. \tag{4.50}$$

In the limit $L \to \infty$, equation (4.50) becomes

$$\begin{aligned} \widetilde{\nu_{eddy}} = \frac{L}{2\,\pi} \sqrt{\frac{B}{2\,k^2}} \Bigg[& \int_{k_{min}}^{k_{max}} p\, dp \left(\frac{k^2 + \mu}{p^2 + \mu}\right) \\ & \times \int_0^{2\,\pi} d\theta \left\{ \frac{\sin^2\theta}{\left(k^2 + p^2 - 2\,p\,k \cos\theta\right)} \right. \\ & \left. \times \frac{\left(k^2 - 2\,p\,k \cos\theta\right)^2}{\left(k^2 + p^2 + \mu - 2\,p\,k \cos\theta\right)\left(k^2 + 2\,p^2 - 2\,p\,k\cos\theta\right)} \right\} \Bigg]^{1/2} , \end{aligned} \tag{4.51}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{p}$.

Up to now, we considered that the eddy viscosity acting on $\zeta_t(\mathbf{k})$ comes from the nonlinear interaction between wave number k and all the other wave numbers, when we are interested in $\zeta_t(\mathbf{k})$. Thus not only smaller scale eddies than k but also larger ones act on $\zeta_t(\mathbf{k})$ as viscosity. However, it is natural to consider that an eddy viscosity is an effective viscosity due to an action of small scale eddies upon large scale eddies (Heisenberg, 1948a; 1948b), because the concept of the eddy viscosity is an analogous one of the molecular viscosity as stated in section 2, that is an effective viscosity due to thermal agitation of molecules, i.e., small scale fluctuation. On the other hand, large scale motions may act on small scales as the advection. Moreover, we must take into account the contribution of wave numbers larger than k in (4.51), if we are interested in a phenomenon whose wave number is k, in order to hold the assumption of the spectrum of the random force to be white, as stated before. In consequence,

$$\nu_{eddy}(k) = \frac{L}{2\pi}\sqrt{\frac{B}{2k^2}}\left[\int_k^{k_{max}} p\,dp\left(\frac{k^2+\mu}{p^2+\mu}\right)\right.$$

$$\times\int_0^{2\pi} d\theta\left\{\frac{\sin^2\theta}{\left(k^2+p^2-2\,p\,k\cos\theta\right)}\right.$$

$$\left.\left.\times\frac{\left(k^2-2\,p\,k\cos\theta\right)^2}{\left(k^2+p^2+\mu-2\,p\,k\cos\theta\right)\left(k^2+2\,p^2-2\,p\,k\cos\theta\right)}\right\}\right]^{1/2}$$

$$= \sqrt{\frac{B\,\kappa_1 L^2}{32\,\pi\,k^4\,\mu}\left\{I(k_{max}) - I(k)\right\}}, \tag{4.52}$$

where

$$I(p) = \left\{\left(\chi(p) - p^2\right) - \left(\kappa_2 - 2\,\mu\right)\log\left[\vartheta(p)\left|\xi(p)+\chi(p)\right|\right]\right.$$

$$+\,2\,\kappa_3\log\left[\left|\frac{\vartheta(p)}{\mu}\cdot\frac{k^4+4\,\mu\,p^2+\kappa_3\,\eta(p)}{k^4-4\,\mu\,p^2+\kappa_3\,\eta(p)}\cdot\frac{1}{\kappa_2\,p^2-\kappa_1{}^2-\mu\,\xi(p)-\kappa_3\,\chi(p)}\right|\right]$$

$$\left.-\,8\mu\log\left[\left|2\,p^2+\eta(p)\right|\right]+\kappa_1\log\left[\mu\,\vartheta(p)\left|\kappa_2\,p^2-\kappa_1{}^2-\kappa_1\,\chi(p)\right|\right]\right\}, \tag{4.53}$$

$$\left.\begin{array}{lll} \kappa_1 \equiv k^2+\mu, & \kappa_2 \equiv k^2-\mu, & \kappa_3 \equiv \sqrt{k^4+4\,\mu^2}, \\ \xi(p) \equiv p^2-k^2+\mu, & & \chi(p) \equiv \sqrt{\left(k^2+p^2+\mu\right)^2-(2\,p\,k)^2}, \\ \vartheta(p) \equiv p^2+\mu & \text{and} & \eta(p) \equiv \sqrt{k^4+4\,p^2}. \end{array}\right\} \tag{4.54}$$

Equation (4.52) is the analytical result of our theory of the eddy viscosity coefficient.

§5. Results and Discussion

5.1 Preliminary Numerical Simulation

We perform a preliminary numerical simulation and examine the validity of the description of the distribution function as (4.28), the vorticity equation (4.6) as the generalized Langevin equation (4.18) and the self-consistent treatment (4.47) in the calculation of the eddy viscosity coefficient.

The *Model C* of Kells and Orszag (1978) is employed for the simulation in the following. The model is the vorticity equation (4.6) isotropically truncated under the condition of $L = 2\pi$ and $(n_x^2 + n_y^2) \leq 5$, i.e.,

$$\frac{d}{dt}\zeta_t(\mathbf{k}) = \sum_{p^2 \leq 5} D(\mathbf{k}, \mathbf{p})\, \zeta_t(\mathbf{p})\, \zeta_t(\mathbf{k} - \mathbf{p}) \,, \tag{5.1}$$

so that it has 20 independent real variables. This is the smallest non-degenerate isotropically truncated system and satisfies the assumption of ergodicity well enough that the time-averaged quantities agree with the ensemble-averaged quantities by making use of (4.28) within a few percentages of error (Kells and Orszag, 1978). This model is numerically integrated using the fourth order Runge-Kutta scheme, with an initial condition chosen randomly but constrained to have $E = 7$ and $Z = 20$ and a time step $\Delta t = 0.0025$, until $t = 3000$. These values of E and Z guarantee that all the modes are significantly excited (Kraichnan, 1975; Bell, 1980). By using (4.40), the energy E and enstrophy Z per unit mass are given by $E = \sum_{k^2 \leq 5} B / \{2(k^2 + \mu)\}$ and $Z = \sum_{k^2 \leq 5} B\, k^2 / \{2(k^2 + \mu)\}$, respectively. Those are the functions of the parameters B and μ so that those can be solved with respect to B and μ under the condition of $E = 7$ and $Z = 20$. Then we have the parameter values $B = 3.56954$ and $\mu = 2.24220$.

The ensemble-averaged second order moments, which are calculated by (4.40), and the time-averaged quantities obtained by the numerical simulation are tabulated in Table I. The relative errors of the quantities by (4.40) to those by the simulation are also shown in Table I. It shows certainly that the ensemble-averaged quantities agree with the time-averaged quantities within a few percentages of error. Hence the distribution function of fluctuation of vorticity is well described by (4.28).

Table I The second order moments evaluated by the formula (4.40) and the time-averaged quantities obtained by the numerical simulation. The relative errors of the results by (4.40) to those by the simulation are shown in the right column.

variables	k^2	second order moment by (4.40)	second order moment by simulation	relative error of (4.40) (%)
Re[$\zeta(1, 0)$]	1	1.10096	1.00455	9.59733
Im[$\zeta(1, 0)$]	1	1.10096	1.05662	4.19640
Re[$\zeta(0, 1)$]	1	1.10096	1.03916	5.94711
Im[$\zeta(0, 1)$]	1	1.10096	1.03333	6.54486
Re[$\zeta(1, 1)$]	2	1.68287	1.89566	- 11.2251
Re[$\zeta(-1, 1)$]	2	1.68287	1.80583	- 6.80906
Im[$\zeta(1, 1)$]	2	1.68287	1.84875	- 8.97255
Im[$\zeta(-1, 1)$]	2	1.68287	1.84203	- 8.64047
Re[$\zeta(2, 0)$]	4	2.28736	2.39260	- 4.39856
Im[$\zeta(2, 0)$]	4	2.28736	2.29845	- 0.482499
Re[$\zeta(0, 2)$]	4	2.28736	2.34787	- 2.57723
Im[$\zeta(0, 2)$]	4	2.28736	2.39843	- 4.63095
Re[$\zeta(1, 2)$]	5	2.46440	2.37611	3.71574
Re[$\zeta(-1, 2)$]	5	2.46440	2.37436	3.79218
Im[$\zeta(1, 2)$]	5	2.46440	2.36229	4.32250
Im[$\zeta(-1, 2)$]	5	2.46440	2.39357	2.95918
Re[$\zeta(2, 1)$]	5	2.46440	2.40491	2.47369
Re[$\zeta(-2, 1)$]	5	2.46440	2.35230	4.76555
Im[$\zeta(2, 1)$]	5	2.46440	2.38998	3.11383
Im[$\zeta(-2, 1)$]	5	2.46440	2.35833	4.49767

The correlation functions obtained by (4.44) and the simulation are graphed in Fig. 5.1. Because of the symmetry of the system, all variables with wave numbers of the same magnitude may have identical statistical properties. The symmetry is numerically confirmed as shown in Table I. Hence one of the graphs for each value of k^2 is shown in Fig. 5.1. Although the theory somewhat underestimates the decay of the correlation functions, the orders of magnitude of it are well predicted. Since we focus our attention on an explanation of the enormousness of the eddy viscosity coefficient by the equation of motion of the fluid dynamics, agreement between the theory and the simulation is satisfactory for our purpose.

We consider that the discrepancy of results obtained by the theory and the simulation is due to a defect in the prediction of oscillations of the system. The oscillations of the system come directly from the oscillation term in (4.22) and indirectly from the long-time correlation of the memory function $\gamma_{\mathbf{k}}(t)$. The former may not affect the present results, because the frequency (4.19) is zero (see (4.31)) for the distribution function (4.28), whose validity is confirmed by Table I. It is generally possible to construct an equation for the memory function which has the same form as the generalized Langevin equation for $\Xi(t)$. Thus we have an infinite chain of equations for a series of the higher order memory functions, the lowest order of which determines the correlation function of interest, $\Xi(t)$. The chain of equations for $\Xi[s]$ has the form of a continued-fraction (Mori, 1965b; Lovesy, 1980, 1986). On the course of derivation of (4.44), we approximate the memory function of Gaussian to that of exponential function. This operation is equivalent to termination of a continued fraction of $\Xi[s]$ at the first level by replacing a memory function appeared in an equation for $\gamma_{\mathbf{k}}(t)$ by a constant, because (4.44) has infinitely continued-fraction form, provided that $\gamma_{\mathbf{k}}(t)$ is Gaussian. Thus the approximation affects the correlation time of $\gamma_{\mathbf{k}}(t)$, and may consequently result the defect of prediction of the oscillations. Let the considerations of hierarchy of equations for memory functions be left as future problems.

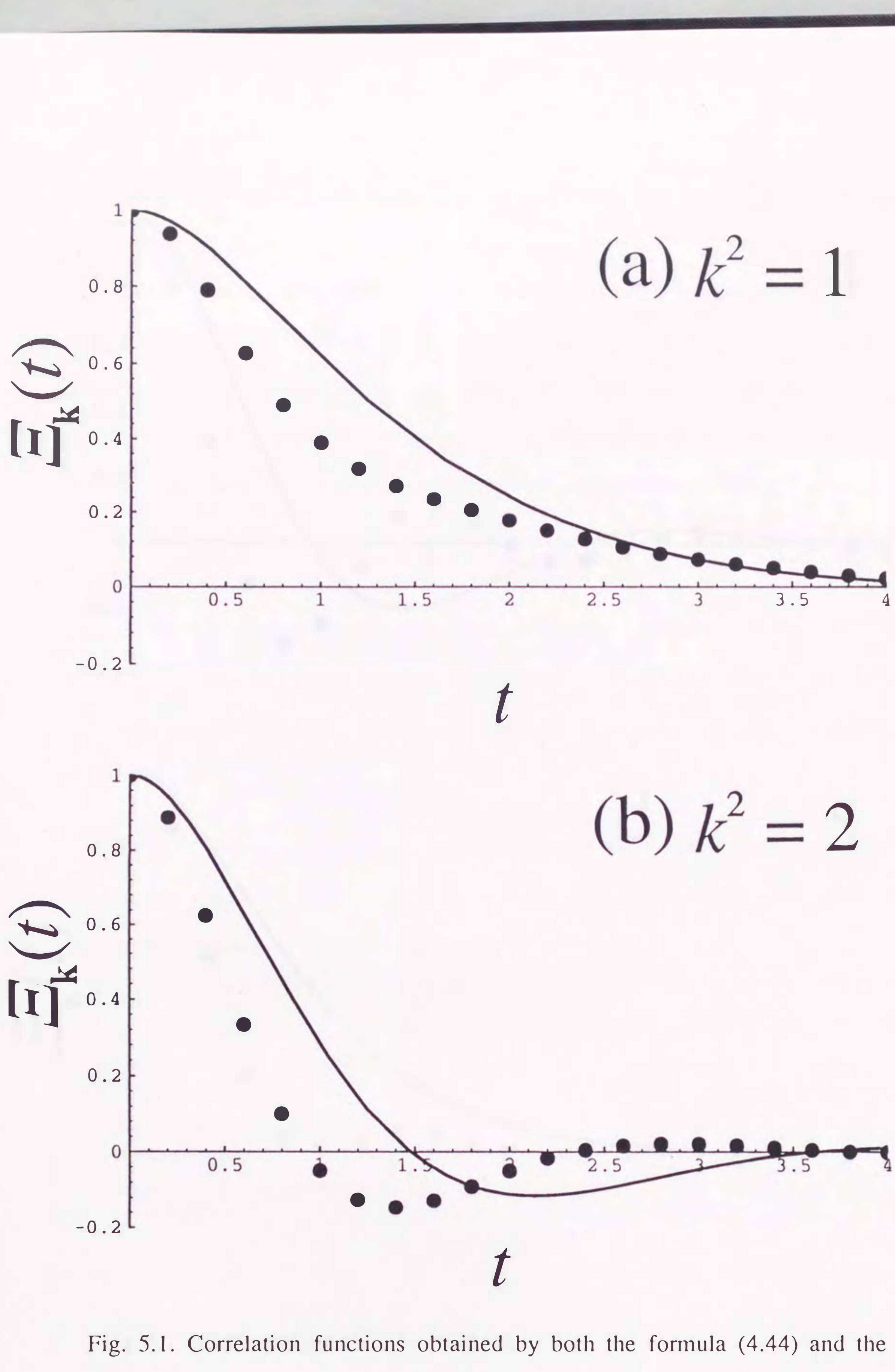
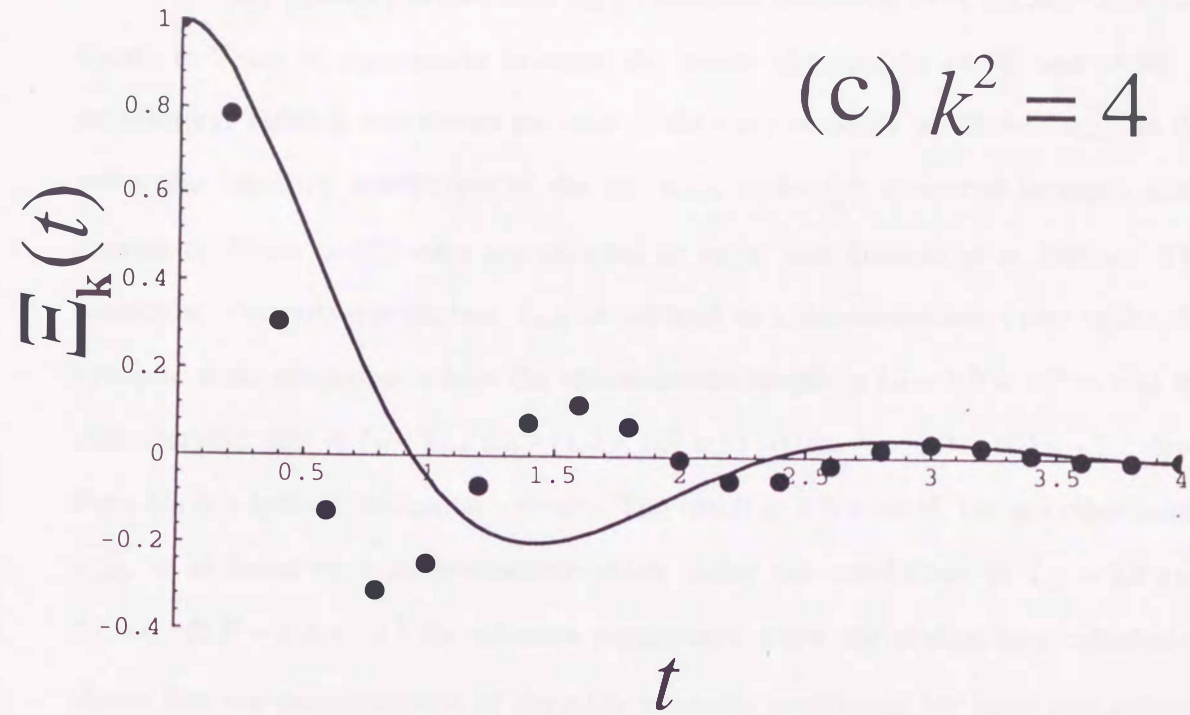


Fig. 5.1. Correlation functions obtained by both the formula (4.44) and the simulation. Solid line and dots indicate the results by (4.44) and the simulation, respectively. The results by the simulation are averages of the correlation functions of the real and imaginary parts of all variables with given k^2, (a) $k^2 = 1$, (b) $k^2 = 2$.



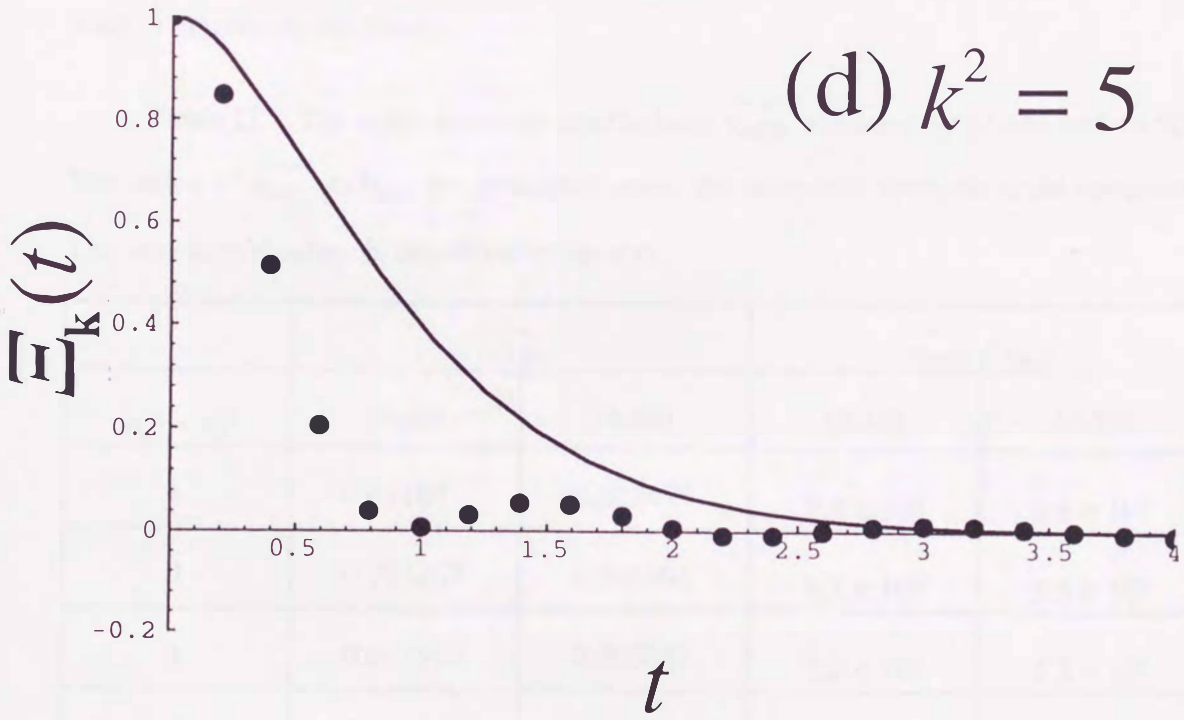


Fig. 5.1. (Continued), (c) $k^2 = 4$, (d) $k^2 = 5$.

The eddy viscosity coefficients $\widetilde{\nu_{eddy}}$ evaluated according to (4.46) and (4.50) are shown in Table II. Agreement between the results obtained by (4.46) and (4.50) is satisfactory. Table II also shows the ratio of the eddy viscosity coefficient $\widetilde{\nu_{eddy}}$ to the molecular viscosity coefficient of the air, ν_{mol}, under the terrestrial synoptic scale condition. These coefficients are reduced to be of non-dimension as follows. The molecular viscosity coefficient ν_{mol} is reduced to a dimensionless value under the synoptic scale condition, where the characteristic length is $L_0 \sim 1.0 \times 10^6$ m and the characteristic time is $T_0 \sim L_0 / U_0 \sim (1.0 \times 10^6 \text{ m}) / (10 \text{ ms}^{-1}) = 1.0 \times 10^5 \text{ s} \sim 1.2$ days. Here U_0 is a typical horizontal velocity. The result is 1.5×10^{-12}. On the other hand, $\widetilde{\nu_{eddy}}$ is reduced to a dimensionless value under the conditions of $L_C = 2\pi$ and $T_C = L / \sqrt{2E} \sim 6.3 \times 10^{-1}$ for selected parameters. Even the preliminary calculation shows that the enormousness of the eddy viscosity coefficient for large atmospheric scale is derived by the theory.

Table II The eddy viscosity coefficients $\widetilde{\nu_{eddy}}$ obtained by (4.46) and (4.50). The ratios of $\widetilde{\nu_{eddy}}$ to ν_{mol} are evaluated under the terrestrial synoptic scale condition. The way to evaluation is described in the text.

	$\widetilde{\nu_{eddy}}$		$\widetilde{\nu_{eddy}} / \nu_{mol}$	
$n_x^2 + n_y^2$	(4.46)	(4.50)	(4.46)	(4.50)
1	0.691071	0.623475	7.3×10^9	6.6×10^9
2	0.785267	0.516991	8.3×10^9	5.5×10^9
4	0.677500	0.503042	7.2×10^9	5.3×10^9
5	0.208142	0.381416	2.2×10^9	4.0×10^9

5.2 Quantitative Considerations

For performing a numerical calculation, we take parameter values as follows. Letting the area of a square domain equal the surface area of a sphere with radius r, we have the side of the domain $L = 2\sqrt{\pi}\, r$. Under the terrestrial condition, $r = 6.4 \times 10^6$ m, then $L = 2.3 \times 10^7$ m. The maximum wave number or the truncation wave number, k_{max}, is taken to be $10^5\, k_{min}$ that corresponds to the minimum wavelength to be $\lambda_{min} = 2.3 \times 10^2$ m. The parameter μ is determined from the form of the power spectrum, because $\sqrt{\mu}$ is the wave number at which the power spectrum is maximum. Comparing the power spectrum $e(k)$ with observed ones (see Fig. 5.2), we select $\sqrt{\mu} = 3\, k_{min}$, for the time being. On the other hand, the parameter B is determined from (4.30). The horizontal velocity scale U_0 is chosen to be 10 ms^{-1}, so that we have energy per unit mass $E = 50\ m^2s^{-2}$. Hence four parameter values, L, k_{max}, μ and U_0 or E, are specified. For 2-D inviscid fluid considered in this study, it's power spectrum $e(k)$ has different shape from that of the atmosphere: for example the dependence on the wave number. However, we show in the following that the shape of the power spectrum has little effect on the enormousness of the eddy viscosity coefficient.

Figure 5.3 shows the numerical result of the theory. The ordinate indicates the ratio of the eddy viscosity coefficient $\nu_{eddy}(k)$ to the molecular viscosity coefficient ν_{mol} of the air. The abscissa indicates the characteristic length of a phenomenon, $2\pi / k$. It shows that the eddy viscosity coefficient depends on the length scale of the phenomenon. For Cheju Island (Cheju-Do) scale, whose mean diameter is about 30 km, the theory derives that the eddy viscosity coefficient is about eight orders of magnitude larger than that of the molecular viscosity coefficient of the air. This value is of the same order as that of the eddy viscosity coefficient obtained by applying the Reynolds' law of similarity to the atmospheric Kármán vortices generated on the leeward of Cheju Island, which was mentioned in section 1.

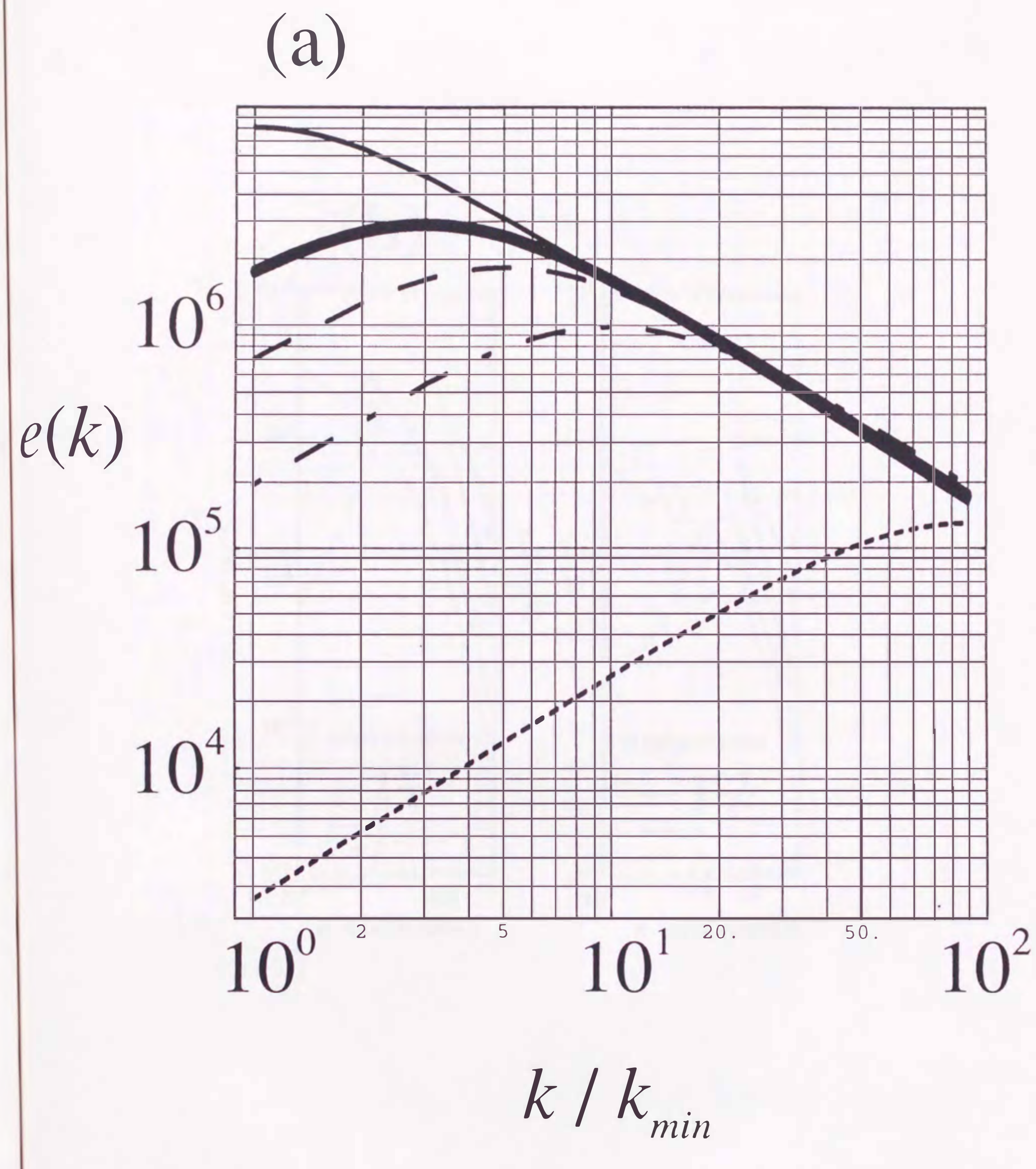


Fig. 5.2 (a) Power spectra as calculated from equation (4.29), for $\sqrt{\mu} = k_{min}$ (thin solid line), $3k_{min}$ (thick solid line), $5k_{min}$ (dashed line), $10k_{min}$ (dash-dotted line) and $100k_{min}$ (dotted line).

(b)

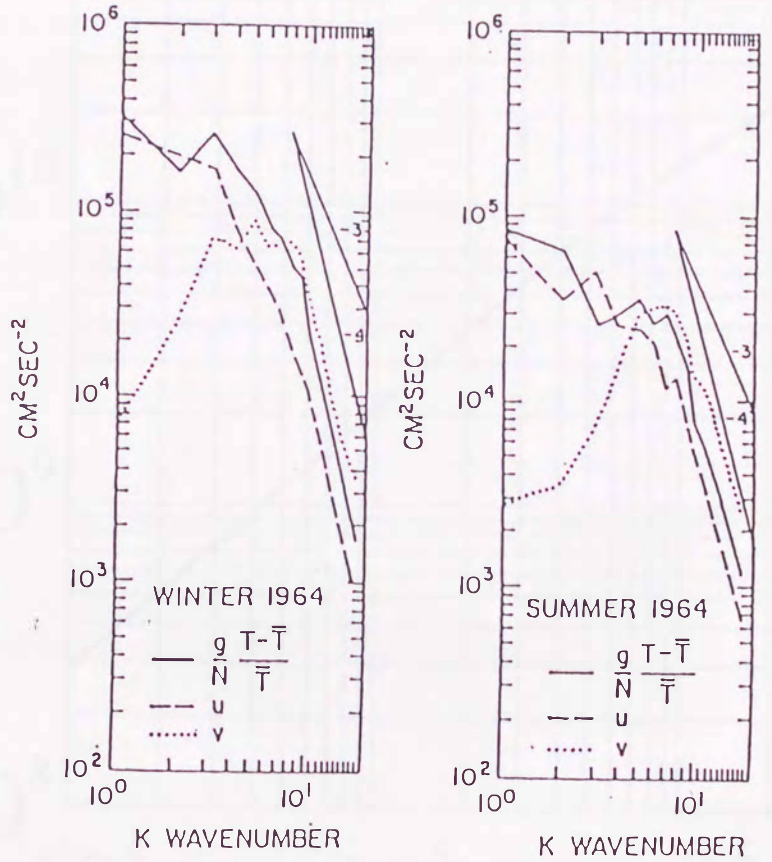


Fig. 5.2 (b) Observed power spectra after Charney (1971).

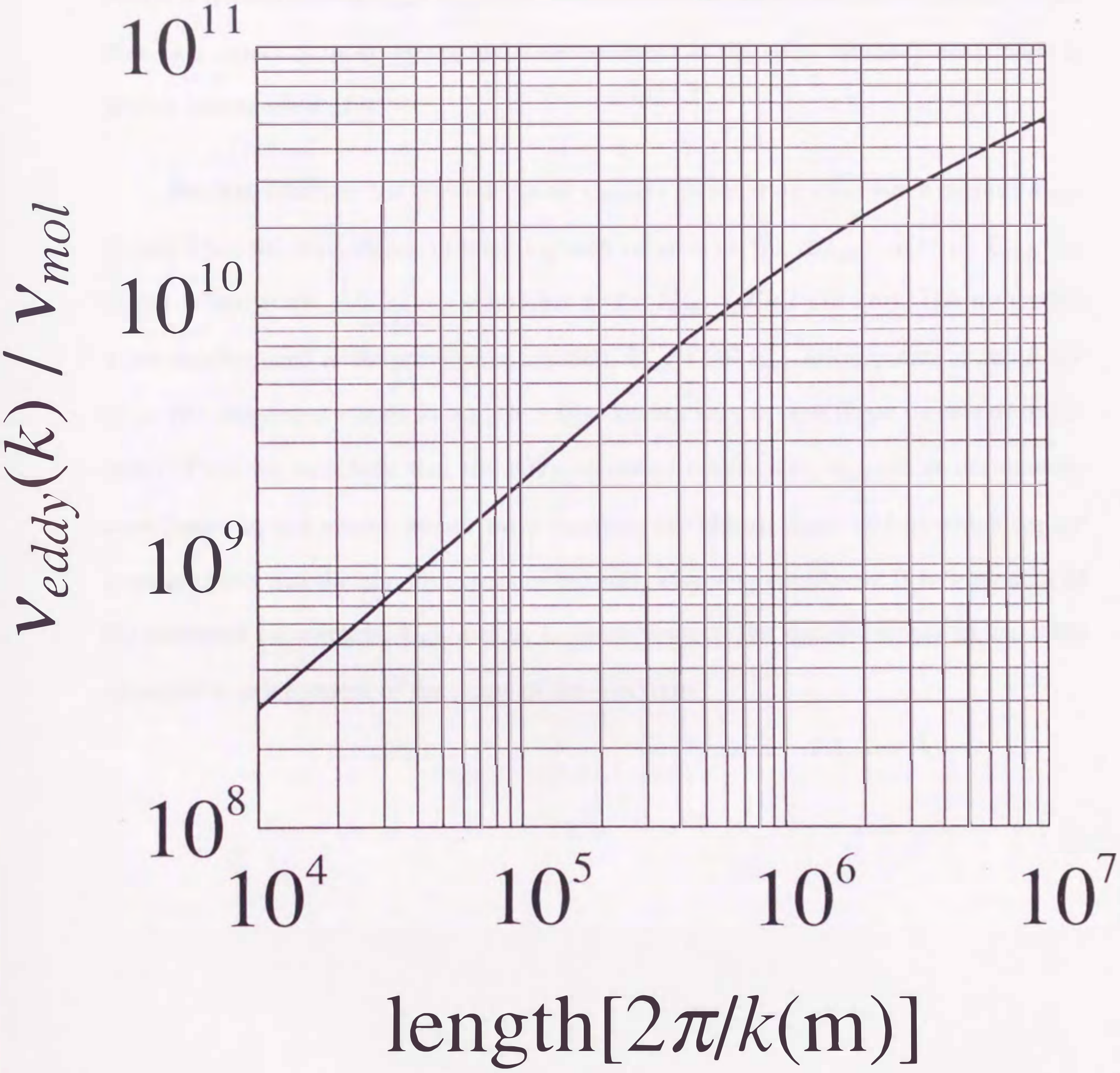


Fig. 5.3. Numerical result as calculated by formula (4.52). The ordinate indicates the ratio of the eddy viscosity coefficient to the molecular viscosity coefficient of the air. The abscissa indicates the characteristic length of a phenomenon, $2\pi/k$.

We examine the dependence of the eddy viscosity coefficient for Cheju Island scale on the parameter μ of the power spectrum (see Fig. 5.4). The ordinate indicates the ratio of the eddy viscosity coefficient to the molecular viscosity coefficient of the air. The abscissa indicates the wave number at which the power spectrum is maximum, which is normalized by k_{min}. Although $\sqrt{\mu}$ changes five orders, the ratio changes by less than two orders at most. Hence the enormousness of the eddy viscosity coefficient is almost independent of μ.

We also examine the dependence of $\nu_{eddy}(k)$ on the truncation wave number k_{max}. Figure 5.5 is the dependence of the integrated value in (4.52), $I(k_{max}) - I(k)$, on k_{max}, for Cheju Island scale, whose wave number is $k = k_{C\text{-}D} = 2\pi / (30 \text{ km})$. The truncation wave number used in the previous estimation, $k_{max} = 10^5 k_{min}$, corresponds to about $10^2 k_{C\text{-}D}$. The integrated values for $k_{max} > 5 k_{C\text{-}D}$ do not vary in significant figures of third-order. Thus we conclude that the eddy viscosity coefficient $\nu_{eddy}(k)$ is dominantly contributed by the modes whose wave numbers are almost equal to k in which we are interested and that the enormousness of the eddy viscosity coefficient is independent of the assumed parameters, k_{max} and μ. In these respect, the enormousness of the eddy viscosity is independent of the shape of the spectrum.

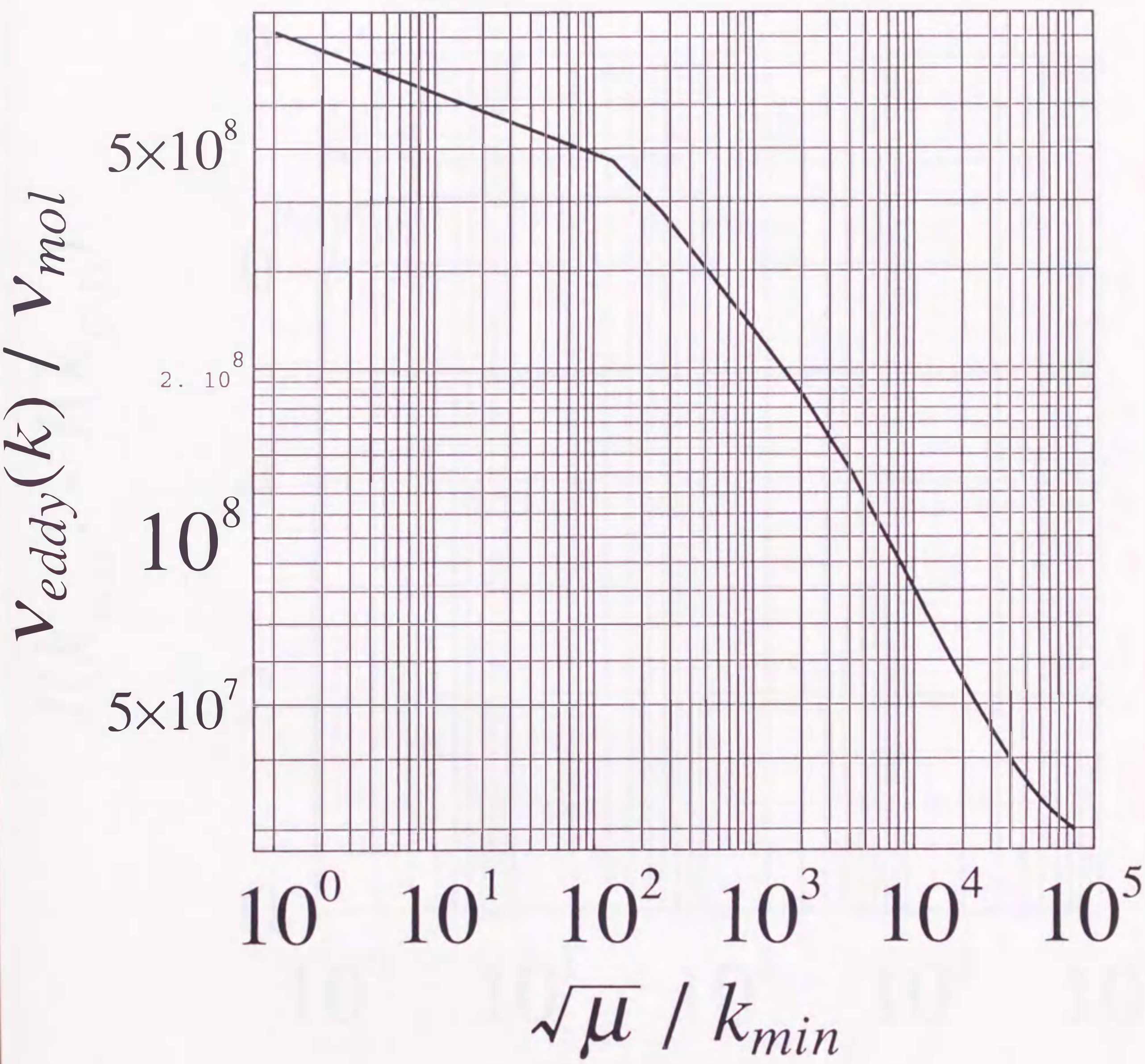


Fig. 5.4. Dependence of the eddy viscosity coefficient on the parameter μ of the power spectrum, for Cheju Island scale. The ordinate indicates the ratio of the eddy viscosity coefficient to the molecular viscosity coefficient of the air. The abscissa indicates the wave number normalized by k_{min}, at which the power spectrum is maximum.

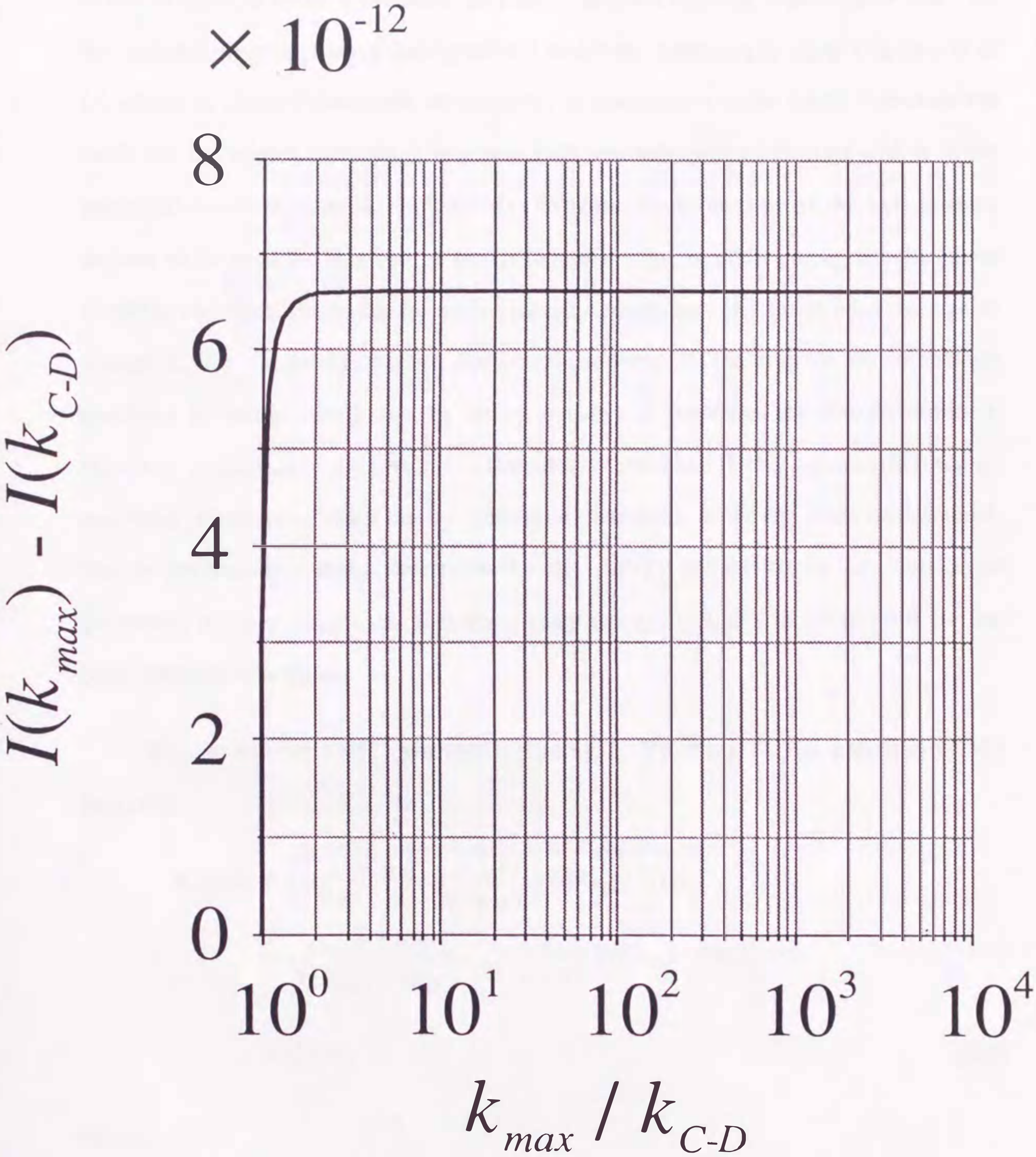


Fig. 5.5. Dependence of the integrated value in (4.52) on the truncation wave number k_{max} for Cheju Island scale, whose wave number corresponds to $k_{C\text{-}D} = 2\,\pi\,/\,(30$ km). The ordinate indicates the integrated value in (4.52). The abscissa indicates the truncation wave number k_{max} divided by $k_{C\text{-}D}$.

Equation (4.52) implies that eddies which have Cheju Island scale recognize the extent of earth, because it includes the scale of domain L. What is physics of that ? As we consider the properties of fluid confined within the earth as a basin of a finite size of L^2, eddies of Cheju Island scale are regarded as phenomena in the basin. Therefore the earth and the eddies correspond to a heat bath and sub-systems contact with it in the statistical-thermodynamics, respectively. Because the properties of the sub-systems depend on those of the heat bath, it stands to reason why the eddies recognize the extent of earth, i.e., the formula for the eddy viscosity coefficient (4.52) includes the size of domain L. For conserved systems, dynamical properties of the systems are completely specified by initial conditions, by which conserved quantities are also determined. However, properties of the systems in statistical mechanical theory are specified not by individual initial conditions but by conserved quantities. Since the scale of domain L and the parameters B and μ determine the total energy and enstrophy, i.e., conserved quantities, it is also reasonable that these quantities are included in the formula for the eddy viscosity coefficient.

We can rewrite (4.52) without including L. By using (4.30), equation (4.52) becomes

$$
\begin{aligned}
\nu_{eddy}(k) &= \sqrt{\left(\frac{L}{2\pi}\right)^2 \frac{\pi B}{2}} \sqrt{\frac{\kappa_1}{4\mu k^4}\{I(k_{max}) - I(k)\}} \\
&= \sqrt{\frac{Z+\mu E}{k_{max}{}^2 - k_{min}{}^2}} \sqrt{\frac{\kappa_1}{4\mu k^4}\{I(k_{max}) - I(k)\}} \\
&= V\, l(k),
\end{aligned}
\tag{5.2}
$$

where

$$
\left.
\begin{aligned}
V &\equiv \sqrt{\frac{Z+\mu E}{k_{max}{}^2 - k_{min}{}^2}}, \\
l(k) &= \sqrt{\frac{\kappa_1}{4\mu k^4}\left\{I(k_{max}) - I(k)\right\}}.
\end{aligned}
\right\}
\tag{5.3}
$$

The former part of (5.2), V, is a constant that is determined by properties of total system, and has the dimension of velocity. The latter part, $l(k)$, is a function of the wave number k, and has the dimension of length. Since the molecular viscosity coefficient ν_{mol} is expressed in terms of the product of the mean free path l of particles and the mean value of fluctuating velocity $\bar{u}$ of them, i.e., $\nu_{mol} \sim \bar{u}\, l$, the function $l(k)$ corresponds to the mean free path of vortices, or the mixing length. The form of $l(k)$ is the same as that of the curve in Fig. 5.3, i.e., $l(k) \sim k^{-1}$ (see Fig. 5.6). For air at atmospheric pressure (1000 hPa) and room temperature (300°K) one has approximately $\bar{u} \sim 10^2$ ms^{-1} and $l \sim 10^{-7}$ m, hence $\nu_{mol} \sim 10^{-5}$ m^{2}s^{-1} (Reif, 1965). On the other hand, for the selected parameter values in this work, we have $V = 1.5$ ms^{-1}. We see by Fig. 5.6 that the enormousness of the eddy viscosity in comparison to the molecular viscosity is due to that of the function $l(k)$, whereas $l(k)$ is the function of the length scale of interest. Thus we conclude that the enormousness of the eddy viscosity coefficient comes from that of the length scale of interest.

We consider what velocity V is. By using (4.30) again, we have

$$V = \sqrt{\frac{E\left(Z/E + \mu\right)}{\left(k_{max}^2 - k_{min}^2\right)}}$$

$$= \sqrt{\frac{E}{\left(k_{max}^2 - k_{min}^2\right)}\left\{\frac{\left(k_{max}^2 - k_{min}^2\right)}{\log\left(\dfrac{k_{max}^2 + \mu}{k_{min}^2 + \mu}\right)} - \mu + \mu\right\}}$$

$$= \sqrt{\frac{E}{\log\left(\dfrac{k_{max}^2 + \mu}{k_{min}^2 + \mu}\right)}}. \tag{5.4}$$

For the enstrophy equipartition state[1], that corresponds to $\mu \to 0$ (Kraichnan, 1975), we have

---

[1] In the enstrophy equipartition state, we have $Z/E = \left(k_{max}^2 + k_{min}^2\right)/2$.

$$V \rightarrow \sqrt{\frac{E}{2 \log \frac{k_{max}}{k_{min}}}} . \tag{5.5}$$

If the truncation wave number $k_{max} = 10^n \, k_{min}$, where n is less than or equal to 7 at most[2], we have

$$V \rightarrow \sqrt{\frac{U_0{}^2}{4 \, \mathrm{n} \log 10}} \sim 0.12 \, U_0 . \tag{5.6}$$

Thus V is approximately the horizontal velocity scale. For the energy equipartition state[3], that corresponds to $\mu \rightarrow \mu_\infty = 0.138033$ for the selected parameter values of L, k_{max} and U_0 in the previous estimation (Kraichnan, 1975), we obtain

$$V \rightarrow 9.5 \, U_0 , \tag{5.7}$$

i.e., V is approximately the horizontal velocity scale or the characteristic speed of wind, again. In consequence, V approximately corresponds to the horizontal velocity scale, for states that all modes are significantly excited, those correspond to be the case of $0 < \mu < \mu_\infty$ (Kraichnan, 1975).

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[2] For the terrestrial condition, $n = 7$ corresponds to the wave length to be approximately 2.3 m. If we are interested in phenomena whose wave lengths are less than that, the assumed conditions in this study, two-dimensional and inviscid fluid, may be violated, i.e., three-dimensionality and the molecular viscosity must be considered.

[3] In this state, we have $Z / E = \left(k_{max}{}^2 - k_{min}{}^2\right) / \left\{2 \log\left(k_{max} / k_{min}\right)\right\}$.

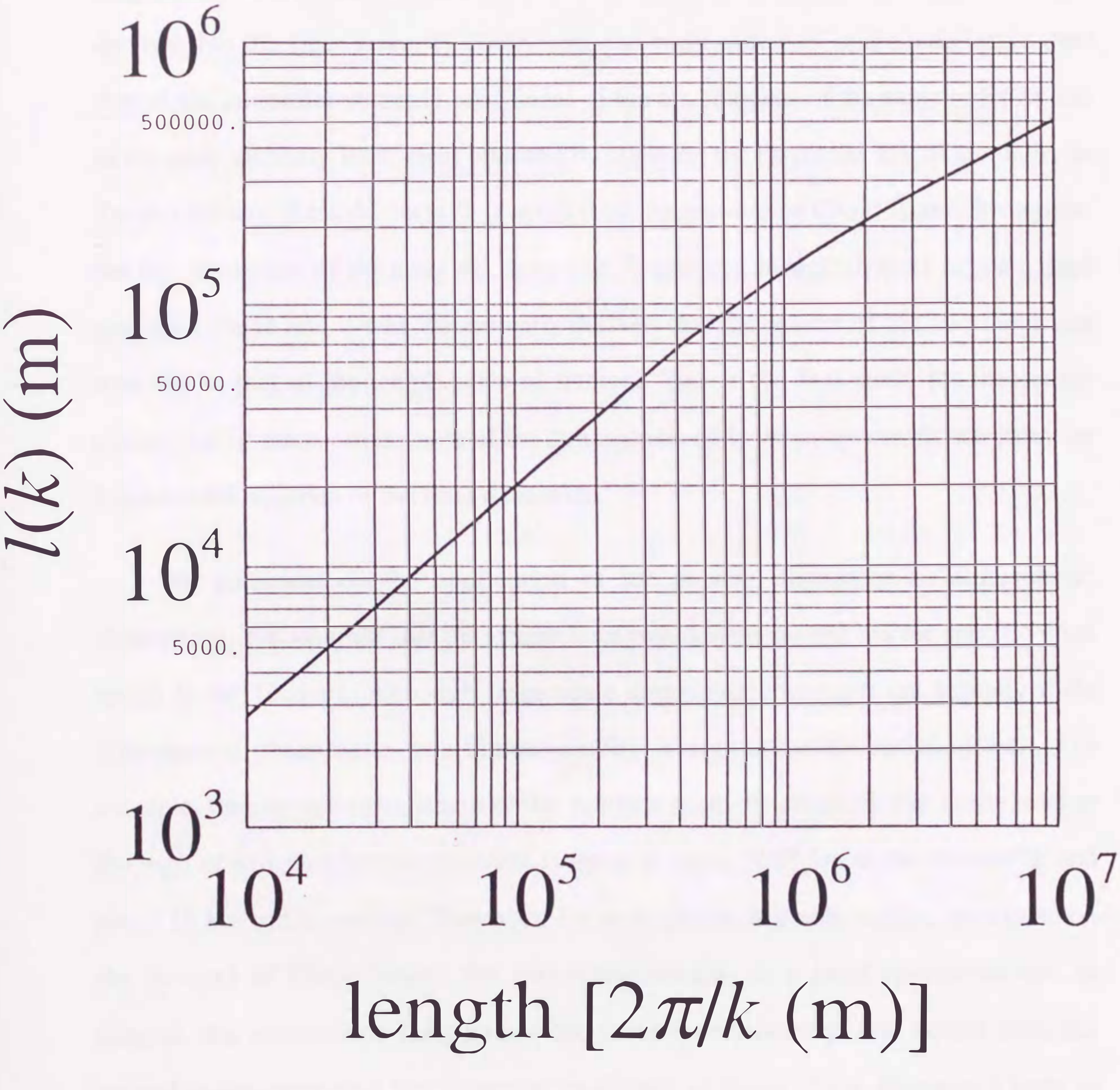

Fig. 5.6. Functional form of $l(k)$. The ordinate indicates the value of $l(k)$. The abscissa indicates the characteristic length, $2\pi/k$.

§6. Concluding Remarks

Applying Mori's theory (Mori, 1965a) to the spectral form of the vorticity equation for the 2-D inviscid barotropic fluid, we derived the formula for the eddy viscosity coefficient. The eddy viscosity coefficient thus obtained depended on the length scale of a phenomenon of interest. For Cheju Island scale, it was theoretically derived that the eddy viscosity coefficient was eight orders of magnitude larger than that of the molecular viscosity coefficient of the air, that was of the same order as that of the eddy viscosity coefficient obtained by applying the Reynolds' law of similarity to the atmospheric Kármán vortices generated on the leeward of Cheju Island. Moreover, the enormousness of the eddy viscosity coefficient was independent of hypothesized spectrum shape and it was theoretically derived that the reason of the enormousness was due to that of the length scale of interest. This is the first study for theoretical elucidation of the enormousness of the atmospheric eddy viscosity coefficient from the fundamental equation of the fluid dynamics.

We comment on the application of the present discussion to atmospheric phenomena. We assumed that the system is of two-dimension and has the specific wind speed to be 10 ms^{-1}. Although large scale atmospheric motions are actually three dimensional phenomena, two-dimensionality is approximately satisfied well. For example, the characteristic length of the synoptic scale disturbances that correspond to the high or low atmospheric pressure systems is about 3000 km in the horizontal and about 15 km in the vertical. Moreover, for atmospheric Kármán vortices generated on the leeward of Cheju Island, the two-dimensionality is a good approximation. In general, the atmospheric temperature decreases monotonically with height from the ground to the tropopause that locates at an altitude of about 15 km. However a layer in which the temperature increases with height is locally observed. This layer is called an inversion layer. The atmospheric Kármán vortices appear on the leeward of Cheju Island when the wind speed is more than 10 knots (~ 5 ms^{-1}) and a well defined inversion layer exists at about 1 km altitude (Tsuchiya, 1969). Then the vertical motion of the atmosphere is suppressed at this altitude. As seen from meteorological satellite photographs [e.g., see Tsuchiya, 1969], the horizontal scale of the atmospheric Kármán vortices is 30 ~ 50 km. Thus the two-dimensionality is a good approximation in this case. In addition, it is seen that the specific wind speed assumed in this study is reasonable.

It is known that the eddy viscosity coefficient of the atmosphere has different value in both horizontal and vertical directions. The vertical eddy viscosity coefficient is 1 ~ 10 m^2s^{-1}, which varies with height, for the mesosphere, thus 5 ~ 6 order larger than the molecular viscosity of the air. That arises from the unstable breakdown of tides and gravity waves whose vertical wavelength is about 10 km (Lindzen, 1981). Although our theory is the one for horizontal system, we can infer by the theory that the smallness of the vertical eddy viscosity coefficient in comparison to the horizontal one comes from the smallness of vertical scale of atmospheric phenomena. This problem for three dimensional inviscid barotropic fluid remains to be discussed.

In this study, we did not consider effects of the terrestrial rotation or the beta effect. For a sufficiently large value of β, a two-dimensional flow on a doubly-periodic beta-plane cannot be ergodic on the phase space of a constant energy and enstrophy (Shepherd, 1987). In consequence, the distribution function of the fluctuation of vorticity used in this study is invalid for such case. Since a distribution function of fluctuation of the vorticity including the beta-effect has not been known yet and the beta-effect violates the isotropy of the system, it is very hard to study the eddy viscosity for the system including the beta-effect, theoretically. This problem also remains to be investigated.

Some parts of this study have been reported in the Progress of Theoretical Physics and the others (Iwayama and Okamoto, 1993a; b; c).

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Professor Michiya Uryu of Kyushu University also considered applications of the fluctuation-dissipation theorem to meteorological systems. Unfortunately, he went to final rest on August 21, 1990. I am very disappointed because I can not discuss this study with him. I would like to dedicate this volume to his soul.

Finally, sincere gratitude is extended to Professor Hiroshi Matsumura of Tokai University for his continuous guidance and encouragement from my student days in Tokai University until now. He firstly noticed the existence of Mori's theory to me.

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Appendix A. Explicit Form of $R(\mathbf{k})$ and $R^{(1)}(\mathbf{k})$

By calculating the inner products, which are listed in Appendix B, we have

$$
\begin{aligned}
R(\mathbf{k}) &= \widehat{Q} \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \\
&= \left(1 - \widehat{P}\right) \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \\
&= \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p}) \left\{ \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) - \frac{\left(\zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}),\, \zeta(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}\, \zeta(\mathbf{k}) \right\} \\
&= \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}). \qquad \text{(A.1)}
\end{aligned}
$$

Next we calculate $R^{(1)}(\mathbf{k})$. We readily see that

$$
\begin{aligned}
R^{(1)}(\mathbf{k}) &= \widehat{Q}\ \Gamma R(\mathbf{k}) \\
&= \widehat{Q} \sum_{\mathbf{k}', \mathbf{q}} D(\mathbf{k}', \mathbf{q})\, \zeta(\mathbf{q})\, \zeta(\mathbf{k}' - \mathbf{q}) \frac{\partial}{\partial \zeta(\mathbf{k}')} \left\{ \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \right\} \\
&= \widehat{Q} \left\{ \sum_{\mathbf{p}, \mathbf{q}} D(\mathbf{p}, \mathbf{q})\, \zeta(\mathbf{q})\, \zeta(\mathbf{p} - \mathbf{q})\, D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{k} - \mathbf{p}) \right. \\
&\qquad \left. + \sum_{\mathbf{p}, \mathbf{q}} D(\mathbf{k} - \mathbf{p}, \mathbf{q})\, \zeta(\mathbf{q})\, \zeta(\mathbf{k} - \mathbf{p} - \mathbf{q})\, D(\mathbf{k}, \mathbf{p})\, \zeta(\mathbf{p}) \right\}. \qquad \text{(A.2)}
\end{aligned}
$$

By replacing $\mathbf{k} - \mathbf{p}$ by $\mathbf{p}$ in the second term of the last expression of (A.2) and noticing that $D(\mathbf{k}, \mathbf{k} - \mathbf{p}) = D(\mathbf{k}, \mathbf{p})$, we obtain

$$
\text{(A.2)} = 2\, \widehat{Q} \sum_{\mathbf{p}, \mathbf{q}} D(\mathbf{k}, \mathbf{p})\, D(\mathbf{p}, \mathbf{q})\, \zeta(\mathbf{k} - \mathbf{p})\, \zeta(\mathbf{p} - \mathbf{q})\, \zeta(\mathbf{q}). \qquad \text{(A.3)}
$$

Moreover, referring (B.4), we have

$$
\begin{aligned}
&\widehat{P} \sum_{\mathbf{p}, \mathbf{q}} D(\mathbf{k}, \mathbf{p})\, D(\mathbf{p}, \mathbf{q})\, \zeta(\mathbf{k} - \mathbf{p})\, \zeta(\mathbf{p} - \mathbf{q})\, \zeta(\mathbf{q}) \\
&\quad = \sum_{\mathbf{p}, \mathbf{q}} D(\mathbf{k}, \mathbf{p})\, D(\mathbf{p}, \mathbf{q}) \frac{\left(\zeta(\mathbf{k} - \mathbf{p})\, \zeta(\mathbf{p} - \mathbf{q})\, \zeta(\mathbf{q}),\, \zeta(\mathbf{k})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}\, \zeta(\mathbf{k})
\end{aligned}
$$

$$= \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p}) \{D(\mathbf{p}, \mathbf{k}) + D(\mathbf{p}, \mathbf{p} - \mathbf{k})\} \left(\zeta(\mathbf{k} - \mathbf{p}), \zeta(\mathbf{k} - \mathbf{p})^* \right) \zeta(\mathbf{k})$$

$$= 2 \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p}) D(\mathbf{p}, \mathbf{k}) \left(\zeta(\mathbf{k} - \mathbf{p}), \zeta(\mathbf{k} - \mathbf{p})^* \right) \zeta(\mathbf{k}) \tag{A.4}$$

and finally obtain

$$R^{(1)}(\mathbf{k}) = 2 \sum_{\mathbf{p}, \mathbf{q}} D(\mathbf{k}, \mathbf{p}) D(\mathbf{p}, \mathbf{q}) \zeta(\mathbf{k} - \mathbf{p}) \zeta(\mathbf{p} - \mathbf{q}) \zeta(\mathbf{q})$$

$$- 4 \sum_{\mathbf{p}} D(\mathbf{k}, \mathbf{p}) D(\mathbf{p}, \mathbf{k}) \left(\zeta(\mathbf{k} - \mathbf{p}), \zeta(\mathbf{k} - \mathbf{p})^* \right) \zeta(\mathbf{k}). \tag{A.5}$$

Appendix B. List of the Inner Products

The inner products or the ensemble averages are given by (4.14) with the distribution function (4.28). It is readily seen that

$$\left\{\zeta(1)\,\zeta(2)\cdots\zeta(\mathbf{m}),\,\zeta(\mathbf{m}+1)^*,\cdots\zeta(\mathbf{m}+\mathbf{n})^*\right\}=0' \tag{B.1}$$

for $m+n$ being odd numbers, where $\mathbf{m}\equiv\mathbf{k}_m$.

When $m+n=4$,

$$\begin{aligned}
&\left(\zeta(1)\,\zeta(2)\,\zeta(3),\,\zeta(4)^*\right)\\
&\quad=\left(\zeta(1),\,\zeta(1)^*\right)\left(\zeta(3),\,\zeta(3)^*\right)\delta_{1,\,-2}\,\delta_{3,\,4}\\
&\quad\quad+\left(\zeta(1),\,\zeta(1)^*\right)\left(\zeta(2),\,\zeta(2)^*\right)\delta_{1,\,-3}\,\delta_{2,\,4}\\
&\quad\quad+\left(\zeta(1),\,\zeta(1)^*\right)\left(\zeta(2),\,\zeta(2)^*\right)\delta_{1,\,4}\,\delta_{2,\,-3}\\
&\quad\quad+\left(\zeta(1)\,\zeta(1),\,\zeta(1)^*\,\zeta(1)^*\right)\times\left(\delta_{1,\,2}\,\delta_{3,\,-4}\,\delta_{1,\,-3}+\delta_{1,\,3}\,\delta_{2,\,-4}\,\delta_{1,\,-2}\right.\\
&\quad\quad\quad\left.+\,\delta_{1,\,-4}\,\delta_{2,\,3}\,\delta_{1,\,-2}\right),
\end{aligned}\tag{B.2}$$

$$\begin{aligned}
&\left(\zeta(1)\,\zeta(2),\,\zeta(3)^*\,\zeta(4)^*\right)\\
&\quad=\left(\zeta(1),\,\zeta(1)^*\right)\left(\zeta(3),\,\zeta(3)^*\right)\delta_{1,\,-2}\,\delta_{3,\,-4}\\
&\quad\quad+\left(\zeta(1),\,\zeta(1)^*\right)\left(\zeta(2),\,\zeta(2)^*\right)\delta_{1,\,3}\,\delta_{2,\,4}\\
&\quad\quad+\left(\zeta(1),\,\zeta(1)^*\right)\left(\zeta(2),\,\zeta(2)^*\right)\delta_{1,\,4}\,\delta_{2,\,3}\\
&\quad\quad+\left(\zeta(1)\,\zeta(1),\,\zeta(1)^*\,\zeta(1)^*\right)\times\left(\delta_{1,\,2}\,\delta_{3,\,4}\,\delta_{1,\,3}+\delta_{1,\,-3}\,\delta_{2,\,-4}\,\delta_{1,\,-2}\right.\\
&\quad\quad\quad\left.+\,\delta_{1,\,-4}\,\delta_{2,\,-3}\,\delta_{1,\,-2}\right).
\end{aligned}\tag{B.3}$$

We show examples of (B.2) and (B.3) for convenience of calculation in Appendix A and C:

$$\left(\zeta(\mathbf{k}-\mathbf{p})\,\zeta(\mathbf{p}-\mathbf{q})\,\zeta(\mathbf{q}),\,\zeta(\mathbf{k})^*\right)$$

$$= \left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}-\mathbf{p}}\,\delta_{\mathbf{q},\,\mathbf{k}}$$

$$+ \left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\delta_{\mathbf{k}-\mathbf{p},\,-\mathbf{q}}\,\delta_{\mathbf{p}-\mathbf{q},\,\mathbf{k}}$$

$$+ \left(\zeta(\mathbf{p}-\mathbf{q}),\,\zeta(\mathbf{p}-\mathbf{q})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{k}}\,\delta_{\mathbf{p}-\mathbf{q},\,-\mathbf{q}}$$

$$+ \left(\zeta(\mathbf{k})\,\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\,\zeta(\mathbf{k})^*\right)\times\left(\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{p}-\mathbf{q}}\,\delta_{\mathbf{q},\,-\mathbf{k}}\,\delta_{\mathbf{p}-\mathbf{q},\,-\mathbf{q}}\right.$$

$$+ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}}\,\delta_{\mathbf{p}-\mathbf{q},\,-\mathbf{k}}\,\delta_{\mathbf{p}-\mathbf{q},\,-\mathbf{q}}$$

$$\left. + \delta_{\mathbf{k}-\mathbf{p},\,-\mathbf{k}}\,\delta_{\mathbf{p}-\mathbf{q},\,\mathbf{q}}\,\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}-\mathbf{p}}\right), \qquad (B.4)$$

$$\left(\zeta(\mathbf{p})\,\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{q})^*\,\zeta(\mathbf{k}-\mathbf{q})^*\right)$$

$$= \left(\zeta(\mathbf{p}),\,\zeta(\mathbf{p})^*\right)\left(\zeta(\mathbf{q}),\,\zeta(\mathbf{q})^*\right)\delta_{\mathbf{p},\,\mathbf{p}-\mathbf{k}}\,\delta_{\mathbf{q},\mathbf{q}-\mathbf{k}}$$

$$+ \left(\zeta(\mathbf{p}),\,\zeta(\mathbf{p})^*\right)\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\delta_{\mathbf{p},\,\mathbf{q}}\,\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{k}-\mathbf{q}}$$

$$+ \left(\zeta(\mathbf{p}),\,\zeta(\mathbf{p})^*\right)\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\delta_{\mathbf{p},\,\mathbf{k}-\mathbf{q}}\,\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}}$$

$$+ \left(\zeta(\mathbf{p})\,\zeta(\mathbf{p}),\,\zeta(\mathbf{p})^*\,\zeta(\mathbf{p})^*\right)\times\left(\delta_{\mathbf{p},\,\mathbf{k}-\mathbf{p}}\,\delta_{\mathbf{q},\,\mathbf{k}-\mathbf{q}}\,\delta_{\mathbf{p},\,\mathbf{q}}\right.$$

$$+ \delta_{\mathbf{p},\,-\mathbf{q}}\,\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}-\mathbf{k}}\,\delta_{\mathbf{p},\,\mathbf{k}-\mathbf{p}}$$

$$\left. + \delta_{\mathbf{p},\,\mathbf{q}-\mathbf{k}}\,\delta_{\mathbf{k}-\mathbf{p},\,-\mathbf{q}}\,\delta_{\mathbf{p},\,\mathbf{p}-\mathbf{k}}\right). \qquad (B.5)$$

We present an example of inner product for $m + n = 6$:

$$\left(\zeta(\mathbf{k}-\mathbf{p})\,\zeta(\mathbf{p}-\mathbf{q})\,\zeta(\mathbf{q}),\,\zeta(\mathbf{k}-\mathbf{p}')^*\,\zeta(\mathbf{p}'-\mathbf{q}')^*\,\zeta(\mathbf{q}')^*\right)$$

$$= 2\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{q}),\,\zeta(\mathbf{k}-\mathbf{p}')^*\,\zeta(\mathbf{p}'-\mathbf{q}')^*\,\zeta(\mathbf{q}')^*\right)\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}-\mathbf{p}}$$

$$+ 2\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{p}-\mathbf{q}),\,\zeta(\mathbf{k}-\mathbf{p}')^*\,\zeta(\mathbf{p}'-\mathbf{q}')^*\,\zeta(\mathbf{q}')^*\right)\delta_{\mathbf{k}-\mathbf{p},\,-\mathbf{q}}$$

$$+\left(\zeta(\mathbf{k}-\mathbf{p}),\ \zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{p}-\mathbf{q})\ \zeta(\mathbf{q}),\ \ \zeta(\mathbf{p}'-\mathbf{q}')^*\ \zeta(\mathbf{q}')^*\right)\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{k}-\mathbf{p}'}$$

$$+2\left(\zeta(\mathbf{k}-\mathbf{p}),\ \zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{p}-\mathbf{q})\ \zeta(\mathbf{q}),\ \ \zeta(\mathbf{k}-\mathbf{p}')^*\ \zeta(\mathbf{q}')^*\right)\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{p}'-\mathbf{q}'}$$

$$+2\left(\zeta(\mathbf{k}-\mathbf{p}),\ \zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{p}-\mathbf{q})\ \zeta(\mathbf{q}),\ \ \zeta(\mathbf{k}-\mathbf{p}')^*\ \zeta(\mathbf{p}'-\mathbf{q}')^*\right)\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}'}$$

$$+\left(\zeta(\mathbf{k}-\mathbf{p})\ \zeta(\mathbf{p}-\mathbf{q})\ \zeta(\mathbf{q}),\ \zeta(\mathbf{k}-\mathbf{p}')^*\right)\left(\zeta(\mathbf{q}),\ \ \zeta(\mathbf{q})^*\right)\delta_{\mathbf{p}'-\mathbf{q}',\,\mathbf{q}'}$$

$$+\left(\zeta(\mathbf{k}-\mathbf{p})\ \zeta(\mathbf{p}-\mathbf{q}),\ \zeta(\mathbf{k}-\mathbf{p}')^*\ \zeta(\mathbf{p}'-\mathbf{q}')^*\right)\left(\zeta(\mathbf{q}),\ \ \zeta(\mathbf{q})^*\right)\delta_{\mathbf{q},\,\mathbf{q}'}$$

$$+\left(\zeta(\mathbf{k}-\mathbf{p})\ \zeta(\mathbf{p}-\mathbf{q}),\ \zeta(\mathbf{k}-\mathbf{p}')^*\ \zeta(\mathbf{q}')^*\right)\left(\zeta(\mathbf{q}),\ \ \zeta(\mathbf{q})^*\right)\delta_{\mathbf{q},\,\mathbf{p}'-\mathbf{q}'}$$

$$+\left(\zeta(\mathbf{k}-\mathbf{p})\ \zeta(\mathbf{q}),\ \zeta(\mathbf{k}-\mathbf{p}')^*\ \zeta(\mathbf{p}'-\mathbf{q}')^*\right)\left(\zeta(\mathbf{p}-\mathbf{q}),\ \ \zeta(\mathbf{p}-\mathbf{q})^*\right)\delta_{\mathbf{p}-\mathbf{q},\,\mathbf{q}'}$$

$$+\left(\zeta(\mathbf{k}-\mathbf{p})\ \zeta(\mathbf{q}),\ \zeta(\mathbf{k}-\mathbf{p}')^*\ \zeta(\mathbf{q}')^*\right)\left(\zeta(\mathbf{p}-\mathbf{q}),\ \ \zeta(\mathbf{p}-\mathbf{q})^*\right)\delta_{\mathbf{p}-\mathbf{q},\,\mathbf{p}'-\mathbf{q}'}$$

$$+\left(\left\{\zeta(\mathbf{k}-\mathbf{p})\right\}^3,\ \left\{\zeta(\mathbf{k}-\mathbf{p})^*\right\}^3\right)$$

$$\times\Big[\delta_{\mathbf{k}-\mathbf{p},\,\mathbf{p}-\mathbf{q}}\ \delta_{\mathbf{p}-\mathbf{q},\,\mathbf{q}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{p}'-\mathbf{q}'}\ \delta_{\mathbf{p}'-\mathbf{q}',\,\mathbf{q}'}\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{k}-\mathbf{p}'}$$

$$+\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{p}-\mathbf{q}}\ \delta_{\mathbf{k}-\mathbf{p},\,-\mathbf{q}}\Big\{\delta_{\mathbf{p}-\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{q},\,\mathbf{q}'-\mathbf{p}'}\ \delta_{\mathbf{p}'-\mathbf{q}',\,\mathbf{q}'}$$

$$+\ \delta_{\mathbf{p}-\mathbf{q},\,\mathbf{q}'-\mathbf{p}'}\ \delta_{\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{q}'}$$

$$+\delta_{\mathbf{p}-\mathbf{q},\,-\mathbf{q}'}\ \delta_{\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{p}'-\mathbf{q}'}\Big\}$$

$$+\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}}\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}-\mathbf{p}}\Big\{\delta_{\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{p}-\mathbf{q},\,\mathbf{q}'-\mathbf{p}'}\ \delta_{\mathbf{p}'-\mathbf{q}',\,\mathbf{q}'}$$

$$+\ \delta_{\mathbf{q},\,\mathbf{q}'-\mathbf{p}'}\ \delta_{\mathbf{p}-\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{q}'}$$

$$+\delta_{\mathbf{q},\,-\mathbf{q}'}\ \delta_{\mathbf{p}-\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{p}'-\mathbf{q}'}\Big\}$$

$$+\ \delta_{\mathbf{p}-\mathbf{q},\,\mathbf{q}}\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}-\mathbf{p}}\Big\{\delta_{\mathbf{q},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{q}'-\mathbf{p}'}\ \delta_{\mathbf{p}'-\mathbf{q}',\,\mathbf{q}'}$$

$$+\ \delta_{\mathbf{q},\,\mathbf{q}'-\mathbf{p}'}\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{q}'}$$

$$+\delta_{\mathbf{q},\,-\mathbf{q}'}\ \delta_{\mathbf{k}-\mathbf{p},\,\mathbf{p}'-\mathbf{k}}\ \delta_{\mathbf{k}-\mathbf{p}',\,\mathbf{p}'-\mathbf{q}'}\Big\}\Big]. \qquad \text{(B.6)}$$

Appendix C. Determination of the Memory Function from the Fluctuation-Dissipation Theorem

From (A.1) and (B.5), we have

$$\left(R(\mathbf{k}),\, R(\mathbf{k})^*\right) = \sum_{\mathbf{p},\,\mathbf{q}} D(\mathbf{k},\mathbf{p})\, D(\mathbf{k},\mathbf{q}) \left(\zeta(\mathbf{p})\,\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{q})^*\,\zeta(\mathbf{k}-\mathbf{q})^*\right)$$

$$= 2\sum_{\mathbf{p}} \{D(\mathbf{k},\mathbf{p})\}^2 \left(\zeta(\mathbf{p}),\, \zeta(\mathbf{p})^*\right)\left(\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{k}-\mathbf{p})^*\right).$$

Hence we obtain

$$\gamma_{\mathbf{k}}^{(0)} = 2\sum_{\mathbf{p}} \{D(\mathbf{k},\mathbf{p})\}^2 \frac{\left(\zeta(\mathbf{p}),\, \zeta(\mathbf{p})^*\right)\left(\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{k}-\mathbf{p})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}. \tag{C.1}$$

Now we define a positive quantity

$$\gamma_{\mathbf{k}}^{\mathbf{p}} \equiv 2\,\{D(\mathbf{k},\mathbf{p})\}^2 \frac{\left(\zeta(\mathbf{p}),\, \zeta(\mathbf{p})^*\right)\left(\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{k}-\mathbf{p})^*\right)}{\left(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\right)}. \tag{C.2}$$

Since $\left(R^{(1)}(\mathbf{k}),\, R(\mathbf{k})^*\right)$ includes only the ensemble average of odd numbers of modes, we have

$$\left(R^{(1)}(\mathbf{k}),\, R(\mathbf{k})^*\right) = 0$$

and

$$\gamma_{\mathbf{k}}^{(1)} = 0. \tag{C.3}$$

Next we see that

$$\left(R^{(1)}(\mathbf{k}),\, R^{(1)}(\mathbf{k})^*\right)$$

$$= 4 \sum_{\mathbf{p},\,\mathbf{q},\,\mathbf{p}',\,\mathbf{q}'} D(\mathbf{k},\mathbf{p})\, D(\mathbf{p},\mathbf{q})\, D(\mathbf{k},\mathbf{p}')\, D(\mathbf{p}',\mathbf{q}')$$

$$\times\left(\zeta(\mathbf{k}-\mathbf{p})\,\zeta(\mathbf{p}-\mathbf{q})\,\zeta(\mathbf{q}),\,\zeta(\mathbf{k}-\mathbf{p}')^*\,\zeta(\mathbf{p}'-\mathbf{q}')^*\,\zeta(\mathbf{q}')^*\right)$$

$$-16\sum_{\mathbf{p},\,\mathbf{p}'}\left[D(\mathbf{k},\mathbf{p})\,D(\mathbf{p},\mathbf{k})\,D(\mathbf{k},\mathbf{p}')\,D(\mathbf{p}',\mathbf{k})\left(\zeta(\mathbf{k}-\mathbf{p}'),\,\zeta(\mathbf{k}-\mathbf{p}')^*\right)\right.$$

$$\left.\times\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\right]$$

$$+16\sum_{\mathbf{p},\,\mathbf{p}'}\left[D(\mathbf{k},\mathbf{p})\,D(\mathbf{p},\mathbf{k})\,D(\mathbf{k},\mathbf{p}')\,D(\mathbf{p}',\mathbf{k})\left(\zeta(\mathbf{k}-\mathbf{p}'),\,\zeta(\mathbf{k}-\mathbf{p}')^*\right)\right.$$

$$\left.\times\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\right]. \qquad \text{(C.4)}$$

By making use of (B.4), the second term of (C.4) reduces to

$$-16\sum_{\mathbf{p},\,\mathbf{p}'}\left[D(\mathbf{k},\mathbf{p})\,D(\mathbf{p},\mathbf{k})\,D(\mathbf{k},\mathbf{p}')\,D(\mathbf{p}',\mathbf{k})\left(\zeta(\mathbf{k}-\mathbf{p}'),\,\zeta(\mathbf{k}-\mathbf{p}')^*\right)\right.$$

$$\left.\times\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\right]. \qquad \text{(C.5)}$$

Using (B.6), (B.2) and (B.3), and noticing that $D(\mathbf{k},\mathbf{k}-\mathbf{p}) = D(\mathbf{k},\mathbf{p})$, the first term of (C.4) becomes

$$\sum_{\mathbf{p},\,\mathbf{q}} 2\,\gamma_{\mathbf{k}}^{\mathbf{p}}\,\gamma_{\mathbf{p}}^{\mathbf{q}}\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)$$

$$+8\sum_{\mathbf{p},\,\mathbf{q}}\{D(\mathbf{k},\mathbf{p}-\mathbf{q})\,D(\mathbf{k}-\mathbf{p}+\mathbf{q},\mathbf{q}) + D(\mathbf{k}-\mathbf{q},\mathbf{p}-\mathbf{q})\,D(\mathbf{k},\mathbf{q})\}$$

$$\times D(\mathbf{k},\mathbf{p})\,D(\mathbf{p},\mathbf{q})\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{p}-\mathbf{q}),\,\zeta(\mathbf{p}-\mathbf{q})^*\right)\left(\zeta(\mathbf{q}),\,\zeta(\mathbf{q})^*\right)$$

$$+16\sum_{\mathbf{p},\,\mathbf{p}'}\left[D(\mathbf{k},\mathbf{p})\,D(\mathbf{p},\mathbf{k})\,D(\mathbf{k},\mathbf{p}')\,D(\mathbf{p}',\mathbf{k})\left(\zeta(\mathbf{k}-\mathbf{p}'),\,\zeta(\mathbf{k}-\mathbf{p}')^*\right)\right.$$

$$\left.\times\left(\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\left(\zeta(\mathbf{k}),\,\zeta(\mathbf{k})^*\right)\right]$$

$$+4\sum_{\mathbf{p}}\left[4\,\{D(\mathbf{k},\mathbf{p})\,D(\mathbf{p},\mathbf{k}-\mathbf{p})\}^2\left(\zeta(\mathbf{k}-\mathbf{p})\,\zeta(\mathbf{k}-\mathbf{p}),\,\zeta(\mathbf{k}-\mathbf{p})^*\,\zeta(\mathbf{k}-\mathbf{p})^*\right)\right.$$

$$\left.\times\left(\zeta(2\,\mathbf{p}-\mathbf{k}),\,\zeta(2\,\mathbf{p}-\mathbf{k})^*\right)\right]. \qquad \text{(C.6)}$$

Replacing $\mathbf{p}-\mathbf{q}$ by $\mathbf{q}$ in the first term in the curly bracket of the second term of (C.6), the second term of (C.6) reduces to

$$16 \sum_{\mathbf{p},\,\mathbf{q}} D(\mathbf{k},\mathbf{q})\, D(\mathbf{k}-\mathbf{q},\mathbf{p}-\mathbf{q})\, D(\mathbf{k},\mathbf{p})\, D(\mathbf{p},\mathbf{q})$$

$$\times \Big(\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{k}-\mathbf{p})^*\Big)\Big(\zeta(\mathbf{p}-\mathbf{q}),\, \zeta(\mathbf{p}-\mathbf{q})^*\Big)\Big(\zeta(\mathbf{q}),\, \zeta(\mathbf{q})^*\Big). \qquad \text{(C.7)}$$

Moreover, replacing $\mathbf{k} - \mathbf{p}$ by $\mathbf{p}$, we obtain

$$\text{(C.7)} = 16 \sum_{\mathbf{p},\,\mathbf{q}} D(\mathbf{k}-\mathbf{p},\mathbf{q})\, D(\mathbf{k}-\mathbf{q},\mathbf{p})\, D(\mathbf{k},\mathbf{p})\, D(\mathbf{k},\mathbf{q})$$

$$\times \Big(\zeta(\mathbf{k}-\mathbf{p}-\mathbf{q}),\, \zeta(\mathbf{k}-\mathbf{p}-\mathbf{q})^*\Big)\Big(\zeta(\mathbf{p}),\, \zeta(\mathbf{p})^*\Big)\Big(\zeta(\mathbf{q}),\, \zeta(\mathbf{q})^*\Big). \qquad \text{(C.8)}$$

The third term of (C.6) is canceled by adding itself with (C.5). With the aid of (C.2), the last term of (C.6) is rewritten as

$$\sum_{\mathbf{p}} 4\, \gamma_{\mathbf{k}}^{\mathbf{p}}\, \gamma_{\mathbf{p}}^{\mathbf{k}-\mathbf{p}} \frac{\Big(\zeta(\mathbf{k}-\mathbf{p})\,\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{k}-\mathbf{p})^*\,\zeta(\mathbf{k}-\mathbf{p})^*\Big)\Big(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\Big)}{\Big(\zeta(\mathbf{k}-\mathbf{p}),\ \zeta(\mathbf{k}-\mathbf{p})^*\Big)^2}. \qquad \text{(C.9)}$$

Therefore we have

$$\gamma_{\mathbf{k}}^{(2)} = \sum_{\mathbf{p},\,\mathbf{q}} 2\, \gamma_{\mathbf{k}}^{\mathbf{p}}\, \gamma_{\mathbf{p}}^{\mathbf{q}}$$

$$+\, 16 \sum_{\mathbf{p},\,\mathbf{q}} D(\mathbf{k}-\mathbf{p},\mathbf{q})\, D(\mathbf{k}-\mathbf{q},\mathbf{p})\, D(\mathbf{k},\mathbf{p})\, D(\mathbf{k},\mathbf{q})$$

$$\times \frac{\Big(\zeta(\mathbf{k}-\mathbf{p}-\mathbf{q}),\, \zeta(\mathbf{k}-\mathbf{p}-\mathbf{q})^*\Big)\Big(\zeta(\mathbf{p}),\, \zeta(\mathbf{p})^*\Big)\Big(\zeta(\mathbf{q}),\, \zeta(\mathbf{q})^*\Big)}{\Big(\zeta(\mathbf{k}),\, \zeta(\mathbf{k})^*\Big)}$$

$$+ \sum_{\mathbf{p}} 4\, \gamma_{\mathbf{k}}^{\mathbf{p}}\, \gamma_{\mathbf{p}}^{\mathbf{k}-\mathbf{p}} \frac{\Big(\zeta(\mathbf{k}-\mathbf{p})\,\zeta(\mathbf{k}-\mathbf{p}),\, \zeta(\mathbf{k}-\mathbf{p})^*\,\zeta(\mathbf{k}-\mathbf{p})^*\Big)}{\Big(\zeta(\mathbf{k}-\mathbf{p}),\ \zeta(\mathbf{k}-\mathbf{p})^*\Big)^2}.$$

$$\text{(C.10)}$$

Appendix D. Integration in (4.52)

The integration for the angular variable in (4.52) is evaluated with the aid of the complex integral:

$$\int_0^{2\pi} d\theta \left\{ \frac{\sin^2\theta}{\left(k^2 + p^2 - 2\,p\,k\cos\theta\right)} \right.$$

$$\left. \times \frac{\left(k^2 - 2\,p\,k\cos\theta\right)^2}{\left(k^2 + p^2 + \mu - 2\,p\,k\cos\theta\right)\left(k^2 + 2\,p^2 - 2\,p\,k\cos\theta\right)} \right\}$$

$$= \int_0^{2\pi} d\theta \frac{1}{2\,p\,k} \frac{\sin^2\theta\left(a - \cos\theta\right)^2}{\left(b - \cos\theta\right)\left(c - \cos\theta\right)\left(d - \cos\theta\right)}$$

$$= \frac{1}{2\,p\,k} \oint_C \Theta(z)dz\,, \tag{D.1}$$

where

$$\left.\begin{array}{ll} a \equiv \dfrac{k^2}{2\,p\,k}, & b \equiv \dfrac{k^2 + p^2}{2\,p\,k}, \\ c \equiv \dfrac{k^2 + p^2 + \mu}{2\,p\,k}, & d \equiv \dfrac{k^2 + 2\,p^2}{2\,p\,k} \end{array}\right\} \tag{D.2}$$

and

$$\Theta(z) \equiv -\frac{i}{2} \frac{\left(z^2 - 1\right)^2\left(z^2 - 2\,a\,z + 1\right)^2}{z^2\left(z^2 - 2\,b\,z + 1\right)\left(z^2 - 2\,c\,z + 1\right)\left(z^2 - 2\,d\,z + 1\right)}. \tag{D.3}$$

Note that $b \geq 1$, $c > 1$ and $d > 1$, because $p > 0$, $k > 0$ and $\mu > 0$. The variable z is the complex variable with magnitude of unity,

$$z = e^{i\,\theta}. \tag{D.4}$$

The last expression of (D.1) is integrated along a curve C which is the unit circle with center at the origin in the z plane.

The poles of integrand of (D.1), $\Theta(z)$, are $z = 0$, $z = b \pm \sqrt{b^2 - 1}$, $z = c \pm \sqrt{c^2 - 1}$ and $z = d \pm \sqrt{d^2 - 1}$. Among them, four poles locate within the unit circle C. These are

$$\left.\begin{aligned} z_1 &= 0, & z_2 &= b - \sqrt{b^2 - 1}, \\ z_3 &= c - \sqrt{c^2 - 1}, & z_4 &= d - \sqrt{d^2 - 1}. \end{aligned}\right\} \tag{D.5}$$

The residues of $\Theta(z)$ at these poles are

$$\mathrm{Res}[z_1] = i\,(2\,a - b - c - d\,) = -\,i\,\frac{4\,p^2 + k^2 + \mu}{2\,p\,k}, \tag{D.6}$$

$$\mathrm{Res}[z_2] = i\,\frac{(a - b)^2\sqrt{b^2 - 1}}{(b - c)\,(b - d)} = i\,\frac{p\left|p^2 - k^2\right|}{2\,\mu\,k}, \tag{D.7}$$

$$\mathrm{Res}[z_3] = i\,\frac{(a - c)^2\sqrt{c^2 - 1}}{(b - c)\,(d - c)} = -\,i\,\frac{\left(p^2 + \mu\right)^2\sqrt{\left(k^2 + p^2 + \mu\right)^2 - 4\,p^2\,k^2}}{2\,\mu\,p\,k\left(p^2 - \mu\right)} \tag{D.8}$$

and

$$\mathrm{Res}[z_4] = i\,\frac{(a - d)^2\sqrt{d^2 - 1}}{(b - d)\,(c - d)} = i\,\frac{2\,p\,\sqrt{k^4 + 4\,p^4}}{k\left(p^2 - \mu\right)}. \tag{D.9}$$

Next, the integration for the radial coordinate is evaluated as follows:

$$\begin{aligned} &\int_k^{k_{max}} dp\,\frac{p}{\left(p^2 + \mu\right)}\int_0^{2\,\pi} d\theta\left\{\frac{\sin^2\theta}{\left(k^2 + p^2 - 2\,p\,k\cos\theta\right)}\right. \\ &\left.\times\frac{\left(k^2 - 2\,p\,k\cos\theta\right)^2}{\left(k^2 + p^2 + \mu - 2\,p\,k\cos\theta\right)\left(k^2 + 2\,p^2 - 2\,p\,k\cos\theta\right)}\right\} \\ &\qquad = \frac{1}{2\,k}\int_k^{k_{max}} dp\,\frac{2\,\pi\,i}{p^2 + \mu}\left\{\mathrm{Res}[z_1] + \mathrm{Res}[z_2] + \mathrm{Res}[z_3] + \mathrm{Res}[z_4]\right\}, \end{aligned} \tag{D.10}$$

$$\int_k^{k_{max}} dp \frac{2\,\pi\,i}{p^2+\mu} \mathrm{Res}[z_1] = \frac{\pi}{k} \int_k^{k_{max}} \frac{4\,p^2+k^2+\mu}{p\left(p^2+\mu\right)} dp$$

$$= \frac{\pi}{2\,\mu\,k} \left\{\kappa_1 \log\left[p^2\right] - \left(\kappa_2 - 2\mu\right) \log\left[\vartheta(p)\right]\right\}\Big|_k^{k_{max}}, \tag{D.11}$$

$$\int_k^{k_{max}} dp \frac{2\,\pi\,i}{p^2+\mu} \mathrm{Res}[z_2] = -\frac{\pi}{\mu\,k} \int_k^{k_{max}} \frac{p\left(p^2-k^2\right)}{\left(p^2+\mu\right)} dp$$

$$= -\frac{\pi}{2\,\mu\,k} \left\{p^2 - \kappa_1 \log\left[\vartheta(p)\right]\right\}\Big|_k^{k_{max}}, \tag{D.12}$$

$$\int_k^{k_{max}} dp \frac{2\,\pi\,i}{p^2+\mu} \mathrm{Res}[z_3] = \frac{\pi}{\mu\,k} \int_k^{k_{max}} \frac{\left(p^2+\mu\right)\sqrt{\left(k^2+p^2+\mu\right)^2 - 4\,p^2\,k^2}}{p\left(p^2-\mu\right)} dp$$

$$= \frac{\pi}{2\,\mu\,k} \left\{\chi(p) \right.$$

$$+ \kappa_1 \log\left[\left|\frac{\kappa_2\,p^2 - \kappa_1{}^2 - \kappa_1\,\chi(p)}{p^2}\right|\right]$$

$$+ 2\,\kappa_3 \log\left[\left|\frac{\left(p^2-\mu\right)}{\kappa_2\,\vartheta(p) - \kappa_1{}^2 - \mu\,p^2 - \kappa_3\,\chi(p)}\right|\right]$$

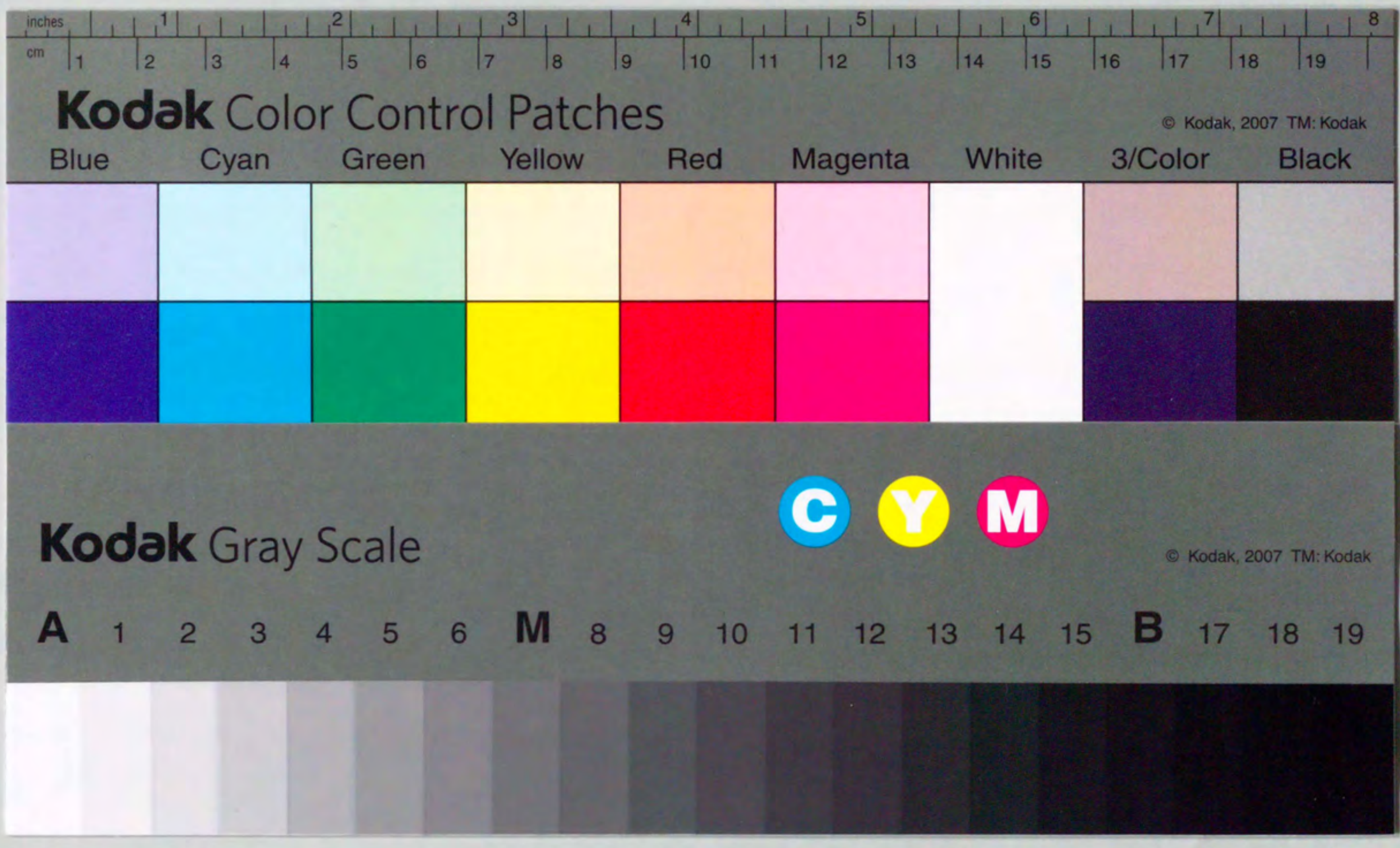
$$\left. - \left(\kappa_2 - 2\,\mu\right) \log\left[\xi(p) + \chi(p)\right]\right\}\Big|_k^{k_{max}}, \tag{D.13}$$

$$\int_k^{k_{max}} dp \frac{2\,\pi\,i}{p^2+\mu} \mathrm{Res}[z_4] = -\frac{4\pi}{k} \int_k^{k_{max}} \frac{p\sqrt{k^4+4\,p^2}}{\left(p^2-\mu\right)\left(p^2+\mu\right)} dp$$

$$= \left\{-\frac{\pi\,\kappa_3}{k\,\mu} \log\left[\left|\frac{\left(p^2-\mu\right)\left(k^4-4\,\mu\,p^2+\kappa_3\,\eta(p)\right)}{\vartheta(p)\left(k^4+4\,\mu\,p^2+\kappa_3\,\eta(p)\right)}\right|\right]\right.$$

$$\left. - \frac{4\,\pi}{k} \log\left[2\,p^2+\eta(p)\right]\right\}\Bigg|_k^{k_{max}}. \tag{D.14}$$

Parameters κ_1, κ_2 and κ_3 and functions $\xi(p)$, $\chi(p)$, $\eta(p)$ and $\vartheta(p)$ are defined in the text.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