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Dedicated to Professor Yuen-da Wang on his 66th Birthday

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§0. Introduction

An important problem in differential geometry is to investigate isometric immersions of complete Riemannian manifolds with constant mean curvature into space forms. The problem involves analysis on manifolds equipped with the moving frame introduced by E.Cartan. It is the purpose of this article to investigate complete submanifolds with constant mean curvature in space forms and space-like submanifolds in indefinite Riemannian manifolds by using the moving frames, where the maximum principle plays an essential role throughout this article. Minimal submanifolds are also discussed.

A classical result due to Myers (see [36]) states that if a closed Riemannian manifold M is minimally immersed into a closed hemi-sphere of the unit m-sphere $S^m(1)$ of constant curvature 1, then it is contained entirely in the great sphere. This may be viewed as a sphere version of the Bernstein problem on complete minimal surface into Euclidean 3-space. The idea of the proof is that the distance from every point on M to the boundary of the closed hemi-sphere is a superharmonic function on M and the maximum principle then implies that the distance is identically zero on M. This type of idea will be employed throughout this paper to obtain constancy of geometric quantities.

The maximum principle was generalized to complete noncompact Riemannian manifolds which was initiated by Omori in [40] and later developed by Yau in [53]. The generalized maximum principle due to Omori-Yau plays an important role throughout this article, and stated as follows. Let $\triangle$ be the Laplacian on M.

Theorem Y-1. *Let M be a complete noncompact Riemannian manifold whose Ricci curvature is bounded below. Let f be a C^2-function on M bounded above. Then there exists for every $\varepsilon > 0$ a point $p \in M$ with the following properties :*

$$\sup_M f < \varepsilon + f(p),$$
$$|grad\, f(p)| < \varepsilon,$$
$$\triangle f(p) < \varepsilon$$

The above theorem is used to generalize Myers' type results on complete noncompact minimal submanifolds as well as the results due to Goldberg (see Theorems 3.1, 3.2 and 3.3). One of our results will be stated as

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Theorem 5.6. *(see [9]) Let M be an n-dimensional complete minimal hypersurface in $S^{n+1}(1)$ which is contained entirely in a closed hemisphere. If the volume of M is finite, then M is totally geodesic.*

Remark. We do not know if the assumption for the volume of M is essential.

An important contribution in the investigation of closed minimal submanifolds in the sphere of constant curvature was established by Simons (see [45]) who derived the linear elliptic second order differential equation satisfied by the second fundamental form of minimal submanifolds. He obtained an important estimate of a lower bound for the index and the nullity of a non-totally geodesic minimal submanifold immersed into $S^n(1)$, and proved the following (see Theorem 5.3.2 ; [45])

The Simons Theorem. *Let M be an n-dimensional closed minimal submanifold in $S^{n+p}(1)$ and if $\mathcal{S}$ is the squared norm of the second fundamental form of M, then*

$$\int_M \{(2-\frac{1}{p})\mathcal{S}-n\}\mathcal{S}\, dv \geq 0,$$

where dv is the volume element of M.

It then follows that if $\mathcal{S} \leq n/(2-\frac{1}{p})$, then we have either $\mathcal{S}=0$ (e.g. M is totally geodesic) or else $\mathcal{S}=n/(2-\frac{1}{p})$. The complete determination of the latter case was obtained by Chern-do Carmo-Kobayashi in [22] as follows. For positive integers m and n with $m<n$ let $M_{m,n-m} := S^m(\frac{n}{m}) \times S^{n-m}(\frac{n}{n-m})$ be the Clifford torus.

Theorem CDK. *The Veronese surface in S^4 and the Clifford torus $M_{m,n-m}$ in S^{n+1} are the only closed minimal submanifolds of dimension n in $S^{n+p}(1)$ satisfying $\mathcal{S}=n/(2-\frac{1}{p})$*

Notice that the second fundamental form of $M_{m,n-m}$ has exactly two eigenvalues with multiplicities m and $n-m$. Conversely, if M is an n-dimensional closed minimal hypersurface in $S^{n+1}(1)$ having two principal curvatures with multiplicities $m>1$ and $n-m>1$, then it is the Riemannian product $M_{m,n-m}$. Otsuki proved that the minimal hypersurfaces with two principal curvatures one of which has multiplicity 1 are determined by a nonlinear second order ordinary differential equation. They have the same topological types as $S^{n-1}\times \boldsymbol{R}$ (when M is noncompact), or $S^{n-1}\times S^1$ (when M is compact), for detail see [42]. There are infinitely many isometrically distinct such minimal hypersurfaces in $S^{n+1}(1)$ on which scalar curvatures are not constant. Every minimal hypersurface of constant scalar curvature in $S^{n+1}(1)$ has the property that $\mathcal{S}$ is constant on it. Chern also conjectured as follows.

Chern's Conjecture. *For closed minimal hypersurfaces in $S^{n+1}(1)$ with constant scalar curvature the values $\{\mathcal{S}\}$ will form a discrete sequence.*

A breakthrough on Chern's Conjecture was made by Peng and Terng in [43], in which they proved the following

Theorem PT. *Let M be a closed minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature. If $\mathcal{S} > n$, then $\mathcal{S} > n + c(n)$ for a positive constant $c(n)$ with $c(n) > \frac{1}{12n}$. In particular, if $n = 3$, then $\mathcal{S} \geq 6$.*

In view of Theorem PT we see that Cartan's example of an isoparametric hypersurface in $S^4(1)$ defined by the following equation (see [8]):

$$2x_5^3 + 3(x_1^2 + x_2^2)x_5 - 6(x_3^2 + x_4^2)x_5 + 3\sqrt{3}(x_1^2 - x_2^2)x_4 + 6\sqrt{3}x_1x_2x_3 = 2$$

has the property that $\mathcal{S} = 6$ which is the twice of its dimension. We thus know that for closed minimal hypersurfaces of constant scalar curvature in $S^{n+1}(1)$, the first and the second constants are 0 and n. It will be expected that the third constant will be $2n$. Thus it will be natural to have the following conjecture (see [43]).

The Peng-Terng Conjecture. *For closed minimal hypersurfaces in $S^{n+1}(1)$ with constant scalar curvature we will have $\mathcal{S} \geq 2n$ if $\mathcal{S} > n$.*

In connection with the conjectures by Chern and Peng-Terng, we shall prove the following

Theorem 4.1. *(see [19]) Let M be a closed minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature. If $\mathcal{S} > n$, then $\mathcal{S} > n + \frac{n}{20}$.*

Further computations will show the following ;

Theorem 4.6. *(see[20]) Under the assumption in Theorem 4.1 we have*

$$\mathcal{S} > n + \frac{2}{7}n - \frac{9}{14} \qquad \text{if } 3 < n \leq 7,$$
$$\mathcal{S} > n + \frac{2n}{7} - \frac{5}{8} \qquad \text{if } 7 < n \leq 17,$$
$$\mathcal{S} > n + \frac{n}{4} \qquad \text{if } n > 17.$$

The above theorem will suggest that the Peng-Terng Conjecture may possibly be solved affirmatively.

In connection with the Chern-do Carmo-Kobayashi theorem, we will prove the following

Theorem 5.2. *Let M be an n-dimensional complete minimal submanifold in $S^{n+p}(1)$ with $p > 1$. If $\mathcal{S} \leq n/(2 - \frac{1}{p})$, then M is either totally geodesic or a Veronese surface in $S^4(1)$.*

In the special case when $n = 3$, we have the following

Corollary 5.4. *(see [10]) Let M be a 3-dimensional complete minimal hypersurface in $S^4(1)$. If $\mathcal{S}$ is constant and if $\mathcal{S} \leq 6$, then $\mathcal{S} = 0, 3$ or 6.*

On the other hand Nomizu and Smyth studied the hypersurfaces of constant mean curvature in the space forms. Let $M^m(c)$ be an m-dimensional space form of constant sectional curvature c. Their results in [38] involve some estimates for $\mathcal{S}$ and are used to determine totally umbilic hypersurfaces such as small spheres in $S^{n+p}(1)$.

Theorem 6.2. *(see [16]) Let M be a complete hypersurface in $M^{n+1}(c)$ with $c \geq 0$ of constant mean curvature H. If*

$$\sup_M \mathcal{S} < [n\{(2(n-1)c + n^2H^2\} - (n-2)n\{n^2H^2 + 4(n-1)c\}^{\frac{1}{2}}|H|]/2(n-1),$$

then M is totally umbilic.

Theorem 6.3. *(see [15]) Let M be a complete hypersurface in $S^4(1)$ with constant mean curvature. If $\mathcal{S}$ is constant and not greater than h^2+6, then we have, by setting $h := 3H$, $\mathcal{S} = \frac{h^2}{3}$, $3 + \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}$, $3 + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2}$, or $h^2 + 6$.*

It should be remarked that in the above theorem 6.3, M is totally umbilical when $\mathcal{S} = \frac{h^2}{3}$, and M an isoparametric hypersurface when $\mathcal{S} = h^2 + 6$ and $h \neq 0$. Moreover isoparametric hypersurfaces in $S^4(1)$ with two distinct principal curvatures have the properties $\mathcal{S} = 3 + \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}$ or $3 + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2}$. However we do not know whether or not the converse of this case in Theorem 6.3 is true.

The Gauss-Kronecker curvature on an n-dimensional hypersurface M is defined to be the determinant of the second fundamental form of M. We shall introduce the notion of the *quasi-Gauss-Kronecker curvature* of M as follows:

$$K = \Pi_i(\lambda_i - H),$$

where λ_i's are the principal curvatures of M. In the case where $n = 3$, the K has the property

$$3K = \sum_{i=1}(\lambda_i - H)^3.$$

With this definition we shall prove the following

Theorem 6.8. *(see [14]) Let M be a closed oriented 3-dimensional hypersurface in $S^4(1)$ with nonzero constant mean curvature H. If the quasi-Gauss-Kronecker curvature K of M is constant and if $K \cdot H \leq 0$, then M is either totally umbilic or else an isoparametric hypersurface with $\mathcal{S} = 3 + \frac{3}{4}h^2 \pm \frac{1}{4}\sqrt{h^4 + 8h^2}$. Here we set $h := 3H$.*

In the second place we shall discuss in §§7 and 8 space-like submanifolds with constant mean curvature in the spaces of indefinite metrices. Calabi [7] and Cheng-Yau [21] investigated that complete space-like hypersurfaces in a Minkowski space $\boldsymbol{R}_1^{n+1}$ possesses remarkable Bernstein property in the maximal case. The classical Bernstein theorem states that a complete minimal surface expressed as a graph of a plane in $\boldsymbol{R}^3$ is isometric to the Euclidean plane. This type of results have been obtained by many people in the case of complete space-like hypersurfaces in the de Sitter space. Among them, Marsden and Tipler pointed out that space-like hypersurfaces with constant mean curvature in arbitrary space-time are interesting from the point of view of relativity theory. In connection with this topic we shall prove the following results.

Theorem 7.1. *(see [16]) Let M be a complete space-like hypersurface with constant mean curvature H in a Lorentzian space form $M_1^{n+1}(c)$ of constant curvature $c \le 0$. Then we have*

$$nH^2 \le \mathcal{S} \le [n\{n^2H^2 - 2(n-1)c\} + (n-2)n\{n^2H^2 - 4(n-1)c\}|H|]^{\frac{1}{2}}/2(n-1).$$

Notice that Theorem 7.1 generalizes the results obtained by Ishihara [29] and Cheng-Yau [21]. In fact, by setting $H = 0$, Theorem 7.1 implies that $0 \le \mathcal{S} \le -nc$. Ishihara obtained this relation for complete maximal space-like hypersurface in $M_1^{n+1}(c)$. A result by Cheng-Yau states that an entire space-like hypersurface of constant mean curvature in R_1^{n+1} satisfies $nH^2 \le \mathcal{S} \le n^2H^2$. This relation is obtained by setting $c = 0$ in Theorem 7.1.

In the special case where the sectional curvature of M is nonnegative we have isometric results as stated below.

Theorem 7.2. *(see [12]) Let M be an n-dimensional complete space-like hypersurface in $S_1^{n+1}(c)$ with constant mean curvature. If the sectional curvature of M is nonnegative everywhere and if the multiplicity of every principal curvature is greater than 1, then M is isometric to $\boldsymbol{R}^n$ or $S^n(c_1)$ for some $0 < c_1 < c$.*

Theorem 7.3. *(see [11]) Let M be a complete space-like hypersurface in $S_1^{n+1}(c)$with nonnegative sectional curvature. If the scalar curvature r of M satisfies $r = k \cdot H$ for some nonnegative constant k and if H assumes its supremum at some point of M, then M is isometric to $\boldsymbol{R}^n$ or to $S^n(c_1)$ for some $0 < c_1 < c$.*

Theorem 7.4. *(see [11]) Let M be a complete space-like hypersurface in $S_1^{n+1}(c)$ of nonnegative sectional curvature. If the scalar curvature r of M satisfies $r = k \cdot H$ for some nonnegative constant k and if the multiplicity of each principal curvature at every point on M is greater than 1, then M is isometric to $\boldsymbol{R}^n$ or to $S^n(c_1)$ for some $0 < c_1 < c$.*

In the case of space-like submanifolds in $S_p^{n+p}(c)$ for an integer $p > 1$, we have the following

Theorem 8.1. *(see [13]) Let M be an n-dimensional complete space-like submanifold in $S_p^{n+p}(c)$ with parallel mean curvature vector. If*

$$H^2 \le c \quad \text{for } n = 2, \tag{8.1}$$

$$n^2H^2 < 4(n-1)c \quad \text{for } n \ge 3, \tag{8.2}$$

then M is totally umbilic.

Remark. The assumptions (8.1) and (8.2) are optimal.

The rest of the article is organized as follows. In §1, we give the basic concepts and formulas used here. We shall discuss the generalized maximum principle in §2 and conformally flat metrices in §3. Submanifolds in Riemannian space forms are discussed in §§4,5 and 6. In §5 we shall provide slight extensions of Theorems CDK and PT to complete manifolds. In §6 we give some results on totally umbilic submanifolds in $S^{n+p}(c)$ of constant curvature c. Complete space-like submanifolds with constant mean curvature in the de Sitter spaces will be discussed in §§7 and 8.

§1. Basic concepts and Formulae

Throughout this paper all manifolds are assumed to be smooth,connected without boundary . We discuss in smooth category. An m-dimensional *semi-Riemannian manifold* (M', g') of index s is by definition an m-dimensional manifold equipped with a nondegenerate symmetric bilinear form g' with index s. This g' will be called a *semi-Riemannian metric* on M'. It is an indefinite Riemannian manifold or simply a Riemannian manifold according as $s > 0$ or $s = 0$. We choose a local field of orthonormal frames $e_1, \ldots, e_m$ adopted to the semi-Riemannian metric on M' and the dual coframes $\omega_1, \ldots, \omega_m$ in such a way that $g'(e_A, e_B) = \varepsilon_A \delta_{AB}$ for $A, B = 1, \ldots, m$ and $\varepsilon_A = -1$ for $A = 1, \ldots, s$, $\varepsilon_A = 1$ for $A = s+1, \ldots, m$. The connection forms with respect to g' will be characterized by the structure equations as stated ;

$$
\begin{aligned}
d\omega_A &= -\sum \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\
d\omega_{AB} &= -\sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \\
\Omega_{AB} &= -\frac{1}{2}\sum \varepsilon_C \varepsilon_D R'_{ABCD} \omega_C \wedge \omega_D,
\end{aligned} \tag{1.1}
$$

where Ω_{AB} and R'_{ABCD} are the semi-Riemannian curvature form and the components of semi-Riemannian curvature tensor R' of M'. The components of Ricci curvature tensor Ric' and the scalar curvature r' are given as

$$R'_{AB} = R'_{BA} = \sum \varepsilon_C R'_{CABC}, \tag{1.2}$$

$$r' = \sum \varepsilon_A R'_{AA} = \sum \varepsilon_A \varepsilon_B R'_{ABBA}. \tag{1.3}$$

A semi-Riemannian manifold M' of constant sectional curvature c is called an *indefinite space form of index* s and denoted by $M^m_s(c)$ or simply a space form and denoted by $M^m(c)$ according to $s > 0$ or $s = 0$. The components of R'_{ABCD} of $M^m_s(c)$ are given by

$$R'_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}). \tag{1.4}$$

Therefore the Ricci curvature Ric' and the scalar curvature r' are written as

$$R'_{AB} = (m-1)c\varepsilon_A \delta_{AB}, \; r' = m(m-1)c. \tag{1.5}$$

In particular, $M^m_1(c)$ is called a *Lorentz space* and if $c = 0$, then it is called a *Minkowski space*.

The standard models of complete semi-space forms are given as follows. In an $(n+p)$-dimensional Euclidean space R^{n+p} with the standard basis, the scalar product $<,>$ is given by

$$< x, y >= -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n+p} x_j y_j,$$

where $x = (x_1, \ldots, x_{n+p})$ and $y = (y_1, \ldots, y_{n+p})$. The $(\boldsymbol{R}^{n+p}, <,>)$ is an indefinite Euclidean space which we denote by $\boldsymbol{R}_p^{n+p}$. Let $S_p^{n+p}(c)$ for $c > 0$ be the hypersurface in $\boldsymbol{R}_p^{n+p+1}$ given as

$$< x, x >= \frac{1}{c} =: r_0^2.$$

Then the $S_p^{n+p}(c)$ inherits a semi-Riemannian metric induced through $\boldsymbol{R}_p^{n+p+1}$ and has constant curvature c. This is called a *de Sitter space* of constant curvature c with the index p.

On the other hand let $H_p^{n+p}(c)$ for $c < 0$ be the hypersurface of $\boldsymbol{R}_{p+1}^{n+p+1}$ defined by

$$< x, x >= \frac{1}{c} =: -r_0^2.$$

Then this $H_p^{n+p}(c)$ induces a semi-Riemannian metric through $\boldsymbol{R}_{p+1}^{n+p+1}$ with respect to which the curvature is negative constant c, and is called an *anti-de Sitter space* of constant curvature c. Detailed discussion on indefinite Riemannian manifolds are refered to O'Neill [41].

From now on let $M' = M_s^{n+p}(c)$ be an $(n+p)$-dimensional semi-Riemannian space form of constant curvature with index s $(s = 0, p)$, and M an n-dimensional submanifold of $M_s^{n+p}(c)$ which has a positive definite induced metric. In the sequel the following convention on the range of indices are used, unless otherwise stated.

$$1 \le A, B, C, \ldots, \le n + p$$
$$1 \le i, j, k, \ldots, \le n$$
$$n + 1 \le \alpha, \beta, \gamma, \ldots, \le n + p$$

And we agree that the repeated indices under a sumation sign without indication are summed over the respective range. The restricted canonical forms ω_A and connection forms ω_{AB} to M are also denoted by the same symbols. We then have

$$\omega_\alpha = 0 \ \text{ for } \alpha = n + 1, \ldots, n + p \tag{1.6}$$

and the induced metric g on M through $M_s^{n+p}(c)$ is given by $g = \sum \omega_i \otimes \omega_j$. We see that $\{e_1, \ldots, e_n\}$ is a local field of orthonormal frames on M with respect to g, and also $\{\omega_1, \ldots, \omega_n\}$ is a local field of its dual frames on M. It follows from (1.1),(1.6) and Cartan's lemma that

$$\omega_{\alpha i} = \sum h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{1.7}$$

The second fundamental form of M is given by

$$\alpha = \sum \varepsilon_\alpha h_{ij}^\alpha \omega_i \omega_j \, e_\alpha.$$

Thus

$$\alpha(e_i, e_j) = \sum \varepsilon_\alpha h_{ij}^\alpha \, e_\alpha. \tag{1.8}$$

The mean curvature vector h and the mean curvature H of M are defined by

$$h = \frac{1}{n} \sum_{\alpha} \varepsilon_\alpha (\sum_i h_{ii}^\alpha) e_\alpha, \tag{1.9}$$

and

$$H^2 = \frac{1}{n^2} \sum (\sum_i h_{ii}^\alpha)^2. \tag{1.10}$$

M is by definition a *minimal submanifold* if the mean curvature of M is identically zero. M is said to be *totally umbilical* if the h_{ij}^α can be simultaneously expressed as a scalar multiplication of the identity matrix for all α at every point on M such that $h_{ij}^{n+1} = H\delta_{ij}$, $h_{ij}^\alpha = 0$ for $\alpha \neq n+1$, where $He_{n+1} = h$.

The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations as follows

$$\begin{aligned} d\omega_i &= -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= -\sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \\ \Omega_{ij} &= -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{aligned} \tag{1.11}$$

where Ω_{ij} (respectively R_{ijkl}) are the Riemannian curvature forms (respectively the components of the Riemannian curvature tensor) of M. It follows from (1.1) and (1.11) that

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum \varepsilon_\alpha (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha). \tag{1.12}$$

The components of the Ricci curvature tensor Ric and the scalar curvature r of M are given by

$$R_{jk} = c(n-1)\delta_{jk} + \sum \varepsilon_\alpha (h_{ii}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{ji}^\alpha), \tag{1.13}$$

and

$$r = n(n-1)c + \sum \varepsilon_\alpha (h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2). \tag{1.14}$$

We also have the structure equations for the normal bundle over M as follows;

$$\begin{aligned} d\omega_\alpha &= -\sum \varepsilon_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \\ d\omega_{\alpha\beta} &= -\sum \varepsilon_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j, \\ R_{\alpha\beta ij} &= \sum (h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta). \end{aligned} \tag{1.15}$$

Now the covariant derivative $\nabla\alpha$ of the second fundamental form α of M with components h^α_{ijk} is given by

$$\sum h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum h^\alpha_{kj}\omega_{ik} + \sum h^\alpha_{ik}\omega_{jk} + \sum \varepsilon_\beta h^\beta_{ij}\omega_{\alpha\beta}. \tag{1.16}$$

Then the exterior derivation of (1.7) together with the structure equation gives

$$\sum h^\alpha_{ijk}\omega_j\omega_k = 0.$$

Thus we obtain the Codazzi equations ;

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{1.17}$$

Similarly we have the covariant derivative $\nabla^2\alpha$ of $\nabla\alpha$ with components h^α_{ijkl} as follows ;

$$\sum h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum h^\alpha_{ljk}\omega_{il} + \sum h^\alpha_{ilk}\omega_{jl} + \sum h^\alpha_{ijl}\omega_{kl} + \sum \varepsilon_\beta h^\beta_{ijk}\omega_{\alpha\beta}. \tag{1.18}$$

Taking the exterior differential of (1.16) we obtain the Ricci formula as follows ;

$$h^\alpha_{ijkl} = h^\alpha_{ijlk} - \sum h^\alpha_{im}R_{mjkl} - \sum h^\alpha_{mj}R_{mikl} - \sum \varepsilon_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{1.19}$$

Similarly we also have

$$h^\alpha_{ijklm} = h^\alpha_{ijkml} - \sum h^\alpha_{tjk}R_{tilm} - \sum h^\alpha_{itk}R_{tjlm} - \sum h^\alpha_{ijt}R_{tklm} - \sum \varepsilon_\beta h^\beta_{ijk}R_{\beta\alpha lm}, \tag{1.20}$$

where h^α_{ijklm} is the component of the covariant derivative $\nabla^3\alpha$ of $\nabla^2\alpha$.

The Laplacian Δh^α_{ij} of the components h^α_{ij} of the second fundamental form α is given by

$$\Delta h^\alpha_{ij} = \sum h^\alpha_{ijkk}.$$

From (1.19) we have

$$\Delta h^\alpha_{ij} = \sum h^\alpha_{kkij} - \sum h^\alpha_{km}R_{mijk} - \sum h^\alpha_{mi}R_{mkjk} - \sum \varepsilon_\beta h^\beta_{ki}R_{\beta\alpha jk}, \tag{1.21}$$

and

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum (h^\alpha_{ijk})^2 + \sum h^\alpha_{ij}\Delta h^\alpha_{ij}\\ &= \sum (h^\alpha_{ijk})^2 + \sum h^\alpha_{ij}h^\alpha_{kkij} - \sum h^\alpha_{ij}h^\alpha_{mk}R_{mijk}\\ &\quad - \sum h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}.\end{aligned} \tag{1.22}$$

The following Cartan's lemma plays an essential role in the study of isoparametric hypersurfaces as well as our investigation.

Cartan's Lemma. *If M is a hypersurface in $S^{n+1}(1)$ with constant distinct principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_p$, each λ_i having the multiplicity m_i, then*

$$\sum_{j\neq i} \frac{m_i(1+\lambda_i\lambda_j)}{\lambda_i - \lambda_j} = 0$$

In particular if $n = 3$, then

$$S = 3H^2,\ 3 + \frac{27}{4}H^2 \pm \frac{3}{4}\sqrt{9H^4 + 8H^2},\ \text{or}\ 9H^2 + 6.$$

§2. Generalized maximum principle

The pioneering work on this topic was made by Omori in [40]. This theorem will be employed in the proof of Theorem 7.3 and stated as:

Theorem O. *Let M be a connected and complete Riemannian manifold whose sectional curvature is bounded below. If $f : M \to \boldsymbol{R}$ is bounded above and smooth, then there exists for any $\varepsilon > 0$ a point $p \in M$ such that*

$$\begin{gathered}\sup_M f < \varepsilon + f(p),\\ |\operatorname{grad} f(p)| < \varepsilon,\\ \nabla^2 f(X,X)(p) < \varepsilon \ \text{ for every unit vector } X.\end{gathered}$$

It was pointed out by Aubin in [5] that a similar result holds under the assumption on the Ricci curvature of M. This fact was later proved by Yau as stated.

Theorem Y-1. *Let M be a connected and complete Riemannian manifold whose Ricci curvature is bounded below. If $f : M \to \boldsymbol{R}$ is bounded above and smooth, then there exists for any $\varepsilon > 0$ a point $p \in M$ such that*

$$\begin{gathered}\sup_M f < \varepsilon + f(p),\\ |\operatorname{grad} f(p)| < \varepsilon,\\ \Delta f(p) < \varepsilon.\end{gathered} \tag{2.1}$$

Theorem Y-2. *Let M be a connected and complete Riemannian manifold whose Ricci curvature is bounded below by a constant L. Let f be a smooth function bounded below such that for some positive constants c_j,*

$$|\Delta f| \le c_1|\nabla f| + c_2(f - \inf_M f), \tag{2.2}$$

and

$$|\nabla(\Delta f)| \le c_3\sqrt{\sum f_{ij}^2} + c_4|\nabla f| + c_5(f - \inf_M f). \tag{2.3}$$

Then there exists a constant c depending only on n, L and c_j such that

$$|\nabla f| \le c(f - \inf_M f). \tag{2.4}$$

Remark. Under the assumptions in Theorem Y-1 it is not possible to replace the first inequality to the following :

$$\sup_M f - \varepsilon < f(p) < \sup_M f - \varepsilon/2.$$

In fact, setting $M := R^2$ and $f(x,y) := -\exp(cx)$ for a constant $c > 2$, we observe that M and f fulfil the assumptions in Theorem Y-1. Suppose that there exists for a decreasing positive sequence $\{\varepsilon_k\}$ converging to zero a sequence $\{p_k\}$, $p_k := (x_k, y_k)$ such that

$$\sup_M f - \varepsilon_k < f(p_k) < \sup_M f - \varepsilon_k/2, \tag{2.5}$$

$$|\mathrm{grad}\, f(p_k)| < \varepsilon_k, \tag{2.6}$$

$$\Delta f(p_k) < \varepsilon_k. \tag{2.7}$$

We then have

$$-\varepsilon_k/c < -\exp(cx_k) < \varepsilon_k/c^2. \tag{2.8}$$

On the other hand it follows from (2.5) and $\sup_M f = 0$ that

$$-\varepsilon_k < f(p_k) < -\varepsilon_k/2. \tag{2.9}$$

This is a contradiction since $c > 2$.

§3. Conformally flat spaces

In this section we shall prove some extensions of the Goldberg Theorem G to not necessarily manifolds whose Ricci curvature is bounded below. The scalar curvature r of M is the trace of the symmetric linear transformation Q defined by the Ricci tensor Ric of M, and is written as

$$r = \mathrm{trace}\, Q.$$

The following result was proved by Goldberg in [26].

Theorem G. *Let M be an n-dimensional complete conformally flat Riemannian manifold ($n > 2$) whose Ricci curvature is bounded below. If the scalar curvature r is positive constant and if $\sup_M$ trace$Q^2 < \frac{r^2}{n-1}$, then M is isometric to a space form.*

We shall prove the following

Theorem 3.1. *(see [18]) Let M be an n-dimensional complete conformally flat Riemannian manifold ($n > 2$). If the scalar curvature r is positive constant and if $\sup_M$ trace$Q^2 < \frac{r^2}{n-1}$, then M is isometric to a space form.*

Proof of Theorem 3.1. In view of Theorem G we only need to show that the Ricci curvature of M is bounded below.

From assumption we have for every unit vector $u \in TM$,

$$|Ric(u,u)| \leq \sqrt{\mathrm{trace}\, Q^2} < \frac{r}{\sqrt{n-1}}.$$

This completes the proof of Theorem 3.1.

Theorem 3.2. *(see [18]) Let M be an n-dimensional complete conformally flat Riemannian manifold ($n > 2$) whose scalar curvature is positive constant. If $\sup_M \operatorname{trace} Q^2 \le \frac{r^2}{n-1}$ and if there is a point $p \in M$ such that $\operatorname{trace} Q^2(p) = \frac{r^2}{n-1}$, then M is isometric to a Riemannian product $M_1 \times N$, where M_1 is a space from and N is 1-dimensional.*

Proof of Theorem 3.2. It follows from assumption that the function $f : M \to \boldsymbol{R}$ defined as $f := \operatorname{trace} Q^2 - \frac{r^2}{n}$ attains its maximum at p. Making use of the Hopf's theorem we conclude $f \equiv \frac{r^2}{n(n-1)}$. This proves Theorem 3.2.

In the special case where $n = 3$, we have the following

Theorem3.3. *Let M be a 3-dimensional complete conformally flat Riemannian manifold of positive constant scalar curvature. If the squared norm of Ricci tensor is constant, then M is isometric to either a space form or else a Riemannian product $M_1 \times N$, where M_1 is a space form and N is 1-dimensional.*

The proof is easy and hence omitted here.

§4. Closed minimal hypersurfaces in $S^{n+1}(1)$

In this section let M be a closed minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature r. It follows from the assumptions that

$$\mathcal{S} = n(n-1) - r. \tag{4.1}$$

The above relation means that $\mathcal{S}$ is intrinsic and independent of the immersion of M into $S^{n+1}(1)$.

We have from (1.21),

$$\Delta h_{ij} = \sum h_{kkij} - \sum h_{mk} R_{mijk} - \sum h_{mi} R_{mkjk}.$$

Since M is minimal we see that

$$\Delta h_{ij} = (n - \mathcal{S}) h_{ij}. \tag{4.2}$$

Therefore we have

$$\frac{1}{2}\Delta \mathcal{S} = \sum h_{ijk}^2 + (n - \mathcal{S})\mathcal{S}. \tag{4.3}$$

Since $\mathcal{S}$ is constant, the assumption implies that the right hand side of (4.3) is zero. Thus we see that if $\mathcal{S} \le n$, then $\mathcal{S} = 0$ or n.

Making use of (1.19) and (1.20) we have

$$\begin{aligned} \frac{1}{2}\Delta\left(\sum h_{ijk}^2\right) = {} & (2n + 3 - \mathcal{S})(\mathcal{S} - n)\mathcal{S} - 3\sum h_{ijk} h_{ijl} h_{km} h_{ml} \\ & + 6\sum h_{ijk} h_{ilm} h_{jl} h_{km} + \sum h_{ijkl}^2. \end{aligned} \tag{4.4}$$

Under our situation $\mathcal{S}$ is constant and (4.3) reduces to

$$\sum h_{ijk}^2 = \mathcal{S}(\mathcal{S}-n). \tag{4.5}$$

For every point $p \in M$ we choose an orthonormal frame field $\{e_1, \ldots, e_n\}$ such that the matrix (h_{ij}) is diagonalized at that point, say,

$$h_{ij} = \lambda_i \delta_{ij}. \tag{4.6}$$

Using this frame field we see that (4.4) becomes

$$\sum h_{ijkl}^2 = \mathcal{S}(\mathcal{S}-n)(\mathcal{S}-2n-3) + 3(A-2B), \tag{4.7}$$

where we set

$$A := \sum \lambda_i^2 h_{ijk}^2, \qquad B = \sum \lambda_i \lambda_j h_{ijk}^2. \tag{4.8}$$

Setting

$$t_{ij} := h_{ijij} - h_{jiji}, \tag{4.9}$$

and using (1.19) we have

$$t_{ij} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j). \tag{4.10}$$

Moreover we obtain

$$\begin{aligned} \sum_{i \neq j} h_{ijij}^2 &= \sum_{i<j} h_{ijij}^2 + \sum_{i<j} (h_{ijij} - t_{ij})^2 \\ &= \frac{1}{4} \sum_{i \neq j} (h_{ijij} + h_{jiji})^2 + \frac{1}{4} \sum_{i \neq j} t_{ij}^2. \end{aligned} \tag{4.11}$$

We shall prove the following

Theorem 4.1. *(see [19]) Let M be an n-dimensional ($n > 2$) closed minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature. If $\mathcal{S} > n$, then there exists a positive constant $c > \frac{1}{20}$ such that*

$$\mathcal{S} > n + cn. \tag{4.12}$$

For the proof of Theorem 4.1 some Lemmas and Propositions will be needed.

Let f_m be the m-th symmetric function of the principal curvatures, i.e.,

$$f_m = \sum \lambda_i^m = \text{trace}[(h_{ij})^m]. \tag{4.13}$$

Lemma 4.2. *Under the assumptions in Theorem 4.1 there exists a point $x_0 \in M$ at which*

$$(\mathcal{S}f_4 - \mathcal{S}^2 - f_3^2) + (2B - A) = 0. \tag{4.14}$$

Proof. A direct calculation implies that

$$2B - A = \sum h_{ijk}(h_{iq}h_{jp}h_{qk} - h_{ij}h_{pq}h_{qk})_p. \tag{4.15}$$

On the other hand,

$$\begin{aligned}
&\sum h_{ijkp}(h_{iq}h_{jp}h_{qk} - h_{ij}h_{pq}h_{qk}) = \sum h_{ijij}(\lambda_i^2\lambda_j - \lambda_j^2\lambda_i) \\
&= \frac{1}{2}\sum(h_{ijij} - h_{jiji})(\lambda_i - \lambda_j)\lambda_i\lambda_j = \frac{1}{2}\sum(\lambda_i - \lambda_j)^2(1 + \lambda_i\lambda_j)\lambda_i\lambda_j \\
&= \mathcal{S}f_4 - \mathcal{S}^2 - f_3^2.
\end{aligned}$$

By iterating (4.15) on M and using the Stokes theorem we obtain

$$\int(2B - A)\,dv = -\int\sum h_{ijkp}(h_{iq}h_{jp}h_{qk} - h_{ij}h_{pq}h_{qk})\,dv = -\int(\mathcal{S}f_4 - \mathcal{S}^2 - f_3^2)\,dv.$$

Thus we have

$$\int[(\mathcal{S}f_4 - \mathcal{S}^2 - f_3^2) + (2B - A)]\,dv = 0.$$

Since the integrand is continuous on M we find the desired point. This proves Lemma 4.2.

We continue further computations. By using (4.11) we get

$$\begin{aligned}
\sum h_{ijkl}^2 &\geq \sum h_{iiii}^2 + 3\sum_{i\neq j} h_{ijij}^2 \\
&= \sum h_{iiii}^2 + \frac{3}{4}[\sum(h_{ijij} + h_{jiji})^2 + \sum t_{ij}^2] \\
&\geq \frac{3}{2}(\mathcal{S}f_4 - \mathcal{S}^2 - f_3^2 - 2\mathcal{S} + n\mathcal{S}).
\end{aligned} \tag{4.16}$$

From (4.7) and (4.16) we derive

$$\mathcal{S}(\mathcal{S} - n)(\mathcal{S} - 2n - 3) + 3(A - 2B) \geq \frac{3}{2}(\mathcal{S}f_4 - 2\mathcal{S}^2 - f_3^2 + n\mathcal{S}). \tag{4.17}$$

If x_0 is a point as obtained in Lemma 4.2, then (4.17) implies at x_0,

$$\mathcal{S}(\mathcal{S} - n)(2n - \mathcal{S}) + \frac{3}{2}[(2B - A) + \mathcal{S}(\mathcal{S} - n)] < 0. \tag{4.18}$$

The following relation was obtained in [43]

$$3(2B - A) \geq -3\mathcal{S}^2(\mathcal{S} - n). \tag{4.19}$$

Lemma 4.3. *Let x_0 be a point as obtained in Lemma 4.2. Assume that two principal curvatures λ_1 and λ_2 at x_0 satisfy*

$$|\lambda_1| = \max_i |\lambda_i|, \qquad |\lambda_2| = \max_{\lambda_j \lambda_1 < 0} |\lambda_j|,$$

then we have

$$-3(2B - A) \leq (\lambda_1^2 - 4\lambda_1\lambda_2)(\mathcal{S} - n)\mathcal{S}. \tag{4.20}$$

Proof. By means of the symmetric property of h_{ijk} we have

$$\begin{aligned} -3(2B - A) &= \sum h_{ijk}^2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_k\lambda_i) \\ &\leq \sum_{i \neq j \neq k} h_{ijk}^2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_k\lambda_i) + 3\sum_{i \neq j} h_{iij}^2(\lambda_j^2 - 4\lambda_i\lambda_j). \end{aligned} \tag{4.21}$$

Without loss of generality we may assume that $\lambda_i \leq \lambda_j \leq \lambda_k$ for $i \neq j \neq k \neq i$. Then $J := \lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2(\lambda_i\lambda_j + \lambda_j\lambda_k + \lambda_k\lambda_i)$ takes its maximum for $\lambda_j = \lambda_i$ or $\lambda_j = \lambda_k$. Thus we get $J \leq \lambda_1^2 - 4\lambda_1\lambda_2$. This together with (4.21) implies that

$$-3(2B - A) \leq (\lambda_1^2 - 4\lambda_1\lambda_2)\sum h_{ijk}^2 = (\lambda_1^2 - 4\lambda_1\lambda_2)\mathcal{S}(\mathcal{S} - n).$$

This proves Lemma 4.3.

In view of (4.18) and (4.19) we see that there exists a constant $t \in [-\frac{1}{2}, 1]$ such that

$$-3(2B - A) - 3\mathcal{S}(\mathcal{S} - n) = 2(1 - t)\mathcal{S}^2(\mathcal{S} - n) \tag{4.22}$$

holds at x_0. If $n < \mathcal{S}$, then Lemma 4.3 and (4.22) imply

$$\lambda_1^2 - 4\lambda_1\lambda_2 \geq 2(1 - t)\mathcal{S} + 3 \tag{4.23}$$

and also

$$-\lambda_1\lambda_2 \geq \frac{1}{4}(1 - 2t)\mathcal{S} + \frac{3}{4}. \tag{4.24}$$

Substituting (4.22) into (4.18) we get at x_0,

$$\mathcal{S}(\mathcal{S} - n)[2n - \mathcal{S} - (1 - t)\mathcal{S}] \leq 0.$$

Thus we have proved the following

Proposition 4.4. *If $\mathcal{S} > n$ and if $t > 0$, then we have*

(4.25)
$$\mathcal{S} \geq n/(1 - \frac{t}{2}).$$

Proposition 4.5. *If $\mathcal{S} > n$ and if $t \in [-\frac{1}{2}, \frac{1}{3}]$, then*

(4.26)
$$\mathcal{S} \geq n + \frac{9}{64}[1 + 3/2(1-t)\mathcal{S}] \cdot [(1-2t)\mathcal{S} - 1]^2/\mathcal{S}.$$

Proof of Proposition 4.5. From Lemma 4.2 and (4.22) we have the following relations at x_0

(4.27)
$$\mathcal{S}f_4 - \mathcal{S}^2 - f_3^2 - \mathcal{S}(\mathcal{S}-n) = A - 2B - \mathcal{S}(\mathcal{S}-n) = \frac{2}{3}(1-t)\mathcal{S}^2(\mathcal{S}-n).$$

On the other hand we have

(4.28)
$$\mathcal{S}f_4 - \mathcal{S}^2 - f_3^2 - \mathcal{S}(\mathcal{S}-n) \geq (\lambda_1 - \lambda_2)^2(1 + \lambda_1\lambda_2)^2.$$

Since (4.23) implies $(\lambda_1 - \lambda_2)^2 \geq \frac{3}{4}(\lambda_1^2 - 4\lambda_1\lambda_2) \geq \frac{3}{4}\{2(1-t)\mathcal{S} + 3\}$ and $-(\lambda_1\lambda_2 + 1) \geq \frac{1}{4}\{(1-2t)\mathcal{S} - 1\}$, we obtain from (4.27) and (4.28)

$$\frac{2}{3}(1-t)\mathcal{S}^2(\mathcal{S}-n) \geq \frac{3}{4}\{2(1-t)\mathcal{S} + 3\} \cdot \frac{1}{16}\{(1-2t)\mathcal{S} - 1\}^2.$$

This proves Proposition 4.5.

Proof of Theorem 4.1. If $t > \frac{2}{21}$, then Proposition 4.4 implies $\mathcal{S} > n + \frac{1}{20}n$, and if $t \in [-\frac{1}{2}, \frac{2}{21}]$, then Proposition 4.5 implies $\mathcal{S} > n + \frac{1}{20}n$. This completes the proof of Theorem 4.1.

A sharper estimate for $\mathcal{S}$ is obtained by using complicated computations to obtain the following

Theorem 4.6. *(see [20]) Let M be a closed minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature. If $\mathcal{S} > n$, then we have*

(4.29)
$$\begin{aligned} \mathcal{S} &> n + \frac{2}{7}n - \frac{9}{14} && \text{if } 3 < n \leq 7, \\ \mathcal{S} &> n + \frac{2}{7}n - \frac{5}{8} && \text{if } 7 < n \leq 17, \\ \mathcal{S} &> n + \frac{1}{4}n && \text{if } n > 17. \end{aligned}$$

For the proof of Theorem 4.6 some Lemmas will be needed.

Lemma 4.7. *Let M satisfy the assumptions in Theorem 4.6. For every c there exists a point $x_c \in M$ such that*

$$(4.30)\qquad (\mathcal{S}-n)(f_3^2 - c\mathcal{S}f_4) = (\sum_{i,j,k} 2\lambda_i h_{ijk}^2)f_3 - c(2A+B)\mathcal{S} + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2$$

holds at that point.

Proof. Consider the function $F := \frac{1}{4}c\mathcal{S}f_4 - \frac{1}{6}f_3^2$. The Green-Stokes theorem implies

$$\int_M \Delta F\, dv = 0.$$

Making use of (4.11) we get

$$(4.31)\qquad \Delta f_4 = -4(\mathcal{S}-n)f_4 + 4(2A+B),$$

$$(4.32)\qquad \Delta f_3 = -3(\mathcal{S}-n)f_3 + 3\sum 2\lambda_i h_{ijk}^2,$$

$$(4.33)\qquad \Delta f_3^2 = -6(\mathcal{S}-n)f_3^2 + 6(\sum 2\lambda_i h_{ijk}^2)f_3 + 18\sum_j(\sum_i \lambda_i^2 h_{iij})^2.$$

The continuity property of functions then implies the existence of a desired point. This proves Lemma 4.7.

Lemma 4.8. *Let x_c be a point as in the previous Lemma. Then we have*

$$(4.34)\qquad \mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S} \geq \frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}(1+\lambda_1\lambda_2)^2\mathcal{S}.$$

Proof. From

$$\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S} = \frac{1}{\mathcal{S}}\sum(\lambda_i^2\mathcal{S} - \lambda_i f_3 - \mathcal{S})^2,$$

we obtain

$$\begin{aligned}\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S} &\geq \frac{1}{\mathcal{S}}[(\lambda_1^2\mathcal{S} - \lambda_1 f_3 - \mathcal{S})^2 + (\lambda_2^2\mathcal{S} - \lambda_2 f_3 - \mathcal{S})^2] \\ &= \frac{1}{\mathcal{S}}\{[(\lambda_1^2-1)^2 + (\lambda_2^2-1)^2]\mathcal{S}^2 + (\lambda_1^2+\lambda_2^2)f_3^2 \\ &\quad - 2[(\lambda_1^2-1)\lambda_1 + (\lambda_2^2-1)\lambda_2]f_3\mathcal{S}\}.\end{aligned}$$

From $x^2 + ax + b \geq b - a^2/4$ we get

$$\begin{aligned}\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S} &\geq [(\lambda_1^2-1)^2 + (\lambda_2^2-1)^2]\mathcal{S} - \frac{[(\lambda_1^2-1)\lambda_1 + (\lambda_2^2-1)\lambda_2]^2\mathcal{S}}{\lambda_1^2+\lambda_2^2} \\ &= \frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2+\lambda_2^2}\cdot(1+\lambda_1\lambda_2)^2\mathcal{S}.\end{aligned}$$

Lemma 4.9. *We have at* x_c,

$$A - B \le \frac{1}{3}(\lambda_1 - \lambda_2)^2(\mathcal{S} - n)(1-\alpha)\mathcal{S}, \tag{4.35}$$

where we set $\alpha(\mathcal{S} - n)\mathcal{S} = \sum h_{iii}^2$.

Proof. Since h_{ijk} is symmetric with respect to all indices,

$$\begin{aligned} A - B &= \sum(\lambda_i^2 - \lambda_i\lambda_j)h_{ijk}^2 \\ &= \frac{1}{3}\sum(\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i\lambda_j - \lambda_j\lambda_k - \lambda_i\lambda_k)h_{ijk}^2 \\ &= \frac{1}{3}\sum_{i\ne j\ne k}(\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i\lambda_j - \lambda_j\lambda_k - \lambda_i\lambda_k)h_{ijk}^2 + \sum_{i\ne j}(\lambda_i - \lambda_j)^2 h_{iij}^2. \end{aligned}$$

Without loss of generality we may consider $\lambda_i \le \lambda_j \le \lambda_k$. The function $f := \lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i\lambda_j - \lambda_j\lambda_k - \lambda_i\lambda_k$ takes its maximum at $\lambda_j = \lambda_i$ or $\lambda_j = \lambda_k$. Since $f|_{\lambda_j=\lambda_i} \le (\lambda_1 - \lambda_2)^2$ and $f|_{\lambda_j=\lambda_k} \le (\lambda_1 - \lambda_2)^2$, we obtain

$$\begin{aligned} A - B &\le \sum_{i\ne j}(\lambda_i - \lambda_j)^2 h_{iij}^2 + \frac{1}{3}\sum_{i\ne j\ne k}(\lambda_1 - \lambda_2)^2 h_{ijk}^2 \\ &\le \frac{1}{3}(\lambda_1 - \lambda_2)^2\left(\sum h_{ijk}^2 - \sum h_{iii}^2\right). \end{aligned}$$

This proves Lemma 4.9.

The terms $\sum_j(\sum_i \lambda_i^2 h_{iij})$ and $\sum_{i,j,k}\lambda_i h_{ijk}^2$ in (4.30) are estimated in the following two lemmas.

Lemma 4.10. *At a point* x_c *we have*

$$\begin{aligned} \sum_j\left(\sum_i \lambda_i^2 h_{iij}\right) &\le \frac{1+2\alpha}{3}\left[\left(\sum \lambda_i^2\mathcal{S} - \lambda_i f_3 - \mathcal{S}\right)^2(\mathcal{S}-n)/\mathcal{S} - (\mathcal{S}-n)^3\mathcal{S}/n\right] \\ &= \frac{1+3\alpha}{3}\left[\{\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S}\}(\mathcal{S}-n) - (\mathcal{S}-n)^3\mathcal{S}/n\right]. \end{aligned} \tag{4.36}$$

Proof. Since $\sum_i \lambda_i h_{iij} = 0$ and $\sum_i h_{iij} = 0$ for every j, we have for every real numbers a and b,

$$\begin{aligned} \sum_j\left(\sum_i \lambda_i^2 h_{iij}\right) &= \sum_j\left[\sum_i(\lambda_i^2 - a\lambda_i - b)h_{iij}\right]^2 \\ &\le \sum_i(\lambda_i^2 - a\lambda_i - b)^2 \cdot \sum h_{iij}^2. \end{aligned} \tag{4.37}$$

From $\sum_{i,j,k} h_{ijk}^2 = \sum_{i\neq j\neq k} h_{ijk}^2 + 3\sum_{i\neq j} h_{iij}^2 + \sum h_{iii}^2$ and the symmetry of h_{ijk} we conclude

$$
\tag{4.38}
\begin{aligned}
\sum h_{iij}^2 &\leq \frac{1}{3}(\sum_{i,j,k} h_{ijk}^2 + 2\sum_i h_{iii}^2) \\
&= \frac{1}{3}(1+2\alpha)(\mathcal{S}-n)\mathcal{S}.
\end{aligned}
$$

Let $a := f_3/\mathcal{S}$ and $b := \mathcal{S}/n$. Then we obtain from (4.37) and (4.38)

$$
\begin{aligned}
\sum_j(\sum_i \lambda_i^2 h_{iij})^2 &\leq \frac{1}{\mathcal{S}^2}\sum_i(\lambda_i^2\mathcal{S} - \lambda_i f_3 - \mathcal{S}^2/n)^2 \sum h_{iij}^2 \\
&\leq \frac{1+2\alpha}{3}\cdot\frac{\mathcal{S}-n}{\mathcal{S}}[\sum_i(\lambda_i^2\mathcal{S} - \lambda_i f_3 - \mathcal{S})^2 - (\mathcal{S}-n)^2\mathcal{S}^2/n] \\
&= \frac{1+2\alpha}{3}[\sum_i(\lambda_i^2\mathcal{S} - \lambda_i f_3 - \mathcal{S})^2(\mathcal{S}-n)/\mathcal{S} - (\mathcal{S}-n)^3\mathcal{S}/n] \\
&= \frac{1+2\alpha}{3}[(f_4\mathcal{S} - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S})(\mathcal{S}-n) - (\mathcal{S}-n)^3\mathcal{S}/n].
\end{aligned}
$$

This completes the proof of Lemma 4.10.

Since we find a unit vector $y = \sum y^k e_k$ such that

$$\sum_j(\sum_i \lambda_i^2 h_{iij})^2 = (\sum_{i,j} \lambda_i^2 h_{iij} y^j)^2$$

we have

Lemma 4.11. *At x_c we have*

$$
\tag{4.39}
(\sum_{i,j,k} \lambda_i h_{ijk}^2)^2 \leq [\frac{A+2B}{3} - \frac{4}{9\mathcal{S}}\sum_j(\sum_i \lambda_i^2 h_{iij})^2]\mathcal{S}(\mathcal{S}-n).
$$

Proof. From the choice fo y it follows

$$\sum h_{ij}h_{ijk}y^k = \sum \lambda_i h_{iik} y^k = 0.$$

Using the symmetric property of h_{ijk} we get

$$
\begin{aligned}
(\sum \lambda_i h_{ijk}^2)^2 &= \frac{1}{9}[\sum\{(\lambda_i+\lambda_j+\lambda_k)h_{ijk} - a h_{ij}y^k\}h_{ijk}]^2 \\
&\leq \frac{1}{9}\sum[(\lambda_i+\lambda_j+\lambda_k)h_{ijk} - a h_{ij}y^k]^2 \cdot \sum h_{ijk}^2 \\
&= [\frac{1}{3}(A+2B) - \frac{4a}{9}\sum \lambda_i^2 h_{iik}y^k + \frac{a^2}{9}\sum \lambda_i^2](\mathcal{S}-n)\mathcal{S} \\
&= [\frac{A+2B}{3} - \frac{4}{9\mathcal{S}}\sum(\sum_i \lambda_i^2 h_{iij})^2](\mathcal{S}-n)\mathcal{S},
\end{aligned}
$$

where we set $a := \frac{2}{\mathcal{S}}\sum \lambda_i^2 h_{iik} y^k$. This completes the proof of Lemma 4.11.

On the other hand $\mathcal{S} = \sum \lambda_i^2$ is constant, and we get

$$0 = \sum h_{ij}\nabla_i\nabla_j \mathcal{S} = 2\sum \lambda_i\lambda_j h_{iijj} + 2\sum \lambda_k h_{ijk}^2.$$

Thus we have

$$\begin{aligned}(\sum 2\lambda_i h_{ijk}^2)^2 &= [\sum \lambda_i\lambda_j(h_{ijij} + h_{jiji})]^2\\ &\le \sum \lambda_i^2\lambda_j^2 \cdot \sum (h_{ijij} + h_{jiji})^2\\ &= \mathcal{S}^2[4\sum_i h_{iiii}^2 + \sum_{i\neq j}(h_{ijij} + h_{jiji})^2]\\ &= 4\mathcal{S}^2[\sum_i h_{iiii}^2 + \frac{1}{4}\sum_{i\neq j}(h_{ijij} + h_{jiji})^2].\end{aligned} \tag{4.40}$$

Proposition 4.12. *Under the assumptions in Theorem 4.6 we have*

$$\begin{aligned}&\mathcal{S} > n + \frac{2}{7}n - \frac{9}{14} \qquad &&\text{if } 3 < n \le 7,\\ &\mathcal{S} > n + \frac{1}{5}n &&\text{if } n > 7.\end{aligned} \tag{4.41}$$

Proof. Taking $c = 1/2$ in Lemma 4.7 we have at x_c,

$$\begin{aligned}0 &= (\mathcal{S}-n)[\frac{1}{2}\mathcal{S}f_4 - f_3^2] + (\sum 2\lambda_i h_{ijk}^2 f_3 - \frac{1}{2}(2A+B)\mathcal{S} + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2\\ &\le (\mathcal{S}-n)[\frac{1}{2}\mathcal{S}f_4 - f_3^2] + \frac{1}{2(\mathcal{S}-n)}(\sum 2\lambda_i h_{ijk}^2)^2 + \frac{\mathcal{S}-n}{2}f_3^2\\ &\quad - \frac{1}{2}(2A+B)\mathcal{S} + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2\\ &\le \frac{\mathcal{S}-n}{2}(\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S}) + \frac{1}{2}\mathcal{S}(\mathcal{S}-n)(2\mathcal{S}-n) - \frac{1}{2}(2A+B)\mathcal{S}\\ &\quad + 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2 + \frac{1}{2}[\frac{4}{3}(A+2B) - \frac{16}{9\mathcal{S}}\sum_j(\sum_i \lambda_i^2 h_{iij})^2]\mathcal{S}\\ &\le \frac{\mathcal{S}-n}{2}(\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S}) + \frac{1}{2}\mathcal{S}(\mathcal{S}-n)(2\mathcal{S}-n) - \frac{\mathcal{S}}{6}(2A-5B)\\ &\quad - \frac{19}{27n}(1+2\alpha)(\mathcal{S}-n)^3\mathcal{S} + \frac{19}{27}(1+2\alpha)(\mathcal{S}-n)(\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S})\\ &= \frac{1}{2}\cdot\frac{65+76\alpha}{27}(\mathcal{S}-n)(\mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S}) - \frac{\mathcal{S}}{6}(2A-5B)\\ &\quad + \frac{1}{2}\mathcal{S}(\mathcal{S}-n)[(\mathcal{S}-n)+\mathcal{S}] - \frac{19}{27n}(1+2\alpha)(\mathcal{S}-n)^3\mathcal{S}.\end{aligned}$$

Setting $t\mathcal{S} = \mathcal{S} - n$ and $f = \mathcal{S}f_4 - f_3^2 - 2\mathcal{S}^2 + n\mathcal{S}$, we obtain

$$-\frac{65}{9}tf \leq -\frac{65}{65+76\alpha}(2A-5B) - \frac{38}{9}\cdot\frac{t^3\mathcal{S}^3}{n} + 3t(1+t)\mathcal{S}^2. \tag{4.42}$$

On the other hand (4.17) implies

$$\frac{3}{2}f \leq t(2t-1-\frac{3}{\mathcal{S}})\mathcal{S}^3 + 3(A-2B). \tag{4.43}$$

From (4.42) and (4.43) it follows that

$$\begin{aligned}\frac{3}{2}(1-\frac{130}{27}t)f &\leq t(2t-1)\mathcal{S}^3 + 3(A-2B) - \frac{65(2A-5B)}{65+76\alpha} + 3t^2\mathcal{S}^2 - \frac{38}{9}t^3\mathcal{S}^2 \\ &\leq t[(2t-1)\mathcal{S} + \frac{1}{3}(4-\frac{195}{65+76\alpha})(1-\alpha)(\lambda_1-\lambda_2)^2 + 3t - \frac{38}{9}t^2]\mathcal{S}^2.\end{aligned}$$

Here we used Lemma 4.9 and $A+2B = \frac{1}{3}\sum(\lambda_i+\lambda_j+\lambda_k)^2h_{ijk}^2 \geq 0$. Setting $\eta := 65+76\alpha$ we obtain

$$(4-\frac{195}{65+76\alpha})(1-\alpha) = \frac{1}{76}(4-\frac{195}{\eta})(141-\eta) \leq \frac{1}{76}(759-4\sqrt{195\times 141}) =: 3\beta_1,$$

where $\beta_1 = 0.4198\ldots < 0.42$.

Suppose now that $t \leq \frac{1}{6}$. Then $3t - \frac{38}{9}t^2 < \frac{1}{2} - \frac{1}{9}$, and the above inequality reduces to

$$t[(2t-1)\mathcal{S} + \beta_1(\lambda_1^2+\lambda_2^2) - 2\beta_1(1+\lambda_1\lambda_2) + 2\beta_1 + \frac{1}{2} - \frac{1}{9}]\mathcal{S}^2 \geq \frac{3}{2}(1-\frac{65}{81})f \geq \frac{8}{27}(1+\lambda_1\lambda_2)^2\mathcal{S}.$$

Since $ax^2 + bx \geq -\frac{b^2}{4a}$ and $\lambda_1^2 + \lambda_2^2 \leq \mathcal{S}$, we have

$$t[(2t-1)\mathcal{S} + \beta_1\mathcal{S} + \frac{27}{8}\beta_1^2\mathcal{S}t + 2\beta_1 + \frac{1}{2} - \frac{1}{9}]\mathcal{S} \geq 0,$$

and thus

$$t > \frac{1}{4.5} - \frac{1}{2\mathcal{S}}.$$

Therefore we get

$$\mathcal{S} > n + \frac{\mathcal{S}}{4.5} - \frac{1}{2},$$

and hence we have

$$\mathcal{S} > n + \frac{2}{7}n - \frac{9}{14}.$$

If $n > 7$, then the above inequality implies $\mathcal{S} > n + \frac{1}{5}n$. This is a contradiction to $t \leq \frac{1}{6}$. Thus the first inequality in (4.41) is shown. In the case where $n > 7$, suppose that $t > \frac{1}{6}$. Then we have $\mathcal{S} > n + \frac{1}{5}n$. This proves Proposition 4.12.

Proof of Theorem 4.6. Let $c := \frac{5}{9}$ be a constant in Lemma 4.7. A little more careful computation than developed in the proof of Proposition 4.12 implies that

$$(4.44)\quad -\frac{298+326\alpha}{45}tf \leq 3t(1+t)\mathcal{S}^3 + \frac{3}{80t\mathcal{S}^2}(2\sum \lambda_i h_{ijk}^2)^2 - (2A-5B) - \frac{163}{45}(1+2\alpha)t^3\mathcal{S}^2.$$

From (4.7),(4.16) and (4.40) we have

$$(4.45)\qquad \frac{3}{2}f + \frac{1}{4\mathcal{S}^2}(2\sum \lambda_i h_{ijk}^2)^2 \leq t\mathcal{S}^3(2t-1-\frac{3}{\mathcal{S}}) + 3(A-2B).$$

Suppose now that $t \leq \frac{1}{5}$ and $n > 7$. From Proposition 4.12 it follows $t > \frac{1}{6}$. If we set $\beta_2 := \frac{2496+894-2\sqrt{4\cdot 3\cdot 298\cdot 624}}{3\cdot 326}$, then we observe from (4.44) and (4.45) that

$$(4.46)\qquad \frac{79}{450}f \leq t\mathcal{S}^2[(2t-1)\mathcal{S} + 0.4552 + \beta_2(\lambda_1-\lambda_2)^2].$$

From Lemma 4.8 and (4.46) we get

$$\begin{aligned} 0 &\leq t\mathcal{S}^2[(2t-1)\mathcal{S} + 1.2783 + \beta_2\mathcal{S} - 2\beta_2(1+\lambda_1\lambda_2)] - \frac{79}{450}(1+\lambda_1\lambda_2)^2\mathcal{S} \\ &\quad + \frac{79\mathcal{S}}{225}\cdot\frac{\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2}(1+\lambda_1\lambda_2)^2 \\ &\leq t\mathcal{S}^2[(2t-1)\mathcal{S} + 1.2783 + \beta_2\mathcal{S} - 2\beta_2(1+\lambda_1\lambda_2)] - \frac{79}{450}\mathcal{S}(1+\lambda_1\lambda_2)^2 + \frac{79}{225}(1+\lambda_1\lambda_2)^3. \end{aligned}$$

For $x > 0$ and $b > 0$ we use $ax - bx^2 \leq \frac{a^2}{4b}$ and $cx - 2bx^3 \leq \frac{2}{3}\sqrt{\frac{c^3}{6b}}$ to obtain

$$0 \leq (2t-1)\mathcal{S} + 1.2783 + \beta_2\mathcal{S} + t\mathcal{S}(\frac{1}{2})^2\frac{450}{4\cdot 79} + \frac{2t\mathcal{S}}{3}\sqrt{\frac{450(2\beta_2-\frac{1}{2})^3}{6\cdot 79t}}.$$

This implies

$$\begin{aligned} t &> \frac{0.5885}{2.6482} - \frac{1.2783}{2.6482}\cdot\frac{1}{\mathcal{S}} \\ &> \frac{1}{4.5} - \frac{1}{2.07\mathcal{S}}. \end{aligned}$$

Thus we have the second inequality of (4.29),

$$\mathcal{S} > n + \frac{2}{7}n - \frac{5}{8}.$$

If $n > 17$, then the above inequality implies $\mathcal{S} > n + \frac{n}{4}$, and a contradiction to $t \leq \frac{1}{5}$. This completes the proof of Theorem 4.6.

Remark 4.1. A similar proof will show that

$$\lim_{n\to\infty}\inf \frac{\mathcal{S}-n}{n} > 0.27.$$

Remark 4.2. Theorems 4.1 and 4.6 will provide a partial answer for the conjectures proposed by Chern and Peng-Terng.

§5. Complete minimal submanifolds in spheres

In this section we shall study complete minimal submanifolds in $S^{n+p}(1)$ by using generalized maximum principle due to Omori-Yau. We want to generalize results by Myers,Chern-do Carmo-Kobayashi and Peng-Terng to complete minimal submanifolds.

Theorem 5.1. *(see [17]) Let M be a complete minimal submanifold of dimension n in $S^{n+p}(1)$. If $\sup_M \mathcal{S} < \frac{n}{2-\frac{1}{p}}$, then M is totally geodesic.*

Proof. Gauss' equation (1.12) implies together with the assumption that the Ricci curvature of M is bounded from below. For a positive constant a, the function $F := (\mathcal{S}+a)^{\frac{1}{2}}$ is bounded since so is $\mathcal{S}$. Computations show that

$$\frac{1}{2}\Delta\mathcal{S} = F\Delta F + |\nabla F|. \tag{5.1}$$

Applying Theorem Y-1 to F, we see that for every $\varepsilon > 0$ there exists a point $p \in M$ such that

$$|\nabla F(p)| < \varepsilon, \quad \Delta F(p) < \varepsilon, \quad \sup_M F - \varepsilon < F(p). \tag{5.2}$$

From (5.1) and (5.2) we have

$$\frac{1}{2}\Delta\mathcal{S}(p) < \varepsilon(\varepsilon + F(p)). \tag{5.3}$$

For a sequence $\{\varepsilon_m\}$ of positive numbers converging to 0 there is a sequence $\{p_m\}$ of points on M satisfying (5.2). Thus (5.3) implies $\{\varepsilon_m(\varepsilon_m + F(p_m))\}$ converges to 0 as $m \to \infty$.

On the other hand (5.2) implies, by taking a subsequence if necessary, that

$$\lim_{m\to\infty} F(p_m) = F_0 \geq \sup_M F.$$

Therefore we have $F_0 = \sup_M F$, and $\lim \mathcal{S}(p_m) = \sup_M \mathcal{S}$. Then a direct computation shows that (see [17] or [22])

$$\frac{1}{2}\Delta\mathcal{S} \geq \mathcal{S}[n - (2 - \frac{1}{p})\mathcal{S}].$$

By means of (5.3) we have $\sup_M \mathcal{S} = 0$. This means that the second fundamental form of M is identically zero, and hence the proof is complete.

Theorem 5.2. *Let M be an n-dimensional complete minimal submanifold in $S^{n+p}(1)$. If $p > 1$, then M is either totally geodesic, or a Veronese surface in $S^4(1)$, or has the property that $\sup_M \mathcal{S} > \frac{n}{2-1/p}$.*

Proof. In view of Theorem 5.1 we only consider the case where $\sup_M \mathcal{S} = \frac{n}{2-1/p}$. In this case we have $\mathcal{S} \leq \frac{n}{2-1/p}$, and from Lemma 6.1

$$Ric(v,v) \geq (n-1)[1 - \frac{1}{2-1/p}] > 0$$

for every unit vector $v \in TM$. The Myers theorem (see [5])then implies that M is compact. Theorem CDK implies that $n = p = 2$ and M is a Veronese surface in $S^4(1)$. This proves Theorem 5.2.

In the special case where $n = 3$ we have the following

Theorem 5.3. *(see [10] Let M be a 3-dimesnional complete minimal hypersurface in $S^4(1)$ with $\mathcal{S}$ being constant. If $\mathcal{S} > 3$, then $\mathcal{S} \geq 6$.*

For the proof of Theorem 5.3 we need the following Sublemma.

Sublemma. *Let $a_1, \ldots, a_n$ be real numbers satisfying $\sum a_i = 0$ and $\sum a_i^2 = k^2$ for $k > 0$. We then have*

$$|\sum a_i^3| \leq (n-2)\{n(n-1)\}^{-1/2}k^3.$$

Proof of Theorem 5.3. Assume that $\sup_M f_3 \cdot \inf_M f_3 = 0$. If $\sup_M f_3 = \inf_M f_3 = 0$, then $f_3 \equiv 0$ and from Lemma 2 in [43] we see that the principal curvatures of M are constant. Thus Theorem 5.3 is valid from Cartan's Lemma.

Assume next that f_3 is not constant and that $\sup_M f_3 \cdot \inf_M f_3 = 0$. We may assume without loss of generality that $\sup_M f_3 = 0$. From the Gauss' equation and $\mathcal{S}$ being constant we see that the Ricci curvature of M is bounded below. Applying Theorem Y-1 to f_3 we have a sequence $\{p_m\}$ of M such that

$$\lim_{m\to\infty} f_3(p_m) = \sup_M f_3 = 0, \quad \lim_{m\to\infty} |\mathrm{grad}\, f_3(p_m)| = 0, \tag{5.4}$$

and also

$$\limsup_{m\to\infty} \Delta f_3(p_m) \leq 0. \tag{5.5}$$

From (4.5) and (4.7) we observe that λ_i,h_{ijk} and h_{ijkl} are all bounded, and hence we may assume that

$$\lim_{m\to\infty} \lambda_i(p_m) = \widetilde{\lambda}_i, \quad \lim_{m\to\infty} h_{ijk}(p_m) = \widetilde{h}_{ijk}, \tag{5.6}$$

and

$$\lim_{m\to\infty} h_{ijkl}(p_m) = \widetilde{h}_{ijkl}. \tag{5.7}$$

Thus we have

$$\begin{aligned} \widetilde{\lambda}_1 + \widetilde{\lambda}_2 + \widetilde{\lambda}_3 &= 0, \\ \widetilde{\lambda}_1^2 + \widetilde{\lambda}_2^2 + \widetilde{\lambda}_3^2 &= \mathcal{S}, \\ \widetilde{\lambda}_1^3 + \widetilde{\lambda}_2^3 + \widetilde{\lambda}_3^3 &= 0. \end{aligned} \tag{5.8}$$

By assuming $\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \widetilde{\lambda}_3$ we get from (5.8),

$$\widetilde{\lambda}_1 = -\sqrt{\frac{\mathcal{S}}{2}}, \ \widetilde{\lambda}_2 = 0, \ \widetilde{\lambda}_3 = \sqrt{\frac{\mathcal{S}}{2}}. \tag{5.9}$$

Taking exterior differentiation, $\sum h_{ii} = 0$ and $\sum h_{ij}^2 = \mathcal{S}$ imply both

$$\sum h_{iik} = 0 \qquad \text{for every } k, \tag{5.10}$$

and

$$\sum h_{iik}\lambda_i = 0 \qquad \text{for every } k. \tag{5.11}$$

From (5.4) we have $\lim_{m\to\infty} |\text{grad}\, f_3(p_m)| = 0$. Since $|\text{grad}\, f_3| = [\sum_k(\sum_j \lambda_j^2 h_{jjk})^2]^{\frac{1}{2}}$, we get for every k,

$$\lim_{m\to\infty} \sum_i h_{iik}\lambda_i^2(p_m) = 0.$$

The above relation together with (5.6) implies that

$$\sum \widetilde{h}_{iik} = 0, \quad \sum \widetilde{h}_{iik}\widetilde{\lambda}_i = 0, \quad \sum \widetilde{h}_{iik}\widetilde{\lambda}_i^2 = 0 \text{ for every } k. \tag{5.12}$$

Since $\widetilde{\lambda}_i$'s are distinct, we get for every i,k,

$$\widetilde{h}_{iik} = 0. \tag{5.13}$$

On the other hand we have

$$\begin{aligned}
\lim_{m\to\infty} 3(A-2B)(p_m) &= \sum(\widetilde{\lambda}_i^2 + \widetilde{\lambda}_j^2 + \widetilde{\lambda}_k^2 - 2\widetilde{\lambda}_i\widetilde{\lambda}_j - 2\widetilde{\lambda}_j\widetilde{\lambda}_k - 2\widetilde{\lambda}_k\widetilde{\lambda}_i)\widetilde{h}_{ijk}^2 \\
&= \sum_{i\neq j\neq k} [2(\widetilde{\lambda}_i^2 + \widetilde{\lambda}_j^2 + \widetilde{\lambda}_k^2) - (\widetilde{\lambda}_i + \widetilde{\lambda}_j + \widetilde{\lambda}_k)^2]\widetilde{h}_{ijk}^2 \\
&\quad + 3\sum_{i\neq k}(\widetilde{\lambda}_k^2 - 4\widetilde{\lambda}_i\widetilde{\lambda}_k)\widetilde{h}_{iik}^2 - 3\sum \widetilde{h}_{iii}^2\widetilde{\lambda}_i^2 \\
&= 2\mathcal{S}\sum \widetilde{h}_{ijk}^2 \qquad \text{by (5.13) and (5.8)} \\
&= 2\mathcal{S}^2(\mathcal{S}-3) \qquad \text{by (4.5).}
\end{aligned} \tag{5.14}$$

From (4.9),(4.11), (5.6) and (5.7) we obtain

$$\sum \widetilde{h}_{ijkl}^2 \geq 3\sum_{i\neq j}(\widetilde{h}_{ijij} - \widetilde{t}_{ij}/2)^2 + \frac{3}{4}\sum \widetilde{t}_{ij}^2 \tag{5.15}$$

where we set $\widetilde{t}_{ij} = \widetilde{h}_{ijij} - \widetilde{h}_{jiji} = (\widetilde{\lambda}_i - \widetilde{\lambda}_j)(1 + \widetilde{\lambda}_i\widetilde{\lambda}_j)$. Then (5.9) implies that

$$\sum_{i\neq j} \widetilde{t}_{ij}^2 = \mathcal{S}^3 - 4\mathcal{S}^2 + 6\mathcal{S}. \tag{5.16}$$

From (4.7),(5.14),(5.15) and (5.16) we get

$$\mathcal{S}(\mathcal{S}-3)(\mathcal{S}-9) + 2\mathcal{S}^2(\mathcal{S}-3) \geq 3\sum_{i\neq j}(\widetilde{h}_{ijij} - \widetilde{t}_{ij}/2)^2 + \frac{3}{4}(\mathcal{S}^3 - 4\mathcal{S}^2 + 6\mathcal{S}). \tag{5.17}$$

Computations show that

$$(5.18) \qquad \sum_{i\neq j}(\widetilde{h}_{ijij} - \widetilde{t}_{ij}/2)^2 \geq (\widetilde{h}_{1212} - \widetilde{h}_{2323})^2,$$

where we used $\widetilde{t}_{12} = \widetilde{t}_{23} = -\sqrt{\mathcal{S}/2}$. Differentiating $\mathcal{S} = \sum h_{ij}^2$, we have

$$(5.19) \qquad \sum_i \widetilde{\lambda}_i \widetilde{h}_{iikk} + \sum_{i,j} \widetilde{h}_{ijk}^2 = 0 \text{ for } k = 1,2,3.$$

Substituting (5.9) into (5.19) we obtain

$$\sqrt{\mathcal{S}/2}(\widetilde{h}_{11kk} - \widetilde{h}_{33kk}) = \sum_{i,j} \widetilde{h}_{ijk}^2 \text{ for } k = 1,2,3.$$

In particular we have

$$\sqrt{\mathcal{S}/2}(\widetilde{h}_{1122} - \widetilde{h}_{3322}) = \sum_{i,j} \widetilde{h}_{ij2}^2 = \frac{1}{3}\mathcal{S}(\mathcal{S}-3).$$

Thus we get

$$(5.20) \qquad \widetilde{h}_{1212} - \widetilde{h}_{2323} = \widetilde{h}_{1212} - \widetilde{h}_{3232} - \widetilde{t}_{23} = \sqrt{2\mathcal{S}}[\frac{\mathcal{S}-3}{3} + \frac{1}{2}].$$

From (5.17), (5.18) and (5.20) we have

$$\mathcal{S}(\mathcal{S}-6)(19\mathcal{S}-42) \geq 0,$$

and the proof in this case is now clear.

Consider now the case where $\inf_M f_3 \cdot \sup_M f_3 \neq 0$. Suppose that f_3 is constant. Then M has constant principal curvatures and Theorem 5.3 is true.

We shall assert that if f_3 is nonconstant, then there is a point $p \in M$ such that $f_3(p) = 0$. Once the above assertion has been established, then the classical Myers theorem (see [36]) together with the previous computations concludes Theorem 5.3.

Finally we shall prove that $\inf_M f_3 \cdot \sup_M f_3 > 0$ does not occur. To see this we may assume that $\sup_M f_3 < 0$. From Sublemma we see

$$(5.21) \qquad -\sqrt{\mathcal{S}^3/6} < \sup_M f_3 < 0,$$

$$(5.22) \qquad \lim_{m\to\infty} f_3(p_m) = \sup_M f_3, \qquad \lim_{m\to\infty} |\nabla f_3(p_m)| = 0,$$

$$(5.23) \qquad \limsup_{m\to\infty} \Delta f_3(p_m) \leq 0,$$

and also

$$\widetilde{\lambda}_1+\widetilde{\lambda}_2+\widetilde{\lambda}_3=0,$$
$$\widetilde{\lambda}_1^2+\widetilde{\lambda}_2^2+\widetilde{\lambda}_3^2=\mathcal{S}, \tag{5.24}$$
$$\widetilde{\lambda}_1^3+\widetilde{\lambda}_2^3+\widetilde{\lambda}_3^3=\sup_M f_3.$$

We observe from (5.21) and (5.24) that $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ and $\widetilde{\lambda}_3$ are distinct. A similar discussion as developed in the first case implies that

$$\widetilde{h}_{iik}=0 \qquad \text{for every } i \text{ and } k. \tag{5.25}$$

It follows from $\Delta f_3 = 3[(3-\mathcal{S})f_3 + 2\sum \lambda_i h_{ijk}^2]$ that

$$3[(3-\mathcal{S})\sup_M f_3 + 2\sum \widetilde{\lambda}_i \widetilde{h}_{ijk}^2] \le 0$$

and also

$$3\sum \widetilde{\lambda}_i \widetilde{h}_{ijk}^2 = \sum (\widetilde{\lambda}_i+\widetilde{\lambda}_j+\widetilde{\lambda}_k)\widetilde{h}_{ijk}^2 = 0.$$

Therefore we have

$$(3-\mathcal{S})\sup_M f_3 \le 0.$$

This is a contradiction. This completes the proof of Theorem 5.3.

As a corollary to Theorems 5.1 and 5.3 we have the following

Corollary 5.4. *(see [10]) Let M be a three dimensional complete minimal hypersurface in $S^4(1)$ with constant scalar curvature. If $\mathcal{S}\le 6$, then we have $\mathcal{S}=0,3,$ or 6.*

It is well known that there exist no closed minimal submanifolds without boundary in a Euclidean space $\boldsymbol{R}^{n+p}$. Similarly, there exist no closed minimal hypersurfaces in an open hemi-sphere of $S^{n+1}(1)$. When M is not necessarily compact, we shall prove the following.

Let $S_+^{n+1}(1)\subset S^{n+1}(1)$ be a closed hemisphere as given ;

$$S_+^{n+1}(1)=\{u\in \boldsymbol{R}^{n+2}; u=(u^1,\ldots,u^{n+2}), \|u\|=1, u^{n+2}\ge 0\}$$

Theorem 5.5. *(see [9]) Let M be a complete minimal hypersurface in $S_+^{n+1}(1)$. If the distance u^{n+2} from points on M to the equator great sphere satisfies*

$$\int_M |\nabla u^{n+2}|\, dv < \infty,$$

then M is the great sphere (and totally geodesic). Here dv is by definition the volume element of M.

Proof. Since M is minimal we have

$$\Delta u^{n+2} = -n\cdot u^{n+2}, \qquad u^{n+2}\ge 0. \tag{5.26}$$

This means that u^{n+2} is superharmonic, and constant by the following Lemma (see [52]). Since $u^{n+2}\equiv 0$, M is totally geodesic. This proves Theorem 5.5.

Yau's Lemma. *If f is a subharmonic function defined on a complete Riemannian manifold M and if*

$$\int_M |\nabla f|\, dv < \infty,$$

then f is harmonic.

Theorem 5.6. *(see [9]) Let M be a complete minimal hypersurface in $S_+^{n+1}(1)$. If the volume of M is finite, then M is totally geodesic.*

Proof. Because M is minimal we have

$$\Delta u^i = -n \cdot u^i, \qquad i = 1, \ldots, n+2,$$
$$\sum_i (u^i)^2 = 1, \qquad \Delta \sum_i (u^i)^2 = \sum_i \Delta (u^i)^2 = 0,$$

and also

$$\Delta (u^i)^2 = 2|\nabla u^i|^2 - 2n(u^i)^2.$$

Therefore we get $\sum_i |\nabla u^i|^2 = n$ and $|\nabla u^{n+2}|^2 \leq n$. Thus

$$\int_M |\nabla u^{n+2}| dv \leq \sqrt{n} \cdot Vol(M) < \infty.$$

This and Theorem 5.5 imply that u^{n+2} is harmonic and M is totally geodesic.

Theorem 5.7. *(see [9]) Let M be a complete minimal hypersurface in $S^{n+1}(1)$ whose Ricci curvature is bounded below by a constant $-L$. If $M \subset S_+^{n+1}(1)$ and if*

$$\int_M u^{n+2} dv < \infty,$$

then M is totally geodesic.

Proof. Applying Theorem Y-1 to a superharmonic function u^{n+2}, we obtain a sequence $\{p_m\}$ of points on M such that

$$\lim_{m \to \infty} u^{n+2}(p_m) = \inf u^{n+2}, \ \lim \inf_{m \to \infty} \Delta u^{n+2}(p_m) \geq 0. \tag{5.27}$$

From (5.26) and (5.27) it follows that $\inf_M u^{n+2} = 0$. Since $|\Delta u^{n+2}| = n|u^{n+2}|$ and $|\nabla(\Delta u^{n+2})| = n|\nabla u^{n+2}|$, Theorem Y-2 implies that

$$|\nabla u^{n+2}| \leq c \cdot u^{n+2}.$$

Thus we have

$$\int_M |\nabla u^{n+2}| dv \leq \int_M c \cdot u^{n+2} dv < \infty,$$

where c is a constant. Theorem 5.5 concludes the proof.

§6. Hypersurfaces in space forms with constant mean curvature

This section is devoted to the study of hypersurfaces in space forms with constant mean curvature. First of all we prove the following

Lemma 6.1. *(see [16]) Let M be a complete hypersurface in $M^{n+1}(c)$ with constant mean curvature, where $c \geq 0$. For every unit vector v tangent to M we have*

$$Ric(v,v) \geq (n-1)(c+2H^2) - \{(n-1)\mathcal{S} + (n-2)|H|[(n-1)(n\mathcal{S} - n^2H^2)]^{\frac{1}{2}}\}/n.$$

Proof. For any point $x \in M$ and for any unit vector v at x we choose a local frame $\{e_1, \ldots, e_n, e_{n+1}\}$ in $M^{n+1}(c)$ such that $e_1, \ldots, e_n$ are tangent to M and $v = e_n$. Then (1.13) gives

$$Ric(v,v) = (n-1)c + nHh_{nn} - \sum_j h_{nj}h_{jn}.$$

By a Lemma due to Cai,Cheng and Nakagawa (see [6] and [16]) we prove Lemma 6.1.

Theorem 6.2. *(see [16]) Let M be a complete hypersurface in $M^{n+1}(c)$, $c \geq 0$ with constant mean curvature H. If*

$$\sup_M \mathcal{S} < \{n[2(n-1)c + n^2H^2] - n(n-2)|H|[n^2H^2 + 4(n-1)c]^{\frac{1}{2}}/2(n-1), \tag{6.1}$$

then M is totally umbilical.

Proof. Making use of (1.22) and Sublemma we obtain

$$\frac{1}{2}\Delta f \geq f^2[nc + nH^2 - n(n-2)|H|f/\{n(n-1)\}^{\frac{1}{2}} - f^2],$$

where $f^2 = \mathcal{S} - nH^2 \geq 0$. By means of Lemma 6.1 and using a similar method as developed in [16], the proof is complete.

Remark 6.1. In the special case where $n \geq 3$, Theorem 6.2 is an extension of Hasanis' theorem, (compare [27]). The restriction of dimension in [27] is inevitable.

Remark 6.2. In the case where $c = 0$, (6.1) is equivalent to $\sup_M \mathcal{S} < \frac{n^2H^2}{n-1}$. Theorem 6.2 also extends Okumura's theorem (compare [39]). Our estimate is optimal because a complete hypersurface obtained as a Riemannian product $M := S^{n-1} \times R$ in R^{n+1} has the property $\mathcal{S} = \frac{n^2H^2}{n-1}$. In the case where $c > 0$ and $n = 2$, (6.1) is equivalent to $\sup_M \mathcal{S} < 2c + n^2H^2$, and this means that the Gaussian curvature is positive. In the case where M is minimal, (6.1) becomes $\sup_M \mathcal{S} < n$. Thus we have shown that Theorem CDK in this case is obtained without assuming compactness of M.

In the special case where $n = 3$, we have the following

Theorem 6.3. *(see [15]) Let M be a complete hypersurface in $S^4(1)$ with constant mean curvature H. If $\mathcal{S}$ is a constant not greater than $9H^2+6$, then M is totally umbilical and $\mathcal{S}=3H^2$, or $\mathcal{S}=3+\frac{27}{4}H^2 \pm \frac{3}{4}\sqrt{9H^4+8H^2}$, or $\mathcal{S}=9H^2+6$. Moreover if $H \neq 0$ and if $\mathcal{S}=9H^2+6$, then M is isoparametric.*

Remark 6.3. It should be remarked that in the above theorem 6.3, M is totally umbilical when $\mathcal{S}=3H^2$, and M an isoparametric hypersurface when $\mathcal{S}=9H^2+6$ and $H \neq 0$. Moreover isoparametric hypersurfaces in $S^4(1)$ with two distinct principal curvatures have the properties $\mathcal{S}=3+\frac{27}{4}H^2-\frac{3}{4}\sqrt{9H^4+8H^2}$ or $3+\frac{27}{4}H^2+\frac{3}{4}\sqrt{9H^4+8H^2}$. However we do not know whether or not the converse of this case in Theorem 6.3 is true.

For the proof of Theorem 6.3 we need some preparations.

For an arbitrary fixed point $p \in M$ we choose a local frame field $e_1, \ldots, e_n$ such that

$$h_{ij}=\lambda_i\delta_{ij}, \tag{6.2}$$

as before in (4.6) and set

$$\mu_i := \lambda_i - H, \qquad B_k := \sum_i \mu_i^k, \qquad h := 3H. \tag{6.3}$$

Then we have

$$B_1=0, \qquad B_2=\mathcal{S}-3H^2, \qquad B_3=f_3-3H\mathcal{S}+6H^3. \tag{6.4}$$

If both $\mathcal{S}$ and H are constant, then we see from a direct calculation,

$$-hf_3=\mathcal{S}(3-\mathcal{S})-h^2+\sum h_{ijk}^2, \tag{6.5}$$

$$\begin{aligned}\frac{1}{2}\Delta\sum h_{ijk}^2 &= (9-\mathcal{S})\sum h_{ijk}^2-3(\sum\lambda_i^2h_{ijk}^2-2\sum\lambda_i\lambda_jh_{ijk}^2)\\ &\quad +3h\sum\lambda_ih_{ijk}^2+\sum h_{ijkl}^2\end{aligned} \tag{6.6}$$

$$-h\Delta f_3=\Delta\sum h_{ijk}^2. \tag{6.7}$$

$$\Delta f_3=6\sum\lambda_ih_{ijk}^2-3\mathcal{S}f_3+3hf_4+9f_3-3h\mathcal{S}, \tag{6.8}$$

$$\Delta f_4=-4\mathcal{S}f_4-4hf_3+4hf_5+12f_4+8\sum\lambda_i^2h_{ijk}^2+4\sum\lambda_i\lambda_jh_{ijk}^2 \tag{6.9}$$

$$f_4=-h^2\mathcal{S}+\frac{1}{6}h^4+\frac{4}{3}hf_3+\frac{1}{2}\mathcal{S}^2, \tag{6.10}$$

$$f_5=\frac{5}{6}(\mathcal{S}+h^2)f_3+\frac{1}{6}h^5-\frac{5}{6}h^3\mathcal{S} \tag{6.11}$$

$$\begin{aligned}&\mathcal{S}^3-\frac{11}{6}h^2\mathcal{S}^2-6\mathcal{S}^2+\mathcal{S}h^4+9\mathcal{S}+6h^2\mathcal{S}-\frac{1}{6}h^6-3h^2-\frac{4}{3}h^4\\ &=(\mathcal{S}-\frac{1}{3}h^2)(\mathcal{S}-3-\frac{3}{4}h^2+\frac{1}{4}\sqrt{h^4+8h^2})(\mathcal{S}-3-\frac{3}{4}h^2-\frac{1}{4}\sqrt{h^4+8h^2}).\end{aligned} \tag{6.12}$$

Lemma 6.4. *Let M be a 3-dimensional hypersurface in $S^4(1)$ with constant mean curvature. If $\mathcal{S}$ is constant, then*

$$\frac{1}{3}\Delta\sum h_{ijk}^2 = -2h\sum\mu_i h_{ijk}^2 - (\mathcal{S}-\frac{2}{3}h^2-3)\sum h_{ijk}^2 \tag{6.13}$$
$$+(\mathcal{S}-\frac{1}{3}h^2)(\mathcal{S}-3-\frac{3}{4}h^2+\frac{1}{4}\sqrt{h^4+8h^2})(\mathcal{S}-3-\frac{3}{4}h^2-\frac{1}{4}\sqrt{h^4+8h^2}),$$

$$\sum h_{ijkl}^2 = -\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2 - 8h\sum\mu_i h_{ijk}^2 + \frac{3}{2}(\mathcal{S}-\frac{2}{3}h^2-3)\sum h_{ijk}^2$$
$$+\frac{3}{2}(\mathcal{S}-\frac{1}{3}h^2)(\mathcal{S}-\frac{3}{4}h^2-3+\frac{1}{4}\sqrt{h^4+8h^2})(\mathcal{S}-3-\frac{3}{4}h^2-\frac{1}{4}\sqrt{h^4+8h^2}). \tag{6.14}$$

Proof. From (6.5) and (6.10) we have

$$\Delta f_4 = \frac{4}{3}h\Delta f_3 = -\frac{4}{3}\Delta\sum h_{ijk}^2. \tag{6.15}$$

From (6.9) and (6.15) we also have

$$2\sum\lambda_i^2h_{ijk}^2+\sum\lambda_i\lambda_jh_{ijk}^2 = -\frac{1}{3}\Delta\sum h_{ijk}^2+\mathcal{S}f_4+hf_3-hf_5-3f_4, \tag{6.16}$$

and from (6.3)

$$\sum\lambda_i^2h_{ijk}^2+2\sum\lambda_i\lambda_jh_{ijk}^2 = \frac{1}{3}h^2\sum h_{ijk}^2+2h\sum\mu_ih_{ijk}^2+\frac{1}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2. \tag{6.17}$$

From (6.3),(6.7) and (6.8) we get

$$-2h\sum\mu_ih_{ijk}^2 = \frac{2}{3}h^2\sum h_{ijk}^2+\frac{1}{3}\Delta\sum h_{ijk}^2 - h\mathcal{S}f_3 - h^2\mathcal{S}+h^2f_4+3hf_5, \tag{6.18}$$

and hence, from (6.5) and (6.10)

$$\frac{1}{3}\Delta\sum h_{ijk}^2 = -2h\sum\mu_ih_{ijk}^2-\frac{2}{3}h^2\sum h_{ijk}^2+h\mathcal{S}f_3+h^2\mathcal{S}-h^2f_4-3hf_3$$
$$= -2h\sum\mu_ih_{ijk}^2-(\mathcal{S}-3-\frac{2}{3}h^2)\sum h_{ijk}^2$$
$$+(\mathcal{S}-\frac{1}{3}h^2)(\mathcal{S}-3-\frac{3}{4}h^2+\frac{1}{4}\sqrt{h^4+8h^2})(\mathcal{S}-\frac{3}{4}h^2-3-\frac{1}{4}\sqrt{h^4+8h^2}).$$

This proves (6.13).

On the other hand (6.6) implies that

$$\begin{aligned}
\sum h_{ijkl}^2 &= \frac{1}{2}\Delta \sum h_{ijk}^2 - 5(\sum \lambda_i^2 h_{ijk}^2 + 2\sum \lambda_i\lambda_j h_{ijk}^2) \\
&\quad + 4(2\sum \lambda_i^2 h_{ijk}^2 + \sum \lambda_i\lambda_j h_{ijk}^2) - 3h\sum \mu_i h_{ijk}^2 + (\mathcal{S} - 9 - h^2)\sum h_{ijk}^2 \\
&= \frac{1}{2}\Delta \sum h_{ijk}^2 - 5[\frac{h^2}{3}\sum h_{ijk}^2 + 2h\sum \mu_i h_{ijk}^2 + \frac{1}{3}\sum(\mu_i + \mu_j + \mu_k)^2 \sum h_{ijk}^2] \\
&\quad + 4(-\frac{1}{3}\Delta \sum h_{ijk}^2 + \mathcal{S} f_4 + hf_3 - hf_5 - 3f_4) - 3h\sum \mu_i h_{ijk}^2 + (\mathcal{S} - 9 - h^2)\sum h_{ijk}^2 \\
&= -\frac{5}{3}\sum(\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h\sum \mu_i h_{ijk}^2 \\
&\quad - \frac{5}{2}(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}) \\
&\quad + 4(\mathcal{S} - 3)[-h^2\mathcal{S} + \frac{1}{6}h^4 + \frac{4}{3}hf_3 + \frac{1}{2}\mathcal{S}^2] + 4hf_3 \\
&\quad - 4h(\frac{5}{6}f_3\mathcal{S} + \frac{5}{6}h^2 f_3 + \frac{1}{6}h^5 - \frac{5}{6}h^3\mathcal{S}) + (\frac{7}{2}\mathcal{S} - \frac{13}{3}h^2 - \frac{33}{2})\sum h_{ijk}^2 \\
&= -\frac{5}{3}\sum(\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h\sum \mu_i h_{ijk}^2 \\
&\quad - \frac{5}{2}(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{3}\sqrt{h^4 + 8h^2}) \\
&\quad + \frac{3}{2}(\mathcal{S} - 3 - \frac{2}{3}h^2)\sum h_{ijk}^2 \\
&\quad + 4[\mathcal{S}^3 - 6\mathcal{S}^2 + 9\mathcal{S} - \frac{11}{6}h^2\mathcal{S}^2 + (6h^2 + h^4)\mathcal{S} - \frac{1}{6}h^6 - 3h^2 - \frac{4}{3}h^4] \\
&= -\frac{5}{3}\sum(\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h\sum \mu_i h_{ijk}^2 + \frac{3}{2}(\mathcal{S} - \frac{2}{3}h^2 - 3)\sum h_{ijk}^2 \\
&\quad + \frac{3}{2}(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}). \qquad (6.19)
\end{aligned}$$

This proves (6.14).

Proof of Theorem 6.3 by assuming Propositions 6.5 and 6.6. In view of Corollary 5.4 we may consider the case where $H > 0$. The proof will be divided into three cases as follows. The proofs of the first and second cases need not Propositions 6.5 and 6.6, but the proof technique developed in the second case will be used in the proof of Proposition 6.6.

In the first case we assume that

$$3H^2 \le \mathcal{S} \le 3 + \frac{27}{4}H^2 - \frac{3}{4}\sqrt{9H^4 + 8H^2}.$$

Since H is constant Theorem 6.2 implies that $\mathcal{S} = 3H^2$ or $\mathcal{S} = 3 + \frac{27}{4}H^2 - \frac{3}{4}\sqrt{9H^4 + 8H^2}$, and the proof in this case is complete.

In the second case we assume that

$$3 + \frac{27}{4}H^2 - \frac{3}{4}\sqrt{9H^4 + 8H^2} < \mathcal{S} \le 3 + \frac{27}{4}H^2 + \frac{3}{4}\sqrt{9H^4 + 8H^2}.$$

If f_3 is constant, then M is of constant principal curvatures, and hence an isoparametric hypersurface in $S^4(1)$. Thus Theorem 6.3 in this case is valid by Cartan's Lemma. If f_3 is not constant, then the constancy of $\mathcal{S}$ implies together with (6.5) and (6.14) that the three functions f_3, $\sum h_{ijk}^2$ and $\sum h_{ijkl}^2$ are bounded. The Ricci curvature of M is bounded below. Let $F := \sum h_{ijk}^2$ and apply Theorem Y-1 to $-F$. Then there exists a sequence $\{p_m\}$ of points in M such that

$$\lim_{m\to\infty} F(p_m) = \inf_M F, \quad \lim_{m\to\infty} |\text{grad } F(p_m)| = 0,$$
$$\lim\inf_{m\to\infty} \Delta F(p_m) \geq 0.$$

Thus we have

$$\lim_{m\to\infty} \sum h_{ijk}^2(p_m) = \inf_M \sum h_{ijk}^2, \quad \lim_{m\to\infty} |\text{grad } \sum h_{ijk}^2| = 0, \tag{6.20}$$

$$\lim\inf_{m\to\infty} \Delta \sum h_{ijk}^2(p_m) \geq 0. \tag{6.21}$$

From (6.4) and (6.5) we get

$$\lim_{m\to\infty} f_3(p_m) = \sup_M f_3, \quad \lim_{m\to\infty} |\text{grad } f_3(p_m)| = 0, \tag{6.22}$$

$$\lim_{m\to\infty} B_3(p_m) = \sup_M B_3, \quad \lim_{m\to\infty} |\text{grad } B_3(p_m)| = 0. \tag{6.23}$$

Suppose that $\lim_{m\to\infty} B_3(p_m) = -\frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}$, then Sublemma implies that

$$B_3 \equiv -\frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}},$$

and hence f_3 is constant. This is a contradiction. This argument shows that

$$\lim_{m\to\infty} B_3(p_m) > -\frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}. \tag{6.24}$$

We shall assert that $\lim_{m\to\infty} B_3(p_m) = \frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}$. To show this suppose that $\lim_{m\to\infty} B_3(p_m) < \frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}$. Then we must have

$$|B_3(p_m)| < \frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}. \tag{6.25}$$

Since λ_i,h_{ijk} and h_{ijkl} are all bounded by a constant $\mathcal{S}$, we may consider, by taking a subsequence of $\{p_m\}$ if necessary, that

$$\lim_{m\to\infty} \lambda_i(p_m) = \widetilde{\lambda}_i, \tag{6.26}$$

$$\lim_{m\to\infty} h_{ijk}(p_m) = \widetilde{h}_{ijk}, \tag{6.27}$$

$$\lim_{m\to\infty} h_{ijkl}(p_m) = \widetilde{h}_{ijkl}. \tag{6.28}$$

Thus we get

$$\lim_{m\to\infty} \mu_i(p_m) = \widetilde{\lambda}_i - H =: \widetilde{\mu}_i. \tag{6.29}$$

Therefore (6.25) implies that

$$\sum \widetilde{\mu}_i = 0,\ \sum \widetilde{\mu}_i^2 = \mathcal{S} - 3H^2,\ |\sum \widetilde{\mu}_i^3| < \frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}.$$

From $\sum \mu_i = 0$ and $\sum \mu_i^2 = \mathcal{S} - 3H^2$ being constant we get

$$\sum_i h_{iik} = 0 \qquad \text{for every } k, \tag{6.30}$$

$$\sum_i \mu_i h_{iik} = 0 \qquad \text{for every } k. \tag{6.31}$$

From (6.29) and (6.27) we get

$$\sum_i \widetilde{h}_{iik} = 0 \qquad \text{for every } k, \tag{6.32}$$

$$\sum_i \widetilde{\mu}_i \widetilde{h}_{iik} = 0 \qquad \text{for every } k. \tag{6.33}$$

According to (6.23) we have

$$\lim_{m\to\infty} |\text{grad } B_3(p_m)|^2 = \lim_{m\to\infty} \sum_k (\sum_i \mu_i^2 h_{iik})^2(p_m) = 0,$$

and hence

$$\sum_i \widetilde{\mu}_i^2\, \widetilde{h}_{iik} = 0 \qquad \text{for every } k. \tag{6.34}$$

Therefore from (6.32), (6.33), (6.34) and from Sublemma we have

$$\widetilde{h}_{iik} = 0 \qquad \text{for every } i \text{ and } k. \tag{6.35}$$

Lemma 6.4 together with (6.20), (6.26), (6.27) and (6.28) implies that

$$(\mathcal{S} - 3 - \frac{2}{3}h^2)\sum \widetilde{h}_{ijk}^2 \leq -2h\sum \widetilde{\mu}_i \widetilde{h}_{ijk}^2 \tag{6.36}$$
$$+(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2})$$

$$\sum \widetilde{h}_{ijkl}^2 = -\frac{5}{3}\sum(\widetilde{\mu}_i + \widetilde{\mu}_j + \widetilde{\mu}_k)^2 \widetilde{h}_{ijk}^2 - 8h\sum \widetilde{\mu}_i \widetilde{h}_{ijk}^2 + \frac{3}{2}(\mathcal{S} - 3 - \frac{2}{3}h^2)\sum \widetilde{h}_{ijk}^2$$
$$+\frac{3}{2}(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}). \tag{6.37}$$

Since $\sum \widetilde{\mu}_i = 0$, (6.35) implies

$$\sum \widetilde{\mu}_i \widetilde{h}^2_{ijk} = 0, \quad \sum (\widetilde{\mu}_i + \widetilde{\mu}_j + \widetilde{\mu}_k)^2 \widetilde{h}^2_{ijk} = 0. \tag{6.38}$$

On the other hand we have, by recalling t_{ij} as in (4.10),

$$\sum h^2_{ijkl} \geq 3 \sum_{i \neq j} h^2_{ijij} = \frac{3}{4} \sum_{i \neq j} (h_{ijij} + h_{jiji})^2 + \frac{3}{4} \sum_{i \neq j} t^2_{ij},$$

and since $\widetilde{\lambda}_i$'s are all distinct from each other

$$\sum \widetilde{h}^2_{ijkl} \geq \frac{3}{4} \sum (\widetilde{\lambda}_i - \widetilde{\lambda}_j)^2 (1 + \widetilde{\lambda}_i \widetilde{\lambda}_j)^2 > 0. \tag{6.39}$$

It follows from the assumption $\mathcal{S} > \frac{3}{4}h^2 + 3 - \frac{1}{4}\sqrt{h^4 + 8h^2}$ in this case and from (6.36),(6.37) and (6.38) that $\sum \widetilde{h}^2_{ijkl} \leq 0$ which is in contradiction with (6.39). Hence we have

$$\lim_{m \to \infty} B_3(p_m) = \frac{1}{\sqrt{6}} (\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}.$$

Thus (6.4) and (6.5) imply

$$\sum \widetilde{h}^2_{ijk} = \mathcal{S}(\mathcal{S} - 3) + h^2 - h \lim_{m \to \infty} f_3(p_m)$$
$$= (\mathcal{S} - \frac{1}{3}h^2)[\sqrt{\mathcal{S} - \frac{1}{3}h^2} - \frac{h}{2\sqrt{6}} + \sqrt{\frac{3}{8}h^2 + 3}][\sqrt{\mathcal{S} - \frac{1}{3}h^2} - \frac{h}{2\sqrt{6}} - \sqrt{\frac{3}{8}h^2 + 3}].$$

From the assumption for $\mathcal{S}$ we see that

$$\mathcal{S} > \frac{1}{3}h^2, \qquad \sqrt{\mathcal{S} - \frac{1}{3}h^2} - \frac{h}{2\sqrt{6}} + \sqrt{\frac{3}{8}h^2 + 3} > 0.$$

Therefore we obtain

$$\mathcal{S} > \frac{1}{3}h^2, \qquad \sqrt{\mathcal{S} - \frac{1}{3}h^2} - \frac{h}{2\sqrt{6}} - \sqrt{\frac{3}{8}h^2 + 3} \geq 0.$$

Thus we have $\mathcal{S} \geq \frac{3}{4}h^2 + 3 + \frac{1}{4}\sqrt{h^4 + 8h^2}$, and we conclude the proof in this case.

In the final case we assume that

$$\frac{3}{4}h^2 + 3 + \frac{1}{4}\sqrt{h^4 + 8h^2} < \mathcal{S} \leq h^2 + 6.$$

The following Propositions 6.5 and 6.6 will assert that in this case we must have $\mathcal{S} = h^2 + 6$ and that M is isoparametric if $H \neq 0$. Thus the proof of Theorem 6.3 is complete if the Propositions 6.5 and 6.6 have been verified.

Proposition 6.5. *A complete hypersurface M of $S^4(1)$ with nonzero constant mean curvature H is an isoparametric hypersurface with $\mathcal{S} = h^2 + 6$ if the following conditions are satisfied.*

$$\mathcal{S} \quad \text{is constant},$$
$$\inf_M B_3 \cdot \sup_M B_3 = 0,$$
$$3 + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2} < \mathcal{S} \leq h^2 + 6.$$

Proposition 6.6. *A complete hypersurface M of $S^4(1)$ with nonzero constant mean curvature H is an isoparametric hypersurface with $\mathcal{S} = h^2 + 6$ if the following conditions are satisfied.*

$$\mathcal{S} \quad \text{is constant},$$
$$\inf_M B_3 \cdot \sup_M B_3 \neq 0,$$
$$3 + \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2} < \mathcal{S} \leq h^2 + 6.$$

Proof of Proposition 6.5. If $\inf_M B_3 = \sup_M B_3 = 0$, then $B_3 \equiv 0$ and in particular (6.4) means that f_3 is constant. Therefore the principal curvatures of M are all constant. Cartan's Lemma concludes the proof.

We shall next assert that B_3 and hence f_3 must be constant. Without loss of generality we may assume that

$$\sup_M B_3 = 0.$$

Apply Theorem Y-1 to B_3 to obtain a sequence $\{p_m\}$ of points on M with the properties that

$$\lim_{m\to\infty} B_3(p_m) = \sup_M B_3, \quad \lim_{m\to\infty} |\text{grad } B_3(p_m)| = 0,$$
$$\lim\sup_{m\to\infty} \ \Delta B_3(p_m) \leq 0. \tag{6.40}$$

By (6.4) we have

$$\lim_{m\to\infty} f_3(p_m) = h\mathcal{S} - \frac{2}{9}h^3,$$
$$\lim_{m\to\infty} |\text{grad } f_3(p_m)| = 0, \ \lim\sup_{m\to\infty} \Delta f_3(p_m) \leq 0.$$

Making use of the same technique as developed in the second case of the proof of Theorem 6.3, we obtain (6.26),(6.27) and (6.28). Thus we have

$$\sum \widetilde{\mu}_i = 0, \ \sum \widetilde{\mu}_i^2 = \mathcal{S} - 3H^2, \ \sum \widetilde{\mu}_i^3 = 0,$$

and hence, setting $B := \sum_i \widetilde{\mu}_i^2$ and $\widetilde{\mu}_1 \le \widetilde{\mu}_2 \le \widetilde{\mu}_3$, we have

$$\widetilde{\mu}_1 = -\sqrt{\frac{B}{2}}, \ \ \widetilde{\mu}_2 = 0, \ \ \widetilde{\mu}_3 = \sqrt{\frac{B}{2}}.$$

Since $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_3$ are all distinct, the same method as in the second case of the proof of Theorem 6.3 implies that

$$\widetilde{h}_{iik} = 0 \qquad \text{for every } i \text{ and } k,$$
$$\sum_{i,j,k} \widetilde{\mu}_i \widetilde{h}_{ijk}^2 = 0.$$

It follows from (6.4) and (6.5) that

$$-h\Delta\, B_3 = \Delta \sum h_{ijk}^2, \tag{6.41}$$

and from (6.40) and (6.13)

$$0 \le -(\mathcal{S} - 3 - \frac{2}{3}h^2) \sum \widetilde{h}_{ijk}^2$$
$$+(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}).$$

Since $\lim_{m\to\infty} f_3(p_m) = h\mathcal{S} - \frac{2}{9}h^3$, we get

$$\sum \widetilde{h}_{ijk}^2 = (\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - \frac{2}{3}h^2 - 3). \tag{6.42}$$

Therefore a contradiction is derived by

$$0 \le -(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - \frac{2}{3}h^2 - 3)^2$$
$$+(\mathcal{S} - \frac{1}{3}h^2)(\mathcal{S} - 3 - \frac{3}{4}h^2 + \frac{1}{4}\sqrt{h^4 + 8h^2})(\mathcal{S} - 3 - \frac{3}{4}h^2 - \frac{1}{4}\sqrt{h^4 + 8h^2}) < 0.$$

This completes the proof of Proposition 6.5.

Proof of Proposition 6.6. If f_3 is constant, then M has constant principal curvatures, for both $\mathcal{S}$ and H are constant. Thus the proof in this case follows from Cartan's lemma.

Now consider the case where f_3 is not constant. We then claim that it cannot occur. It then follows from $\inf_M B_3 \cdot \sup_M B_3 \ne 0$ that;

(1) If $\inf_M B_3 \cdot \sup_M B_3 < 0$, then there exists a point $p \in M$ such that

$$B_3(p) = 0. \tag{6.43}$$

(2) If $\inf_M B_3 \cdot \sup_M B_3 > 0$, then $\inf_M B_3$ and $\sup_M B_3$ have the same sign.

In the second case we may assume without loss of generality that

$$\sup_M B_3 < 0.$$

From Sublemma we see that

$$0 > \sup_M B_3 > -\frac{1}{\sqrt{6}}(\mathcal{S} - \frac{1}{3}h^2)^{\frac{3}{2}}. \tag{6.44}$$

Applying Theorem Y-1 to B_3 we see that there exists a sequence $\{p_m\}$ of points on M such that

$$\lim_{m\to\infty} B_3(p_m) = \sup_M B_3,\ \lim_{m\to\infty} |\mathrm{grad}\, B_3(p_m)| = 0,\ \lim\sup_{m\to\infty} \Delta B_3(p_m) \leq 0. \tag{6.45}$$

By the same discussion as developed in the second case of the proof of Theorem 6.3, we see that

$$\sum \widetilde{\mu}_i = 0,\ \sum \widetilde{\mu}_i^2 = B = \mathcal{S} - 3H^2,\ \sum \widetilde{\mu}_i^3 = \sup_M B_3.$$

Then the above relations together with (6.44) imply that $\widetilde{\mu}_1$,$\widetilde{\mu}_2$ and $\widetilde{\mu}_3$ are distinct from each other. By the same technique as in the second case of Theorem 6.3 we see that (6.35) and (6.38) are valid. From (6.4),(6.5) and (6.8) we have

$$\begin{aligned}\Delta f_3 \geq 6\sum \mu_i h_{ijk}^2 - 3(\mathcal{S} - \frac{2}{3}h^2 - 3)(h\mathcal{S} - \frac{2}{9}h^3)\\ + h(\frac{7}{2}\mathcal{S}^2 - 9\mathcal{S} + 2h^2 - 3h^2\mathcal{S} + \frac{1}{2}h^4).\end{aligned}$$

This means that

$$0 \geq h(\mathcal{S} - \frac{1}{3}h^2)^2 > 0.$$

This is impossible and the second case does not occur.

Finally we shall show that the first case does not occur. Since $B_3(p) = 0$ we see

$$f_3(p) = h\mathcal{S} - \frac{2}{9}h^3.$$

At the point p we have

$$\begin{aligned}\mu_1 + \mu_2 + \mu_3 = 0,\\ \mu_1^2 + \mu_2^2 + \mu_3^2 = B = \mathcal{S} - \frac{1}{3}h^2,\\ \mu_1^3 + \mu_2^3 + \mu_3^3 = 0.\end{aligned}$$

Hence we have

$$\mu_1 = -\sqrt{\frac{B}{2}},\ \mu_2 = 0,\ \mu_3 = \sqrt{\frac{B}{2}},$$

$$\lambda_1 = H + \sqrt{\frac{B}{2}},\ \lambda_2 = H,\ \lambda_3 = H - \sqrt{\frac{B}{2}}. \tag{6.46}$$

It follows from $\sum \lambda_i = 3H$ and $\sum \lambda_i^2 = \mathcal{S}$ that

$$(6.47) \qquad \sum_i h_{iik} = 0, \quad \sum_i \lambda_i h_{iik} = 0 \quad \text{for every } k.$$

Thus we get

$$h_{11k} = h_{33k}, \quad h_{22k} = -2h_{11k} \qquad \text{for every } k.$$

On the other hand, because $B_3(p) = 0$ we have

$$(6.48) \qquad \sum h_{ijk}^2 = B(\mathcal{S} - \frac{2}{3}h^2 - 3).$$

Then (6.47) implies that

$$\sum h_{ijk}^2 = 6h_{123}^2 + 16h_{111}^2 + \frac{5}{2}h_{222}^2 + 16h_{333}^2 = B(\mathcal{S} - \frac{2}{3}h^2 - 3).$$

Therefore we have

$$\sum h_{ij2}^2 = 2h_{123}^2 + 8h_{111}^2 + \frac{3}{2}h_{222}^2 + 8h_{333}^2 \geq \frac{1}{3}\sum h_{ijk}^2 = \frac{1}{3}B(\mathcal{S} - \frac{2}{3}h^2 - 3).$$

$$\sum_{i \neq j} t_{ij}^2 = \sum(\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j)^2 = 2B(\frac{1}{2}\mathcal{S}^2 - \frac{1}{2}h^2\mathcal{S} + \frac{4}{27}h^4 - 2\mathcal{S} + \frac{4}{3}h^2 + 3),$$

$$\frac{1}{4}\sum_{i \neq j}(h_{ijij} + h_{jiji})^2 \geq [h_{1212} - h_{3232} - t_{23} - \frac{1}{2}(t_{12} - t_{23})]^2.$$

By differentiating $\mathcal{S} = \sum h_{ij}^2$, we obtain $\sum_i h_{iikk}\lambda_i + \sum_{ij} h_{ijk}^2 = 0$ for any $k = 1, 2, 3$. Therefore

$$\sqrt{\frac{B}{2}}(h_{3232} - h_{1212}) = \sum h_{ij2}^2. \qquad t_{12} - t_{23} = \frac{1}{3}hB,$$

$$t_{23} = \sqrt{\frac{B}{2}} + \frac{1}{9}h^2\sqrt{\frac{B}{2}} - \frac{1}{6}hB.$$

$$(6.49) \sum h_{ijkl}^2 \geq \frac{3}{2}B(\frac{1}{2}\mathcal{S}^2 - \frac{1}{2}h^2\mathcal{S} + \frac{4}{27}h^4 - 2\mathcal{S} + \frac{4}{3}h^2 + 3) + 6B(\frac{1}{3}\mathcal{S} - \frac{1}{2} - \frac{1}{6}h^2)^2.$$

(6.14), (6.48) and (6.49) yield

$$0 \leq -\frac{5}{3}\sum(\mu_i + \mu_j + \mu_k)^2 h_{ijk}^2 - 8h\sum \mu_i h_{ijk}^2$$
$$+[\frac{19}{12}\mathcal{S}^2 - (\frac{17}{6}h^2 + 13)\mathcal{S} + \frac{37}{36}h^4 + 9h^2 + 21]B.$$

The above inequality will be treated in the following two cases.

Suppose in the first case that $-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2 - 8h\sum\mu_i h_{ijk}^2 \le 0$. Then $\frac{19}{12}\mathcal{S}^2 - (\frac{17}{6}h^2+13)\mathcal{S} + \frac{37}{36}h^4 + 9h^2 + 21 \ge 0$. According to the assumption of Proposition 6.6,we get $\mathcal{S} > h^2+6$. This is a contradiction.

Suppose next that $-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2 - 8h\sum\mu_i h_{ijk}^2 > 0$. Then $-\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2 \le -20Bh_{333}^2$ and $-8h\sum\mu_i h_{ijk}^2 \le 32h\sqrt{\frac{B}{2}}h_{333}^2$ imply

$$(6.50)\quad -\frac{5}{3}\sum(\mu_i+\mu_j+\mu_k)^2h_{ijk}^2 - 8h\sum\mu_i h_{ijk}^2 \le B(-\frac{3}{4}\mathcal{S}^2+\frac{9}{4}\mathcal{S}+\frac{7}{4}h^2\mathcal{S}-\frac{5}{6}h^4-\frac{15}{4}h^2).$$

(6.49) and (6.50) imply

$$\frac{5}{6}\mathcal{S}^2 - (\frac{13}{12}h^2+\frac{43}{4})\mathcal{S} + \frac{7}{36}h^4 + \frac{21}{4}h^2 + 21 \ge 0.$$

Let $F(\mathcal{S}) := \frac{5}{6}\mathcal{S}^2 - (\frac{13}{12}h^2+\frac{43}{4})\mathcal{S} + \frac{7}{36}h^4 + \frac{21}{4}h^2 + 21$. As a function of $\mathcal{S}$, $F(\mathcal{S})$ reaches its minimum at $\mathcal{S} = \frac{3}{5}(\frac{13}{12}h^2+\frac{43}{4})$. Without loss of generality we may assume

$$\frac{3}{4}h^2+3+\frac{1}{4}\sqrt{h^4+8h^2} \le \frac{3}{5}(\frac{13}{12}h^2+\frac{43}{4}) < h^2+6.$$

Since $F(\frac{3}{4}h^2+3+\frac{1}{4}\sqrt{h^4+8h^2}) < 0$, $F(h^2+6) < 0$ and $F(\frac{3}{5}(\frac{13}{12}h^2+\frac{43}{4}))$ is only a minimum of F, we obtain $F<0$ If $\frac{3}{4}h^2+3+\frac{1}{4}\sqrt{h^4+8h^2} < \mathcal{S} \le h^2+6$. This is a contradiction.Hence we complete the proof of Proposition 6.6.

It is proved by de Almeida (see [3]) that the Clifford torus $S^1\times S^2$ is the only closed oriented minimal hypersurface in $S^4(1)$ with constant scalar curvature and nonvanishing Gauss-Kronecker curvature. de Almeida and Brito [4] discussed closed oriented minimal hypersurfaces in $S^4(1)$ with constant Gauss-Kronecker curvature, and proved that if a closed oriented minimal hypersurface M in $S^4(1)$ with constant Gauss-Kronecker curvature has its second fundamental form being non-zero everywhere, then M is the Clifford torus $S^1\times S^2$ or the boundary of tube of a minimal 2-dimensional surface in $\mathcal{S}^4(1)$.

We now discuss on the quasi-Gauss-Kronecker curvature on $M\subset S^4(1)$. We know from definition that if M is minimal, then the quasi-Gauss-Kronecker curvature of M is nothing but the Gauss-Kronecker curvature.

Theorem 6.7. *(see [14]) Let M be a closed oriented hypersurface in $S^4(1)$ with non-zero constant mean curvature. If the quasi-Gauss-Kronecker curvature K of M is identically zero, then M is totally umbilical.*

Proof. Since $K=\mu_1\mu_2\mu_3$ and $\mu_1+\mu_2+\mu_3=0$, we have

$$(6.51)\qquad 3K = \mu_1^3+\mu_2^3+\mu_3^3 =: B_3.$$

A direct computations show that

$$(6.52)\qquad B_3 = 6H^3 - 3H\mathcal{S} + f_3, B = \mu_1^2+\mu_2^2+\mu_3^2 = \mathcal{S}-3H^2,$$

$$(6.53)\qquad \Delta\mathcal{S} = 2\mathcal{S}(3-\mathcal{S}) - 18H^2 + 6Hf_3 + 2\sum h_{ijk}^2,$$

$$(6.54)\qquad \Delta f_3 = 6\sum\lambda_i h_{ijk}^2 - 3(\mathcal{S}-3)f_3 + 9Hf_4 - 9H\mathcal{S},$$

$$(6.55)\qquad f_4 = -9H^2\mathcal{S} + \frac{27}{2}H^4 + 4Hf_3 + \mathcal{S}^2/2.$$

Since $\mu_i = \lambda_i - H$, we get from (6.54) and (6.55)

$$\begin{aligned}\Delta f_3 &= 6\sum \mu_i h_{ijk}^2 + 6H\sum h_{ijk}^2 - 3(\mathcal{S} - 3 - 12H^2)f_3 \\ &\quad + 9H(-9H^2\mathcal{S} + \frac{27}{2}H^4 + \frac{1}{2}\mathcal{S}^2 - \mathcal{S}).\end{aligned} \tag{6.56}$$

Since $B_3 = 3K = 0$ and H is constant, (6.52) implies that

$$\Delta f_3 = 3H\Delta\mathcal{S}. \tag{6.57}$$

Thus we get

$$\begin{aligned}3H\Delta\mathcal{S} &= 6\sum \mu_i h_{ijk}^2 + 6H\sum h_{ijk}^2 - 3(\mathcal{S} - 3 - 12H^2)f_3 \\ &\quad + 9H(-9H^2\mathcal{S} + \frac{27}{2}H^4 + \frac{1}{2}\mathcal{S}^2 - \mathcal{S}).\end{aligned} \tag{6.58}$$

From (6.52), (6.53) and (6.58) we obtain

$$\begin{aligned}-6\sum \mu_i h_{ijk}^2 &= -3(\mathcal{S} - 3 - 6H^2)f_3 + 9H(-9H^2\mathcal{S} + \frac{27}{2}H^2 + \frac{7}{6}\mathcal{S}^2 - 3\mathcal{S} + 6H^2) \\ &= \frac{3}{2}H(\mathcal{S} - 3H^2)^2.\end{aligned} \tag{6.59}$$

Let $p \in M$ be a maximal point of $\mathcal{S}$. Then $B = \mathcal{S} - 3H^2$ also attains its maximum at p. Since $K = 0$, we may assume without loss of generality that $\mu_2 = 0$ at p.

In the first place we assume that $\mu_1(p) = \mu_3(p) = 0$. Then clearly we have $\max\mathcal{S} = \min\mathcal{S} = \mathcal{S}(p)$ and $\mathcal{S}$ is constant. Namely, $\lambda_i = H$ for all $i = 1, 2, 3$ and hence M is totally umbilical.

Assume in the next place that $\mu_1(p) = -\mu_3(p) \neq 0$. We then assert that this cannot occur. In fact, $\mu_1 = -\mu_3 \neq 0$ implies that $\mu_2 = 0$, and therefore μ_1, μ_2, μ_3 are distinct at p. Thus $\lambda_1, \lambda_2, \lambda_3$ are distinct at p. Since $3H = \sum_i h_{ii}$ is constant, we have

$$\sum_i h_{iik} = 0 \qquad \text{for all } k. \tag{6.60}$$

On the other hand from the choice of p we obtain

$$\begin{aligned}&\sum_i \lambda_i h_{iik} = 0 &&\text{for all } k, \\ &\nabla_k f_3 = 3H\nabla_k\mathcal{S} = 0 &&\text{for all } k.\end{aligned} \tag{6.61}$$

Namely we have

$$\sum_i \lambda_i^2 h_{iik} = 0 \qquad \text{for all } k. \tag{6.62}$$

Because principal curvatures are all distinct,(6.60),(6.61) and (6.62) imply

$$h_{iik} = 0 \qquad \text{for all } k \text{ and } i. \tag{6.63}$$

The symmetric property of h_{ijk} implies together with (6.63) that

$$\sum \mu_i h_{ijk}^2 = \frac{1}{3}\sum_{i,j,k}(\mu_i + \mu_j + \mu_k)h_{ijk}^2 = 0. \tag{6.64}$$

Thus (6.64) and (6.59) imply

$$\frac{3}{2}H(\mathcal{S} - 3H^2)^2 = 0.$$

Since $\max \mathcal{S} = \mathcal{S}(p) = 3H^2 = \min \mathcal{S}$ implies $\mu_i = 0$, this is a contradiction.

This completes the proof of Theorem 6.7.

Theorem 6.8. *(see [15]) Let M be a closed oriented hypersurface in $S^4(1)$ with non-zero constant mean curvature H. If the quasi-Gauss-Kronecker curvature K is constant and if $K \cdot H \leq 0$, then M is either totally umbilical or an isoparametric hypersurface with $\mathcal{S} = 3 + \frac{27}{4}H^2 \pm \frac{3}{4}\sqrt{9H^4 + 8H^2}$.*

Proof. Since $B_3 = 3K$ is constant, the same technique as developed in the proof of Theorem 6.7 implies that

$$-6\sum \mu_i h_{ijk}^2 = -3B_3(\mathcal{S} - 3 - 6H^2) + 3H(\mathcal{S} - 3H^2)^2/2. \tag{6.65}$$

If $B_3 = 0$, then we see from Theorem 6.7 that M is totally umbilical.

We only need to discuss the case where $B_3 \neq 0$. Let $p \in M$ be a maximum point of $\mathcal{S}$. Then $B = \mathcal{S} - 3H^2$ attains its maximum at p. Computations will be carried out at p which are divided into three cases as follows.

In the first case assume that μ_1, μ_2, μ_3 are all distinct at p. As is discussed in the proof of Theorem 6.7, we get (6.60),(6.61) and (6.62), and hence $h_{iik} = 0$ for all i and for all k. Thus we have

$$\sum \mu_i h_{ijk}^2 = 0. \tag{6.66}$$

From (6.65) and (6.66) we have

$$-3B_3(\mathcal{S} - 3 - 6H^2) + 3H(\mathcal{S} - 3H^2)^2 = 0. \tag{6.67}$$

Since $B_3 \cdot H < 0$ we see

$$\mathcal{S} < 3 + 6H^2. \tag{6.68}$$

We may assume without loss of generality that $H > 0$ and $B_3 < 0$. Then,

$$0 < -B_3 < (\mathcal{S} - 3H^2)^{\frac{3}{2}}/\sqrt{6}. \tag{6.69}$$

Thus we get

$$\mathcal{S} \geq 3 + \frac{27}{4}H^2 + \frac{3}{4}\sqrt{9H^4 + 8H^2}$$

or

$$\mathcal{S} \leq 3 + \frac{27}{4}H^2 - \frac{3}{4}\sqrt{9H^4 + 8H^2}.$$

From (6.68) we obtain $\mathcal{S} \leq 3 + \frac{27}{4}H^2 - \frac{3}{4}\sqrt{9H^4 + 8H^2}$. Theorem 6.2 and closedness of M then imply that $\mathcal{S} = 3H^2$ or $\mathcal{S} = 3 + \frac{27}{4}H^2 - \frac{3}{4}\sqrt{9H^4 + 8H^2}$. Thus the proof in this case is complete.

In the second case assume that $\mu_1(p) = \mu_2(p) = \mu_3(p)$. Then, $\max \mathcal{S} = 3H^2 = \min \mathcal{S}$, and hence $\mathcal{S} \equiv 3H^2$. Thus M in this case is totally umbilical.

In the final case assume that two of the three are equal, e.g., assume that $\mu_1(p) = \mu_2(p) = -\mu_3(p)/2$. Then we see that

$$\max B = B(p) = \mathcal{S}(p) - 3H^2 = (54K^2)^{\frac{1}{3}}.$$

Making use of the Lagrange multiplier to the minimizing function $F = \mu_1^2 + \mu_2^2 + \mu_3^2$ with constraints $\sum \mu_i = 0$ and $\mu_1 \cdot \mu_2 \cdot \mu_3 = K$, we have

$$F \geq (54K^2)^{\frac{1}{3}}.$$

Therefore $\mathcal{S}$ is constant on M, and M is an isoparametric hypersurface with two distinct principal curvatures. A Lemma due to E.Cartan as stated in §1 implies in our case that $\mathcal{S} = 3 + \frac{27}{4}H^2 \pm \frac{3}{4}\sqrt{9H^4 + 8H^2}$, and the proof is complete.

§7. Complete space-like hypersurfaces in indefinite space forms

Let M be a complete space-like hypersurface in $M_1^{n+1}(c)$. For simplicity we write $h_{ij} = h_{ij}^{n+1}$. For an arbitrary fixed point $p \in M$ we choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ as before in such a way that

$$h_{ij} = \lambda_i \delta_{ij}.$$

Let

$$\mu_i := \lambda_i - H,$$

where $H = \frac{1}{n}\sum \lambda_i$ is the mean curvature of M. We then have

$$\sum \mu_i = 0, \tag{7.1}$$

$$\sum \mu_i^2 = \mathcal{S} - nH^2, \tag{7.2}$$

$$\sum \mu_i^3 = f_3 - 3H\sum \mu_i^2 - nH^3, \tag{7.3}$$

where $f_m = \sum \lambda_i^m$ and $\mathcal{S}$ is the squared norm of the second fundamental form of M.

By means of (1.10) and (1.21) we have

$$\begin{aligned}\frac{1}{2}\Delta \mathcal{S} = \sum h_{ijk}^2 + \sum h_{ij}h_{kkij} + c(n\mathcal{S} - n^2H^2) \\ - nHf_3 + \mathcal{S}^2.\end{aligned}$$

By setting $f^2 := \sum \mu_i^2 = \mathcal{S} - nH^2$ we see that if H is constant, then

$$\frac{1}{2}\Delta f^2 \geq ncf^2 - nHf_3 + (f^2 + nH^2)^2.$$

From (7.2) and (7.3) we get

$$\begin{aligned}\frac{1}{2}\Delta f^2 &\geq f^2(nc - nH^2 + f^2) - nH\sum \mu_i^3 \\ &\geq f^2[f^2 - (n-2)\{n(n-1)\}^{-1/2}n|H|f + nc - nH^2].\end{aligned} \tag{7.4}$$

Theorem 7.1. *(see [16]) Let M be a complete space-like hypersurface with constant mean curvature in a Lorentz space form $M_1^{n+1}(c)$ with $c \leq 0$. Then $\mathcal{S}$ satisfies*

$$nH^2 \leq \mathcal{S} \leq [n\{n^2H^2 - 2(n-1)c\} + n(n-2)|H|\{n^2H^2 - 4(n-1)c\}^{1/2}]/2(n-1). \tag{7.5}$$

Proof. For an arbitrary given positive number a we define a function $F := -(f^2 + a)^{-\frac{1}{2}}$. Since M is space-like, the Ricci tensor of M is given by

$$\begin{aligned}R_{ij} &= (n-1)c\delta_{ij} - nHh_{ij} + \sum h_{ik}h_{kj} \\ &= [(n-1)c - nH\lambda_i + \lambda_i^2]\delta_{ij}.\end{aligned} \tag{7.6}$$

This equality shows that the Ricci curvature of M is bounded below. Since F is bounded, we apply Theorem Y-1 to F. For any positive ε there exists a point $p \in M$ at which

$$\sup_M F - \varepsilon < F(p),$$
$$|\nabla F(p)| < \varepsilon,$$
$$\Delta F(p) < \varepsilon. \tag{7.7}$$

Consequently a straightforward computation leads to

$$\frac{1}{2} F(p)^4 \Delta f^2(p) < 3\varepsilon^2 - \varepsilon F(p). \tag{7.8}$$

For a monotone decreasing sequence $\{\varepsilon_m\}$ of positive numbers converging to 0 there exists a sequence $\{p_m\}$ of points on M such that $\{F(p_m)\}$ converges to a number F_0. From (7.7) we see that $F_0 = \sup_M F$, and hence $\{f(p_m)\}$ also converges to $f_0 = \sup_M f$.

On the other hand we have from (7.8),

$$\frac{1}{2} F(p_m)^4 \Delta f^2(p_m) < 3\varepsilon_m^2 - \varepsilon_m F(p_m), \tag{7.9}$$

and the right hand side of the above inequality converges to 0 since F is bounded. Thus, for a sufficiently small positive ε there exists an integer m such that

$$F(p_m)^4 \Delta f^2(p_m) < \varepsilon.$$

This and (7.4) imply

$$0 > (2-\varepsilon) f(p_m)^4 - 2(n-2)\{n(n-1)\}^{-\frac{1}{2}} n|H| f(p_m)^3 + 2(nc - nH^2 - \varepsilon a) f(p_m)^2 - \varepsilon a^2.$$

This inequality means that $\{f(p_m)\}$ is bounded and hence

$$F_0 = \sup_M F \neq 0.$$

Moreover we see that

$$\lim_{m\to\infty} \sup \Delta f^2(p_m) \le 0.$$

Therefore f_0 must satisfy

$$f_0^2[f_0^2 - (n-2)\{n(n-1)\}^{-\frac{1}{2}} n|H| f_0 + (nc - nH^2)] \le 0. \tag{7.10}$$

From this relation we have

$$0 \le f_0 \le \frac{1}{2}(n-2)\{n(n-1)\}^{-\frac{1}{2}} n|H| + \frac{1}{2}[\frac{n(n-2)^2H^2}{n-1} - 4(nc - nH^2)]^{\frac{1}{2}}.$$

We conclude the proof by $S = f^2 + nH^2$.

Remark 7.1. From $f_0 = 0$ it follows that f is identically zero and that M is totally umbilical. The above discussion also gives a proof of a result due to Akutagawa [2] and Ramanathan [44] as stated in §8.

Remark 7.2. Notice that Theorem 7.1 generalizes the results obtained by Ishihara [29] and Cheng-Yau [21]. In fact, by setting $H = 0$, Theorem 7.1 implies that $0 \leq \mathcal{S} \leq -nc$. Ishihara obtained this relation for complete maximal space-like hypersurface in $M_1^{n+1}(c)$. A result by Cheng-Yau states that an entire space-like hypersurface of constant mean curvature in $\boldsymbol{R}_1^{n+1}$ satisfies $nH^2 \leq \mathcal{S} \leq n^2H^2$. This relation is obtained by setting $c = 0$ in Theorem 7.1.

We next discuss complete space-like hypersurfaces in a de Sitter space.

Theorem 7.2. *(see [11]) Let M be a space-like hypersurface with constant mean curvature in $S_1^{n+1}(c)$ and $n \geq 3$. If the sectional curvature of M is nonnegative and if the multiplicity of each principal curvature is greater than 1, then M is isometric to either a Euclidean space $\boldsymbol{R}^n$ or a sphere $S^n(c_1)$ with $0 < c_1 < c$.*

Proof. For an arbitrary fixed point $p \in M$ we choose a local frame field as before such that

$$h_{ij}(p) = \lambda_i \delta_{ij}. \tag{7.11}$$

At this point we have, by choosing e_{n+1} time-like vector,

$$nH = \sum \lambda_i, \quad \mathcal{S} = \sum \lambda_i^2, \quad f_3 = \sum \lambda_i^3.$$

Thus we get

$$\begin{aligned} &- c(n\mathcal{S} - n^2H^2) + nHf_3 - \mathcal{S}^2 \\ &= c[-n\sum \lambda_i^2 + (\sum \lambda_i)^2] + \sum \lambda_i \cdot \sum \lambda_i^3 - (\sum \lambda_i^2)^2 \\ &= \frac{1}{2}\sum(-c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2. \end{aligned} \tag{7.12}$$

According to (1.10),(1.21) and (7.12) we get

$$\begin{aligned} \frac{1}{2}\Delta\mathcal{S} &= \sum h_{ijk}^2 + c(n\mathcal{S} - n^2H^2) - nHf_3 + \mathcal{S}^2 \\ &= \sum h_{ijk}^2 + \frac{1}{2}\sum(c - \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2. \end{aligned} \tag{7.13}$$

From (7.6) we see that the Ricci curvature of M is bounded from below by $(n-1)c - \frac{n^2H^2}{4}$. Setting $F := (\mathcal{S} + a)^{-\frac{1}{2}}$ for any fixed positive number a, we see $-F < 0$ and a similar technique as developed in the proof of Theorem 7.1 shows that there exists for a decreasing sequence $\{\varepsilon_m\}$ of positive numbers converging to zero a sequence $\{p_m\}$ of points on M such that

$$\lim_{m\to\infty} F(p_m) := F_0 = \inf_M F, \tag{7.14}$$

$$\lim_{m\to\infty} \mathcal{S}(p_m) = \sup_M \mathcal{S}, \tag{7.15}$$

$$\frac{1}{2}F(p_m)^4\Delta\mathcal{S}(p_m) < 3\varepsilon_m^2 - \varepsilon_m F(p_m). \tag{7.16}$$

The right hand side of (7.16) converges to zero since F is bounded. From assumption for the sectional curvature we have $K(e_i, e_j) = c - \lambda_i\lambda_j \geq 0$, and (7.13) implies that

$$\Delta\mathcal{S}(p_m) \geq \sum(c - \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 \geq 0. \tag{7.17}$$

For a sufficiently small positive ε there is a number m_0 such that

$$F(p_m)^4\Delta\mathcal{S}(p_m) < \varepsilon, \qquad \text{for all } m > m_0. \tag{7.18}$$

On the other hand $\{F(p_m)\}$ is bounded from below by a positive constant. In fact we have $f_4\mathcal{S} - f_3^2 = \frac{1}{2}\sum\lambda_i^2\lambda_j^2(\lambda_i - \lambda_j)^2 \geq 0$ and $f_4 - \mathcal{S}^2 = -\sum_{j\neq k}(\lambda_j\lambda_k)^2 \leq 0$. Hence we have $-\mathcal{S}^{\frac{3}{2}} \leq f_3 \leq \mathcal{S}^{\frac{3}{2}}$. Thus (7.13) implies that

$$\frac{1}{2}\Delta\mathcal{S} \geq c(n\mathcal{S} - n^2H^2) - n|H|\mathcal{S}^{\frac{3}{2}} + \mathcal{S}^2.$$

The above inequality together with (7.18) yields

$$0 > (2-\varepsilon)\mathcal{S}(p_m)^2 - 2n|H|\mathcal{S}^{\frac{3}{2}}(p_m) + 2(nc - a\varepsilon)\mathcal{S}(p_m) - a^2\varepsilon - 2cn^2H^2.$$

Therefore the sequence $\{\mathcal{S}(p_m)\}$ is bounded from above. From the above facts we get

$$\lim_{m\to\infty}\Delta\mathcal{S}(p_m) = 0. \tag{7.19}$$

Thus (7.17) and (7.19) imply

$$\lim_{m\to\infty}(c - \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2(p_m) = 0 \qquad \text{for all distinct indices } i \text{ and } j. \tag{7.20}$$

Now the following Assertions (1) and (2) will be derived from (7.20) :

(1) For every j the sequence $\{\lambda_j(p_m)\}$ is bounded.
(2) For every distinct indices i and j there exists a subsequence $\{p_k\}$ of $\{p_m\}$ such that $\lim_{k\to\infty}(\lambda_i - \lambda_j)(p_k) = 0$.

The assertion (1) is clear from $\{\mathcal{S}(p_m)\}$ being bounded.

For the proof of the second assertion we set $a_m := (c-\lambda_i\lambda_j)(p_m)$ and $b_m := (\lambda_i-\lambda_j)(p_m)$. Both of the sequences $\{a_m\}$ and $\{b_m\}$ are bounded from Assertion (1). (7.20) means that the sequence $\{a_mb_m\}$ converges to zero.

Suppose in the first place that there is a subsequence $\{p_k\}$ of $\{p_m\}$such that $\{a_k\}$ converges to a non-zero a. Then $\lim_{k\to\infty} b_k = 0$.

Suppose finally that there is a subsequence $\{p_k\}$ such that $\lim_{k\to\infty} a_k = 0$. An inductive argument is employed here. Suppose that the sequence $\{\lambda_1\lambda_2(p_k)\}$ converges to c. Since $\{\lambda_1(p_k)\}$ is bounded, the limit λ_1^* of its converging subsequence is non-zero because $\lim\lambda_1\lambda_2(p_k) = c \neq 0$. Because $\lambda_1^* \neq 0$, we have

$$|\lambda_1^*| \cdot |\lambda_2(p_k) - \frac{c}{\lambda_1^*}| \leq |\lambda_2(p_k)| \cdot |\lambda_1(p_k) - \lambda_1^*| + |\lambda_1\lambda_2(p_k) - c| \longrightarrow 0,$$

and hence $\{\lambda_2(p_k)\}$ tends to $\frac{c}{\lambda_1^*} =: \lambda_2^*$. As a consequence two limits have the same sign. We may assume without loss of generality that they are positive. From our assumption for the multiplicity of each principal curvature and for the sectional curvature of M it follows that $c - \lambda_1(p_k)^2 \geq 0$ and $c - \lambda_2(p_k)^2 \geq 0$, and hence $0 < \lambda_1^*, \lambda_2^* \leq \sqrt{c}$. From what is supposed we see that $\lim_{k\to\infty} \lambda_1^*(p_k) = \lim_{k\to\infty} \lambda_2^*(p_k) = \sqrt{c}$, and hence $\lim_{k\to\infty}(\lambda_1 - \lambda_2)(p_k) = 0$. The same argument holds for λ_1 and λ_3. The iteration of this procedure completes the proof of assertion (2) in this case.

We now continue the proof of Theorem 7.2. Now the inequality

$$n\mathcal{S} - n^2H^2 = \frac{1}{2}\sum(\lambda_i - \lambda_j)^2 \geq 0$$

implies that $\mathcal{S}$ is bounded from below by nH^2, and from Assertion (2) we see that

$$\lim_{k\to\infty} \mathcal{S}(p_k) = nH^2 = \inf_M \mathcal{S}.$$

This and (7.15) imply that $\sup_M \mathcal{S} = \inf_M \mathcal{S} = nH^2$, and hence $\mathcal{S} \equiv nH^2$. Thus M is totally umbilical and the proof of Theorem 7.2 is complete.

Remark 7.3. For constants $c_1 < 0, c_2 > 0$ and $c > 0$ with $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$ and for $k = 1, \ldots, n-1$, a *hyperbolic cylinder* $\boldsymbol{H}^k(c_1) \times S^{n-k}(c_2)$ is introduced in [1] as follows:

$$\begin{aligned}&\boldsymbol{H}^k(c_1) \times S^{n-k}(c_2)\\ &= \{(x,y) \in S_1^{n+1}(c) \subset \boldsymbol{R}_1^{k+1} \times \boldsymbol{R}^{n-k+1}; |x|^2 = -\frac{1}{c_1}, |y|^2 = \frac{1}{c_2}\}.\end{aligned}$$

We see that $\boldsymbol{H}^1(c_1) \times S^{n-1}(c_2)$ has two principal curvatures with multiplicities $n-1$ and 1 and is not totally umbilical. Therefore the assumption on the principal curvatures in Theorem 7.2 cannot be omitted.

Remark 7.4. A hypersurface of a Minkowski space $\boldsymbol{R}_1^{n+1}$ is said to be *entire* if it is expressed as the graph of a function defined over the whole $\boldsymbol{R}^n$. Cheng and Yau proved in [21] that the Ricci curvature of every space-like entire hypersurface with constant mean curvature in $\boldsymbol{R}_1^{n+1}$ is nonpositive everywhere. Stumbles [46] and Treibergs [47] showed that there are many entire space-like hypersurfaces of $\boldsymbol{R}_1^{n+1}$ with constant mean curvature which are different from hyperboloids.

Remark 7.5. In [34] Milnor investigated surfaces of Minkowski 3-space on which the mean curvature H and Gaussian curvature K are linearly related. She showed that if a space-like surface in $\boldsymbol{R}_1^3$ has the property that $\alpha + \beta H + \gamma K = 0$ for some constants α, β, γ with $\beta^2 \neq 4\alpha\gamma$, then it is an equidistant surface from some surface whose mean curvature or Gaussian curvature is constant.

Now we shall study complete space-like hypersurfaces of nonnegative sectional curvature in a de Sitter space with the property that

$$r = kH,$$

where k is a nonnegative constant and r the scalar curvature of M.

Theorem 7.3. *(see [12]) Let M be a complete space-like hypersurface with nonnegative sectional curvature in the de Sitter space $S_1^{n+1}(c)$ of constant curvature $c > 0$. If H attains its maximum at some point on M and if*

$$r = kH,$$

then M is isometric to $\boldsymbol{R}^n$ or the sphere $S^n(c_1)$ of constant curvature $c_1 \in (0, c)$.

Theorem 7.4. *(see [12]) Let M be a complete space-like hypersurface with nonnegative sectional curvature in the de Sitter space $S_1^{n+1}(c)$ of constant curvature $c > 0$. If*

$$r = kH,$$

for a constant $k \geq 0$ and if the multiplicity of each principal curvature is greater than one at every point on M, then M is isometric to $\boldsymbol{R}^n$ or the sphere $S^n(c_1)$ of constant curvature $c_1 \in (0, c)$.

Two propositions are needed for the proof of the above theorems. For a smooth function f defined on M let $\Box f$ be the operator $\Box f := \sum(nH\delta_{ij} - h_{ij})f_{ij}$.

Proposition 7.5. *Let M be a space-like hypersurface with nonnegative sectional curvature in $\boldsymbol{S}_1^{n+1}(c)$. If for a positive constant k,*

$$r = kH$$

is satisfied, then the operator

$$L = \Box + \frac{k}{2n}\Delta \tag{7.21}$$

is elliptic. In particular we have $r > 0$.

Proof of Proposition 7.5. $r \geq 0$ follows from the assumption for the sectional curvature. By choosing as before an orthonormal local frame field around an arbitrary fixed point on M, we have

$$r = n(n-1)c - n^2H^2 + \sum \lambda_i^2, \tag{7.22}$$

$$\sum \lambda_i^2 = kH + n^2H^2 - n(n-1)c. \tag{7.23}$$

From above relations we derive $r > 0$. In fact, suppose that there is a point at which $r = 0$, then the assumption implies $H = 0$ (by $k > 0$). From (7.22) we see $\sum \lambda_i^2 + n(n-1)c = 0$. This is a contradiction, and hence $r > 0$ and $H > 0$.

Next, for every i we have

$$
\begin{aligned}
&(nH-\lambda_i+\frac{k}{2n})\\
&=\sum\lambda_j-\lambda_i+\frac{1}{2nH}[\sum\lambda_j^2-n^2H^2+n(n-1)c]\\
&=\frac{1}{nH}[(\sum\lambda_j)^2-\lambda_i\sum\lambda_j-\frac{1}{2}\sum_{m\neq j}\lambda_m\lambda_j+\frac{1}{2}n(n-1)c]\\
&=\frac{1}{nH}[\sum\lambda_j^2+\frac{1}{2}\sum_{m\neq j}\lambda_m\lambda_j-\lambda_i\sum\lambda_j+\frac{1}{2}n(n-1)c]\\
&=\frac{1}{nH}[\sum_{j\neq i}\lambda_j^2+\frac{1}{2}\sum_{i\neq m\neq j\neq i}\lambda_m\lambda_j+\frac{1}{2}n(n-1)c]\\
&=\frac{1}{2nH}[\sum_{j\neq i}\lambda_j^2+(\sum_{j\neq i}\lambda_j)^2+n(n-1)c]>0. \qquad (7.24)
\end{aligned}
$$

This proves Proposition 7.5.

Proposition 7.6. *Let M be a complete Riemannian manifold with nonnegative sectional curvature. Let f be a C^2-function on M which is bounded from above. Then there exists a sequence $\{p_m\}$ of points on M such that*

$$\lim_{m\to\infty} f(p_m)=\sup_M f,\quad \lim_{m\to\infty}|\nabla f(p_m)|=0, \qquad (7.25)$$

$$\lim\sup_{m\to\infty} Lf(p_m)\leq 0. \qquad (7.26)$$

Proof. Theorem O states that there exists a sequence $\{p_m\}$ of points on M satisfying (7.25) and $\limsup_{m\to\infty}\nabla_i\nabla_i f(p_m)\leq 0$ holds for every i. Setting

$$Lf=\sum_i b_i f_{ii},$$

we see that $b_i\geq 0$ is bounded, and hence $\{b_i(p_m)\}$ is convergent (by taking a subsequence if necessary). This proves (7.26).

Proof of Theorem 7.3. Since the sectional curvature of M is nonnegative, we have $r\geq 0$.

If $k=0$, then $r=0$, and hence M is flat and $h_{ij}=\sqrt{c}\delta_{ij}$.

If $k>0$, then L is elliptic by Proposition 7.5, and we have $r>0$, $H>0$ and

$$\frac{1}{2}n^2\Delta H^2=n^2|\nabla H|^2+n^2H\Delta H. \qquad (7.27)$$

On the other hand, from (1.10),(1.21) and (7.12) we get

$$
\begin{aligned}
\frac{1}{2}n^2\Delta H^2&=\frac{1}{2}\Delta\sum h_{ij}^2-\frac{1}{2}\Delta r \qquad (7.28)\\
&=\sum h_{ijk}^2+\sum h_{ij}\Delta h_{ij}-\frac{1}{2}\Delta r\\
&=\sum h_{ijk}^2+n\sum\lambda_i H_{ii}+\frac{1}{2}\sum(\lambda_i-\lambda_j)^2(c-\lambda_i\lambda_j)-\frac{1}{2}\Delta r,
\end{aligned}
$$

and

$$\begin{aligned} nLH &= n[\Box H + \frac{k}{2n}\Delta H] = n\Box H + \frac{1}{2}\Delta r \\ &= n^2 H\Delta H - n\sum \lambda_i H_{ii} + \frac{1}{2}\Delta r \\ &= \frac{1}{2}n^2\Delta H^2 - n^2|\nabla H|^2 - n\sum \lambda_i H_{ii} + \frac{1}{2}\Delta r \\ &= \sum h_{ijk}^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum(\lambda_i - \lambda_j)^2(c - \lambda_i\lambda_j). \end{aligned} \tag{7.29}$$

From (7.22) and $r = kH$ we have

$$k\nabla_i H = -2n^2 H\nabla_i H + 2\sum h_{kj}h_{kji}, \tag{7.30}$$

$$(\frac{k}{2} + n^2 H)^2|\nabla H|^2 \leq \sum h_{ij}^2 \cdot \sum h_{kji}^2. \tag{7.31}$$

Thus from (7.29) and (7.31) we have

$$\begin{aligned} n\sum h_{ij}^2 LH &\geq [(\frac{k}{2} + n^2 H)^2 - n^2\sum h_{ij}^2]|\nabla H|^2 + \frac{1}{2}\sum h_{ij}^2 \sum(\lambda_i - \lambda_j)^2(c - \lambda_i\lambda_j) \\ &= [\frac{k^2}{4} + n^3(n-1)c]|\nabla H|^2 + \frac{1}{2}\sum h_{ij}^2 \sum(\lambda_i - \lambda_j)^2(c - \lambda_i\lambda_j) \\ &\geq 0. \end{aligned} \tag{7.32}$$

Because L is elliptic and H attains its maximum, we conclude by (7.32) that H is constant. This proves Theorem 7.3.

Proof of Theorem 7.4. Since the sectional curvature of M is nonnegative, $c - \lambda_i\lambda_j \geq 0$ for every $i \neq j$. Since the multiplicity of the principal curvature at each point of M is greater than one, we see that $c - \lambda_i^2 \geq 0$ for all i, and hence both H and $\sum \lambda_i^2$ are bounded.

If $k = 0$, then $r = 0$ and M is flat.

If $k > 0$, then L is elliptic and (7.32) implies that

$$\lim_{m\to\infty} \sum(\lambda_i - \lambda_j)^2(c - \lambda_i\lambda_j)(p_m) = 0, \tag{7.33}$$

$$\lim_{m\to\infty} H(p_m) = \sup_M H. \tag{7.34}$$

By a similar technique as developed in the proof of Theorem 7.2, we have

$$\lim_{m\to\infty} (\lambda_i - \lambda_j)(p_m) = 0. \tag{7.35}$$

From (7.23) and (7.34) we obtain

$$\sup_M \sum h_{ij}^2 = k\sup_M H + n^2 \sup_M H^2 - n(n-1)c. \tag{7.36}$$

On the other hand we see

$$\frac{1}{2}(n\sum h_{ij}^2 - n^2H^2) = \sum(\lambda_i - \lambda_j)^2. \tag{7.37}$$

Therefore we have $\lim_{m\to\infty}(n\sum h_{ij}^2 - n^2H^2)(p_m) = 0$. Namely we see

$$\sup_M \sum h_{ij}^2 = n \sup_M H^2.$$

Thus (7.35) and (7.37) imply

$$\inf_M(\sum h_{ij}^2 - nH^2) = 0. \tag{7.38}$$

Since $\sum h_{ij}^2 - nH^2 = kH + n(n-1)H^2 - n(n-1)c$, we have from (7.38)

$$k \inf_M H + n(n-1)(\inf_M H)^2 - n(n-1)c = 0.$$

Thus we have

$$\inf_M H = \frac{-k + \sqrt{k^2 + 4n^2(n-1)^2c}}{2n(n-1)}.$$

It follows from (7.36) and $\sup \sum h_{ij}^2 = n \sup H^2$, we observe that

$$\sup_M H = \frac{-k + \sqrt{k^2 + 4n^2(n-1)^2c}}{2n(n-1)},$$

and hence

$$\inf_M H = \sup_M H.$$

Thus H is constant and Theorem 7.2 concludes the proof of Theorem 7.4.

§8. Complete space-like submanifolds in a de Sitter space

A complete space-like hypersurface of a Minkowski space R_1^{n+1} possesses a remarkable Bernstein property in the maximal case. Namely, the mean curvature of such a hypersurface is identically zero, for details see Calabi [7] and Cheng,S.Y-Yau [21]. As a study of the Bernstein property, Ishihara discussed in [29] complete space-like maximal submanifolds under certain conditions. It was proved that if c is nonnegative, then M is totally geodesic. The Bernstein type property was studied by Nishikawa in [37] from a different point of view. K.Milnor in [33] and Yamada in [48] gave a characterization of the hyperboloid cylinder in R_1^3. Recently, Akutagawa [2] and Ramanathan [44] investigated complete space-like hypersurfaces of a de Sitter space S_1^{n+1} with constant mean curvature. They proved the following

Theorem AR. *A complete space-like hypersurface of a de Sitter space $S_1^{n+1}(c)$ with constant mean curvature is totally umbilical if the following conditions are satisfied :*

$$(8.1) \qquad n^2H^2 \leq 4c \qquad \text{when } n = 2,$$

$$(8.2) \qquad n^2H^2 < 4(n-1)c \qquad \text{when } n \geq 3.$$

In this section we generalize a result obtained independently by Akutagawa and Ramanathan to higher codimensions. It was independently shown by Akutagawa and Ramanathan and show that the condition (8.1) is optimal. We observe from the following example that condition (8.2) is also optimal.

Let c_1, c_2 be constants such that

$$c_1 := (2-n)c, \qquad c_2 := \frac{n-2}{n-1} \cdot c$$

and consider a hyperbolic cylinder in $S_1^{n+1}(c)$ given by

$$\begin{aligned} &H^1(c_1) \times S^{n-1}(c_2) \\ &= \{(x,y) \in S_1^{n+1}(c) \subset R_1^{n+2} = R_1^2 \times R^n,\ |x|^2 = -\frac{1}{c_1},\ |y|^2 = \frac{1}{c_2}\} \end{aligned}$$

We see that the above is a complete space-like hypersurface of a de Sitter space $S_1^{n+1}(c)$ with constant mean curvature. This is not totally umbilical and satisfies $n^2H^2 = 4(n-1)c$ when $n \geq 3$.

We shall prove the following

Theorem 8.1. *(see [13]) Let M be a complete space-like submanifold of dimension n in $S_p^{n+p}(c)$ with parallel mean curvature vector. M is totally umbilical if (8.1) and (8.2) are satisfied.*

Proof. From (1.10) and (1.21) we get

(8.3)

$$\begin{aligned} \Delta h_{ij}^\alpha &= \sum h_{kkij}^\alpha + nch_{ij}^\alpha - c\sum h_{kk}^\alpha \delta_{ij} + \sum h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta \\ &- 2\sum h_{mk}^\alpha h_{mj}^\beta h_{ik}^\beta + \sum h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta - \sum h_{mi}^\alpha h_{mj}^\beta h_{kk}^\beta + \sum h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta. \end{aligned}$$

Since the mean curvature vector is parallel H is constant.

If $H = 0$, then we see from a result in [29] that M is totally geodesic.

It $H \neq 0$, then we choose e_{n+1} to be the mean curvature vector, e.g., $h = H \cdot e_{n+1}$. We then have

$$\omega_{\beta,n+1} = 0, \tag{8.4}$$

$$H^\alpha H^{n+1} = H^{n+1} H^\alpha, \tag{8.5}$$

$$\operatorname{trace} H^{n+1} = nH, \ \ \operatorname{trace} H^\alpha = 0, \ \ \alpha \neq n+1, \tag{8.6}$$

where H^α denotes the matrix (h^α_{ij}). By setting

$$\mu_{ij} = h^{n+1}_{ij} - H\delta_{ij}, \qquad \tau^\alpha_{ij} = h^\alpha_{ij}, \ \alpha \neq n+1 \tag{8.7}$$

we have

$$|\mu|^2 = \operatorname{trace}(\mu)^2 = \sum \mu^2_{ij} = \operatorname{trace}(H^{n+1})^2 - nH^2, \tag{8.8}$$

$$|\tau|^2 = \sum_{\beta \neq n+1} \operatorname{trace}(\tau^\beta)^2 = \sum_{\beta \neq n+1} (\tau^\beta_{ij})^2 = \sum_{\beta \neq n+1} (h^\beta_{ij})^2, \tag{8.9}$$

$$\operatorname{trace} \mu = 0, \ \ \operatorname{trace}(\tau^\beta) = 0, \ \ \beta \neq n+1, \tag{8.10}$$

$$\mathcal{S} = |\mu|^2 + |\tau|^2 + nH^2. \tag{8.11}$$

From the above relations we see that $|\mu|^2$ and $|\tau|^2$ are independent of the choice of local frame fields and globally defined scalars on M.

A submanifold is said to be *pseudo umbilical* if it is umbilical with respect to the mean curvature vector, i.e., $h^{n+1}_{ij} = H\delta_{ij}$ holds at every point on M. From (8.7) to (8.11) we see that M is pseudo umbilical if and only if $|\mu|^2 = 0$ and totally umbilical if and only if $|\tau|^2 = 0$. Making use of $H^{n+1}H^\beta = H^\beta H^{n+1}$, we see that

$$\begin{aligned} \Delta h^{n+1}_{ij} &= nch^{n+1}_{ij} - ncH\delta_{ij} + \sum h^{n+1}_{km} h^\beta_{mk} h^\beta_{ij} - 2\sum h^{n+1}_{km} h^\beta_{mj} h^\beta_{ki} \\ &\quad + \sum h^{n+1}_{mi} h^\beta_{mk} h^\beta_{kj} - nH \sum h^{n+1}_{mi} h^{n+1}_{mj} + \sum h^{n+1}_{jm} h^\beta_{mk} h^\beta_{ki}, \end{aligned} \tag{8.12}$$

$$\begin{aligned} \Delta h^\alpha_{ij} &= nch^\alpha_{ij} + \sum h^\alpha_{km} h^\beta_{mk} h^\beta_{ij} - 2\sum h^\alpha_{mk} h^\beta_{mj} h^\beta_{ik} \\ &\quad + \sum h^\alpha_{mi} h^\beta_{mk} h^\beta_{kj} - nH \sum h^\alpha_{mi} h^{n+1}_{mj} + \sum h^\alpha_{jm} h^\beta_{mk} h^\beta_{ki}, \ \alpha \neq n+1. \end{aligned} \tag{8.13}$$

$$\begin{aligned} \frac{1}{2}\Delta|\mu|^2 &= \sum (h^{n+1}_{ijk})^2 + nc\sum (h^{n+1}_{ij})^2 - n^2cH^2 + \sum h^{n+1}_{mk} h^\beta_{mk} h^\beta_{ij} h^{n+1}_{ij} \\ -2\sum h^{n+1}_{mk} h^\beta_{mj} h^\beta_{ik} h^{n+1}_{ij} &+ \sum h^{n+1}_{im} h^\beta_{mk} h^\beta_{kj} h^{n+1}_{ij} - nH \sum h^{n+1}_{mi} h^{n+1}_{mj} h^{n+1}_{ij} + \sum h^{n+1}_{jm} h^\beta_{mk} h^\beta_{ki} h^{n+1}_{ij} \\ &= \sum (h^{n+1}_{ijk})^2 + nc\sum (h^{n+1}_{ij})^2 - n^2cH^2 - nH\operatorname{trace}(H^{n+1})^3 \\ &\quad + \sum_{\beta \neq n+1} \operatorname{trace}(H^{n+1}H^\beta)^2 + [\operatorname{trace}(H^{n+1})^2]^2, \end{aligned} \tag{8.14}$$

and

$$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq n+1}(\tau^\alpha_{ijk})^2 + nc|\tau|^2 + \sum_{\alpha\neq n+1} h^\alpha_{km}h^\beta_{mk}h^\beta_{ij}h^\alpha_{ij}$$
$$-2\sum_{\alpha\neq n+1} h^\alpha_{km}h^\beta_{mj}h^\beta_{ik}h^\alpha_{ij} + \sum_{\alpha\neq n+1} h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj}h^\alpha_{ij}$$
$$-nH\sum_{\alpha\neq n+1} h^\alpha_{mi}h^\alpha_{ij}h^{n+1}_{mj} + \sum_{\alpha\neq n+1} h^\alpha_{jm}h^\beta_{mk}h^\beta_{ki}h^\alpha_{ij}. \tag{8.15}$$

On the other hand we have

$$\operatorname{trace}(H^{n+1})^3 = \operatorname{trace}\mu^3 + 3H[\operatorname{trace}(H^{n+1})^2 - nH^2] + nH^3. \tag{8.16}$$

It follows from (8.14) and (8.16) that

$$\frac{1}{2}\Delta|\mu|^2 \geq |\mu|^2[|\mu|^2 + nc - nH^2] - nH\operatorname{trace}(\mu)^3. \tag{8.17}$$

Since $\operatorname{trace}\mu = 0$, we apply Sublemma to the eigenvalues of μ to obtain

$$|\operatorname{trace}(\mu)^3| \leq (n-2)\{n(n-1)\}^{-\frac{1}{2}}|\mu|^3. \tag{8.18}$$

Therefore we have

$$\frac{1}{2}\Delta|\mu|^2 \geq |\mu|^2[|\mu|^2 - n|H|(n-2)\{n(n-1)\}^{-\frac{1}{2}}|\mu| + nc - nH^2]. \tag{8.19}$$

From (1.13) we see that the Ricci curvature of M is bounded from below. Setting

$$F := -(|\mu|^2 + a)^{-\frac{1}{2}},$$

we see that F is bounded. Since M is space-like we apply Theorem Y-1 to F and using the same technique as developed in the proof of Theorem 7.1, we obtain

$$\sup_M |\mu|^2[\sup_M |\mu|^2 - (n-2)\{n(n-1)\}^{-\frac{1}{2}}n|H|\cdot\sup_M|\mu| + nc - nH^2] \leq 0.$$

Now, in the case where $n=2$ and $H^2 \leq c$, the above inequality implies that $\sup_M|\mu|^2 = 0$, and hence M is pseudo umbilical. Thus the proof in this case is complete. We proceed the proof of Theorem 8.1 in the case where $n \geq 3$ and $n^2H^2 < 4(n-1)c$. We then have $|\mu|^2 \equiv 0$, and hence M is pseudo umbilical. From this property and (8.15) we have

$$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq n+1}(\tau^\alpha_{ijk})^2 + \sum_{\alpha,\beta\neq n+1}[\operatorname{trace}(H^\alpha H^\beta)]^2 + (nc - nH^2)|\tau|^2$$
$$-2\sum_{\alpha,\beta\neq n+1} h^\alpha_{mk}h^\beta_{mj}h^\beta_{ik}h^\alpha_{ij}$$
$$+\sum_{\alpha,\beta\neq n+1} h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj}h^\alpha_{ij} + \sum_{\alpha,\beta\neq n+1} h^\alpha_{jm}h^\beta_{mk}h^\beta_{ki}h^\alpha_{ij}. \tag{8.20}$$

Setting $\mathcal{S}_{\alpha,\beta} := \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta$ for $\alpha, \beta \neq n+1$, we observe that $(\mathcal{S}_{\alpha,\beta})$ is a symmetric $(p-1)\times(p-1)$-matrix, which can be diagonalized at an arbitrary fixed point by choosing a suitable normal frame $\{e_{n+2}, \dots, e_{n+p}\}$. By setting

$$\mathcal{S}_\alpha := \mathcal{S}_{\alpha,\alpha}$$

we have $|\tau|^2 = \sum_{\alpha\neq n+1} \mathcal{S}_\alpha$.

For a matrix $A = (a_{ij})$ we define $\mathcal{N}(A) := \mathrm{trace}(A^t A)$. We then get from (8.20)

$$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq n+1} (\tau_{ijk}^\alpha)^2 + (nc - nH^2)|\tau|^2 + \sum_{\alpha\neq n+1} \mathcal{S}_\alpha^2 + \sum_{\alpha,\beta\neq n+1} \mathcal{N}(H^\alpha H^\beta - H^\beta H^\alpha).$$

From Lemma 1 in [22] we get

$$\mathcal{N}(H^\alpha H^\beta - H^\beta H^\alpha) \geq 0.$$

Setting σ_1 and σ_2 by

$$(p-1)\sigma_1 = \sum_{\alpha\neq n+1} \mathcal{S}_\alpha = |\tau|^2, \quad \frac{(p-1)(p-2)\sigma_2}{2} = \sum_{\alpha<\beta,\alpha,\beta\neq n+1} \mathcal{S}_\alpha \mathcal{S}_\beta,$$

we have

$$\sum_{\alpha\neq n+1} \mathcal{S}_\alpha^2 = (p-1)\sigma_1^2 + (p-1)(p-2)(\sigma_1^2 - \sigma_2),$$

$$(p-1)^2(p-2)(\sigma_1^2 - \sigma_2) = \sum_{\alpha<\beta,\alpha,\beta\neq n+1} (\mathcal{S}_\alpha - \mathcal{S}_\beta)^2.$$

This implies that

$$\sum_{\alpha\neq n+1} \mathcal{S}_\alpha^2 = \frac{1}{p-1}|\tau|^4 + \frac{1}{p-1}\sum_{\alpha<\beta,\alpha,\beta\neq n+1} (\mathcal{S}_\alpha - \mathcal{S}_\beta)^2.$$

Thus we have

$$\frac{1}{2}\Delta|\tau|^2 \geq (nc - nH^2)|\tau|^2 + \frac{1}{p-1}|\tau|^4.$$

A similar argument as before implies that $|\tau|^2 \equiv 0$, and therefore M is totally umbilical. Thus the proof of Theorem 8.1 is complete.

Remark 8.1. As was pointed out by Akutagawa [2] and Ramanathan [44] the condition (8.1) in the case of $n = 2$ is optimal. We observe from the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ as stated in the previous section that the condition (8.2) is also best possible in the case of general dimensions.

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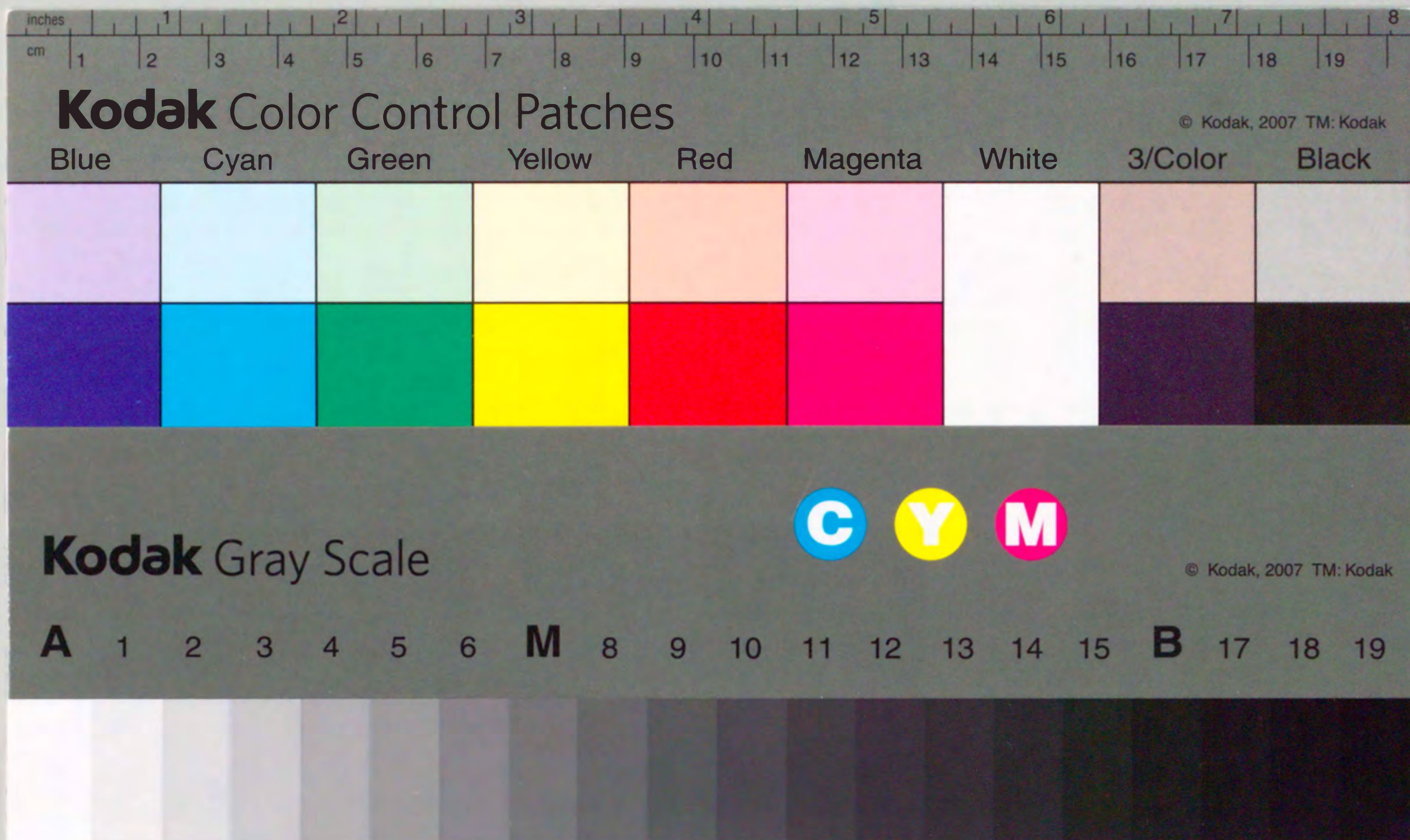
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