GENERATING MAPPING CLASS GROUPS OF SURFACES BY TORSION ELEMENTS

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概要

古典的な群論において、与えられた群に対して、生成系、あるいは有限位数の元のみからなる生成系を具体的に与える問題がある。写像類群に関しても古くからこの問題についての結果がある。

この論文では、有向曲面と非有向曲面の写像類群について、有限位数の元のみからなる生成集合を考える。

有向曲面の場合、Lanier (2018)は、$k \geq 6$、種数が$(k-1)^2+1$以上の時について、写像類群が位数kの元3つで生成されることを示している。また、彼は$k \geq 8$または$k=6$の時に、非負整数a, bについて種数が$ak+b(k-1)>0$に等しい時に位数kの元3つ、種数が$ak+1$ $(a \geq 1)$に等しい時に位数kの元4つで写像類群が生成されることを示した。本論文の最初の主結果は、位数6の元のみからなる写像類群の生成系を新しく構成し、彼の結果を$k=6$に限定した時に種数が7,8,9,13,14,19である場合について改善したことである。

非有向曲面の場合、Szepietowski (2004)が involutions(位数 2 の元)のみからなる点付き写像類群の生成系を構成したが、彼の生成系の個数は種数と点の個数に依存する。involutions のみからなる生成系で生成元の個数が種数や点の個数に依存しないようなものが構成できるかという問題が考えられる。点の個数が0の場合、Szepietowski (2006)は写像類群が4つの involutions で生成できることを示し、この問題に肯定的な解答を与えた。点の個数が1以上の場合、この問題に対する解答は知られていなかった。これに対して、本論文では点付きの写像類群が、種数が奇数かつ13以上の場合に8個、種数が偶数かつ14以上の場合に11個の involutions で生成できることを示し、肯定的な回答を与える。

GENERATING MAPPING CLASS GROUPS OF SURFACES BY TORSION ELEMENTS

KAZUYA YOSHIHARA

Abstract. Let $\Sigma_{g,n}$ (resp. $N_{g,n}$) denote the closed orientable (resp. non-orientable) surface of genus g with n punctures and let $\mathrm{Mod}(\Sigma_{g,n})$ (resp. $\mathrm{Mod}(N_{g,n})$) denote the mapping class group of $\Sigma_{g,n}$ (resp. $N_{g,n}$).

In this thesis, we consider finite generating sets for the mapping class groups $\mathrm{Mod}(\Sigma_{g,n})$ and $\mathrm{Mod}(N_{g,n})$ which consist of elements of finite order.

In the orientable case, Lanier proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by three elements of order k for $k \geq 6$ and $g \geq (k-1)^2+1$. For $k \geq 8$ or $k = 6$ and non-negative integers a and b, he also showed that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by three (resp. four) elements of order k if $g = ak + b(k-1)$ (resp. $g = ak + 1$ $(a \geq 1)$). In this thesis, we construct a new finite generating set for $\mathrm{Mod}(\Sigma_{g,0})$ which consits only of elements of order six. When we restict Lanier's theorem to $k = 6$, we improve his theorem for $g = 7, 8, 9, 13, 14$, and 19.

In the non-orientable case, Szepietowski showed that $\mathrm{Mod}(N_{g,n})$ is generated by finitely many involutions. The number of elements in his generating set depends linearly on g and n. In the case of $n = 0$, Szepietowski found an involution generating set in such a way that the number of its elements does not depend on g, showing that $\mathrm{Mod}(N_{g,0})$ is generated by four involutions. As our second main theorem of this thesis, for $n \geq 0$, we prove that $\mathrm{Mod}(N_{g,n})$ is generated by eight involutions if $g \geq 13$ is odd and by eleven involutions if $g \geq 14$ is even.

1. Introduction

For $n \geq 0$, let $\Sigma_{g,n}$ (resp. $N_{g,n}$) denote the closed connected orientable (resp. non-orientable) surface of genus g with arbitrarily chosen n distinct points which we call *punctures*. The *mapping class group* $\mathrm{Mod}(\Sigma_{g,n})$ (resp. $\mathrm{Mod}(N_{g,n})$) is the group of isotopy classes of orientation preserving diffeomorphisms (resp. diffeomorphisms) of $\Sigma_{g,n}$ (resp. $N_{g,n}$) which preserve the set of punctures. Denote by $\mathrm{PMod}(\Sigma_{g,n})$ (resp. $\mathrm{PMod}(N_{g,n})$) the subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ (resp. $\mathrm{Mod}(N_{g,n})$) consisting of the isotopy classes of diffeomorphisms which fix each puncture.

In the orientable case, Dehn [De] and Lickorish [Li1] first proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by Dehn twists. Lickorish [Li2] showed that certain $3g-1$ Dehn twists generate $\mathrm{Mod}(\Sigma_{g,0})$ for $g \geq 1$. This number was improved to be $2g+1$ by Humphries [Hu] for $g \geq 3$. Moreover, Humphries showed that $\mathrm{Mod}(\Sigma_{g,0})$ cannot be generated by $2g$ (or less) Dehn twists for any $g \geq 2$. Johnson [J] proved that $2g+1$ Dehn twists also generate $\mathrm{Mod}(\Sigma_{g,1})$. If we allow generators other than Dehn twists, then we can obtain smaller generating sets for $\mathrm{Mod}(\Sigma_{g,n})$. Lu [Lu] found a generating set of $\mathrm{Mod}(\Sigma_{g,0})$ which consists of three elements, where two of the generators are of finite order. For $n = 0, 1$, Wajnryb showed that the group $\mathrm{Mod}(\Sigma_{g,n})$ is generated by two elements, one of which has finite order [W2].

It has been extensively studied the problem of finding smaller sets of generators and torsion generators for finite groups and mapping class groups. The study of finding torsion generating sets for $\mathrm{Mod}(\Sigma_{g,n})$ was started by Maclachlan [Ma]. He proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by torsion elements and used this result to show that the moduli space of Riemann surfaces of genus g is simply connected as a topology space. Patterson [P] showed that $\mathrm{Mod}(\Sigma_{g,n})$ is generated by torsion elements for $g \geq 3$ and $n \geq 1$. Korkmaz [Ko2] showed that $\mathrm{Mod}(\Sigma_{g,n})$ is generated by two elements of order $4g+2$ for $g \geq 3$ and $n = 0, 1$. McCarthy and Papadopoulos [MP] proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by infinitely many conjugates of a certain involution. Luo [Luo] showed that $\mathrm{Mod}(\Sigma_{g,n})$ is generated by $12g + 1$ involutions for $g \geq 3, n \leq 1$. In his paper, Luo asked the following question: *Is there a unversal upper bound which is independent of g and n for the number of torsion elements necessary to generate* $\mathrm{Mod}(\Sigma_{g,n})$? Brendle and Farb [BF] gave a positive answer to Luo's question for $n = 0, 1$. They found a generating set for $\mathrm{Mod}(\Sigma_{g,0})$ which consists of six involutions. Moreover, they showed that $\mathrm{Mod}(\Sigma_{g,n})$ can be realized as a quotient of a Coxeter group on six generators. For every $n \geq 0$, Kassabov [Ka] proved that $\mathrm{Mod}(\Sigma_{g,n})$ is generated by four (resp. five or six) involutions if $g \geq 8$ (resp. if $g \geq 6$ or if $g \geq 4$). Monden [Mo1] proved that $\mathrm{Mod}(\Sigma_{g,n})$ is generated by four (resp. five) involutions if $g \geq 7$ (resp. if $g \geq 5$). He also showed the following theorem ([Mo2]).

Theorem 1.1 (Monden, 2011)**.** *For $g \geq 3$,* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order three and by four elements of order four.*

Recently, Lanier showed the following theorems ([La]).

Theorem 1.2 (Lanier, 2018)**.** *For $k \geq 6$ and $g \geq (k-1)^2 + 1$,* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order k. Also,* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order* 5 *when $g \geq 8$.*

Theorem 1.3 (Lanier, 2018)**.** (1) *Let $k \geq 5$ and let $g > 0$ be of the form $ak+b(k-1)$ with non-negative integer a and b or of the form $ak + 1$ with integer $a > 0$. Then* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order k.* (2) *Let $k \geq 8$ or $k = 6$ and let $g > 0$ be of the form $ak + b(k-1)$ with non-negative integer a and b. Then* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order k. If instead $k = 7$ and g is of the form $7+7a+6b$ with integer $a, b > 0$, then three elements of order* 7 *also suffice.*

In this paper, we first construct a generating set of $\mathrm{Mod}(\Sigma_{g,0})$ which consists of elements of order six. For $g = 7, 8, 9, 13, 14$, and 19, our generating set improves Lanier's theorem if $k = 6$.

Theorem 1.4. (1) *For $g \geq 7$,* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order six.* (2) *For $g = 5, 6$,* $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order six.*

The idea of proof is as follows: By using lantern relation, we write one of elements of Humphries's generator set as a product of elements of order six. And, we construct mapping classes of order six which map the simple closed curves corresponding to above element to simple closed curves corresponding another generator. Although the basic idea is similar to the cases of order two, three, and four, the consutructions for mapping classes of order six are more complicated. The presentations of $\mathrm{Mod}(\Sigma_{g,0})$ are given by Wajnryb ([W1]). But a presentations of this groups with only torsion generators are not known except Korkmaz's one. Since the generators in Korkmaz's presentation depend on g, it is not known such a presentation that generators are independent of g. Using Theorem 1.4 to Wajnryb's presentation, we expect to get such a presentaion.

In the non-orientable case, Lickorish [Li3] first proved that $\mathrm{Mod}(N_{g,0})$ is generated by Dehn twists and Y-homeomorphisms. Chillingworth [C] found a finite set of generators of this group. Korkmaz [Ko1] found finite generating sets for the groups $\mathrm{Mod}(N_{g,n})$ and $\mathrm{PMod}(N_{g,n})$. The number of Chillingworth's generators is improved to $g+1$ by Szepietowski [S2]. Hirose [Hi] proved that his generating set is the minimal generating set by Dehn twists and Y-homemorphisms. Szepietowski [S1] proved that $\mathrm{Mod}(N_{g,n})$ is generated by involutions. The cardinality of his generating set of involutions depends linearly on g and n. We can consider Luo's problem for $\mathrm{Mod}(N_{g,n})$: *Is there a unversal upper bound which is independent of g and n for the number of torsion elements necessary to generate* $\mathrm{Mod}(N_{g,n})$? In the case $n=0$, Szepietowski gave a positive answer and found four involutions which generate $\mathrm{Mod}(N_{g,0})$ for $g \geq 4$ [S3]. But, in the case $n \neq 0$, it is not known. We will gave a positive answer for this problem.

Theorem 1.5. *Let n be a non-negative integer. Then, for g odd with $g \geq 13$, $\mathrm{Mod}(N_{g,n})$ is generated by eight involutions. For g even with $g \geq 14$, $\mathrm{Mod}(N_{g,n})$ is generated by eleven involutions.*

The idea of proof is as follows: First, we consider Korkmaz's generating set for $\mathrm{PMod}(N_{g,n})$ which consists of Dehn twists, Y-homeomorphism, and puncture slides. We write one of Dehn twists, one of puncture slides, and Y-homeomorphism as products of involutions which are allowed permutation of punctures. Next, we construct involutions to map simple closed curves corresponding to above Dehn twist and puncture slide to simple closed curves corresponding other Dehn twist and other puncture slide in Korkmaz's generating set, respectively. Then, a subgroup G generated by these involutions includes $\mathrm{PMod}(N_{g,n})$. There is a surjection from $\mathrm{Mod}(N_{g,n})$ to a symmetric group on n letters by an action of $\mathrm{Mod}(N_{g,n})$ on n puctures. We note that we construct involutions as in which a restriction this surjection to G is also surjection onto the symmetric group. It is well known that the abelianization of $\mathrm{Mod}(N_{g,n})$ is isomorphic to $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ for $g \geq 7$. By Theorem 1.5, a minimal number of involutions which need to generate $\mathrm{Mod}(N_{g,n})$ is three or more and eight (resp. eleven) or less if g is odd (resp. even). Presentations of $\mathrm{Mod}(N_{g,n})$ are given by Szepietowski, Omori, Paris-Szepietowski, and Stukow ([S4],[O],[PS],[St2]). But a presentations of $\mathrm{Mod}(N_{g,n})$ with only torsion generators are not known. Theorem 1.5 is one of the approaches for obtaining such presentations. As a Corollary of Theorem 1.5, there is a surjection from the Coxeter group with 8 or 11 generators onto $\mathrm{Mod}(N_{g,n})$ for $g \geq 13$, $n \geq 0$. If this kernel is finitely generated, we can get a presentation of $\mathrm{Mod}(N_{g,n})$ with generating set which only consist of involutions. As a Corollary of Theorem 1.5, a Dehn twist along a non-separating simple closed curve, a Y-homeomorphism, and a puncture slide are products of two involutions. Generally, we have the question of whether there is a number C such that every element in $\mathrm{Mod}(N_{g,n})$ can be written as a product of at most C involutions. But this is not known.

The paper is organized as follows. In Section 2 we recall the properties of Dehn twists, Y-homeomorphisms and puncture slides. In Section 3 in order to prove the Theorem 1.4, we construct elements of order six and show a single Dehn twist is written as a product of elements of order six. In Section 4 we construct involutions of $\mathrm{Mod}(N_{g,n})$ and prove the theorem 1.5. Finally, in Section 5, We note that Theorem 1.5 implies that $\mathrm{Mod}(N_{g,n})$ is the quotient of 8 or 11 generator Coxeter groups. And we consider some problems for Theorem 1.5.

2. Preliminaries

2.1. **Orientable surfaces.**

Let $\Sigma_{g,n}$ denote a closed oriented surface of genus g with n punctures. The set of orientation preserving diffeomorphisms of $\Sigma_{g,n}$ which preserve the set of punctures obviously forms a group, which we denote by $\mathrm{Diff}^+(\Sigma_{g,n})$. Let $\mathrm{Diff}_0^+(\Sigma_{g,n})$ be the subset consisting of all elements of $\mathrm{Diff}^+(\Sigma_{g,n})$ that are isotopic to the identity, where the isotopies fix *punctures*. It is immediately seen that $\mathrm{Diff}_0^+(\Sigma_{g,n})$ is a normal subgroup of $\mathrm{Diff}^+(\Sigma_{g,n})$. The mapping class group of $\Sigma_{g,n}$, denoted by $\mathrm{Mod}(\Sigma_{g,n})$, is the quotient group $\mathrm{Diff}^+(\Sigma_{g,n})/\mathrm{Diff}_0^+(\Sigma_{g,n})$. Usually we identify a diffeomorphism with its isotopy class. We assign the orientation of $\Sigma_{g,n}$ as in Fig. 1. For a simple closed curve a on $\Sigma_{g,n}$, the *right handed Dehn twist* t_a along a is the isotopy class of the diffeomorphism obtained by cutting $\Sigma_{g,n}$ along a, twisting one of the sides by 2π to the right and gluing the two sides of a back to each other (see Fig. 1). We recall the following lemmas and theorems. These are well known (see [FM]).

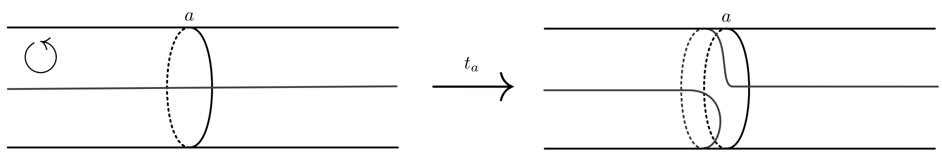

Figure 1. Dehn twist along a simple closed curve a

Lemma 2.1. *Let a be a simple closed curve on $\Sigma_{g,n}$ and let f be any element in $\mathrm{Mod}(\Sigma_{g,n})$. Then we have*
$$f t_a f^{-1} = t_{f(a)}.$$

Lemma 2.2. *Let a and b be simple closed curves on $\Sigma_{g,n}$.*

(1) *If a is disjoint from b, then we have*
$$t_a t_b = t_b t_a.$$

(2) *If a and b intersect transversely at one point, then we have*
$$t_a t_b t_a = t_b t_a t_b.$$

Lemma 2.3 (lantern relation)**.** *Let S be a four-holed sphere and x_1, x_2, x_3, y_1, y_2, y_3 and y_4 be simple closed curves in S as shown in Fig. 2. Then we have*
$$t_{x_1} t_{x_2} t_{x_3} = t_{y_1} t_{y_2} t_{y_3} t_{y_4}.$$

Lantern relation was discovered by Dehn, and later by Johnson. We say that an ordered set $c_1, c_2, \ldots, c_n$ of simple closed curves on Σ_g forms an *n-chain* if c_i and c_{i+1} intersect transversely at one point for $i = 1, 2, \ldots, n-1$ and c_i is disjoint from c_j if $\mid i - j \mid \geq 2$.

Lemma 2.4 (chain relation)**.** *Let $c_1, c_2, \ldots, c_n$ be an n-chain. For n odd, we have*
$$(t_{c_1} t_{c_2} \ldots t_{c_n})^{n+1} = t_{d_1} t_{d_2},$$
and for n even, we have
$$(t_{c_1} t_{c_2} \ldots t_{c_n})^{2n+2} = t_d,$$

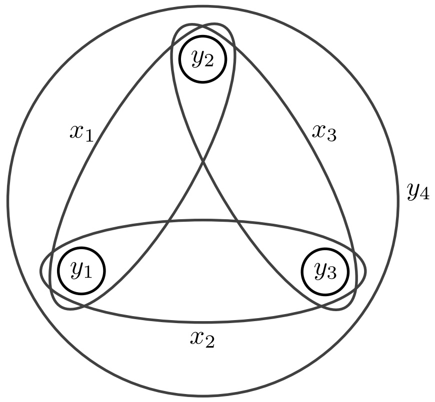

FIGURE 2. Simple closed curves x_1, x_2, x_3, y_1, y_2, y_3 and y_4 on four-holed sphere

where d_1 and d_2 (resp. d) are the boundary components of the regular neighborhood of this n-chain if n is odd (resp. even).

For $i = 1, 2, \ldots, g$ and $j = 1, 2, \ldots, g-1$, a_i, b_i and c_j are simple closed curves on $\Sigma_{g,0}$ as in Fig. 3.

Lickorish proved the following theorem.

Theorem 2.5. *For $g \geq 3$, $\mathrm{Mod}(\Sigma_{g,0})$ is generated by $3g-1$ Dehn twists t_{a_1}, t_{a_2}, $\ldots$, t_{a_g}, t_{r_1}, t_{r_2}, $\ldots$, $t_{r_{g-1}}$, t_{b_1}, t_{b_2}, $\ldots$, t_{b_g}.*

Humphries reduced Lickorish's system of generators for $\mathrm{Mod}(\Sigma_{g,0})$ as follows.

Theorem 2.6. *For $g \geq 3$, $\mathrm{Mod}(\Sigma_{g,0})$ is generated by $2g+1$ Dehn twists t_{a_1}, t_{a_2}, t_{r_1}, t_{r_2}, $\ldots$, $t_{r_{g-1}}$, t_{b_1}, t_{b_2}, $\ldots$, t_{b_g}.*

We call the curves $a_1, a_2, r_1, r_2, \ldots, r_{g-1}, b_1, b_2, \ldots, b_g$ *Humphries's curves.*

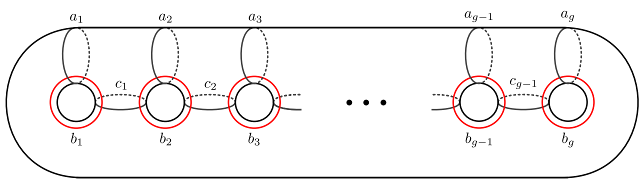

FIGURE 3. Simple closed curves $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, and $c_1, \ldots, c_{g-1}$

2.2. **Non-orientable surfaces.**

Let $N_{g,n}$ be the closed non-orientable surface of genus g with n punctures and let Δ be the set of punctures of $N_{g,n}$. We represent the surface $N_{g,n}$ as a connected sum of an orientable surface and one or two projective planes (one for g odd and two for g even). In Figs. 4 and 5, each encircled cross mark represents a crosscap: the interior of the encircled disk is to be removed and each pair of antipodal points on the boundary are to be identified.

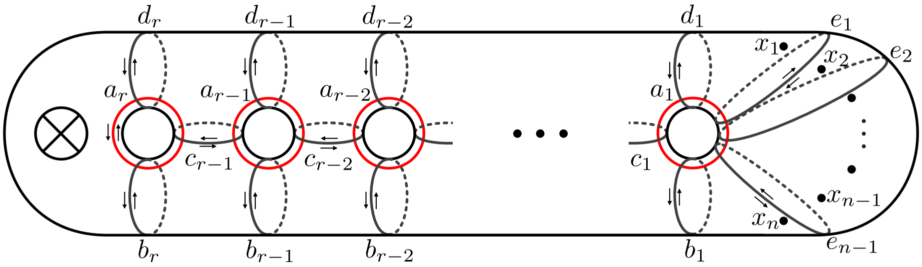

FIGURE 4. Surface $N_{g,n}$ for $g = 2r + 1$ and its simple closed curves

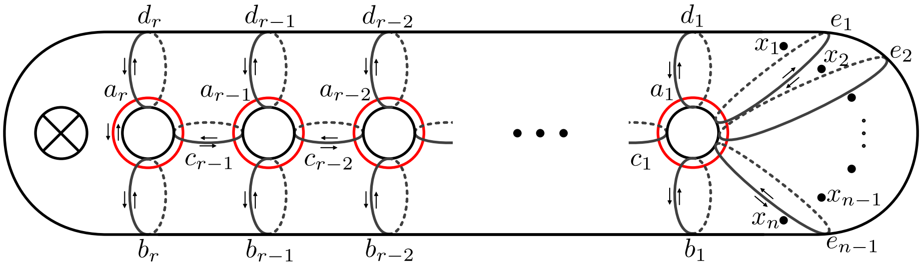

FIGURE 5. Surface $N_{g,n}$ for $g = 2r + 2$ and its simple closed curves

The set of all diffeomorphisms of $N_{g,n}$ which preserve the set of punctures obviously forms a group, which we denote by $\mathrm{Diff}(N_{g,n})$. Let $\mathrm{Diff}_0(N_{g,n})$ be the subset consisting of all elements of $\mathrm{Diff}(N_{g,n})$ that are isotopic to the identity, where the isotopies fix Δ. It is immediately seen that $\mathrm{Diff}_0(N_{g,n})$ is a normal subgroup of $\mathrm{Diff}(N_{g,n})$. The mapping class group of $N_{g,n}$, denoted by $\mathrm{Mod}(N_{g,n})$, is the quotient group $\mathrm{Diff}(N_{g,n})/\mathrm{Diff}_0(N_{g,n})$. We denote by $\mathrm{PMod}(N_{g,n})$ the subgroup of $\mathrm{Mod}(N_{g,n})$ consisting of the isotopy classes of diffeomorphisms which fix each puncture. Let Sym_n be a symmetric group on n letters. Clearly we have the exact sequence

$$1 \to \mathrm{PMod}(N_{g,n}) \to \mathrm{Mod}(N_{g,n}) \xrightarrow{\pi} Sym_n \to 1,$$

where the last projection is given by the restriction of homeomorphism to its action on the puncture points. Let c be a simple closed curve on $N_{g,n}$. If the regular neighborhood of c, denoted by N_c, is an annulus (resp. a Möbius band), we call c *two-sided* (resp. *one-sided*) simple closed curve. Let a be a two-sided simple closed curve on $N_{g,n}$. By the definition, the regular neighborhood of a is an annulus, and it has two possible orientation. Now, we fix one of its two possible orientations. For two sided simple closed curve a, we can also define the Dehn twist t_a.

It is well known that $\mathrm{Mod}(N_{g,n})$ is not generated by Dehn twists. We need another class of diffeomorphisms, called Y-homeomorphism, to generate $\mathrm{Mod}(N_{g,n})$. A Y-homeomorphism is defined as follow. For a one-sided simple closed curve m and a two-sided oriented simple closed curve a which intersects m transversely in one point, the regular neighborhood K of $m \cup a$ is homomeomorphic to the Klein bottle with one hole. Let M be the regular neighborhood of m. Then the *Y-homeomorphism* $Y_{m,a}$ is the isotopy class of the diffeomorphism obtained by pushing M once along a keeping the boundary of K fixed (see Fig. 6).

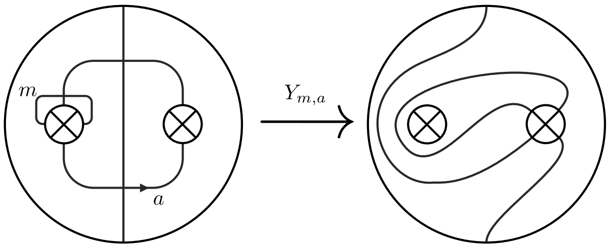

FIGURE 6. Y-homeomorphism on K

Furthermore, to generate the groups $\mathrm{Mod}(N_{g,n})$ and $\mathrm{PMod}(N_{g,n})$ we need a puncture slide. A puncuter slide is defined as follow. Let M denote a Möbius band with a puncture x embedded in $N_{g,n}$. For a one-sided simple closed curve α based at x on M, we push the puncture x once along α keeping the boundary of M fixed. Then a *puncture slide* on M is described as the result.

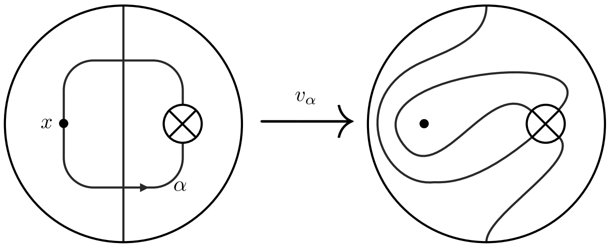

FIGURE 7. Puncture slide on M

These diffeomorphisms have the following properties.

Lemma 2.7. *For any diffeomorphism f of the surface $N_{g,n}$ and a two-sided simple closed curve a, we have*

$$t^{\epsilon}_{f(a)} = f t_a f^{-1},$$

where if $f\mid_{N_a}$ is an orientation preserving diffeomorphism (resp. orientation reversing diffeomorphism), then $\epsilon = 1$ (resp. $\epsilon = -1$).

Lemma 2.8. *For a one-sided simple closed curve m and a two-sided simple closed curve a, we have the following.*
(1) $Y_{m^{-1},a} = Y_{m,a}$.
(2) $Y_{m,a^{-1}} = Y^{-1}_{m,a}$.
(3) *For any element f in $\mathrm{Mod}(N_{g,n})$, we have $fY_{m,a}f^{-1} = Y_{f(m),f(a)}$.*

Lemma 2.9. *Let v be a puncture slide of x along a one-sided simple closed curve α.*
For any element f in $\mathrm{Mod}(N_{g,n})$, fvf^{-1} is the puncture slide of $f(x)$ along $f(\alpha)$.

3. Proof of Theorem 1.4

In this section, we prove that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by elements of order six. Let m be a positive integer.

3.1. **Construction of elements of order six.**

We construct two elements of order six.

3.1.1. *Case of $g = 5m$ for some integer $m \geq 2$.*

We construct an element f_1 in $\mathrm{Mod}(\Sigma_{g,0})$ which has order six as follows. We cut the surface $\Sigma_{g,0}$ along the curves a_3, c_1, c_2, ϵ_1, c_4, c_5, a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1} ($i = 2, 3, \ldots, m-1$) as shown in Fig. 8 and obtain $m-1$ surfaces $L_{1,1}$, $L_{1,2}$, $\ldots, L_{1,m-1}$. The surface $L_{1,1}$ is a surface of genus 4 with $6m$ boundary components, $L_{1,i}$ is a sphere with 6 boundary components bounded by a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i} and a_{5i+1} ($i = 2, 3, \ldots, m-1$). Let $L'_{1,1}$ be a subsurface of genus 4 in $L_{1,1}$ bounded by δ_{g-1}. Let $f_{1,1}$, $f_{1,2}$, $\ldots$, $f_{1,m-1}$ be the $\pi/3$ rotation as shown in Fig. 9. Note that in this picture δ_{g-4} is on the back side and the map $f_{1,1}$ keeps the subsurface $L'_{1,1}$ fixed. We found that $(f_{1,1})^6$ produces a twsit $t_{\delta_{g-4}}$. In order to cancel the twist $t_{\delta_{g-4}}$, we define $f'_{1,1}$ as a composition of $f_{1,1}$ and $f_{1,m}$ which defined as follow.

$$f_{1,m} = (t_{a_{g-3}} t_{b_{g-3}} t_{c_{g-3}} t_{b_{g-2}} t_{a'_{g-2}})^{-1} (t_{a_{g-1}} t_{b_{g-1}} t_{c_{g-1}} t_{b_g} t_{a_g}).$$

Since the diffeomorphisms $f'_{1,1}$, $f_{1,2}$, $\ldots$, $f_{1,m-1}$ coincide on the boundaries, they define a diffeomorphism $f_1 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

We construct an element h_1 in $\mathrm{Mod}(\Sigma_{g,0})$ of order six. We cut the surface $\Sigma_{g,0}$ along the curves a_1, a_2, c_2, c_3, ϵ_2, ϵ_3, a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} ($i = 2, 3, \ldots, m$) as shown in Fig. 10 and obtain $m+1$ surfaces $M_{1,1}$, $M_{1,2}$, $\ldots$, $M_{1,m+1}$. The surface $M_{1,1}$ is a surface with $6m$ boundary components, $M_{1,i}$ is a sphere with 6 boundary components bounded by a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} ($i = 2, 3, \ldots, m$) and $M_{1,m+1}$ is a sphere with 6 boundary components bounded by a_1, a_2, c_2, c_3, ϵ_2, ϵ_3. Let $h_{1,1}$, $h_{1,2}$, $\ldots$, $h_{1,m+1}$ be $\pi/3$ rotation as shown in Fig. 11.

Since the diffeomorphisms $h_{1,1}$, $h_{1,2}$, $\ldots$, $h_{1,m+1}$ coincide on the boundaries, they define a diffeomorphism $h_1 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

The diffeomorphism f_1 acts on the curves on $\Sigma_{g,0}$ as follows:

$(f_1)^5(a_3) = (f_1)^4(c_5) = (f_1)^3(c_1) = (f_1)^2(c_4) = (f_1)(c_2) = \epsilon_1$,

$(f_1)^5(a_{5i-3}) = (f_1)^4(c_{5i-3}) = (f_1)^3(c_{5i-2}) = (f_1)^2(c_{5i-1}) = (f_1)(c_{5i}) = a_{5i+1}$,

$(f_1)^4(b_{5i-3}) = (f_1)^3(b_{5i-2}) = (f_1)^2(b_{5i-1}) = (f_1)(b_{5i}) = b_{5i+1}$ $(i = 2, 3, \ldots, m-1)$,

$(f_1)^4(a_{g-1}) = (f_1)^3(b_{g-1}) = (f_1)^2(c_{g-1}) = (f_1)(b_g) = a_g$.

The diffeomorphism h_1 acts on the curves on $\Sigma_{g,0}$ as follows:

$(h_1)^5(a_1) = (h_1)^2(c_3) = (h_1)(c_2) = a_2$,

$(h_1)^4(b_1) = (h_1)^3(b_g) = (h_1)^2(b_4) = (h_1)(b_3) = b_2$,

$(h_1)^5(a_{5i-5}) = (h_1)^4(c_{5i-5}) = (h_1)^3(c_{5i-4}) = (h_1)^2(c_{5i-3}) = (h_1)(c_{5i-2}) = a_{5i-1}$,

$(h_1)^4(b_{5i-5}) = (h_1)^3(b_{5i-4}) = (h_1)^2(b_{5i-3}) = (h_1)(b_{5i-2}) = b_{5i-1}$ $(i = 2, 3, \ldots, m)$.

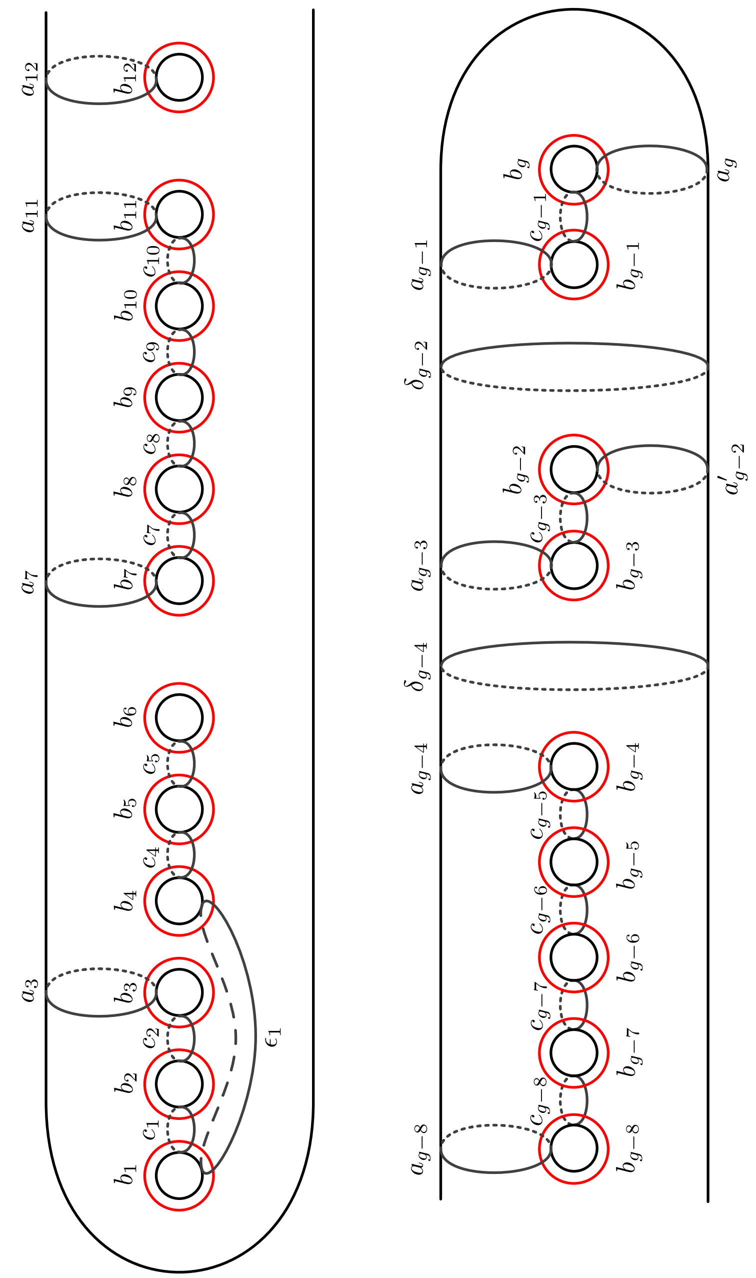

FIGURE 8. Cutting the surface I

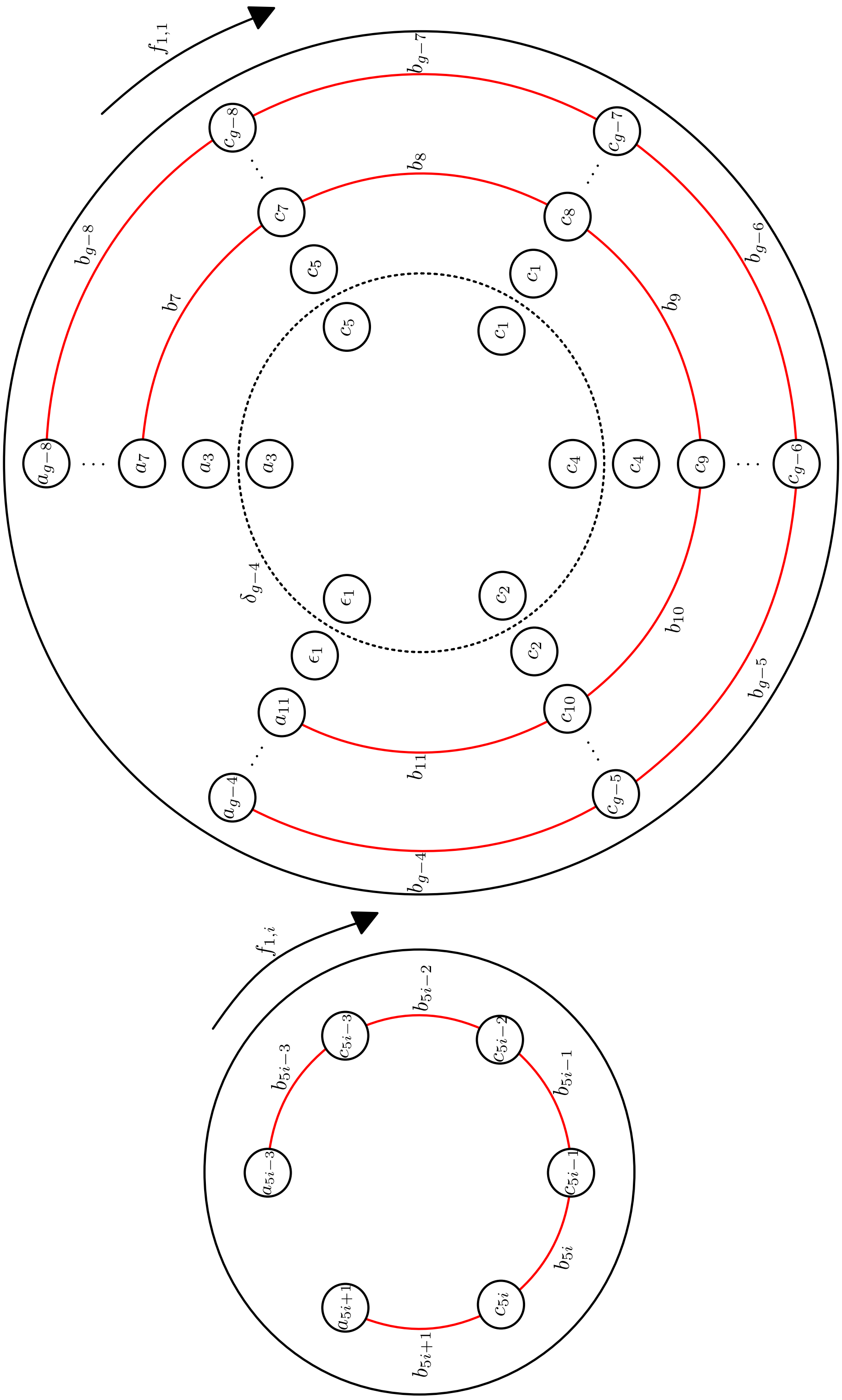

FIGURE 9. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$ I

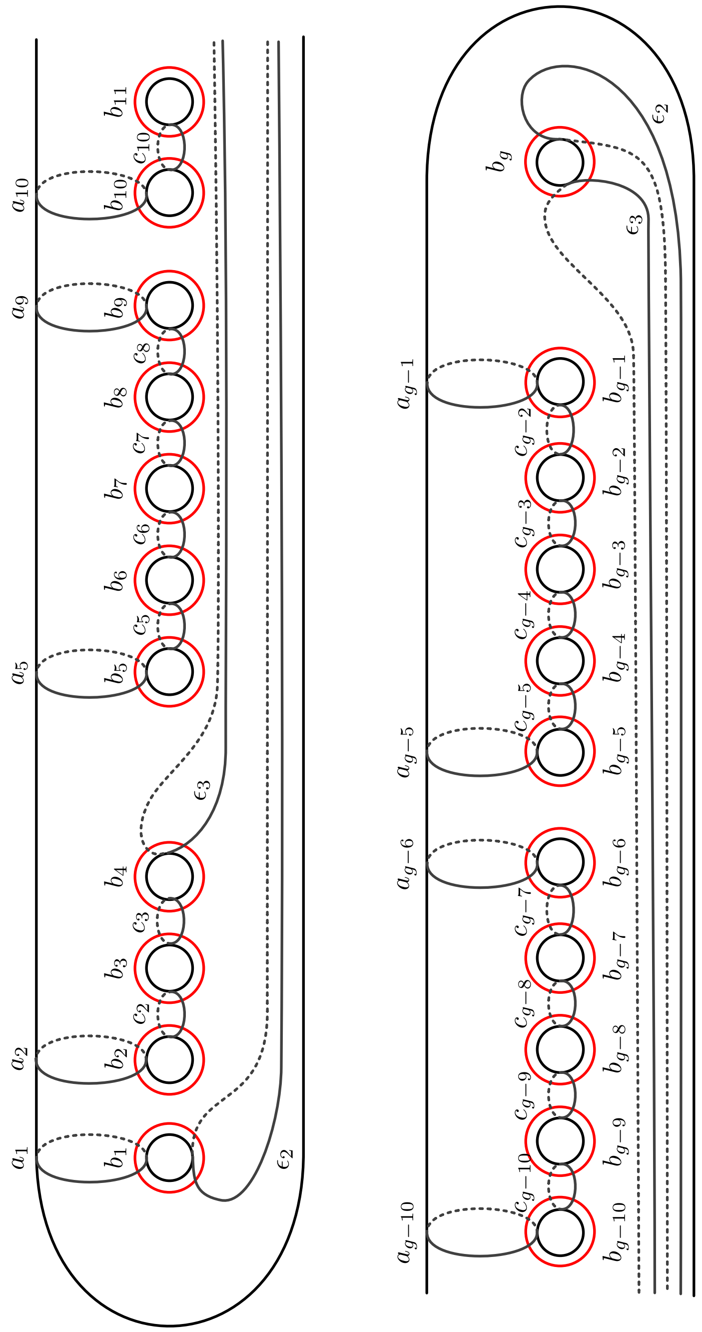

FIGURE 10. Cutting the surface II

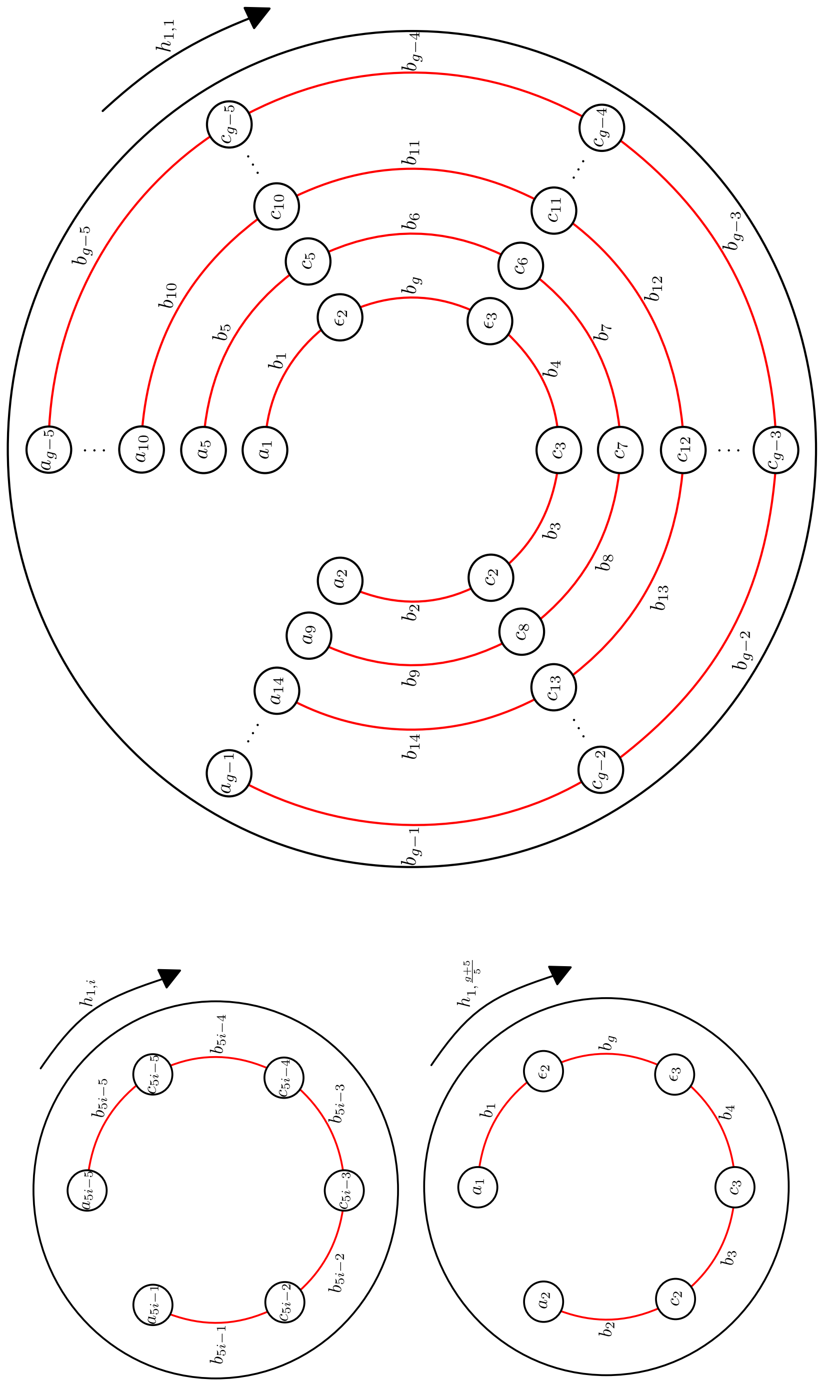

FIGURE 11. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$ II

3.1.2. *Case of $g = 5m + 1$ for some integer $m \geq 2$.*

We construct an element f^2 in $\mathrm{Mod}(\Sigma_{g,0})$ which has order six as follows. We cut the surface $\Sigma_{g,0}$ along the curves a_3, c_1, c_2, ϵ_1, c_4, c_5, a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1} $(i = 2, 3, \dots, m)$ as shown in Fig. 12 and obtain m surfaces $L_{2,1}$, $L_{2,2}$, $\dots, L_{2,m}$. The surface $L_{2,1}$ is a surface with $6m+6$ boundary components, $L_{2,i}$ is a sphere with 6 boundary components bounded by a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i} and a_{5i+1} $(i = 2, 3, \dots, m)$. Let $f_{2,1}$, $f_{2,2}$, $\dots$, $f_{2,m}$ be $\pi/3$ rotation as shown in Fig. 13.

Since the diffeomorphisms $f_{2,1}$, $f_{2,2}$, $\dots$, $f_{2,m}$ coincide on the boundaries, they define a diffeomorphism $f_2 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

We construct an element h_2 in $\mathrm{Mod}(\Sigma_{g,0})$ of order six. We cut the surface $\Sigma_{g,0}$ along the curves a_1, a_2, c_2, c_3, ϵ_4, ϵ_5, a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} $(i = 2, 3, \dots, m)$ as shown in Fig. 14 and obtain $m+1$ surfaces $M_{2,1}$, $M_{2,2}$, $\dots, M_{2,m+1}$. The surface $M_{2,1}$ is a torus with $6m$ boundary components, $M_{2,i}$ is a sphere with 6 boundary components bounded by a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} $(i = 2, 3, \dots, m)$, $M_{2,m+1}$ is a sphere with 6 boundary components bounded by a_1, a_2, c_2, c_3, ϵ_4, ϵ_5. Let $M'_{2,1}$ be a subsurface of genus 1 in the surface $M_{2,1}$ bounded by δ_{g-1}. Let $h_{2,1}$, $h_{2,2}$, $\dots$, $h_{2,m}$ be $\pi/3$ rotation as shown in Fig. 15. Note that in this picture δ_{g-1} is on the back side and the map $h_{2,1}$ keeps $M'_{2,1}$ fixed. We found that $(h_{2,1})^6$ produces a twist $t_{\delta_{g-1}}$. In order to cancel the twist $t_{\delta_{g-1}}$, we define $h'_{2,1}$ as a composition of $h_{2,1}$ and $h_{2,m+2}$ which defined as follow.

$$h_{2,m+2} = (t_{a_g} t_{b_g})^{-1}.$$

Since the diffeomorphisms $h'_{2,1}$, $h_{2,2}$, $\dots$, $h_{2,m}$ coincide on the boundaries, they define a diffeomorphism $h_2 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

For $i = 2, 3, \dots, m$, f_2 acts on the curves on $\Sigma_{g,0}$ as follows:

$$(f_2)^5(a_3) = (f_2)^4(c_5) = (f_2)^3(c_1) = (f_2)^2(c_4) = (f_2)(c_2) = \epsilon_1,$$
$$(f_2)^5(a_{5i-3}) = (f_2)^4(c_{5i-3}) = (f_2)^3(c_{5i-2}) = (f_2)^2(c_{5i-1}) = (f_2)(c_{5i}) = a_{5i+1},$$
$$(f_2)^4(b_{5i-3}) = (f_2)^3(b_{5i-2}) = (f_2)^2(b_{5i-1}) = (f_2)(b_{5i}) = b_{5i+1}.$$

For $i = 2, 3, \dots, m$, h_2 acts on the curves on $\Sigma_{g,0}$ as follows:

$$(h_2)^5(a_1) = (h_2)^2(c_3) = (h_2)(c_2) = a_2,$$
$$(h_2)^4(b_1) = (h_2)^3(b_{g-1}) = (h_2)^2(b_4) = (h_2)(b_3) = b_2,$$
$$(h_2)^5(a_{5i-5}) = (h_2)^4(c_{5i-5}) = (h_2)^3(c_{5i-4}) = (h_2)^2(c_{5i-3}) = (h_2)(c_{5i-2}) = a_{5i-1},$$
$$(h_2)^4(b_{5i-5}) = (h_2)^3(b_{5i-4}) = (h_2)^2(b_{5i-3}) = (h_2)(b_{5i-2}) = b_{5i-1}$$
$$h_2(b_g) = a_g.$$

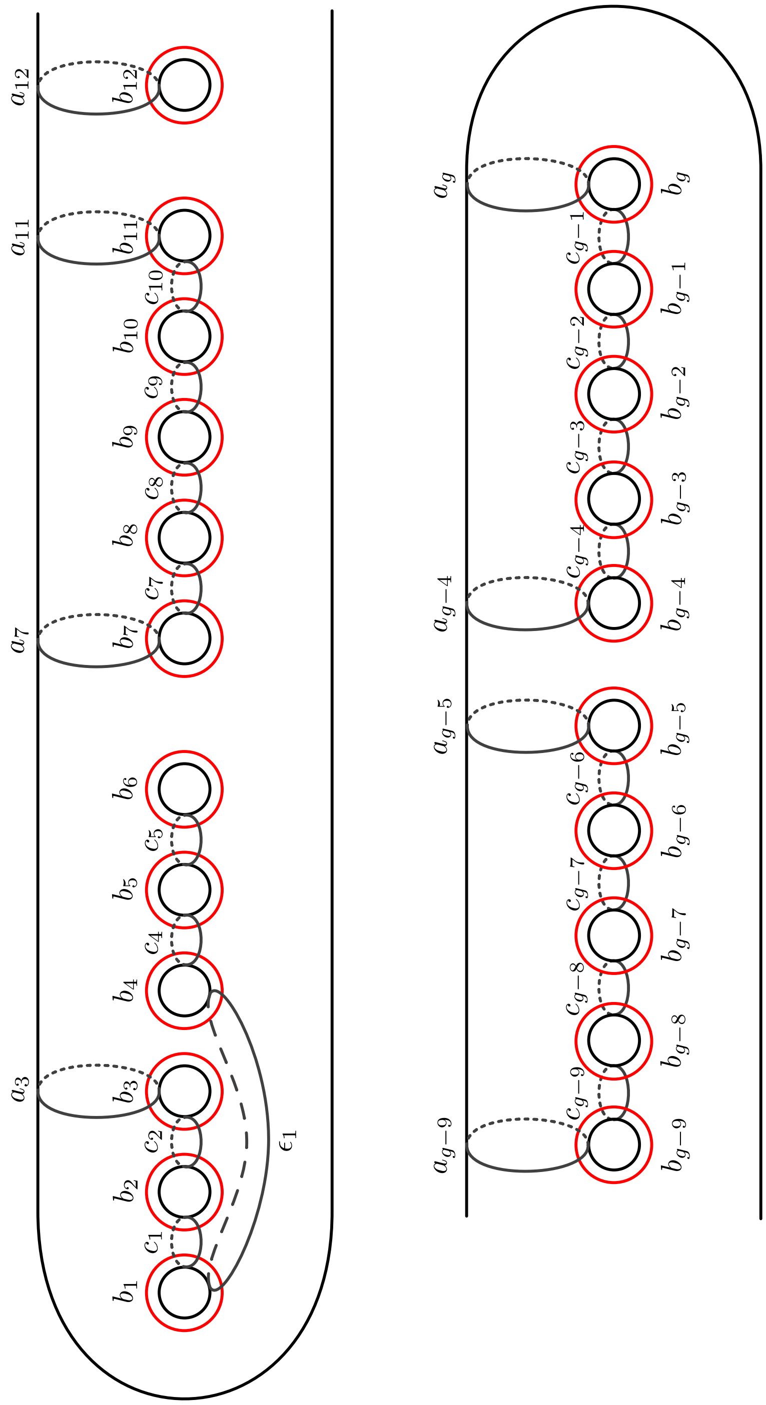

FIGURE 12. Cutting the surface III

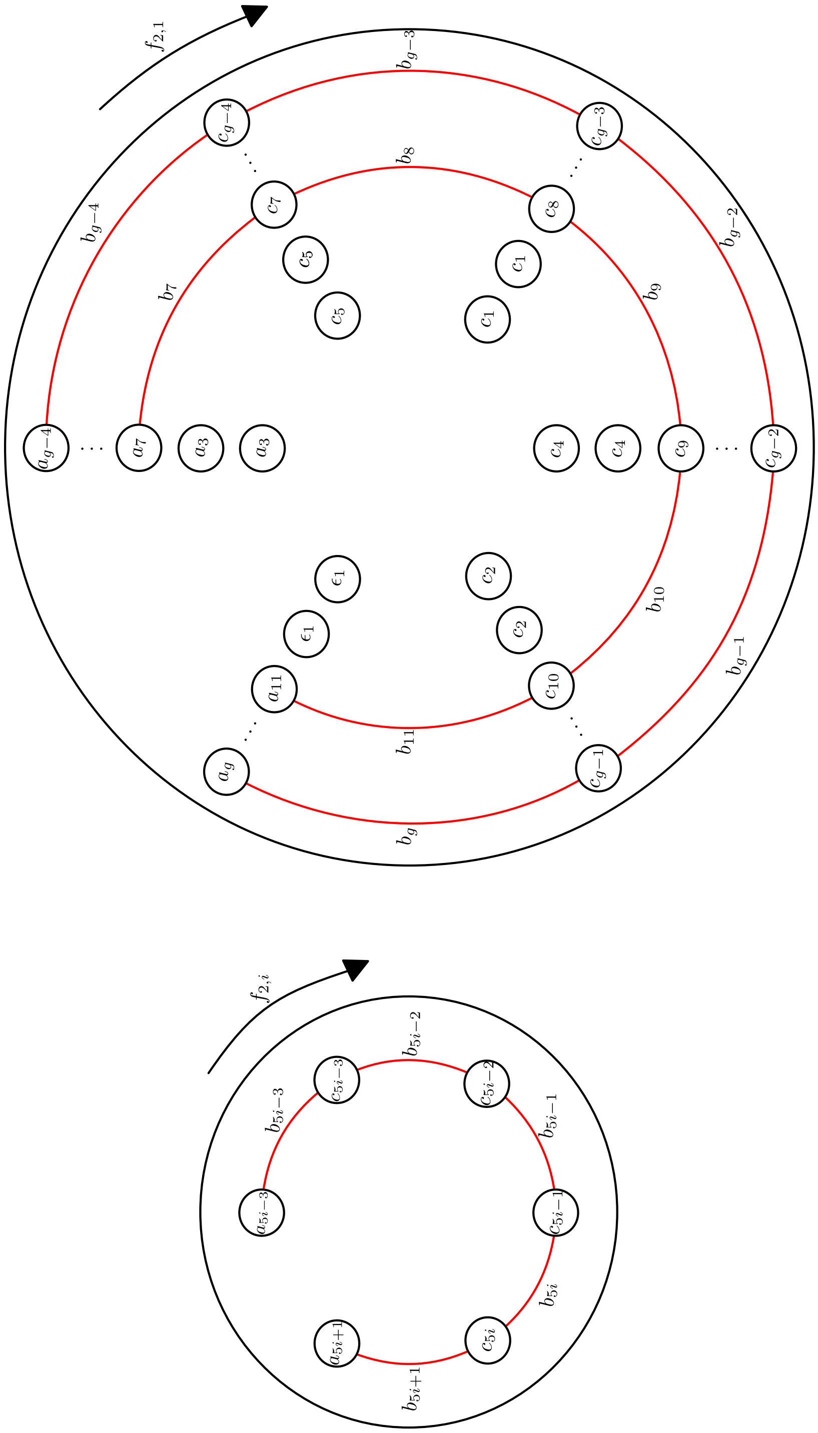

FIGURE 13. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,III

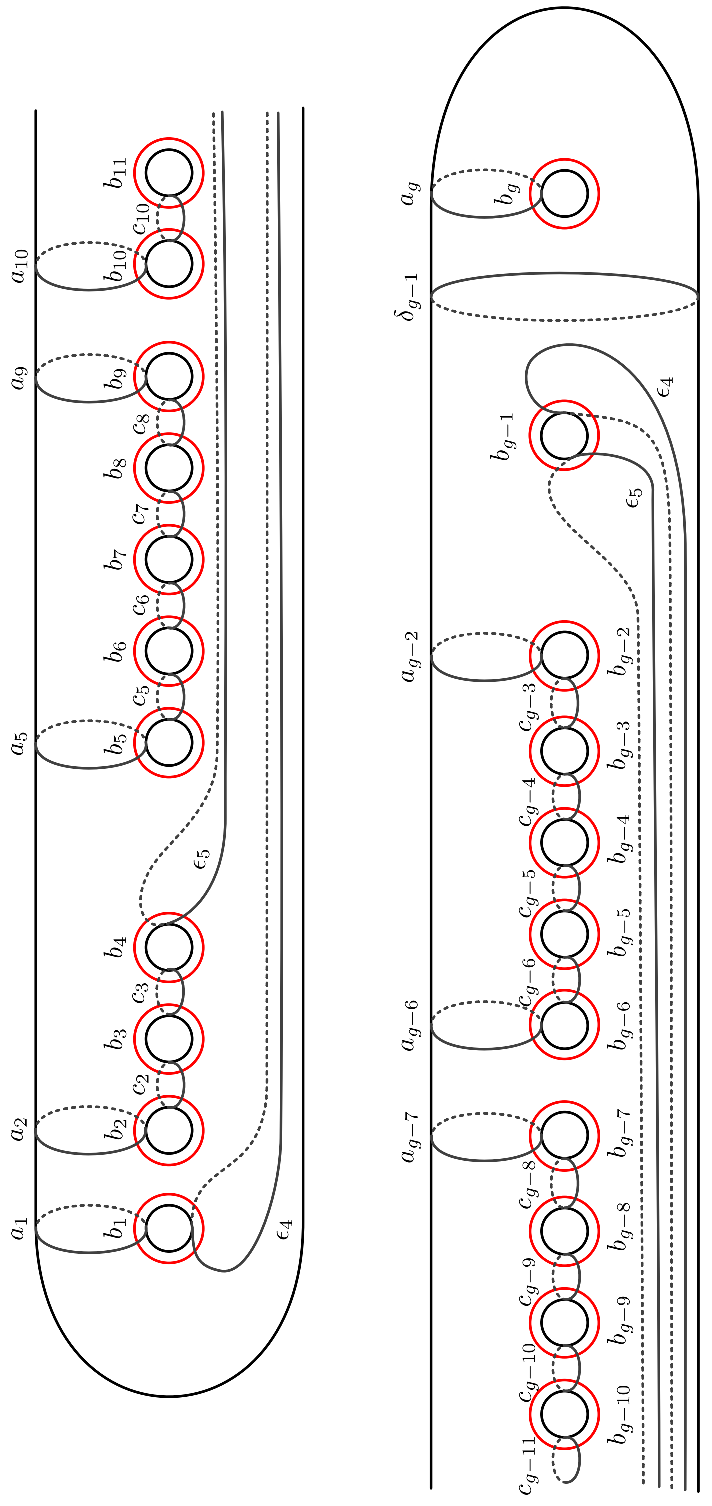

Figure 14. Cutting the surface IV

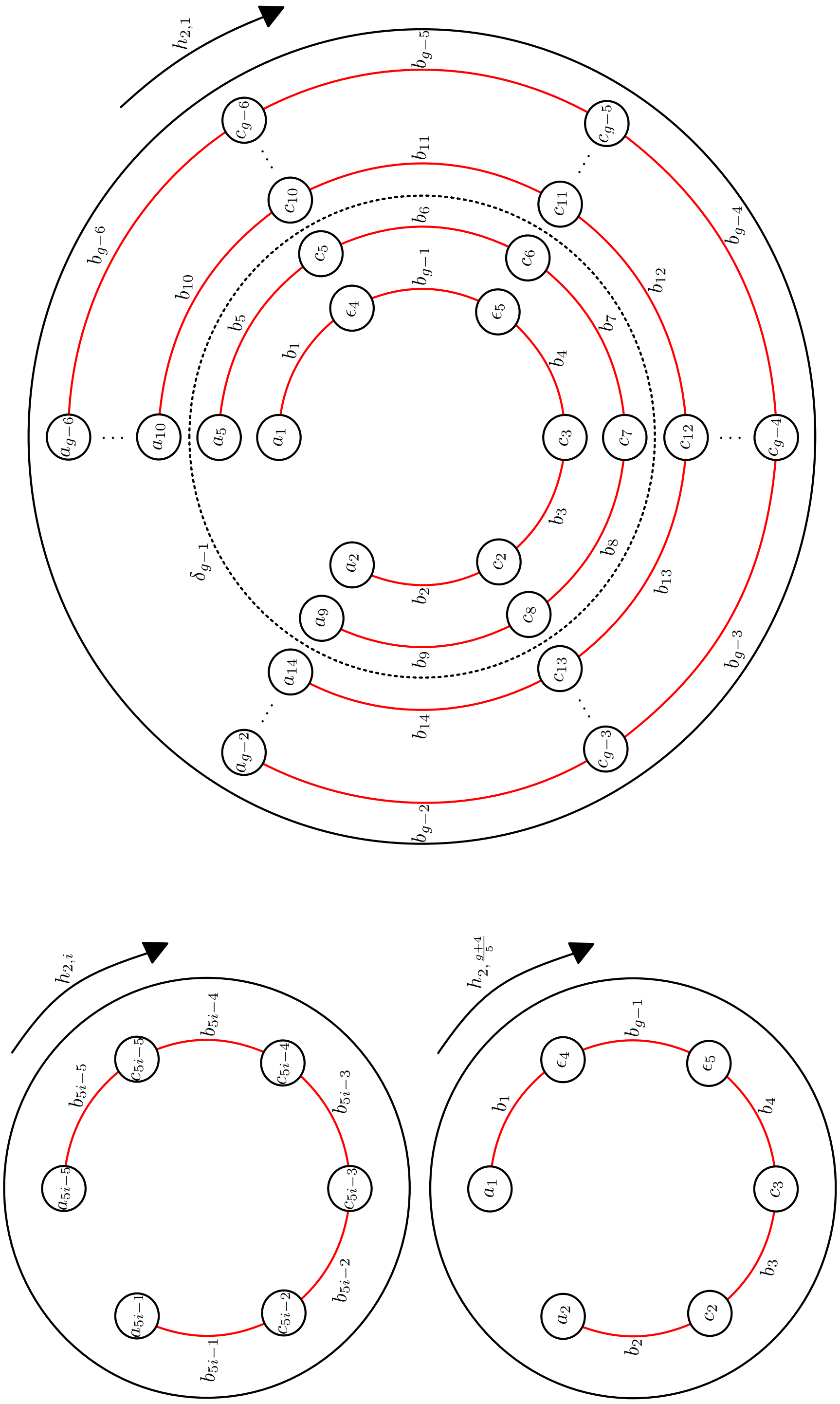

FIGURE 15. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,IV

3.1.3. *Case of $g = 5m + 2$ for some integer $m \geq 1$.*

We construct an element f_3 in $\mathrm{Mod}(\Sigma_{g,0})$ which has order six as follows. We cut the surface $\Sigma_{g,0}$ along the curves a_3, c_1, c_2, ϵ_1, c_4, c_5, a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1} $(i = 2, 3, \ldots, m)$ as shown in Fig. 16 and obtain m surfaces $L_{3,1}$, L^3_2, $\ldots, L^3_m$. The surface $L_{3,1}$ is a torus with $6m+6$ boundary components, $L_{3,i}$ is a sphere with 6 boundary components bounded by a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i} and a_{5i+1} $(i = 2, 3, \ldots, m)$. Let $L'_{3,1}$ be a subsurface of genus 1 in $L_{3,1}$ bounded by δ_{g-1}. Let $f_{3,1}$, $f_{3,2}$, $\ldots$, $f_{3,m}$ be $\pi/3$ rotation as shown in Fig. 17. Note that in this picture δ_{g-1} is on the back side and the map $f_{3,1}$ keeps $L'_{3,1}$ fixed. We found that $(f_{3,1})^6$ produces a twist $t_{\delta_{g-1}}$. In order to cancel the twist $t_{\delta_{g-1}}$, we define $f'_{3,1}$ as a composition of $f_{3,1}$ and $f_{3,m+1}$ which defined as follow.

$$f_{3,m+1} = (t_{a_g} t_{b_g})^{-1}.$$

Since the diffeomorphisms $f'_{3,1}$, $f_{3,2}$, $\ldots$, $f_{3,m}$ coincide on the boundaries, they define a diffeomorphism $f_3 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

We construct an element h_3 in $\mathrm{Mod}(\Sigma_{g,0})$ of order six as follows. We cut the surface $\Sigma_{g,0}$ along the curves a_1, a_2, c_2, c_3, ϵ_6, ϵ_7, a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} $(i = 2, 3, \ldots, m)$ as shown in Fig. 18 and obtain $m+1$ surfaces $M_{3,1}$, $M_{3,2}$, $\ldots, M_{3,m+1}$. The surface $M_{3,1}$ is a surface of genus 2 with $6m$ boundary components, $M_{3,i}$ is a sphere with 6 boundary components bounded by a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} $(i = 2, 3, \ldots, m)$, $M_{3,m+1}$ is a sphere with 6 boundary components bounded by a_1, a_2, c_2, c_3, ϵ_6, ϵ_7. Let $M'_{3,1}$ be a subsurface of genus 2 in $M_{3,1}$ bounded by δ_{g-2}. Let $h_{3,1}$, $h_{3,2}$, $\ldots$, $h_{3,m+1}$ be $\pi/3$ rotation as shown in Fig. 19. Note that in this picture δ_{g-2} is on the back side and the map $h_{3,1}$ keeps $M'_{3,1}$ fixed. We found that $(h_{3,1})^6$ produces a twist $t_{\delta_{g-2}}$. In order to cancel the twist $t_{\delta_{g-2}}$, we define $h'_{3,1}$ as a composition of $h_{3,1}$ and $h_{3,m+2}$ which defined as follow.

$$h_{3,m+2} = (t_{a_{g-1}} t_{b_{g-1}} t_{c_{g-1}} t_{b_g} t_{a_g})^{-1}.$$

Since the diffeomorphisms $h'_{3,1}$, $h_{3,2}$, $\ldots$, $h_{3,m+1}$ coincide on the boundaries, they define a diffeomorphism $h_3 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

For $i = 2, 3, \ldots, m$, f_3 acts on the curves on $\Sigma_{g,0}$ as follows:

$$(f_3)^5(a_3) = (f_3)^4(c_5) = (f_3)^3(c_1) = (f_3)^2(c_4) = (f_3)(c_2) = \epsilon_1,$$
$$(f_3)^5(a_{5i-3}) = (f_3)^4(c_{5i-3}) = (f_3)^3(c_{5i-2}) = (f_3)^2(c_{5i-1}) = (f_3)(c_{5i}) = a_{5i+1},$$
$$(f_3)^4(b_{5i-3}) = (f_3)^3(b_{5i-2}) = (f_3)^2(b_{5i-1}) = (f_3)(b_{5i}) = b_{5i+1}.$$

For $i = 2, 3, \ldots, m$, h_3 acts on the curves on $\Sigma_{g,0}$ as follows:

$$(h_3)^5(a_1) = (h_3)(c_3) = a_2,$$
$$(h_3)^4(b_1) = (h_3)^3(b_{g-2}) = (h_3)^2(b_4) = (h_3)(b_3) = b_2,$$
$$(h_3)^5(a_{5i-5}) = (h_3)^4(c_{5i-5}) = (h_3)^3(c_{5i-4}) = (h_3)^2(c_{5i-3}) = (h_3)(c_{5i-2}) = a_{5i-1},$$
$$(h_3)^4(b_{5i-5}) = (h_3)^3(b_{5i-4}) = (h_3)^2(b_{5i-3}) = (h_3)(b_{5i-2}) = b_{5i-1},$$
$$(h_3)^{-3}(b_{g-1}) = (h_3)^{-2}(c_{g-1}) = (h_3)^{-1}(b_g) = a_g.$$

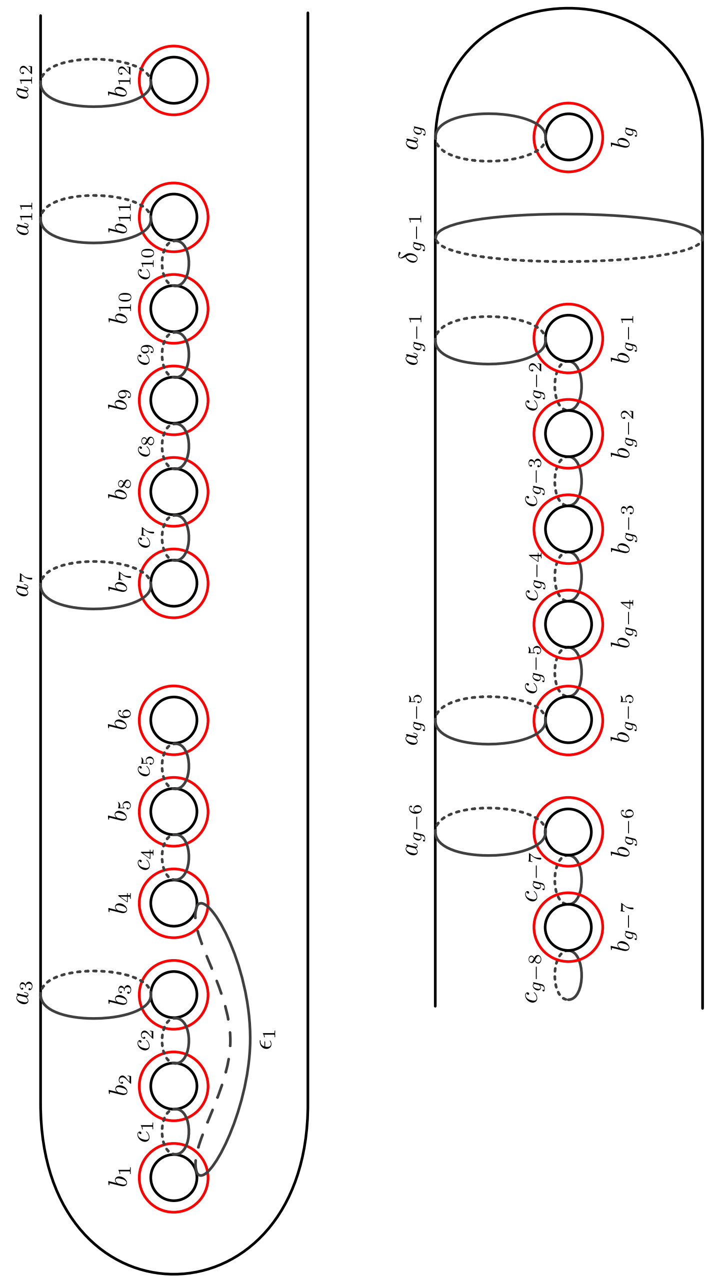

Figure 16. Cutting the surface V

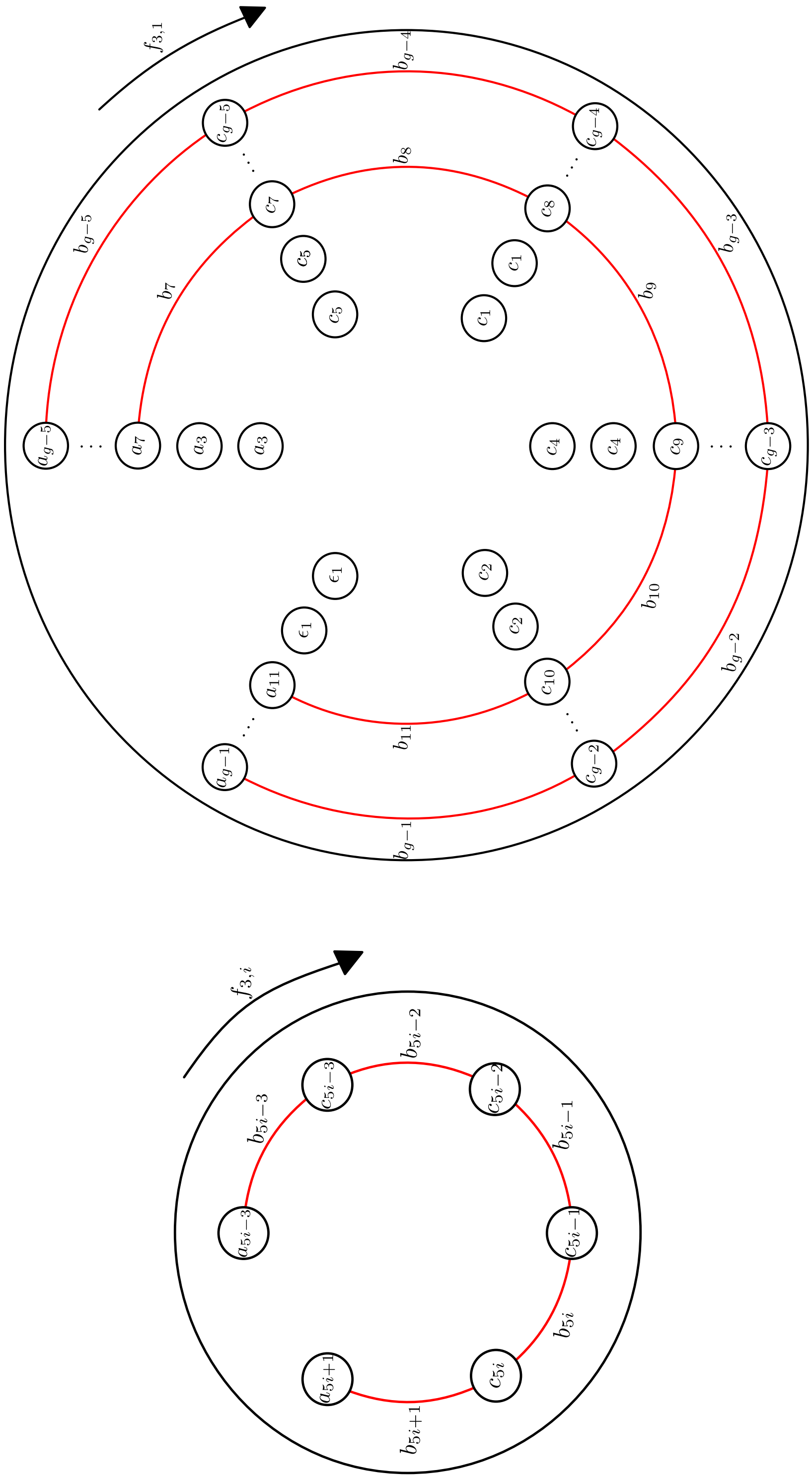

FIGURE 17. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,V

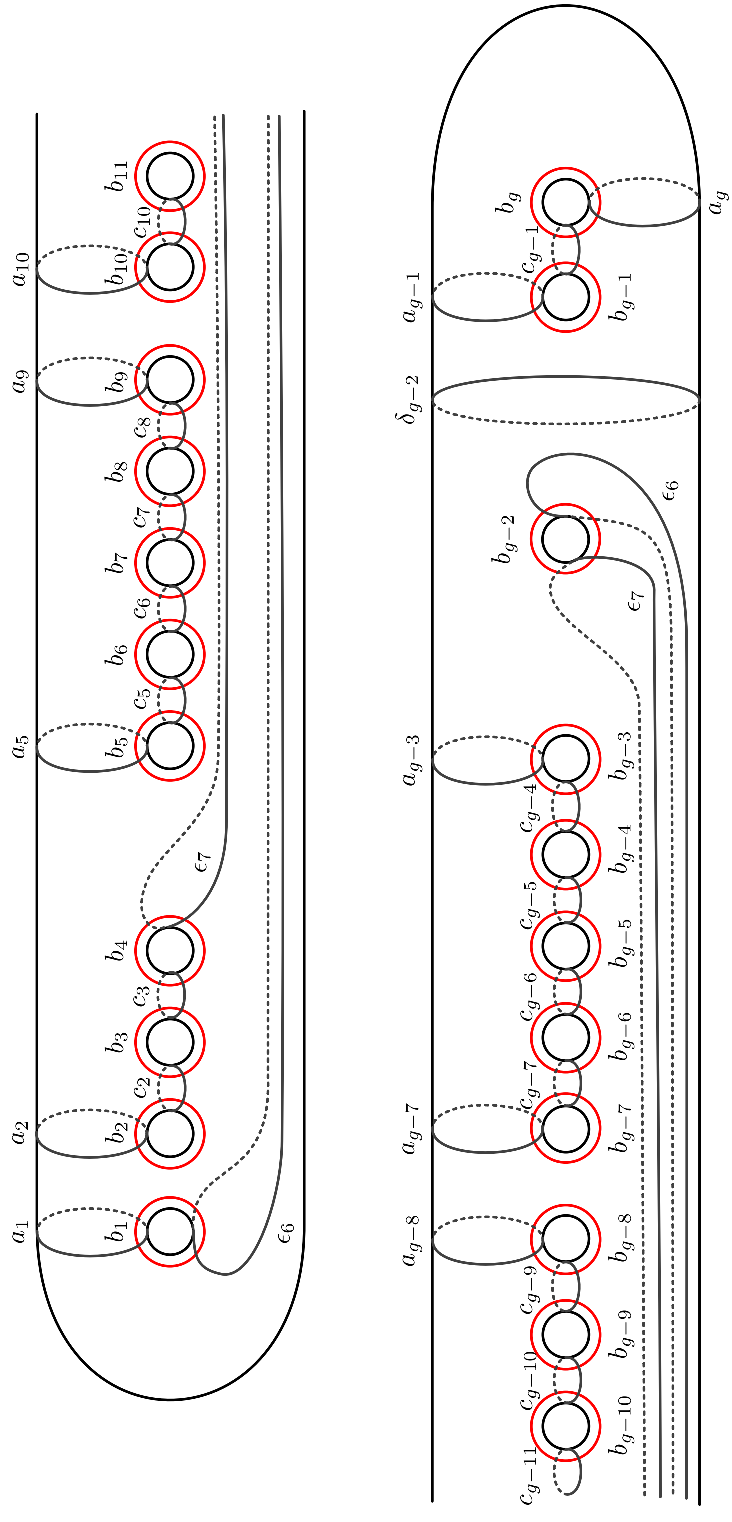

Figure 18. Cutting the surface VI

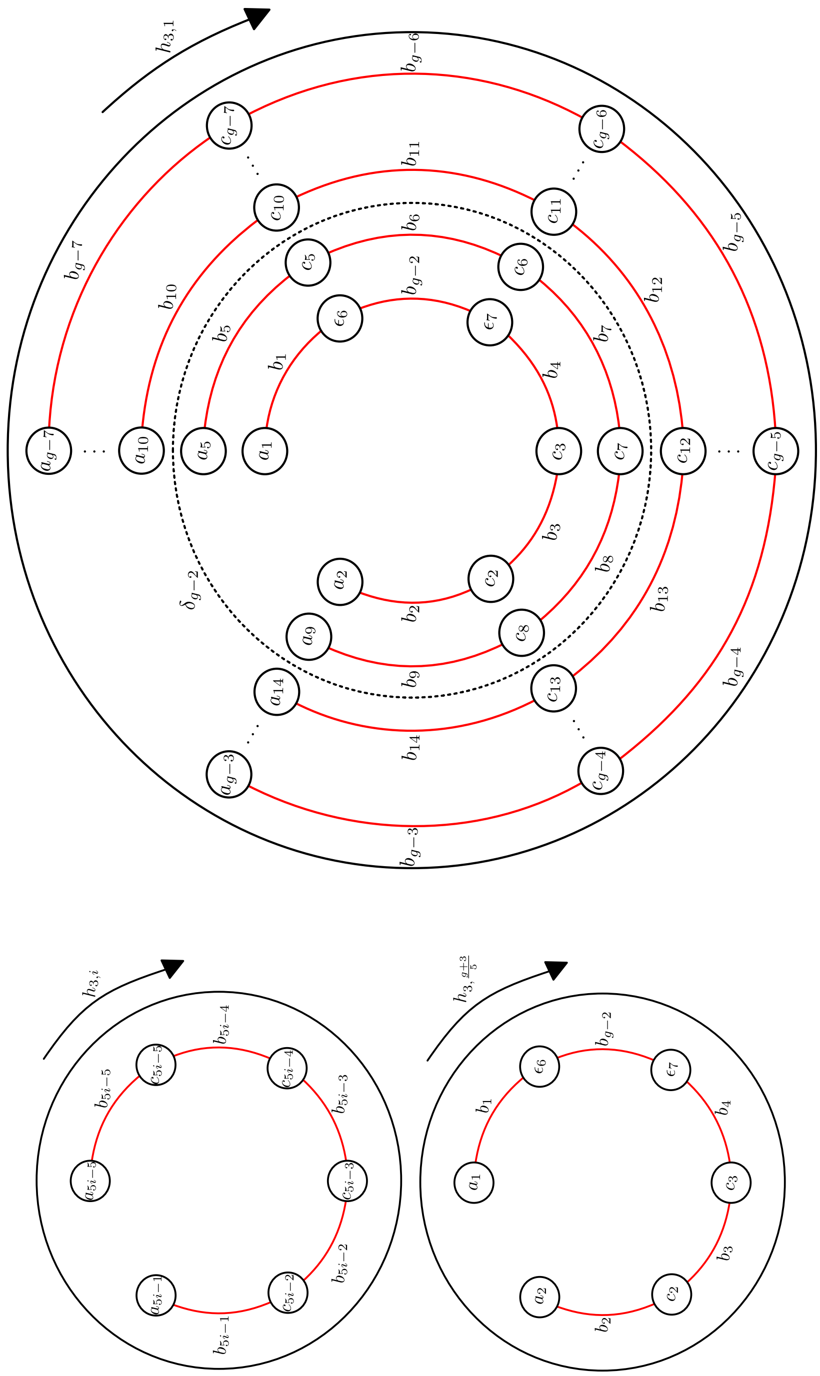

FIGURE 19. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,VI

3.1.4. *Case of $g = 5m + 3$ for some integer $m \geq 1$.*

We construct an element f_4 in $\mathrm{Mod}(\Sigma_{g,0})$ which has order six as follows. We cut the surface $\Sigma_{g,0}$ along the curves a_3, c_1, c_2, ϵ_1, c_4, c_5, a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1} $(i = 2, 3, \ldots, m)$ as shown in Fig. 20 and obtain m surfaces $L_{4,1}$, $L_{4,2}$, $\ldots, L_{4,m}$. The surface $L_{4,1}$ is a surface of genus 2 with $6m+6$ boundary components, $L_{4,i}$ is a sphere with 6 boundary components bounded by a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i} and a_{5i+1} $(i = 2, 3, \ldots, m)$. Let $L'_{4,1}$ be a subsurface of genus 2 in $L_{4,1}$ bounded by δ_{g-2}. Let $f_{4,1}$, $f_{4,2}$, $\ldots$, $f_{4,m}$ be $\pi/3$ rotation as shown in Fig. 21. Note that in this picture δ_{g-2} is on the back side and the map $f_{4,1}$ keeps $L'_{4,1}$ fixed. We found that $(f_{4,1})^6$ produces a twist $t_{\delta_{g-2}}$. In order to cancel the twist $t_{\delta_{g-2}}$, we define $f'_{4,1}$ as a composition of $f_{4,1}$ and $f_{4,m+1}$ which defined as follow.

$$f_{4,m+1} = (t_{a_{g-1}} t_{b_{g-1}} t_{c_{g-1}} t_{b_g} t_{a_g})^{-1}.$$

Since the diffeomorphisms $f'_{4,1}$, $f_{4,2}$, $\ldots$, $f_{4,m}$ coincide on the boundaries, they define a diffeomorphism $f_4 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

We construct an element h_4 in $\mathrm{Mod}(\Sigma_{g,0})$ of order six as follows. We cut the surface $\Sigma_{g,0}$ along the curves a_1, a_2, c_2, c_3, ϵ_8, ϵ_9, a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} $(i = 2, 3, \ldots, m)$ as shown in Fig. 22 and obtain $m + 1$ surfaces $M_{4,1}$, $M_{4,2}$, $\ldots, M_{4,m+1}$. The surface $M_{4,1}$ is a surface of genus 3 with $6m$ boundary components, $M_{4,i}$ is a sphere with 6 boundary components bounded by a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} $(i = 2, 3, \ldots, m)$, $M_{4,m+1}$ is a sphere with 6 boundary components bounded by a_1, a_2, c_2, c_3, ϵ_8, ϵ_9. Let $M'_{4,1}$ be a subsurface of genus 3 in $M_{4,1}$ bounded by δ_{g-3}. Let $h_{4,1}$, $h_{4,2}$, $\ldots$, $h_{4,m+1}$ be $\pi/3$ rotation as shown in Fig. 23. Note that in this picture δ_{g-3} is on the back side and the map $h_{4,1}$ keeps $M'_{4,1}$ fixed. We found that $(h_{4,1})^6$ produces a twist $t_{\delta_{g-3}}$. In order to cancel the twist $t_{\delta_{g-3}}$, we define $h'_{4,1}$ as a compostion of $h_{4,1}$ and $h_{4,m+2}$ which defined as follow.

$$h_{4,m+2} = (t_{a_{g-2}} t_{b_{g-2}} t_{c_{g-2}} t_{b_{g-1}} t_{a'_{g-1}})^{-1} (t_{a_g} t_{b_g}).$$

Since the diffeomorphisms $h'_{4,1}$, $h_{4,2}$, $\ldots$, $h_{4,m+1}$ coincide on the boundaries, they define a diffeomorphism $h_4 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

For $i = 2, 3, \ldots, m$, f_4 acts on the curves on $\Sigma_{g,0}$ as follows:

$(f_4)^5(a_3) = (f_4)^4(c_5) = (f_4)^3(c_1) = (f_4)^2(c_4) = (f_4)(c_2) = \epsilon_1$,

$(f_4)^5(a_{5i-3}) = (f_4)^4(c_{5i-3}) = (f_4)^3(c_{5i-2}) = (f_4)^2(c_{5i-1}) = (f_4)(c_{5i}) = a_{5i+1}$,

$(f_4)^4(b_{5i-3}) = (f_4)^3(b_{5i-2}) = (f_4)^2(b_{5i-1}) = (f_4)(b_{5i}) = b_{5i+1}$,

$(f_4)^{-3}(b_{g-1}) = (f_4)^{-2}(c_{g-1}) = (f_4)^{-1}(b_g) = a_g$.

For $i = 2, 3, \ldots, m$, h_4 acts on the curves on $\Sigma_{g,0}$ as follows:

$(h_4)^5(a_1) = (h_4)^2(c_3) = h_4(c_2) = a_2$,

$(h_4)^4(b_1) = (h_4)^3(b_{g-3}) = (h_4)^2(b_4) = (h_4)(b_3) = b_2$,

$(h_4)^5(a_{5i-5}) = (h_4)^4(c_{5i-5}) = (h_4)^3(c_{5i-4}) = (h_4)^2(c_{5i-3}) = (h_4)(c_{5i-2}) = a_{5i-1}$,

$(h_4)^4(b_{5i-5}) = (h_4)^3(b_{5i-4}) = (h_4)^2(b_{5i-3}) = (h_4)(b_{5i-2}) = b_{5i-1}$,

$(h_4)^{-2}(b_{g-2}) = (h_4)^{-1}(c_{g-2}) = b_{g-1}, (h_4)^{-1}(b_g) = a_g$.

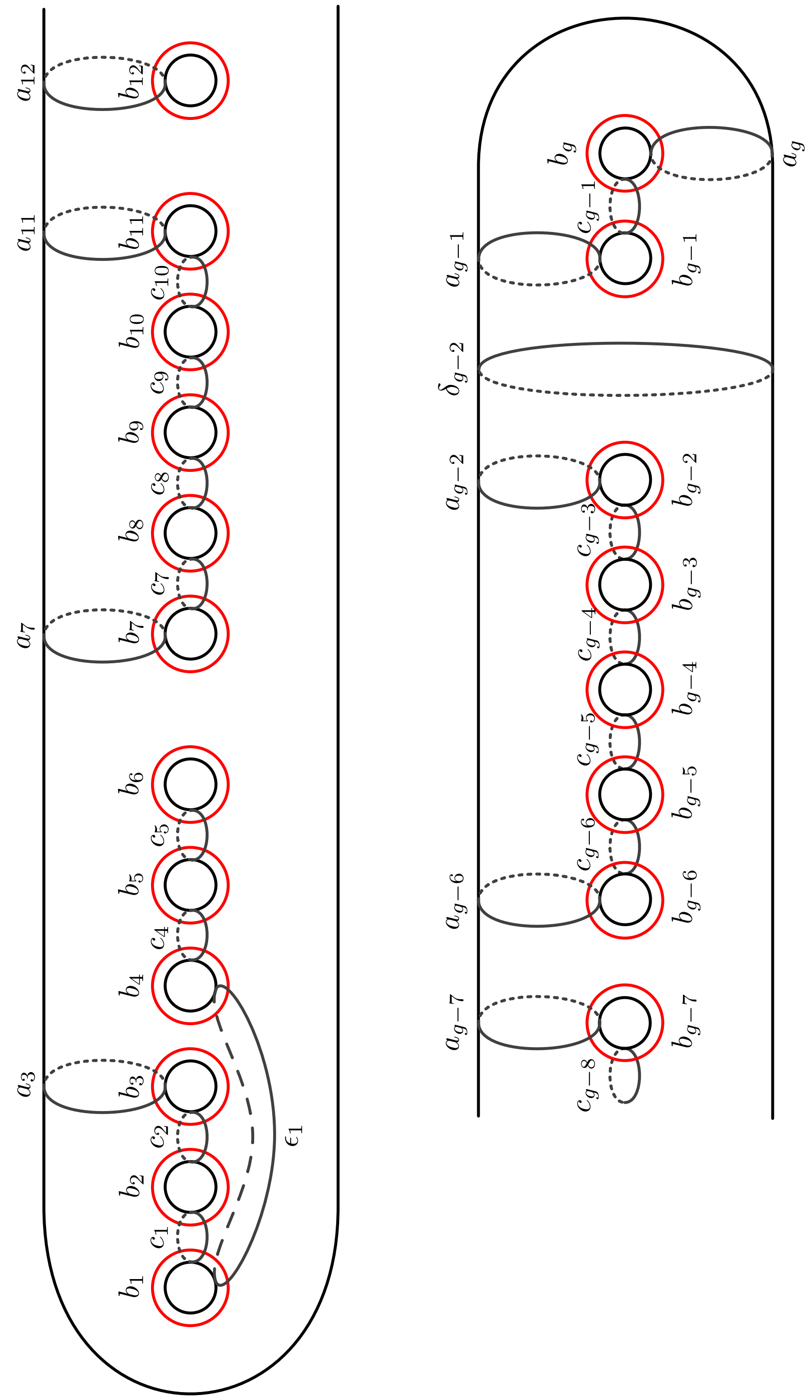

FIGURE 20. Cutting the surface VII

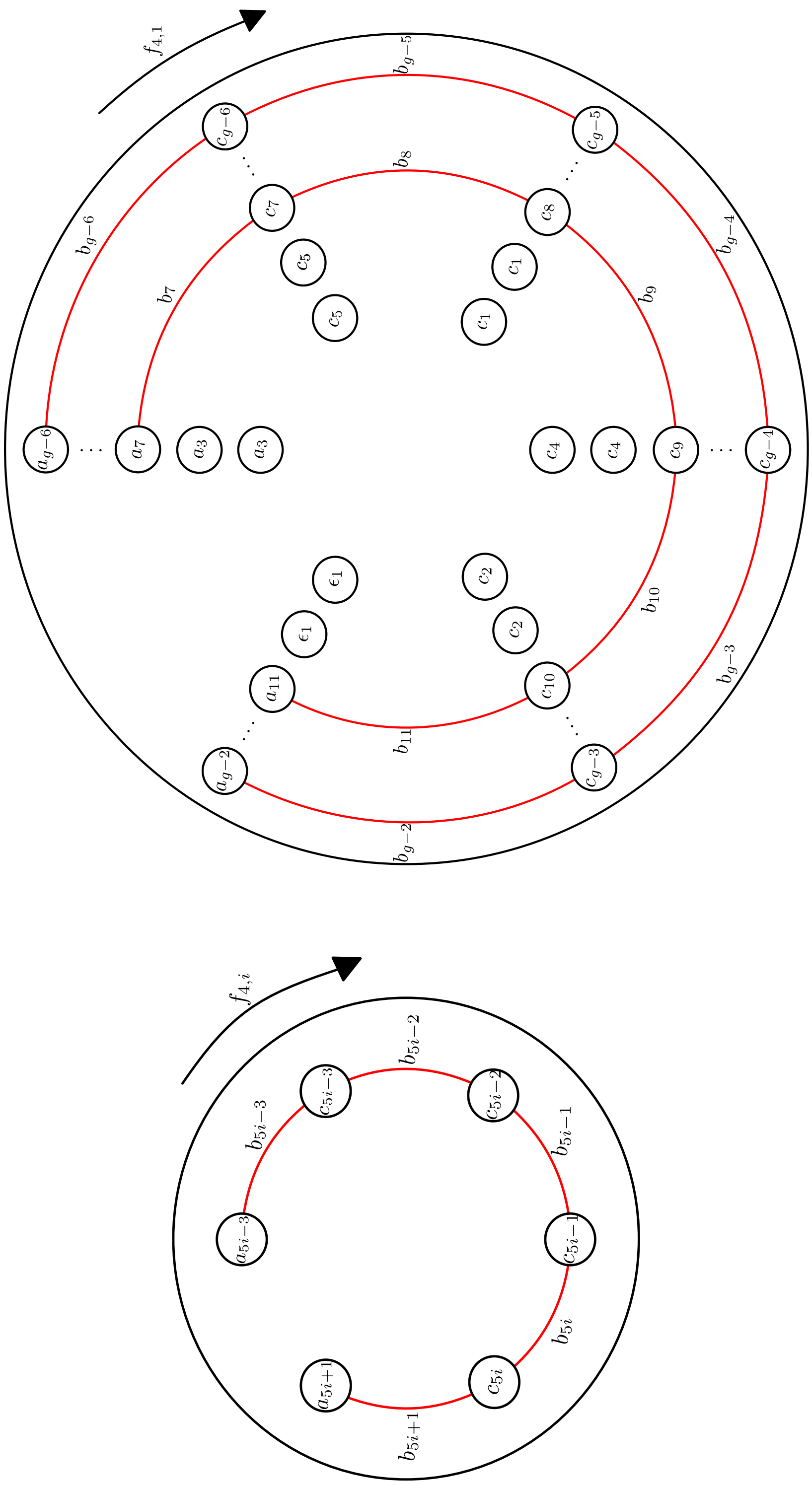

FIGURE 21. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,VII

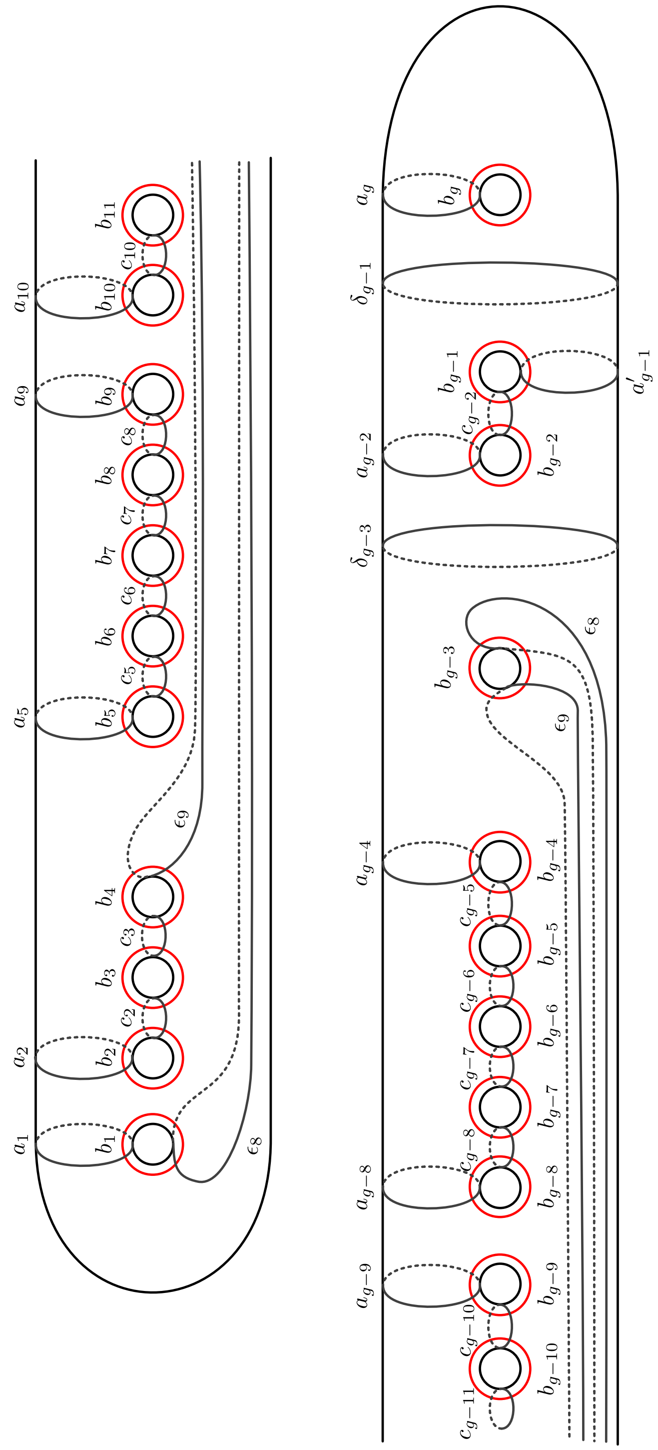

FIGURE 22. Cutting the surface VIII

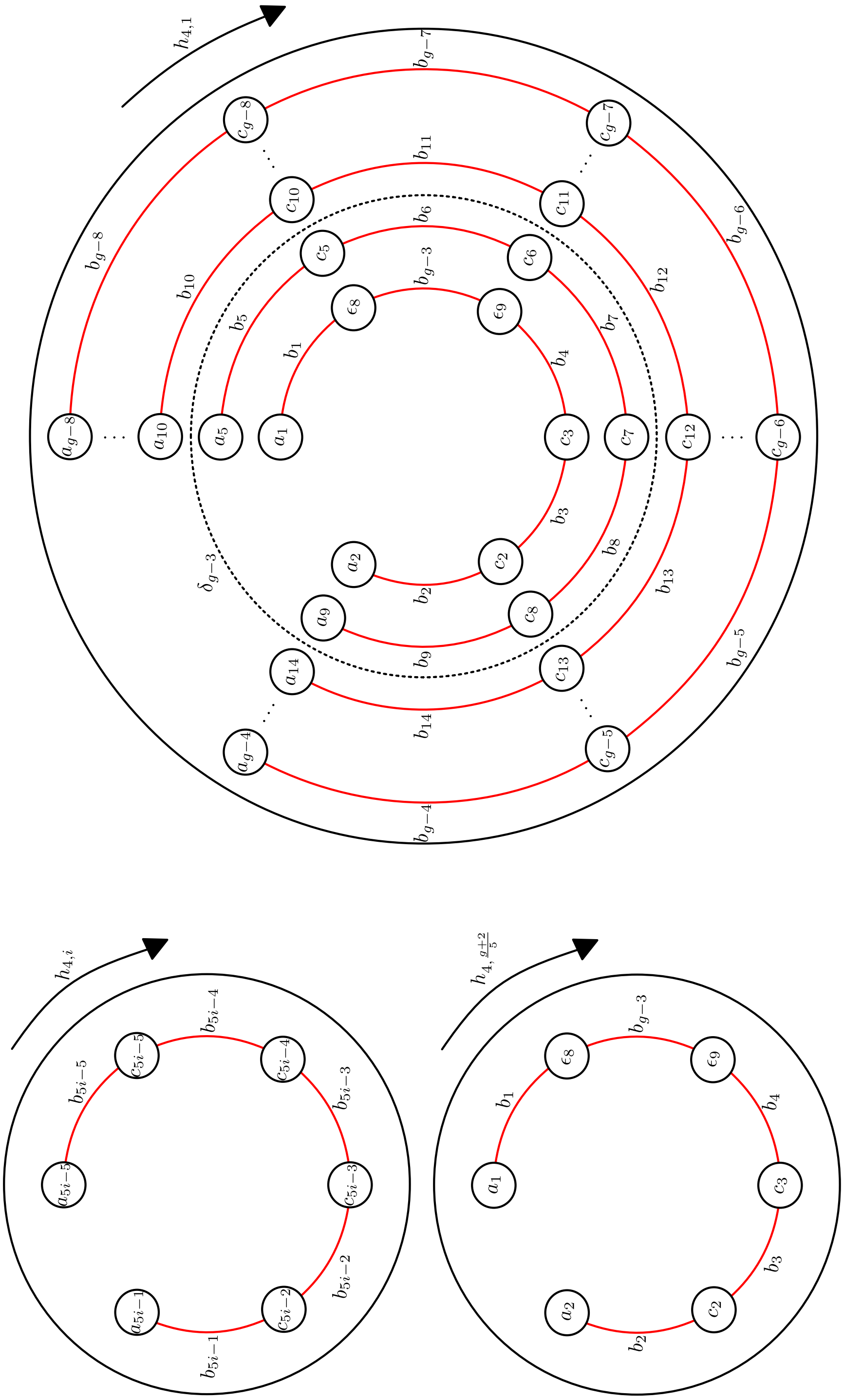

FIGURE 23. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,VIII

3.1.5. *Case of $g = 5m + 4$ for some integer $m \geq 1$.*

We construct an element f_5 in $\mathrm{Mod}(\Sigma_{g,0})$ which has order six as follows. For $i = 2, 3, \ldots, m$, we cut the surface $\Sigma_{g,0}$ along the curves a_3, c_1, c_2, ϵ_1, c_4, c_5, a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1} as shown in Fig. 24 and obtain m surfaces $L_{5,1}$, $L_{5,2}$, $\ldots, L_{5,m}$. The surface $L_{5,1}$ is a surface of genus 3 with $6m+6$ boundary components, $L_{5,i}$ is a sphere with 6 boundary components bounded by a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1}. Let $L'_{5,1}$ be a subsurface of genus 3 in $L_{5,1}$ bounded by δ_{g-3}. Let $f_{5,1}$, $f_{5,2}$, …, $f_{5,m}$ be $\pi/3$ rotation as shown in Fig. 25. Note that in this picture δ_{g-3} is on the back side and the map $f_{5,1}$ keeps $L'_{5,1}$ fixed. We found that $(f_{5,1})^6$ produces a twist $t_{\delta_{g-3}}$. In order to cancel the twist $t_{\delta_{g-3}}$, we define $f'_{5,1}$ as a composition of $f_{5,1}$ and $f_{5,m+1}$ which defined as follow.

$$f_{5,m+1} = (t_{a_{g-2}} t_{b_{g-2}} t_{c_{g-2}} t_{b_{g-1}} t_{a'_{g-1}})^{-1} (t_{a_g} t_{b_g}).$$

Since the diffeomorphisms $f'_{5,1}$, $f_{5,2}$, …, $f_{5,m}$ coincide on the boundaries, they define a diffeomorphism $f_5 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

We construct an element h_5 in $\mathrm{Mod}(\Sigma_{g,0})$ of order six as follows. For $i = 2, 3, \ldots, m$, we cut the surface $\Sigma_{g,0}$ along the curves a_1, a_2, c_2, c_3, ϵ_{10}, ϵ_{11}, a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1} as shown in Fig. 26 and obtain $m+1$ surfaces $M_{5,1}$, $M_{5,2}$, $\ldots, M_{5,m+1}$. The surface $M_{5,1}$ is a surface of genus 4 with $6m$ boundary components, $M_{5,i}$ is a sphere with 6 boundary components bounded by a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}, a_{5i-1}, $M_{5,m+1}$ is a sphere with 6 boundary components bounded by a_1, a_2, c_2, c_3, ϵ_{10}, ϵ_{11}. Let $M'_{5,1}$ be a subsurface of genus 4 in $M_{5,1}$ bounded by δ_{g-4}. Let $h_{5,1}$, $h_{5,2}$, …, $h_{5,m+1}$ be $\pi/3$ rotation as shown in Fig. 27. Note that in this picture δ_{g-4} is on the back side and the map $h_{5,1}$ keeps $M'_{5,1}$ fixed. We found that $(h_{5,1})^6$ produces a twist $t_{\delta_{g-4}}$. In order to cancel the twist $t_{\delta_{g-4}}$, we define $h'_{5,1}$ as a composition of $h_{5,1}$ and $h_{5,m+2}$ which defined as follow.

$$h_{5,m+2} = (t_{a_{g-3}} t_{b_{g-3}} t_{c_{g-3}} t_{b_{g-2}} t_{a'_{g-2}})^{-1} (t_{a_{g-1}} t_{b_{g-1}} t_{c_{g-1}} t_{b_g} t_{a_g}).$$

Since the diffeomorphisms $h'_{5,1}$, $h_{5,2}$, …, $h_{5,m+1}$ coincide on the boundaries, they define a diffeomorphism $h_5 : \Sigma_{g,0} \to \Sigma_{g,0}$ of order six.

For $i = 2, 3, \ldots, m$, f_5 acts on the curves on $\Sigma_{g,0}$ as follows:

$(f_5)^5(a_3) = (f_5)^4(c_5) = (f_5)^3(c_1) = (f_5)^2(c_4) = (f_5)(c_2) = \epsilon_1$,

$(f_5)^5(a_{5i-3}) = (f_5)^4(c_{5i-3}) = (f_5)^3(c_{5i-2}) = (f_5)^2(c_{5i-1}) = (f_5)(c_{5i}) = a_{5i+1}$,

$(f_5)^4(b_{5i-3}) = (f_5)^3(b_{5i-2}) = (f_5)^2(b_{5i-1}) = (f_5)(b_{5i}) = b_{5i+1}$,

$(f_5)^{-2}(b_{g-2}) = (f_5)^{-1}(c_{g-2}) = b_{g-1}, (f_5)^{-1}(b_g) = a_g$.

For $i = 2, 3, \ldots, m$, h_5 acts on the curves on $\Sigma_{g,0}$ as follows:

$(h_5)^5(a_1) = (h_5)^2(c_3) = h_5(c_2) = a_2$,

$(h_5)^4(b_1) = (h_5)^3(b_{g-4}) = (h_5)^2(b_4) = (h_5)(b_3) = b_2$,

$(h_5)^5(a_{5i-5}) = (h_5)^4(c_{5i-5}) = (h_5)^3(c_{5i-4}) = (h_5)^2(c_{5i-3}) = (h_5)(c_{5i-2}) = a_{5i-1}$,

$(h_5)^4(b_{5i-5}) = (h_5)^3(b_{5i-4}) = (h_5)^2(b_{5i-3}) = (h_5)(b_{5i-2}) = b_{5i-1}$,

$(h_5)^{-2}(b_{g-3}) = (h_5)^{-1}(c_{g-3}) = b_{g-2}$,

$(h_5)^3(b_{g-1}) = (h_5)^2(c_{g-1}) = (h_5)(b_g) = a_g$.

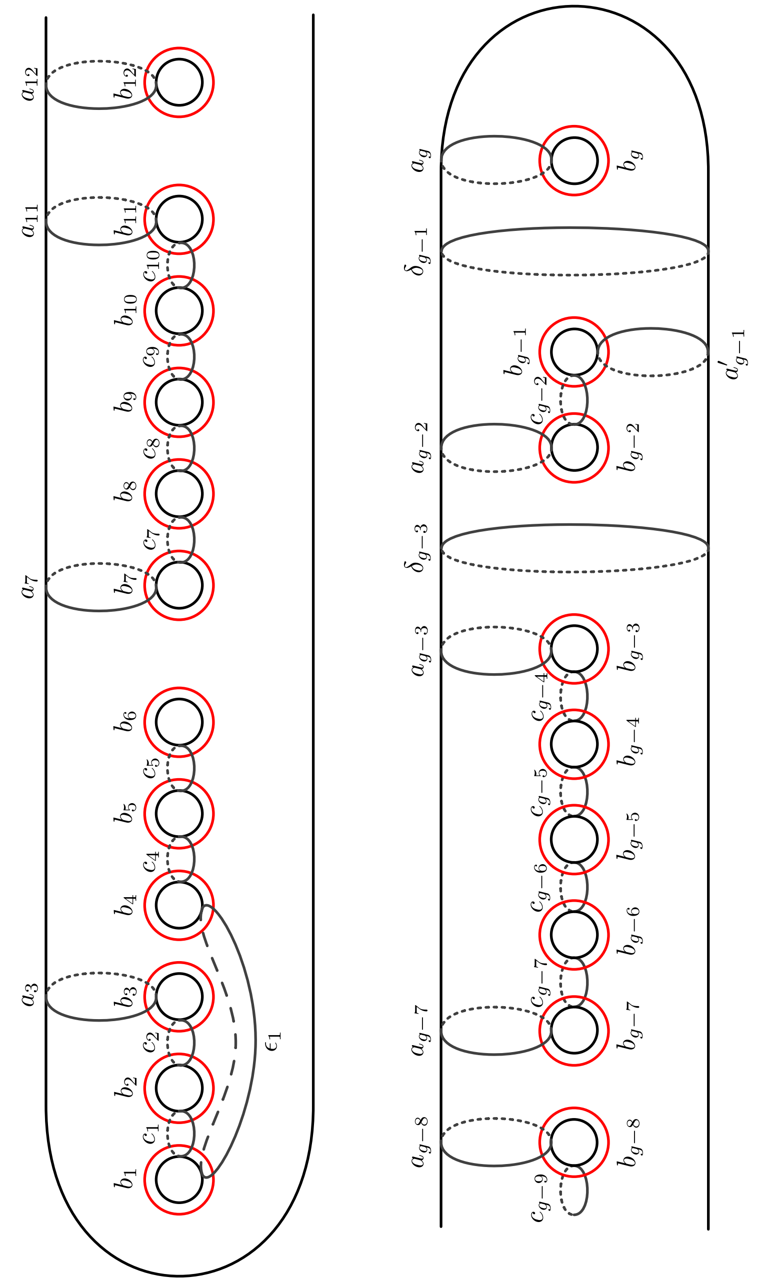

FIGURE 24. Cutting the surface IX

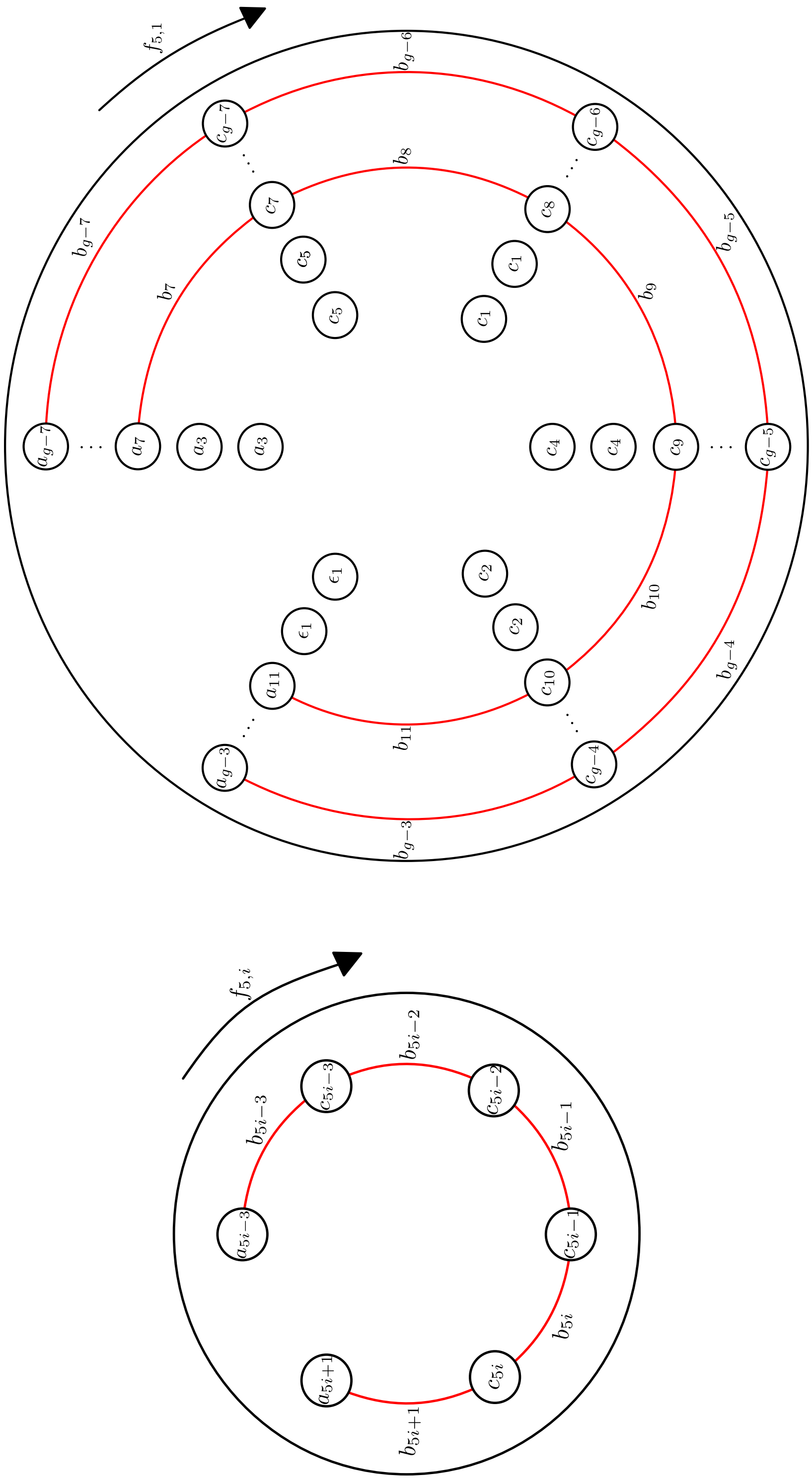

FIGURE 25. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,IX

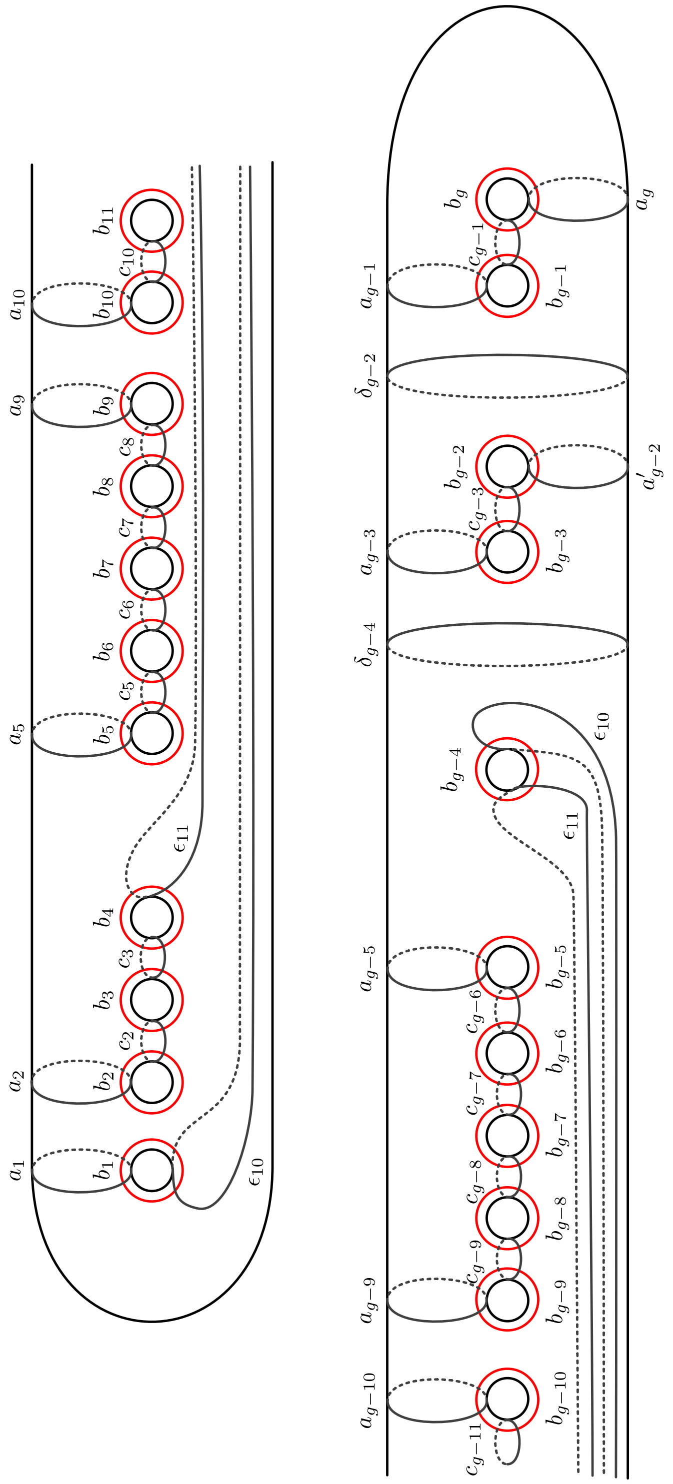

FIGURE 26. Cutting the surface X

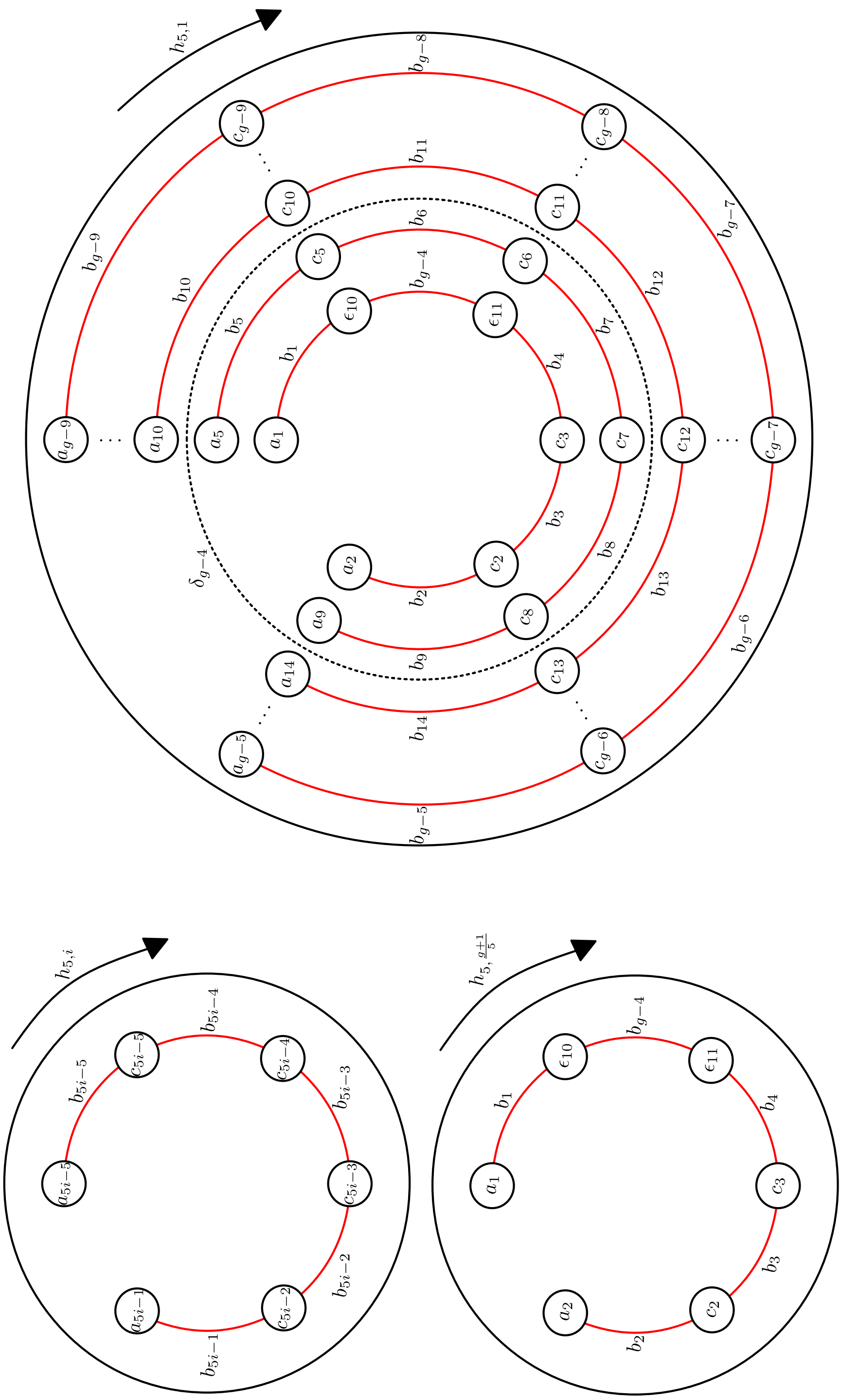

FIGURE 27. $\mathbb{Z}_6$-symmetry of $\Sigma_{g,0}$,X

3.1.6. *Case of* $g = 5$.

We construct an element f_6 in $\mathrm{Mod}(\Sigma_{5,0})$ which has order six as follows. We cut the surface $\Sigma_{5,0}$ along the curves $a_3, a_5, c_1, c_2, c_4, \epsilon_1$ as shown in Fig. 28 and obtain 2 six holed spheres $L_{6,1}$ and $L_{6,2}$.

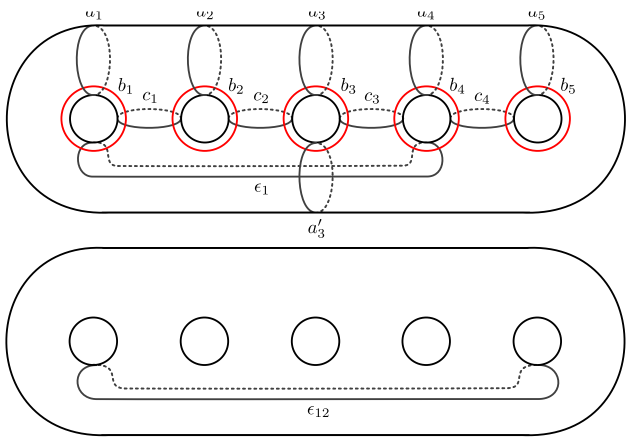

FIGURE 28. Simple Closed Curves on $\Sigma_{5,0}$

Let $f_{6,1}$ and $f_{6,2}$ be $\pi/3$ rotation as shown in Fig. 29. Since the diffeomorphisms $f_{6,1}$ and $f_{6,2}$ coincide on the boundaries, they define a diffeomorphism $f_6 : \Sigma_{5,0} \to \Sigma_{5,0}$ of order six.

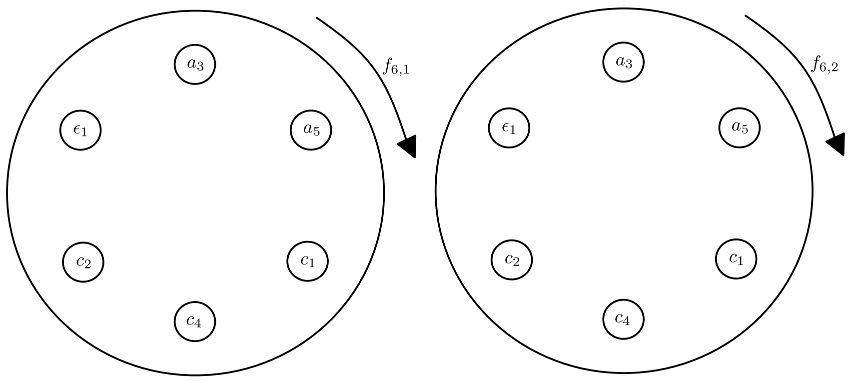

FIGURE 29. $\mathbb{Z}_6$-symmetry of $\Sigma_{5,0}$,XI

We construct an element h_6 in $\mathrm{Mod}(\Sigma_{5,0})$ which has order six. We cut the surface $\Sigma_{5,0}$ along the curves $a_1, a_2, c_2, c_3, c_4, \epsilon_{12}$ as shown in Fig. 28 and obtain two spheres with 6 boundary components $M_{6,1}$ and $M_{6,2}$. Let $h_{6,1}$ and $h_{6,2}$ be $\pi/3$ rotation as shown in Fig. 30.

Since the diffeomorphisms $h_{6,1}$ and $h_{6,2}$ coincide on the boundaries, they define a diffeomorphism $h_6 : \Sigma_{5,0} \to \Sigma_{5,0}$ of order six. In this case, for $i = 1, 2, \ldots, 4$ and

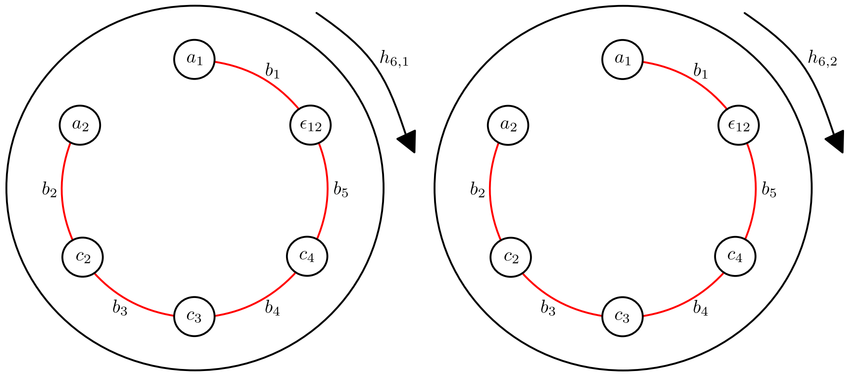

FIGURE 30. $\mathbb{Z}_6$-symmetry of $\Sigma_{5,0}$,XII

$j = 1, 2, \ldots, 5$, since there is no element which maps from a_i and c_i to b_j, we need such element. we define r_6 as follow:

$$r_6 = (a_1 b_1)(a_2 b_2 c_2 b_3 a_3')^{-1}(a_4 b_4 c_4 b_5 a_5).$$

By chain relation, the element r_6 has order six.

f_6 acts on the curves on $\Sigma_{5,0}$ as follows:

$$(f_6)^5(a_3) = (f_6)^4(c_5) = (f_6)^3(c_1) = (f_6)^2(c_4) = (f_6)(c_2) = \epsilon_1,$$
$$(f_6)^2(b_5) = b_4.$$

h_6 acts on the curves on $\Sigma_{5,0}$ as follows:

$$(h_6)^5(a_1) = (h_6)^4(\epsilon_{12}) = (h_6)^3(c_4) = (h_6)^2(c_3) = (h_6)(c_2) = a_2,$$
$$(h_6)^4(b_1) = (h_6)^3(b_5) = (h_6)^2(b_4) = (h_6)(b_3) = b_2.$$

r_6 acts on the curves on $\Sigma_{5,0}$ as follows:

$$(r_6)(a_1) = b_1,$$
$$(r_6)^3(a_2) = (r_6)^2(b_2) = (r_6)(c_2) = b_3,$$
$$(r_6)^4(a_4) = (r_6)^3(b_4) = (r_6)^2(c_4) = (r_6)(b_5) = a_5.$$

3.1.7. *Case of $g = 6$.*

We construct an element f_7 in $\mathrm{Mod}(\Sigma_{6,0})$ which has order six. We cut the surface $\Sigma_{6,0}$ along the curves $a_3, c_1, c_2, c_4, c_5, \epsilon_1$ as shown in Fig. 31 and obtain a sphere with 12 boundary components.

Let $f_{7,1}$ be $\pi/3$ rotation as shown in Fig. 32 and let f_7 be a diffeomorphism which is obtained from $f_{7,1}$ by bluing each boundary.

We construct an element h_7 in $\mathrm{Mod}(\Sigma_{6,0})$ which has order six. We cut the surface $\Sigma_{6,0}$ along the curves $a_1, a_2, c_2, c_3, c_4, \epsilon_{12}$ as shown in Fig. 31 and obtain a sphere with 6 boundary components $M_{7,1}$, a torus with 6 boundary components $M_{7,2}$. Let $M_{7,2}'$ be a subsurface of genus 1 in $M_{7,2}$ bounded by δ_5.

Let $h_{7,1}$ and $h_{7,2}$ be $\pi/3$ rotation as shown in Fig. 33 . Note that in this picture δ_5 is on the back side and the map $h_{7,2}$ keeps $M_{7,2}'$ fixed. We found that $(h_{7,2})^6 = t_{\delta_5}$.

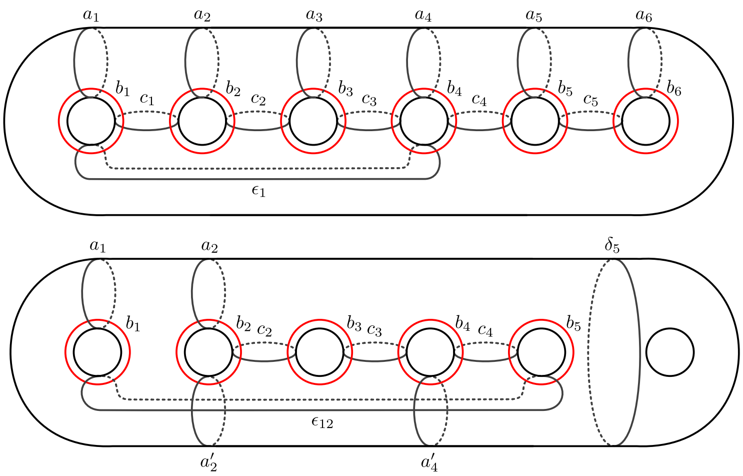

FIGURE 31. Simple Closed Curves on $\Sigma_{6,0}$

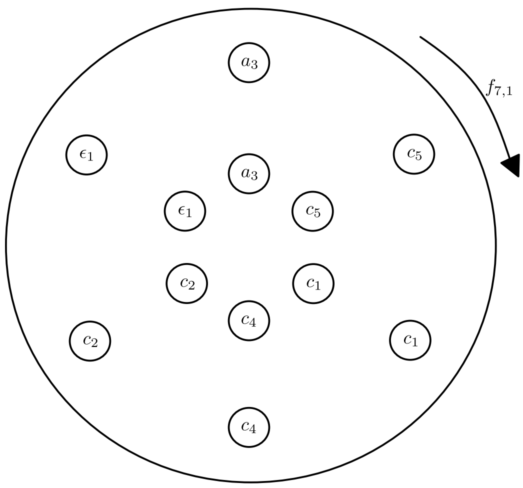

FIGURE 32. $\mathbb{Z}_6$-symmetry of $\Sigma_{6,0}$,XIII

In order to cancel the twist t_{δ_5}, we define $h'_{7,2}$ as a composition of $h_{7,2}$ and $h_{7,3}$ which defined as follow.

$$h_{7,3} = (t_{a_6} t_{b_6})^{-1}.$$

Since the diffeomorphisms $h_{7,1}$ and $h'_{7,2}$ coincide on the boundaries, they define a diffeomorphism $h_7 : \Sigma_{6,0} \to \Sigma_{6,0}$ of order six. In this case, for $i = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, 6$, since there is no element which maps from a_i and c_i to b_j, we need such element. we define r_7 as follow:

$$r_7 = (a_1 b_1 c_1 b_2 a'_2)(a_3 b_3 c_3 b_4 a'_4)^{-1}(a_5 b_5 c_5 b_6 a_6).$$

By chain relation, the element r_7 has order six.

The diffeomorphism f_7 acts on the curves on $\Sigma_{6,0}$ as follows:

$$(f_7)^5(a_3) = (f_7)^4(c_5) = (f_7)^3(c_1) = (f_7)^2(c_4) = (f_7)(c_2) = \epsilon_1,$$

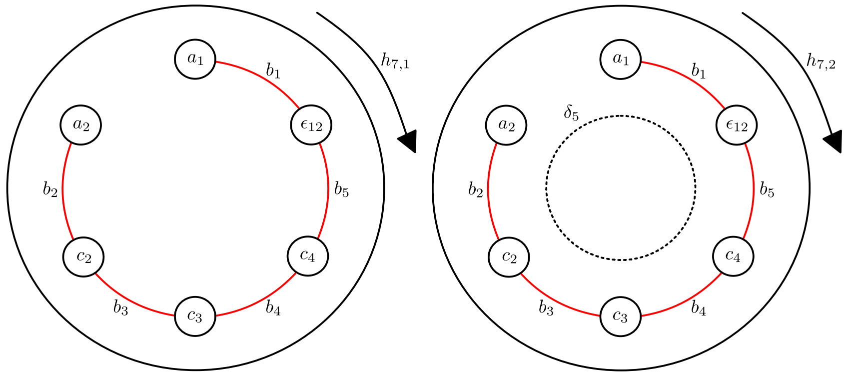

FIGURE 33. $\mathbb{Z}_6$-symmetry of $\Sigma_{6,0}$,XIV

The diffeomorphism h_7 acts on the curves on $\Sigma_{6,0}$ as follows:

$$(h_7)^5(a_1) = (h_7)^4(\epsilon_{12}) = (h_7)^3(c_4) = (h_7)^2(c_3) = (h_7)(c_2) = a_2,$$
$$(h_7)^4(b_1) = (h_7)^3(b_5) = (h_7)^2(b_4) = (h_7)(b_3) = b_2.$$

The diffeomorphism r_7 acts on the curves on $\Sigma_{6,0}$ as follows:

$$(r_7)^3(a_1) = (r_7)^2(b_1) = (r_7)(c_1) = b_2,$$
$$(r_7)^4(a_3) = (r_7)^3(b_3) = (r_7)^2(c_3) = (r_7)(b_4) = c_4,$$
$$(r_7)^4(a_5) = (r_7)^3(b_5) = (r_7)^2(c_5) = (r_7)(b_6) = a_6.$$

3.2. **Generating a Dehn twist by elements of order six.**

In this subsection, we use the lantern relation in order to generate the Dehn twist by 3 elements of order 6. We embed the four-holed sphere S in $\Sigma_{g,0}$ as shown in Fig. 34.

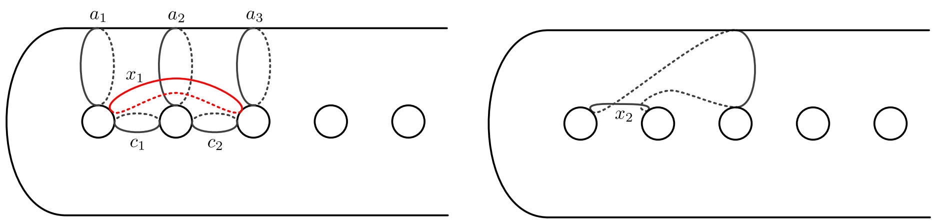

FIGURE 34. Curves x_1 and x_2.

By Lantern relation, we have

$$t_{a_1}t_{c_1}t_{c_2}t_{a_3} = t_{x_1}t_{x_2}t_{a_2},$$

where the curves $a_1, a_2, c_1, c_2, a_3, x_1$ and x_2 are shown in Fig. 34. By contructions f_i and h_i $(i = 1, 2, \ldots, 7)$, we have

$$(f_i)^4(a_2) = x_1,\ (f_i)^2(a_2) = x_2,$$
$$(f_i)^4(c_2) = c_1,\ (f_i)^2(c_2) = a_3,$$
$$(h_i)(c_2) = a_2.$$

Now, we put k_i as a product $t_{c_2}(h_i)^{-1}t_{c_2}^{-1}$. We remark that k_i has a six order. We see that

$$\begin{aligned} t_{a_2}t_{c_2}^{-1} &= t_{h_i(c_2)}t_{c_2}^{-1} = h_i t_{c_2}(h_i)^{-1}t_{c_2}^{-1} = h_i k_i, \\ t_{x_1}t_{c_1}^{-1} &= t_{(f_i)^4(a_2)}(t_{(f_i)^4(c_2)})^{-1} = (f_i)^4 t_{a_2}t_{c_2}^{-1}(f_i)^{-4} = (f_i)^4 h_i k_i (f_i)^{-4}, \\ t_{x_2}t_{a_3}^{-1} &= t_{(f_i)^2(a_2)}(t_{(f_i)^2(c_2)})^{-1} = (f_i)^2 t_{a_2}t_{c_2}^{-1}(f_i)^{-2} = (f_i)^2 h_i k_i (f_i)^{-2}. \end{aligned}$$

Hence, by the lantern relation and above equations, we have

$$t_{a_1} = ((f_i)^4 h_i k_i (f_i)^{-4})((f_i)^2 h_i k_i (f_i)^{-2})(h_i k_i).$$

3.3. **Generating mapping class groups by elements of order six.**

Now we begin the proof of the theorem 1.4. Let G_i denote the subgroup of $\mathrm{Mod}(\Sigma_{g,0})$ generated by f_i, h_i and k_i for $i = 1, 2, \ldots, 5$ and let G_j denote the subgroup of $\mathrm{Mod}(\Sigma_{g,0})$ generated by f_j, h_j, k_i, and r_j for $j = 6, 7$. In previous subsection, we can find t_{a_1} is in G_i for $i = 1, 2, \ldots, 7$. Let a and b be simple closed curves on $\Sigma_{g,0}$. For $f \in G_i$, the symbol $a \overset{f}{\longleftrightarrow} b$ means that $f(a) = b$ or $f^{-1}(a) = b$.

In the case of $g = 5m$, f_1 and h_1 can map a_1 to all b_i and c_i as shown in Fig. 35. Hence, we have, for all i, t_{b_i} and t_{c_i} are in G_1. Since we have $(h_1)^5(a_1) = a_2$, t_{a_2} is in G_1. Therefore, all Humphries's generators are in G_1. As is the case with $g = 5m$, in the case of $g = 5m+1$, $g = 5m+2$, $g = 5m+3$, $g = 5m+4$ and $g = 5, 6$ for $j = 2, 3, \ldots, 7$, f_j and h_j can map a_1 to all b_i and c_i as shown in Fig. 36, 37, 38, 39, 40 and 41 respectively. Hence, we have ,for all i, t_{b_i} and t_{c_i} are in G_j. Since we have $(h_j)^5(a_1) = a_2$, t_{a_2} is in G_j. Therefore, all Humphries's generators are in G_j. We prove that G_i is equal to $\mathrm{Mod}(\Sigma_{g,0})$ for $g \geq 7$ and $i = 1, 2, \ldots, 7$.

$$\begin{tikzcd}
a_1 \arrow[r, leftrightarrow, "(h_1)^3"] & c_3 \arrow[r, leftrightarrow, "h_1"] & c_2 \arrow[r, leftrightarrow, "h_1"] \arrow[d, leftrightarrow, "f_1"'] & a_2 & & & & & & & & & \\
& & c_4 \arrow[r, leftrightarrow, "f_1"] & c_1 \arrow[r, leftrightarrow, "f_1"] & c_5 \arrow[r, leftrightarrow, "h_1"] & c_6 \arrow[r, leftrightarrow, "h_1"] & c_7 \arrow[r, leftrightarrow, "f_1"] & c_8 \arrow[r, leftrightarrow, "f_1"] & \cdots \arrow[r, leftrightarrow, "f_1"] & c_{g-5} \arrow[r, leftrightarrow, "h_1"] & c_{g-4} \arrow[r, leftrightarrow, "h_1"] & c_{g-3} \arrow[r, leftrightarrow, "h_1"] & c_{g-2} \arrow[d, leftrightarrow, "h_1"] \\
& & & & b_{g-10} \arrow[r, leftrightarrow, "f_1"] \arrow[d, leftrightarrow, "f_1"'] & \cdots \arrow[r, leftrightarrow, "f_1"] & b_7 \arrow[r, leftrightarrow, "h_1"] & b_6 \arrow[r, leftrightarrow, "h_1"] & b_5 & & & & a_{g-1} \arrow[d, leftrightarrow, "f_1"] \\
& & & & b_{g-9} \arrow[r, leftrightarrow, "h_1"] & b_{g-8} \arrow[r, leftrightarrow, "f_1"] & b_{g-7} \arrow[r, leftrightarrow, "f_1"] & b_{g-6} \arrow[r, leftrightarrow, "f_1"] & b_{g-5} \arrow[r, leftrightarrow, "h_1"] & b_{g-4} \arrow[r, leftrightarrow, "h_1"] & b_{g-3} \arrow[r, leftrightarrow, "h_1"] & b_{g-2} \arrow[r, leftrightarrow, "h_1"] & b_{g-1} \arrow[d, leftrightarrow, "f_1"] \\
& & & & & & & & b_2 \arrow[r, leftrightarrow, "h_1"] & b_3 \arrow[r, leftrightarrow, "h_1"] & b_4 \arrow[r, leftrightarrow, "h_1"] & b_g \arrow[r, leftrightarrow, "f_1"] \arrow[d, leftrightarrow, "h_1"] & c_{g-1} \\
& & & & & & & & & & & b_1 &
\end{tikzcd}$$

Figure 35

$$\begin{tikzcd}
a_1 \arrow[r, leftrightarrow, "(h_2)^3"] & c_3 \arrow[r, leftrightarrow, "h_2"] & c_2 \arrow[r, leftrightarrow, "h_2"] \arrow[d, leftrightarrow, "f_2"'] & a_2 \\
& & c_4 \arrow[r, leftrightarrow, "f_2"] & c_1 \arrow[r, leftrightarrow, "f_2"] & c_5 \arrow[r, leftrightarrow, "h_2"] & c_6 \arrow[r, leftrightarrow, "h_2"] & c_7 \arrow[r, leftrightarrow, "f_2"] & c_8 \arrow[r, leftrightarrow, "f_2"] & \cdots \arrow[r, leftrightarrow, "h_2"] & c_{g-5} \arrow[r, leftrightarrow, "h_2"] & c_{g-4} \arrow[r, leftrightarrow, "f_2"] & c_{g-3} \arrow[r, leftrightarrow, "f_2"] & c_{g-2} \arrow[d, leftrightarrow, "f_2"] \\
& & & & & & & & & & & & c_{g-1} \arrow[d, leftrightarrow, "f_2"] \\
& & & b_{g-10} \arrow[r, leftrightarrow, "f_2"] \arrow[d, leftrightarrow, "h_2"'] & \cdots \arrow[r, leftrightarrow, "f_2"] & b_7 \arrow[r, leftrightarrow, "h_2"] & b_6 \arrow[r, leftrightarrow, "h_2"] & b_5 & & & & b_1 \arrow[d, leftrightarrow, "h_2"] & a_g \arrow[d, leftrightarrow, "h_2"] \\
& & & b_{g-9} \arrow[r, leftrightarrow, "f_2"] & b_{g-8} \arrow[r, leftrightarrow, "f_2"] & b_{g-7} \arrow[r, leftrightarrow, "f_2"] & b_{g-6} \arrow[r, leftrightarrow, "f_2"] & b_{g-5} \arrow[r, leftrightarrow, "h_2"] & b_{g-4} \arrow[r, leftrightarrow, "f_2"] & b_{g-3} \arrow[r, leftrightarrow, "f_2"] & b_{g-2} \arrow[r, leftrightarrow, "f_2"] & b_{g-1} \arrow[r, leftrightarrow, "f_2"] \arrow[d, leftrightarrow, "h_2"] & b_g \\
& & & & & & & & & b_2 \arrow[r, leftrightarrow, "h_2"] & b_3 \arrow[r, leftrightarrow, "h_2"] & b_4
\end{tikzcd}$$

FIGURE 36

$$\begin{tikzcd}
a_1 \arrow[r, leftrightarrow, "(h_3)^3"] & c_3 \arrow[r, leftrightarrow, "h_3"] & c_2 \arrow[r, leftrightarrow, "h_3"] \arrow[d, leftrightarrow, "f_3"'] & a_2 \\
& & c_4 \arrow[r, leftrightarrow, "f_3"] & c_1 \arrow[r, leftrightarrow, "f_3"] & c_5 \arrow[r, leftrightarrow, "f_3"] & c_6 \arrow[r, leftrightarrow, "f_3"] & c_7 \arrow[r, leftrightarrow, "f_3"] & c_8 \arrow[r, leftrightarrow, "h_3"] & \cdots \arrow[r, leftrightarrow, "h_3"] & c_{g-5} \arrow[r, leftrightarrow, "h_3"] & c_{g-4} \arrow[r, leftrightarrow, "f_3"] & c_{g-3} \arrow[r, leftrightarrow, "f_3"] & c_{g-2} \arrow[d, leftrightarrow, "f_3"] \\
& & & & b_{g-10} \arrow[r, leftrightarrow, "h_3"] \arrow[d, leftrightarrow, "f_3"'] & \cdots \arrow[r, leftrightarrow, "f_3"] & b_7 \arrow[r, leftrightarrow, "h_3"] & b_6 \arrow[r, leftrightarrow, "h_3"] & b_5 & & & b_1 \arrow[d, leftrightarrow, "h_3"] & a_{g-1} \arrow[d, leftrightarrow, "h_3"] \\
& & & & b_{g-9} \arrow[r, leftrightarrow, "f_3"] & b_{g-8} \arrow[r, leftrightarrow, "f_3"] & b_{g-7} \arrow[r, leftrightarrow, "f_3"] & b_{g-6} \arrow[r, leftrightarrow, "h_3"] & b_{g-5} \arrow[r, leftrightarrow, "f_3"] & b_{g-4} \arrow[r, leftrightarrow, "f_3"] & b_{g-3} \arrow[r, leftrightarrow, "f_3"] & b_{g-2} \arrow[r, leftrightarrow, "f_3"] \arrow[d, leftrightarrow, "h_3"] & b_{g-1} \arrow[d, leftrightarrow, "h_3"] \\
& & & & & & & & & b_2 \arrow[r, leftrightarrow, "h_3"] & b_3 \arrow[r, leftrightarrow, "h_3"] & b_4 & c_{g-1} \arrow[d, leftrightarrow, "h_3"] \\
& & & & & & & & & & & & b_g
\end{tikzcd}$$

FIGURE 37

$$\begin{tikzcd}
a_1 \arrow[r, leftrightarrow, "(h_4)^3"] & c_3 \arrow[r, leftrightarrow, "h_4"] & c_2 \arrow[r, leftrightarrow, "h_4"] \arrow[d, leftrightarrow, "f_4"'] & a_2 \\
& & c_4 \arrow[r, leftrightarrow, "f_4"] & c_1 \arrow[r, leftrightarrow, "f_4"] & c_5 \arrow[r, leftrightarrow, "h_4"] & c_6 \arrow[r, leftrightarrow, "h_4"] & c_7 \arrow[r, leftrightarrow, "f_4"] & c_8 \arrow[r, leftrightarrow, "f_4"] & \cdots \arrow[r, leftrightarrow, "h_4"] & c_{g-5} \arrow[r, leftrightarrow, "f_4"] & c_{g-4} \arrow[r, leftrightarrow, "f_4"] & c_{g-3} \arrow[r, leftrightarrow, "f_4"] & c_{g-2} \arrow[d, leftrightarrow, "f_4"] \\
& & & & b_{g-10} \arrow[r, leftrightarrow, "h_4"] \arrow[d, leftrightarrow, "f_4"'] & \cdots \arrow[r, leftrightarrow, "f_4"] & b_7 \arrow[r, leftrightarrow, "h_4"] & b_6 \arrow[r, leftrightarrow, "h_4"] & b_5 & & & b_1 \arrow[d, leftrightarrow, "h_4"] & a_{g-1} \arrow[d, leftrightarrow, "h_4"] \\
& & & & b_{g-9} \arrow[r, leftrightarrow, "f_4"] & b_{g-8} \arrow[r, leftrightarrow, "f_4"] & b_{g-7} \arrow[r, leftrightarrow, "f_4"] & b_{g-6} \arrow[r, leftrightarrow, "h_4"] & b_{g-5} \arrow[r, leftrightarrow, "f_4"] & b_{g-4} \arrow[r, leftrightarrow, "f_4"] & b_{g-3} \arrow[r, leftrightarrow, "f_4"] & b_{g-2} \arrow[r, leftrightarrow, "f_4"] \arrow[d, leftrightarrow, "h_4"] & b_{g-1} \arrow[d, leftrightarrow, "h_4"] \\
& & & & & & & & & b_2 \arrow[r, leftrightarrow, "h_4"] & b_3 \arrow[r, leftrightarrow, "h_4"] & b_4 & c_{g-1} \arrow[d, leftrightarrow, "h_4"] \\
& & & & & & & & & & & & b_g
\end{tikzcd}$$

FIGURE 38

$$
\begin{tikzcd}
a_1 \arrow[r, leftrightarrow, "(h_5)^3"] & c_3 \arrow[r, leftrightarrow, "h_5"] & c_2 \arrow[r, leftrightarrow, "h_5"] \arrow[d, leftrightarrow, "f_5"'] & a_2 \\
& & c_4 \arrow[r, leftrightarrow, "f_5"] & c_1 \arrow[r, leftrightarrow, "f_5"] & c_5 \arrow[r, leftrightarrow, "h_5"] & c_6 \arrow[r, leftrightarrow, "h_5"] & c_7 \arrow[r, leftrightarrow, "f_5"] & c_8 \arrow[r, leftrightarrow, "f_5"] & \cdots \arrow[r, leftrightarrow, "f_5"] & c_{g-5} \arrow[r, leftrightarrow, "f_5"] & c_{g-4} \arrow[r, leftrightarrow, "f_5"] & c_{g-3} \arrow[r, leftrightarrow, "f_5"] & a_{g-2} \arrow[d, leftrightarrow, "h_5"] \\
& & & & & & & & b_g \arrow[r, leftrightarrow, "f_5"] & c_{g-1} \arrow[r, leftrightarrow, "f_5"] & b_{g-1} \arrow[r, leftrightarrow, "h_5"] & c_{g-2} \arrow[r, leftrightarrow, "h_5"] & b_{g-2} \arrow[d, leftrightarrow, "f_5"] \\
& & & & & b_{g-9} \arrow[r, leftrightarrow, "f_5"] \arrow[d, leftrightarrow, "f_5"'] & b_{g-8} \arrow[r, leftrightarrow, "f_5"] & b_{g-7} \arrow[r, leftrightarrow, "h_5"] & b_{g-6} \arrow[r, leftrightarrow, "f_5"] & b_{g-5} \arrow[r, leftrightarrow, "f_5"] & b_{g-4} \arrow[rr, leftrightarrow, "f_5"] & & b_{g-3} \arrow[r, leftrightarrow, "h_5"] \arrow[d, leftrightarrow, "h_5"] & b_1 \\
& & & & & b_{g-10} \arrow[r, leftrightarrow, "f_5"] & \cdots \arrow[r, leftrightarrow, "f_5"] & b_7 \arrow[r, leftrightarrow, "h_5"] & b_6 \arrow[r, leftrightarrow, "h_5"] & b_5 & b_2 \arrow[r, leftrightarrow, "h_5"] & b_3 \arrow[r, leftrightarrow, "h_5"] & b_4
\end{tikzcd}
$$

FIGURE 39

$$
\begin{tikzcd}
& & & & & a_2 \arrow[d, leftrightarrow, "r_6"] \\
a_1 \arrow[r, leftrightarrow, "r_6"] & b_1 \arrow[r, leftrightarrow, "h_6"] & b_5 \arrow[r, leftrightarrow, "h_6"] \arrow[d, leftrightarrow, "r_6"] & b_4 \arrow[r, leftrightarrow, "h_6"] \arrow[d, leftrightarrow, "r_6"] & b_3 \arrow[r, leftrightarrow, "h_6"] & b_2 \arrow[d, leftrightarrow, "r_6"] \\
& & a_5 \arrow[d, leftrightarrow, "f_6"] & c_4 & & c_2 \arrow[d, leftrightarrow, "r_6"] \\
& & c_1 & & & c_3
\end{tikzcd}
$$

FIGURE 40

$$
\begin{tikzcd}
& & & & & c_4 \arrow[d, leftrightarrow, "h_7"] \\
a_1 \arrow[r, leftrightarrow, "r_7"] & b_1 \arrow[r, leftrightarrow, "r_7"] & c_1 \arrow[r, leftrightarrow, "r_7"] & b_2 \arrow[r, leftrightarrow, "h_7"] & b_3 \arrow[r, leftrightarrow, "r_7"] & c_3 \arrow[r, leftrightarrow, "r_7"] \arrow[d, leftrightarrow, "h_7"] & b_4 \arrow[r, leftrightarrow, "h_7"] & b_5 \arrow[r, leftrightarrow, "r_7"] & c_5 \arrow[r, leftrightarrow, "r_7"] & b_6 \\
& & & & & c_2 \arrow[d, leftrightarrow, "h_7"] \\
& & & & & a_2
\end{tikzcd}
$$

FIGURE 41

4. Proof of Theorem 1.5

In this section, we proof theorem 1.5. We use the following lemma in order to prove theorem 1.5.

Lemma 4.1. *Let G and N be groups and let H and K be subgroups of G. Suppose that the sequence*

$$1 \to H \xrightarrow{i} G \xrightarrow{\pi} N \to 1$$

is exact. If K contains $i(H)$ and the restriction of π to K is a surjection onto N, then we have that $K = G$.

Proof. Let g be any element of G. If g is in $i(H)$, then K contains g by the assumption $i(H) \subset K$. We suppose that g is not in $i(H)$. Since the restriction $\pi\ |_K$ is surjection, there exists $k \in K$ such that $\pi(g) = \pi(k)$. Since $\pi(gk^{-1}) = e$, we see that $gk^{-1} \in Ker\ \pi = Im\ i$. Therefore, there exists $h \in H$ such that $gk^{-1} = i(h)$. Since $i(h) \in K$, we have $g = i(h)k \in K$. Hence, $G \subset K$. □

Since we have the following exact sequence

$$1 \to \mathrm{PMod}(N_{g,n}) \to \mathrm{Mod}(N_{g,n}) \xrightarrow{\pi} \mathrm{Sym}_n \to 1,$$

we have following corollary.

Corollary 4.2. *Let K denote the subgroup of $\mathrm{Mod}(N_{g,n})$. If K contains $\mathrm{PMod}(N_{g,n})$ and the restriction π to K is a surjection to Sym_n, then K is equal $\mathrm{Mod}(N_{g,n})$.*

We recall the Korkmaz's generating set for $\mathrm{PMod}(N_{g,n})$. Let Λ be the set of simple closed curves indicated in Fig. 4 for $g = 2r+1$, and in Fig. 5 for $g = 2r+2$. Hence

$$\Lambda = \{a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_{r-1}, d_1, d_2, \dots, d_r, e_1, e_2, \dots, e_{n-1}\}$$

for $g = 2r+1$, and

$$\Lambda = \{a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_{r+1}, c_1, c_2, \dots, c_r, d_1, d_2, \dots, d_r, e_1, e_2, \dots, e_{n-1}\}$$

fro $g = 2r+2$. In the figures, we choose orientations of local neighbourhoods of simple closed curves in *lambda*, the orientation is that the arrow points to the right if we approach the curve. Therefore for the simple closed curve a in Λ, the Dehn twist about a is determined by this particular choice of orientation.

Let α_i be the one-sided simple closed curve based at x_i for $i = 1, 2, \dots, n$ as in Fig. 42. If $g = 2r+2$, let β_i be the one-sided simple closed curve based at x_i as in Fig. 42. For $i = 1, 2, \dots, n$, let v_i and w_i be puncture slides along α_i and β_i, respectively.

Let y be a crosscap slide such that y^2 is the Dehn twist along ξ.

Theorem 4.3. *For $g \geq 3$, the pure mapping class group $\mathrm{PMod}(N_{g,n})$ is generated by*

(i) $\{t_l \mid l \in \Lambda\} \cup \{v_i \mid 1 \leq i \leq n\} \cup \{y\}$ *if g is odd, and*

(ii) $\{t_l \mid l \in \Lambda\} \cup \{v_i, w_i \mid 1 \leq i \leq n\} \cup \{y\}$ *if g is even.*

The following theorem can be deduced from Korkmaz's generating set by using the method of Humphries. Set

$$\Lambda' = \{a_1, a_2, \dots, a_r, b_1, b_2, c_1, c_2, \dots, c_{r-1}, d_1, d_2, e_1, e_2, \dots, e_{n-1}\}$$

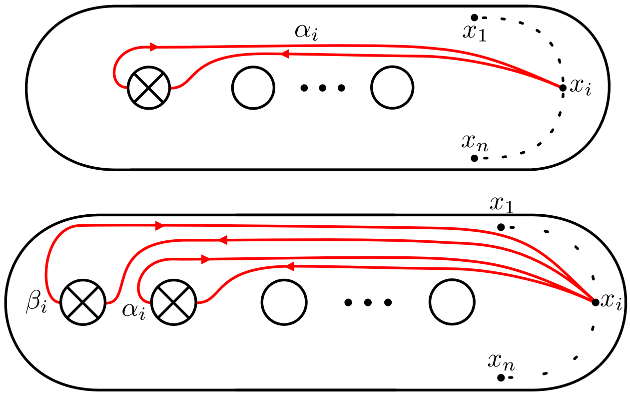

FIGURE 42. Simple closed curves $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$.

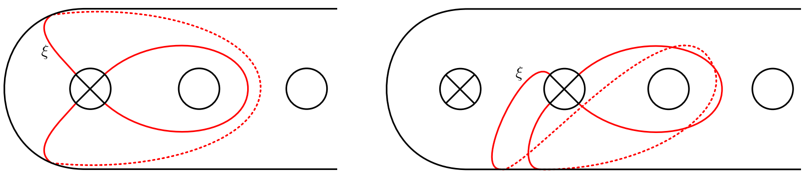

FIGURE 43. Simple closed curve ξ.

for $g = 2r + 1$, and

$$\Lambda' = \{a_1, a_2, \ldots, a_r, b_1, b_2, b_{r+1}, c_1, c_2, \ldots, c_r, d_1, d_2, e_1, e_2, \ldots, e_{n-1}\}$$

fro $g = 2r + 2$.

Theorem 4.4. *For $g \geq 3$, the pure mapping class group* $\mathrm{PMod}(N_{g,n})$ *is generated by*

(i) $\{t_l, v_i, y \mid l \in \Lambda', 1 \leq i \leq n\}$ *if g is odd and*

(i) $\{t_l, v_i, w_i, y \mid l \in \Lambda', 1 \leq i \leq n\}$ *if g is even.*

4.1. **In the case of odd genus.** In this subsection, we suppose that $g = 2r + 1$ for a positive integer $r \geq 6$. Let us consider the two models of $N_{g,b}$ as shown in Fig. 44 and 45. (In these pictures, we will suppose that $r = 2k$ and the number of punctures $b = 2l + 1$ is odd for a interger $l \geq 0$.) We deform the surface in Fig. 44 from the surface in Fig. 4 by diffeomorphism ψ such that the simple closed curves and the punctures in Fig. 4 map to the curves and punctures with same label in Fig. 44, and the deformed surface is symmetrical about a plane across the central of this surface, which we call mirror. Let σ' be a reflection of this surface in the mirror and let σ be a product $\psi^{-1}\sigma\psi$. Then σ is involution in $\mathrm{Mod}(N_{g,n})$. In the same way, we can define a involution τ as a reflection in a mirror in Fig. 45.

We will construct the third involution I. We cut the surface $N_{g,n}$ along $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$ to obtain the surfaces S_1 and S_2.(see Fig.46) S_1 is a sphere bounded by $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$ and S_2 is a non-orientable surface of genus $g - 8$ with b punctures and 5 boundaries. Fig. 47 gives the involutions $\overline{I}$ and $\widetilde{I}$ on

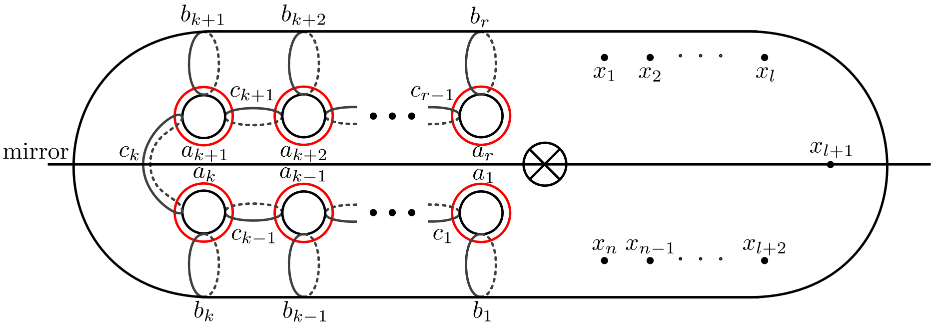

FIGURE 44. Involution $\sigma : N_{g,n} \to N_{g,n}$

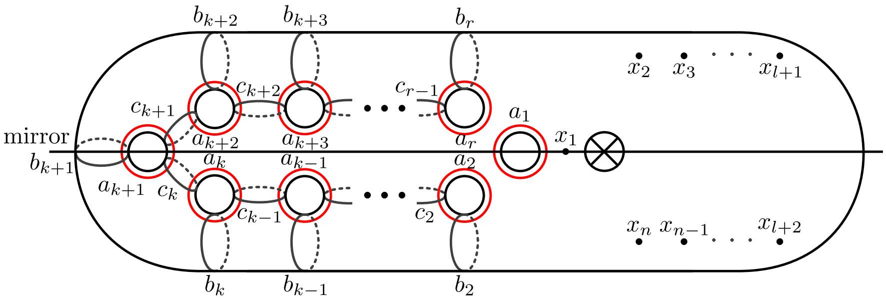

FIGURE 45. Involution $\tau : N_{g,n} \to N_{g,n}$

S_1 and S_2, respectively. Since $\overline{I}$ and $\widetilde{I}$ coincide on the boundaries, they natually define a involution $I : N_{g,n} \to N_{g,n}$.

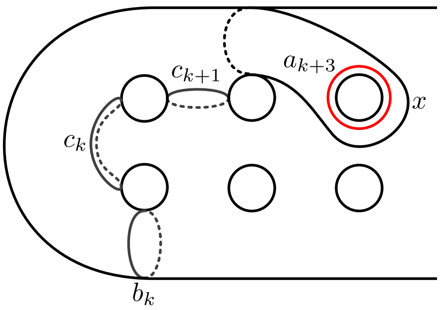

FIGURE 46. The curves $a_{k+3}, b_k, c_k, c_{k+1}$ and x

From the construction of I, we see the following:

$$I(a_{k+3}) = c_{k+1}, I(c_k) = b_k,$$
$$I(b_1) = d_1, I(b_2) = d_2.$$

Let ρ_1 be the product τt_{a_1}. Since τ fixes a_1 and the restriction $\tau\mid_{N_{a_1}}$ reverses the orientation, by Lemma 2.7, we see that

$$\tau t_{a_1} \tau = t_{a_1}^{-1}.$$

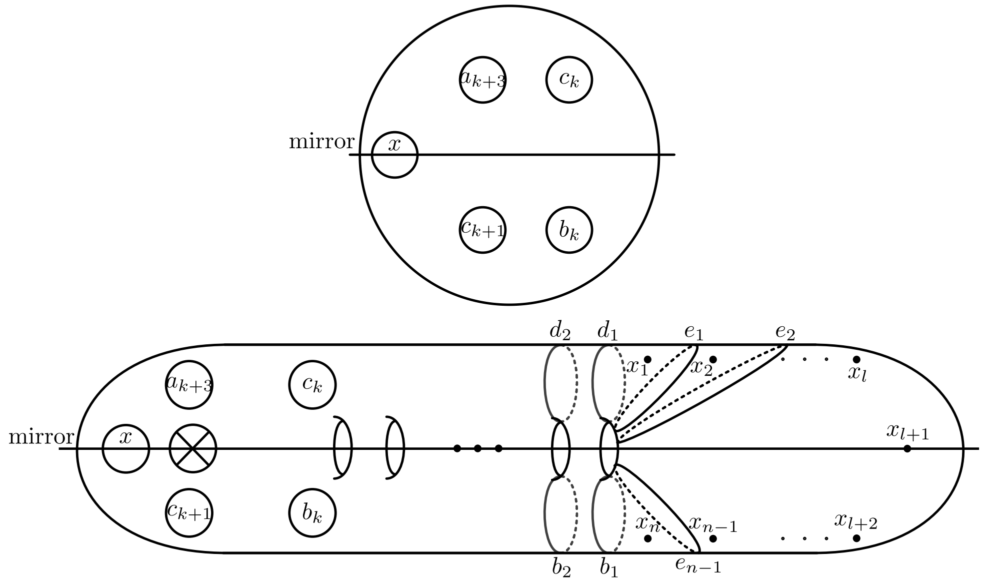

FIGURE 47. Involutions $\overline{I}$ and $\widetilde{I}$

Hence, τ is an involution. Then we can get following lemma.

Lemma 4.5. *Dehn twists $t_{a_1}, t_{a_2}, \ldots, t_{a_r}, t_{b_1}, t_{b_2}$, $t_{c_1}, t_{c_2}, \ldots, t_{c_{r-1}}$,$t_{d_1}$ and t_{d_2} are products of involutions σ,τ,ρ_1 and I.*

Proof. Let R be the product $\tau\sigma$. We can see that R acts as following by Fig. 44 and Fig. 45.

$$(1) R(a_1) = a_2, R(a_2) = a_3, \ldots, R(a_k) = a_{k+1}, R(a_{k+1}) = a_{k+2}, \ldots, R(a_{r-1}) = a_r.$$

$$(2) R(b_1) = b_2, R(b_2) = b_3, \ldots, R(b_k) = b_{k+1}, R(b_{k+1}) = b_{k+2}, \ldots, R(b_{r-1}) = b_r.$$

$$(3) R(c_1) = c_2, R(c_2) = c_3, \ldots, R(c_k) = c_{k+1}, R(c_{k+1}) = c_{k+2}, \ldots, R(c_{r-2}) = c_{r-1}.$$

Clearly, we can see that t_{a_1} is a product of τ and ρ_1. By (1) and Lemma 2.7,

$$t_{a_i} = R t_{a_{i-1}} R^{-1}. (i = 2, 3, \ldots, r)$$

So $t_{a_1}, t_{a_2}, \ldots, t_{a_r}$ are products of σ, τ, and ρ_1.

By construction of I and Lemma 2.7, we have

$$t_{c_{k+1}} = I t_{a_{k+3}}^{-1} I.$$

By (3) and Lemman 2.7, we see that

$$t_{c_j} = R t_{c_{j-1}} R^{-1}, \qquad (j = k+2, k+3, \ldots, r-1)$$

$$t_{c_j} = R^{-1} t_{c_{j+1}} R. \qquad (j = 1, 2, \ldots, k)$$

Hence, $t_{c_1}, t_{c_2}, \ldots, t_{c_{r-1}}$ are products of σ, τ, ρ_1, and I.

Also, we have

$$t_{b_k} = I t_{c_k}^{-1} I.$$

Similar to the above, by (2) and Lemma 2.7, we see that

$$t_{b_i} = R t_{b_{i-1}} R^{-1}, \qquad (i = k+1, k+3, \ldots, r)$$

$$t_{b_i} = R^{-1}t_{b_{i+1}}R. \qquad (i = 1, 2, \ldots, k-1)$$

Hence, $t_{b_1}, t_{b_2}, \ldots, t_{b_r}$ are product of σ, τ, ρ_1, and I.

Finally, Since $I(b_1) = d_1$ and $I(b_2) = d_2$, we have

$$t_{d_1} = It_{b_1}^{-1}I, t_{d_2} = It_{b_2}^{-1}I.$$

t_{d_1} and t_{d_2} are products of σ, τ, ρ_1, and I. □

τ maps α_1 to itself but reverses the orientation of α_1. By Lemma 2.9, we see that

$$\tau v_1 \tau = v_1^{-1}.$$

Now let ρ_2 denote a product of τv_1.Then ρ_2 is a involution.

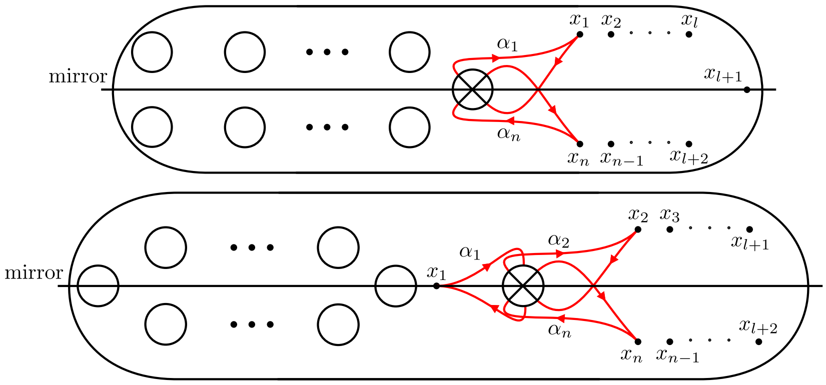

FIGURE 48. Involutions σ and τ

Lemma 4.6. *Puncture slides $v_i (i = 1, 2, \ldots, n)$ is a products of involutions σ, τ and ρ_2.*

Proof. v_1 is a product of τ and ρ_2. In Fig. 48, we fucus the figures which define σ and τ on α_i. $R = \tau\sigma$ acts on α_i as follow.

(4) $R(\alpha_1) = \alpha_2, R(\alpha_2) = \alpha_3, \ldots, R(\alpha_l) = \alpha_{l+1}, R(\alpha_{l+1}) = \alpha_{l+2}, \ldots, R(\alpha_{n-1}) = \alpha_n$.

By (4) and Lemma 2.9, we see that

$$v_j = Rv_{j-1}R^{-1}. \qquad (j = 2, 3, \ldots, n)$$

Hence, v_i is a product of involution σ, τ and ρ_2. □

We consider the diffeomorphism Φ on $N_{g,n}$ which satisfies $\Phi y \Phi^{-1} = Y_{m,a}$ and fixes each puncuters. The right figure in Fig. 49 gives the involution w. Since w fixes m and a but reverses the orientation of m and a, we can see that $wY_{m,a}w = Y_{m^{-1},a^{-1}} = Y_{m,a}^{-1}$.

Let W be a product of $\Phi^{-1}w\Phi$ and let ρ_3 be a product of Wy. Clearly, we can see that W is an involution. Since we have

$$\begin{aligned} WyW &= \Phi^{-1}w(\Phi y \Phi^{-1})w\Phi \\ &= \Phi^{-1}(wY_{m,a}w)\Phi \\ &= \Phi^{-1}Y_{m,a}^{-1}\Phi = y^{-1}, \end{aligned}$$

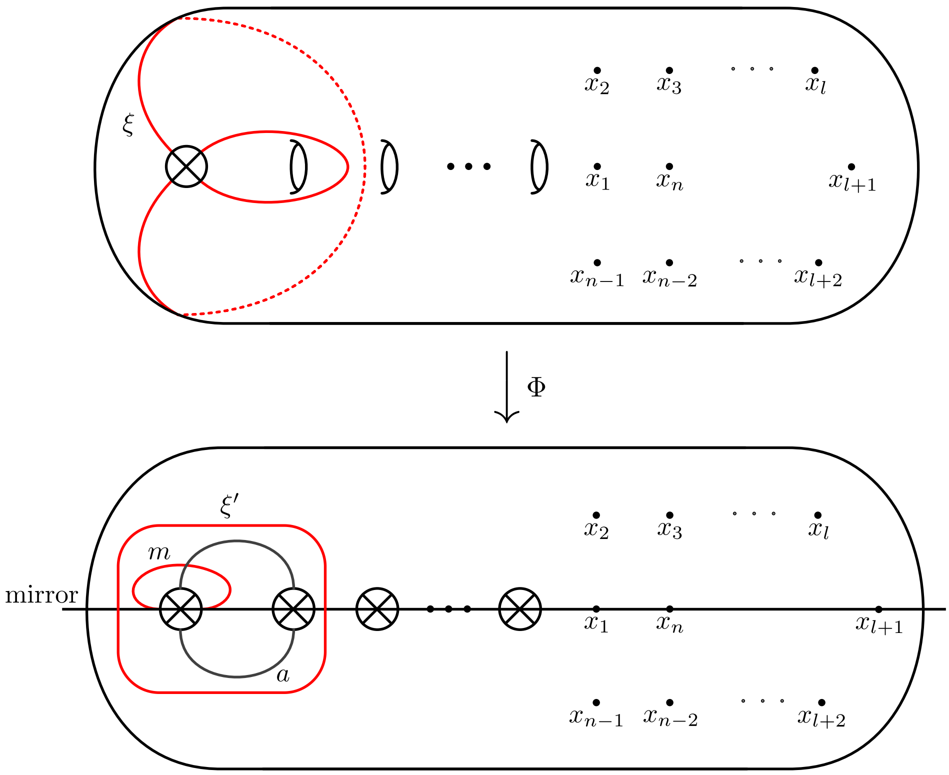

FIGURE 49. Diffeomorphism Φ

ρ_3 is a involution. So we can get the following lemma.

Lemma 4.7. *The Y-homeomorphism y is the product of involutions W and ρ_3.*

We need the another involution to generate $t_{e_1}, t_{e_2}, \ldots, t_{e_{n-1}}$. Fig. 50 gives the involution J which is a reflection in the mirror.

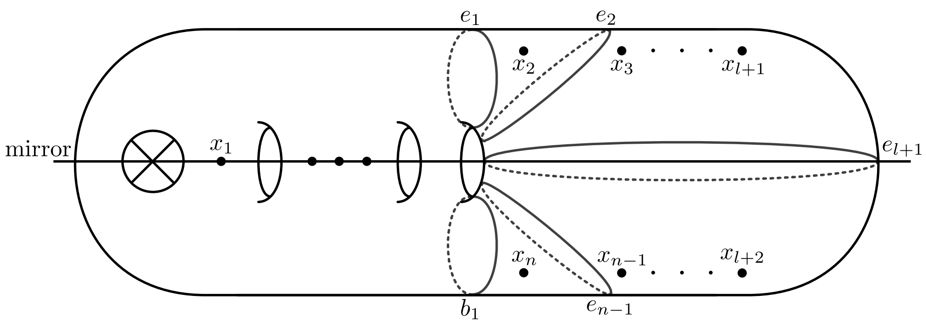

FIGURE 50. Involution J

Lemma 4.8. $t_{e_1}, t_{e_2}, \ldots, t_{e_{n-1}}$ *are products of involutions σ, τ, I, J and ρ_1.*

Proof. Since we have $J(n_1) = e_1$, $t_{e_1} = Jt_{n_1}^{-1}J$. t_{e_1} is the product of σ, τ, I, J, ρ_1. Let T denote the product of JI. We see that T acts as following.

$$(5) T(e_1) = e_2, T(e_2) = e_3, \ldots, T(e_l) = e_{l+1}, T(e_{l+1}) = e_{l+2}, \ldots, T(e_{n-2}) = e_{n-1}.$$

Hence, for $(i = 2, 3, \ldots, n-1)$, we can see that $t_{e_i} = Tt_{e_{i-1}}T^{-1}$. So, t_{e_i} is a product of σ, τ, I, J and ρ_1. □

Let the subgroup G of $\mathrm{Mod}(N_{g,n})$ be generated by $\sigma, \tau, W, I, J, \rho_1, \rho_2$ and ρ_3.

Proof of Theorem 1.5 for genus $g = 2r + 1$. We see that G contains $\mathrm{PMod}(N_{g,n})$ since all Korkmaz's generators for $\mathrm{PMod}(N_{g,n})$ are in G by Lemma 4.5, 4.6, 4.7 and 4.8.

When we consider the actions of σ, τ and W on the punctures, we can see that

$$\pi(\sigma) = (1, n)(2, n-1)\ldots(l, l+2)(l+1),$$
$$\pi(\tau) = (2, n)(3, n-1)\ldots(l+1, l+2)(1),$$
$$\pi(W) = (2, n-1)(3, n-2)\ldots(l, l+2)(1)(l+1)(n).$$

By the following lemma, the restriction $\pi|_G \colon G \to \mathrm{Sym}_n$ is a surjection. Hence, we can see that $G = \mathrm{Mod}(N_{g,n})$ by Lemma 4.1. □

Lemma 4.9. *The group* Sym_n *is generated by folllowing elements,*

$$r_1 = (1, b)(2, n-1)\ldots(l, l+2)(l+1),$$
$$r_2 = (2, b)(3, n-1)\ldots(l+1, l+2)(1),$$
$$r_3 = (2, n-1)(3, n-2)\ldots(l, l+2)(1)(l+1)(n).$$

4.2. **In the case of even genus.** In this section, We suppose that $g = 2r + 2$. Similar to odd case, let us consider the two models of $N_{g,n}$ as shown in Fig. 51 and 52. (In these pictures, we will suppose that $r = 2k+1$ and the number of punctures $b = 2l$ is even.) Each pictures gives a involution of the $N_{g,n}$, which is the reflection in the mirror.

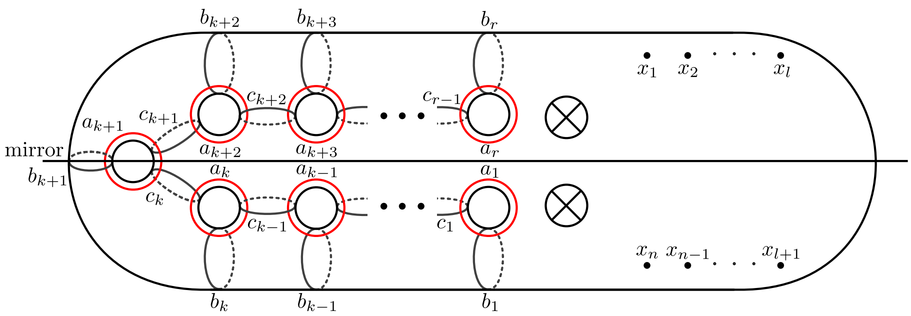

FIGURE 51. Involution $\sigma : N_{g,n} \to N_{g,n}$

We will construct third involution I. We cut the surface $N_{g,n}$ along $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$ to obtain the surfaces S_1 and S_2.(see Fig.53) S_1 is a sphere bounded by $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$ and S_2 is a non-orientable surface of genus $g - 8$ with b punctures and 5 boundaries. Fig.54 gives the involutions $\overline{I}$ and $\widetilde{I}$ on S_1 and S_2, respectively. Since $\overline{I}$ and $\widetilde{I}$ coincide on the boundaries, they natually define a involution $I : N_{g,n} \to N_{g,n}$.

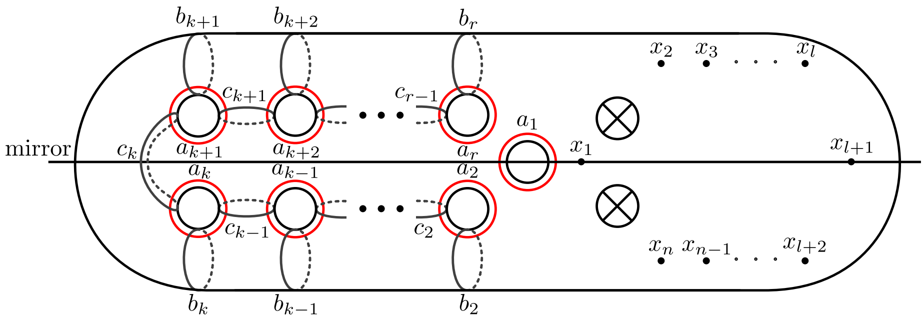

FIGURE 52. Involution $\tau : N_{g,n} \to N_{g,n}$

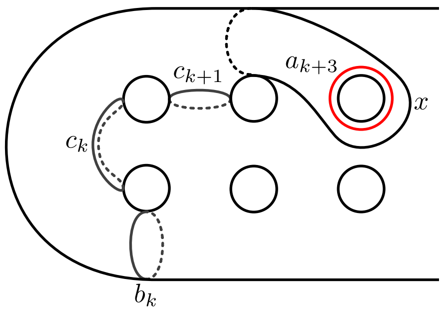

FIGURE 53. The curves $a_{k+3}, b_k, c_k, c_{k+1}$ and x

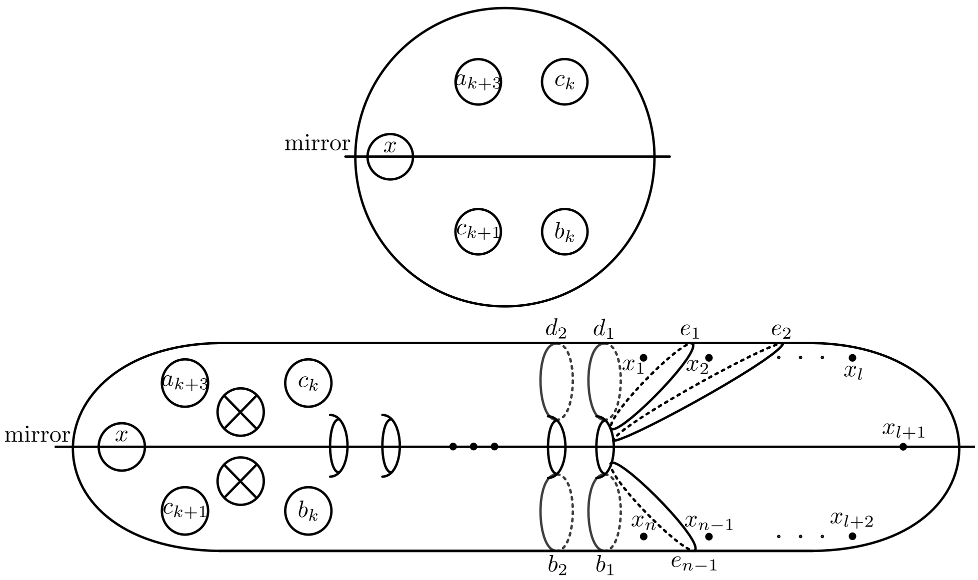

FIGURE 54. involutions $\overline{I}$ and $\widetilde{I}$

From the construction of I, we see the following:

$$I(a_{k+3}) = c_{k+1}, I(c_k) = b_k,$$

$$I(b_1) = d_1, I(b_2) = d_2.$$

Let ρ_1 be the product τt_{a_1}. As in the odd genus case, ρ_1 is an involution. We will prepare three involutions to prove following Lemma. Fig .55 gives the involution J which is a reflection in the mirror. Let ρ_4 and ρ_5 be the products $Jt_{b_{r+1}}$ and Jt_{c_r}, respectively. We can found that ρ_4 and ρ_5 are involutions.

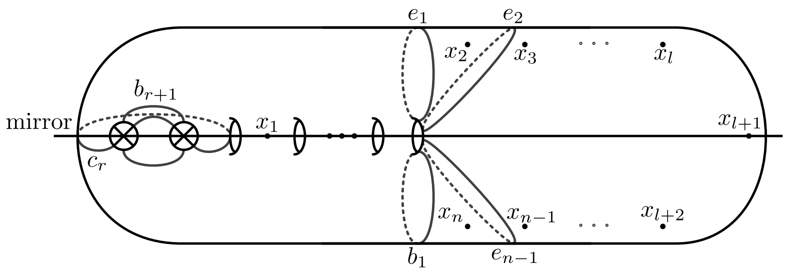

FIGURE 55. Involution J

Lemma 4.10. *Dehn twists t_{a_1}, t_{a_2}, ..., t_{a_r}, t_{b_1}, t_{b_2}, $t_{b_{r+1}}$, t_{c_1}, t_{c_2}, ..., t_{c_r}, t_{d_1}, t_{d_2}, t_{e_1}, t_{e_2}, ..., $t_{e_{n-1}}$ are products of involutions σ, τ, ρ_1, ρ_4, ρ_5, I, and J.*

Proof. Let R be the product $\tau\sigma$. We can see that R acts as following by Fig. 51 and Fig. 52.

$$(1) R(a_1) = a_2, R(a_2) = a_3, \dots, R(a_k) = a_{k+1}, R(a_{k+1}) = a_{k+2}, \dots, R(a_{r-1}) = a_r.$$

$$(2) R(b_1) = b_2, R(b_2) = b_3, \dots, R(b_k) = b_{k+1}, R(b_{k+1}) = b_{k+2}, \dots, R(b_{r-1}) = b_r.$$

$$(3) R(c_1) = c_2, R(c_2) = c_3, \dots, R(c_k) = c_{k+1}, R(c_{k+1}) = c_{k+2}, \dots, R(c_{r-2}) = c_{r-1}.$$

Clearly, we can see that t_{a_1} is a product of τ and ρ_1. By (1) and Lemma 2.7,

$$t_{a_i} = Rt_{a_{i-1}}R^{-1}.(i = 2, 3, \dots, r)$$

So $t_{a_1}, t_{a_2}, \dots, t_{a_r}$ are products of σ, τ, and ρ_1.

By construction of I and Lemma 2.7, we have

$$t_{c_{k+1}} = It_{a_{k+3}}^{-1}I.$$

By (3) and Lemman 2.7, we see that

$$t_{c_j} = Rt_{c_{j-1}}R^{-1}, \qquad (j = k+2, k+3, \dots, r-1)$$

$$t_{c_j} = R^{-1}t_{c_{j+1}}R. \qquad (j = 1, 2, \dots, k)$$

Hence, $t_{c_1}, t_{c_2}, \dots, t_{c_{r-1}}$ are products of σ, τ, ρ_1, and I.

Also, we have

$$t_{b_k} = It_{c_k}^{-1}I$$

. Similar to the above, by (2) and Lemma 2.7, we see that

$$t_{b_i} = Rt_{b_{i-1}}R^{-1}, \qquad (i = k+1, k+3, \dots, r)$$

$$t_{b_i} = R^{-1}t_{b_{i+1}}R. \qquad (i = 1, 2, \dots, k-1)$$

Hence, $t_{b_1}, t_{b_2}, \dots, t_{b_r}$ are product of σ, τ, ρ_1, and I.

By the constructions about ρ_4 and ρ_5,we have $t_{b_{r+1}} = J\rho_4$ and $t_{c_r} = J\rho_5$.

Since $I(b_1) = d_1$ and $I(b_2) = d_2$, we have

$$t_{d_1} = It_{b_1}^{-1}I, t_{d_2} = It_{b_2}^{-1}I.$$

t_{d_1} and t_{d_2} are products of σ, τ, ρ_1, and I. □

We want to generate puncture slides $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ by involutions. we will construct an involution K which fixes α_1 and reverses the orientation of α_1. The involution K is a reflection in the mirror in Fig. 56. Let ρ_2 be the product Kv_1.

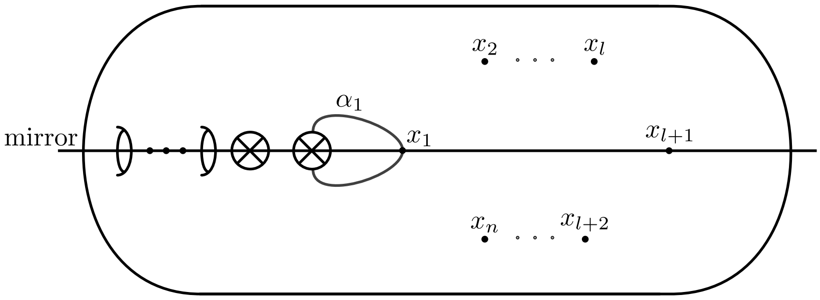

FIGURE 56. Involution K

Lemma 4.11. *Puncture slides* v_i *and* $w_i (i = 1, 2, \ldots, n)$ *are products of involutions* σ, τ, K *and* ρ_2.

Proof. Since v_1 is equal to $K\rho_2$, we can write v_1 as a product of two involutions. Let S and R be products $\tau\sigma$ and $\sigma\tau$, respectively. By the constructions of σ and τ, we have

$$S(\alpha_1) = \alpha_2, S(\alpha_2) = \alpha_3, \ldots, S(\alpha_{n-1}) = \alpha_n,$$

$$R(\beta_n) = \beta_{n-1}, R(\beta_{n-1}) = \beta_{n-2}, \ldots, R(\beta_2) = \beta_1,$$

$$\sigma(\alpha_1) = \beta_n.$$

By lemma 2.9, we can prove this lemma. □

We will write y as a product of involutions. We consider the diffeomorphism $\Phi : N_{g,n} \to N_{g,n}$ which satisfies $\Phi y \Phi^{-1} = Y_{m,a}$ and fixes each punctures as shown Fig. 57.

Let ω be reflection in the mirror as shown bottom figure in Fig. 57. Since ω fixes m and a but reverses the orientation of m and a, we can see that $\omega Y_{m,a} \omega = Y_{m,a}^{-1}$. Let W be the product $\Phi^{-1}\omega\Phi$ and let ρ_3 be the product Wy. We can see that W and ρ_3 are involutions. We can see the following lemma.

Lemma 4.12. *The Y-homeomorphism* y *is the product of involutions* W *and* ρ_3.

Let G be the subgroup of $\mathrm{Mod}(N_{g,n})$ generated by $\sigma, \tau, W, I, J, K, \rho_1, \rho_2, \rho_3, \rho_4$ and ρ_5.

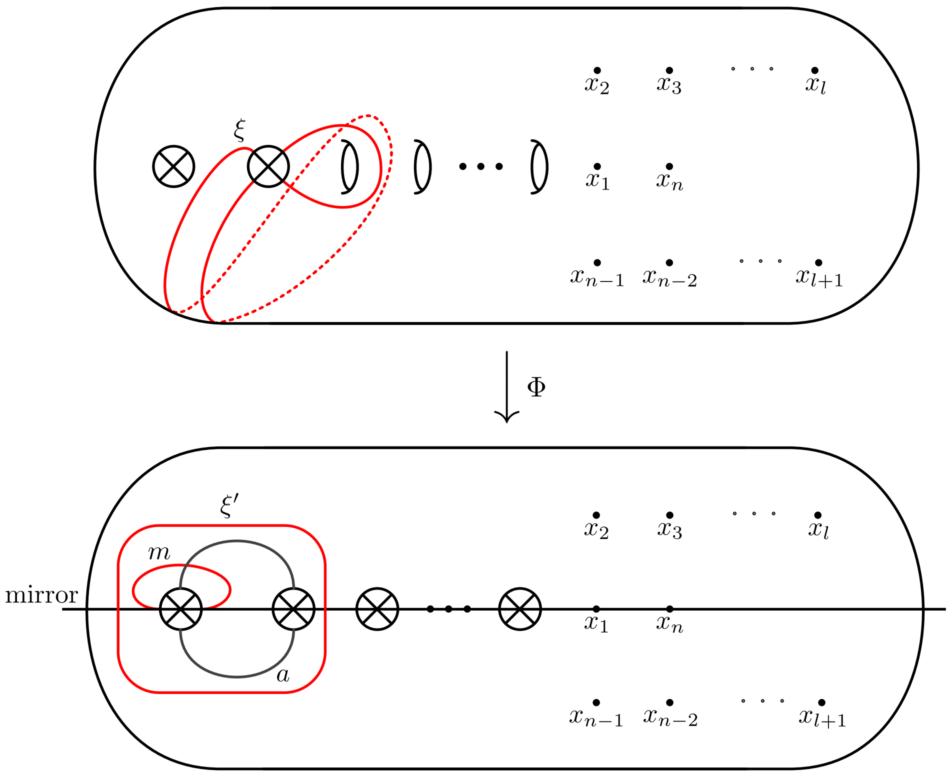

FIGURE 57. Diffeomorphism Φ

Proof of Theorem 1.5 for genus $g = 2r + 2$. We see that G contains $\mathrm{PMod}(N_{g,n})$ since all Korkmaz's generators for $\mathrm{PMod}(N_{g,n})$ are in G by Lemma 4.10, 4.11 and 4.12.

When we consider the actions of σ, τ and W on the punctures, we can see that

$$\pi(\sigma) = (1, n)(2, n-1) \ldots (l, l+1),$$
$$\pi(\tau) = (2, n)(3, n-1) \ldots (l, l+2)(1)(l+1),$$
$$\pi(W) = (2, n-1)(3, n-2) \ldots (l, l+1)(1)(n).$$

By the following lemma, the restriction $\pi\mid_G : G \to \mathrm{Sym}_n$ is a surjection. Hence, we can see that $G = \mathrm{Mod}(N_{g,n})$ by Lemma 4.1. □

Lemma 4.13. *The group Sym_n is generated by folllowing elements,*

$$r_1 = (1, n)(2, n-1) \ldots (l, l+1),$$
$$r_2 = (2, n)(3, n-1) \ldots (l, l+2)(1)(l+1),$$
$$r_3 = (2, n-1)(3, n-2) \ldots (l, l+1)(1)(n).$$

5. CONCLUDING REMARKS

Sezpietowski showed that $\mathrm{Mod}(N_{g,0})$ is generated by four involutions, but the number of involution generators in Theorem 1.5 is more than Sezpietowski's one. Then we can consider following problem:

Problem 5.1. *For $g \geq 4$ and $n \geq 1$, can the mapping class group* $\mathrm{Mod}(N_{g,n})$ *be generated by* 4 *involutions?*

The Coxter group C is defined as a group with the presentation

$$\langle x_1, x_2, \ldots, x_n \mid (x_i x_j)^{m_{ij}=1} \rangle$$

where $m_{ii} = 1$, $m_{ij} \geq 2$ for $i \neq j$ and m_{ij} means no relation between x_i and x_j. Let C_n be the coxter group with following presentation

$$\langle x_1, x_2, \ldots, x_n \mid (x_i)^2 = 1(i = 1, 2, \ldots, n) \rangle.$$

By theorem 1.1, we have the following epimorphisms:

$$\Pi : C_8 \to \mathrm{Mod}(N_{g,n}) \text{ if } g \geq 13 \text{ and } g \text{ is odd, and}$$

$$\Pi : C_{11} \to \mathrm{Mod}(N_{g,n}) \text{ if } g \geq 14 \text{ and } g \text{ is even.}$$

Corollary 5.2. *For an odd $g \geq 13$,* $\mathrm{Mod}(N_{g,n})$ *can be realized as a quotient of a Coxter group on* 8 *generators.*
For an even $g \geq 14$, $\mathrm{Mod}(N_{g,n})$ *can be realized as a quotient of a Coxter group on* 11 *generators.*

A presentation of $\mathrm{Mod}(N_{g,n})$ which consists of involutions as generators are isn't known. If $ker\Pi$ is finite (normally) generated, we have such a presentation.

Problem 5.3. *For $g \geq 13$ and $n \geq 1$, can the kernel $ker\Pi$ be finite generated?*

We have the following corollary by construction of involutions in theorem 1.5:

Corollary 5.4. *Let c be a two-sided simple closed curve. The Dehn twist t_c, a Y-homeomorphism, and a puncture slide are products of two involutions.*

We have the following question.

Problem 5.5. *Whether there is a number C such that f can be written as a product of at most C involutions for any f in* $\mathrm{Mod}(N_{g,n})$*?*

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