

## Study on the stability of stationary solutions of the artificial compressible system

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# Study on the stability of stationary solutions of the artificial compressible system

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## Abstract

Stability of stationary solutions of the incompressible Navier-Stokes system and the corresponding artificial compressible system is considered. Both systems have the same sets of stationary solutions and the incompressible system is obtained from the artificial compressible one in the zero limit of the artificial Mach number  $\epsilon$  which is a singular limit. It is proved that if a stationary solution of the incompressible system is asymptotically stable and the velocity field of the stationary solution satisfies an energy-type stability criterion by variational method with admissible functions being only potential flow parts of velocity fields, then it is also stable as a solution of the artificial compressible one for sufficiently small  $\epsilon$ . The result is applied to the Taylor problem. In general, the range of  $\epsilon$  for which the above mentioned stability result holds shrinks when the spectrum of the linearized operator for the incompressible system approaches to the imaginary axis. This can happen when a stationary bifurcation occurs. It is proved that when a stationary bifurcation from a simple eigenvalue occurs, the range of  $\epsilon$  can be taken uniformly near the bifurcation point to conclude the stability of the bifurcating solution as a solution of the artificial compressible system.

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# 1 Introduction

This thesis studies the stability of stationary solutions of the artificial compressible system which originates from the incompressible Navier-Stokes system

$$\operatorname{div} \mathbf{v} = 0, \quad (1.1)$$

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{g}. \quad (1.2)$$

Here  $\mathbf{v} = {}^\top(v^1(x, t), v^2(x, t), v^3(x, t))$  and  $p = p(x, t)$  denote the unknown velocity field and pressure, respectively, at time  $t > 0$  and position  $x \in \Omega$ , where  $\Omega$  is a bounded domain of  $\mathbb{R}^3$  with smooth boundary  $\partial\Omega$ ;  $\mathbf{g} = \mathbf{g}(x)$  is a given external force. The artificial compressible system for (1.1)–(1.2) then takes the form

$$\epsilon^2 \partial_t p + \operatorname{div} \mathbf{v} = 0, \quad (1.3)$$

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{g}. \quad (1.4)$$

Here  $\epsilon > 0$  is a small parameter, called the artificial Mach number.

We consider (1.1)–(1.2) and (1.3)–(1.4) under the boundary condition

$$\mathbf{v}|_{\partial\Omega} = \mathbf{v}_*. \quad (1.5)$$

Here  $\mathbf{v}_*$  is a given velocity field satisfying  $\int_{\partial\Omega} \mathbf{v}_* \cdot \mathbf{n} \, dS = 0$ , where  $\mathbf{n}$  denotes the unit outward normal to  $\partial\Omega$ .

In this thesis we are interested in the relation of stability properties between stationary solutions of (1.1)–(1.2) and (1.3)–(1.4) under the boundary condition (1.5).

The artificial compressible system (1.3)–(1.4) was proposed by A. Chorin ([3, 4, 5]) and R. Temam ([23, 24]). In [3, 4, 5], the system (1.3)–(1.4) was introduced to find numerically stationary solutions of the incompressible Navier-Stokes equation (1.1)–(1.2). The idea is as follows. Obviously, the set of stationary solutions of (1.1)–(1.2) is the same as that of (1.3)–(1.4). If solutions of the artificial compressible system (1.3)–(1.4) converge to a function  $u_s = {}^\top(p_s, \mathbf{v}_s)$  as  $t \rightarrow \infty$ , then the limit  $u_s$  is a stationary solution of (1.3)–(1.4), and thus,  $u_s$  is a stationary solution of (1.1)–(1.2). By using this method, Chorin numerically obtained stationary cellular convection patterns of the Bénard convection problem described by the Oberbeck-Boussinesq

equation

$$\operatorname{div} \mathbf{v} = 0, \quad (1.6)$$

$$\operatorname{Pr}^{-1} (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \Delta \mathbf{v} + \nabla p - \sqrt{\operatorname{Ra}} \theta \mathbf{e}_3 = \mathbf{0}, \quad (1.7)$$

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta - \Delta \theta - \sqrt{\operatorname{Ra}} \mathbf{v} \cdot \mathbf{e}_3 = 0 \quad (1.8)$$

in the infinite layer  $\{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\}$ . Here  $\theta(x, t)$  is the temperature deviation from the heat conductive state;  $\mathbf{e}_3 = {}^\top(0, 0, 1) \in \mathbb{R}^3$ ;  $\operatorname{Pr} > 0$  and  $\operatorname{Ra} > 0$  are non-dimensional parameters, called the Prandtl and Rayleigh numbers, respectively

Since the limit function  $u_s$  in Chorin's method is a large time limit of solutions of (1.3)–(1.4),  $u_s$  is stable as a solution of (1.3)–(1.4). The following questions were then addressed in [19]:

- (a) whether  $u_s$  is stable as a solution of (1.1)–(1.2) with (1.5), in other words, whether  $u_s$  represents an observable stationary flow in the real world,
- (b) conversely, what kind of stationary flows can be computed by Chorin's method.

These questions were formulated in [19] as stability problems of stationary solutions of the incompressible system and the corresponding artificial compressible one. Since the incompressible system (1.1)–(1.2) is obtained from the artificial compressible one (1.3)–(1.4) as the limit  $\epsilon \rightarrow 0$ , one could expect that solutions of (1.1)–(1.2) would be approximated by solutions of (1.3)–(1.4) with  $\epsilon \ll 1$ . However, the limiting procedure is a singular limit, so it is not straightforward to conclude that stability properties of  $u_s$  as a solution of (1.1)–(1.2) are the same as those as a solution of (1.3)–(1.4) even if  $0 < \epsilon \ll 1$ .

The convergence of solutions as  $\epsilon \rightarrow 0$  was discussed in [23, 24, 25] for the system (1.3)–(1.4) with the additional stabilizing nonlinear term  $+\frac{1}{2}(\operatorname{div} \mathbf{v})\mathbf{v}$  on the left of (1.4); and it was shown that there exists a weak solution  ${}^\top(p_\epsilon, \mathbf{v}_\epsilon)$  for each  $\epsilon > 0$  such that  $\mathbf{v}_{\epsilon'} \rightarrow \mathbf{v}$  in  $L^2(0, T; L^2(\Omega)^3)$  and  $\nabla p_{\epsilon'} \rightarrow \nabla p$  weakly in  $H^{-1}(\Omega \times (0, T))$  for all  $T > 0$  along a sequence  $\epsilon' \rightarrow 0$ , where  ${}^\top(p, \mathbf{v})$  is a weak solution of (1.1)–(1.2). We also mention interesting works by Donatelli [9, 10] and Donatelli Marcati [11, 12] where similar convergence results were obtained in the case of unbounded domains by using dispersive estimates for the linear semigroup therein. For the stability questions, we need to investigate the spectrum of the linearized operators around a stationary solution.

The purpose of this thesis is to investigate whether (1.3)–(1.4) gives a good approximation of (1.1)–(1.2), when  $0 < \epsilon \ll 1$ , from the view point of the stability of stationary solutions.

In [19], the above questions were considered for the Oberbeck-Boussinesq equation (1.6)–(1.8) in the infinite layer under the boundary condition  $\mathbf{v} = \mathbf{0}$ ,  $\theta = 0$  on  $\{x_3 = 0, 1\}$  and a periodic boundary condition in  $x' = (x_1, x_2)$ . The results can be restated for the systems (1.1)–(1.2) and (1.3)–(1.4) in the following way.

We introduce the linearized operators around a stationary solution  $u_s = {}^\top(p_s, \mathbf{v}_s)$  associated with the systems (1.1)–(1.2) and (1.3)–(1.4) under (1.5). Here and in what follows  ${}^\top \cdot$  stands for the transposition. Let  $\mathbb{L} : L_\sigma^2(\Omega) \rightarrow L_\sigma^2(\Omega)$  be the operator defined by

$$\mathbb{L} = -\nu \mathbb{P} \Delta + \mathbb{P}(\mathbf{v}_s \cdot \nabla + {}^\top(\nabla \mathbf{v}_s))$$

with domain  $D(\mathbb{L}) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \cap L_\sigma^2(\Omega)$ . Here  $H^k(\Omega)$  denotes the  $k$  th order  $L^2$ -Sobolev space on  $\Omega$ ,  $\mathbb{P}$  is the orthogonal projection, called the Helmholtz projection, from  $L^2(\Omega)^3$  to  $L_\sigma^2(\Omega)$ , and  $L_\sigma^2(\Omega)$  denotes the set of all  $L^2$ -vector fields  $\mathbf{w}$  on  $\Omega$  satisfying  $\operatorname{div} \mathbf{w} = 0$  and  $\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0$ , where  $\mathbf{n}$  denotes the unit outward normal to  $\partial\Omega$ . We define the operator  $L_\epsilon : H_*^1(\Omega) \times L^2(\Omega)^3 \rightarrow H_*^1(\Omega) \times L^2(\Omega)^3$ , acting on  $u = {}^\top(p, \mathbf{w})$ , by

$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2} \operatorname{div} \\ \nabla & -\nu \Delta + \mathbf{v}_s \cdot \nabla + {}^\top(\nabla \mathbf{v}_s) \end{pmatrix}$$

with domain  $D(L_\epsilon) = H_*^1(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]^3$ . Here  $H_*^1(\Omega)$  denotes the set of  $H^1$  functions on  $\Omega$  that have zero mean value over  $\Omega$ .

As for the question (a), it was proved in [19] that if there exists a positive number  $b_0$  such that  $\rho(-L_{\epsilon_n}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\}$  for some sequence  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists a positive constant  $b_1$  such that  $\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_1\}$ . Therefore, a stationary solution obtained by Chorin's method with  $0 < \epsilon \ll 1$  is stable as a solution of the incompressible system (1.1)–(1.2). Furthermore, the instability result was proved: if  $\sigma(-\mathbb{L}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \neq \emptyset$ , then  $\sigma(-L_\epsilon) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \neq \emptyset$  for  $0 < \epsilon \ll 1$ . This shows that unstable stationary solutions of (1.1)–(1.2) cannot be obtained by Chorin's method with  $0 < \epsilon \ll 1$ .

As for the question (b), it was shown in [19] that if  $\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\}$  for some positive constant  $b_0$ , then there exist positive constants  $\kappa_0$  and  $b_1$  such that  $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_1\}$  for  $0 < \epsilon \ll 1$ ,

provided that

$$\inf_{\mathbf{w} \in H_0^1(\Omega)^3, \mathbf{w} \neq \mathbf{0}} \frac{\operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w})_{L^2}}{\|\nabla \mathbf{w}\|_{L^2}^2} \geq -\kappa_0. \quad (1.9)$$

This gives a sufficient condition for  $u_s$  to be computed by Chorin's method with  $0 < \epsilon \ll 1$ . The corresponding result for the Oberbeck-Boussinesq system (1.6)–(1.8) is stated exactly in the same form; and the result is applicable to stable bifurcating cellular convective patterns of the system (1.6)–(1.8), such as roll pattern, hexagonal pattern and etc., since they bifurcate from  $\mathbf{v} = \mathbf{0}$ ,  $\theta = 0$ , and hence, the condition (1.9) is satisfied near the bifurcation point. (Observe that the condition (1.9) is independent of  $\theta_s$  and  $\theta$ .) However, the condition (1.9) is stringent since most of its applications might be limited to stationary flows whose velocity fields are sufficiently small. We note that the condition (1.9) seems to be the standard energy stability criterion; but this is not the case; the standard energy stability criterion for the Oberbeck-Boussinesq system (1.6)–(1.8) should be formulated in order for the functional

$$\|\nabla \mathbf{w}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 - \operatorname{Re} \{ 2\sqrt{\operatorname{Ra}}(\mathbf{w} \cdot \mathbf{e}_3, \theta)_{L^2} - \operatorname{Pr}^{-1}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w})_{L^2} - (\mathbf{w} \cdot \nabla \theta_s, \theta)_{L^2} \}$$

to be positive definite. Here  $u_s = {}^\top(p_s, \mathbf{v}_s, \theta_s)$  is a stationary solution of (1.6)–(1.8). In fact, the condition (1.9) arises from a *compressible* aspect of the system (1.3)–(1.4), more precisely, eigenvalues related to diffusion waves whose imaginary parts are of order  $O(\epsilon^{-1})$ . This suggests us one direction to investigate the nature of the condition (1.9).

In this thesis we consider the following two subjects:

- (i) improvement of the condition (1.9),
- (ii) stability of bifurcating stationary solutions.

The aim of the first subject (i) is to improve the condition (1.9) in such a way that we could treat stationary flows whose velocity fields are not necessarily small and also could see some of compressible aspects of the system (1.3)–(1.4) with (1.5). We show that the condition (1.9) can be replaced by

$$\inf_{\mathbf{w} \in H_0^1(\Omega)^3, \mathbf{w} \neq \mathbf{0}} \frac{\operatorname{Re}((\mathbb{Q}\mathbf{w}) \cdot \nabla \mathbf{v}_s, \mathbb{Q}\mathbf{w})_{L^2}}{\|\nabla \mathbb{Q}\mathbf{w}\|_{L^2}^2} \geq -\kappa_0. \quad (1.10)$$

Here  $\mathbb{Q} = I - \mathbb{P}$  is the orthogonal projection from  $L^2(\Omega)^3$  to the space  $G^2(\Omega) = \{\nabla p; p \in H_*^1(\Omega)\}$  which is the orthogonal complement of  $L_\sigma^2(\Omega)$ . More precisely, if  $\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\}$  for some positive constant  $b_0$ , then there exist positive numbers  $\epsilon_0$ ,  $\kappa_0$  and  $b_1$  such that  $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_1\}$  for  $0 < \epsilon \leq \epsilon_0$ , provided that the condition (1.10) holds. See Theorem 3.1 below.

The same result also holds for the case of the Oberbeck-Boussinesq system (1.6)–(1.8). For simplicity, in this thesis, we consider the above improvement only in the context of the Navier-Stokes system (1.1)–(1.2).

As an application, we consider the Taylor problem, namely, a flow between two concentric infinite cylinders, whose inner cylinder rotates with a uniform speed and outer one is at rest. If the rotation speed is sufficiently small, then a laminar flow, called the Couette flow, is stable. When the rotation speed increases, beyond a certain value of the rotation speed, the Couette flow is getting unstable, and a vortex pattern is observed. The vortex pattern is periodic in the direction of the axis of the cylinders and it is called the Taylor vortex. This phenomenon has been studied mathematically as a bifurcation problem for the incompressible system (1.1)–(1.2) (see [6, 16, 17, 20, 26]). The velocity field near the bifurcation point of the Taylor vortex is not necessarily small, but one can show that the condition (1.10) is satisfied with  $\mathbf{v}_s$  being the Taylor vortex under *axi-symmetric perturbations* (i.e.,  $\mathbf{w}$  in (1.10) are *axi-symmetric*). In section 5, we discuss not only the stability of the Taylor vortex but also the stability and instability of the Couette flow by applying the main result of this thesis and the instability result of [19].

As was shown in [19], the spectrum of  $-L_\epsilon$  near the imaginary axis is divided into two parts. One part locates in a region with imaginary part of  $O(1)$  as  $\epsilon \rightarrow 0$ , and this part is obtained by small perturbations of eigenvalues of  $-\mathbb{L}$  when  $0 < \epsilon \ll 1$ . The other part locates in a region with imaginary part of  $O(\epsilon^{-1})$ , and this part consists of eigenvalues arising from a *compressible* aspect of  $-L_\epsilon$ . In [19] the condition (1.9) was used to show that the eigenvalues in this region have negative real parts. The idea to obtain the condition (1.10) instead of (1.9) is as follows. By using the Helmholtz decomposition,  $\mathbf{w}$  in (1.9) is written as a sum of the incompressible part  $\mathbb{P}\mathbf{w}$  and the potential flow part  $\nabla\phi = \mathbb{Q}\mathbf{w}$ . Since  $-\mathbb{L}$  is sectorial, we see that  $\|(\lambda + \mathbb{L})^{-1}\mathbf{F}\|_{L^2} = O(|\operatorname{Im} \lambda|^{-1})\|\mathbf{F}\|_{L^2}$  as  $|\operatorname{Im} \lambda| \rightarrow \infty$ . So, if  $|\operatorname{Im} \lambda| = O(\epsilon^{-1})$ , then we expect that the incompressible part  $\mathbb{P}\mathbf{w}$  becomes small in  $L^2$  as  $\epsilon \rightarrow 0$  since  $|\operatorname{Im} \lambda|^{-1} = O(\epsilon) \rightarrow 0$ , which implies that the parts including  $\mathbb{P}\mathbf{w}$  of the numerator in (1.9) is getting smaller when  $\epsilon \rightarrow 0$ . In fact, we can show that

if  $\mathbf{w}$  is the velocity component of  $(\lambda + L_\epsilon)^{-1}F$  with  $\lambda$  near the imaginary axis and  $|\operatorname{Im} \lambda| = O(\epsilon^{-1})$ , then

$$\|\mathbb{P}\mathbf{w}\|_{L^2(\Omega)} \leq C\{\epsilon\|F\|_{H^1(\Omega) \times L^2(\Omega)} + \epsilon^{\frac{1}{4}}\|\mathbb{Q}\mathbf{w}\|_{L^2(\Omega)}\}. \quad (1.11)$$

Using this estimate, one can show that the condition (1.9) can be replaced by the condition (1.10). To obtain (1.11), we establish the estimate

$$\begin{aligned} & |\lambda|\|\mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{v}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \\ & \leq C\{\|\mathbf{g}\|_{L^2(\Omega)} + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\} \end{aligned} \quad (1.12)$$

for a solution  ${}^\top(p, \mathbf{v})$  of the Stokes system with nonhomogeneous boundary data:

$$\begin{cases} \operatorname{div} \mathbf{v} &= 0, \\ \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla p &= \mathbf{g}, \\ \mathbf{v}|_{\partial\Omega} &= \boldsymbol{\psi}, \end{cases} \quad (1.13)$$

where  $\lambda \in \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \pi - a\}$  for some  $0 < a < \frac{\pi}{2}$ ,  $\mathbf{g} \in L^2(\Omega)$  and  $\boldsymbol{\psi} \in H^{\frac{3}{2}}(\partial\Omega)$  with  $\boldsymbol{\psi} \cdot \mathbf{n}|_{\partial\Omega} = 0$ . See Lemma 3.12 below.

As for the second subject (ii), we note that  $\epsilon_0$  in Theorem 3.1 depends on  $b_0$  and it may occur that  $\epsilon_0 \rightarrow 0$  as  $b_0 \rightarrow 0$ . So if  $b_0$  approaches to zero, we have to take the range of  $\epsilon$  smaller and smaller. This is inconvenient to consider the stability of a bifurcating stationary solution near the bifurcation point; the range of  $\epsilon$  shrinks when the bifurcation parameter approaches its critical value. We show that the range of  $\epsilon$  can be taken uniformly near the bifurcation point when the stability of a bifurcating solution from a simple eigenvalue is considered.

The properties of the eigenvalues of the linearized operator around bifurcating solution near the bifurcation point is well known. Let  $\mathbf{v}_\eta$  be a basic stationary solution with a bifurcation parameter  $\eta$ , and let  $\mathbb{L}_\eta$  denote the linearized operator around  $\mathbf{v}_\eta$  for (1.1)–(1.2). We assume that  $\rho(-\mathbb{L}_\eta) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\tilde{b}_0\} \setminus \{\lambda_\eta\}$  uniformly in  $\eta \in [-\eta_0, \eta_0]$  for some positive constants  $\tilde{b}_0$  and  $\eta_0$ . Here  $\lambda_\eta$  is a simple eigenvalue of  $-\mathbb{L}_\eta$  and  $\lambda_\eta$  crosses the origin when  $\eta$  crosses 0. Then a stationary bifurcation occurs at  $\eta = 0$  and there exists a nontrivial solution branch  $\{\eta(\delta), \tilde{\mathbf{v}}(\delta)\}$ , where  $\eta(\delta)$  and  $\tilde{\mathbf{v}}(\delta)$  are analytic in  $\delta$  ( $0 < |\delta| \leq \delta_0$ ). We denote by  $\mathbb{L}(\delta)$  the linearized operator around  $\tilde{\mathbf{v}}(\delta)$ .

As for the spectrum of  $-\mathbb{L}(\delta)$ , it holds that

$$\rho(-\mathbb{L}(\delta)) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\frac{3}{4}\tilde{b}_0\} \setminus \{\lambda(\delta)\}$$

for  $0 < |\delta| \leq \delta_0$ . Here  $\lambda(\delta)$  is analytic in  $\delta$  and satisfies  $\lambda(0) = 0$ . Therefore, the stability of  $\tilde{\mathbf{v}}(\delta)$  as a solution of (1.1)–(1.2) is determined by  $\text{sgn}(\lambda(\delta))$  the sign of  $\lambda(\delta)$ .

We denote by  $L(\epsilon, \delta)$  the linearized operator around  $\tilde{\mathbf{v}}(\delta)$  for the artificial compressible system (1.3)–(1.4). We will show, by a perturbation argument, that the spectrum of  $-L(\epsilon, \delta)$  near the origin is given by a simple eigenvalue  $\lambda(\epsilon, \delta)$  which satisfies

$$\lambda(\epsilon, \delta) = c_1(\epsilon^2, \delta)\lambda(\delta),$$

where  $c_1(\epsilon^2, \delta)$  satisfies  $c_1(\epsilon^2, \delta) \geq \frac{1}{2}$  uniformly for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq \delta_0$ . As a consequence, we have

$$\text{sgn}(\lambda(\epsilon, \delta)) = \text{sgn}(\lambda(\delta))$$

for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq \delta_0$ . This implies that if  $\tilde{\mathbf{v}}(\delta)$  is unstable as a solution of (1.1)–(1.2). Then it is also unstable as a solution of (1.3)–(1.4) for  $0 < \epsilon \leq \epsilon_1$ . Furthermore, if  $\text{sgn}(\lambda(\delta)) > 0$  and (1.10) is satisfied with  $\mathbf{v}_s = \tilde{\mathbf{v}}(\delta)$  for  $0 < |\delta| \leq \delta_0$ , then  $\tilde{\mathbf{v}}(\delta)$  is stable as a solution of (1.3)–(1.4) for  $0 < \epsilon \leq \epsilon_1$ .

This thesis is organized as follows. In section 2 we introduce notations used in this thesis. In section 3 we give an improvement of the condition (1.9). In section 3.1 we state the result about the stability criterion for stationary solutions (Theorem 3.1). We apply Theorem 3.1 to the Taylor problem in section 3.2. In section 3.3 we prove Theorem 3.1. We show that the condition (1.9) can be replaced by the condition (1.10). In section 4 we consider the stability of bifurcating stationary solutions. In section 4.1 we state the result about the stability of bifurcating stationary solutions (Theorems 4.5 and 4.6). Section 4.2 is devoted to the proofs of Theorems 4.5 and 4.6. Section 5 is devoted to a derivation of a solution formula for the problem (1.13) with  $\boldsymbol{\psi} = \mathbf{0}$ .

## 2 Notation

We first introduce notation used in this thesis. For  $1 \leq p \leq \infty$  we denote by  $L^p(D)$  the usual Lebesgue space over  $D$  and its norm is denoted by  $\|\cdot\|_{L^p(D)}$ . The  $m$ th order  $L^2$  Sobolev space over  $D$  is denoted by  $H^m(D)$ , and its norm is denoted by  $\|\cdot\|_{H^m(D)}$ . When  $D = \Omega$ , we simply denote these norms by

$\|\cdot\|_p, \|\cdot\|_{H^m}$ . The inner product of  $L^2(D)$  is denoted by  $(\cdot, \cdot)_{L^2(D)}$ , i.e.,

$$(f, g)_{L^2(D)} = \int_D f(x) \overline{g(x)} dx.$$

Here  $\bar{z}$  denotes the complex conjugate of  $z \in \mathbb{C}$ . When  $D = \Omega$  we simply denote  $(\cdot, \cdot)_{L^2(D)}$  by  $(\cdot, \cdot)$ . We also defined the weighted inner product  $\langle\langle \cdot, \cdot \rangle\rangle_\epsilon$  by

$$\langle\langle u_1, u_2 \rangle\rangle_\epsilon = \epsilon^2(p_1, p_2) + (\mathbf{w}_1, \mathbf{w}_2)$$

for  $u_j = {}^\top(p_j, \mathbf{w}_j)$ ,  $j = 1, 2$ .

We set

$$\begin{aligned} H_0^1(D) &= \text{the } H^1(D)\text{-closure of } C_0^\infty(D), \\ H^{-1}(D) &= \text{the dual space of } H_0^1(D), \\ \dot{H}^1(D) &= \{f \in L_{loc}^2(D) : \|\nabla f\|_{L^2(D)} < \infty\}, \\ \dot{H}^{-1}(D) &= \text{the dual space of } \dot{H}^1(D). \end{aligned}$$

We define  $L_*^2(\Omega)$  and  $H_*^k(\Omega)$  by

$$L_*^2(\Omega) = \{f \in L^2(\Omega); \int_\Omega f(x) dx = 0\},$$

$$H_*^k(\Omega) = H^k(\Omega) \cap L_*^2(\Omega) \quad (k \geq 1).$$

We set

$$L_\sigma^2(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

Here and in what follows,  $\mathbf{n}$  denotes the unit outward normal to  $\partial\Omega$ . It is known that  $(L^2(\Omega))^3 = L_\sigma^2(\Omega) \oplus G^2(\Omega)$ , where  $G^2(\Omega) = \{\nabla p; p \in H_*^1(\Omega)\}$  is orthogonal complement of  $L_\sigma^2(\Omega)$ .

The orthogonal projection  $\mathbb{P}$  from  $L^2(\Omega)^3$  onto  $L_\sigma^2(\Omega)$  is called the Helmholtz projection. We set  $\mathbb{Q} = I - \mathbb{P}$ .

We denote the resolvent set of an operator  $A$  by  $\rho(A)$  and the spectrum of  $A$  by  $\sigma(A)$ . Let  $X$  and  $Y$  be Banach spaces. We denote by  $B(X, Y)$  the set of bounded linear operators from  $X$  to  $Y$ .

We introduce the linearized operators for the Navier-Stokes and the corresponding artificial compressible systems. Let  $u_s = {}^\top(p_s, \mathbf{v}_s)$  be a smooth

stationary solution of (1.1)–(1.2), (1.5). The equations for the perturbation is then written as

$$\operatorname{div} \mathbf{w} = 0, \quad (2.1)$$

$$\partial_t \mathbf{w} - \nu \Delta \mathbf{w} + \mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla p = \mathbf{0}. \quad (2.2)$$

The boundary condition is the non-slip one:

$$\mathbf{w}|_\Omega = \mathbf{0}. \quad (2.3)$$

Applying the Helmholtz projection  $\mathbb{P}$  we have

$$\frac{d\mathbf{w}}{dt} + \mathbb{L}\mathbf{w} + \mathbb{P}(\mathbf{w} \cdot \nabla \mathbf{w}) = \mathbf{0}. \quad (2.4)$$

Here  $\mathbb{L}$  is the linearized operator around  $\mathbf{v}_s$  on  $L_\sigma^2(\Omega)$  defined by

$$\begin{aligned} D(\mathbb{L}) &= (H^2(\Omega) \cap H_0^1(\Omega))^3 \cap L_\sigma^2(\Omega), \\ \mathbb{L}\mathbf{w} &= -\nu \mathbb{P} \Delta \mathbf{w} + \mathbb{P}(\mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s), \quad (\mathbf{w} \in D(\mathbb{L})). \end{aligned}$$

The corresponding artificial system is written as

$$\frac{du}{dt} + L_\epsilon u + N(u, u) = 0. \quad (2.5)$$

Here  $u = {}^\top(p, \mathbf{w})$ ;  $L_\epsilon$  is the linearized operator around  $u_s$  on  $H_*^1(\Omega) \times L^2(\Omega)^3$  defined by

$$\begin{aligned} D(L_\epsilon) &= H_*^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))^3, \\ L_\epsilon &= \begin{pmatrix} 0 & \frac{1}{\epsilon^2} \operatorname{div} \\ \nabla & -\nu \Delta + \mathbf{v}_s \cdot \nabla + {}^\top(\nabla \mathbf{v}_s) \end{pmatrix}; \end{aligned}$$

and  $N(u, u)$  is the nonlinear operator given by  $N(u, u) = {}^\top(0, \mathbf{w} \cdot \nabla \mathbf{w})$  ( $u = {}^\top(p, \mathbf{w})$ ).

### 3 Stability criterion for stationary solutions

In this section, we consider what kind of stationary flows can be computed by Chorin's method and improve the stability result given in [19].

### 3.1 Stability criterion

We give a sufficient condition in order for a stable stationary solution of (1.1)–(1.2) to be stable as a solution of (1.3)–(1.4) for  $0 < \epsilon \ll 1$ .

**Theorem 3.1.** *Suppose that  $\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\}$  for some positive constant  $b_0$ . Then there exist positive constants  $\epsilon_0$ ,  $\kappa_0$  and  $b_1$  such that if*

$$\inf_{\mathbf{w} \in H_0^1(\Omega)^3, \mathbf{w} \neq \mathbf{0}} \frac{\operatorname{Re}((\mathbb{Q}\mathbf{w}) \cdot \nabla \mathbf{v}_s, \mathbb{Q}\mathbf{w})}{\|\nabla \mathbb{Q}\mathbf{w}\|_2^2} \geq -\kappa_0, \quad (3.1)$$

then  $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_1\}$  for all  $0 < \epsilon \leq \epsilon_0$ .

**Remark 3.2.** *If we consider flows on an unbounded domain  $\Omega$  which is translation invariant in the unbounded directions of  $\Omega$  such as an infinite layer and a cylindrical domain, then we can consider a stationary flow which is periodic in the unbounded directions of  $\Omega$ . For definiteness, let us consider a cylindrical domain whose unbounded direction is the  $x_1$ -axis. The problem is then formulated in the basic period domain  $\Omega_{\text{per}}$  under the periodic boundary condition in  $x_1$ . In this case, if the stationary solution  $\mathbf{v}_s$  which is periodic in  $x_1$  satisfies  $\partial_{x_1} \mathbf{v}_s \neq \mathbf{0}$ , then 0 is always an eigenvalue of  $-\mathbb{L}$ , due to the translation invariance in  $x_1$ . As in [19], one can prove the following result instead of Theorem 3.1.*

*Suppose that  $\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\} \setminus \{0\}$  for some positive constant  $b_0$  and 0 is a simple eigenvalue with  $\operatorname{Ker}(-\mathbb{L}) = \operatorname{span}\{\partial_{x_1} \mathbf{v}_s\}$ . Then there exist positive constants  $\epsilon_0$ ,  $\kappa_0$  and  $b_1$  such that if*

$$\inf_{\mathbf{w} \in H_{0,\text{per}}^1(\Omega_{\text{per}})^3, \mathbf{w} \neq \mathbf{0}} \frac{\operatorname{Re}((\mathbb{Q}\mathbf{w}) \cdot \nabla \mathbf{v}_s, \mathbb{Q}\mathbf{w})}{\|\nabla \mathbb{Q}\mathbf{w}\|_2^2} \geq -\kappa_0,$$

then  $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_1\} \setminus \{0\}$  for all  $0 < \epsilon \leq \epsilon_0$  and 0 is a simple eigenvalue with  $\operatorname{Ker}(-L_\epsilon) = \operatorname{span}\{\partial_{x_1} u_s\}$ . Here  $H_{0,\text{per}}^1(\Omega_{\text{per}})$  denotes the set of all  $H^1$  functions on  $\Omega_{\text{per}}$  which vanish on  $\partial\Omega$  and satisfy the periodic boundary condition in  $x_1$ ; and  $u_s = {}^\top(p_s, \mathbf{v}_s)$  with  $p_s$  being the corresponding pressure.

**Remark 3.3.** *One can easily see from the proofs of Theorem 3.1 and [19, Theorem 3.3] that the same result also holds for the case of the Oberbeck-Boussinesq system (1.6)–(1.8).*

As an application of Theorem 3.1 we will consider the Taylor problem.

## 3.2 Application to the Taylor problem

We consider the Navier-Stokes equation

$$\operatorname{div} \mathbf{v} = 0, \quad (3.2)$$

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho_*} \nabla p = \mathbf{0}, \quad (3.3)$$

in a domain  $\Omega_{R_1, R_2}$  between two concentric cylinders with radii  $R_1$  and  $R_2$ ,  $R_1 < R_2$ . In the cylindrical coordinates  $(r, \theta, z)$ , the domain  $\Omega_{R_1, R_2}$  is represented as

$$\Omega_{R_1, R_2} = \{(r, \theta, z) : R_1 < r < R_2, \theta \in [0, 2\pi], z \in \mathbb{R}\},$$

and the velocity field  $\mathbf{v}$  is given by

$$\mathbf{v} = v^r \mathbf{e}_r + v^\theta \mathbf{e}_\theta + v^z \mathbf{e}_z$$

with  $\mathbf{e}_r = {}^\top(\cos \theta, \sin \theta, 0)$ ,  $\mathbf{e}_\theta = {}^\top(-\sin \theta, \cos \theta, 0)$ ,  $\mathbf{e}_z = {}^\top(0, 0, 1)$ . Here  $v^r$ ,  $v^\theta$  and  $v^z$  are the  $r$ ,  $\theta$  and  $z$ -components of  $\mathbf{v}$ , respectively. The inner and outer cylinders rotate with constant angular velocities  $\omega_1$  and  $\omega_2$ , respectively. We assume that  $\omega_1 > 0$ .

The boundary conditions are given by

$$v^r|_{r=R_1, R_2} = v^z|_{r=R_1, R_2} = 0, \quad v^\theta|_{r=R_1} = \omega_1 R_1, \quad v^\theta|_{r=R_2} = \omega_2 R_2. \quad (3.4)$$

We impose a periodic boundary condition in  $z$ , i.e.,

$$\mathbf{v}, p \text{ are } \frac{2\pi}{\alpha_*}\text{-periodic in } z, \quad (3.5)$$

where  $\alpha_* > 0$  is a given wave number.

We rewrite the problem (3.2)–(3.5) in a non-dimensional form. We introduce the following non-dimensional variables:

$$\mathbf{x} = d\tilde{\mathbf{x}}, \quad \mathbf{v} = \omega_1 R_1 \tilde{\mathbf{v}}, \quad t = \frac{d^2}{\nu} \tilde{t}, \quad p = \frac{\rho_* \nu \omega_1 R_1}{d} \tilde{p},$$

where  $d = R_2 - R_1$ . Then (3.2) and (3.3) are transformed into

$$\operatorname{div}_{\tilde{\mathbf{x}}} \tilde{\mathbf{v}} = 0, \quad (3.6)$$

$$\partial_{\tilde{t}} \tilde{\mathbf{v}} - \Delta_{\tilde{\mathbf{x}}} \tilde{\mathbf{v}} + \mathcal{R} \tilde{\mathbf{v}} \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{v}} + \nabla_{\tilde{\mathbf{x}}} \tilde{p} = \mathbf{0}, \quad (3.7)$$

where  $\mathcal{R} = \frac{\omega_1 R_1 d}{\nu}$  is the Reynolds number. The domain  $\Omega_{R_1, R_2}$  is transformed into  $\Omega$ :

$$\Omega = \{(\tilde{r}, \tilde{\theta}, \tilde{z}) : \frac{\eta}{1-\eta} < \tilde{r} < \frac{1}{1-\eta}, \tilde{\theta} \in [0, 2\pi], \tilde{z} \in \mathbb{R}\}.$$

Here  $\mathcal{R} = \frac{dR_1\omega_1}{\nu}$  is the Reynolds number; and  $\eta = \frac{R_1}{R_2}$ . The boundary conditions (3.4) and (3.5) become

$$\tilde{v}^{\tilde{r}}|_{\tilde{r}=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = \tilde{v}^{\tilde{z}}|_{\tilde{r}=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = 0, \quad \tilde{v}^{\tilde{\theta}}|_{\tilde{r}=\frac{\eta}{1-\eta}} = 1, \quad \tilde{v}^{\tilde{\theta}}|_{\tilde{r}=\frac{1}{1-\eta}} = \frac{\omega}{\eta}, \quad (3.8)$$

where  $\omega = \frac{\omega_2}{\omega_1}$ , and

$$\tilde{v}, \tilde{p} \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } \tilde{z}, \quad (3.9)$$

where  $\alpha > 0$  is a wave number. In what follows we omit the tildes "  $\sim$  " in the non-dimensional quantities  $\tilde{\mathbf{x}}, \tilde{t}, \tilde{\mathbf{v}}, \tilde{p}$ , and simply write them as  $\mathbf{x}, t, \mathbf{v}, p$ .

In the cylindrical coordinates  $(r, \theta, z)$  and  $\mathbf{v} = v^r \mathbf{e}_r + v^\theta \mathbf{e}_\theta + v^z \mathbf{e}_z$ , the problem (3.6)–(4.21) is written as

$$\begin{cases} \operatorname{div} \mathbf{v} &= 0, \\ \partial_t v^r + \mathcal{R}[(\mathbf{v} \cdot \nabla) v^r - \frac{1}{r}(v^\theta)^2] &= -\partial_r p + (\Delta v^r - \frac{1}{r^2} v^r - \frac{2}{r^2} \partial_\theta v^\theta), \\ \partial_t v^\theta + \mathcal{R}[(\mathbf{v} \cdot \nabla) v^\theta + \frac{1}{r} v^r v^\theta] &= -\frac{1}{r} \partial_\theta p + (\Delta v^\theta - \frac{1}{r^2} v^\theta + \frac{2}{r^2} \partial_\theta v^r), \\ \partial_t v^z + \mathcal{R}(\mathbf{v} \cdot \nabla) v^z &= -\partial_z p + \Delta v^z, \end{cases}$$

$$v^r|_{r=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = v^z|_{r=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = 0, \quad v^\theta|_{r=\frac{\eta}{1-\eta}} = 1, \quad v^\theta|_{r=\frac{1}{1-\eta}} = \frac{\omega}{\eta},$$

and

$$\mathbf{v}, p \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } z.$$

Here  $\operatorname{div} \mathbf{v} = \frac{1}{r} \partial_r(r v^r) + \frac{1}{r} \partial_\theta v^\theta + \partial_z v^z$ ,  $\mathbf{v} \cdot \nabla = v^r \partial_r + \frac{1}{r} v^\theta \partial_\theta + v^z \partial_z$  and  $\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \partial_z^2$ .

This problem has a stationary solution  ${}^\top(p_C, \mathbf{v}_C)$ , the Couette flow, of the form

$$\begin{aligned} \mathbf{v}_C &= v_C^\theta(r) \mathbf{e}_\theta, \quad v_C^\theta(r) = ar + \frac{b}{r}, \\ p_C &= p_C(r) = \mathcal{R} \int^r \frac{\{v_C^\theta(s)\}^2}{s} ds, \end{aligned}$$

where

$$a = \frac{\omega - \eta^2}{\eta(1 + \eta)}, \quad b = \frac{\eta(1 - \omega)}{(1 - \eta)(1 - \eta^2)}.$$

The perturbation  ${}^\top(q, \mathbf{w}) = (p - p_C, \mathbf{v} - \mathbf{v}_C)$  is governed by

$$\begin{aligned} \operatorname{div} \mathbf{w} &= 0, \\ \partial_t \mathbf{w} - \Delta \mathbf{w} + \mathcal{R}(\mathbf{v}_C \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_C) + \mathcal{R} \mathbf{w} \cdot \nabla \mathbf{w} + \nabla q &= \mathbf{0}, \end{aligned} \quad (3.10)$$

$$\mathbf{w}|_{r=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = \mathbf{0}, \quad (3.11)$$

and

$$\mathbf{w}, q \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } z. \quad (3.12)$$

We introduce function spaces. We denote the basic period domain by  $\Omega_\alpha$ :

$$\Omega_\alpha = \{(r, \theta, z) : \frac{\eta}{1-\eta} < r < \frac{1}{1-\eta}, \theta \in [0, 2\pi], -\frac{\pi}{\alpha} < z < \frac{\pi}{\alpha}\}.$$

The symbol  $C_{per}^\infty$  stands for the space of restrictions to  $\Omega_\alpha$  of functions in  $C^\infty(\bar{\Omega})$  which are  $\frac{2\pi}{\alpha}$ -periodic in  $z$ ; and  $C_{0,per}^\infty$  denotes the space of restrictions to  $\Omega_\alpha$  of functions in  $C^\infty$  which are  $\frac{2\pi}{\alpha}$ -periodic in  $z$  and vanish near  $r = \frac{\eta}{1-\eta}, \frac{1}{1-\eta}$ .

We set

$$L_{per}^2 = \text{the } L^2(\Omega_\alpha)\text{-closure of } C_{0,per}^\infty,$$

$$H_{per}^k = \text{the } H^k(\Omega_\alpha)\text{-closure of } C_{per}^\infty,$$

$$H_{0,per}^1 = \text{the } H^1(\Omega_\alpha)\text{-closure of } C_{0,per}^\infty.$$

We note that if  $f \in H_{0,per}^1$ , then  $f|_{z=-\pi/\alpha} = f|_{z=\pi/\alpha}$  and  $f|_{r=\frac{\eta}{1-\eta}, \frac{1}{1-\eta}} = 0$ .

The inner product of  $f_j \in L_{per}^2$  ( $j = 1, 2$ ) is denoted by

$$(f_1, f_2) = \int_{\Omega_\alpha} f_1 \bar{f}_2 r dr d\theta dz,$$

where  $\bar{f}$  denotes the complex conjugate of  $f$ . The mean value of a function  $f$  over  $\Omega_\alpha$  is denoted by  $\langle f \rangle$ :

$$\langle f \rangle = \frac{1}{|\Omega_\alpha|} \int_{\Omega_\alpha} f r dr d\theta dz.$$

The set of all  $f \in L_{per}^2$  with  $\langle f \rangle = 0$  is denoted by  $L_{per,*}^2$ , i.e.,

$$L_{per,*}^2 = \{f \in L_{per}^2 : \langle f \rangle = 0\}.$$

Furthermore, we set

$$H_{per,*}^k = H_{per}^k \cap L_{per,*}^2.$$

We denote by  $C_{0,per,\sigma}^\infty$  the set of all vector fields  $\mathbf{v}$  in  $(C_{0,per}^\infty)^3$  with  $\text{div } \mathbf{v} = 0$ . We set

$$L_{per,\sigma}^2 = \text{the } L^2(\Omega_\alpha)^3\text{-closure of } C_{0,per,\sigma}^\infty.$$

It is known that  $(L_{per}^2)^3 = L_{per,\sigma}^2 \oplus G_{per}^2$ , where  $G_{per}^2 = \{\nabla p; p \in H_{per,*}^1\}$  is the orthogonal complement of  $L_{per,\sigma}^2$ .

Applying the Helmholtz projection  $\mathbb{P}$  to (3.10) we have

$$\frac{d}{dt} \mathbf{w} + \mathbb{L}_{\mathcal{R},C} \mathbf{w} = -\mathcal{R} \mathbb{P}(\mathbf{w} \cdot \nabla \mathbf{w}).$$

Here  $\mathbb{L}_{\mathcal{R},C}$  is the linearized operator around  $\mathbf{v}_C$  which is defined by

$$\mathbb{L}_{\mathcal{R},C} : L_{\sigma,per}^2 \rightarrow L_{\sigma,per}^2,$$

$$D(\mathbb{L}_{\mathcal{R},C}) = (H_{per}^2 \cap H_{0,per}^1)^3 \cap L_{\sigma,per}^2,$$

$$\mathbb{L}_{\mathcal{R},C} \mathbf{w} = -\mathbb{P} \Delta \mathbf{w} + \mathcal{R} \mathbb{P}(\mathbf{v}_C \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_C) \quad (\mathbf{w} \in D(\mathbb{L}_{\mathcal{R},C})).$$

The corresponding artificial compressible problem are written as

$$\frac{d}{dt} u + L_{\mathcal{R},C,\epsilon} u = -\mathcal{R} \begin{pmatrix} 0 \\ \mathbf{w} \cdot \nabla \mathbf{w} \end{pmatrix},$$

where  $u = {}^\top(q, \mathbf{w})$  and  $L_{\mathcal{R},C,\epsilon}$  is the linearized operator around the Couette flow defined by

$$L_{\mathcal{R},C,\epsilon} : H_{per,*}^1 \times (L_{per}^2)^3 \rightarrow H_{per,*}^1 \times (L_{per}^2)^3,$$

$$D(L_{\mathcal{R},C,\epsilon}) = H_{per,*}^1 \times (H_{per}^2 \cap H_{0,per}^1)^3,$$

$$L_{\mathcal{R},C,\epsilon} = \begin{pmatrix} 0 & \frac{1}{\epsilon^2} \text{div} \\ \nabla & -\Delta + \mathcal{R}(\mathbf{v}_C \cdot \nabla + {}^\top(\nabla \mathbf{v}_C)) \end{pmatrix}.$$

Since the Couette flow and Taylor vortices are axisymmetric, in what follows, we restrict the consideration to axisymmetric functions, namely, we consider vector fields  $\mathbf{v} = v^r \mathbf{e}_r + v^\theta \mathbf{e}_\theta + v^z \mathbf{e}_z$  and scalar functions  $q$  with  $v^r, v^\theta, v^z$

and  $q$  independent of  $\theta$ . For simplicity, function spaces of axisymmetric ones and the linearized operators restricted to such function spaces are denoted by the same symbols.

The instability of the Couette flow and the occurrence of the formation of Taylor vortex patterns are stated mathematically in the following way. We fix  $\alpha$ ,  $\omega$  and  $\eta$  and assume that  $\omega \geq 0$ .

There exists a critical number  $\mathcal{R}_c > 0$  such that when  $\mathcal{R} < \mathcal{R}_c$ , the Couette flow  $\mathbf{v}_C$  is asymptotically stable, whereas, when  $\mathcal{R} > \mathcal{R}_c$ , the Couette flow  $\mathbf{v}_C$  becomes unstable; in other words, if  $\mathcal{R} < \mathcal{R}_c$ , then  $\rho(-\mathbb{L}_{\mathcal{R},C}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\}$  for some positive constant  $b_0 = b_0(\mathcal{R})$ , whereas, if  $\mathcal{R} > \mathcal{R}_c$ , then  $\sigma(-\mathbb{L}_{\mathcal{R},C}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \neq \emptyset$ .

When  $\mathcal{R} > \mathcal{R}_c$ , for each  $\mathcal{R}$  with  $0 < \mathcal{R} - \mathcal{R}_c \ll 1$ , there exists an axisymmetric stationary solution (the Taylor vortex)  $\mathbf{v}_T = \mathbf{v}_C + \tilde{\mathbf{v}}_T$  with  $\partial_z \tilde{\mathbf{v}}_T \neq 0$  which bifurcates from  $\mathbf{v}_C$  at  $\mathcal{R} = \mathcal{R}_c$  supercritically. The bifurcating solution is unique for  $\mathcal{R} \sim \mathcal{R}_c$  up to translations in  $z$ . Furthermore,  $\mathbf{v}_T$  is orbitally stable, more precisely, the linearized operator  $\mathbb{L}_{\mathcal{R},T}$  around  $\mathbf{v}_T$  satisfies  $\rho(-\mathbb{L}_{\mathcal{R},T}) \supset \{\lambda; \operatorname{Re} \lambda \geq -\tilde{b}_0\} \setminus \{0\}$  for some positive constant  $\tilde{b}_0 = \tilde{b}_0(\mathcal{R})$  and 0 is a simple eigenvalue due to the translation invariance in  $z$ , the eigenspace is spanned by  $\partial_z \mathbf{v}_T$ .

The existence of the bifurcating branch of Taylor vortices  $\mathbf{v}_T$  was shown by Velte [26], Iudovich [16], and Kirchgässner and Sorger [20]; and the supercritical bifurcation of  $\mathbf{v}_T$  and spectral properties of  $\mathbb{L}_{\mathcal{R},T}$  are shown by numerical computations for a certain range of values of  $\alpha$ ,  $\eta$ ,  $\omega$  and  $\mathcal{R}$  when  $\omega \geq 0$  (even when  $\omega$  is slightly negative); see the book [6] by Chossat and Iooss. Hereafter we assume that the above mentioned instability of  $\mathbf{v}_C$ , supercritical bifurcation of  $\mathbf{v}_T$  and the spectral properties of  $\mathbb{L}_{\mathcal{R},T}$  hold true.

A direct computation gives  $\operatorname{Re}((\mathbb{Q}\mathbf{w}) \cdot \nabla \mathbf{v}_C, \mathbb{Q}\mathbf{w}) = 0$  for  $\mathbf{w} = w^r(r, z)\mathbf{e}_r + w^\theta(r, z)\mathbf{e}_\theta + w^z(r, z)\mathbf{e}_z$ , since  $\mathbb{Q}\mathbf{w} = \partial_r \phi_r + \partial_z \phi_z$  for such  $\mathbf{w}$  and  $\mathbf{v}_C = \mathbf{v}_\theta(r)\mathbf{e}_\theta$ . Applying Theorem 3.1 and the instability result [19, Theorem 3.2] to  $\mathbb{L}_{\mathcal{R},C}$ , we have the following result.

**Theorem 3.4.** *If  $\mathcal{R} < \mathcal{R}_c$ , then  $\rho(-L_{\mathcal{R},C,\epsilon}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_1\}$  with some positive constant  $b_1 = b_1(\mathcal{R})$  for sufficiently small  $\epsilon > 0$ ; and if  $\mathcal{R} > \mathcal{R}_c$ , then  $\sigma(-L_{\mathcal{R},C,\epsilon}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \neq \emptyset$  for sufficiently small  $\epsilon > 0$ .*

We next consider the stability of the Taylor vortex  $\mathbf{v}_T$ . Since  $\mathbf{v}_T = \mathbf{v}_C + \tilde{\mathbf{v}}_T$  bifurcates from  $\mathbf{v}_C$  at  $\mathcal{R}_c$ , we see that  $\|\nabla \tilde{\mathbf{v}}_T\|_\infty \rightarrow 0$  as  $\mathcal{R} \rightarrow \mathcal{R}_c$ . We then

find that

$$\frac{\operatorname{Re}((\mathbb{Q}\mathbf{w}) \cdot \nabla \mathbf{v}_T, \mathbb{Q}\mathbf{w})}{\|\nabla \mathbb{Q}\mathbf{w}\|_2^2} = \frac{\operatorname{Re}((\mathbb{Q}\mathbf{w}) \cdot \nabla \tilde{\mathbf{v}}_T, \mathbb{Q}\mathbf{w})}{\|\nabla \mathbb{Q}\mathbf{w}\|_2^2} \geq -C\|\nabla \tilde{\mathbf{v}}_T\|_\infty \geq -\kappa_0$$

for  $\mathcal{R}$ ,  $0 < \mathcal{R} - \mathcal{R}_c \ll 1$ . We apply Theorem 3.1 (cf., Remark 3.2) to obtain the following result.

**Theorem 3.5.** *If  $\mathcal{R} > \mathcal{R}_c$  and  $\mathcal{R} - \mathcal{R}_c \ll 1$ , then  $\rho(-L_{\mathcal{R},T,\epsilon}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\tilde{b}_1\} \setminus \{0\}$  with some positive constant  $\tilde{b}_1 = \tilde{b}_1(\mathcal{R})$  for sufficiently small  $\epsilon > 0$  and 0 is a simple eigenvalue with  $\operatorname{Ker}(-L_{\mathcal{R},T,\epsilon}) = \operatorname{span}\{\partial_z u_T\}$ . Here  $u_T = {}^\top(p_T, \mathbf{v}_T)$  with  $p_T$  being the pressure of the Taylor vortex.*

### 3.3 Proof of Theorem 3.1

In this section we give a proof of Theorem 3.1. We consider the resolvent problem for  $-L_\epsilon$ :

$$\lambda u + L_\epsilon u = F, \quad u = {}^\top(p, \mathbf{w}) \in D(L_\epsilon), \quad (3.13)$$

where  $F = {}^\top(f, \mathbf{g}) \in H_*^1(\Omega) \times L^2(\Omega)^3$  is given.

For simplicity we set  $\nu = 1$ . The problem (3.13) is then written as

$$\epsilon^2 \lambda p + \operatorname{div} \mathbf{w} = \epsilon^2 f, \quad (3.14)$$

$$\lambda \mathbf{w} - \Delta \mathbf{w} + \mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s + \nabla p = \mathbf{g}, \quad (3.15)$$

$$\mathbf{w}|_{\partial\Omega} = \mathbf{0}. \quad (3.16)$$

Throughout this section we assume that

$$\rho(-\mathbb{L}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\} \quad (3.17)$$

for some positive constant  $b_0$ .

The following two propositions imply that the part of the spectrum  $\sigma(-L_\epsilon)$  near the imaginary axis can lie only in a region  $\operatorname{Im} \lambda = O(\epsilon^{-1})$  under the assumption (3.17).

**Proposition 3.6.** *There exist positive constants  $a$  and  $b$  such that  $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -a\epsilon^2|\operatorname{Im} \lambda|^2 + b\} \subset \rho(-L_\epsilon)$  for all  $0 < \epsilon \leq 1$ .*

Proposition 3.6 can be proved by the standard Matsumura-Nishida energy method as in the proof of [19, Proposition 6.1]. We omit the proof.

**Proposition 3.7.** *There exist positive numbers  $\epsilon_1$  and  $a_1$  such that*

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0, |\lambda| \leq a_1 \epsilon^{-1}\} \subset \rho(-L_\epsilon)$$

for all  $0 < \epsilon \leq \epsilon_1$ .

**Proof.** Similarly to the proof of [19, Proposition 4.3 (ii)], by using the solvability result for the nonhomogeneous Stokes problem (see, e.g., [13, 15]), one can show that  $0 \in \rho(-L_\epsilon)$ , i.e.,  $R(L_\epsilon)$  (= the range of  $L_\epsilon$ ) =  $H_*^1(\Omega) \times L^2(\Omega)^3$ . Proposition 3.7 can then be proved by the same perturbation argument as in the proof of [19, Proposition 6.3]. This completes the proof.  $\square$

**Remark 3.8.** *Under the situation of Remark 3.2, Proposition 3.7 is stated as*

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0, |\lambda| \leq a_1 \epsilon^{-1}\} \setminus \{0\} \subset \rho(-L_\epsilon) \quad (3.18)$$

and 0 is a simple eigenvalue of  $L_\epsilon$ . In fact, by [19, Proposition 4.3], we see that 0 is a simple eigenvalue of  $L_\epsilon$ . Furthermore, by [19, Proposition 6.2], one can see that if  $\lambda \in \Sigma \setminus \{0\}$ , then the argument of the proof of [19, Proposition 6.3] works well without restricting to the subspace  $Y_{1,\epsilon} = R(L_\epsilon)$  to conclude (3.18), with the replacement of the estimate in the end of the proof of [19, Proposition 6.3] by

$$\|(\lambda + L_\epsilon)^{-1} F\|_{H^1 \times L^2 \times L^2} \leq \frac{C}{|\lambda|} \{\epsilon \|f\|_{H^1} + \|\mathbf{F}\|_2\}.$$

The argument in the proof of [19, Proposition 6.3] to obtain the Neumann series on  $Y_{1,\epsilon} = R(L_\epsilon)$  does not work since  $Y_{1,\epsilon} = R(L_\epsilon)$  is not invariant under  $\mathcal{L}_{\epsilon,\lambda}^{-1} J$ ; but the Neumann series can be obtained just by removing the restriction to  $Y_{1,\epsilon} = R(L_\epsilon)$ ; and, with this modification, the proof of [19, Proposition 6.3] can be completed.

Theorem 3.1 follows from Propositions 3.6 and 3.7 without the condition (3.1) if  $\sqrt{b/a} < a_1$ . In the case  $\sqrt{b/a} \geq a_1$ , there exists some range of  $\lambda$  near the imaginary axis with  $\operatorname{Im} \lambda = O(\epsilon^{-1})$  which should be proved to be in  $\rho(-L_\epsilon)$  for  $0 < \epsilon \ll 1$ .

To complete the proof of Theorem 3.1, it suffices to deduce the a priori estimate for solutions of (3.13) uniformly for  $\lambda = \mu + i\frac{\eta}{\epsilon}$  with  $-\mu_0 \leq \mu \leq \mu_1$  and  $a_1/2 \leq |\eta| \leq 2\sqrt{b/a}$ , where  $\mu_0$  and  $\mu_1$  are some positive constants. In fact, if we obtain such a uniform a priori estimate, then it follows that

$\{\lambda = \mu + i\frac{\eta}{\epsilon}; -\mu_0 \leq \mu \leq \mu_1, a_1/2 \leq |\eta| \leq 2\sqrt{b/a}\} \subset \rho(-L_\epsilon)$  by a standard continuation argument since  $\lambda = \pm i\frac{a_1}{\epsilon} \in \rho(-L_\epsilon)$  for  $0 < \epsilon \leq \epsilon_1$  by Proposition 3.7. We will establish an appropriate a priori estimate under the condition (3.1).

We begin with estimating the  $H^1$ -norm of the velocity component  $\mathbf{w}$  of a solution  $u = {}^\top(p, \mathbf{w})$  of (3.13) for  $\lambda$  near the imaginary axis with  $\text{Im } \lambda = O(\epsilon^{-1})$ .

**Proposition 3.9.** *Let  $\lambda = \mu + i\frac{\eta}{\epsilon}$  with  $\mu, \eta \in \mathbb{R}$ . Suppose that  $u = {}^\top(p, \mathbf{w}) \in D(L_\epsilon)$  is a solution of (3.13). For given positive numbers  $\mu_1$  and  $\eta_*$  there exist positive constants  $\delta_1$  and  $C' = C'(\|\mathbf{v}_s\|_{C^1}, \beta, \Omega)$  such that if*

$$\inf \left\{ \frac{\text{Re}(\nabla \varphi \cdot \nabla \mathbf{v}_s, \nabla \varphi)}{\|\Delta \varphi\|_2^2}; \varphi \in H_*^2(\Omega), \varphi \neq 0, \frac{\partial \varphi}{\partial \mathbf{n}}|_{\partial \Omega} = 0 \right\} \geq -\delta_1$$

and

$$-\frac{\beta^2}{128} \leq \mu \leq \mu_1, \quad \eta_* \leq \eta \leq C'\epsilon^{-1},$$

then

$$(\eta^3 + \beta^2 \eta) \|\mathbf{w}\|_2^2 + \eta \|\nabla \mathbf{w}\|_2^2 \leq C \left\{ \left( \eta + \frac{\epsilon^2}{\eta} \right) \|\mathbf{g}\|_2^2 + \frac{\epsilon^2}{\eta} \|f\|_{H^1}^2 \right\}$$

for all  $0 < \epsilon \leq C' \min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_* \mu_*^{-\frac{2}{3}}, \sqrt{\frac{1}{\mu_*}}\}$  with  $\mu_* = \max\{\frac{\beta^2}{128}, \mu_1\}$ .

To prove Proposition 3.9, we prepare several estimates for a solution of (3.13).

The following estimate can be proved in a similar manner to the proof of [19, Proposition 6.5]. For the convenience of the readers we give its proof.

**Proposition 3.10.** *Let  $\mu_1$  and  $\eta_*$  be given positive numbers. Let  $u = {}^\top(p, \mathbf{w}) \in D(L_\epsilon)$  be a solution of (3.13) with  $\lambda = \mu + i\frac{\eta}{\epsilon}$ ,  $\mu, \eta \in \mathbb{R}$ . There exists a positive constant  $C' = C'(\|\mathbf{v}_s\|_{C^1}, \beta, \Omega)$  such that if*

$$\epsilon \leq C' \min\{1, \eta_*, \frac{\eta_*}{\mu_*}, \frac{1}{\sqrt{\mu_1}}\}, \quad -\frac{\beta^2}{128} \leq \mu \leq \mu_1, \quad \eta_* \leq \eta \leq \frac{1}{4\epsilon}$$

with  $\mu_* = \max\{\frac{\beta^2}{128}, \mu_1\}$ , then

$$(\eta^3 + 2\beta^2 \eta) \|\mathbf{w}\|_2^2 + \eta \|\nabla \mathbf{w}\|_2^2 \leq -64\eta \text{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + C(\epsilon^2 \eta^2 + \epsilon) \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2.$$

Here  $\mathbf{G}_\lambda = \lambda \mathbf{g} - \nabla f$ ; and  $C$  is a positive constant depending only on  $\|\mathbf{v}_s\|_{C^1}$  and  $\Omega$ .

**Proof.** Let  $u = {}^\top(p, \mathbf{v}) \in D(L_\epsilon)$  be a solution of (3.13). Then, by (3.14), we have

$$p = -\frac{1}{\epsilon^2 \lambda} \operatorname{div} \mathbf{w} + \frac{1}{\lambda} f.$$

Substituting this into (3.15), we obtain

$$\epsilon^2 \lambda^2 \mathbf{w} - \epsilon^2 \lambda \Delta \mathbf{w} - \nabla \operatorname{div} \mathbf{w} + \epsilon^2 \lambda (\mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s) = \epsilon^2 \mathbf{G}_\lambda. \quad (3.19)$$

We take the inner product of (3.19) with  $\mathbf{w}$ . It follows that

$$\epsilon^2 \lambda^2 \|\mathbf{w}\|_2^2 + \epsilon^2 \lambda \|\nabla \mathbf{w}\|_2^2 + \|\operatorname{div} \mathbf{w}\|_2^2 = -\epsilon^2 \lambda (\mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + \epsilon^2 (\mathbf{G}_\lambda, \mathbf{w}). \quad (3.20)$$

Since  $\lambda^2 = \left(\mu^2 - \frac{\eta^2}{\epsilon^2}\right) + 2i\frac{\mu\eta}{\epsilon}$ , the real part of (3.20) yields

$$\begin{aligned} & (\epsilon^2 \mu^2 - \eta^2) \|\mathbf{w}\|_2^2 + \epsilon^2 \mu \|\nabla \mathbf{w}\|_2^2 + \|\operatorname{div} \mathbf{w}\|_2^2 \\ &= -\epsilon^2 \mu \operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + \epsilon \eta \operatorname{Im}(\mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + \epsilon^2 \operatorname{Re}(\mathbf{G}_\lambda, \mathbf{w}). \end{aligned} \quad (3.21)$$

Therefore,

$$\begin{aligned} (\eta^2 - \epsilon^2 \mu^2) \|\mathbf{w}\|_2^2 &\leq (\epsilon^2 \mu + 3 + \epsilon \eta) \|\nabla \mathbf{w}\|_2^2 \\ &\quad + (\epsilon^2 \mu \|\nabla \mathbf{v}_s\|_\infty + \epsilon \eta (\|\mathbf{v}_s\|_\infty^2 + \|\nabla \mathbf{v}_s\|_\infty)) \|\mathbf{w}\|_2^2 \\ &\quad + \epsilon^2 \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2. \end{aligned} \quad (3.22)$$

The imaginary part of (3.20) yields

$$\begin{aligned} & 2\mu\eta \|\mathbf{w}\|_2^2 + \eta \|\nabla \mathbf{w}\|_2^2 \\ &= -\epsilon\mu \operatorname{Im}(\mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) - \eta \operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + \epsilon \operatorname{Im}(\mathbf{G}_\lambda, \mathbf{w}) \\ &\leq -\eta \operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + \left( \epsilon^2 \frac{\mu^2}{2\eta} \|\mathbf{v}_s\|_\infty^2 + \epsilon |\mu| \|\nabla \mathbf{v}_s\|_\infty \right) \|\mathbf{w}\|_2^2 \\ &\quad + \frac{1}{2} \eta \|\nabla \mathbf{w}\|_2^2 + \epsilon \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2, \end{aligned}$$

and hence,

$$\begin{aligned} & 2\mu\eta \|\mathbf{w}\|_2^2 + \frac{1}{2} \eta \|\nabla \mathbf{w}\|_2^2 \\ &\leq -\eta \operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) + \left( \epsilon^2 \frac{\mu^2}{\eta} \|\mathbf{v}_s\|_\infty^2 + \epsilon |\mu| \|\nabla \mathbf{v}_s\|_\infty \right) \|\mathbf{w}\|_2^2 + \epsilon \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2. \end{aligned} \quad (3.23)$$

By (3.22) and (3.23), we have

$$\begin{aligned}
& \left( \frac{\eta^3}{12} - \epsilon^2 \frac{\eta \mu^2}{12} + 2\mu\eta \right) \|\mathbf{w}\|_2^2 + \frac{\eta}{4} (1 - \epsilon^2 \mu - \epsilon\eta) \|\nabla \mathbf{w}\|_2^2 \\
& \leq -\eta \operatorname{Re}(\mathbf{w} \cdot \mathbf{v}_s, \mathbf{w}) \\
& \quad + C \left\{ (\epsilon^2 \mu \eta + \epsilon \eta^2 + \epsilon |\mu|) \|\nabla \mathbf{v}_s\|_\infty + (\epsilon \eta^2 + \epsilon^2 \frac{\mu^2}{\eta}) \|\mathbf{v}_s\|_\infty^2 \right\} \|\mathbf{w}\|_2^2 \\
& \quad + C(\epsilon^2 \eta + \epsilon) \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2,
\end{aligned}$$

and consequently, if  $\epsilon \leq \frac{1}{2\sqrt{\mu_*}}$  and  $\eta \leq \frac{1}{4\epsilon}$ , then

$$\begin{aligned}
& \left( \frac{\eta^3}{12} - \epsilon^2 \frac{\eta \mu^2}{12} + 2\mu\eta + \frac{1}{16} \beta^2 \eta \right) \|\mathbf{w}\|_2^2 + \frac{1}{16} \eta \|\nabla \mathbf{w}\|_2^2 \\
& \leq -\eta \operatorname{Re}(\mathbf{w} \cdot \mathbf{v}_s, \mathbf{w}) \\
& \quad + C \left\{ (\epsilon^2 \mu \eta + \epsilon \eta^2 + \epsilon |\mu|) \|\nabla \mathbf{v}_s\|_\infty + (\epsilon \eta^2 + \epsilon^2 \frac{\mu^2}{\eta}) \|\mathbf{v}_s\|_\infty^2 \right\} \|\mathbf{w}\|_2^2 \\
& \quad + C(\epsilon^2 \eta + \epsilon) \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2.
\end{aligned}$$

Therefore, there exists a positive constant  $C' = C'(\|\mathbf{v}_s\|_{C^1}, \beta)$  such that if

$$\epsilon \leq C' \min\left\{1, \eta_*, \frac{\eta_*}{\mu_1}, \frac{1}{\sqrt{\mu_1}}\right\}, \quad -\frac{\beta^2}{128} \leq \mu \leq \mu_*, \quad \eta_* \leq \eta \leq \frac{1}{4\epsilon},$$

then

$$\frac{1}{32} (\eta^3 + \beta^2 \eta) \|\mathbf{w}\|_2^2 + \frac{1}{16} \eta \|\nabla \mathbf{w}\|_2^2 \leq -\eta \operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{w}_s, \mathbf{w}) + C(\epsilon^2 \eta^2 + \epsilon) \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2.$$

This completes the proof.  $\square$

We next estimate incompressible part of  $\mathbf{w}$ .

**Proposition 3.11.** *Let  $\mu_0, \mu_1$  and  $\eta_*$  be given positive numbers. Let  $u = {}^\top(p, \mathbf{w})$  be a solution of (3.13) with  $\lambda = \mu + i\frac{\eta}{\epsilon}$ ,  $-\mu_0 \leq \mu \leq \mu_1$ ,  $\eta \geq \eta_*$ . If  $\mathbf{w} = \mathbf{v} + \nabla \varphi$  is the Helmholtz decomposition of  $\mathbf{w}$ , then*

$$\begin{aligned}
\|\mathbf{v}\|_2^2 \leq & C \left\{ \frac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} \|\nabla \varphi\|_{H^1}^2 + \frac{\epsilon^2}{\eta^2} \|\nabla \varphi\|_{H^2}^2 + \frac{\epsilon^2}{\eta^2} \|\mathbf{g}\|_2^2 \right. \\
& \left. + \frac{\epsilon^2}{\eta^2} \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 + \frac{\epsilon^2}{\eta^2} \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 \right\},
\end{aligned}$$

$$\|\mathbf{v}\|_{H^2}^2 \leq C \left\{ \frac{\eta^{\frac{3}{2}}}{\epsilon^{\frac{3}{2}}} \|\nabla \varphi\|_{H^1}^2 + \|\nabla \varphi\|_{H^2}^2 + \|\mathbf{g}\|_2^2 + \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 + \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 \right\}.$$

To prove Proposition 3.11 we make use of the following estimate for the Stokes system with nonhomogeneous boundary data.

**Lemma 3.12.** *Suppose that  ${}^\top(p, \mathbf{v}) \in H_*^1(\Omega) \times H^2(\Omega)$  is a solution of*

$$\begin{cases} \operatorname{div} \mathbf{v} &= 0, \\ \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla p &= \mathbf{g}, \\ \mathbf{v}|_{\partial\Omega} &= \boldsymbol{\psi}, \end{cases} \quad (3.24)$$

with  $\lambda \in \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \pi - \omega\}$  for some  $0 < \omega < \frac{\pi}{2}$ ,  $\mathbf{g} \in L^2(\Omega)$  and  $\boldsymbol{\psi} \in H^{\frac{3}{2}}(\partial\Omega)$  satisfying  $\boldsymbol{\psi} \cdot \mathbf{n}|_{\partial\Omega} = 0$ . Then there exists a positive constant  $C = C(\omega, \Omega)$  such that

$$|\lambda| \|\mathbf{v}\|_2 + \|\mathbf{v}\|_{H^2} + \|p\|_{H^1} \leq C \{ \|\mathbf{g}\|_2 + |\lambda|^{\frac{3}{4}} \|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)} \}.$$

Lemma 3.12 will be proved in section 4.

**Proof of Proposition 3.11.** Let  $u = {}^\top(p, \mathbf{w}) \in D(L_\epsilon)$  be a solution of (3.14)–(3.16) and let  $\mathbf{w} = \mathbf{v} + \nabla \varphi$  be the Helmholtz decomposition of  $\mathbf{w}$ . Then,  $\operatorname{div} \mathbf{v} = 0$ ,  $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$ ,  $\frac{\partial \varphi}{\partial \mathbf{n}}|_{\partial\Omega} = \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega}$  and  $\int_\Omega \varphi dx = 0$ . Since  $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$ , we see that

$$\frac{\partial \varphi}{\partial \mathbf{n}} \Big|_{\partial\Omega} = 0$$

and

$$\begin{cases} \operatorname{div} \mathbf{v} &= 0, \\ \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla q &= \mathbf{g} - (\mathbf{v}_s \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_s), \\ \mathbf{v}|_{\partial\Omega} &= -\nabla \varphi|_{\partial\Omega}. \end{cases}$$

Here

$$q = \lambda \varphi - \Delta \varphi + p.$$

Note that

$$\int_\Omega q dx = \int_\Omega (\lambda \varphi - \Delta \varphi + p) dx = - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\sigma = 0.$$

By Lemma 3.12 with  $\boldsymbol{\psi}$  replaced by  $-\nabla\varphi|_{\partial\Omega}$ , we have

$$\begin{aligned} & |\lambda| \|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{H^2} + \|q\|_{H^1} \\ & \leq C \{ |\lambda|^{\frac{3}{4}} \|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|\boldsymbol{g}\|_2 + \|\boldsymbol{v}_s\|_\infty \|\nabla \boldsymbol{w}\|_2 + \|\nabla \boldsymbol{v}_s\|_\infty \|\boldsymbol{w}\|_2 \} \\ & \leq C \{ |\lambda|^{\frac{3}{4}} \|\nabla\varphi\|_{H^1} + \|\nabla\varphi\|_{H^2} + \|\boldsymbol{g}\|_2 + \|\boldsymbol{v}_s\|_\infty \|\nabla \boldsymbol{w}\|_2 + \|\nabla \boldsymbol{v}_s\|_\infty \|\boldsymbol{w}\|_2 \}. \end{aligned}$$

Therefore, if  $\lambda = \mu + i\frac{\eta}{\epsilon}$ ,  $-\mu_0 \leq \mu \leq \mu_1$  and  $\eta \geq \eta_*$ , then

$$\begin{aligned} \|\boldsymbol{v}\|_2^2 & \leq C \{ |\lambda|^{-\frac{1}{2}} \|\nabla\varphi\|_{H^1}^2 + |\lambda|^{-2} \|\nabla\varphi\|_{H^2}^2 + |\lambda|^{-2} \|\boldsymbol{g}\|_2^2 \\ & \quad + |\lambda|^{-2} (\|\boldsymbol{v}_s\|_\infty^2 \|\nabla \boldsymbol{w}\|_2^2 + \|\nabla \boldsymbol{v}_s\|_\infty^2 \|\boldsymbol{w}\|_2^2) \} \\ & \leq C \left\{ \frac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} \|\nabla\varphi\|_{H^1}^2 + \frac{\epsilon^2}{\eta^2} \|\nabla\varphi\|_{H^2}^2 + \frac{\epsilon^2}{\eta^2} \|\boldsymbol{g}\|_2^2 \right. \\ & \quad \left. + \frac{\epsilon^2}{\eta^2} (\|\boldsymbol{v}_s\|_\infty^2 \|\nabla \boldsymbol{w}\|_2^2 + \|\nabla \boldsymbol{v}_s\|_\infty^2 \|\boldsymbol{w}\|_2^2) \right\} \end{aligned}$$

and

$$\begin{aligned} \|\boldsymbol{v}\|_{H^2}^2 + \|q\|_{H^1}^2 & \leq C \left\{ \eta^{\frac{3}{2}} \epsilon^{-\frac{3}{2}} \|\nabla\varphi\|_{H^1}^2 + \|\nabla\varphi\|_{H^2}^2 + \|\boldsymbol{g}\|_2^2 \right. \\ & \quad \left. + \|\boldsymbol{v}_s\|_\infty^2 \|\nabla \boldsymbol{w}\|_2^2 + \|\nabla \boldsymbol{v}_s\|_\infty^2 \|\boldsymbol{w}\|_2^2 \right\}. \end{aligned}$$

This completes the proof.  $\square$

The potential flow part  $\nabla\varphi$  satisfies the following estimates.

**Proposition 3.13.** *Let  $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$  be as in Proposition 3.11. Then there exists a positive constant  $C' = C'(\|\boldsymbol{v}_s\|_{C^1})$  such that if  $0 < \epsilon \leq C' \min\{1, \frac{\eta_*}{\mu_*}, \eta_*\}$  with  $\mu_* = \max\{\mu_0, \mu_1\}$ , the following estimates*

$$\|\Delta\varphi\|_2^2 \leq C_1 \left\{ \eta^2 \|\boldsymbol{w}\|_2^2 + \epsilon\eta \|\nabla \boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta^2} \|\boldsymbol{G}_\lambda\|_2^2 \right\},$$

$$\frac{1}{\eta^2} \|\nabla \Delta\varphi\|_2^2 \leq C_1 \left\{ \eta^2 \|\boldsymbol{w}\|_2^2 + \epsilon\eta \|\nabla \boldsymbol{w}\|_2^2 + \epsilon^2 \|\Delta \boldsymbol{v}\|_2^2 + \frac{\epsilon^4}{\eta^2} \|\boldsymbol{G}_\lambda\|_2^2 \right\},$$

hold with  $C_1 > 0$  independent of  $\mu_0, \mu_1, \eta_*, \epsilon$ , and  $\Omega$ .

**Proof.** Let  $\mathbf{w} = \mathbf{v} + \nabla\varphi$  be the Helmholtz decomposition of  $\mathbf{w}$ . Since  $\operatorname{div} \mathbf{w} = \Delta\varphi$ , we see from (3.21) that

$$\begin{aligned}\|\Delta\varphi\|_2^2 &= -(\epsilon^2\mu^2 - \eta^2)\|\mathbf{w}\|_2^2 - \epsilon^2\mu\|\nabla\mathbf{w}\|_2^2 \\ &\quad - \epsilon^2\mu\operatorname{Re}(\mathbf{w} \cdot \nabla\mathbf{v}_s, \mathbf{w}) + \epsilon\eta\operatorname{Im}(\mathbf{v}_s \cdot \nabla\mathbf{w} + \mathbf{w} \cdot \nabla\mathbf{v}_s, \mathbf{w}) \\ &\quad + \epsilon^2\operatorname{Re}(\mathbf{G}_\lambda, \mathbf{w}) \\ &\leq \eta^2\|\mathbf{w}\|_2^2 + \epsilon^2\mu_0\|\nabla\mathbf{w}\|_2^2 + \epsilon^2\mu_1\|\nabla\mathbf{v}_s\|_\infty\|\mathbf{w}\|_2^2 \\ &\quad + \epsilon\eta\|\nabla\mathbf{w}\|_2^2 + \epsilon\eta(\|\mathbf{v}_s\|_\infty^2 + \|\nabla\mathbf{v}_s\|_\infty)\|\mathbf{w}\|_2^2 \\ &\quad + \epsilon^2\operatorname{Re}(\mathbf{G}_\lambda, \mathbf{w}),\end{aligned}$$

and hence,

$$\begin{aligned}\|\Delta\varphi\|_2^2 &\leq \eta^2\|\mathbf{w}\|_2^2 + (\epsilon^2\mu_0 + \epsilon\eta)\|\nabla\mathbf{w}\|_2^2 \\ &\quad + \{\epsilon^2\mu_1\|\nabla\mathbf{v}_s\|_\infty + \epsilon\eta(\|\mathbf{v}_s\|_\infty^2 + \|\nabla\mathbf{v}_s\|_\infty)\}\|\mathbf{w}\|_2^2 \\ &\quad + \epsilon^2\operatorname{Re}(\mathbf{G}_\lambda, \mathbf{w}).\end{aligned}$$

It follows that if  $\epsilon \leq \frac{\eta_*}{\mu_*}$ , then

$$\begin{aligned}\|\Delta\varphi\|_2^2 &\leq (2\eta^2 + 2\epsilon\eta(\|\mathbf{v}_s\|_\infty^2 + \|\nabla\mathbf{v}_s\|_\infty))\|\mathbf{w}\|_2^2 \\ &\quad + 2\epsilon\eta\|\nabla\mathbf{w}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\mathbf{G}_\lambda\|_2^2.\end{aligned}\tag{3.25}$$

Therefore there exists a positive constant  $C' = C'(\|\mathbf{v}_s\|_{C^1})$  such that if  $\epsilon \leq C' \min\left\{\frac{\eta_*}{\mu_*}, \eta_*\right\}$ , then

$$\|\Delta\varphi\|_2^2 \leq C_1 \left\{ \eta^2\|\mathbf{w}\|_2^2 + \epsilon\eta\|\nabla\mathbf{w}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\mathbf{G}_\lambda\|_2^2 \right\}.$$

We next estimate  $\|\nabla\Delta\varphi\|_2^2$ . We take the inner product of (3.19) with  $-\nabla\Delta\varphi$  to obtain

$$\begin{aligned}& -\epsilon^2\lambda^2(\mathbf{w}, \nabla\Delta\varphi) + \epsilon^2\lambda(\Delta\mathbf{w}, \nabla\Delta\varphi) + \|\nabla\Delta\varphi\|_2^2 \\ &= \epsilon^2\lambda(\mathbf{v}_s \cdot \nabla\mathbf{w} + \mathbf{w} \cdot \nabla\mathbf{v}_s, \nabla\Delta\varphi) - \epsilon^2(\mathbf{G}_\lambda, \nabla\Delta\varphi).\end{aligned}\tag{3.26}$$

Since  $\mathbf{w}|_{\partial\Omega} = 0$  and  $\operatorname{div} \mathbf{w} = \Delta\varphi$ , we have

$$-\epsilon^2\lambda^2(\mathbf{w}, \nabla\Delta\varphi) = \epsilon^2\lambda^2(\operatorname{div} \mathbf{w}, \Delta\varphi) = \epsilon^2\lambda^2\|\Delta\varphi\|_2^2,$$

$$\epsilon^2 \lambda (\Delta \mathbf{w}, \nabla \Delta \varphi) = \epsilon^2 \lambda (\Delta \mathbf{v}, \nabla \Delta \varphi) + \epsilon^2 \lambda \|\nabla \Delta \varphi\|_2^2.$$

Taking the real part of (3.26), we thus have

$$\begin{aligned} & (\epsilon^2 \mu^2 - \eta^2) \|\Delta \varphi\|_2^2 + \epsilon^2 \mu \|\nabla \Delta \varphi\|_2^2 + \|\nabla \Delta \varphi\|_2^2 \\ & \leq \frac{1}{2} \|\nabla \Delta \varphi\|_2^2 + 3\epsilon^4 |\lambda|^2 \{ \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 + \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 \} \\ & \quad + \frac{3}{2} \epsilon^4 \|\mathbf{G}_\lambda\|_2^2 + \frac{3}{2} \epsilon^4 |\lambda|^2 \|\Delta \mathbf{v}\|_2^2. \end{aligned}$$

This implies that, if  $\lambda = \mu + i \frac{\eta}{\epsilon}$  with  $-\mu_0 \leq \mu \leq \mu_1$ ,  $\eta \geq \eta_*$ , then

$$\begin{aligned} & \epsilon^2 \mu^2 \|\Delta \varphi\|_2^2 + \left(\frac{1}{2} - \epsilon^2 \mu_*\right) \|\nabla \Delta \varphi\|_2^2 \\ & \leq \eta^2 \|\Delta \varphi\|_2^2 + \frac{3}{2} (\epsilon^4 \mu^2 + \epsilon^2 \eta^2) \|\Delta \mathbf{v}\|_2^2 \\ & \quad + 3(\epsilon^4 \mu^2 + \epsilon^2 \eta^2) \{ \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 + \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 \} + \frac{3}{2} \epsilon^4 \|\mathbf{G}_\lambda\|_2^2. \end{aligned} \tag{3.27}$$

By (3.25) and (3.27), if  $\epsilon \leq \frac{\eta_*}{\mu_*}$  and  $\epsilon^2 \leq \frac{1}{4\mu_*}$ , then

$$\begin{aligned} & \epsilon^2 \frac{\mu^2}{\eta^2} \|\Delta \varphi\|_2^2 + \frac{1}{4\eta^2} \|\nabla \Delta \varphi\|_2^2 \\ & \leq 2\eta^2 \|\mathbf{w}\|_2^2 + 2\epsilon\eta \|\nabla \mathbf{w}\| + 2\epsilon\eta (\|\mathbf{v}_s\|_\infty^2 + \|\nabla \mathbf{v}_s\|_\infty) \|\mathbf{w}\|_2^2 \\ & \quad + \frac{3}{2} (\epsilon^4 \frac{\mu^2}{\eta^2} + \epsilon^2) \|\Delta \mathbf{v}\|_2^2 + 3(\epsilon^4 \frac{\mu^2}{\eta^2} + \epsilon^2) \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 \\ & \quad + 3(\epsilon^4 \frac{\mu^2}{\eta^2} + \epsilon^2) \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 + \frac{5}{2} \frac{\epsilon^4}{\eta^2} \|\mathbf{G}_\lambda\|_2^2. \end{aligned}$$

Therefore, there exists a positive constant  $C' = C'(\|\mathbf{v}_s\|_{C^1})$  such that if

$\epsilon \leq C' \min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_* \mu_*^{-\frac{2}{3}}, \frac{1}{\sqrt{\mu_*}}\}$ , then

$$\begin{aligned}
& \frac{1}{\eta^2} \|\nabla \Delta \varphi\|_2^2 \\
& \leq \left( 8\eta^2 + 12(\epsilon^4 \frac{\mu^2}{\eta^2} + \epsilon^2) \|\nabla \mathbf{v}_s\|_\infty^2 + 8\epsilon\eta(\|\mathbf{v}_s\|_\infty^2 + \|\nabla \mathbf{v}_s\|_\infty) \right) \|\mathbf{w}\|_2^2 \\
& \quad + \left( 8\epsilon\eta + 12(\epsilon^4 \frac{\mu^2}{\eta^2} + \epsilon^2) \|\mathbf{v}_s\|_\infty^2 \right) \|\nabla \mathbf{w}\|_2^2 \\
& \quad + 6(\epsilon^4 \frac{\mu^2}{\eta^2} + \epsilon^2) \|\Delta \mathbf{v}\|_2^2 + \frac{10\epsilon^4}{\eta^2} \|\mathbf{G}_\lambda\|_2^2 \\
& \leq C\{\eta^2 \|\mathbf{w}\|_2^2 + \epsilon\eta \|\nabla \mathbf{w}\|_2^2 + \epsilon^2 \|\Delta \mathbf{v}\|_2^2 + \frac{\epsilon^4}{\eta^2} \|\mathbf{G}_\lambda\|_2^2\}.
\end{aligned}$$

This completes the proof.  $\square$

We are now in a position to prove Proposition 3.9.

**Proof of Proposition 3.9.** Let  $\mathbf{w} = \mathbf{v} + \nabla \varphi$  be the Helmholtz decomposition of  $\mathbf{w}$ . Then

$$\begin{aligned}
-\eta \operatorname{Re}(\mathbf{w} \cdot \nabla \mathbf{v}_s, \mathbf{w}) & \leq -\eta \operatorname{Re}(\nabla \varphi \cdot \nabla \mathbf{v}_s, \nabla \varphi) + \eta\{|\operatorname{Re}(\mathbf{v} \cdot \nabla \mathbf{v}_s, \nabla \varphi)| \\
& \quad + |\operatorname{Re}(\nabla \varphi \cdot \nabla \mathbf{v}_s, \mathbf{v})| + |\operatorname{Re}(\mathbf{v} \cdot \nabla \mathbf{v}_s, \mathbf{v})|\} \\
& \leq -\eta \operatorname{Re}(\nabla \varphi \cdot \nabla \mathbf{v}_s, \nabla \varphi) \\
& \quad + \kappa\eta \|\nabla \mathbf{v}_s\|_\infty \|\nabla \varphi\|_2^2 + (1 + \frac{1}{\kappa})\eta \|\nabla \mathbf{v}_s\|_\infty \|\mathbf{v}\|_2^2
\end{aligned}$$

for any  $\kappa > 0$ . Choose  $\kappa = \frac{\beta^2}{64\|\nabla \mathbf{v}_s\|_\infty}$ . Then, since  $\|\mathbf{w}\|_2^2 = \|\mathbf{v}\|_2^2 + \|\nabla \varphi\|_2^2$ , we see from Proposition 3.10 that

$$\begin{aligned}
& (\eta^3 + \beta^2\eta) \|\mathbf{w}\|_2^2 + \eta \|\nabla \mathbf{w}\|_2^2 \\
& \leq -c_0\eta \operatorname{Re}(\nabla \varphi \cdot \nabla \mathbf{v}_s, \nabla \varphi) + C\eta \|\nabla \mathbf{v}_s\|_\infty \|\mathbf{v}\|_2^2 + C(\epsilon^2\eta^2 + \epsilon) \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2,
\end{aligned} \tag{3.28}$$

where  $c_0 = 64$ .

We estimate the second term on the right-hand side of (3.28). By Propositions 3.11 and 3.13, there exists a positive constant  $C = C(\Omega)$  such that

$$\begin{aligned}
\frac{1}{\eta^2} \|\nabla \Delta \varphi\|_2^2 & \leq C\left\{ \eta^2 \|\mathbf{w}\|_2^2 + \epsilon\eta \|\nabla \mathbf{w}\|_2^2 + \epsilon^{\frac{1}{2}}\eta^{\frac{3}{2}} \|\nabla \varphi\|_{H^1}^2 + \epsilon^2 \|\nabla \Delta \varphi\|_2^2 \right. \\
& \quad + \epsilon^2 \|\mathbf{g}\|_2^2 + \epsilon^2 \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 + \epsilon^2 \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 \\
& \quad \left. + \frac{\epsilon^4}{\eta^2} \|\mathbf{G}_\lambda\|_2^2 \right\}.
\end{aligned}$$

This implies that if  $\eta^2 \leq \frac{1}{2C\epsilon^2}$ , then

$$\begin{aligned} \frac{1}{\eta^2} \|\nabla \Delta \varphi\|_2^2 \leq & C \{ (\eta^2 + \epsilon^2 \|\nabla \mathbf{v}_s\|_\infty^2) \|\mathbf{w}\|_2^2 + (\epsilon \eta + \epsilon^2 \|\mathbf{v}_s\|_\infty^2) \|\nabla \mathbf{w}\|_2^2 \\ & + \epsilon^{\frac{1}{2}} \eta^{\frac{3}{2}} \|\nabla \varphi\|_{H^1}^2 + \epsilon^2 \|\mathbf{g}\|_2^2 + \frac{\epsilon^4}{\eta^2} \|\mathbf{G}_\lambda\|_2^2 \}. \end{aligned} \quad (3.29)$$

By (3.29), Proposition 3.11 and the elliptic estimates:  $\|\nabla \varphi\|_{H^k} \leq C \|\nabla^{k-1} \Delta \varphi\|_2$  ( $k = 1, 2$ ), we have

$$\begin{aligned} \|\mathbf{v}\|_2^2 \leq & C \left\{ \left( \frac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} + \epsilon^{\frac{5}{2}} \eta^{\frac{3}{2}} \right) \|\Delta \varphi\|_2^2 + \epsilon^2 (\eta^2 + \epsilon^2 \|\nabla \mathbf{v}_s\|_\infty^2) \|\mathbf{w}\|_2^2 \right. \\ & + \epsilon^2 (\epsilon \eta + \epsilon^2 \|\mathbf{v}_s\|_\infty^2) \|\nabla \mathbf{w}\|_2^2 + \epsilon^4 \|\mathbf{g}\|_2^2 + \frac{\epsilon^6}{\eta^2} \|\mathbf{G}_\lambda\|_2^2 \\ & \left. + \frac{\epsilon^2}{\eta^2} \|\mathbf{g}\|_2^2 + \frac{\epsilon^2}{\eta^2} \|\mathbf{v}_s\|_\infty^2 \|\nabla \mathbf{w}\|_2^2 + \frac{\epsilon^2}{\eta^2} \|\nabla \mathbf{v}_s\|_\infty^2 \|\mathbf{w}\|_2^2 \right\}. \end{aligned} \quad (3.30)$$

Furthermore, Proposition 3.13 gives

$$\eta \|\Delta \varphi\|_2^2 \leq C_1 \left\{ \eta^3 \|\mathbf{w}\|_2^2 + \epsilon \eta^2 \|\nabla \mathbf{w}\|_2^2 + \frac{\epsilon^4}{\eta} \|\mathbf{G}_\lambda\|_2^2 \right\}. \quad (3.31)$$

It then follows from (3.28), (3.30) and (3.31) that

$$\begin{aligned} & (\eta^3 + \beta^2 \eta) \|\mathbf{w}\|_2^2 + \eta \|\nabla \mathbf{w}\|_2^2 + \frac{\eta}{2C_1} \|\Delta \varphi\|_2^2 \\ & \leq -c_0 \eta \operatorname{Re} (\nabla \varphi \cdot \nabla \mathbf{v}_s, \nabla \varphi) \\ & \quad + \frac{1}{2} \eta^3 \|\mathbf{w}\|_2^2 + \frac{\epsilon}{2} \eta^2 \|\nabla \mathbf{w}\|_2^2 + \frac{\epsilon^4}{2\eta} \|\mathbf{G}_\lambda\|_2^2 \\ & \quad + C \eta \|\nabla \mathbf{v}_s\|_\infty \left[ \left( \epsilon^{\frac{1}{2}} \eta^{-\frac{1}{2}} + \epsilon^{\frac{5}{2}} \eta^{\frac{3}{2}} \right) \|\Delta \varphi\|_2^2 \right. \\ & \quad + \epsilon^2 (\eta^2 + \epsilon^2 \|\nabla \mathbf{v}_s\|_\infty^2) \|\mathbf{w}\|_2^2 \\ & \quad + \epsilon^2 (\epsilon \eta + \epsilon^2 \|\mathbf{v}_s\|_\infty^2) \|\nabla \mathbf{w}\|_2^2 + \frac{\epsilon^2}{\eta^2} \|\mathbf{v}_s\|_\infty \|\nabla \mathbf{w}\|_2^2 \\ & \quad \left. + \frac{\epsilon^2}{\eta^2} \|\nabla \mathbf{v}_s\|_\infty \|\mathbf{w}\|_2^2 + \epsilon^4 \|\mathbf{g}\|_2^2 + \frac{\epsilon^2}{\eta^2} \|\mathbf{g}\|_2^2 + \frac{\epsilon^6}{\eta^2} \|\mathbf{G}_\lambda\|_2^2 \right] \\ & \quad + C (\epsilon^2 \eta^2 + \epsilon) \|\mathbf{G}_\lambda\|_2 \|\mathbf{w}\|_2. \end{aligned}$$

From this we see that there exists a positive constant  $C' = C'(\|\mathbf{v}_s\|_{C^1}, \beta, \Omega)$

such that if  $\epsilon \leq C' \min\{1, \eta_*\}$ ,  $\eta \leq \frac{C}{\epsilon}$ , then

$$\begin{aligned}
& \frac{1}{4}(\eta^3 + \beta^2\eta)\|\mathbf{w}\|_2^2 + \frac{1}{2}\eta\|\nabla\mathbf{w}\|_2^2 + \frac{1}{4C_1}\eta\|\Delta\varphi\|_2^2 \\
& \leq -c_0\eta\operatorname{Re}(\nabla\varphi \cdot \nabla\mathbf{v}_s, \nabla\varphi) + C\epsilon^4\eta\|\mathbf{g}\|_2^2 + C(\epsilon^2\eta^2 + \epsilon)\|\mathbf{G}_\lambda\|_2\|\mathbf{w}\|_2 \\
& \quad + C\frac{\epsilon^4}{\eta}\|\mathbf{G}_\lambda\|_2^2 + C\frac{\epsilon^2}{\eta}\|\mathbf{g}\|_2^2. \\
& \leq \frac{1}{8}(\eta^3 + \beta^2\eta)\|\mathbf{w}\|_2^2 - c_0\eta\operatorname{Re}(\nabla\varphi \cdot \nabla\mathbf{v}_s, \nabla\varphi) \\
& \quad + C\left(\frac{\epsilon^4}{\eta} + \epsilon^4\eta + \frac{\epsilon^2}{\beta^2\eta}\right)\|\mathbf{G}_\lambda\|_2^2 + C\left(\epsilon^4\eta + \frac{\epsilon^2}{\eta}\right)\|\mathbf{g}\|_2^2.
\end{aligned}$$

Therefore, we conclude that if

$$\inf \left\{ \frac{\operatorname{Re}(\nabla\varphi \cdot \nabla\mathbf{v}_s, \nabla\varphi)}{\|\Delta\varphi\|_2^2}; \varphi \in H_*^2(\Omega), \varphi \neq 0, \frac{\partial\varphi}{\partial\mathbf{n}} \Big|_{\partial\Omega} = 0 \right\} \geq -\frac{1}{8c_0C_1},$$

then

$$(\eta^3 + \beta^2\eta)\|\mathbf{w}\|_2^2 + \eta\|\nabla\mathbf{w}\|_2^2 + \eta\|\Delta\varphi\|_2^2 \leq C\left\{\left(\eta + \frac{\epsilon^2}{\eta}\right)\|\mathbf{g}\|_2^2 + \frac{\epsilon^2}{\eta}\|\nabla f\|_2^2\right\}.$$

This completes the proof.  $\square$

It remains to derive the estimates for  $\|p\|_{H^1}$  and  $\|\partial_x^2\mathbf{w}\|_2$ .

**Proposition 3.14.** *Under the assumption of Proposition 3.9 (with  $C'$  suitably replaced),*

$$\|\partial_x^2\mathbf{w}\|_2 + \|\nabla p\|_2 \leq C\left\{\left(1 + \frac{\eta}{\epsilon}\right)\|\mathbf{g}\|_2 + \|f\|_{H^1}\right\}$$

for all  $0 < \epsilon \leq C' \min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_*\mu_*^{-\frac{2}{3}}, \frac{1}{\sqrt{\mu_*}}\}$ .

**Proof.** We see from (3.14)–(3.16) that

$$\begin{aligned}
\operatorname{div} \mathbf{w} &= \epsilon^2 f - \epsilon^2 \lambda p, \\
-\Delta \mathbf{w} + \nabla p &= \mathbf{g} - \lambda \mathbf{w} - \mathbf{v}_s \cdot \nabla \mathbf{w} - \mathbf{w} \cdot \nabla \mathbf{v}_s, \\
\mathbf{w}|_{\partial\Omega} &= \mathbf{0}.
\end{aligned}$$

The elliptic estimate for the Stokes system ([15, 22]) then gives

$$\begin{aligned} \|\partial_x^2 \mathbf{w}\|_2 + \|\nabla p\|_2 &\leq C\{\epsilon^2 \|f\|_{H^1} + \epsilon^2 |\lambda| \|p\|_{H^1} \\ &\quad + \|\mathbf{g}\|_2 + |\lambda| \|\mathbf{w}\|_2 + \|\mathbf{v}_s\|_\infty \|\nabla \mathbf{w}\|_2 + \|\nabla \mathbf{v}_s\|_\infty \|\mathbf{w}\|_2\}. \end{aligned}$$

By the Poincaré inequality, we have

$$\epsilon^2 |\lambda| \|p\|_{H^1} \leq C(\epsilon^2 \mu_* + \epsilon \eta) \|\nabla p\|_2$$

with  $C = C(\Omega) > 0$ . Therefore, if  $\epsilon \leq \frac{C}{\sqrt{\mu_*}}$  and  $\eta \leq \frac{C}{\epsilon}$ , then

$$\begin{aligned} \|\partial_x^2 \mathbf{w}\|_2 + \|\nabla p\|_2 &\leq C\{\epsilon^2 \|f\|_{H^1} + \|\mathbf{g}\|_2 + |\lambda| \|\mathbf{w}\|_2 + \|\mathbf{v}_s\|_\infty \|\nabla \mathbf{w}\|_2 + \|\nabla \mathbf{v}_s\|_\infty \|\mathbf{w}\|_2\}. \end{aligned}$$

This, together with Proposition 3.9, implies the desired conclusion. This completes the proof.  $\square$

We give a proof of Lemma 3.12. We first estimate a solution  ${}^\top(p, \mathbf{v})$  of the following problem on the half-space  $\mathbb{R}_+^3 = \{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, x_3 > 0\}$ :

$$\begin{cases} \operatorname{div} \mathbf{v} &= f, \\ \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla p &= \mathbf{g}, \\ \mathbf{v}|_{x_3=0} &= \boldsymbol{\psi}. \end{cases} \quad (3.32)$$

Here  $\lambda \in \mathbb{C}$  is a parameter;  $f$  and  $\mathbf{g}$  are given functions on  $\mathbb{R}_+^3$ ; and  $\boldsymbol{\psi} = {}^\top(\psi^1(x'), \psi^2(x'), \psi^3(x'))$  is a given function on  $\mathbb{R}^2$  satisfying  $\boldsymbol{\psi} \cdot \mathbf{n}|_{\partial \mathbb{R}_+^3} = -\psi^3(x') = 0$  ( $x' \in \mathbb{R}^2$ ). We denote by  $\Sigma_\omega$  the sector defined by

$$\Sigma_\omega = \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \pi - \omega\}.$$

We have the following estimate.

**Lemma 3.15.** *Let  $\omega$  be a positive number satisfying  $0 < \omega < \frac{\pi}{2}$ . If  $\lambda \in \Sigma_\omega \setminus \{0\}$ , then*

$$\begin{aligned} &|\lambda| \|\mathbf{v}\|_{L^2(\mathbb{R}_+^3)} + \|\partial_x^2 \mathbf{v}\|_{L^2(\mathbb{R}_+^3)} + \|\partial_x p\|_{L^2(\mathbb{R}_+^3)} \\ &\leq C_\omega \{|\lambda| \|f\|_{\dot{H}^{-1}(\mathbb{R}_+^3)} + \|\partial_x f\|_{L^2(\mathbb{R}_+^3)} + \|\mathbf{g}\|_{L^2(\mathbb{R}_+^3)} \\ &\quad + |\lambda|^{\frac{3}{4}} \|\boldsymbol{\psi}\|_{L^2(\mathbb{R}^2)} + \|\boldsymbol{\psi}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)}\}. \end{aligned}$$

Here  $C_\omega$  is a positive constant depending only on  $\omega$ .

To prove Lemma 3.15, we introduce some notation. For  $\lambda \in \mathbb{C} \setminus (-\infty, -|\xi'|^2]$ ,  $\lambda \neq 0$  and  $\xi' \in \mathbb{R}^2$ ,  $\xi' \neq 0$ , we denote the principal branch of the square root of  $\lambda + |\xi'|^2$  by  $\mu_1 = \mu_1(\lambda, \xi')$ , i.e.,

$$\mu_1 = \mu_1(\lambda, \xi') = \sqrt{\lambda + |\xi'|^2}$$

with  $\operatorname{Re} \mu_1 > 0$ . For a complex number  $\mu$ , we define functions  $g_\mu^{(\pm)}(x_3, y_3)$  and  $h_\mu(x_3)$  by

$$g_\mu^{(\pm)}(x_3, y_3) = \frac{1}{2\mu} \{e^{-\mu|x_3-y_3|} \pm e^{-\mu(x_3+y_3)}\}$$

and

$$h_\mu(x_3) = \frac{1}{\mu} e^{-\mu x_3}.$$

We also introduce functions  $\tilde{g}^{(\pm)}(x_3, y_3)$  and  $\tilde{h}(x_3)$  which are defined by

$$\tilde{g}^{(\pm)}(x_3, y_3) = g_{\mu_1}^{(\pm)}(x_3, y_3) - g_{|\xi'|}^{(\pm)}(x_3, y_3)$$

and

$$\tilde{h}(x_3) = h_{\mu_1}(x_3) - h_{|\xi'|}(x_3).$$

We denote by  $\delta(x_3)$  the Dirac delta function.

For a function  $K(x_3, y_3)$  on  $(0, \infty) \times (0, \infty)$  we denote by  $Kf$  the integral operator  $\int_0^\infty K(x_3, y_3) f(y_3) dy_3$ . The Fourier transform in  $x'$  variable is denoted by  $\mathcal{F}'$ :

$$(\mathcal{F}'f)(\xi') = \int_{\mathbb{R}^2} f(x') e^{i\xi' \cdot x'} dx', \quad \xi' \in \mathbb{R}^2,$$

and its inverse is denoted by  $\mathcal{F}'^{-1}$ . We also employ the notation  $\hat{f}(\xi')$  for the Fourier transform  $(\mathcal{F}'f)(\xi')$ . Then the solution  ${}^\top(p, \mathbf{v})$  of (3.32) is written as

$$\begin{aligned} \begin{pmatrix} p(x) \\ \mathbf{v}(x) \end{pmatrix} &= \mathcal{F}'^{-1} \hat{\Gamma} \mathcal{F}' F(x) + \mathcal{F}'^{-1} \hat{\Gamma}_B \mathcal{F}' \begin{pmatrix} 0 \\ \boldsymbol{\psi} \end{pmatrix} (x) \\ &= \mathcal{F}'^{-1} \left[ \int_0^\infty \hat{\Gamma}(\lambda, \xi', x_3, y_3) \hat{F}(\xi, y_3) dy_3 \right] (x') \\ &\quad + \mathcal{F}'^{-1} \left[ \hat{\Gamma}_B(\lambda, \xi', x_3) \begin{pmatrix} 0 \\ \hat{\boldsymbol{\psi}} \end{pmatrix} \right] (x') \end{aligned} \quad (3.33)$$

with  $F = {}^\top(f, \mathbf{g})$ ,  $\mathbf{g} = {}^\top(\mathbf{g}', g^3)$ . Here

$$\hat{\Gamma}(\lambda, \xi', x_3, y_3) = \hat{G}(\lambda, \xi', x_3, y_3) + \hat{H}(\lambda, \xi', x_3, y_3)$$

is the integral kernel of the solution operator for the problem (3.32) with  $\psi = \mathbf{0}$ , and

$$\hat{\Gamma}_B(\lambda, \xi', x_3) = \partial_{y_3} \hat{G}(\lambda, \xi', x_3, 0) + \partial_{y_3} \hat{H}(\lambda, \xi', x_3, 0),$$

where

$$\begin{aligned} & \hat{G}(\lambda, \xi', x_3, y_3) \\ &= \delta(x_3 - y_3) Q_0 \\ &+ \begin{pmatrix} \lambda g_{|\xi'|}^{(+)}(x_3, y_3) & -i^\top \xi' g_{|\xi'|}^{(+)}(x_3, y_3) & -\partial_{x_3} g_{|\xi'|}^{(-)}(x_3, y_3) \\ -i \xi' g_{|\xi'|}^{(+)}(x_3, y_3) & 0 & 0 \\ -\partial_{x_3} g_{|\xi'|}^{(+)}(x_3, y_3) & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{\mu_1}^{(-)}(x_3, y_3) I_2 & 0 \\ 0 & 0 & g_{\mu_1}^{(-)}(x_3, y_3) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\xi'^\top \xi'}{\lambda} \tilde{g}^{(+)}(x_3, y_3) & -\frac{i \xi'}{\lambda} \partial_{x_3} \tilde{g}^{(-)}(x_3, y_3) \\ 0 & -\frac{i^\top \xi'}{\lambda} \partial_{x_3} \tilde{g}^{(+)}(x_3, y_3) & -\frac{1}{\lambda} \partial_{x_3}^2 \tilde{g}^{(-)}(x_3, y_3) \end{pmatrix} \end{aligned}$$

with  $Q_0$  being the  $4 \times 4$  diagonal matrix  $Q_0 = \text{diag}(1, 0, 0, 0)$ , and

$$\begin{aligned} & \hat{H}(\lambda, \xi', x_3, y_3) \\ &= \begin{pmatrix} 0 & i^\top \xi' h_{|\xi'|}(x_3) e^{-\mu_1 y_3} & 0 \\ 0 & -\frac{\xi'^\top \xi'}{\lambda} \tilde{h}(x_3) e^{-\mu_1 y_3} & 0 \\ 0 & \frac{i^\top \xi'}{\lambda} \partial_{x_3} \tilde{h}(x_3) e^{-\mu_1 y_3} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ h_{\mu_1}(x_3) \beta_0(y_3) & h_{\mu_1}(x_3) B(y_3) & h_{\mu_1}(x_3) \beta_3(y_3) \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} -i^\top \xi' h_{|\xi'|}(x_3) \beta_0(y_3) & -i^\top \xi' h_{|\xi'|}(x_3) B(y_3) & -i^\top \xi' h_{|\xi'|}(x_3) \beta_3(y_3) \\ \frac{\xi'^\top \xi'}{\lambda} \tilde{h}(x_3) \beta_0(y_3) & \frac{\xi'^\top \xi'}{\lambda} \tilde{h}(x_3) B(y_3) & \frac{\xi'^\top \xi'}{\lambda} \tilde{h}(x_3) \beta_3(y_3) \\ -\frac{i^\top \xi'}{\lambda} \partial_{x_3} \tilde{h}(x_3) \beta_0(y_3) & -\frac{i^\top \xi'}{\lambda} \partial_{x_3} \tilde{h}(x_3) B(y_3) & -\frac{i^\top \xi'}{\lambda} \partial_{x_3} \tilde{h}(x_3) \beta_3(y_3) \end{pmatrix}, \end{aligned}$$

with  $\mu_1 = \mu_1(\lambda, \xi')$  and

$$\begin{aligned} \beta_0(y_3) &= \beta_0(\lambda, \xi', y_3) = \frac{i\lambda\xi'}{|\xi'|(\mu_1 - |\xi'|)} e^{-|\xi'|y_3}, \\ B(y_3) &= B(\lambda, \xi', y_3) = -\frac{\xi'^\top \xi'}{|\xi'|(\mu_1 - |\xi'|)} (e^{-\mu_1 y_3} - e^{-|\xi'|y_3}), \end{aligned}$$

and

$$\beta_3(y_3) = \beta_3(\lambda, \xi', y_3) = \frac{i\xi'}{\mu_1 - |\xi'|} (e^{-\mu_1 y_3} - e^{-|\xi'|y_3}).$$

The solution formula (3.33) will be derived in the end of this section.

To estimate  ${}^\top(p, \mathbf{v})$ , we prepare elementary inequalities.

**Lemma 3.16.** *Let  $\omega \in (0, \frac{\pi}{2})$ . Then there exists a positive constant  $C$  depending only on  $\omega$  such that the following inequalities hold uniformly for  $\lambda \in \Sigma_\omega \setminus \{0\}$ :*

- (i)  $C^{-1}(|\lambda|^{\frac{1}{2}} + |\xi'|) \leq |\mu_1| \leq C(|\lambda|^{\frac{1}{2}} + |\xi'|),$
- (ii)  $C^{-1}|\mu_1| \leq \text{Re } \mu_1 \leq C|\mu_1|,$
- (iii)  $|e^{-\mu_1 x_3} - e^{-|\xi'|x_3}| \leq C|\mu_1 - |\xi'|| \left( \frac{1}{|\xi'|} e^{-\frac{1}{2}\text{Re } \mu_1 x_3} + \frac{1}{\text{Re } \mu_1} e^{-\frac{1}{2}|\xi'|x_3} \right).$

Inequalities (i) and (ii) can be proved by an elementary computation; and estimate (iii) can be proved by using the mean value theorem. (Cf., [18, Lemma 3.15].)

We will use the following generalized Young inequality.

**Lemma 3.17.** *Let  $K(x_3, y_3)$  be a function on  $(0, \infty) \times (0, \infty)$  satisfying*

$$\sup_{y_3 > 0} \int_0^\infty |K(x_3, y_3)| dx_3 \leq M_1 \quad \text{and} \quad \sup_{x_3 > 0} \int_0^\infty |K(x_3, y_3)| dy_3 \leq M_2$$

for some constants  $M_1 > 0$  and  $M_2 > 0$ . Set

$$KF(x_3) = \int_0^\infty K(x_3, y_3) F(y_3) dy_3.$$

Then

$$\|KF\|_{L^2(0, \infty)} \leq M_1^{\frac{1}{2}} M_2^{\frac{1}{2}} \|F\|_{L^2(0, \infty)}.$$

See, e.g., [14] for a proof of Lemma 3.17.

**Proof of Lemma 3.15.** The estimate for  $\mathcal{F}^{-1} \hat{\Gamma} \mathcal{F} F$  is well-known. In fact, one can obtain the desired estimate for  $\mathcal{F}^{-1} \hat{\Gamma} \mathcal{F} F$  by using Lemmas 3.16 and 3.17. As for  $\mathcal{F}^{-1} \hat{\Gamma}_B \mathcal{F} [\top(0, \psi)]$ , we set  $\top(p_B, \mathbf{v}_B) = \mathcal{F}^{-1} \hat{\Gamma}_B \mathcal{F} [\top(0, \psi)]$ . Then, by a direct computation, we see from (3.33) that

$$\begin{aligned} \hat{p}_B &= -i(\mu_1 + |\xi'|) h_{|\xi'|}(x_3) \xi' \cdot \hat{\psi}', \\ \hat{\mathbf{v}}_B &= \begin{pmatrix} \mu_1 h_{\mu_1}(x_3) \hat{\psi}' + \frac{\xi'}{|\xi'|} h_{\mu_1}(x_3) \xi' \cdot \hat{\psi}' + \frac{\xi'}{\lambda} (\mu_1 + |\xi'|) \tilde{h}(x_3) \xi' \cdot \hat{\psi}' \\ -\frac{i}{\lambda} (\mu_1 + |\xi'|) \partial_{x_3} \tilde{h}(x_3) \xi' \cdot \hat{\psi}' \end{pmatrix}. \end{aligned}$$

The desired estimates can be obtained by applying Lemma 3.16. This completes the proof.  $\square$

We turn to the proof of Lemma 3.12. We fix  $\omega \in (0, \frac{\pi}{2})$ . Let  $C_\omega$  be the positive constant in Lemma 3.15. Then for each  $\bar{x} \in \partial\Omega$  there exist an open neighborhood  $\mathcal{O}_{\bar{x}}$  of  $\bar{x}$ , a local coordinate system with coordinates  $\tilde{x} = (\tilde{x}', \tilde{x}_3)$  in  $\bar{x}$  and a smooth function  $h(\tilde{x}')$  such that

$$\mathcal{O}_{\bar{x}} \cap \Omega = \{(\tilde{x}', \tilde{x}_3); h(\tilde{x}') < \tilde{x}_3 < h(\tilde{x}') + \gamma, |\tilde{x}'| < r\},$$

$$\mathcal{O}_{\bar{x}} \cap \partial\Omega = \{(\tilde{x}', \tilde{x}_3); \tilde{x}_3 = h(\tilde{x}'), |\tilde{x}'| < r\},$$

$$(0, h(0)) = \bar{x}, \quad \nabla_{\tilde{x}'} h(0) = 0, \quad \sup_{|\tilde{x}'| < r} |\nabla_{\tilde{x}'} h(\tilde{x}')| \leq \frac{1}{24C_\omega}$$

for some positive constants  $\gamma$  and  $r$ . By a translation and a rotation of the coordinate system we can write  $\tilde{x}$  as  $x$ . We set  $\Phi(x) = {}^\top(x', x_3 - h(x'))$ . Then  $y = \Phi(x)$  is a diffeomorphism from  $\mathcal{O}_{\tilde{x}}$  to an open set  $\tilde{\mathcal{O}}_x$  of  $\mathbb{R}_+^3$  and

$$\begin{cases} \Phi(\mathcal{O}_{\tilde{x}} \cap \Omega) = \{y = (y', y_3); |y'| < r, 0 < y_3 < \gamma\}, \\ \Phi(\mathcal{O}_{\tilde{x}} \cap \partial\Omega) = \{y = (y', y_3); |y'| < r, y_3 = 0\}, \\ |\nabla_{y'} h(y')| \leq \frac{C_\omega}{24} \text{ for } |y'| < r. \end{cases} \quad (3.34)$$

Since  $\partial\Omega$  is compact, there exist  $x^{(1)}, \dots, x^{(N)} \in \partial\Omega$  such that  $\partial\Omega \subset \cup_{m=1}^N \mathcal{O}_m$ ,  $\mathcal{O}_m = \mathcal{O}_{x^{(m)}}$  and diffeomorphisms  $\Phi_m : \mathcal{O}_m \rightarrow \tilde{\mathcal{O}}_m$ ,  $\Phi_m(x) = {}^\top(x', x_3 - h_m(x'))$ ,  $m = 1, \dots, N$ , which satisfy (3.34) with  $\mathcal{O}_{\tilde{x}}$  replaced by  $\mathcal{O}_m$  for some  $h = h_m(x')$ ,  $r = r_m$ , and  $\gamma = \gamma_m$ .

We take an open set  $\mathcal{O}_0$  such that  $\bar{\mathcal{O}}_0 \subset \Omega$  and  $\bar{\Omega} \subset \cup_{m=0}^N \mathcal{O}_m$ , and introduce a partition of unity  $\{\chi_m\}_{m=0}^N$  subordinate to  $\{\mathcal{O}_m\}_{m=0}^N$  satisfying  $\text{supp}\chi_m \subset \mathcal{O}_m$ ,  $\sum_{m=0}^N \chi_m = 1$  on  $\bar{\Omega}$ .

We make the following transformation of  $\chi_m p$  and  $\chi_m \mathbf{v}$  :

$$\begin{aligned} \tilde{p}_m(y) &= (\chi_m p)(\Phi_m^{-1}(y)), \\ \tilde{\mathbf{v}}_m(y) &= {}^\top(\tilde{\mathbf{v}}'_m(y), \tilde{v}_m^3(y)), \\ \tilde{\mathbf{v}}'_m(y) &= (\chi_m \mathbf{v}')(\Phi_m^{-1}(y)), \\ \tilde{v}_m^3(y) &= -\nabla_{y'} h(y') \cdot (\chi_m \mathbf{v}')(\Phi_m^{-1}(y)) + (\chi_m v^3)(\Phi_m^{-1}(y)). \end{aligned}$$

Furthermore,  $\chi_m \mathbf{g}$  is transformed into  $\tilde{\mathbf{g}}_m = {}^\top(\tilde{\mathbf{g}}'_m(y), \tilde{g}_m^n(y))$  in a similar manner.

Under these transformation the localized problem for (3.24) on  $\Omega_m = \mathcal{O}_m \cap \Omega$  is transformed into the following one on  $\mathbb{R}_+^n$ :

$$\begin{cases} \text{div } \tilde{\mathbf{v}}_m &= \widetilde{\mathbf{v} \cdot \nabla \chi_m}, \\ \lambda \tilde{\mathbf{v}}_m - \Delta \tilde{\mathbf{v}}_m + \nabla \tilde{p}_m &= \tilde{\mathbf{G}}_m, \\ \tilde{\mathbf{v}}'_m|_{y_3=0} &= \tilde{\psi}'_m, \quad \tilde{v}_m^3|_{y_3=0} = 0. \end{cases} \quad (3.35)$$

Here  $\widetilde{\mathbf{v} \cdot \nabla \chi_m}$  denotes  $(\mathbf{v} \cdot \nabla \chi_m)(\Phi_m^{-1}(y))$  and  $\tilde{\mathbf{G}}_m = {}^\top(\tilde{\mathbf{G}}'_m, \tilde{G}_m^3)$ , where

$$\begin{aligned} \tilde{\mathbf{G}}'_m &= (p \widetilde{\nabla_x \chi_m}) - 2 \widetilde{\nabla_x \chi_m \cdot \nabla_x \mathbf{v}'} - \widetilde{\mathbf{v}' \Delta_x \chi_m} + \tilde{\mathbf{g}}'_m + (\nabla_{y'} h) \partial_{y_3} \tilde{p}_m \\ &\quad - 2(\nabla_{y'} h) \cdot \nabla_{y'} \partial_{y_3} \tilde{\mathbf{v}}'_m - (\Delta_{y'} h) \partial_{y_3} \tilde{\mathbf{v}}'_m - |\nabla_{y'} h|^2 \partial_{y_3}^2 \tilde{\mathbf{v}}'_m, \end{aligned}$$

$$\begin{aligned}
\tilde{G}_m^3 &= -\nabla_{y'} h \cdot \tilde{\mathbf{G}}_m' + 2(\nabla_{y'}(\nabla_{y'} h)) \cdot \nabla_{y'} \tilde{\mathbf{v}}_m' + (\Delta_{y'} \nabla_{y'} h) \cdot \tilde{\mathbf{v}}_m' + \nabla_{y'} h \cdot \nabla_{y'} \tilde{p}_m \\
&\quad + \widetilde{(p\partial_{x_3}\chi_m)} - 2\widetilde{\nabla_x \chi_m \cdot \nabla_x v^3} - \widetilde{v^3 \Delta_x \chi_m} + \tilde{g}_m^3 - 2(\nabla_{y'} h) \cdot \nabla_{y'} \partial_{y_3} \tilde{w}_m^3 \\
&\quad - (\Delta_{y'} h) \partial_{y_3} \tilde{w}_m^3 - |\nabla_{y'} h|^2 \partial_{y_3}^2 \tilde{w}_m^3.
\end{aligned}$$

Here

$$\tilde{w}_m^3 = \tilde{v}_m^3 + \nabla' h \cdot \tilde{\mathbf{v}}_m'.$$

Applying Lemma 3.15 to (3.35), we have the following estimate.

**Proposition 3.18.** *Let  $\lambda \in \Sigma_\omega \setminus \{0\}$ . Then*

$$\begin{aligned}
&|\lambda| \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y^2 \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{p}_m\|_{L^2(\mathbb{R}_+^3)} \\
&\leq C'_\omega \{ \|\lambda \mathbf{v} \cdot \widetilde{\nabla \chi_m}\|_{\dot{H}^{-1}(\mathbb{R}_+^3)} + \|\nabla(\mathbf{v} \cdot \widetilde{\nabla \chi_m})\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\widetilde{p \nabla_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\nabla_x \chi_m \cdot \nabla_x \mathbf{v}}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\mathbf{v} \Delta_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\tilde{\mathbf{g}}_m\|_{L^2(\mathbb{R}_+^3)} + |\lambda|^{\frac{3}{4}} \|\tilde{\boldsymbol{\psi}}_m\|_{L^2(\mathbb{R}^2)} + \|\tilde{\boldsymbol{\psi}}_m\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)} \},
\end{aligned}$$

where  $C'_\omega$  depends only on  $\omega$ .

**Proof.** We have

$$\begin{aligned}
\|\tilde{\mathbf{G}}_m\|_{L^2(\mathbb{R}_+^3)} &\leq 12\|\nabla_{y'} h\|_\infty \{ \|\partial_y \tilde{p}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y^2 \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} \} \\
&\quad + C \{ \|\widetilde{p \nabla_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\nabla_x \chi_m \cdot \nabla_x \mathbf{v}}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\mathbf{v} \Delta_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\tilde{\mathbf{g}}_m\|_{L^2(\mathbb{R}_+^3)} \}.
\end{aligned}$$

Noting that  $12\|\nabla_{y'} h\|_\infty \leq \frac{1}{2}$  and applying Lemma 3.15 to (3.35), we obtain

$$\begin{aligned}
&|\lambda| \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y^2 \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{p}_m\|_{L^2(\mathbb{R}_+^3)} \\
&\leq \frac{1}{2} \{ \|\partial_y \tilde{p}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y^2 \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} \} \\
&\quad + C''_\omega \{ \|\lambda \mathbf{v} \cdot \widetilde{\nabla \chi_m}\|_{\dot{H}^{-1}(\mathbb{R}_+^3)} + \|\nabla(\mathbf{v} \cdot \widetilde{\nabla \chi_m})\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\widetilde{p \nabla_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\nabla_x \chi_m \cdot \nabla_x \mathbf{v}}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\mathbf{v} \Delta_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} \\
&\quad + \|\tilde{\mathbf{g}}_m\|_{L^2(\mathbb{R}_+^3)} + |\lambda|^{\frac{3}{4}} \|\tilde{\boldsymbol{\psi}}_m\|_{L^2(\mathbb{R}^2)} + \|\tilde{\boldsymbol{\psi}}_m\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)} \},
\end{aligned}$$

and hence,

$$\begin{aligned}
& |\lambda| \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y^2 \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{p}_m\|_{L^2(\mathbb{R}_+^3)} \\
& \leq 2C_\omega'' \{ \|\lambda \widetilde{\mathbf{v} \cdot \nabla \chi_m}\|_{\dot{H}^{-1}(\mathbb{R}_+^3)} + \|\nabla(\widetilde{\mathbf{v} \cdot \nabla \chi_m})\|_{L^2(\mathbb{R}_+^3)} \\
& \quad + \|\widetilde{p \nabla_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} + \|\nabla_x \chi_m \cdot \nabla_x \mathbf{v}\|_{L^2(\mathbb{R}_+^3)} + \|\widetilde{\mathbf{v} \Delta_x \chi_m}\|_{L^2(\mathbb{R}_+^3)} \\
& \quad + \|\tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y \tilde{\mathbf{v}}_m\|_{L^2(\mathbb{R}_+^3)} \\
& \quad + \|\tilde{\mathbf{g}}_m\|_{L^2(\mathbb{R}_+^3)} + |\lambda|^{\frac{3}{4}} \|\tilde{\boldsymbol{\psi}}_m\|_{L^2(\mathbb{R}^2)} + \|\tilde{\boldsymbol{\psi}}_m\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)} \}.
\end{aligned}$$

This completes the proof.  $\square$

We then have the following estimate.

**Proposition 3.19.** *Let  $\lambda \in \Sigma_\omega \setminus \{0\}$  with  $|\lambda| \geq \delta$  for some  $\delta > 0$ . Then*

$$\begin{aligned}
& |\lambda| \|\mathbf{v}\|_2 + \|\mathbf{v}\|_{H^2} + \|\partial_x p\|_{H^1} \\
& \leq C_1 \{ |\lambda| \|\mathbf{v}\|_{H^{-1}} + \|\mathbf{v}\|_{H^1} + \|p\|_2 + \|\mathbf{g}\|_2 + |\lambda|^{\frac{3}{4}} \|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{\dot{H}^{\frac{3}{2}}(\partial\Omega)} \}
\end{aligned}$$

**Proof.** By Proposition 3.18 we have

$$\begin{aligned}
& |\lambda| \|\mathbf{v}\|_2 + \|\partial_x^2 \mathbf{v}\|_2 + \|\partial_x p\|_2 \\
& \leq C \{ |\lambda| \|\mathbf{v}\|_{H^{-1}} + \|\mathbf{v}\|_{H^1} + \|p\|_2 + \|\mathbf{g}\|_2 + |\lambda|^{\frac{3}{4}} \|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{\dot{H}^{\frac{3}{2}}(\partial\Omega)} \}.
\end{aligned}$$

Since  $|\lambda| \geq \delta$ , using the Poincaré inequality:  $\|p\|_{L^2(\Omega)} \leq \|\partial_x p\|_{L^2(\Omega)}$  for  $p \in H_*^1(\Omega)$ , we see that

$$\begin{aligned}
& |\lambda| \|\mathbf{v}\|_2 + \|\mathbf{v}\|_2 + \|\partial_x^2 \mathbf{v}\|_2 + \|p\|_{H^1} \\
& \leq C \{ |\lambda| \|\mathbf{v}\|_{H^{-1}} + \|\mathbf{v}\|_{H^1} + \|p\|_2 + \|\mathbf{g}\|_2 + |\lambda|^{\frac{3}{4}} \|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{\dot{H}^{\frac{3}{2}}(\partial\Omega)} \}. \quad (3.36)
\end{aligned}$$

Furthermore, since

$$\|\nabla \mathbf{v}\|_2^2 = \int_{\partial\Omega} \mathbf{v} \frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{n}} d\sigma - \int_{\Omega} \mathbf{v} \Delta \bar{\mathbf{v}} dx,$$

we have

$$\|\nabla \mathbf{v}\|_2 \leq C \{ \|\mathbf{v}\|_{L^2(\partial\Omega)} + \|\mathbf{v}\|_2 + \|\partial_x^2 \mathbf{v}\|_2 \}.$$

This, together with (3.36), gives the desired estimate.  $\square$

Lemma 3.12 follows from Proposition 3.19 by a compactness argument as given in [13].

We finally give a proof of solution formula (3.33). To this end, we introduce the integral kernel of the generalized resolvent for the adjoint problem for (3.32) with  $\boldsymbol{\psi} = \mathbf{0}$ :

$$\begin{cases} -\operatorname{div} \mathbf{v}^* &= q, \\ \bar{\lambda} \mathbf{v}^* - \Delta \mathbf{v}^* - \nabla p^* &= \mathbf{f}, \\ \mathbf{v}^*|_{x_3=0} &= \mathbf{0}. \end{cases} \quad (3.37)$$

The solution of (3.37) has a representation similar to (3.33). In fact, it holds that

$$\begin{pmatrix} p^*(x) \\ \mathbf{v}^*(x) \end{pmatrix} = \begin{pmatrix} q \\ \mathbf{f} \end{pmatrix} \Gamma^*(x),$$

where  $\Gamma^*(x, y) = (\mathcal{F}^{-1} \hat{\Gamma}^* \mathcal{F})(x, y)$  with  $\hat{\Gamma}^*(\lambda, \xi', x_3, y_3) = \overline{{}^\top \hat{\Gamma}(\bar{\lambda}, \xi', y_3, x_3)}$ . We denote the integral kernel of  $\mathcal{F}^{-1} \hat{\Gamma} \mathcal{F}$  by  $\Gamma(x, y) = \int_{\mathbb{R}^2} e^{i(x'-y') \cdot \xi'} \hat{\Gamma}(\lambda, \xi', x_3, y_3) d\xi'$  by

$$\Gamma(x, y) = \begin{pmatrix} \Gamma_{11}(x, y) & \Gamma_{12}(x, y) \\ \Gamma_{21}(x, y) & \Gamma_{22}(x, y) \end{pmatrix}$$

where  $\Gamma_{12}(x, y)$  is an  $1 \times n$  matrix;  $\Gamma_{21}(x, y)$  is an  $n \times 1$  matrix; and  $\Gamma_{22}(x, y)$  is an  $n \times n$  matrix. Similarly we write

$$\Gamma^*(x, y) = \begin{pmatrix} \Gamma_{11}^*(x, y) & \Gamma_{12}^*(x, y) \\ \Gamma_{21}^*(x, y) & \Gamma_{22}^*(x, y) \end{pmatrix}.$$

It follows that

$$p^*(x) = \Gamma_{11}^* q(x) + \Gamma_{12}^* \mathbf{f}(x), \quad \mathbf{v}^*(x) = \Gamma_{21}^* q(x) + \Gamma_{22}^* \mathbf{f}(x).$$

Let  $q \in C_0^\infty(\mathbb{R}_+^3)$  and  $\mathbf{f} \in C_0^\infty(\mathbb{R}_+^3)^3$ . Since  $\mathbf{v} \cdot \mathbf{n}|_{\partial \mathbb{R}_+^3} = \boldsymbol{\psi} \cdot \mathbf{n} = -\psi^3 = 0$ ,

we have

$$\begin{aligned}
& \int_{\mathbb{R}_+^3} (p(x)\overline{q(x)} + \mathbf{v}(x)\overline{\mathbf{f}(x)})dx \\
&= - \int_{\mathbb{R}_+^3} p(x)\overline{\operatorname{div}(\Gamma_{21}^*q(x) + \Gamma_{22}^*\mathbf{f}(x))}dx + \int_{\mathbb{R}_+^3} \mathbf{v}(x)(\bar{\lambda} - \Delta)(\overline{\Gamma_{21}^*q(x) + \Gamma_{22}^*\mathbf{f}(x)})dx \\
&\quad - \int_{\mathbb{R}_+^3} \mathbf{v}(x) \cdot \overline{\nabla(\Gamma_{11}^*q(x) + \Gamma_{12}^*\mathbf{f}(x))}dx \\
&= \int_{\mathbb{R}_+^3} \nabla p(x)\overline{(\Gamma_{21}^*q(x) + \Gamma_{22}^*\mathbf{f}(x))}dx \\
&\quad + \int_{\mathbb{R}_+^3} (\lambda - \Delta)\mathbf{v}(x)\overline{(\Gamma_{21}^*q(x) + \Gamma_{22}^*\mathbf{f}(x))}dx - \int_{\partial\mathbb{R}_+^3} \mathbf{v}(x)\overline{\left(\frac{\partial\Gamma_{21}^*q}{\partial\mathbf{n}_x}(x) + \frac{\partial\Gamma_{22}^*\mathbf{f}}{\partial\mathbf{n}_x}(x)\right)}dS_x \\
&\quad + \int_{\mathbb{R}_+^3} \operatorname{div} \mathbf{v}(x)\overline{(\Gamma_{11}^*q(x) + \Gamma_{12}^*\mathbf{f}(x))}dx \\
&= \int_{\mathbb{R}_+^3} \mathbf{g}(x) \cdot \overline{(\Gamma_{21}^*q(x) + \Gamma_{22}^*\mathbf{f}(x))}dx - \int_{\partial\mathbb{R}_+^3} \psi(x)\overline{\left(\frac{\partial\Gamma_{21}^*q}{\partial\mathbf{n}_x}(x) + \frac{\partial\Gamma_{22}^*\mathbf{f}}{\partial\mathbf{n}_x}(x)\right)}dS_x \\
&\quad + \int_{\mathbb{R}_+^3} f(x)\overline{(\Gamma_{11}^*q(x) + \Gamma_{12}^*\mathbf{f}(x))}dx \\
&= \left( \int_{\mathbb{R}_+^3} \Gamma_{12}(y, x)\mathbf{g}(x)dx - \int_{\partial\mathbb{R}_+^3} \frac{\partial\Gamma_{12}}{\partial\mathbf{n}_x}(y, x)\psi(x)dS_x + \int_{\mathbb{R}_+^3} \Gamma_{11}(y, x)f(x)dx, q \right)_{L_y^2} \\
&\quad + \left( \int_{\mathbb{R}_+^3} \Gamma_{22}(y, x)\mathbf{g}(x)dx - \int_{\partial\mathbb{R}_+^3} \frac{\partial\Gamma_{22}}{\partial\mathbf{n}_x}(y, x)\psi(x)dS_x + \int_{\mathbb{R}_+^3} \Gamma_{21}(y, x)f(x)dx, \mathbf{f} \right)_{L_y^2}.
\end{aligned}$$

We thus obtain

$$p(x) = \Gamma_{11}f(x) + \Gamma_{12}\mathbf{g}(x) - \int_{\partial\mathbb{R}_+^3} \frac{\partial\Gamma_{12}}{\partial\mathbf{n}_y}(x, y)\psi(y)dS_y, \quad (3.38)$$

$$\mathbf{v}(x) = \Gamma_{21}f(x) + \Gamma_{22}\mathbf{g}(x) - \int_{\partial\mathbb{R}_+^3} \frac{\partial\Gamma_{22}}{\partial\mathbf{n}_y}(x, y)\psi(y)dS_y. \quad (3.39)$$

Since  $\frac{\partial}{\partial\mathbf{n}_y} = -\partial_{y_3}$  on  $\partial\mathbb{R}_+^3 = \{(y', y_3); y' \in \mathbb{R}^2, y_3 = 0\} = \mathbb{R}^2$ , we obtain the solution formula (3.33) from (3.38) and (3.39).

## 4 Stability of bifurcating stationary solutions

Our interest in this section is in the stability of a stationary solution bifurcating from a basic stationary flow. In Theorem 3.1,  $\epsilon_0$  depends on  $b_0$  and it may occur that  $\epsilon_0 \rightarrow 0$  as  $b_0 \rightarrow 0$ . Therefore, the range of  $\epsilon$  for which Theorem 3.1 holds shrinks as the spectrum  $\sigma(-L)$  approaches to the imaginary axis. This can happen when a bifurcation occurs as a control parameter approaches its critical value. In this section we show that the range of  $\epsilon$  can be taken uniformly near the bifurcation point when the stability of a bifurcating solution from a simple eigenvalue is considered.

### 4.1 Formulation and results

We consider the following situation. Let  $\mathbf{v}_{\mathcal{R}}$  be a basic stationary flow with a bifurcation parameter  $\mathcal{R}$  (e.g.,  $\mathcal{R}$  is the Reynolds number) and let the linearized operator around  $\mathbf{v}_{\mathcal{R}}$  be written as

$$\mathbb{L} = -\mathbb{P}\Delta + \mathcal{R}\mathbb{P}\mathbb{M}[\mathbf{v}_{\mathcal{R}}],$$

with domain  $D(\mathbb{L}) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \cap L_\sigma^2(\Omega)$ , where

$$\mathbb{M}[\mathbf{v}]\mathbf{w} = \mathbf{v} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}.$$

We assume that a stationary bifurcation occurs in the following way.

- (A0) There exists a positive number  $\mathcal{R}_c$  such that if  $\mathcal{R}$  is smaller than  $\mathcal{R}_c$ , then  $\mathbf{v}_{\mathcal{R}}$  is stable; and if  $\mathcal{R}$  is larger than  $\mathcal{R}_c$ , then  $\mathbf{v}_{\mathcal{R}}$  is unstable and a stationary bifurcation occurs at  $\mathcal{R} = \mathcal{R}_c$ .

We thus introduce a bifurcation parameter  $\eta = \mathcal{R} - \mathcal{R}_c$  and write  $\mathbf{v}_{\mathcal{R}}$  as  $\mathbf{v}_\eta$ , and the linearized operator  $\mathbb{L}$  as  $\mathbb{L}_\eta$ :

$$\begin{aligned} \mathbb{L}_\eta &= -\mathbb{P}\Delta + (\mathcal{R}_c + \eta)\mathbb{P}(\mathbf{v}_\eta \cdot \nabla + (\nabla \mathbf{v}_\eta)^\top) \\ &= -\mathbb{P}\Delta + (\mathcal{R}_c + \eta)\mathbb{P}\mathbb{M}[\mathbf{v}_\eta], \end{aligned}$$

We denote by  $\mathbb{L}_\eta^*$  the adjoint operator of  $\mathbb{L}_\eta$ ,

$$\mathbb{L}_\eta^* = -\mathbb{P}\Delta + (\mathcal{R}_c + \eta)\mathbb{P}\mathbb{M}^*[\mathbf{v}_\eta]$$

with domain  $D(\mathbb{L}_\eta) = D(\mathbb{L})$ , where

$$\mathbb{M}^*[\mathbf{v}]\mathbf{w} = -\mathbf{v} \cdot \nabla \mathbf{w} + (\nabla \mathbf{v})\mathbf{w}.$$

We make the following assumptions.

(A1)  $\mathbf{v}_\eta$  is a smooth stationary solution.

(A2)  $\mathbf{v}_\eta$  is analytic in  $\eta$  in  $H^2(\Omega)^3$ .

(A3) 0 is a simple eigenvalue of  $-\mathbb{L}_0$  with  $\text{Ker}(\mathbb{L}_0) = \text{span}\{\mathbf{w}_0\}$ .

The eigenprojection for the eigenvalue 0 is  $P_0\mathbf{w} = \langle \mathbf{w} \rangle \mathbf{w}_0$ . Here and in what follows the symbol  $\langle \mathbf{w} \rangle$  for  $\mathbf{w} \in L^2(\Omega)^3$  is defined by

$$\langle \mathbf{w} \rangle = (\mathbf{w}, \mathbf{w}_0^*),$$

where  $\mathbf{w}_0^*$  is the eigenfunction for the eigenvalue 0 of  $\mathbb{L}_0^*$  satisfying  $\langle \mathbf{w}_0 \rangle = 1$ .

(A4)  $\langle \mathbb{M}[\mathbf{v}_0]\mathbf{w}_0 + \mathcal{R}_c\mathbb{M}[\mathbf{v}_1]\mathbf{w}_0 \rangle \neq 0$ , where  $\mathbf{v}_1 = \partial_\eta \mathbf{v}_\eta|_{\eta=0}$ .

(A5) There exists a positive constant  $\tilde{b}_0 > 0$  such that

$$\{\lambda \in \mathbb{C}; \text{Re } \lambda \geq -\tilde{b}_0\} \setminus \{0\} \subset \rho(-\mathbb{L}_0)$$

We look for a solution branch  $\{\eta, \mathbf{w}_\eta\}$ ,  $\mathbf{w}_\eta \neq \mathbf{0}$ , of

$$(\text{NS})_\eta \quad \mathbb{L}_\eta \mathbf{w}_\eta + (\mathcal{R}_c + \eta)\mathbb{PN}(\mathbf{w}_\eta, \mathbf{w}_\eta) = 0.$$

Here  $\mathbb{N}(\mathbf{w}_\eta, \mathbf{w}_\eta) = \mathbf{w}_\eta \cdot \nabla \mathbf{w}_\eta$ . Note that  $\mathbf{w}_\eta = \mathbf{0}$  is a solution of  $(\text{NS})_\eta$  for all  $\eta$ . Under (A1)-(A4) we have a nontrivial solution branch. In fact, by applying the standard bifurcation theory ([7]), one can prove the following proposition.

**Proposition 4.1.** *Assume (A1)-(A4). There exist a positive constant  $\delta_0$  and a solution branch  $\{\eta(\delta), \mathbf{w}_\eta(\delta)\}$  of  $(\text{NS})_\eta$  with  $\eta = \eta(\delta)$  of the form*

$$\eta(\delta) = \delta\sigma(\delta),$$

$$\mathbf{w}_\eta(\delta) = \delta(\mathbf{w}_0 + \delta\mathbf{w}_1(\delta)),$$

where  $\sigma(\delta)$  is analytic in  $\delta$  ( $|\delta| \leq \delta_0$ ), and  $\mathbf{w}_1(\delta)$  is analytic in  $\delta$  in  $H^2(\Omega)$  ( $|\delta| \leq \delta_0$ ).

We next consider the stability of  $\tilde{\mathbf{v}}(\delta) = \mathbf{v}_{\eta(\delta)} + \mathbf{w}_\eta(\delta)$ . The linearized operator around  $\tilde{\mathbf{v}}(\delta)$  is denoted by

$$\mathbb{L}(\delta) = -\mathbb{P}\Delta + (\mathcal{R}_c + \eta(\delta))\mathbb{M}[\tilde{\mathbf{v}}(\delta)].$$

As for the spectrum of  $-\mathbb{L}(\delta)$ , we have the following proposition.

**Proposition 4.2.** *Assume (A1)-(A5). There exists positive number  $\delta_0$  such that*

$$\rho(-\mathbb{L}(\delta)) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\frac{3}{4}\tilde{b}_0, |\lambda| > \frac{\tilde{b}_0}{4}\},$$

$$\sigma(-\mathbb{L}(\delta)) \cap \{\lambda \in \mathbb{C}; |\lambda| \leq \frac{\tilde{b}_0}{4}\} = \{\lambda(\delta)\},$$

for all  $\delta \in (-\delta_0, \delta_0)$ . Here  $\lambda(\delta)$  is a simple eigenvalue given by

$$\lambda(\delta) = -\alpha(\delta)\delta \frac{d\eta}{d\delta}(\delta),$$

where  $\alpha(\delta)$  is an analytic function of  $\delta \in (-\delta_0, \delta_0)$  satisfying

$$\alpha(0) = -\langle \mathbb{M}[\mathbf{v}_0]\mathbf{w}_0 + \mathcal{R}_c\mathbb{M}[\mathbf{v}_1]\mathbf{w}_0 \rangle (\neq 0).$$

Proposition 4.2 was obtained by Crandall-Rabinowitz [8] (See also [1]).

Concerning the stability of  $\tilde{\mathbf{v}}(\delta)$ , we have the following proposition.

**Proposition 4.3.** *Assume (A0)-(A5).*

(i)  $\alpha(0) = -\langle \mathbb{M}[\mathbf{v}_0]\mathbf{w}_0 + \mathcal{R}_c\mathbb{M}[\mathbf{v}_1]\mathbf{w}_0 \rangle > 0$ .

(ii)  $\lambda(\delta) = \lambda_k \delta^k + \mathcal{O}(\delta^{k+1})$  if and only if  $\eta(\delta) = \eta_k \delta^k + \mathcal{O}(\delta^{k+1})$ . In this case, it follows that  $\lambda_k = -k\alpha(0)\eta_k$ . Therefore,  $\operatorname{sgn}(\lambda(\delta)) = -\operatorname{sgn}(\eta(\delta))$  for  $0 < |\delta| \ll 1$ .

See Figure 1.

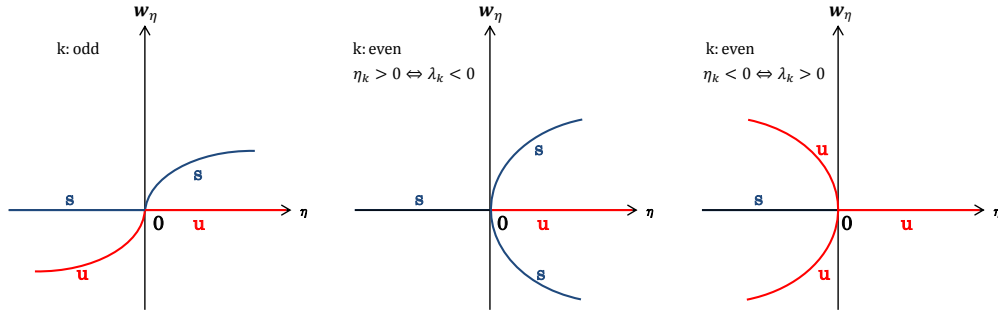


Figure 1: Bifurcation diagram of the incompressible system  $(\text{NS})_\eta$

We next consider relations between  $\lambda^{(l)}$  and  $\eta^{(l)}$ .

**Proposition 4.4.** *The following (a)-(c) are equivalent:*

- (a)  $\lambda^{(l)}(0) = 0$  for  $l = 1, \dots, k$ .
- (b)  $\eta^{(l)}(0) = 0$  for  $l = 1, \dots, k$ .
- (c)  $\sigma^{(l-1)}(0) = 0$  for  $l = 1, \dots, k$ .

**Proof.** We will prove the proposition by induction on  $k$ . Differentiating  $\lambda(\delta)$  we have

$$\partial_\delta \lambda(\delta) = \dot{\alpha}(\delta) \delta \dot{\eta}(\delta) + \alpha(\delta) \dot{\eta}(\delta) + \alpha(\delta) \delta \ddot{\eta}(\delta),$$

and so

$$\partial_\delta \lambda(0) = \alpha(0) \dot{\eta}(0).$$

Therefore, we see that

$$\partial_\delta \lambda(0) = 0 \text{ if and only if } \sigma(0) = \dot{\eta}(0) = 0.$$

This shows that the proposition holds for  $k = 1$ . Let  $k \geq 2$  and suppose that the proposition is true for  $k - 1$ . Then

$$\begin{aligned} \partial_\delta^k \lambda(\delta) &= \sum_{l=0}^k \binom{k}{l} \{\delta\}^{(l)} \partial_\delta^{k-l} \{\alpha(\delta) \dot{\eta}(\delta)\} \\ &= \delta \partial_\delta^k \{\alpha(\delta) \dot{\eta}(\delta)\} + k \partial_\delta^{k-1} \{\alpha(\delta) \dot{\eta}(\delta)\}. \end{aligned} \quad (4.1)$$

Since,

$$\partial_\delta^{k-1} \{\alpha(\delta) \dot{\eta}(\delta)\} = \sum_{m=0}^{k-1} \binom{k-1}{m} \alpha^{(m)}(\delta) \dot{\eta}^{(k-1-m)}(\delta),$$

we see from (4.1) that

$$\lambda^{(k)}(0) = k \alpha(0) \dot{\eta}^{(k-1)}(0) = -k \alpha(0) \eta^{(k)}(0).$$

This shows that the proposition is true for  $k$  and the proof of the proposition is complete.  $\square$

We next consider the stability of the bifurcating solution  $\tilde{\mathbf{v}}(\delta)$  as a solution of the artificial compressible system (1.3)–(1.4). We denote by  $L(\epsilon, \delta)$  the

linearized operator around  $\tilde{\mathbf{v}}(\delta)$  which is an operator on  $H_*^1(\Omega) \times L^2(\Omega)^3$  given by

$$L(\epsilon, \delta) = \begin{pmatrix} 0 & \frac{1}{\epsilon^2} \operatorname{div} \\ \nabla & -\Delta + (\mathcal{R}_c + \eta(\delta))\mathbb{M}[\tilde{\mathbf{v}}(\delta)] \end{pmatrix}$$

with domain  $D(L(\epsilon, \delta)) = D$ . We also introduce  $\mathbb{M}(\delta)$  and  $M(\delta)$  defined by

$$\mathbb{M}(\delta) = (\mathcal{R}_c + \eta(\delta))\mathbb{M}[\tilde{\mathbf{v}}(\delta)] - \mathcal{R}_c\mathbb{M}[\mathbf{v}_0],$$

$$M(\delta) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{M}(\delta) \end{pmatrix}.$$

It follows from Proposition 4.1 that  $\mathbb{M}(\delta)$  and  $M(\delta)$  can be expanded as

$$\mathbb{M}(\delta) = \sum_{k=1}^{\infty} \delta^k \mathbb{M}_k,$$

$$M(\delta) = \sum_{k=1}^{\infty} \delta^k M_k, \quad M_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{M}_k \end{pmatrix}.$$

Here  $\mathbb{M}_k$  satisfies the estimate

$$\|\mathbb{M}_k \mathbf{w}\|_2 \leq c_k \|\mathbf{w}\|_{H^1} \quad (4.2)$$

uniformly for  $\mathbf{w} \in H^1(\Omega)$  with positive constant  $c_k$  satisfying  $\sum_{k=1}^{\infty} c_k \delta^k < \infty$  for  $|\delta| \leq \delta_1$ .

The result is stated as follows.

**Theorem 4.5.** *Let  $\lambda(\delta) = \lambda_k \delta^k + \mathcal{O}(\delta^{k+1})$  with  $\lambda_k \neq 0$  for some  $k \geq 1$ . Then there exist positive constants  $\delta_1 = \delta_1(\tilde{\mathbf{b}}_0, \mathbf{v}_0)$  and  $\epsilon_1 = \epsilon_1(\tilde{\mathbf{b}}_0, \mathbf{v}_0)$  such that*

$$\sigma(-L(\epsilon, \delta)) \cap \{\lambda \in \mathbb{C}; |\lambda| \leq \frac{\tilde{b}_0}{4}\} = \{\lambda(\epsilon, \delta)\},$$

$$\lambda(\epsilon, \delta) = \delta^k ((1 + c_1(\epsilon^2))\lambda_k + \Lambda_k(\epsilon, \delta))$$

with some  $\Lambda_k(\epsilon, \delta) = \mathcal{O}(\delta)$  uniformly for  $0 < \epsilon \leq \epsilon_1$ ,  $0 < |\delta| \leq \delta_1$ . Here  $c_1(\epsilon^2)$  satisfies  $|c_1(\epsilon^2)| \leq \frac{1}{2}$  for  $0 < \epsilon \leq \epsilon_1$ .

Applying Theorem 3.1 and Theorem 4.5 we have the following result on the stability of the bifurcating solution  $\tilde{\mathbf{v}}(\delta)$  as a solution of the artificial compressible system (1.3)–(1.4).

**Theorem 4.6.** *Assume that (A0)–(A5). There exist positive constants  $\epsilon_1 = \epsilon_1(\tilde{b}_0, \mathbf{v}_0)$  and  $\delta_1 = \delta_1(\tilde{b}_0, \mathbf{v}_0)$  such that the following assertions hold true for  $0 < |\delta| \leq \delta_1$ .*

(i) *If  $\tilde{\mathbf{v}}(\delta)$  is unstable as a solution of (1.1)–(1.2) then so is  $\tilde{\mathbf{v}}(\delta)$  as a solution of (1.3)–(1.4) for  $0 < \epsilon \leq \epsilon_1$ .*

(ii) *Let  $\tilde{\mathbf{v}}(\delta)$  be stable as a solution of (1.1)–(1.2). Then there exists  $\kappa > 0$  such that if*

$$\inf_{\mathbf{w}_\eta \in H_0^1(\Omega)^3, \mathbf{w}_\eta \neq \mathbf{0}} \frac{\operatorname{Re}(\mathbb{Q}\mathbf{w}_\eta \cdot \nabla \tilde{\mathbf{v}}(\delta), \mathbb{Q}\mathbf{w}_\eta)_{L^2}}{\|\nabla \mathbb{Q}\mathbf{w}_\eta\|_{L^2}^2} \geq -\kappa, \quad (4.3)$$

*then there exists  $\epsilon_2 = \epsilon_2(\tilde{b}_0, \mathbf{v}_0) > 0$  such that  $\tilde{\mathbf{v}}(\delta)$  is stable as a solution of (1.3)–(1.4) for  $0 < \epsilon \leq \epsilon_2$ .*

Similarly to the proof of Theorems 4.5 and 4.6, it is possible to prove the stability and instability of the basic flow  $\mathbf{v}_\eta$ . In fact, one can show that the spectrum of the linearized operator  $\mathbb{L}_\eta$  satisfies

$$\sigma(-\mathbb{L}_\eta) = \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\frac{3}{4}\tilde{b}_0\} \cup \{\lambda_\eta\}, \quad \eta \in [-\eta_0, \eta_0]$$

for some positive constant  $\eta_0$ . Here  $\lambda_\eta$  is a simple eigenvalue of  $-\mathbb{L}_\eta$  and satisfies

$$\lambda_\eta = \alpha(0)\eta + \mathcal{O}(\eta^2).$$

Let  $L_{\epsilon, \eta}$  be the linearized operator around  $u_\eta = {}^\top(p_\eta, \mathbf{v}_\eta)$  of the artificial compressible system. Here  $p_\eta$  is the pressure corresponding to  $\mathbf{v}_\eta$ . As in the proof of Theorem 4.5, we have the following result.

**Theorem 4.7.** *There exist positive constants  $\tilde{\eta}_1 = \tilde{\eta}_1(\tilde{b}_0, \mathbf{v}_0)$  and  $\epsilon_3 = \epsilon_3(\tilde{b}_0, \mathbf{v}_0)$  such that*

$$\sigma(-L_{\epsilon, \eta}) \cap \{\lambda \in \mathbb{C}; |\lambda| \leq \frac{\tilde{b}_0}{4}\} = \{\lambda_{\epsilon, \eta}\}$$

$$\lambda_{\epsilon, \eta} = \eta(c_1(\epsilon^2)\alpha(0) + \Lambda_{\epsilon, \eta})$$

*with some  $\Lambda_{\epsilon, \eta} = \mathcal{O}(\eta)$  uniformly for  $0 < \epsilon \leq \epsilon_3$  and  $0 < |\eta| \leq \tilde{\eta}_1$ .*

Combining Theorems 4.5 and 4.7, we can see that the same exchange of stability as in the case of (1.1)–(1.2) holds for the case of (1.3)–(1.4) uniformly for small  $\epsilon$ . For definiteness, we consider the case where  $k$  is even and  $\eta_k$  is positive in Proposition 4.3 (ii). In this case we have the following result.

**Theorem 4.8.** *Let  $k$  be even and  $\eta_k$  be positive in Proposition 4.3 (ii). Then there exist positive constants  $\epsilon_4$  and  $\delta_2$  such that*

- (i) *The basic flow  $\mathbf{v}_{\eta(\delta)}$  is unstable for  $0 < |\delta| \leq \delta_2$  and  $0 < \epsilon \leq \epsilon_4$ .*
- (ii) *There exist positive constants  $\epsilon_5$ ,  $\delta_3$ ,  $\tilde{\eta}_2$  and  $\tilde{\kappa}$  such that if*

$$\inf_{\mathbf{w} \in H_0^1(\Omega)^3, \mathbf{w} \neq \mathbf{0}} \frac{\operatorname{Re}(\mathbb{Q}\mathbf{w} \cdot \nabla \mathbf{v}_0, \mathbb{Q}\mathbf{w})}{\|\nabla \mathbb{Q}\mathbf{w}\|^2} \geq -\tilde{\kappa},$$

*then  $\mathbf{v}_\eta$  is stable for  $-\tilde{\eta}_2 \leq \eta < 0$  and  $0 < \epsilon \leq \epsilon_5$  and  $\tilde{\mathbf{v}}(\delta)$  is stable for  $0 < |\delta| \leq \delta_3$  and  $0 < \epsilon \leq \epsilon_5$ .*

The other cases where  $k$  is odd or  $\eta_k$  is negative, one can obtain similar results. See Figure 2.

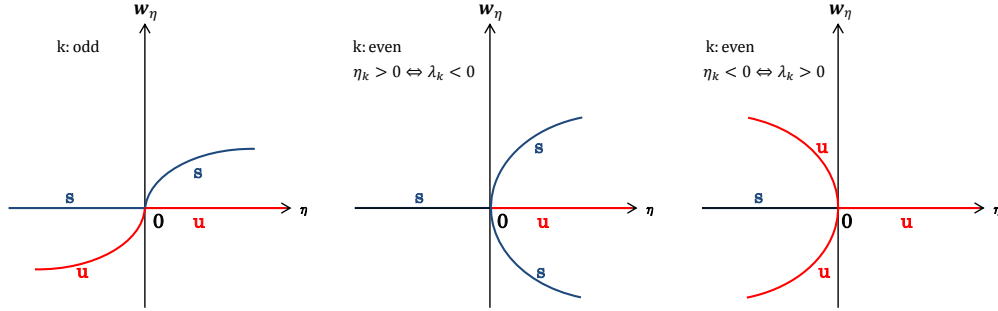


Figure 2: Bifurcating diagram of the artificial compressible system

## 4.2 Proofs of Theorems 4.5 and 4.6

To prove Theorem 4.5 we introduce the operator  $\mathcal{L}_{\epsilon, \lambda}$  on  $H_*^1(\Omega) \times L^2(\Omega)^3$  defined by

$$D(\mathcal{L}_{\epsilon, \lambda}) = D := H_*^1(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]^3,$$

$$\mathcal{L}_{\epsilon, \lambda} = \begin{pmatrix} 0 & \frac{1}{\epsilon^2} \operatorname{div} \\ \nabla & \lambda - \Delta + \mathcal{R}_c \mathbb{M}[\mathbf{v}_0] \end{pmatrix}.$$

We also introduce the operator  $\mathbf{A}_\lambda(\delta)$ :

$$\mathbf{A}_\lambda(\delta)\mathbf{w} = -\Delta\mathbf{w} + \mathcal{R}_c \mathbb{M}[\mathbf{v}_0]\mathbf{w} + \mathbb{M}(\delta)\mathbf{w}.$$

We give an expression for  $\mathcal{L}_{\epsilon,\lambda}^{-1}$  in terms of  $(\lambda + \mathcal{L})^{-1}$ . To do so, we prepare the following lemma.

**Lemma 4.9.** ([19, Lem.4.2]) *There exists a bounded linear operator  $\mathbf{V} : H_*^1(\Omega) \rightarrow [H^2(\Omega) \cap H_0^1(\Omega)]^3$  such that*

$$\operatorname{div} \mathbf{V} f = f, \quad \|\mathbf{V} f\|_{H^2} \leq C \|f\|_{H^1}$$

for  $f \in H_*^1(\Omega)$ .

This lemma follows from the solvability for the nonhomogeneous Stokes problem  $\operatorname{div} \mathbf{v} = f$ ,  $-\Delta \mathbf{v} + \nabla p = 0$  under the boundary conditions  $v|_{\partial\Omega} = \mathbf{0}$ . See, e.g., [15].

The following expression and estimate for  $\mathcal{L}_{\epsilon,\lambda}^{-1}$  were obtained in [19].

**Lemma 4.10.** ([19, Prop.6.2]) *Let  $\lambda \in \Sigma_0 := \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\tilde{b}_0\} \setminus 0$ . Then*

$$\mathcal{L}_{\epsilon,\lambda}^{-1} F = \begin{pmatrix} p_\lambda [\mathbf{g} - \epsilon^2 \mathbf{A}_\lambda \mathbf{V} f] \\ (\lambda + \mathbb{L})^{-1} \mathbb{P} [\mathbf{g} - \epsilon^2 \mathbf{A}_\lambda \mathbf{V} f] + \epsilon^2 \mathbf{V} f \end{pmatrix},$$

$$\|\mathcal{L}_{\epsilon,\lambda}^{-1} F\|_{H^1 \times H^2} \leq C \left( \frac{1}{|\lambda|} + 1 \right) \{ \epsilon^2 \|f\|_{H^1} + \|\mathbf{g}\|_2 \}$$

for  $F = {}^\top(f, \mathbf{g}) \in H_*^1(\Omega) \times L^2(\Omega)^3$ . Here  $p_\lambda = p_\lambda[\mathbf{g}] \in H_*^1(\Omega)$  is the unique function satisfying

$$\lambda \mathbf{w} - \Delta \mathbf{w} + \mathcal{R}_c \mathbb{M}[\mathbf{v}_0] \mathbf{w} + \mathbb{M}(\delta) \mathbf{w} + \nabla p_\lambda = \mathbf{g}$$

with  $\mathbf{w} = (\lambda + \mathbb{L}(\delta))^{-1} \mathbb{P} \mathbf{g}$ ; and  $C$  is a positive constant independent of  $\lambda \in \Sigma_0$  and  $F$ .

We first show that the spectrum of  $-L(\epsilon, \delta)$  near the origin consists of a simple eigenvalue which is analytic in  $\epsilon^2$  and  $\delta$ .

**Lemma 4.11.** *Let  $\lambda(\delta) = \lambda_k \delta^k + \mathcal{O}(\delta^{k+1})$  with  $\lambda_k \neq 0$  for some  $k \geq 1$ . Then there exist positive constants  $\delta_1 = \delta_1(\tilde{b}_0, \mathbf{v}_0)$  and  $\epsilon_1 = \epsilon_1(\tilde{b}_0, \mathbf{v}_0)$  such that*

$$\sigma(-L(\epsilon, \delta)) \cap \{\lambda \in \mathbb{C}; |\lambda| \leq \frac{\tilde{b}_0}{4}\} = \{\lambda(\epsilon, \delta)\}$$

for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| < \delta_1$ . Here  $\lambda(\epsilon, \delta)$  is a simple eigenvalue of  $-L(\epsilon, \delta)$  and is analytic in  $\epsilon^2$  and  $\delta$ .

**Proof.** By Lemma 4.10, if  $|\lambda| = r_0$ ,  $0 < r_0 \leq \frac{3}{4}\tilde{b}_0$ , then

$$\|\mathcal{L}_{\epsilon,\lambda}^{-1}F\|_{H^1 \times H^2} \leq C \frac{1}{r_0} \{\epsilon^2 \|f\|_{H^1} + \|\mathbf{g}\|_2\} \quad (4.4)$$

for  $F = {}^\top(f, \mathbf{g}) \in H_*^1(\Omega) \times L^2(\Omega)^3$ . We set

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It then follows from (4.4) and (4.2) that

$$\|\lambda \mathcal{L}_{\epsilon,\lambda}^{-1} J F\|_{H^1 \times H^2} \leq C \epsilon^2 \|f\|_{H^1}, \quad (4.5)$$

$$\|\mathcal{L}_{\epsilon,\lambda}^{-1} M(\delta) F\|_{H^1 \times H^2} \leq C \frac{\delta}{r_0} \|\mathbf{g}\|_{H^1} \leq C \frac{\delta}{r_0} \|\mathbf{g}\|_{H^2}. \quad (4.6)$$

We see from (4.5) and (4.6) that there exist positive constants  $\epsilon_0$  and  $\delta_0$  which depend only on  $\tilde{b}_0$  and  $\mathbf{v}_0$  such that if  $0 \leq \epsilon \leq \epsilon_0$  and  $0 \leq |\delta| \leq \delta_0$ , then

$$\|(\lambda \mathcal{L}_{\epsilon,\lambda}^{-1} J + \mathcal{L}_{\epsilon,\lambda}^{-1} M(\delta)) F\|_{H^1 \times H^2} \leq \frac{1}{2} \|F\|_{H^1 \times H^2}$$

for all  $F \in D(L(\epsilon, \delta))$  uniformly for  $\lambda$  with  $\frac{\tilde{b}_0}{4} \leq |\lambda| \leq \frac{3}{4}\tilde{b}_0$ . This implies that there exists a bounded inverse  $(I + \lambda \mathcal{L}_{\epsilon,\lambda}^{-1} J + \mathcal{L}_{\epsilon,\lambda}^{-1} M(\delta))^{-1}$  on  $D(L(\epsilon, \delta))$  for  $0 < \epsilon \leq \epsilon_0$ ,  $0 < |\delta| \leq \delta_0$  and  $\frac{\tilde{b}_0}{4} \leq |\lambda| \leq \frac{3}{4}\tilde{b}_0$ , with estimate

$$\|(I + \lambda \mathcal{L}_{\epsilon,\lambda}^{-1} J + \mathcal{L}_{\epsilon,\lambda}^{-1} M(\delta))^{-1} F\|_{H^1 \times H^2} \leq 2 \|F\|_{H^1 \times H^2}.$$

It then follows that  $\lambda + L(\epsilon, \delta)$  has the inverse

$$(\lambda + L(\epsilon, \delta))^{-1} = (I + \lambda \mathcal{L}_{\epsilon,\lambda}^{-1} J + \mathcal{L}_{\epsilon,\lambda}^{-1} M(\delta))^{-1} \mathcal{L}_{\epsilon,\lambda}^{-1} \quad (4.7)$$

which is bounded on  $H_*^1(\Omega) \times L^2(\Omega)^3$  and is also bounded from  $H_*^1(\Omega) \times L^2(\Omega)^3$  to  $D(L(\epsilon, \delta))$ . Furthermore,  $(\lambda + L(\epsilon, \delta))^{-1}$  is given by the Neumann series:

$$(\lambda + L(\epsilon, \delta))^{-1} = \sum_{N=0}^{\infty} (-1)^N (\lambda \mathcal{L}_{\epsilon,\lambda}^{-1} J + \mathcal{L}_{\epsilon,\lambda}^{-1} M(\delta))^N \mathcal{L}_{\epsilon,\lambda}^{-1},$$

and hence, by Lemma 4.10, it is analytic in  $\epsilon^2$ ,  $\delta$  and  $\lambda$ . We set

$$\begin{aligned} P(\epsilon, \delta) &= \frac{1}{2\pi i} \int_{|\lambda|=\frac{\tilde{b}_0}{2}} (\lambda + L(\epsilon, \delta))^{-1} d\lambda \\ &= \sum_{N=0}^{\infty} \frac{(-1)^N}{2\pi i} \int_{|\lambda|=\frac{\tilde{b}_0}{2}} (\lambda \mathcal{L}_{\epsilon, \lambda}^{-1} J + \mathcal{L}_{\epsilon, \lambda}^{-1} M(\delta))^N \mathcal{L}_{\epsilon, \lambda}^{-1} d\lambda. \end{aligned} \quad (4.8)$$

Then, by [19, Thm.5.2 (ii)],  $P(\epsilon, \delta)$  is the total eigenprojection to the eigenvalues lying inside of  $\{\lambda; |\lambda| = \frac{\tilde{b}_0}{2}\}$  and it is analytic in  $\epsilon^2$  and  $\delta$  in  $B(X, Y)$  with  $X = H_*^1(\Omega) \times L^2(\Omega)^3$  and  $Y = H_*^1(\Omega) \times H^2(\Omega)^3$ . If  $\delta = 0$ , then  $P(\epsilon, 0)$  is an eigenprojection for the eigenvalue 0 of  $-L(\epsilon, 0)$ . Since 0 is a simple eigenvalue of  $-L(0)$ , we see from [19, Prop.4.3] that 0 is a simple eigenvalue of  $-L(\epsilon, 0)$  and the eigenprojection  $P(\epsilon, 0)$  for 0 is given by

$$P(\epsilon, 0)u = \langle \langle u, u_0^* \rangle \rangle_{\epsilon} u_{0, \epsilon} \quad (u \in H_*^1(\Omega) \times L^2(\Omega)^3).$$

Here

$$\begin{aligned} u_{0, \epsilon} &= c_0(\epsilon) \begin{pmatrix} p_0 \\ \mathbf{w}_0 \end{pmatrix}, \quad c_0(\epsilon^2) = (1 + \epsilon^2(p_0, p_0^*))^{-1}, \\ u_0^* &= \begin{pmatrix} p_0^* \\ \mathbf{w}_0^* \end{pmatrix}, \end{aligned}$$

where  $p_0$  and  $p_0^*$  are the pressures corresponding to  $\mathbf{w}_0$  and  $\mathbf{w}_0^*$ , respectively, i.e.,  $L(\epsilon, 0)u_0 = L^*(\epsilon, 0)u_0^* = 0$ . Note that  $u_0^*$  is independent of  $\epsilon$  and  $\langle \langle u_{0, \epsilon}, u_0^* \rangle \rangle_{\epsilon} = 1$ . Since  $P(\epsilon, \delta)$  is continuous in  $\epsilon$  and  $\delta$ , we find from the perturbation theory that  $\dim R(P(\epsilon, \delta)) = \dim(P(\epsilon, 0)) = 1$ , and hence,

$$\sigma(-L(\epsilon, \delta)) \cap \{\lambda; |\lambda| < \frac{\tilde{b}_0}{2}\} = \{\lambda(\epsilon, \delta)\},$$

where  $\lambda(\epsilon, \delta)$  is a simple eigenvalue of  $-L(\epsilon, \delta)$ .

We now show that  $\lambda(\epsilon, \delta)$  is analytic in  $\epsilon^2$  and  $\delta$ . Let

$$u(\epsilon, \delta) = \begin{pmatrix} p(\epsilon, \delta) \\ \mathbf{w}(\epsilon, \delta) \end{pmatrix} = P(\epsilon, \delta)u_{0, \epsilon}.$$

Then  $u(\epsilon, \delta)$  is analytic in  $\epsilon^2$  and  $\delta$  in  $H_*^1(\Omega) \times H^2(\Omega)^3$  and

$$-L(\epsilon, \delta)u(\epsilon, \delta) = \lambda(\epsilon, \delta)u(\epsilon, \delta). \quad (4.9)$$

Taking  $\langle\langle \cdot, \cdot \rangle\rangle_\epsilon$  of (4.9) with  $u_0^*$  we have

$$\lambda(\epsilon, \delta) \langle\langle u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon = -\langle\langle L(\epsilon, \delta)u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon.$$

Clearly,  $\langle\langle u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon$  is analytic in  $\epsilon^2$  and  $\delta$ . We also have

$$\begin{aligned} & \langle\langle L(\epsilon, \delta)u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon \\ &= \epsilon^2 \left( \frac{1}{\epsilon^2} \operatorname{div} \mathbf{w}(\epsilon, \delta), p_0^* \right)_{L^2} + (\mathbf{A}_0(\delta) \mathbf{w}(\epsilon, \delta) + \nabla p(\epsilon, \delta), \mathbf{w}_0^*) \\ &= (\operatorname{div} \mathbf{w}(\epsilon, \delta), p_0^*)_{L^2} + (\mathbf{A}_0(\delta) \mathbf{w}(\epsilon, \delta), \mathbf{w}_0^*). \end{aligned}$$

This shows that  $\langle\langle L(\epsilon, \delta)u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon$  is analytic in  $\epsilon^2$  and  $\delta$ . Furthermore, since  $\langle\langle u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon|_{\epsilon=\delta=0} = \langle\mathbf{w}_0\rangle = 1$ , we see that  $\langle\langle u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon \neq 0$  for sufficiently small  $\epsilon$  and  $\delta$ . Therefore, we have

$$\lambda(\epsilon, \delta) = -\frac{\langle\langle L(\epsilon, \delta)u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon}{\langle\langle u(\epsilon, \delta), u_0^* \rangle\rangle_\epsilon}, \quad (4.10)$$

and hence, conclude that  $\lambda(\epsilon, \delta)$  is analytic in  $\epsilon^2$  and  $\delta$ .  $\square$

From the proof of Lemma 4.11 one can see that  $\lambda(\epsilon, \delta)$  is given by a small perturbation of  $\lambda(\delta)$ . In fact, we have the following relation.

**Lemma 4.12.** *It holds that  $\lambda(0, \delta) = \lambda(\delta)$ .*

**Proof.** We see from [19, Prop.5.1(ii)] that

$$P(\epsilon, \delta)F \rightarrow P(0, \delta)F \text{ in } H_*^1(\Omega) \times H^2(\Omega)^3 \text{ as } \epsilon \rightarrow 0. \quad (4.11)$$

Here

$$P(0, \delta)F = \begin{pmatrix} \Pi(\delta)\mathbf{g} \\ \mathbb{P}(\delta)\mathbb{P}\mathbf{g} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi i} \int_{|\lambda|=\frac{\tilde{b}_0}{2}} p_\lambda[\mathbf{g}] d\lambda \\ \frac{1}{2\pi i} \int_{|\lambda|=\frac{\tilde{b}_0}{2}} (\lambda + \mathbb{L}(\delta))^{-1} \mathbb{P}\mathbf{g} d\lambda \end{pmatrix}.$$

Since  $\mathbb{P}(\delta)$  is the total eigenprojection for the eigenvalues of  $-\mathbb{L}(\delta)$  lying in  $|\lambda| = \frac{\tilde{b}_0}{2}$ ,  $\mathbb{P}(\delta)$  is the eigenprojection for the eigenvalue  $\lambda(\delta)$ . Therefore,

$$u(\epsilon, \delta) \rightarrow u(0, \delta) = \begin{pmatrix} p(0, \delta) \\ \mathbf{w}(0, \delta) \end{pmatrix} \text{ in } H^1(\Omega) \times H^2(\Omega)^3 \text{ as } \epsilon \rightarrow 0,$$

where  $\mathbf{w}(0, \delta)$  is an eigenfunction of  $-\mathbb{L}(\delta)$  and  $p(0, \delta)$  is the corresponding pressure, i.e.,  $(\mathcal{L}_{1, \lambda(\delta)} + M(\delta))u(0, \delta) = 0$ . (In [19], the convergence in (5.9) is written in  $H_*^1(\Omega) \times L^2(\Omega)^3$ ; but it is easily verified that the convergence in (5.9) also holds in  $H_*^1(\Omega) \times H^2(\Omega)$ . In fact, it follows from (4.6), (4.7) and (4.8).) In (4.10) we let  $\epsilon \rightarrow 0$ . Then

$$\begin{aligned} \lambda(0, \delta) &= -\frac{(\operatorname{div} \mathbf{w}(0, \delta), p_0^*)_{L^2} + (\mathbf{A}_0(\delta) \mathbf{w}(0, \delta) + \nabla p(0, \delta), \mathbf{w}_0^*)}{(\mathbf{w}(0, \delta), \mathbf{w}_0^*)_{L^2}} \\ &= -\frac{(\mathbb{L}(\delta) \mathbf{w}(0, \delta), \mathbf{w}_0^*)}{(\mathbf{w}(0, \delta), \mathbf{w}_0^*)} \\ &= \frac{\lambda(\delta)(\mathbf{w}(0, \delta), \mathbf{w}_0^*)}{(\mathbf{w}(0, \delta), \mathbf{w}_0^*)} \\ &= \lambda(\delta). \end{aligned}$$

This completes the proof.  $\square$

To complete the proof of Theorem 4.5, we prepare the following lemma.

**Lemma 4.13.** *Let  $\lambda(\delta)$  be expanded as  $\lambda(\delta) = \sum_{l=0}^{\infty} \lambda_l \delta^l$  and let  $k \in \mathbb{N}$ . If  $\lambda_l = 0$  for  $l = 0, \dots, k-1$ , then*

$$\lambda_k = -\sum_{m=0}^{k-1} \langle \mathbb{M}_{m+1} \mathbf{W}_{k-m-1} \rangle$$

and

$$U_k = \begin{pmatrix} p_k \\ \mathbf{W}_k \end{pmatrix} = -L(1, 0)^{-1} \begin{pmatrix} 0 \\ \lambda_k \mathbf{w}_0 + \sum_{m=0}^{k-1} \mathbb{M}_{m+1} \mathbf{W}_{k-m-1} \end{pmatrix}.$$

Here  $p_k$  is the pressure corresponding to  $\mathbf{W}_k$ .

The proof of Lemma 4.13 will be given in the end of this section. We expand  $\lambda(\epsilon, \delta)$  in  $\delta$  in the following way.

$$\lambda(\epsilon, \delta) = \sum_{k=0}^{\infty} \delta^k \lambda_k(\epsilon, 0).$$

We then have the following relation.

**Lemma 4.14.** *If  $\lambda_l = 0$  for  $l = 0, \dots, k-1$ , then*

$$\lambda_k(\epsilon, 0) = c_0(\epsilon^2)\lambda_k(0, 0) = c_0(\epsilon^2)\lambda_k$$

*and furthermore, if  $\lambda_k = 0$ , then*

$$u_k(\epsilon, 0) = c_0(\epsilon^2)U_k.$$

**Proof.** By Lemma 4.12, we have

$$\lambda(0, \delta) = \sum_{k=0}^{\infty} \delta^k \lambda_k(0, 0) = \sum_{k=0}^{\infty} \delta^k \lambda_k.$$

Therefore, by assumption,  $\lambda_l(0, 0) = 0$  for  $l = 0, \dots, k-1$ , and we also have  $\lambda_k(0, 0) = \lambda_k$ . Since  $u(\epsilon, \delta)$  is analytic in  $\epsilon^2$  and  $\delta$ , we have

$$u(\epsilon, \delta) = \sum_{k=0}^{\infty} \delta^k u_k(\epsilon, 0).$$

We recall that  $L(\epsilon, \delta)$  is written as

$$L(\epsilon, \delta) = L(\epsilon, 0) + M(\delta) = L(\epsilon, 0) + \sum_{k=1}^{\infty} \delta^k M_k,$$

where  $M(\delta) = \sum_{k=1}^{\infty} \delta^k M_k = \sum_{k=1}^{\infty} \delta^k \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{M}_k \end{pmatrix}$ . Since

$$(\lambda(\epsilon, \delta) + L(\epsilon, \delta))u(\epsilon, \delta) = 0,$$

we have

$$\sum_{m,l=0}^{\infty} \delta^{m+l} \lambda_m(\epsilon, 0) u_l(\epsilon, 0) + \sum_{l=0}^{\infty} \delta^l L(\epsilon, 0) u_l(\epsilon, 0) + \sum_{m,l=0}^{\infty} \delta^{m+1+l} M_{m+1} u_l(\epsilon, 0) = 0.$$

We equate the coefficient of each power of  $\delta$  to 0. As for the coefficient of  $\delta^0$ , we have

$$\lambda_0(\epsilon, 0)u_0(\epsilon, 0) + L(\epsilon, 0)u_0(\epsilon, 0) = 0,$$

and so

$$\lambda_0(\epsilon, 0) = 0, \quad u_0(\epsilon, 0) = P(\epsilon, 0)u_{0,\epsilon} = u_{0,\epsilon} = c_0(\epsilon^2)u_0.$$

From the coefficient of  $\delta^1$ , we see that

$$(\lambda_0(\epsilon, 0) + L(\epsilon, 0))u_1(\epsilon, 0) + \lambda_1(\epsilon, 0)u_0(\epsilon, 0) + M_1u_0(\epsilon, 0) = 0,$$

and hence,

$$\lambda_1(\epsilon, 0) = -\langle \langle M_1u_0(\epsilon, 0), u_0^* \rangle \rangle_\epsilon = -c_0(\epsilon^2)\langle \mathbb{M}_1 \mathbf{W}_0 \rangle.$$

This, together with Lemma 4.13, implies that  $\lambda_1(\epsilon, 0) = c_0(\epsilon^2)\lambda_1$ . We thus find that the lemma holds for  $k = 1$ .

Suppose that lemma holds for  $k = m$ . Then  $\lambda_l(\epsilon, 0) = c_0(\epsilon^2)\lambda_l(0, 0) = 0$  and  $u_l(\epsilon, 0) = c_0(\epsilon^2)U_l(0, 0)$  for  $l = 0, \dots, m$ . On the other hand, as for the coefficient of  $\delta^{m+1}$ , we have

$$(\lambda_0(\epsilon, 0) + L(\epsilon, 0))u_{m+1}(\epsilon, 0) + \sum_{l=1}^{m+1} \lambda_l(\epsilon, 0)u_{m+1-l}(\epsilon, 0) + \sum_{l=0}^m M_{l+1}u_{m-l}(\epsilon, 0) = 0,$$

and hence,

$$\begin{aligned} \lambda_{m+1}(\epsilon, 0) &= -\sum_{l=0}^m \langle \langle M_{l+1}u_{m-l}(\epsilon, 0), u_0^* \rangle \rangle_\epsilon \\ &= -c_0(\epsilon^2) \sum_{l=0}^m \langle \mathbb{M}_{l+1} \mathbf{W}_{m-l} \rangle. \end{aligned}$$

Therefore, by Lemma 4.13, we have

$$\lambda_{m+1}(\epsilon, 0) = c_0(\epsilon^2)\lambda_{m+1}.$$

Furthermore, if  $\lambda_{m+1} = 0$ , then

$$\begin{aligned} u_{m+1}(\epsilon, 0) &= -L(\epsilon, 0)^{-1} \sum_{l=0}^m M_{l+1}u_{m-l} \\ &= -L(1, 0)^{-1} \sum_{l=0}^m M_{l+1}u_{m-l} \\ &= c_0(\epsilon^2)U_{m+1}. \end{aligned}$$

Here we used the fact that  $L(\epsilon, 0)^{-1}F = L(1, 0)^{-1}F$  for all  $F$  of the form  $F = {}^\top(0, \mathbf{g})$ . This shows that the lemma holds for  $k = m + 1$  and the proof of the lemma is complete.  $\square$

We are now in a position to prove Theorem 4.5

**Proof of Theorem 4.5.** By the assumption of Theorem 4.5, we have  $\lambda_l = 0$  for  $l = 0, 1, \dots, k-1$ . Therefore, we see from Lemma 4.14 that  $\lambda_l(\epsilon, 0) = 0$  for  $l = 1, \dots, k-1$  and  $\lambda_k(\epsilon, 0) = c_0(\epsilon)\lambda_k$ . It then follows that

$$\begin{aligned}\lambda(\epsilon, \delta) &= \delta^k c_0(\epsilon) \lambda_k(0, 0) + \sum_{l=k+1}^{\infty} \delta^l \lambda_l(\epsilon, 0) \\ &= \delta^k \left( c_0(\epsilon) \lambda_k + \sum_{l=1}^{\infty} \delta^l \lambda_{k+l}(\epsilon, 0) \right).\end{aligned}$$

Since  $c_0(\epsilon^2) - 1 = \mathcal{O}(\epsilon^2)$ , setting  $c_1(\epsilon^2) = c_0(\epsilon^2) - 1$ , we have the desired result. This completes the proof.  $\square$

We finally give a proof of Theorem 4.6.

**Proof of Theorem 4.6.** Since  $\lambda(\delta)$  is analytic in  $\delta$ , we see that  $\lambda(\delta)$  has the form  $\lambda(\delta) = \lambda_k \delta^k + \mathcal{O}(\delta^{k+1})$  with  $\lambda_k \neq 0$  for some  $k \in \mathbb{N}$ . By Proposition 4.2, the stability of  $\tilde{\mathbf{v}}(\delta)$  as a solution of (1.1)–(1.2) is determined by  $\text{sgn} \lambda(\delta)$ . We see that there exist a positive constant  $\delta_0$  such that  $\text{sgn}(\lambda(\delta)) = \text{sgn}(\lambda_k \delta^k)$  for  $|\delta| \leq \delta_0$ . On the other hand, by Theorem 4.5, we find that the spectrum near the origin consists of a simple eigenvalue  $\lambda(\epsilon, \delta)$ :

$$\lambda(\epsilon, \delta) = \delta^k ((1 + c_1(\epsilon^2)) \lambda_k + \Lambda_k(\epsilon, \delta)).$$

Since  $|c_1(\epsilon^2)| \leq \frac{1}{2}$  for  $0 < \epsilon \leq \epsilon_1$ , we see that there exists a positive constant  $\delta_1$  such that

$$\text{sgn}(\lambda(\epsilon, \delta)) = \text{sgn}(\lambda_k \delta^k) \tag{4.12}$$

for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq \delta_1$ .

If  $\tilde{\mathbf{v}}(\delta)$  is unstable as a solution of (1.1)–(1.2), then  $\text{sgn}(\lambda(\delta)) = \text{sgn}(\lambda_k \delta^k) = 1$ , and hence, by (4.12),  $\text{sgn}(\lambda(\delta)) = 1$  for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq |\delta_1|$ . This shows (i).

As for (ii), we see from the arguments in [19, Sect.6] and section 3 of this thesis that  $\sigma(-L(\epsilon, \delta)) \cap \{\lambda; \text{Re } \lambda \geq -\frac{3}{4}\tilde{b}_0\}$  may consist of  $\lambda(\epsilon, \delta)$  and a part of the spectrum in a region with  $\text{Im } \lambda = \mathcal{O}(\epsilon^{-1})$ . If  $\tilde{\mathbf{v}}(\delta)$  is stable as a solution of (1.1)–(1.2), then  $\text{sgn}(\lambda(\delta)) = \text{sgn}(\lambda_k \delta^k) = -1$ , and hence, by (4.12),  $\text{sgn}(\lambda(\epsilon, \delta)) = -1$  for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq \delta_1$ . Concerning the part of the spectrum in a region with  $\text{Im } \lambda = \mathcal{O}(\epsilon^{-1})$ , we see from Theorem 3.1 (and

its proof) that it lies in the left-half plane strictly away from the imaginary axis uniformly in  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq \delta_1$  (by taking  $\epsilon_1 > 0$  smaller if necessary), provided that the condition (4.12) holds. As a consequence, there exists a positive constant  $b_0$  such that  $\sigma(-L(\epsilon, \delta)) \subset \{\lambda; \operatorname{Re} \lambda \leq -b_0\}$  for  $0 < \epsilon \leq \epsilon_1$  and  $0 < |\delta| \leq \delta_1$ . This completes the proof.  $\square$

It remains to prove Lemma 4.13. To this end we employ the following lemma.

**Lemma 4.15.** *Let  $F = {}^\top(0, \mathbf{g})$  with  $\mathbf{g} \in L^2(\Omega)^3$ . Then  $L(1, 0)u = F$  is solvable if and only if  $\langle \mathbf{g} \rangle = 0$ . In this case, there exists a unique solution  $u = {}^\top(p, \mathbf{w}) \in H_*^1(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]^3$  of  $L(1, 0)u = F$  satisfying  $\langle \mathbf{w} \rangle = 0$ .*

**Proof.** If  $F = {}^\top(0, \mathbf{g})$ , then  $L(1, 0)u = F$  is written as

$$\begin{cases} \operatorname{div} \mathbf{w} &= f, \\ -\Delta \mathbf{w} + \mathcal{R}_c \mathbb{M}[\mathbf{v}_0] \mathbf{w} + \nabla p &= \mathbf{g}, \\ \mathbf{w}|_{\partial\Omega} &= \mathbf{0}, \end{cases}$$

which is reduced to  $\mathbb{L}(0)\mathbf{w} = \mathbb{P}\mathbf{g}$ . Since 0 is a simple eigenvalue of  $-\mathbb{L}(0)$ , we see that  $\mathbb{L}(0)\mathbf{w} = \mathbb{P}\mathbf{g}$  is solvable if and only if  $\langle \mathbb{P}\mathbf{g} \rangle = \langle \mathbf{g} \rangle = 0$ . In this case there exists a unique solution  $\mathbf{w} \in D(\mathbb{L}(0))$  of  $\mathbb{L}(0)\mathbf{w} = \mathbb{P}\mathbf{g}$  satisfying  $\langle \mathbf{w} \rangle = 0$ . It then follows that there exists a unique  $p \in H_*^1(\Omega)$  such that

$$\nabla p = \mathbf{g} - (-\Delta \mathbf{w} + \mathcal{R}_c \mathbb{M}[\mathbf{v}_0] \mathbf{w}).$$

This completes the proof.  $\square$

**Proof of Lemma 4.13.** Let  $\Pi(\delta) = \frac{1}{2\pi i} \int_{|\lambda|=\frac{\bar{b}_0}{4}} (\lambda + \mathbb{L}(\delta))^{-1} d\lambda$ . Then  $\Pi(\delta)$  is the eigenprojection for the eigenvalue  $\lambda(\delta)$ . Set  $\mathbf{W}(\delta) = \Pi(\delta)\mathbf{w}_0$ . Then  $\mathbf{W}(\delta)$  is an eigenfunction for the eigenvalue  $\lambda(\delta)$  and is analytic in  $\delta$  in  $H^2(\Omega)^3$  satisfying  $\mathbf{W}(0) = \mathbf{w}_0$ . We write  $(\lambda(\delta) + \mathbb{L}(\delta))\mathbf{W}(\delta) = \mathbf{0}$  as

$$(\lambda(\delta) + \mathbb{L}_0 + \sum_{k=1}^{\infty} \delta^k \mathbb{P}\mathbb{M}_k) \mathbf{W}(\delta) = \mathbf{0}.$$

It follows that

$$\sum_{l,m=0}^{\infty} \delta^{l+m} \lambda_l \mathbf{W}_m + \sum_{m=0}^{\infty} \delta^m \mathbb{L}_0 \mathbf{W}_m + \sum_{l,m=0}^{\infty} \delta^{l+1+m} \mathbb{P}\mathbb{M}_{l+1} \mathbf{W}_m = \mathbf{0}.$$

We equate the coefficient of each power of  $\delta$  to  $\mathbf{0}$ . Then we have

$$(\lambda_0 + \mathbb{L}_0)\mathbf{W}_m + \sum_{l=1}^m \lambda_l \mathbf{W}_{m-l} + \sum_{l=0}^{m-1} \mathbb{P}\mathbb{M}_{l+1} \mathbf{W}_{m-l-1} = \mathbf{0} \quad (4.13)$$

for  $m = 0, 1, 2, \dots$ . For  $m = 0$ , we have  $\lambda_0 \mathbf{W}_0 + \mathbb{L}_0 \mathbf{W}_0 = \mathbf{0}$ , and hence

$$\lambda_0 = 0, \mathbf{W}_0 = \mathbf{W}(0) = \mathbf{w}_0.$$

We prove the lemma by induction on  $k$ . Let  $k = 1$ . We see from (4.13) with  $m = 1$  that

$$(\lambda_0 + \mathbb{L}_0)\mathbf{W}_1 + \lambda_1 \mathbf{W}_0 + \mathbb{P}\mathbb{M}_1 \mathbf{W}_0 = \mathbf{0}.$$

Taking  $\langle \cdot \rangle$  of this equation, we have  $\lambda_1 + \langle \mathbb{M}_1 \mathbf{W}_0 \rangle = 0$ , and so

$$\lambda_1 = -\langle \mathbb{M}_1 \mathbf{w}_0 \rangle.$$

Furthermore, by Lemma 4.15, we have

$$u_1 = -L(1, 0)^{-1} \begin{pmatrix} 0 \\ \lambda_1 \mathbf{w}_0 + \mathbb{M}_1 \mathbf{w}_0 \end{pmatrix}.$$

This shows that the lemma holds for  $k = 1$ . Suppose that the lemma holds for  $k = m$ . We will show that the lemma also holds for  $k = m + 1$ . Assume that  $\lambda_l = 0$  for  $l = 0, \dots, m$ . Then, by induction assumption, we have

$$U_k = \begin{pmatrix} p_k \\ \mathbf{W}_k \end{pmatrix} = -L(1, 0)^{-1} \begin{pmatrix} 0 \\ \sum_{l=0}^{k-1} \mathbb{M}_{l+1} \mathbf{W}_{k-l-1} \end{pmatrix}$$

for  $k = 1, \dots, m$ . Furthermore, it follows from (4.13) with  $m$  replaced by  $m + 1$  that  $\lambda_{m+1} + \sum_{l=0}^m \langle \mathbb{M}_{l+1} \mathbf{W}_{m-l} \rangle = 0$ , and hence,

$$\lambda_{m+1} = -\sum_{l=0}^m \langle \mathbb{M}_{l+1} \mathbf{W}_{m-l} \rangle.$$

By Lemma 4.15, we see that the equation

$$L(1, 0)U = -\begin{pmatrix} 0 \\ \lambda_{m+1} \mathbf{w}_0 + \sum_{l=0}^m \mathbb{M}_{l+1} \mathbf{W}_{m-l} \end{pmatrix}$$

has a unique solution  $U_{m+1} = \begin{pmatrix} p_{m+1} \\ \mathbf{W}_{m+1} \end{pmatrix} \in H_*^1(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]^3$  with  $\langle \mathbf{W}_{m+1} \rangle = 0$ . This completes the proof.  $\square$

### 4.3 Applications

We can apply Theorems 4.6 and 4.8 to the Taylor problem discussed in section 3.2 to see that the structure of the bifurcation and stability of the artificial compressible system coincides with that of the incompressible system uniformly for small  $\epsilon$ . In this section, we consider another example, the Rayleigh-Bénard convection problem. For simplicity, we consider the two-dimensional problem. The Rayleigh-Bénard convection phenomenon is governed by the Oberbeck-Boussinesq equation

$$\operatorname{div} \mathbf{v} = 0, \quad (4.14)$$

$$\operatorname{Pr}^{-1} (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \Delta \mathbf{v} + \nabla p - \sqrt{\operatorname{Ra}} \theta \mathbf{e}_2 = \mathbf{0}, \quad (4.15)$$

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta - \Delta \theta - \sqrt{\operatorname{Ra}} \mathbf{v} \cdot \mathbf{e}_2 = 0. \quad (4.16)$$

The corresponding artificial compressible system is written as

$$\epsilon^2 \partial_t p + \operatorname{div} \mathbf{v} = 0, \quad (4.17)$$

$$\operatorname{Pr}^{-1} (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \Delta \mathbf{v} + \nabla p - \sqrt{\operatorname{Ra}} \theta \mathbf{e}_2 = \mathbf{0}, \quad (4.18)$$

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta - \Delta \theta - \sqrt{\operatorname{Ra}} \mathbf{v} \cdot \mathbf{e}_2 = 0. \quad (4.19)$$

Here  $\theta = \theta(x, t)$  denote the unknown temperature deviation from the heat conductive state at time  $t > 0$  and position  $x \in \mathbb{R}^2$ ;  $\mathbf{e}_2 = {}^\top(0, 1) \in \mathbb{R}^2$ ;  $\operatorname{Pr} > 0$  and  $\operatorname{Ra} > 0$  are non-dimensional parameters, called Prandtl and Rayleigh numbers, respectively. These systems are considered in the two-dimensional infinite layer  $\Omega$ :

$$\Omega = \{x = (x_1, x_2); x_1 \in \mathbb{R}, 0 < x_2 < 1\}.$$

The Oberbeck-Boussinesq equation (4.14)–(4.16) describes convection phenomena of viscous fluid occupying  $\Omega$  heated from below (heated at  $x_2 = 0$ ) under the gravitational force. It is well known ([2, 16, 17, 21]) that under the boundary conditions

$$\mathbf{v}|_{x_2=0,1} = \mathbf{0}, \quad \theta|_{x_2=0,1} = 0, \quad (4.20)$$

there exists a critical number  $\operatorname{Ra}_c > 0$  such that when  $\operatorname{Ra} < \operatorname{Ra}_c$ , the heat conductive state  $\mathbf{v} = \mathbf{0}$ ,  $\theta = 0$  is stable, while, when  $\operatorname{Ra} > \operatorname{Ra}_c$ , the heat conductive state is unstable and convective cellular stationary solutions bifurcate from the heat conductive state.

To be more precise, we consider (4.14)–(4.16) and (4.17)–(4.19) under the boundary conditions (4.20) and the periodicity condition

$$p, \mathbf{v} \text{ and } \theta \text{ are } \frac{2\pi}{\alpha}\text{-periodic in } x_1, \quad (4.21)$$

where  $\alpha$  is a positive constant. We also impose the following condition of reflection symmetry:

$$\begin{cases} p \text{ and } v^1 \text{ are even in } x_1, \\ v^2 \text{ and } \theta \text{ are odd in } x_1. \end{cases} \quad (4.22)$$

Let  $u_s = {}^\top(p_s, \mathbf{v}_s, \theta_s) = {}^\top(0, \mathbf{0}, 0)$  be the trivial solution of (4.14)–(4.16) which represents the heat conductive state. We introduce the linearized operator around  $\mathbf{U}_s = {}^\top(\mathbf{v}_s, \theta_s) = {}^\top(\mathbf{0}, 0)$  associated with problem (4.14)–(4.16) under (4.20)–(4.22). We define the operator  $L_{\text{Ra}}: X \rightarrow X$  by

$$L_{\text{Ra}} = \begin{pmatrix} -\text{Pr}\mathbb{P}\Delta & -\text{Pr}\sqrt{\text{Ra}}\mathbb{P}\mathbf{e}_2 \\ -\sqrt{\text{Ra}}^\top\mathbf{e}_2 & -\Delta \end{pmatrix}$$

with domain  $D(L) = \{[(H_{\text{per}}^2 \cap H_{0,\text{per}}^1)^2] \times [H_{\text{per}}^2 \cap H_{0,\text{per}}^1]\} \cap X$ . Here

$$X = \{{}^\top(\mathbf{v}, \theta) \in L_{\text{per},\sigma}^2 \times L_{\text{per}}^2; v^1, v^2 \text{ and } \theta \text{ satisfy (4.22)}\}.$$

We also introduce the linearized operator around  $u_s = {}^\top(0, \mathbf{0}, 0)$  associated with (4.17)–(4.19) under (4.20)–(4.22). We define the operator  $L_\epsilon: Y \rightarrow Y$  by

$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\text{div} & 0 \\ \text{Pr}\nabla & -\text{Pr}\Delta & -\text{Pr}\sqrt{\text{Ra}}\mathbf{e}_2 \\ 0 & -\sqrt{\text{Ra}}^\top\mathbf{e}_3 & -\Delta \end{pmatrix}$$

with domain  $D(L_\epsilon) = \{H_{\text{per},*}^1 \times [H_{\text{per}}^2 \cap H_{0,\text{per}}^1]^2 \times [H_{\text{per}}^2 \cap H_{0,\text{per}}^1]\} \cap Y$ . Here

$$Y = \{{}^\top(p, \mathbf{v}, \theta) \in H_{\text{per},*}^1 \times (L_{\text{per}}^2)^2 \times L_{\text{per}}^2; p, v^1, v^2 \text{ and } \theta \text{ satisfy (4.22)}\}.$$

The arguments in sections 3 and 4 are also valid for the Oberbeck-Boussinesq settings (4.14)–(4.16) and (4.17)–(4.19); and the results in sections 3 and 4 also hold. It is known ([2], [16], [17], [21]) that there exists a positive number  $\text{Ra}_c$  such that the assumptions (A0)–(A5) are satisfied by replacing  $\mathcal{R}$ ,  $\mathcal{R}_c$  and  $\mathbb{L}_\eta$  with  $\text{Ra}$ ,  $\text{Ra}_c$  and  $L_{\text{Ra}_c+\eta}$ ; and so, the following result holds for the incompressible system.

**Proposition 4.16.** *There exist a positive constant  $\delta_0$  and a solution branch  $\{\eta(\delta), \mathbf{U}_\eta(\delta)\}$  of (4.14)–(4.16), (4.20), (4.21) with  $\text{Ra} = \text{Ra}_c + \eta$ ,  $\eta = \eta(\delta)$  of the form*

$$\begin{aligned}\eta(\delta) &= \delta\sigma(\delta), \\ \mathbf{U}_\eta(\delta) &= \delta\tilde{\mathbf{U}}_1(\delta),\end{aligned}$$

where  $\sigma(\delta)$  is analytic in  $\delta$  ( $|\delta| \leq \delta_0$ ), and  $\tilde{\mathbf{U}}_1(\delta)$  is analytic in  $\delta$  in  $H^2(\Omega) \times H^2(\Omega)$  ( $|\delta| \leq \delta_0$ ). Furthermore,  $\mathbf{U}_\eta(\delta)$  is stable.

Since  $\mathbf{U}_s = {}^\top(\mathbf{0}, 0)$  and  $\|\mathbf{U}_\eta(\delta)\|_{H^2 \times H^2} \leq C\delta$ , condition (4.3) is satisfied by both the velocity fields of  $\mathbf{U}_s$  and  $\mathbf{U}_\eta(\delta)$  if  $|\delta| \leq \tilde{\delta}_0$  for some small positive number  $\tilde{\delta}_0$ . We can apply Theorems 4.6 and 4.8 to obtain the following result.

**Theorem 4.17.** *There exist positive constants  $\epsilon_1$  and  $\delta_1$  such that the following assertions hold true for  $0 < |\delta| \leq \delta_1$ .*

- (i) *The heat conductive state  $\mathbf{U}_s = {}^\top(0, \mathbf{0}, 0)$  is stable when  $\text{Ra} < \text{Ra}_c$  and unstable when  $\text{Ra} = \text{Ra}_c + \eta(\delta)$  for  $0 < |\delta| \leq \delta_1$  and  $0 < \epsilon \leq \epsilon_1$ .*
- (ii) *The bifurcating solution  $\mathbf{U}_\eta(\delta) = {}^\top(p_\eta(\delta), \mathbf{w}_\eta(\delta), \theta_\eta(\delta))$  with  $\eta = \eta(\delta)$  is stable for  $0 < |\delta| \leq \delta_1$  and  $0 < \epsilon \leq \epsilon_1$ . Here  $p_\eta(\delta)$  is the pressure corresponding to  $\mathbf{U}_\eta(\delta)$ .*

## 5 Derivation of solution formula (3.33) with $\boldsymbol{\psi} = \mathbf{0}$

In this section we derive the solution formula (3.33) for the problem (3.32) with  $\boldsymbol{\psi} = \mathbf{0}$ . The solution formula (3.33) holds for the general  $n$ -dimensional problem on  $\mathbb{R}_+^n = \{x = (x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\}$ ,  $n \geq 2$ , with obvious modifications:  $\xi' \in \mathbb{R}^2 \rightarrow \xi' \in \mathbb{R}^{n-1}$ ,  $x_3 \rightarrow x_n$ ,  $y_3 \rightarrow y_n$ . We thus give a derivation of the solution formula for the general  $n$ -dimensional problem.

Let us consider the problem on the half-space:

$$\left\{ \begin{array}{lcl} \operatorname{div} \mathbf{v} & = & f, \\ \lambda \mathbf{v} - \Delta \mathbf{v} + \nabla p & = & \mathbf{g}, \\ \mathbf{v}|_{x_n=0} & = & 0 \end{array} \right. \quad (5.1)$$

We denote the Fourier cosine and sine transforms in  $x_n \in (0, \infty)$  by  $\mathcal{F}_c$  and  $\mathcal{F}_s$ , respectively:

$$\mathcal{F}_c = \int_0^\infty f(x_n) \cos \xi_n x_n dx_n,$$

$$\mathcal{F}_s = \int_0^\infty f(x_n) \sin \xi_n x_n dx_n.$$

For  ${}^\top(p(x_n), \mathbf{v}'(x_n), v^n(x_n))$  ( $x_n \in (0, \infty)$ ) we define  $\mathcal{F}({}^\top(p, \mathbf{v}', v^n))$  by

$$\mathcal{F} \begin{pmatrix} p \\ \mathbf{v}' \\ v^n \end{pmatrix} (\xi_n) = \begin{pmatrix} \mathcal{F}_c p \\ \mathcal{F}_c \mathbf{v}' \\ \mathcal{F}_s v^n \end{pmatrix} (\xi_n), \quad \xi_n \in (0, \infty).$$

For simplicity in notation we denote  $\mathcal{F}_c \mathcal{F}' f$  and  $\mathcal{F}_c \mathcal{F}' g$  by  $f$  and  $g$ , respectively.

Applying  $\mathcal{F}' \mathcal{F}$  to (5.1) we have

$$\begin{cases} i\xi' \cdot \mathbf{v}' + \xi_n v^n = f, \\ (\lambda + \nu|\xi|^2) \mathbf{v}' + \nu \partial_{x_n} \mathbf{v}'(0) + i\xi' p = \mathbf{g}', \\ (\lambda + \nu|\xi|^2) v^n - \xi_n p = g^n, \\ \mathbf{v}|_{x_n=0} = \mathbf{0}. \end{cases} \quad (5.2)$$

This problem is written as

$$A \begin{pmatrix} p \\ \mathbf{v}' \\ v^n \end{pmatrix} = \begin{pmatrix} f \\ \mathbf{g}' \\ g^n \end{pmatrix} - \nu \begin{pmatrix} 0 \\ \partial_{x_n} \mathbf{v}'(0) \\ 0 \end{pmatrix} \quad \left. \begin{pmatrix} \mathbf{v}' \\ v^n \end{pmatrix} \right|_{x_n=0} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & i^\top \xi' & \xi_n \\ i\xi' & \lambda + \nu|\xi'|^2 & 0 \\ -\xi_n & 0 & \lambda + \nu|\xi'|^2 \end{pmatrix}.$$

Multiplying the second and third equations of (5.2) by  $i\xi'$  and  $\xi_n$ , respectively, and using the first equation (5.2), we have

$$-(|\xi'|^2 + \xi_n^2)p = i\xi' \cdot \mathbf{g}' + \xi_n g^n - (\lambda + \nu|\xi'|^2)f - \nu i\xi' \cdot \partial_{x_n} \mathbf{v}'(0),$$

and hence,

$$p = \frac{\lambda + \nu|\xi'|^2}{|\xi|^2}f - \frac{i\xi'}{|\xi|^2} \cdot (g' - \nu\partial_{x_n}\mathbf{v}'(0)) - \frac{\xi_n}{|\xi|^2}g^n. \quad (5.3)$$

Here and in what follows we write  $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ ,  $\xi' \in \mathbb{R}^{n-1}$ , and  $|\xi| = \sqrt{|\xi'|^2 + \xi_n^2}$ .

We set  $\lambda_0 = \lambda + \nu|\xi|^2$ . Substituting (5.3) into (5.2), we obtain

$$\begin{aligned} \mathbf{v}' &= \frac{1}{\lambda_0}(g' - \nu\partial_{x_n}\mathbf{v}'(0)) - \frac{i}{\lambda_0}\xi'p \\ &= -\frac{i\xi'}{|\xi|^2}f + \left(\frac{1}{\lambda_0} - \frac{\xi'^\top \xi'}{\lambda_0|\xi|^2}\right)(g' - \nu\partial_{x_n}\mathbf{v}'(0)) + \frac{i\xi_n\xi'}{\lambda_0|\xi|^2}g^n \end{aligned}$$

and

$$\begin{aligned} v^n &= \frac{1}{\lambda_0}g^n + \frac{\xi_n}{\lambda_0}p \\ &= \frac{\xi_n}{|\xi|^2}f - \frac{i\xi_n^\top \xi'}{\lambda_0|\xi|^2}(g' - \nu\partial_{x_n}\mathbf{v}'(0)) + \left(\frac{1}{\lambda_0} - \frac{\xi_n^2}{\lambda_0|\xi|^2}\right)g^n. \end{aligned}$$

It follows that

$$A^{-1} = \begin{pmatrix} \frac{\lambda_0}{|\xi|^2} & -\frac{i^\top \xi'}{|\xi|^2} & -\frac{\xi_n}{|\xi|^2} \\ -\frac{i\xi'}{|\xi|^2} & 0 & 0 \\ \frac{\xi_n}{|\xi|^2} & 0 & 0 \end{pmatrix} + \frac{1}{\lambda_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{n-1} - \frac{\xi'^\top \xi'}{|\xi|^2} & \frac{i\xi_n\xi'}{|\xi|^2} \\ 0 & -\frac{i\xi_n^\top \xi'}{|\xi|^2} & 1 - \frac{\xi_n^2}{|\xi|^2} \end{pmatrix}$$

and

$$\begin{pmatrix} p \\ \mathbf{v}' \\ v_n \end{pmatrix} = A^{-1} \begin{pmatrix} f \\ \mathbf{g}' \\ g^n \end{pmatrix} - A^{-1} \begin{pmatrix} 0 \\ \nu\partial_{x_n}\mathbf{v}'(0) \\ 0 \end{pmatrix}.$$

Our next task is to represent  $\partial_{x_n}\mathbf{v}'(0)$  in terms of  $\lambda$ ,  $\nu$ ,  $f$  and  $\mathbf{g}$ . To this end, we consider an *adjoint problem*:

$$\begin{cases} i\xi' \cdot \mathbf{u}' - \partial_{x_n}u^n &= 0, \\ \lambda\mathbf{u}' + \nu(|\xi'|^2 - \partial_{x_n}^2)\mathbf{u}' + i\xi'q &= 0, \\ \lambda u^n + \nu(|\xi'|^2 - \partial_{x_n}^2)u^n - \partial_{x_n}q &= 0. \end{cases} \quad (5.4)$$

We obtain

$$\begin{cases} \partial_{x_n}^2 q &= |\xi'|^2 q, \\ \partial_{x_n}^2 \mathbf{u}' &= \frac{\lambda + \nu |\xi'|^2}{\nu} \mathbf{u}' + \frac{i\xi'}{\nu} q, \\ \partial_{x_n}^2 u^n &= \frac{\lambda + \nu |\xi'|^2}{\nu} u^n - \frac{1}{\nu} \partial_{x_n} q. \end{cases} \quad (5.5)$$

Set

$$\begin{cases} Q &= \partial_{x_n} q, \\ \mathbf{U}' &= \partial_{x_n} \mathbf{u}' , \\ U^n &= \partial_{x_n} u^n \end{cases} , \quad V = \begin{pmatrix} p \\ \mathbf{v}' \\ u^n \\ Q \\ \mathbf{U}' \\ U^n \end{pmatrix} \quad (5.6)$$

It then follows from (5.4) and (5.5) that

$$\frac{d}{dx_n} V = BV,$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ |\xi'|^2 & 0 & 0 & 0 & 0 & 0 \\ \frac{i\xi'}{\nu} & \frac{\lambda + \nu |\xi'|^2}{\nu} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda + \nu |\xi'|^2}{\nu} & -\frac{1}{\nu} & 0 & 0 \end{pmatrix}.$$

We write  $B$  as

$$B = \left( \begin{array}{c|c} O & I_{n+1} \\ \hline A_1 & A_2 \end{array} \right), \quad A_1 = \begin{pmatrix} |\xi'|^2 & 0 & 0 \\ \frac{i\xi'}{\nu} & \frac{\lambda + \nu |\xi'|^2}{\nu} & 0 \\ 0 & 0 & \frac{\lambda + \nu |\xi'|^2}{\nu} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\nu} & 0 & 0 \end{pmatrix},$$

where  $I_k$  denotes the  $k \times k$  identity matrix. We then compute

$$\begin{aligned}
|\mu - B| &= \left| \begin{pmatrix} \mu I_{n+1} & -I_{m+1} \\ -A_1 & \mu - A_2 \end{pmatrix} \right| \\
&= |\mu(\mu - A_2) - A_1| \\
&= \left| \begin{pmatrix} \mu^2 & 0 & 0 \\ 0 & \mu^2 I_{n-1} & 0 \\ \frac{1}{\nu}\mu & 0 & \mu^2 \end{pmatrix} - \begin{pmatrix} |\xi'|^2 & 0 & 0 \\ \frac{i}{\nu}\xi' & \frac{\lambda'_0}{\nu}I_n - 1 & 0 \\ 0 & 0 & \frac{\lambda'_0}{\nu} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} \mu^2 - |\xi'|^2 & 0 & 0 \\ -\frac{i}{\nu}\xi' & (\mu^2 - \frac{\lambda'_0}{\nu})I_{n-1} & 0 \\ \frac{1}{\nu}\mu & 0 & \mu^2 - \frac{\lambda'_0}{\nu} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} \mu^2 - |\xi'|^2 & 0 & 0 & 0 \\ -\frac{i}{\nu}|\xi'| & \mu^2 - \frac{\lambda'_0}{\nu} & 0 & 0 \\ 0 & 0 & (\mu^2 - \frac{\lambda'_0}{\nu})I_{n-2} & 0 \\ \frac{1}{\nu}\mu & 0 & 0 & \mu^2 - \frac{\lambda'_0}{\nu} \end{pmatrix} \right| \\
&= (\mu^2 - |\xi'|^2)(\mu^2 - \frac{\lambda'_0}{\nu})^n,
\end{aligned}$$

where  $\lambda'_0 = \frac{\lambda + \nu|\xi'|^2}{\nu}$ . Therefore, the eigenvalues of  $B$  are given by  $\mu = \pm\mu_0, \pm|\xi'|$ . Here and in what follows we set

$$\mu_0 := \sqrt{\frac{\lambda'_0}{\nu}},$$

and take the principal branch of  $\sqrt{\frac{\lambda'_0}{\nu}}$  in such away that  $\sqrt{\frac{\lambda'_0}{\nu}} > 0$  for  $\lambda > -\nu|\xi'|^2$ . We look for solutions that decay as  $x_n \rightarrow \infty$ , and so consider the eigenvectors for the eigenvalues  $\mu = -\mu_0, -|\xi'|$ .

As for  $\mu = -\mu_0$  we see that

$$B \begin{pmatrix} X \\ Y \end{pmatrix} = -\mu_0 \begin{pmatrix} X \\ Y \end{pmatrix},$$

which can be rewritten as

$$\begin{cases} Y = -\mu_0 X, \\ A_1 X + A_2 Y = -\mu_0 Y. \end{cases}$$

This gives

$$A_1 X - \mu_0 A_2 X - \mu_0^2 X = 0.$$

We thus obtain

$$\begin{pmatrix} |\xi'|^2 - \mu_0^2 & 0 & 0 \\ \frac{i}{\nu} \xi' & 0 & 0 \\ \frac{1}{\nu} \mu_0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_0 \\ X' \\ X^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Noting that  $|\xi'|^2 - \mu_0^2 = -\frac{\lambda}{\nu}$ , we see that if  $\lambda \neq 0$  or  $\xi' \neq 0$ , then  $X_0 = 0$ . Since  $i\xi' \cdot X' - Y^n = 0$ , we have  $i\xi' \cdot X' + \mu_0 X^n = 0$ . We thus see that the eigenspace for the eigenvalue  $-\mu_0$  is spanned by  $w_1, \dots, w_{n-1}$ , where

$$w_1 = \begin{pmatrix} 0 \\ -\mu_0 \\ 0 \\ \vdots \\ i\xi_1 \end{pmatrix} e^{-\mu_0 x_n}, \dots, w_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ -\mu_0 \\ i\xi'_{n-1} \end{pmatrix} e^{-\mu_0 x_n}.$$

We next compute the eigenspace for the eigenvalue  $-|\xi'|$ . We consider

$$B \begin{pmatrix} X \\ Y \end{pmatrix} = -|\xi'| \begin{pmatrix} X \\ Y \end{pmatrix}.$$

This can be written as

$$\begin{cases} Y = -|\xi'| X, \\ A_1 X + A_2 Y = -|\xi'| Y, \end{cases}$$

and hence,

$$A_1 X - |\xi'| A_2 X - |\xi'|^2 X = 0.$$

We thus obtain

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{i}{\nu} \xi' & (\frac{\lambda_0}{\nu} - |\xi'|^2) I & 0 \\ \frac{1}{\nu} |\xi'| & 0 & \frac{\lambda_0}{\nu} - |\xi'|^2 \end{pmatrix} \begin{pmatrix} X^0 \\ X' \\ X^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{cases} \frac{i\xi'}{\nu} X_0 + \frac{\lambda}{\nu} X' = 0, \\ \frac{1}{\nu} X_0 + \frac{\lambda}{\nu} X^n = 0. \end{cases}$$

Since  $i\xi' \cdot X' - Y^n = 0$ , we have  $i\xi' \cdot X' + |\xi'|X^n = 0$ . Therefore the eigenspace for the eigenvalue  $-|\xi'|$  is spanned by  $w_n$ , where

$$w_n = \begin{pmatrix} \lambda \\ -i\xi' \\ -|\xi'| \end{pmatrix} e^{-|\xi'|x_n}.$$

We denote the real inner product of  $L^2(0, \infty)$  by  $(f, g)$ :

$$(f, g) = \int_0^\infty f(x_n)g(x_n) dx_n.$$

Let  ${}^\top(p, \mathbf{v})$ ,  $\mathbf{v} = {}^\top(\mathbf{v}', v^n)$ , be a solution of (5.2) and let  ${}^\top(q, \mathbf{u})$ ,  $\mathbf{u} = {}^\top(\mathbf{u}', u^n)$ , be a solution of (5.4). It then follows that

$$\begin{aligned} & (\lambda + \nu|\xi'|^2)(\mathbf{v}, \mathbf{u}) - \nu(\partial_{x_n}^2 \mathbf{v}, \mathbf{u}) + (i\xi' p, \mathbf{u}') + (\partial_{x_n} p, u^n) \\ & + (i\xi' \cdot \mathbf{v}' + \partial_{x_n} v^n, q) = (f, q) + (\mathbf{g}, u). \end{aligned}$$

By integration by parts, we have

$$\nu \partial_{x_n} \mathbf{v}(0) \cdot \mathbf{u}(0) - p(0)u^n(0) = (f, q) + (\mathbf{g}, u).$$

This implies that

$${}^\top \mathbf{u}(0) \begin{pmatrix} \nu \partial_{x_n} \mathbf{v}'(0) \\ P(0) \end{pmatrix} = {}^\top w F,$$

where

$$P(0) = \nu \partial_{x_n} v^n(0) - p(0), \quad w = \begin{pmatrix} q \\ \mathbf{u} \end{pmatrix}, \quad F = \begin{pmatrix} f \\ \mathbf{g} \end{pmatrix}.$$

We set

$$S = \begin{pmatrix} {}^\top u_1(0) \\ \vdots \\ {}^\top u_n(0) \end{pmatrix}.$$

We then find that

$$\begin{pmatrix} \nu \partial_{x_n} \mathbf{v}'(0) \\ P(0) \end{pmatrix} = S^{-1} \begin{pmatrix} (w_1, F) \\ \vdots \\ (w_n, F) \end{pmatrix} = \int_0^\infty S^{-1} \begin{pmatrix} {}^\top w_1(y_n) F(y_n) \\ \vdots \\ {}^\top w_n(y_n) F(y_n) \end{pmatrix} dy_n.$$

Here

$$S = \begin{pmatrix} -\mu_0 I_{n-1} & i\xi' \\ -i^\top \xi' & -|\xi'| \end{pmatrix}.$$

Since

$$\begin{aligned} S^{-1} &= \begin{pmatrix} \mu_0^{-1} I & 0 \\ 0 & 0 \end{pmatrix} + \frac{\mu_0}{|\xi'|(|\xi'| - \mu_0)} \begin{pmatrix} -i\mu_0^{-1}\xi'(-\mu_0^{-1})(-i^\top \xi') & \mu_0^{-1}i\xi' \\ -\mu_0^{-1}i^\top \xi' & 1 \end{pmatrix} \\ &= -\frac{1}{\mu_0} \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{|\xi'|(|\xi'| - \mu_0)} \begin{pmatrix} \frac{1}{\mu_0}\xi'^\top \xi' & i\xi' \\ -i^\top \xi' & \mu_0 \end{pmatrix} \end{aligned}$$

and

$$W(y_n) := \begin{pmatrix} {}^\top w_1(y_n) \\ \vdots \\ {}^\top w_n(y_n) \end{pmatrix} = \begin{pmatrix} 0 & -\mu_0 I_{n-1} e^{-\mu_0 y_n} & i\xi' e^{-\mu_0 y_n} \\ \lambda e^{-|\xi'| y_n} & -i^\top \xi' e^{-|\xi'| y_n} & -|\xi'| e^{-|\xi'| y_n} \end{pmatrix},$$

we have

$$\begin{aligned} S^{-1\top} W(y_n) &= \begin{pmatrix} 0 & e^{-\mu_0 y_n} I_{n-1} & -\frac{i\xi'}{\mu_0} e^{-\mu_0 y_n} \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{|\xi'|(|\xi'| - \mu_0)} \\ &\times \begin{pmatrix} i\lambda \xi' e^{-|\xi'| y_n} & -\xi'^\top \xi' (e^{-\mu_0 y_n} - e^{-|\xi'| y_n}) & i\xi' |\xi'|^2 (\frac{1}{\mu_0} e^{-\mu_0 y_n} - \frac{1}{|\xi'|} e^{-|\xi'| y_n}) \\ \lambda \mu_0 e^{-|\xi'| y_n} & i\mu_0^\top \xi' (e^{-\mu_0 y_n} - e^{-|\xi'| y_n}) & \mu_0 |\xi'| (\frac{1}{\mu_0} e^{-\mu_0 y_n} - \frac{1}{|\xi'|} e^{-|\xi'| y_n}) \end{pmatrix}. \end{aligned}$$

We thus obtain

$$\nu \partial_{x_n} v'(0) = \int_0^\infty (S^{-1\top} W(y_n))' F(y_n) dy_n.$$

We compute

$$\begin{aligned} (S^{-1\top} W(y_n))' &= (0 \ I_{n-1} \ 0) e^{-\mu_0 y_n} + \frac{\lambda}{|\xi'|(|\xi'| - \mu_0)} (i\xi' \ 0 \ 0) e^{-|\xi'| y_n} \\ &+ \frac{\lambda}{|\xi'|(|\xi'| - \mu_0)} (0 \ -\frac{\xi'^\top \xi'}{\lambda} \ \frac{i|\xi'| \xi'}{\lambda}) (e^{-\mu_0 y_n} - e^{-|\xi'| y_n}), \end{aligned}$$

and hence,

$$\begin{aligned}
& - \begin{pmatrix} 0 \\ (S^{-1\top} W(y_n))' \\ 0 \end{pmatrix} \\
& = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -I_{n-1} e^{-\mu_0 y_n} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \beta_0(\xi', y_n) & B(\xi', y_n) & \beta_n(\xi', y_n) \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Here

$$\begin{aligned}
\beta_0(\xi', y_n) &= -\frac{i\lambda\xi'}{|\xi'|(\mu_0 - |\xi'|)} e^{-|\xi'|y_n}, \\
B(\xi', y_n) &= -\frac{\xi'^\top \xi'}{|\xi'|(\mu_0 - |\xi'|)} (e^{-\mu_0 y_n} - e^{-|\xi'|y_n}), \\
\beta_n(\xi', y_n) &= \frac{i\xi'}{\mu_0 - |\xi'|} (e^{-\mu_0 y_n} - e^{-|\xi'|y_n}).
\end{aligned}$$

Since

$$\mathcal{F}_c^{-1} \left[ \frac{1}{\xi_n^2 + a^2} \right] = \frac{1}{a} e^{-ax_n}, \quad \mathcal{F}_s^{-1} \left[ \frac{\xi_n}{\xi_n^2 + a^2} \right] = -\partial_{x_n} \left( \frac{1}{a} e^{-ax_n} \right)$$

and

$$\begin{aligned}
\mathcal{F}_c^{-1} \left[ \frac{1}{\lambda + \nu|\xi|^2} \frac{1}{|\xi|^2} \right] &= \mathcal{F}_c^{-1} \left[ -\frac{1}{\lambda} \left( \frac{1}{\mu_0^2 + \xi_n^2} - \frac{1}{|\xi'|^2 + \xi_n^2} \right) \right] \\
&= -\frac{1}{\lambda} \left( \frac{1}{\mu_0} e^{-\mu_0 x_n} - \frac{1}{|\xi'|} e^{-|\xi'|x_n} \right),
\end{aligned}$$

we have

$$\mathcal{F}^{-1} A^{-1} = \begin{pmatrix} \lambda \frac{1}{|\xi'|} e^{-|\xi'|x_n} + \nu\delta & -\frac{i^\top \xi'}{|\xi'|} e^{-|\xi'|x_n} & -\partial_{x_n} \left( \frac{1}{|\xi'|} e^{-|\xi'|x_n} \right) \\ -\frac{i\xi'}{|\xi'|} e^{-|\xi'|x_n} & \frac{1}{\nu} \frac{1}{\mu_0} e^{-\mu_0 x_n} + \frac{\xi'^\top \xi'}{\lambda} h_{\mu_0, |\xi'|} & -\frac{i\xi'}{\lambda} \partial_{x_n} h_{\mu_0, |\xi'|} \\ -\partial_{x_n} \left( \frac{1}{|\xi'|} e^{-|\xi'|x_n} \right) & -\frac{i^\top \xi'}{\lambda} \partial_{x_n} h_{\mu_0, |\xi'|} & \frac{1}{\nu} \frac{1}{\mu_0} e^{-\mu_0 x_n} - \frac{1}{\lambda} \partial_{x_n}^2 h_{\mu_0, |\xi'|} \end{pmatrix},$$

where

$$h_{\mu_0, |\xi'|}(x_n) = \frac{1}{\mu_0} e^{-\mu_0 x_n} - \frac{1}{|\xi'|} e^{-|\xi'|x_n}.$$

It then follows that

$$\begin{aligned}
& -\mathcal{F}^{-1}A^{-1}\begin{pmatrix} 0 \\ S^{-1\top}W' \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{\nu}\frac{1}{\mu_0}e^{-\mu_0(x_n+y_n)} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i^\top\xi'h_{|\xi'|}(x_n)e^{-\mu_0y_n} & 0 \\ 0 & -\frac{\xi'^\top\xi'}{\lambda}h_{\mu_0,|\xi'|}(x_n)e^{-\mu_0y_n} & 0 \\ 0 & \frac{i^\top\xi'}{\lambda}\partial_{x_n}h_{\mu_0,|\xi'|}(x_n)e^{-\mu_0y_n} & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\nu}h_{\mu_0}(x_n)\beta_0(y_n) & \frac{1}{\nu}h_{\mu_0}(x_n)B(y_n) & \frac{1}{\nu}h_{\mu_0}(x_n)\beta_n(y_n) \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} -i^\top\xi'h_{|\xi'|}(x_n)\beta_0(y_n) & -i^\top\xi'h_{|\xi'|}(x_n)B(y_n) & -i^\top\xi'h_{|\xi'|}(x_n)\beta_n(y_n) \\ \frac{\xi'^\top\xi'}{\lambda}h_{\mu_0,|\xi'|}(x_n)\beta_0(y_n) & \frac{\xi'^\top\xi'}{\lambda}h_{\mu_0,|\xi'|}(x_n)B(y_n) & \frac{\xi'^\top\xi'}{\lambda}h_{\mu_0,|\xi'|}(x_n)\beta_n(y_n) \\ -\frac{i^\top\xi'}{\lambda}\partial_{x_n}h_{\mu_0,|\xi'|}(x_n)\beta_0(y_n) & -\frac{i^\top\xi'}{\lambda}\partial_{x_n}h_{\mu_0,|\xi'|}(x_n)B(y_n) & -\frac{i^\top\xi'}{\lambda}\partial_{x_n}h_{\mu_0,|\xi'|}(x_n)\beta_n(y_n) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{\nu}\frac{1}{\mu_0}e^{-\mu_0(x_n+y_n)} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \hat{H}(\lambda, \xi', x_n, y_n).
\end{aligned}$$

To compute  $\mathcal{F}^{-1}[A^{-1\top}(f, \mathbf{g}', g^n)]$ , we use the following relations:

$$\frac{1}{\mu^2 + \xi_n^2}\mathcal{F}_c f = \mathcal{F}_c \left[ \frac{1}{\mu}e^{-\mu x_n} \right] \mathcal{F}_c f = \mathcal{F}_c[g_\mu^{(+)}f], \quad (5.7)$$

$$\frac{1}{\mu^2 + \xi_n^2}\mathcal{F}_s f = \mathcal{F}_c \left[ \frac{1}{\mu}e^{-\mu x_n} \right] \mathcal{F}_s f = \mathcal{F}_s[g_\mu^{(-)}f], \quad (5.8)$$

$$\frac{\xi_n}{\mu^2 + \xi_n^2}\mathcal{F}_c f = \xi_n \mathcal{F}_c \left[ \frac{1}{\mu}e^{-\mu x_n} \right] \mathcal{F}_c f = \xi_n \mathcal{F}_c[g_\mu^{(+)}f] = -\mathcal{F}_s[\partial_{x_n}(g_\mu^{(+)}f)], \quad (5.9)$$

$$\frac{\xi_n}{\mu^2 + \xi_n^2}\mathcal{F}_s f = \xi_n \mathcal{F}_c \left[ \frac{1}{\mu}e^{-\mu x_n} \right] \mathcal{F}_s f = \xi_n \mathcal{F}_s[g_\mu^{(-)}f] = \mathcal{F}_c[\partial_{x_n}(g_\mu^{(-)}f)]. \quad (5.10)$$

Here we used the fact  $(g_\mu^{(-)}f)|_{x_n=0} = 0$  to obtain the relation (5.10). By

(5.7)–(5.10), we have

$$\begin{aligned}\mathcal{F}^{-1}A^{-1}\begin{pmatrix} f \\ \mathbf{g}' \\ g^n \end{pmatrix} &= \int_0^\infty \hat{G}(\lambda, \xi', x_n, y_n) \begin{pmatrix} \mathcal{F}'f(\xi', y_n) \\ \mathcal{F}'\mathbf{g}'(\xi', y_n) \\ \mathcal{F}'g^n(\xi', y_n) \end{pmatrix} dy_n \\ &\quad + \int_0^\infty \frac{1}{\nu} \frac{1}{\mu_0} e^{-\mu_0(x_n+y_n)} \begin{pmatrix} 0 \\ \mathcal{F}'\mathbf{g}'(\xi', y_n) \\ 0 \end{pmatrix} dy_n.\end{aligned}$$

Therefore, we obtain the desired solution formula

$$\begin{aligned}\begin{pmatrix} \mathcal{F}'p \\ \mathcal{F}'\mathbf{v}' \\ \mathcal{F}'v^n \end{pmatrix} &= \int_0^\infty \hat{G}(\lambda, \xi', x_n, y_n) \begin{pmatrix} \mathcal{F}'f(\xi', y_n) \\ \mathcal{F}'\mathbf{g}'(\xi', y_n) \\ \mathcal{F}'g^n(\xi', y_n) \end{pmatrix} dy_n \\ &\quad + \int_0^\infty \hat{H}(\lambda, \xi', x_n, y_n) \begin{pmatrix} \mathcal{F}'f(\xi', y_n) \\ \mathcal{F}'\mathbf{g}'(\xi', y_n) \\ \mathcal{F}'g^n(\xi', y_n) \end{pmatrix} dy_n.\end{aligned}$$

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