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Equilibrium and Non-Equilibrium Steady States on Boson Systems with BEC

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Abstract

In the present paper, we consider Bose–Einstein condensation (BEC) of free bosons on graphs and non-equilibrium steady states (NESS), in the sense of D. Ruelle [Commun. Math. Phys. **224**, 3–16 (2001)], of boson system with BEC. The Hamiltonian is the second quantization of transient adjacency operators.

In the first part of the paper, we prove equivalence of BEC and non-factoriality of the quasi-free state. Moreover, quasi-free states with BEC are decomposed into generalized coherent states. For completeness, we include results of quasi-free states (M. Shiraishi and H. Araki [Publ. Res. Inst. Math. Sci. **7**, 105–120 (1971/72)], H. Araki [Publ. Res. Inst. Math. Sci. **7**, 121–152 (1971/72)], and H. Araki and S. Yamagami [Publ. Res. Inst. Math. Sci. **18**(2), 703–758 (1982)]). We obtain necessary and sufficient conditions for faithfulness, factoriality, and purity of a generalized coherent state and quasi-equivalence of generalized coherent states.

In the second part of the paper, we consider NESS of boson systems with BEC. The model consists of a quantum particle and several bosonic reservoirs. We show that the mean entropy production rate is strictly positive, independent of phase differences provided that the temperatures or the chemical potentials of reservoirs are different. Moreover, Josephson currents occur without entropy production, even if the temperatures and the chemical potentials of reservoirs are identical.

Keywords: CCR algebra, generalized coherent states, quasi-equivalence, BEC, NESS, Mourre estimate, Spectrum of the adjacency operator of graphs.
AMS subject classification: 82B10

Contents

List of Notations

$\mathfrak{h}$: A subspace of a Hilbert space.
G : Graphs.
A_G : the adjacency operator of a graph G. v : PF weights. $\mathcal{D}(A)$: The domain of an operator A.
(V, σ) : A symplectic space. $\mathcal{W}(\mathfrak{h})$: The Weyl CCR algebra over $\mathfrak{h}$.
ω : A state on $\mathcal{W}(\mathfrak{h})$.
$a(f), a^\dagger(f)$: The annihilation and the creation operators.
α_t : A time evolution.
π : A representation. π_ω : A representation with respect to a state ω.
$\Psi(f)$: Field operators.
$W(f)$: Weyl operators.
Part I, Sections 3.
$\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$: CCR algebra, p9.
S, S_K : A covariance operator, p10, (3.7).
ω_S : A quasi-free state with S, p10, (3.4).
Part I, Sections 4.
q : A linear functional.
$\omega_{S,q}$: A generalized coherent state with S and q, p12, (4.4).
Part I, Section 5.
$\omega_{q,D}$: A quasi-free state with q and $D \geq 0$, p16, (5.1).
ϕ_{s_1,s_2} : A generalized coherent state, $s_1, s_2 \in \mathbb{R}$, p18, (5.20).
ω_D : A quasi-free state, $D \geq 0$, p19, (5.22).
Part II.
h_0 : A one-particle hamiltonian of an uncoupled system, p22, (6.2).
h : A one-particle hamiltonian of a coupled system, p22, (6.3).
α_t : A coupled time evolution, p22, (6.4).
$h_{0,0} = \bigoplus_{k=1}^N h_{0,k}$: A one-particle hamiltonian of reservoirs.
Conditions (Abs), (A), and (B) are in p23.
η : A complex valued function, p23, (6.6).
$\mathfrak{h}(g)$: p23, (6.9).
$F(v; f)$: A measurable function, p23.
$\varphi(f)$: A vector, p23.
ω_0 : An initial state of the system, p27, (7.1).
Conditions (C) and (D) are in p28.
ω_+ : A NESS, p29, (7.15) and (7.16).
J_l, E_l : Currents, p32, (8.1) and (8.2).
$\mathrm{Ep}(\omega_+)$: Entropy production rate at a NESS ω_+, p32, (8.4).

1 Introduction

The mathematical studies of Bose–Einstein condensation (BEC for short.) have a long history. (cf. [37].) In the case of $\mathbb{R}^d$, J. T. Lewis and J. V. Pulè suggested that equilibrium states with BEC are non-factor in [22]. In [26], T. Matsui studied BEC in terms of the random walk on graphs. In [11, 12, 13], F. Fidaleo and their coworkers studied hidden spectrum of the adjacency operator on graphs and BEC. They obtained a criterion for BEC on graphs. Factoriality of equilibrium states of the system is not studied completely. Thus, in the first part of this paper, we study equilibrium states with BEC and prove equivalence of BEC and non-factoriality of the quasi-free state. Moreover, we give a concrete factor decomposition of equilibrium states with BEC into generalized coherent states (Theorem 5.9). Generalized coherent states are generalization of coherent states in the following sense. Let $\mathfrak{h}$ be a subspace of a Hilbert space. A coherent state ω on the Weyl CCR algebra $\mathcal{W}(\mathfrak{h})$, specified in Section 2.1, is given by

$$\omega(W(f)) = \exp\left(-\|f\|^2 + i\mathrm{Re}q(f)\right) \tag{1.1}$$

for each $f \in \mathfrak{h}$, where $W(f)$, $f \in \mathfrak{h}$, are the Weyl operators which generate $\mathcal{W}(\mathfrak{h})$, $\|\cdot\|$ is the norm induced by the inner product on $\mathfrak{h}$, and q is a $\mathbb{C}$-linear functional on $\mathfrak{h}$. (See [15, Theorem 3.1].) A state is said to be generalized coherent, if there exist a sesquilinear form S on $\mathfrak{h}$ and an $\mathbb{R}$-linear functional $q : \mathfrak{h} \to \mathbb{R}$ such that

$$\omega(W(f)) = \exp\left(-S(f,f)/4 + iq(f)\right), \quad f \in \mathfrak{h}. \tag{1.2}$$

Faithfulness, factoriality, and purity of a quasi-free state and quasi-equivalence of quasi-free states are studied in [1, 2, 3, 23, 24, 36]. By using the results in [1, 2, 3], we obtain necessary and sufficient conditions of faithfulness, factoriality, and purity of a generalized coherent state and conditions of quasi-equivalence of generalized coherent states (Theorems 4.2, 4.3, 4.4, and 4.7).

In the second part of this paper, we study non-equilibrium steady states (NESS for short.) of a model, which consists of a quantum particle and several Bosonic reservoirs with BEC. The reservoirs consist of free Bose particles on $\mathbb{R}^d$ or on graphs. We denote the annihilation and the creation operators by a and $a^\dagger$ (resp. $a_{x,k}$ and $a^\dagger_{x,k}$). These operators satisfy canonical commutation relations (CCR)

$$\left[a, a^\dagger\right] = 1, \quad \left[a_{x,k}, a_{y,l}\right] = \delta_{k,l}\delta(x-y), \quad k,l = 1,\dots,N, \tag{1.3}$$

where N is the number of reservoirs. In the case of $\mathbb{R}^d$, the Hamiltonian H of our coupled model is formally given by

$$H = H_0 + \lambda \sum_{k=1}^{N} W_k, \tag{1.4}$$

where $\lambda > 0$ and

$$H_0 = \Omega a^\dagger a + \sum_{k=1}^{N} \int_{\mathbb{R}^d} dp \frac{|p|^2}{2} a^\dagger_{p,k} a_{p,k}, \quad W_k = \int_{\mathbb{R}^d} dp \left\{\overline{g_k(p)} a^\dagger a_{p,k} + g_k(p) a a^\dagger_{p,k}\right\}. \tag{1.5}$$

When we consider the case of graphs, we replace the integral part of (1.5) by the sum over the set of vertices of graphs and $|p|^2/2$ by the adjacency operator of graphs. Following D. Ruelle [31], we say that a state is a NESS, if it is a weak $*$-limit point of the net

$$\left\{ \frac{1}{T} \int_0^T \omega_0 \circ \alpha_t dt \,\middle|\, T > 0 \right\}, \tag{1.6}$$

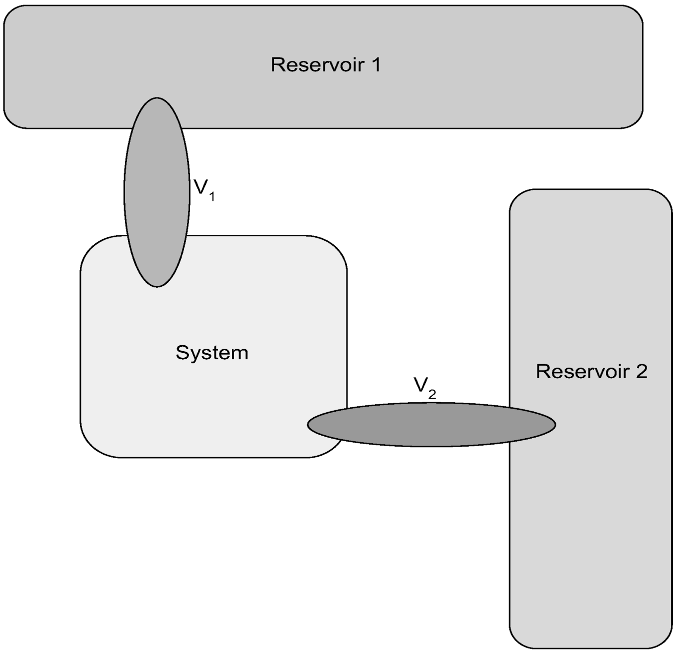

Figure 1: An example of coupled model

where ω_0 is the initial state and α_t is the Heisenberg time evolution of our coupled model defined by $\alpha_t(Q) = e^{itH}Qe^{-itH}$ for a quantum observable Q. The initial state is given by a product state of a state of a finite system and condensed states with different temperatures. We obtain explicit formulas of NESS, currents, and the mean entropy production rate. We prove that the mean entropy production rate is strictly positive, if the temperatures or the chemical potentials of reservoirs are different and if there exists an open channel, specified in Section 8. Moreover, we show that Josephson currents occur without entropy production, if the temperatures and the chemical potentials of reservoirs are identical. V. Jakšić, C.-A. Pillet, and their co-workers investigated various aspects of NESS (to take a few example, [4, 5, 8, 9, 16, 17, 19]). The case of bosonic reservoir without BEC was studied in [27] and [33]. However, the case of bosonic reservoirs with BEC was hardly studied before except for the study of S. Tasaki and T. Matsui [34].

This paper is organized as follows. In Section 2, we recall the definition of the Weyl CCR algebras and the notations of infinite graphs. Part I consists of Sections 2, 3, and 4 and the results in this part have been already published [20]. In Section 3, we review works of M. Shiraishi and H. Araki [1], H. Araki [2], and H. Araki and S. Yamagami [3]. In Section 4, we consider generalized coherent states on the Weyl CCR algebras. We prove necessary and sufficient conditions for faithfulness, factoriality, and purity of a generalized coherent state and quasi-equivalences of generalized coherent states. Moreover, we give an explicit factor decomposition of a non-factor generalized coherent state. In [14], R. Honegger obtained a decomposition of gauge-invariant quasi-free states. In the present paper, we only assume that a state on the Weyl CCR algebras is quasi-free or generalized coherent. In Section 5, we review works of F. Fidaleo [13] and consider the non-factoriality of quasi-free states with BEC. We show that quasi-free states with BEC are non-factor and such state is decomposed into generalized coherent states.

Part II consists of Sections 6, 7, 8, and 9, and the results in this part have been already published [21]. In Section 6, we have an explicit formula of our coupled time evolution (Theorem 6.3). In Section 7, we give the initial state on the Weyl CCR algebra and obtain an explicit formula of NESS (Theorem 7.3). Section 8 contains the main results in part II: explicit formulas of currents and the (strict) positivity of the mean entropy production rate (Corollary 8.2, Proposition 8.3, and Theorem 8.4). In Section 9, we verify the assumptions of Theorem 7.3 in the case of $\mathbb{R}^d$, $d \geq 3$, and of graphs. For our purpose, we used Mourre estimate techniques due to [25]. After introduction of notations, we consider typical examples of graphs: periodic graphs and comb graphs.

2 Preliminaries

In this section, we recall the definition of the Weyl CCR algebras and organize the notation of graphs.

2.1 Weyl Operators and Weyl CCR Algebra

Let $\mathfrak{h}$ be a subspace of a Hilbert space $\mathfrak{H}$. Then, on the Boson–Fock space $\mathcal{F}_+(\mathfrak{h})$, we can define the annihilation operators $a(f)$, $f \in \mathfrak{h}$, and the creation operators $a^\dagger(f)$, $f \in \mathfrak{h}$. (See e.g. [7].) The operators $a(f)$ and $a^\dagger(f)$, $f \in \mathfrak{h}$, are closed and satisfy CCRs:

$$[a(f), a(g)] = 0 = \left[a^\dagger(f), a^\dagger(g)\right], \quad \left[a(f), a^\dagger(g)\right] = \langle f, g\rangle \mathbb{1}, \quad f, g \in \mathfrak{h}, \tag{2.1}$$

where $[A, B] = AB - BA$, is the commutator. The field operators $\Psi(f)$, $f \in \mathfrak{h}$, are defined by

$$\Psi(f) = \frac{1}{\sqrt{2}} \overline{\{a(f) + a^\dagger(f)\}}^{\text{op.cl.}}, \tag{2.2}$$

where $\overline{A}^{\text{op.cl.}}$ means the closure of operator A. Then $\Psi(f)$, $f \in \mathfrak{h}$, are (unbounded) self-adjoint operators and satisfy

$$[\Psi(f), \Psi(g)] = \mathrm{Im}\langle f, g\rangle \mathbb{1}, \quad f, g \in \mathfrak{h}. \tag{2.3}$$

The equation (2.3) is called CCR. The Weyl operator $W(f)$ is defined by

$$W(f) = \exp(i\Psi(f)), \quad f \in \mathfrak{h}, \tag{2.4}$$

and satisfy the following equations:

$$W(0) = \mathbb{1}, \quad W(f)^* = W(-f), \quad W(f)W(g) = e^{-i\frac{\sigma(f,g)}{2}} W(f+g), \quad f, g \in \mathfrak{h}, \tag{2.5}$$

where $\sigma(f, g) = \mathrm{Im}\langle f, g\rangle$, $f, g \in \mathfrak{h}$. The Weyl CCR algebra $\mathcal{W}(\mathfrak{h})$ is the unital C*-algebra generated by unitaries $W(f)$, $f \in \mathfrak{h}$. Generally, the Weyl CCR algebra $\mathcal{W}(\mathfrak{h})$ is the unital universal C*-algebra generated by unitaries $W(f)$, $f \in \mathfrak{h}$, which satisfy (2.5). (See e.g. [7, Theorem 5.2.8.].)

Next, we consider the Weyl CCR algebra over a symplectic space (V, σ). Let V be an $\mathbb{R}$-linear space with a symplectic form $\sigma : V \times V \to \mathbb{R}$, i.e., σ is a bilinear form on V and satisfies the following relations:

$$\sigma(f, g) = -\sigma(g, f), \quad f, g \in V. \tag{2.6}$$

We assume that there exists an operator J on V with the properties

$$\sigma(Jf, g) = -\sigma(f, Jg), \quad J^2 = -1. \tag{2.7}$$

Then V is a $\mathbb{C}$-linear space with scalar multiplication defined by

$$(c_1 + ic_2)f = c_1 f + c_2 Jf, \quad c_1, c_2 \in \mathbb{R},\ f \in V. \tag{2.8}$$

Then we define the complexification $V^{\mathbb{C}}$ of V by (2.8) and an inner product $\langle \cdot, \cdot \rangle$ on V by

$$\langle f, g \rangle = \sigma(f, Jg) + i\sigma(f, g). \tag{2.9}$$

Then $(V^{\mathbb{C}}, \langle \cdot, \cdot \rangle)$ becomes an inner product space. By the same discussion in the case of a subspace of a Hilbert space, we can define the Weyl CCR algebra over $(V^{\mathbb{C}}, \langle \cdot, \cdot \rangle)$ and denote the Weyl CCR algebra over $(V^{\mathbb{C}}, \langle \cdot, \cdot \rangle)$ by $\mathcal{W}(V, \sigma)$. See [7, Theorem 5.2.8.] for details.

For a C*-algebra $\mathcal{A}$ and a state ω on $\mathcal{A}$, there exists the GNS-representation space $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$, where $\mathfrak{H}_\omega$ is a Hilbert space, π_ω is a representation of $\mathcal{A}$, and Ω_ω is a cyclic vector for $\pi_\omega(\mathcal{A})$. We denote the commutant of $\pi_\omega(\mathcal{A})$ by $\pi_\omega(\mathcal{A})'$, i.e.,

$$\pi_\omega(\mathcal{A})' = \{ A \in \mathcal{B}(\mathfrak{H}_\omega) \mid AB = BA, \forall B \in \pi(\mathcal{A}) \}. \tag{2.10}$$

A state ω on a C*-algebra $\mathcal{A}$ is said to be factor, if on the GNS representation space $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$, the center of $\pi_\omega(\mathcal{A})''$ is equal to $\mathbb{C}\mathbb{1}$, i.e., $\pi_\omega(\mathcal{A})'' \cap \pi_\omega(\mathcal{A})' = \mathbb{C}\mathbb{1}$.

2.2 Graphs

Let $G = (VG, EG)$ be an undirected graph, where VG is the set of all vertices in G and EG is the set of all edges in G. Two vertices $x, y \in VG$ are said to be adjacent if there exists an edge $(x, y) \in EG$ joining x and y, and we write $x \sim y$. In the present paper, we assume that VX is countable. Let $\ell^2(VG)$ be the set of all square summable sequence labeled by the vertices in VG. Let A_G be the adjacency operator of G defined by

$$\left\langle \delta_x, A_G \delta_y \right\rangle = \begin{cases} 1 & (x \sim y), \\ 0 & (x \nsim y), \end{cases} \quad x, y \in VG, \tag{2.11}$$

where δ_x is the delta function such that $\delta_x(y) = 0$ for any $y \neq x$ and $\delta_x(x) = 1$. In addition, for any $x \in VG$, we set the degree of x by $\deg_G(x)$ and

$$\deg_G := \sup_{x \in VG} \deg_G(x). \tag{2.12}$$

In this paper, we only consider graphs which are connected with $\deg_G < \infty$, with no loop, and with no multiple edges and has countable vertices. Then, the adjacency operator A_G acting on $\ell^2(VG)$ is bounded. If for any δ_x, $x \in VG$, A_G satisfies the condition

$$\lim_{\lambda \searrow \|A_G\|} \left\langle \delta_x, (\lambda \mathbb{1} - A_G)^{-1} \delta_x \right\rangle < \infty, \tag{2.13}$$

then A_G is said to be transient.

A bounded operator B acting on $\ell^2(VG)$ is called a positivity preserving operator, if $B_{x,y} := \left\langle \delta_x, B\delta_y \right\rangle \geq 0$ for any $x, y \in VG$. Fix a positivity preserving operator B. The sequence $v := \{v(x)\}_{x \in VG}$ is called a Perron-Frobenius weight (PF weight for short) if it has positive entries and

$$\sum_{y \in VG} B_{xy} v(y) = \mathrm{spr}(B) v(x), \quad x \in VG, \tag{2.14}$$

where "spr" stands for spectral radius. If such a vector v belongs to $\ell^2(VG)$ it is a standard eigenvector for B. For the adjacency operator A_G of G, the existence of v is proved in [11, Proposition 4.1]. We regard a PF weight v as a densely defined linear functional on $\ell^2(VG)$. We define the domain $\mathcal{D}(v)$ of v by

$$\mathcal{D}(v) = \left\{ \psi \in \ell^2(VG) \;\middle|\; \sum_{x \in VG} v(x)|\psi(x)| < \infty \right\}, \tag{2.15}$$

where $\psi(x) = \langle \delta_x, \psi \rangle$. If $\psi \in \mathcal{D}(v)$, we denote $\sum_{x \in VG} v(x)\psi(x)$ by $\langle v, \psi \rangle$. If the adjacency operator A_G of a graph G has an eigenvalue $\|A_G\|$, then an eigenvalue $\|A_G\|$ is simple and a PF weight (vector) exists uniquely. However, if G is a comb graph $\mathbb{Z}^d \dashv \mathbb{Z}$, specified in Section 9, then A_G does not have an eigenvalue $\|A_G\|$ and there exist two PF weights. (See e.g. [11], [13], and [28].) When v is a PF weight for A_G, we say that v is a PF weight for $\|A_G\|\mathbb{1} - A_G$.

Part I
Non-factoriality of Quasi-free States with BEC

In this part, we show the non-factoriality of quasi-free states with BEC and give an explicit factor decomposition of quasi-free states with BEC. We review works of H. Araki and their co-workers [1, 2, 3]. By using their results, we give necessary and sufficient conditions of faithfulness, factoriality, and purity of a generalized coherent state and conditions of quasi-equivalence of generalized coherent states. By using these result, we show that quasi-free states with BEC are non-factor and give an explicit factor decomposition of quasi-free states with BEC.

3 Some Properties of Quasi-free States

In this section, we review works of H. Araki and M. Shiraishi [1], H. Araki [2], and H. Araki and S. Yamagami [3]. In [1], H. Araki and M. Shiraishi and in [2], H. Araki considered quasi-free states on the CCR algebras and obtained necessary and sufficient conditions of factoriality, purity, and faithfulness of a quasi-free state. In [3], H. Araki and S. Yamagami obtained necessary and sufficient conditions of quasi-equivalence of quasi-free states. We use facts presented in the this section to consider necessary and sufficient conditions of factoriality, purity, and faithfulness of a generalized coherent state and conditions of quasi-equivalence of generalized coherent states and to prove non-factoriality of quasi-free states exhibiting BEC.

Let $\tilde{K}$ be a $\mathbb{C}$-linear space and $\gamma_{\tilde{K}} : \tilde{K} \times \tilde{K} \to \mathbb{C}$ be a sesquilinear form. Let $\Gamma_{\tilde{K}}$ be an anti-linear involution ($\Gamma_{\tilde{K}}^2 = \mathbb{1}$) satisfying $\gamma_{\tilde{K}}(\Gamma_{\tilde{K}} f, \Gamma_{\tilde{K}} g) = -\gamma_{\tilde{K}}(g, f)$. A CCR algebra $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ over $(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ is the quotient of the complex $*$-algebra generated by $B(f)$, $f \in \tilde{K}$, its adjoint $B(f)^*$, $f \in \tilde{K}$ and an identity over the following relations:

1. $B(f)$ is complex linear in f,
2. $B(f)^* B(g) - B(g) B(f)^* = \gamma_{\tilde{K}}(f, g)\mathbb{1}$,
3. $B(\Gamma_{\tilde{K}} f)^* = B(f)$.

Any linear operator P on $\tilde{K}$ satisfying

1. $P^2 = P$,
2. $\gamma_{\tilde{K}}(Pf, g) > 0$ for any $g \in \tilde{K}$, if $Pf \neq 0$,
3. $\gamma_{\tilde{K}}(Pf, g) = \gamma_{\tilde{K}}(f, Pg)$,
4. $\Gamma_{\tilde{K}} P \Gamma_{\tilde{K}} = 1 - P$,

is called a basis projection.

Let $\mathfrak{h}$ be a complex pre-Hilbert space. A CCR ($*$-)algebra $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ over $\mathfrak{h}$ is the quotient of the $*$-algebra generated by $a^\dagger(f)$ and $a(f)$, $f \in \mathfrak{h}$, and an identity by the following relations:

1. $a^\dagger(f)$ is complex linear in f,
2. $(a^\dagger(f))^* = a(f)$,
3. $\left[a(f), a^\dagger(g)\right] = \langle f, g\rangle_{\mathfrak{h}} \mathbb{1}$ and $\left[a^\dagger(f), a^\dagger(g)\right] = 0 = [a(f), a(g)]$.

Let P be a basis projection. Then the mapping $\alpha(P)$ from $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ to $\mathcal{A}_{\mathrm{CCR}}(P\tilde{K})$ defined by

$$\alpha(P)(B(f_1)B(f_2)\cdots B(f_n)) = (\alpha(P)B(f_1))(\alpha(P)B(f_2))\cdots(\alpha(P)B(f_n))$$
$$\alpha(P)B(f) = a^{\dagger}(Pf) + a(P\Gamma_{\tilde{K}}f) \tag{3.1}$$

is a $*$-isomorphism of $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ onto $\mathcal{A}_{\mathrm{CCR}}(P\tilde{K})$.

Let $\mathcal{A}$ be a $*$-algebra with identity. A linear functional ω on $\mathcal{A}$ is said to be a state, if ω satisfies $\omega(A^*A) \geq 0$, $A \in \mathcal{A}$, and $\omega(\mathbb{1}) = 1$. For a state ω on $\mathcal{A}$, we have the GNS-representation space $(\mathfrak{H}_\omega, \pi_\omega, \xi_\omega)$ associated with ω. We set $\mathrm{Re}\tilde{K} := \{ f \in \tilde{K} \mid \Gamma_{\tilde{K}} f = f \}$. Then $f \in \mathrm{Re}\tilde{K}$ if and only if $B(f)^* = B(f)$.

For $f \in \mathrm{Re}\tilde{K}$, the operators $B(f)$ correspond to field operators. Moreover, $a^{\dagger}(f)$ and $a(f)$ correspond to the creation operators and the annihilation operators, respectively. We give examples of $\tilde{K}$, $\gamma_{\tilde{K}}$, and $\Gamma_{\tilde{K}}$ in Sections 4 and 5.

Let ω be a state on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ such that $\pi_\omega(B(f))$ is essentially self-adjoint for all $f \in \mathrm{Re}\tilde{K}$. Then we put $W_\omega(f) = \exp(i\pi_\omega(B(f)))$, $f \in \mathrm{Re}\tilde{K}$. Such state ω is said to be regular if $W_\omega(f)$ satisfies the Weyl–Segal relations:

$$W_\omega(f)W_\omega(g) = \exp(-\gamma_{\tilde{K}}(f,g)/2)W_\omega(f+g), \quad f,g \in \mathrm{Re}\tilde{K}. \tag{3.2}$$

A state ω on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ is said to be quasi-free, if ω satisfies the following equations:

$$\omega(B(f_1)\cdots B(f_{2n-1})) = 0,$$
$$\omega(B(f_1)\cdots B(f_{2n})) = \sum \prod_{j=1}^{n} \omega(B(f_{s(j)})B(f_{s(j+n)})), \tag{3.3}$$

where $n \in \mathbb{N}$ and the sum is over all permutations s satisfying $s(1) < s(2) < \cdots < s(n)$, $s(j) < s(j+n)$, $j = 1, 2, \cdots, n$. For any quasi-free state ω over $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$, the sesquilinear form $S_{\tilde{K}} : \tilde{K} \times \tilde{K} \to \mathbb{C}$ defined by

$$S_{\tilde{K}}(f,g) = \omega(B(f)^*B(g)), \quad f,g \in \tilde{K} \tag{3.4}$$

is positive semi-definite and satisfies

$$\gamma_{\tilde{K}}(f,g) = S_{\tilde{K}}(f,g) - S_{\tilde{K}}(\Gamma g, \Gamma f), \quad f,g \in \tilde{K}. \tag{3.5}$$

(See [1, Lemma 3.2.].) Any quasi-free state on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ determines the positive semi-definite sesquilinear form S, which satisfies the equation (3.5). Conversely, for any positive semi-definite sesquilinear form $S_{\tilde{K}}$ on $\tilde{K} \times \tilde{K}$ satisfying (3.5), there exists a unique quasi-free state ω satisfying (3.4) and ω is regular. (See [1, Lemma 3.5.].) Thus, there exists a one-to-one correspondence between a positive semi-definite sesquilinear form $S_{\tilde{K}}$ on $\tilde{K} \times \tilde{K}$ and a quasi-free state ω on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$. We denote the quasi-free state on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ determined by a positive semi-definite sesquilinear form $S_{\tilde{K}}$ by ω_S. We define a positive semi-definite form $\langle \cdot, \cdot \rangle_S$ on $\tilde{K} \times \tilde{K}$ by the following equation:

$$\langle f, g \rangle_S := S_{\tilde{K}}(f,g) + S_{\tilde{K}}(\Gamma_{\tilde{K}} g, \Gamma_{\tilde{K}} f), \quad f,g \in \tilde{K}. \tag{3.6}$$

We set $N_S := \{ f \in \tilde{K} \mid \|f\|_S = 0 \}$, where $\|f\|_S = \langle f, f \rangle_S^{1/2}$. We denote the completion of $\tilde{K}/N_S$ with respect to the norm $\|\cdot\|_S$ by K. Since $S_{\tilde{K}}(f,f) \leq \|f\|_S^2$, $\left|\gamma_{\tilde{K}}(f,f)\right| \leq \|f\|_S^2$, and $\left\|\Gamma_{\tilde{K}} f\right\|_S = \|f\|_S$ for any $f \in \tilde{K}$, we can extend the sesquilinear form $S_{\tilde{K}}$ and $\gamma_{\tilde{K}}$ to the sesquilinear form on $K \times K$ and the operator $\Gamma_{\tilde{K}}$ to the operator on K. We denote the extensions of $S_{\tilde{K}}$, $\gamma_{\tilde{K}}$, and $\Gamma_{\tilde{K}}$ by S_K, γ_K, and Γ_K, respectively. We define bounded operators S_K and γ_K on K by the following equations:

$$\langle \xi, S_K \eta \rangle_S = S_K(\xi, \eta), \tag{3.7}$$
$$\langle \xi, \gamma_K \eta \rangle_S = \gamma_K(\xi, \eta), \quad \xi, \eta \in K. \tag{3.8}$$

A quasi-free state ω_S is said to be Fock type if $N_S = \{0\}$ and the spectrum of S_K is contained in $\{0, 1/2, 1\}$. For any positive semi-definite sesquilinear form $S_{\tilde{K}}$ on $\tilde{K} \times \tilde{K}$, we can construct a Fock type state as follows. Let $\tilde{L} = K \oplus K$. For $\xi_1, \xi_2, \eta_1, \eta_2 \in K$, we set

$$\gamma_L(\xi_1 \oplus \xi_2, \eta_1 \oplus \eta_2) = \langle \xi_1, \gamma_K \eta_1 \rangle_S - \langle \xi_2, \gamma_K \eta_2 \rangle_S, \tag{3.9}$$

$$\widetilde{\Gamma_L} = \Gamma_K \oplus \Gamma_K, \tag{3.10}$$

$$\begin{aligned}\langle \xi_1 \oplus \xi_2, \eta_1 \oplus \eta_2 \rangle_L = \langle \xi_1, \eta_1 \rangle_S + \langle \xi_2, \eta_2 \rangle_S + 2\Big\langle \xi_1, S_K^{1/2}(\mathbb{1} - S_K)^{1/2}\eta_2 \Big\rangle_S \\ + 2\Big\langle \xi_2, S_K^{1/2}(\mathbb{1} - S_K)^{1/2}\eta_1 \Big\rangle_S.\end{aligned} \tag{3.11}$$

Let $N_L = \{\, \xi \in \tilde{L} \mid \langle \xi, \xi \rangle_L = 0 \,\}$. Then we denote the completion of $\tilde{L}/N_L$ with respect to the norm $\|\cdot\|_L$ by L. We define bounded operators γ_L and Π_L on L by

$$\langle \xi, \gamma_L \eta \rangle_L = \gamma_L(\xi, \eta), \quad \xi, \eta \in L, \tag{3.12}$$

$$\Pi_L = \frac{1}{2}(\mathbb{1} + \gamma_L). \tag{3.13}$$

Then the spectrum of Π_L is contained in $\{0, 1/2, 1\}$. (See [1, Lemma 5.8.] and [1, Lemma 6.1.].) Moreover the following three lemmas hold:

Lemma 3.1. [1, Corollary 6.2.] *The map $f \in \tilde{K} \mapsto [f] \in L$, where $[f] := (f \oplus 0) + N_L$, induces a $*$-homomorphism $\tau_{\tilde{K}}$ of $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ into $\mathcal{A}(L, \gamma_L, \Gamma_L)$. The restriction of a Fock type state ω_{Π_L} of $\mathcal{A}(L, \gamma_L, \Gamma_L)$ to $\tau_{\tilde{K}}(\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}}))$ gives a quasi-free state ω_S of $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ through $\omega_{\Pi_L}(\tau_{\tilde{K}}(A)) = \omega_S(A)$.*

Lemma 3.2. [2, Lemma 2.3.] *Let R_S be the von Neumann algebra generated by spectral projections of all $\pi_{\Pi_L}(B(f))$, $f \in \mathrm{Re}\tilde{K}$, on the GNS representation space $(\mathfrak{H}_{\Pi_L}, \pi_{\Pi_L}, \xi_{\Pi_L})$ of $\mathcal{A}(L, \gamma_L, \Gamma_L)$ associated with ω_{Π_L}. Then the following conditions are equivalent:*

1. *The GNS cyclic vector ξ_{Π_L} is cyclic for R_S.*
2. *The GNS cyclic vector ξ_{Π_L} is separating for R_S.*
3. *The operator S_K on K does not have an eigenvalue* 0.
4. *The operator S_K on K does not have an eigenvalue* 1.

Lemma 3.3. [2, Lemma 2.4.] *Let R_S be the von Neumann algebra defined in Lemma 3.2. The center of R_S is generated by* $\exp(i\pi_{\Pi_L}(B(h)))$, *$h \in \mathrm{Re}\overline{(E_0 K \oplus 0)}^L$, where E_0 is the spectral projection of S_K for* $1/2$ *and $\overline{(E_0 K \oplus 0)}^L$ is the closure of $E_0 K \oplus 0$ with respect to the norm $\|\cdot\|_L$. In particular, R_S is factor if and only if $K_0 = E_0 K = \{0\}$.*

3.1 Quasi-equivalence of Quasi-free States

We recall the definitions of quasi-equivalence of representations and states.

Definition 3.4. [2, Definition 6.1.] *Let π_{S_1} and π_{S_2} be representations associated with quasi-free states ω_{S_1} and ω_{S_2} on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$, respectively. The representations π_{S_1} and π_{S_2} are said to be quasi-equivalent, if there exists an isomorphism τ from R_{S_1} onto R_{S_2} such that*

$$\tau(W_{S_1}(f)) = W_{S_2}(f), \quad f \in \mathrm{Re}\tilde{K}, \tag{3.14}$$

where $R_{S_j} = \{\, W_{S_j}(f) \mid f \in \mathrm{Re}\tilde{K} \,\}''$ and $W_{S_j}(f) = \exp(i\pi_{S_j}(B(f)))$, $i = 1, 2$. Let ω_{S_1} and ω_{S_2} be quasi-free states on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$. The states ω_{S_1} and ω_{S_2} are said to be quasi-equivalent, if for each GNS-representations $(\mathfrak{H}_{S_i}, \pi_{S_i})$, $i = 1, 2$ associated with ω_{S_i}, respectively, are quasi-equivalent.

This definition is equivalent to the definition of quasi-equivalence of states on a C*-algebra. (See [6, Definition 2.4.25.] and [6, Theorem 2.4.26.].)

In [3], H. Araki and S. Yamagami showed the following theorem:

Theorem 3.5. [3, Theorem] *Two quasi-free states ω_{S_1} and ω_{S_2} on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ are quasi-equivalent if and only if the following conditions hold:*

1. *The topologies induced by $\|\cdot\|_{S_1}$ and $\|\cdot\|_{S_2}$ are equal.*
2. *Let K be the completion of $\tilde{K}$ with respect to the topology $\|\cdot\|_{S_1}$ or $\|\cdot\|_{S_2}$. Then $S_1^{1/2} - S_2^{1/2}$ is in the Hilbert–Schmidt class on K, where S_1 and S_2 are operators on K defined in* (3.7).

4 Generalized Coherent States

In this section, we consider generalized coherent states on the Weyl CCR algebras. By using facts in the previous section, we give necessary and sufficient conditions of factoriality, purity, and faithfulness of a generalized coherent state and conditions of quasi-equivalence of generalized coherent states.

Let (V, σ) be a symplectic space with an operator J on V satisfying (2.7). We define the operation $*$ on $V^{\mathbb{C}}$ by $(f + ig)^* = f - ig$ for $f, g \in V$. We put $\tilde{K} = V^{\mathbb{C}}$,

$$\begin{aligned} \Gamma_{\tilde{K}} f &= f^*, \quad f \in \tilde{K}, \\ \gamma_{\tilde{K}}(f, g) &= \frac{1}{2}\{\sigma(f, Jg) + i\sigma(f, g) - \sigma(g^*, Jf^*) - i\sigma(g^*, f^*)\}, \quad f, g \in \tilde{K}. \end{aligned} \tag{4.1}$$

Then on the GNS-representation space $(\mathfrak{H}_\omega, \pi_\omega)$ associated with a regular state ω on $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$, $\mathcal{W}(\mathrm{Re}\tilde{K}, \gamma_{\tilde{K}}) = \mathcal{W}(V, \sigma)$. Moreover, $\pi_\omega(B(f))$, $f \in \mathrm{Re}\tilde{K}$, correspond to filed operators. We define the annihilation operators $a_\omega(f)$ and the creation operators $a_\omega^\dagger(f)$ on $\mathfrak{H}_\omega$ by the following equation:

$$a_\omega(f) := \{\pi_\omega(B(f)) + i\pi_\omega(B(if))\}/\sqrt{2}, \quad a_\omega^\dagger(f) := \{\pi_\omega(B(f)) - i\pi_\omega(B(if))\}/\sqrt{2}, \tag{4.2}$$

for any $f \in \mathrm{Re}\tilde{K}$.

In this section, we identify the Weyl CCR algebra $\mathcal{W}(V, \sigma)$ with a regular state ω and $\mathcal{A}(\tilde{K}, \gamma_{\tilde{K}}, \Gamma_{\tilde{K}})$ with ω, where $\tilde{K}$, $\gamma_{\tilde{K}}$ and $\Gamma_{\tilde{K}}$ defined in (4.1).

For an $\mathbb{R}$-linear functional $q : V \to \mathbb{R}$, there exists a $*$-automorphism τ_q on $\mathcal{W}(V, \sigma)$ defined by

$$\tau_q(W(f)) := e^{iq(f)} W(f), \quad f \in V. \tag{4.3}$$

Let ω_S be a quasi-free state on $\mathcal{W}(V, \sigma)$. Then we define the generalized coherent state $\omega_{S,q}$ by the following equation:

$$\omega_{S,q}(W(f)) := \omega_S \circ \tau_q(W(f)) = e^{iq(f)} \omega_S(W(f)), \quad f \in V. \tag{4.4}$$

We set $N_S = \{ f \in V^{\mathbb{C}} \mid \|f\|_S = 0 \}$, where $\|\cdot\|_S = (\cdot, \cdot)_S^{1/2}$ is the semi-norm defined in (3.6) and $V_S^{\mathbb{C}}$ is the completion of $V^{\mathbb{C}}/N_S$ by the norm $\|\cdot\|_S$. We denote the GNS-representation space with respect to ω_S and $\omega_{S,q}$ by $(\mathfrak{H}_S, \pi_S, \xi_S)$ and $(\mathfrak{H}_{S,q}, \pi_{S,q}, \xi_{S,q})$, respectively.

Lemma 4.1. *Let ω_S and $\omega_{S,q}$ be a quasi-free state and a generalized coherent state on $\mathcal{W}(V, \sigma)$, respectively. Then*

$$R_S = R_{S,q}, \tag{4.5}$$

where R_S and $R_{S,q}$ is the von Neumann algebra generated by $\{ \pi_S(W(f)) \mid f \in V \}$ and $\{ \pi_{S,q}(W(f)) \mid f \in V \}$, respectively.

Proof. Since ω_S is regular, there exist self-adjoint operators $\Psi_S(f)$, $f \in V$ such that $\pi_S(W(f)) = \exp(i\Psi_S(f))$. By the definition of generalized coherent states, we have $\pi_{S,q}(W(f)) = e^{iq(f)}\pi_S(W(f))$ and $(\mathfrak{H}_{S,q}, \pi_{S,q}, \xi_{S,q}) = (\mathfrak{H}_S, \pi_{S,q}, \xi_S)$. On $\mathfrak{H}_S$, we obtain

$$\{\pi_S(W(f)) \mid f \in V\}'' = \{e^{iq(f)}\pi_S(W(f)) \mid f \in V\}'' = \{\pi_{S,q}(W(f)) \mid f \in V\}''. \tag{4.6}$$

Thus, $R_S = R_{S,q}$ by the double commutant theorem. ∎

Theorem 4.2. *Let $\omega_{S,q}$ be a generalized coherent state on $\mathcal{W}(V,\sigma)$. Then $\omega_{S,q}$ is faithful if and only if S does not have an eigenvalue 0 on $V_S^{\mathbb{C}}$.*

Proof. Note that both ω_S and $\omega_{S,q}$ have the same GNS cyclic vector space ξ_{Π_L}. By Lemma 3.2, $\omega_{S,q}$ is faithful if and only if S does not have an eigenvalue 0 on $V_S^{\mathbb{C}}$. ∎

Theorem 4.3. *Let $\omega_{S,q}$ be a generalized coherent state on $\mathcal{W}(V,\sigma)$. Then $\omega_{S,q}$ is factor if and only if S does not have an eigenvalue $1/2$ on $V_S^{\mathbb{C}}$.*

Proof. By Lemma 3.3 and Lemma 4.1, we have the statement. ∎

Theorem 4.4. *Let (V,σ) be a non-degenerate symplectic space and $\omega_{S,q}$ be a generalized coherent state on $\mathcal{W}(V,\sigma)$. Then $\omega_{S,q}$ is pure if and only if S is a basis projection.*

Proof. If S is a basis projection, then by Lemma 4.1 and [1, Lemma 5.5.] ω_S is pure.

We use the notation in Section 3. Thus, $\tilde{K} = V^{\mathbb{C}}$, $K = V_S^{\mathbb{C}}$, and L is the completion of $V_S^{\mathbb{C}} \oplus V_S^{\mathbb{C}}/N_L$ with respect to the norm $\|\cdot\|_L$ defined in (3.11). If $\omega_{S,q}$ is pure, then by Theorem 4.3, S does not have an eigenvalue $1/2$. Then Π_L defined in (3.13) does not have an eigenvalue $1/2$ since the eigenspace of Π_L associated with the eigenvalue $1/2$ is the completion of the set $\{f \oplus f \mid f \in E_0 K\}$ with respect to the norm $\|\cdot\|_L$, where E_0 is the spectral projection of S onto $\ker(S - 1/2)$. (See also the proof of (4) of [1, Lemma 6.1.].) Thus, Π_L is a basis projection. We have $R_S = R_{\Pi_L}(H_1)$, with $H_1 = [\mathrm{Re}\tilde{K}] \oplus 0 \subset L$ and $\overline{H_1} = \overline{\mathrm{Re}K \oplus 0}^L \oplus 0$. If $\Pi_L \neq S$, then $K \neq L$. Thus, we have $R_{\Pi_L}(H_1)' = R_{\Pi_L}(H_1^{\perp})$ by [1, Lemma 5.5.] and $H_1^{\perp} \neq \{0\}$, where $H_1^{\perp}$ is the orthogonal complement with respect to the inner product $(\cdot,\cdot)_L$ defined in (3.11). It leads $R_S' \neq \mathbb{C}\mathbb{1}$, but it contradicts to the purity of ω_S. Thus, S is a basis projection. ∎

Lemma 4.5. *Let $\omega_{S,q}$ be a generalized coherent state on $\mathcal{W}(V,\sigma)$. Then $f \in N_S$ if and only if $\pi_{S,q}(W(f)) = e^{iq(f)}\mathbb{1}$.*

Proof. If $f \in N_S$, then $\omega_S(W(tf)) = 1$ for any $t \in \mathbb{R}$. Thus, by regularity of ω_S, $\pi_S(W(f)) = \mathbb{1}$. By the definition of generalized coherent state, $\pi_{S,q}(W(f)) = e^{iq(f)}\mathbb{1}$.

If $\pi_{S,q}(W(f)) = e^{iq(f)}\mathbb{1}$, $f \in V$, then $\pi_S(W(f)) = \mathbb{1}$. Since $g^* = g$ for any $g \in V$, we have that $(f,f)_S = 0$. ∎

Lemma 4.6. *Let ω_{S_1,q_1} and ω_{S_2,q_2} be generalized coherent states on $\mathcal{W}(V,\sigma)$. If ω_{S_1,q_1} and ω_{S_2,q_2} are quasi-equivalent, then $N_{S_1} = N_{S_2}$.*

Proof. Since ω_{S_1,q_1} and ω_{S_2,q_2} are quasi-equivalent, then there exists $\tau : \pi_{S_1,q_1}(\mathcal{W}(V,\sigma))'' \to \pi_{S_2,q_2}(\mathcal{W}(V,\sigma))''$ such that

$$\tau(\pi_{S_1,q_1}(A)) = \pi_{S_2,q_2}(A), \quad A \in \mathcal{W}(V,\sigma). \tag{4.7}$$

If $N_{S_1} \neq N_{S_2}$, then there exists $f \in V^{\mathbb{C}}$ such that $f \in N_{S_1}$ and $f \notin N_{S_2}$. Put $h = f + f^*$. Then $h \in V = \mathrm{Re}V^{\mathbb{C}}$ and $h \in N_{S_1}$ and $h \notin N_{S_2}$. We have

$$\pi_{S_1,q_1}(W(h)) = e^{iq_1(h)}\mathbb{1} \tag{4.8}$$

by Lemma 4.5. However, the following equation holds:

$$\pi_{S_2,q_2}(W(h)) = e^{iq_2(h)}\pi_2(W(h)) = \tau(\pi_{S_1,q_1}(W(h))) = e^{iq_1(h)}\mathbb{1}, \tag{4.9}$$

but it contradicts to Lemma 4.5. ∎

Theorem 4.7. *Let ω_{S_1,q_1} and ω_{S_2,q_2} be generalized coherent states on $\mathcal{W}(V,\sigma)$. Then ω_{S_1,q_1} and ω_{S_2,q_2} are quasi-equivalent if and only if the following conditions hold:*

1. *$\|\cdot\|_{S_1}$ and $\|\cdot\|_{S_2}$ induce the same topology,*
2. *$S_1^{1/2} - S_2^{1/2}$ is a Hilbert–Schmidt class operator,*
3. *$q_1 = q_2$ on $N_{S_1} = N_{S_2}$,*
4. *$q_1 - q_2$ is continuous with respect to the norm $\|\cdot\|_{S_1}$ and $\|\cdot\|_{S_2}$.*

Proof. Assume that the topologies induced by $\|\cdot\|_{S_1}$ and $\|\cdot\|_{S_2}$ are equivalent, $S_1^{1/2} - S_2^{1/2}$ is Hilbert-Schmidt class, $q_1 - q_2$ is continuous with respect to $\|\cdot\|_{S_1}$, and $q_1 = q_2$ on $N_{S_1} = N_{S_2}$. Then ω_{S_1} and ω_{S_2} are quasi-equivalent by [3, Theorem] and ω_{S_1,q_1} and ω_{S_2,q_2} are quasi-equivalent by continuity of $q_1 - q_2$ and $q_1 = q_2$ on $N_{S_1} = N_{S_2}$.

Next, we assume that ω_{S_1,q_1} and ω_{S_2,q_2} are quasi-equivalent. The quasi-equivalence of ω_{S_1,q_1} and ω_{S_2,q_2} induces the quasi-equivalence of ω_{S_1,q_1-q_2} and ω_{S_2}. Put $q := q_1 - q_2$. Then there exists a $*$-isomorphism τ from $\pi_{S_1,q}(\mathcal{W}(V,\sigma))''$ onto $\pi_{S_2}(\mathcal{W}(V,\sigma))''$ such that

$$\tau(\pi_{S_1,q}(A)) = \pi_{S_2}(A), \quad A \in \mathcal{W}(V,\sigma). \tag{4.10}$$

For any $f \in V$,

$$\exp(iq(f) - S_1(f,f)/2) = \left\langle \xi_{S_1}, \tau^{-1}(\pi_{S_2}(W(f)))\xi_{S_1} \right\rangle = \left\langle \xi_{S_1}, \tau^{-1}(\pi_{S_2}(W(f)))\xi_{S_1} \right\rangle \tag{4.11}$$

is $\|\cdot\|_{S_2}$-continuous in $f \in V$. Thus, q and S_1 are $\|\cdot\|_{S_2}$-continuous. By symmetry, q and S_2 are $\|\cdot\|_{S_1}$-continuous as well. By Lemma 4.5, $N_S := N_{S_1} = N_{S_2}$. If $q \neq 0$ on N_S, then there exists $f \in N_S \setminus \{0\}$ such that $q(f) \neq 0$. If $q(f) = 2n\pi$ for some $n \in \mathbb{Z}$, then we replace f by f/π. For such f, we have

$$e^{iq(f)}\mathbb{1} = \tau(\pi_{S_1,q}(W(f))) = \pi_{S_2}(W(f)) = \mathbb{1} \tag{4.12}$$

by Lemma 4.5. It contradicts to the quasi-equivalence of $\omega_{S_1,q}$ and ω_{S_2}. Thus, $q = 0$ on N_S. Let τ' be the map from $\pi_{S_1,q}(\mathcal{W}(V,\sigma))$ to $\pi_{S_1}(\mathcal{W}(V,\sigma))$ defined by

$$\tau'(\pi_{S_1,q}(A)) = \pi_{S_1}(A), \quad A \in \mathcal{W}(V,\sigma). \tag{4.13}$$

Since q is continuous with respect to the norm $\|\cdot\|_{S_1}$ and $q = 0$ on N_S, then we can extend τ' to a map from $\pi_{S_1,q}(\mathcal{W}(V,\sigma))''$ onto $\pi_{S_1}(\mathcal{W}(V,\sigma))''$. Then τ' induces the quasi-equivalence of $\omega_{S_1,q}$ and ω_{S_1}. Thus, ω_{S_1} and ω_{S_2} are quasi-equivalent and by Theorem 3.5, we have the statement. ■

Remark 4.8. *In* [41], *S. Yamagami obtained quasi-equivalence conditions of (generalized) coherent states in terms of the transition amplitude. For applications to concrete models Hilbert-Schmidt conditions in Theorem* 4.7 *are easier to handle. Let ω_{S_1,q_1} and ω_{S_2,q_2} be generalized coherent states on the Weyl CCR algebra $\mathcal{W}(V,\sigma)$. Assume that ω_{S_1} and ω_{S_2} are quasi-equivalent. If $q_1 - q_2$ is not continuous in $\|\cdot\|_{S_1}$ or $\|\cdot\|_{S_2}$ or $q_1 \neq q_2$, then the transition amplitude $(\omega_{S_1,q_1}^{1/2}, \omega_{S_2,q_2}^{1/2}) = 0$, where $\omega_{S_1}^{1/2}$ and $\omega_{S_2,q_2}^{1/2}$ are GNS-vectors in the universal representation space $L^2(\mathcal{W}(V,\sigma)^{**})$.* (See [41, Theorem 5.3.].)

Factor decompositions of quasi-free states are given in [14, 29, 40], e.t.c. For the convenience of readers, we give an explicit form of a factor decomposition of a non-factor generalized coherent state. We recall the definition of the disjointness of states. (See also [6, Definition 4.1.20.] and [6, Lemma 4.2.8.].)

Definition 4.9. *Let ω_1 and ω_2 be positive linear functionals on a* C^**-algebra $\mathcal{A}$. The positive linear functionals ω_1 and ω_2 are said to be disjoint, if for $\omega = \omega_1 + \omega_2$, there is a projection $P \in \pi_\omega(\mathcal{A})'' \cap \pi_\omega(\mathcal{A})'$ such that*

$$\omega_1(A) = (\xi_\omega, P\pi_\omega(A)\xi_\omega), \quad \omega_2(A) = (\xi_\omega, (\mathbb{1} - P)\pi_\omega(A)\xi_\omega), \quad A \in \mathcal{A}, \tag{4.14}$$

where π_ω is the GNS-representation and ξ_ω is the GNS-cyclic vector associated with ω.

Note that factor states are either quasi-equivalent or disjoint. (See e.g. [6, Proposition 2.4.22.], [6, Theorem 2.4.26. (1)], and [6, Proposition 2.4.27.].) We denote the spectral projection of S associated with an eigenvalue $1/2$ by $E_{1/2}$.

Theorem 4.10. *Let $\omega_{S,q}$ be a generalized coherent state on $\mathcal{W}(V, \sigma)$. If $\omega_{S,q}$ is non-factor, then there exists a probability measure μ on $\mathbb{R}^{2I}$ and $\omega_{S,q}$ has a factor decomposition of the form*

$$\omega_{S,q} = \int_{\mathbb{R}^{2I}} \omega_{SE_{1/2}^\perp, x\cdot\rho+q} d\mu(x), \tag{4.15}$$

where $\omega_{SE_{1/2}^\perp, x\cdot\rho+q}(W(f)) = \exp(-S(E_{1/2}^\perp f, E_{1/2}^\perp f)/4 + ix\cdot\rho(f) + iq(f))$ and $\rho(f) = (\mathrm{Re}(e_k, f)_S, \mathrm{Im}(e_k, f)_S)_{k\in I} \in \mathbb{R}^{2I}$. Moreover, $\omega_{SE_{1/2}^\perp, x\cdot\rho+q}$ and $\omega_{SE_{1/2}^\perp, y\cdot\rho+q}$ are disjoint unless $x = y$, $x, y \in \mathbb{R}^{2I}$.

Proof. If a generalized coherent state $\omega_{S,q}$ on $\mathcal{W}(V, \sigma)$ is non-factor, then on $V_S^{\mathbb{C}}$, S has the spectral decomposition

$$Sf = SE_{1/2}^\perp f + \frac{1}{2}\sum_{k\in I}(e_k, f)_S e_k, \quad f \in V_S^{\mathbb{C}}, \tag{4.16}$$

where I is an index set such that $|I| = \dim\ker(S - 1/2)$ and $\{e_k\}_{k\in I}$ is an orthonormal basis for $\ker(S - 1/2)$. Thus, for any $W(f)$, $f \in V$, we have

$$\begin{aligned}\omega_{S,q}(W(f)) =& \exp\left(-\frac{S(E_{1/2}^\perp f, E_{1/2}^\perp f)}{4} + iq(f)\right)\exp\left(-\frac{\sum_{k\in I}|(e_k, f)_S|^2}{8}\right) \\ =& \omega_{SE_{1/2}^\perp}(W(f))\exp\left(-\frac{\sum_{k\in I}|(e_k, f)_S|^2}{8}\right)\end{aligned} \tag{4.17}$$

By a theorem of Bochner–Minlos (See e.g. [32, Theorem 2.2.]), there exists a probability measure μ on $\mathbb{R}^{2I}$ such that

$$\exp\left(-\frac{\sum_k |(e_k, f)_S|^2}{8}\right) = \int_{\mathbb{R}^{2I}} \exp(ix\cdot\rho(f))d\mu(x), \tag{4.18}$$

where $\rho(f) = (\mathrm{Re}(e_k, f)_S, \mathrm{Im}(e_k, f)_S)_{k\in I} \in \mathbb{R}^{2I}$. For $\omega_{SE_{1/2}^\perp, x\cdot\rho+q}$, we have $N_{SE_{1/2}^\perp} = E_{1/2}V^{\mathbb{C}} \neq \{0\}$. Since $E_{1/2}V^{\mathbb{C}} \neq \{0\}$, there exists $f \in V$ such that $\mathrm{Re}(e_k, f)_S \neq 0$ or $\mathrm{Im}(e_k, f)_S \neq 0$. We put $f_n := E_{1/2}f + 1/nE_{1/2}^\perp f$. Then $\|f_n\|_{SE_{1/2}^\perp} \to 0$ and $\mathrm{Re}(e_k, f_n)_S \not\to 0$ or $\mathrm{Im}(e_k, f_n)_S \not\to 0$ as $n \to \infty$. Thus, generalized coherent states $\omega_{SE_{1/2}^\perp, x\cdot\rho+q}$ and $\omega_{SE_{1/2}^\perp, y\cdot\rho+q}$, $x, y \in \mathbb{R}^{2I}$ are not quasi-equivalent unless $x = y$ by Theorem 4.7. Since $\|\cdot\|_S$ and $\|\cdot\|_{SE_{1/2}^\perp}$ induce the same topology on $V^{\mathbb{C}}$ and $SE_{1/2}^\perp$ on $V_{SE_{1/2}^\perp}^{\mathbb{C}}$ does not have an eigenvalue $1/2$, $\omega_{SE_{1/2}^\perp, x\cdot\rho+q}$ is factor and $\omega_{SE_{1/2}^\perp, x\cdot\rho+q}$ and $\omega_{SE_{1/2}^\perp, y\cdot\rho+q}$ are disjoint unless $x = y$, $x, y \in \mathbb{R}^{2I}$. ∎

5 BEC and Non-factor States

In this section, we consider quasi-free states on $\mathcal{W}(\mathfrak{h},\sigma)$, where $\mathfrak{h}$ is a pre-Hilbert space over $\mathbb{C}$ with an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ and $\sigma(f,g) = \mathrm{Im}\langle f,g\rangle_{\mathfrak{h}}$, $f,g \in \mathfrak{h}$. We give an explicit decomposition of quasi-free states on $\mathcal{W}(\mathfrak{h},\sigma)$ into generalized coherent states which are mutually disjoint. In the case of 2-dimensional Ferromagnetic Ising models at low temperatures, Gibbs measures with free boundary conditions (mixed phase) are decomposed into states with ± boundary conditions (pure phase). We consider similar decompositions for generalized coherent states and factoriality of states is a non-commutative analogue of decomposition to pure phase [6].

5.1 General Properties

In this subsection, we use the following notations. Let $\mathfrak{h}$ be a subspace of a Hilbert space over $\mathbb{C}$. We assume that $\mathfrak{h}$ is equipped with positive definite inner products $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ and $\langle\cdot,\cdot\rangle_0$. Let q be a linear functional on $\mathfrak{h}$. We consider the quasi-free state $\omega_{q,D}$, $D \geq 0$, on $\mathcal{W}(\mathfrak{h},\sigma)$ defined by

$$\omega_{q,D}(a^\dagger(f)a(g)) = \langle g,f\rangle_0 + D\overline{q(g)}q(f), \tag{5.1}$$

where $a(f)$ and $a^\dagger(f)$, $f \in \mathfrak{h}$, are the annihilation operators and the creation operators on the GNS representation space $\mathfrak{H}_{\omega_{q,D}}$, respectively. Note that the annihilation operators $a(f)$ and the creation operators $a^\dagger(f)$ satisfy the following equation:

$$\left[a(f),a^\dagger(g)\right] = \langle f,g\rangle_{\mathfrak{h}}, \quad [a(f),a(g)] = 0 = \left[a^\dagger(f),a^\dagger(g)\right], \quad f,g \in \mathfrak{h}. \tag{5.2}$$

Our aim is to show that if q is not continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$ defined in (5.9) and $D > 0$, then $\omega_{q,D}$ is non-factor and to get a factor decomposition of $\omega_{q,D}$. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis on a Hilbert space which is contained in $\mathfrak{h}$. Fix $\{e_n\}_{n\in\mathbb{N}}$. We set

$$\overline{f} = \sum_{n\in\mathbb{N}} \overline{f_n} e_n \tag{5.3}$$

for $f = \sum_{n\in\mathbb{N}} f_n e_n \in \mathfrak{h}$, where $f_n \in \mathbb{C}$, $n \in \mathbb{N}$ and $\overline{f_n}$ is the complex conjugate of f_n. For a linear functional q and $D \geq 0$, we put $\tilde{K}_{q,D} = \mathfrak{h} \oplus \mathfrak{h}$. For $f_1, f_2, g_1, g_2 \in \mathfrak{h}$, we sets

$$\gamma_D(f_1 \oplus f_2, g_1 \oplus g_2) = \frac{1}{2}\Big(\langle f_1,g_1\rangle_{\mathfrak{h}} - \langle f_2,g_2\rangle_{\mathfrak{h}}\Big), \tag{5.4}$$

$$\Gamma(f_1 \oplus f_2) = \overline{f_2} \oplus \overline{f_1}, \tag{5.5}$$

$$B(f_1 \oplus f_2) = \frac{1}{\sqrt{2}}\Big(a^\dagger(f_1) + a(\overline{f_2})\Big), \tag{5.6}$$

$$\begin{aligned} S_{q,D}(f_1 \oplus f_2, g_1 \oplus g_2) &= \omega_{q,D}(B(f_1 \oplus f_2)^* B(g_1 \oplus g_2)) \\ &= \frac{1}{2}\langle f_1,g_1\rangle_{\mathfrak{h}} + \frac{1}{2}\langle f_1,g_1\rangle_0 + \frac{1}{2}\langle f_2,g_2\rangle_0 + \frac{D}{2}\overline{q(f_1)}q(g_1) + \frac{D}{2}\overline{q(f_2)}q(g_2). \end{aligned} \tag{5.7}$$

We define an inner product on $\tilde{K}_{q,D}$ by

$$\langle f_1 \oplus f_2, g_1 \oplus g_2\rangle_{q,D} = \frac{1}{2}\langle f_1,g_1\rangle_{\mathfrak{h}} + \frac{1}{2}\langle f_2,g_2\rangle_{\mathfrak{h}} + \langle f_1,g_1\rangle_0 + \langle f_2,g_2\rangle_0 + D\overline{q(f_1)}q(g_1) + D\overline{q(f_2)}q(g_2). \tag{5.8}$$

Let $N_{K_{q,D}} = \{\, f \in \tilde{K}_{q,D} \mid \|f\|_{q,D} = 0 \,\}$. Then we denote the completion of $\tilde{K}_{q,D}/N_{K_{q,D}}$ with respect to the norm $\|\cdot\|_{q,D}$ by $K_{q,D}$. In this case, $\|f_1 \oplus f_2\|_{q,D} = 0$ leads $f_1 = 0$ and $f_2 = 0$. Thus, $N_{K_{q,D}} = \{0\}$.

We put

$$\langle f, g\rangle_{\mathfrak{K}} = \frac{1}{2}\langle f, g\rangle_{\mathfrak{h}} + \langle f, g\rangle_0, \quad f, g \in \mathfrak{h}, \tag{5.9}$$

and $\|\cdot\|_{\mathfrak{K}} = \langle\cdot,\cdot\rangle_{\mathfrak{K}}^{1/2}$. We define the Hilbert space $\mathfrak{K}$ by the completion of $\mathfrak{h}$ with respect to the norm $\|\cdot\|_{\mathfrak{K}}$.

Lemma 5.1. *The space $K_{q,D}$ has the following form:*

1. *If $D > 0$ and q is not continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$, then we have*

$$K_{q,D} = \mathbb{C} \oplus \mathfrak{K} \oplus \mathbb{C} \oplus \mathfrak{K}. \tag{5.10}$$

2. *If $D = 0$ or q is continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$, then we have*

$$K_{q,D} = \mathfrak{K} \oplus \mathfrak{K}. \tag{5.11}$$

Proof. We consider the case of $D > 0$ and q is not continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$. It suffices to show that $\mathbb{C} \oplus \mathfrak{K} = \bar{\mathfrak{h}}$, where $\bar{\mathfrak{h}}$ is the completion of $\mathfrak{h}$ with respect to the norm $\|\cdot\|'$ defined by

$$(\|f\|')^2 = \|f\|_{\mathfrak{K}}^2 + D|q(f)|^2, \quad f \in \mathfrak{h}. \tag{5.12}$$

We define $\pi : \mathfrak{h} \to \mathbb{C} \oplus \mathfrak{K}$ by

$$\pi(f) = q(f) \oplus f. \tag{5.13}$$

Since q is not continuous, for any $f \in \mathfrak{h}$, there exists a sequence f_n in $\mathfrak{h}$ such that $\lim_{n\to\infty} \|f_n - f\|_{\mathfrak{K}} = 0$ and $\lim_{n\to\infty} q(f_n) = 0$. For such f_n and f, we have

$$\pi(f_n - f) \to q(f) \oplus 0, \quad \pi(f_n) \to 0 \oplus f. \tag{5.14}$$

If $D = 0$, then $\|f\|' = \|f\|_{\mathfrak{K}}$ for any $f \in \mathfrak{h}$. We assume that $D > 0$ and q is continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$. By continuity of q, the norms $\|\cdot\|'$ and $\|\cdot\|_{\mathfrak{K}}$ induce the same topology. ■

When a quasi-free state is non-factor, the spontaneous $U(1)$ symmetry breaking occurs. The next theorem corresponds to non-factoriality of quasi-free states with BEC.

Theorem 5.2. *Assume that a linear space $\mathfrak{h}$ has positive definite inner products $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ and $\langle\cdot,\cdot\rangle_0$. If $D > 0$ and q is not continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$, then the two-point function $\omega_{q,D}$ defined in* (5.1) *is a non-factor state on $\mathcal{W}(\mathfrak{h}, \sigma)$.*

Proof. By Lemmas 3.1 and 3.2, it suffices to show that $1/2 \in \sigma_P(S_{q,D})$. By Lemma 5.1, an element of $K_{q,D}$ has the form (a_1, f_1, a_2, f_2), $a_1, a_2 \in \mathbb{C}$, $f_1, f_2 \in \mathfrak{K}$. For any $(a_1 \oplus f_1 \oplus a_2 \oplus f_2), (b \oplus 0 \oplus 0 \oplus 0) \in K_{q,D}, b \in \mathbb{C}$, the operator $S_{q,D}$ satisfies

$$\Big\langle (a_1 \oplus f_1 \oplus a_2 \oplus f_2), S_{q,D}(b_1 \oplus 0 \oplus 0 \oplus 0) \Big\rangle_{q,D} = \frac{D}{2}\overline{a_1} b = \frac{1}{2}\langle (a_1 \oplus f_1 \oplus a_2 \oplus f_2), (b \oplus 0 \oplus 0 \oplus 0)\rangle_{q,D}. \tag{5.15}$$

Thus, we have $S_{q,D}(b \oplus 0 \oplus 0 \oplus 0) = 1/2(b \oplus 0 \oplus 0 \oplus 0)$ for any $b \in \mathbb{C}$ and $1/2 \in \sigma_P(S_{q,D})$. ■

Proposition 5.3. *For a linear space $\mathfrak{h}$ with positive definite inner products $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ and $\langle\cdot,\cdot\rangle_0$, if $D = 0$ or q is continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$, the two-point function $\omega_{q,D}$ defined in* (5.1) *is a factor state on $\mathcal{W}(\mathfrak{h}, \sigma)$.*

Proof. If q is continuous with respect to the norm $\|\cdot\|_{\mathfrak{K}}$, then $\omega_{q,D}$ is quasi-equivalent to $\omega_{0,0}$ by Theorem 4.7. Thus, it suffice to show the case of $D = 0$. There exists the positive contraction operator A on $\mathfrak{K}$ such that $\langle \xi, A\eta \rangle_{\mathfrak{K}} = \langle \xi, \eta \rangle_{\mathfrak{h}}/2$ and $\langle \xi, (\mathbb{1} - A)\eta \rangle_{\mathfrak{K}} = \langle \xi, \eta \rangle_0$, $\xi, \eta \in \mathfrak{K}$. Then $S_{0,0}$ has the following form:

$$S_{0,0}(\eta_1 \oplus \eta_2) = (A + (\mathbb{1} - A)/2)\eta_1 \oplus \frac{\mathbb{1} - A}{2}\eta_2 = \frac{\mathbb{1} + A}{2}\eta_1 \oplus \frac{\mathbb{1} - A}{2}\eta_2, \tag{5.16}$$

for $\eta_1, \eta_2 \in \mathfrak{K}$. If $1/2 \in \sigma_P(S_{0,0})$, then $(\mathbb{1} + A)\eta_1 = \eta_1$ and $(\mathbb{1} - A)\eta_2 = \eta_2$. Thus, $\eta_1, \eta_2 \in \ker A$. Since the positive definiteness of $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ and $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{h}$, $\mathfrak{h} \cap \ker A = \{0\}$. Thus, $\ker A = \{0\}$ and $\omega_{0,0}$ is factor. ■

Next, we consider a factor decomposition of $\omega_{q,D}$, if q is not continuous in $\|\cdot\|_{\mathfrak{K}}$. Let $(\mathfrak{H}_0, \pi_0, \xi_0)$ be the GNS-representation space with respect to $\omega_0 := \omega_{q,0} = \omega_{0,D}$. Since ω_0 is regular state on $\mathcal{W}(\mathfrak{h}, \sigma)$, there exist self-adjoint operators $\Psi_0(f)$, $f \in \mathfrak{h}$, such that

$$\pi_0(W(f)) = \exp(i\Psi_0(f)). \tag{5.17}$$

Now we define the field operators $\Psi_{s_1,s_2}(f)$, $s_1, s_2 \in \mathbb{R}$, $f \in \mathfrak{h}$, on $\mathfrak{H}_0$ by

$$\Psi_{s_1,s_2}(f) = \Psi_0(f) + s_1 D^{1/2}\mathrm{Re}q(f)\mathbb{1} + s_2 D^{1/2}\mathrm{Im}q(f)\mathbb{1}, \quad f \in \mathfrak{h}. \tag{5.18}$$

Let π_{s_1,s_2} be the representation of $\mathcal{W}(\mathfrak{h}, \sigma)$ on $\mathfrak{H}_0$ defined by

$$\pi_{s_1,s_2}(W(f)) = \exp(i\Psi_{s_1,s_2}(f)), \quad f \in \mathfrak{h}. \tag{5.19}$$

Using π_{s_1,s_2}, we define the state ϕ_{s_1,s_2} on $\mathcal{W}(\mathfrak{h}, \sigma)$ by

$$\phi_{s_1,s_2}(A) = \langle \xi_0, \pi_{s_1,s_2}(A)\xi_0 \rangle, \quad A \in \mathcal{W}(\mathfrak{h}, \sigma). \tag{5.20}$$

Then we have the following theorem.

Theorem 5.4. *If q is not continuous in $\|\cdot\|_{\mathfrak{K}}$, then for each $s_1, s_2, t_1, t_2 \in \mathbb{R}$, ϕ_{s_1,s_2} and ϕ_{t_1,t_2} are factor and disjoint unless $t_1 = s_1$ and $t_2 = s_2$.*

Proof. By Lemma 4.1 and Proposition 5.3, ϕ_{s_1,s_2} and ϕ_{t_1,t_2} are factor. Since q is not continuous with respect to the norm, ϕ_{s_1,s_2} and ϕ_{t_1,t_2} are disjoint unless $t_1 = s_1$ and $t_2 = s_2$ by Theorem 4.7. ■

Finally, we obtain a factor decomposition of $\omega_{q,D}$.

Theorem 5.5. *If q is not continuous in $\|\cdot\|_{\mathfrak{K}}$, then for any $D > 0$, a factor decomposition of $\omega_{q,D}$ defined in* (5.1) *is given by*

$$\omega_{q,D} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \phi_{s_1,s_2} e^{-\frac{s_1^2+s_2^2}{2}} ds_1 ds_2. \tag{5.21}$$

Proof. By Theorem 4.10, we are done. ■

5.2 On Graphs

In [13], F. Fidaleo considered BEC on graphs and showed the following two results.

Proposition 5.6. [13, Proposition 4.1.] *Let G be an undirected graph. Let A_G be the adjacency operator of G on $\ell^2(VG)$ and h be the Hamiltonian defined by $h = \|A_G\|\mathbb{1} - A_G$. Let $\mathfrak{h}$ be a subspace of $\ell^2(VG)$ satisfying the following three conditions: For each $\beta > 0$,*

1. $e^{ith}\mathfrak{h} = \mathfrak{h}$, $t \in \mathbb{R}$;
2. *For each entire function f, $f(h)\mathfrak{h} \subset \mathcal{D}((e^{\beta h} - 1)^{-1/2})$;*

3. $\sum_{x\in VG} |(f(h)u)(x)|v(x) < \infty$, *and* $\langle f(h)u, v\rangle = \overline{f(0)}\langle u, v\rangle$, *where v is a PF weight for A_G.*

Then for $D \geq 0$, the two–point function

$$\omega_D(a^*(f_1)a(f_2)) = \left\langle (e^{\beta h} - \mathbb{1})^{-1} f_2, f_1\right\rangle_{\ell^2} + D\langle f_2, v\rangle\langle v, f_1\rangle \tag{5.22}$$

satisfies the KMS condition at inverse temperature $\beta > 0$ on the Weyl CCR algebra $\mathcal{W}(\mathfrak{h}, \sigma)$ with respect to the dynamics generated by the Bogoliubov transformations

$$\mathfrak{h} \ni f \mapsto e^{ith} f, \quad t \in \mathbb{R}. \tag{5.23}$$

By the above proposition and [26, Proposition 1.1.], we say BEC occur in the case $D > 0$ and BEC does not occur in the case $D = 0$.

Theorem 5.7. [13, Theorem 4.5.] *Suppose that A_G is transient. Let $\mathfrak{h}_1$ be the subspace of $\ell^2(VG)$ defined by*

$$\mathfrak{h}_1 = \{\, e^{ith}\delta_x \mid t \in \mathbb{R}, x \in VG \,\}. \tag{5.24}$$

Then $\mathfrak{h}_1$ satisfies the conditions 1, 2, *and* 3 *in Proposition* 5.6. *Thus, for $\mathfrak{h}_1$ and any $D \geq 0$, the two-point function given in* (5.22) *defines KMS state on the Weyl CCR algebra $\mathcal{W}(\mathfrak{h}_1, \sigma)$.*

Let $\mathcal{P}(\mathbb{R})$ be the set of all $\mathbb{C}$-coefficient polynomials on $\mathbb{R}$. Let $\mathfrak{h}_2$ be the subspace defined by

$$\mathfrak{h}_2 = \left\{ \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{ith}\delta_x dt \;\middle|\; p \in \mathcal{P}(\mathbb{R}), a \in \mathbb{R}, b > 0, x \in VG \right\}. \tag{5.25}$$

Lemma 5.8. *The space $\mathfrak{h}_2$ satisfies the following conditions;*

1′. $e^{ith}\mathfrak{h}_2 = \mathfrak{h}_2$, $t \in \mathbb{R}$;

2′. $e^{\beta h}\mathfrak{h}_2 \subset \mathcal{D}((e^{\beta h} - 1)^{-1/2})$;

3′. $\sum_{x\in VG} \left|(e^{\beta h}u)(x)\right| < \infty$, *and* $\left\langle e^{\beta h}u, v\right\rangle = \langle u, v\rangle$, $u \in \mathfrak{h}_2$.

Proof. First, we consider the condition 1′. For a generator of $\mathfrak{h}_2$, we see that

$$e^{ish} \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{ith}\delta_x dt = \int_{\mathbb{R}} p(t-s)e^{-(t-s-a)^2/b} e^{ith}\delta_x dt. \tag{5.26}$$

Thus, we have $e^{ish}\mathfrak{h}_2 \subset \mathfrak{h}_2$. Moreover, for any $p \in \mathcal{P}(\mathbb{C})$ and $a, s \in \mathbb{R}$, we put $p'(t) = p(t-s)$ and $a' = a + s$. Then we obtain

$$\mathfrak{h}_2 \ni \int_{\mathbb{R}} p'(t)e^{-(t-a')/b} e^{ith}\delta_x dt = e^{ish} \int_{\mathbb{R}} p(t)e^{-(t-a)^2/b} e^{ith}\delta_x dt \tag{5.27}$$

and $\mathfrak{h}_2 \subset e^{ish}\mathfrak{h}_2$.

Now we prove the condition 2′, $e^{\beta h}\mathfrak{h}_2 \subset \mathcal{D}((e^{\beta h} - \mathbb{1})^{-1/2})$. Note that $(e^{\beta x} - 1)^{-1} - (\beta x)^{-1}$ is continuous on $[0, \infty)$. Thus, it is enough to show that $e^{\beta h}\mathfrak{h}_2 \subset \mathcal{D}(h^{-1/2})$. Since A_G is transient and $p(t)e^{-(t-a)^2/b}$ is a rapidly decreasing function on $\mathbb{R}$, for a generator of $\mathfrak{h}_2$, $\int_{\mathbb{R}} p(t)e^{-\frac{(t-a)^2}{b}} e^{ith}\delta_x dt$, we have

$$\begin{aligned}
&\left\langle (\lambda\mathbb{1} - A_G)^{-1} e^{\beta h} \int_{\mathbb{R}} p(t)e^{-\frac{(t-a)^2}{b}} e^{ith}\delta_x dt, e^{\beta h} \int_{\mathbb{R}} p(t)e^{-\frac{(t-a)^2}{b}} e^{ith}\delta_x dt \right\rangle \\
=&\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{p(t)}p(s)e^{-\frac{(t-a)^2}{b}} e^{-\frac{(s-a)^2}{b}} \left\langle (\lambda\mathbb{1} - A_G)^{-1} e^{\beta h} e^{ith}\delta_x, e^{\beta h} e^{ish}\delta_x \right\rangle dt ds \right| \\
=&\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{p(t)}p(s)e^{-\frac{(t-a)^2}{b}} e^{-\frac{(s-a)^2}{b}} \int_{\sigma(A_G)} \frac{e^{i(s-t)a} e^{2\beta(\|A_G\|\mathbb{1} - a)}}{\lambda - a} d\langle \delta_x, E(a)\delta_x\rangle dt ds \right| \\
\leq& C_1 e^{4\beta\|A_G\|} \left\langle (\lambda\mathbb{1} - A_G)^{-1}\delta_x, \delta_x \right\rangle \nearrow C_1 e^{4\beta\|A_G\|} \left\langle (\|A_G\|\mathbb{1} - A_G)^{-1}\delta_x, \delta_x \right\rangle < \infty,
\end{aligned} \tag{5.28}$$

where C_1 is a positive constant satisfying

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\overline{p(t)}p(s)e^{-\frac{(t-a)^2}{b}}e^{-\frac{(s-a)^2}{b}}\right|dtds < C_1. \tag{5.29}$$

Next, we show that $\sup_{n\in\mathbb{N}}\sum_{x\in VG_n}\left|(e^{\beta h}u)(x)\right|v(x) < \infty$, $u \in \mathfrak{h}_2$, where G_n is a subsequence of a finite subgraphs of G such that $G_n \nearrow G$. Let C_R be a circle centered at the origin with radius $R > \|A_G\|$. We have

$$\begin{aligned}
&\left|\left\langle e^{\beta h}\int_{\mathbb{R}}p(t)e^{-\frac{(t-a)^2}{b}}e^{ith}\delta_x dt, \delta_y\right\rangle\right| \le \int_{\mathbb{R}}|p(t)|e^{-\frac{(t-a)^2}{b}}\left|\left\langle e^{\beta h}e^{ith}\delta_x, \delta_y\right\rangle\right|dt \\
&= \int_{\mathbb{R}}|p(t)|e^{-\frac{(t-a)^2}{b}}\left|\frac{1}{2\pi i}\oint_{C_R}e^{\beta z}e^{itz}\left\langle(z\mathbb{1}-A_G)^{-1}\delta_x, \delta_y\right\rangle dz\right|dt \\
&\le Re^{\beta R}\int_{\mathbb{R}}|p(t)|e^{-\frac{(t-a)^2}{b}}e^{tR}dt\left\langle(R\mathbb{1}-A_G)^{-1}\delta_x, \delta_y\right\rangle \le C_2\left\langle(R\mathbb{1}-A_G)^{-1}\delta_x, \delta_y\right\rangle,
\end{aligned} \tag{5.30}$$

for any $x, y \in VG$, where C_2 is a positive constant satisfying

$$Re^{\beta R}\int_{\mathbb{R}}|p(t)|e^{-\frac{(t-a)^2}{b}}e^{tR}dt < C_2. \tag{5.31}$$

By (5.30), we get

$$\begin{aligned}
&\sum_{y\in VG_n}\left|\left\langle e^{\beta h}\int_{\mathbb{R}}p(t)e^{-\frac{(t-a)^2}{b}}e^{ith}\delta_x dt, \delta_y\right\rangle\right|v(y) \le C_2\sum_{y\in VG_n}\left\langle(R\mathbb{1}-A_G)^{-1}\delta_x, \delta_y\right\rangle v(y) \\
&=C_2\left\langle(R\mathbb{1}-A_G)^{-1}\delta_x, v\restriction_{VG_n}\right\rangle = C_2\sum_{k=0}^{\infty}\frac{\left\langle A_G^k\delta_x, v\restriction_{V\Lambda_n}\right\rangle}{R^{k+1}} \le C_2(R-\|A_G\|)^{-1}v(x).
\end{aligned} \tag{5.32}$$

Finally, we show the second part of the condition 3′. For any $f \in \mathfrak{h}_2$, by definition of v,

$$\left\langle e^{\beta h}f, v\right\rangle = \langle f, v\rangle. \tag{5.33}$$

Thus, the proof is complete. ■

At the end of this part, we have that a quasi-free state with BEC is non-factor and an explicit decomposition of quasi-free states with BEC. An explicit decomposition of mixed states with BEC into factor states (pure phases) is described as follows.

Theorem 5.9. *Suppose that the adjacency operator A_G of a graph G is transient. For $D > 0$, the two-point function ω_D defined in* (5.22) *is a non-factor KMS state on $\mathcal{W}(\mathfrak{h}_1, \sigma)$ and $\mathcal{W}(\mathfrak{h}_2, \sigma)$. Moreover, we have a factor decomposition of ω_D into extremal KMS states*

$$\omega_D = \frac{1}{2\pi}\int_{\mathbb{R}^2}\phi_{s_1,s_2}e^{-\frac{s_1^2+s_2^2}{2}}ds_1ds_2. \tag{5.34}$$

Proof. Since $\left\langle\cdot, (e^{\beta h}+\mathbb{1})(e^{\beta h}-\mathbb{1})^{-1}\cdot\right\rangle$ is a positive definite inner product on $\mathfrak{h}_1$ and $\mathfrak{h}_2$, it suffice to show that $\langle v, f\rangle$, $f \in \mathfrak{h}_1$ or $f \in \mathfrak{h}_2$ is not continuous with respect to the norm $\left\langle\cdot, (e^{\beta h}+\mathbb{1})(e^{\beta h}-\mathbb{1})^{-1}\cdot\right\rangle$ by Theorems 5.4 and 5.5. Let p_n be the polynomial defined by

$$p_n(x) = \sum_{k=0}^{n}\frac{(-nx)^k}{k!}. \tag{5.35}$$

For any $f \in \mathfrak{h}_1$, $(\mathbb{1} - p_n(h))f \in \mathfrak{h}_1$. Put $f_n = (\mathbb{1} - p_n(h))f$. Then

$$\left\langle f_n - f, (e^{\beta h} + \mathbb{1})(e^{\beta h} - \mathbb{1})^{-1}(f_n - f)\right\rangle \to 0, \quad (n \to \infty) \tag{5.36}$$

and

$$\langle v, f_n \rangle = 0 \tag{5.37}$$

for any $n \in \mathbb{N}$. Thus, we see that $\langle v, \cdot \rangle$ is not continuous.

For any $f \in \mathfrak{h}_2$, we put $f_n = (\mathbb{1} - p_n(h))f$. Then for any $f \in \mathfrak{h}_2$, we have $f_n \in \mathfrak{h}_2$, (5.36), and (5.37) as well. ∎

Part II
A Model of Josephson Junctions on Boson Systems

In this part, we consider a model of Josephson junctions on Boson systems. We give an explicit formula of the coupled time evolutions and NESS in the sense of D. Ruelle [31]. By using these formula, we obtain explicit formulas of currents and entropy production rates. We introduce typical examples of a model: $\mathbb{R}^d$, $\mathbb{Z}^d$, and Comb graphs.

6 Time Evolutions

In this section, we give an explicit formula of the coupled time evolution. The model is defined on the Boson–Fock space $\mathcal{F}_+(\mathcal{K})$ over the Hilbert space $\mathcal{K} := \mathbb{C} \oplus (\bigoplus_{k=1}^N \mathfrak{K}_k)$ equipped with the inner product

$$\left\langle \begin{pmatrix} c^{(1)} \\ \psi_1^{(1)} \\ \vdots \\ \psi_N^{(1)} \end{pmatrix}, \begin{pmatrix} c^{(2)} \\ \psi_1^{(2)} \\ \vdots \\ \psi_N^{(2)} \end{pmatrix} \right\rangle = \overline{c^{(1)}} c^{(2)} + \sum_{k=1}^N \left\langle \psi_k^{(1)}, \psi_k^{(2)} \right\rangle_k, \tag{6.1}$$

where $c^{(1)}, c^{(2)} \in \mathbb{C}$, for each $k = 1, \ldots, N$, $\mathfrak{K}_k$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_k$, and $\psi_k^{(1)}, \psi_k^{(2)} \in \mathfrak{K}_k$. The free Hamiltonian H_0 on $\mathcal{F}_+(\mathcal{K})$ is given by $H_0 = d\Gamma(h_0)$, where $d\Gamma$ is the second quantization (see e.g. [7, Section 5.2]), h_0 is the positive self-adjoint operator on $\mathcal{K}$ defined by

$$h_0 \begin{pmatrix} c \\ \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} = \begin{pmatrix} \Omega c \\ h_{0,1}\psi_1 \\ \vdots \\ h_{0,N}\psi_N \end{pmatrix}, \tag{6.2}$$

$\Omega > 0$, $c \in \mathbb{C}$, $h_{0,k}$ is the positive one-particle Hamiltonian on each reservoirs, $k = 1, \ldots, N$, and ψ_k is a vector in the domain of $h_{0,k}$. The Hamiltonian H of our coupled model is given by $H = d\Gamma(h)$, where h is the self-adjoint operator on $\mathcal{K}$ defined by

$$h \begin{pmatrix} c \\ \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} = \begin{pmatrix} \Omega c + \lambda \sum_{k=1}^N \langle g_k, \psi_k \rangle \\ h_{0,1}\psi_1 + \lambda c g_1 \\ \vdots \\ h_{0,N}\psi_N + \lambda c g_N \end{pmatrix} =: (h_0 + \lambda V) \begin{pmatrix} c \\ \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{6.3}$$

$\lambda > 0$, and $g_k \in \mathfrak{K}_k$, $k = 1, \ldots, N$. On the Weyl CCR algebra $\mathcal{W}(\mathcal{K})$, the map α_t, $t \in \mathbb{R}$, defined by

$$\alpha_t(W(f)) = e^{itd\Gamma(h)} W(f) e^{-itd\Gamma(h)} = W(e^{ith} f), \quad f \in \mathcal{K}, \tag{6.4}$$

is a one-parameter group of automorphisms on $\mathcal{W}(\mathcal{K})$.

For simplicity, we denote vectors ${}^t(\psi_1, \ldots, \psi_N)$, ${}^t(g_1, \ldots, g_N)$, and the self-adjoint operator $\bigoplus_{k=1}^N h_{0,k}$ by ψ, g, and $h_{0,0}$, respectively.

To obtain an explicit formula of the coupled time evolution, we need some conditions.

(Abs) For $k = 1, \dots, N$, a pair (ψ, ξ) of vectors $\psi, \xi \in \mathfrak{K}_k$ satisfies

$$\sup_{\nu\in\mathbb{R},\varepsilon>0} \left| \left\langle \psi, (\nu - h_{0,k} \pm i\varepsilon)^{-1}\xi \right\rangle \right| < \infty. \tag{6.5}$$

For simplicity, if vectors $\psi_k, \xi_k \in \mathfrak{K}_k$, $k = 1, \dots, N$, satisfy condition (Abs), then we say that (ψ, ξ) has condition (Abs).

(A) The form factor g defined in (6.3) satisfies condition (Abs), i.e., (g, g) has condition (Abs).

(B) We define the function $\eta(z)$ by

$$\eta(z) := z - \Omega - \lambda^2 \int_{\sigma_0} \frac{1}{z-\nu} d\langle g, E_0(\nu) g\rangle, \tag{6.6}$$

where E_0 is the spectral measure of $h_{0,0}$ and σ_0 is the spectrum of $h_{0,0}$. Then $1/\eta_+ \in L^\infty(\mathbb{R})$, where $\eta_+(x) = \lim_{\varepsilon\searrow 0} \eta(x + i\varepsilon)$.

Remark 6.1. *By condition* (A)*, there exists a constant $C_g > 0$ such that*

$$\sup_{\nu\in\mathbb{R},\varepsilon>0} \left| \left\langle g, (\nu - h_{0,0} \pm i\varepsilon)^{-1} g \right\rangle \right| < C_g. \tag{6.7}$$

If $\Omega \in \sigma_0$, λ is sufficiently small, and there exists a constant $C > 0$ such that

$$\frac{d\langle g, E_0(\nu) g\rangle}{d\nu} > C \tag{6.8}$$

for a.e. $\nu \in [\Omega - 2\lambda^2 C_g, \Omega + 2\lambda^2 C_g]$, then the function η satisfies condition (B).

We define the sets $\mathfrak{h}_k(g_k)$ and $\mathfrak{h}(g)$ by

$$\mathfrak{h}_k(g_k) = \{ \psi \in \mathfrak{K}_k \mid (\psi, g_k) \text{ has condition (Abs)} \}, \quad \mathfrak{h}(g) = \left\{ {}^t(\psi_1, \dots, \psi_N) \,\middle|\, \psi_k \in \mathfrak{h}_k(g_k) \right\}. \tag{6.9}$$

For any $c \in \mathbb{C}$ and any $\psi \in \mathfrak{h}(g)$, we put $f = {}^t(c, \psi)$,

$$F(\nu; f) := c + \lambda \left\langle g, (\nu - h_{0,0} - i0)^{-1}\psi \right\rangle \left(= c + \lambda \lim_{\varepsilon\searrow 0} \left\langle g, (\nu - h_{0,0} - i\varepsilon)^{-1}\psi \right\rangle \right), \quad \text{a.e. } \nu \in \mathbb{R},$$

$$\varphi_l(f) := \psi_l + \lambda \frac{F(h_{0,0}; f)}{\eta_-(h_{0,0})} g_l, \quad \varphi(f) := \psi + \lambda \frac{F(h_{0,0}; f)}{\eta_-(h_{0,0})} g.$$

Let $\mathfrak{H}$ be a Hilbert space. For any $\xi, \zeta, \psi \in \mathfrak{H}$, we set

$$(\xi \otimes \zeta)\psi = \langle \zeta, \psi\rangle \xi, \tag{6.10}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathfrak{H}$.

Proposition 6.2. *Let h_0 and h be the operators defined in* (6.2) *and* (6.3)*. Then we have*

$$(z - h)^{-1} = (z - h_0)^{-1} + B(z)(z - h_0)^{-1} \tag{6.11}$$

for $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$*, where*

$$\begin{aligned} B(z) =& \lambda^2 \frac{\left\langle g, (z - h_{0,0})^{-1} g \right\rangle}{\eta(z)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \frac{z-\Omega}{\eta(z)} \begin{pmatrix} 0 \\ (z - h_{0,0})^{-1} g \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ \frac{\lambda}{\eta(z)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ g \end{pmatrix} + \frac{\lambda^2}{\eta(z)} \begin{pmatrix} 0 \\ (z - h_{0,0})^{-1} g \end{pmatrix} \otimes \begin{pmatrix} 0 \\ g \end{pmatrix} \end{aligned} \tag{6.12}$$

and the function $\eta(z)$ is defined in (6.6)*.*

Proof. By resolvent formula, we have

$$(z-h)^{-1} = (z-h_0)^{-1} + B(z)(z-h_0)^{-1}. \tag{6.13}$$

Since V is a finite rank operator, $B(z)$ has the form of

$$B(z) = \xi_1(z) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_2(z) \otimes \begin{pmatrix} 0 \\ g \end{pmatrix} \tag{6.14}$$

with some $\xi_1(z), \xi_2(z) \in \mathfrak{H}$. By multiplying the equation (6.13) by $z-h$ from the right, we have

$$B(z) = \lambda(z-h_0)^{-1}V + \lambda B(z)(z-h_0)^{-1}V. \tag{6.15}$$

By (6.14), we obtain the equation

$$\begin{aligned}\xi_1(z) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_2(z) \otimes \begin{pmatrix} 0 \\ g \end{pmatrix} =& \lambda \begin{pmatrix} (z-\Omega)^{-1} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ g \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ (z-h_{0,0})^{-1}g \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ \lambda \left\langle g, (z-h_{0,0})^{-1}g \right\rangle \xi_2(z) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\lambda}{z-\Omega} \xi_1(z) \otimes \begin{pmatrix} 0 \\ g \end{pmatrix}\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}\xi_1(z) =& \lambda \frac{z-\Omega}{\eta(z)} \begin{pmatrix} 0 \\ (z-h_{0,0})^{-1}g \end{pmatrix} + \lambda^2 \frac{\left\langle g, (z-h_{0,0})^{-1}g \right\rangle}{\eta(z)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & (6.17) \\ \xi_2(z) =& \frac{\lambda}{\eta(z)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\lambda^2}{\eta(z)} \begin{pmatrix} 0 \\ (z-h_{0,0})^{-1}g \end{pmatrix}. & (6.18)\end{aligned}$$

Thus, we get (6.12). ■

By using the above proposition, we give an explicit formula of e^{ith}, which we will use later. We are not aware of any literature presenting this formula else where.

Theorem 6.3. *Assume that $h_{0,0}$ is bounded. Under conditions* (A) *and* (B)*, for any $c, d \in \mathbb{C}$ and any $\psi, \xi \in \mathfrak{h}(g)$, which (ψ, ξ) has condition* (Abs)*, e^{ith} has the following form:*

$$\left\langle \begin{pmatrix} d \\ \xi \end{pmatrix}, e^{ith} \begin{pmatrix} c \\ \psi \end{pmatrix} \right\rangle = dc(t) + \langle \xi, \psi(t) \rangle, \tag{6.19}$$

where

$$\begin{aligned}c(t) =& \lambda \left\langle g, \frac{e^{ith_{0,0}}}{\eta_+(h_{0,0})} \varphi(f) \right\rangle, & (6.20) \\ \langle \xi, \psi(t) \rangle =& \left\langle \xi, e^{ith_{0,0}} \varphi(f) \right\rangle - \lambda^2 \int_{\sigma_0} \frac{e^{it\nu}}{\eta_+(\nu)} \left\langle \xi, (h_{0,0} - \nu - i0)^{-1} g \right\rangle d\langle g, E_0(\nu) \varphi(f) \rangle. & (6.21)\end{aligned}$$

To prove the above theorem, we will use the following lemma.

Lemma 6.4. *Assume conditions* (A) *and* (B)*. For any $R > \left\| h_{0,0} \right\|$ and any $\zeta, \xi \in \mathfrak{h}(g)$, which (ζ, ξ) has* *condition* (Abs)*, we have the following equations:*

$$\frac{1}{2\pi i}\lim_{\varepsilon\searrow 0}\int_{-R}^{R}\left\{\frac{e^{it(x-i\varepsilon)}}{\eta(x-i\varepsilon)}-\frac{e^{it(x+i\varepsilon)}}{\eta(x+i\varepsilon)}\right\}dx=\lambda^2\left\langle g,\frac{e^{ith_{0,0}}}{\left|\eta_-(h_{0,0})\right|^2}g\right\rangle,\tag{6.22}$$

$$\frac{1}{2\pi i}\lim_{\varepsilon\searrow 0}\int_{-R}^{R}\left\{\frac{e^{it(x-i\varepsilon)}\left\langle\zeta,(x-h_{0,0}-i\varepsilon)^{-1}\xi\right\rangle}{\eta(x-i\varepsilon)}-\frac{e^{it(x+i\varepsilon)}\left\langle\zeta,(x-h_{0,0}+i\varepsilon)^{-1}\xi\right\rangle}{\eta(x+i\varepsilon)}\right\}dx$$

$$=\left\langle\zeta,\frac{e^{ith_{0,0}}}{\eta_+(h_{0,0})}\xi\right\rangle+\lambda^2\int_{\sigma_0}\frac{e^{it\nu}\left\langle\zeta,(\nu-h_{0,0}-i0)^{-1}\xi\right\rangle}{|\eta_-(\nu)|^2}d\langle g,E_0(\nu)g\rangle\tag{6.23}$$

$$=\left\langle\zeta,\frac{e^{ith_{0,0}}}{\eta_-(h_{0,0})}\xi\right\rangle-\lambda^2\int_{\sigma_0}\frac{e^{it\nu}\left\langle\zeta,(h_{0,0}-\nu-i0)^{-1}\xi\right\rangle}{|\eta_-(\nu)|^2}d\langle g,E_0(\nu)g\rangle,\tag{6.24}$$

$$\frac{1}{2\pi i}\lim_{\varepsilon\searrow 0}\int_{-R}^{R}\left\{\frac{e^{it(x-i\varepsilon)}}{\eta(x-i\varepsilon)}\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}-i\varepsilon)^{-1}g\right\rangle\right.$$
$$\left.-\frac{e^{it(x+i\varepsilon)}}{\eta(x+i\varepsilon)}\left\langle g,(x-h_{0,0}+i\varepsilon)^{-1}\zeta\right\rangle d\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle\right\}dx$$
$$=\int_{\sigma_0}\frac{e^{it\nu}}{\eta_-(\nu)}\left\langle g,(\nu-h_{0,0}-i0)^{-1}\zeta\right\rangle d\langle\xi,E_0(\nu)g\rangle+\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}\left\langle\xi,(\nu-h_{0,0}+i0)^{-1}g\right\rangle d\langle g,E_0(\nu)\zeta\rangle$$
$$+\int_{\sigma_0}\frac{\lambda^2e^{it\nu}}{|\eta_-(\nu)|^2}\left\langle g,(\nu-h_{0,0}-i0)^{-1}\zeta\right\rangle\left\langle\xi,(\nu-h_{0,0}+i0)^{-1}g\right\rangle d\langle g,E_0(\nu)g\rangle.\tag{6.25}$$

Proof. Since the equations (6.22), (6.23), (6.24), and (6.25) can be shown by similar computations, we only prove (6.25). For the left hand side of the equation (6.25), we obtain

$$\int_{-R}^{R}\left\{\frac{e^{it(x-i\varepsilon)}}{\eta(x-i\varepsilon)}\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}-i\varepsilon)^{-1}g\right\rangle\right.$$
$$\left.-\frac{e^{it(x+i\varepsilon)}}{\eta(x+i\varepsilon)}\left\langle g,(x-h_{0,0}+i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle\right\}dx$$
$$=\int_{-R}^{R}\frac{e^{itx}}{|\eta(x-i\varepsilon)|^2}\left\{e^{t\varepsilon}\eta(x+i\varepsilon)\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}-i\varepsilon)^{-1}g\right\rangle\right.$$
$$\left.-e^{-t\varepsilon}\eta(x-i\varepsilon)\left\langle g,(x-h_{0,0}+i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle\right\}dx.\tag{6.26}$$

The integrand in (6.26) has the form of

$$\begin{aligned}&e^{t\varepsilon}\eta(x+i\varepsilon)\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}-i\varepsilon)^{-1}g\right\rangle\\&-e^{-t\varepsilon}\eta(x-i\varepsilon)\left\langle g,(x-h_{0,0}+i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle\\=&e^{t\varepsilon}\eta(x+i\varepsilon)\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle\left(\left\langle\xi,(x-h_{0,0}-i\varepsilon)^{-1}g\right\rangle-\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle\right)\\&+e^{-t\varepsilon}\eta(x-i\varepsilon)\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle\left(\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle-\left\langle g,(x-h_{0,0}+i\varepsilon)^{-1}\zeta\right\rangle\right)\\&+\left(e^{t\varepsilon}\eta(x+i\varepsilon)-e^{-t\varepsilon}\eta(x-i\varepsilon)\right)\left\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\right\rangle\left\langle\xi,(x-h_{0,0}+i\varepsilon)^{-1}g\right\rangle.\end{aligned}\tag{6.27}$$

By conditions (Abs), (A), and (B) and the equation (6.27), we see that

$$\frac{1}{2\pi i}\lim_{\varepsilon\searrow 0}\int_{-R}^{R}\left\{\frac{e^{it(x-i\varepsilon)}}{\eta(x-i\varepsilon)}\Big\langle g,(x-h_{0,0}-i\varepsilon)^{-1}\zeta\Big\rangle\Big\langle \xi,(x-h_{0,0}-i\varepsilon)^{-1}g\Big\rangle\right.$$
$$\left.-\frac{e^{it(x+i\varepsilon)}}{\eta(x+i\varepsilon)}\Big\langle g,(x-h_{0,0}+i\varepsilon)^{-1}\zeta\Big\rangle d\Big\langle \xi,(x-h_{0,0}+i\varepsilon)^{-1}g\Big\rangle\right\}dx$$
$$=\int_{\sigma_0}\frac{e^{it\nu}}{\eta_-(\nu)}\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\zeta\Big\rangle d\langle \xi,E_0(\nu)g\rangle+\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}\Big\langle \xi,(\nu-h_{0,0}+i0)^{-1}g\Big\rangle d\langle g,E_0(\nu)\zeta\rangle$$
$$+\int_{\sigma_0}\frac{\lambda^2 e^{it\nu}}{|\eta_-(\nu)|^2}\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\zeta\Big\rangle\Big\langle \xi,(\nu-h_{0,0}+i0)^{-1}g\Big\rangle d\langle g,E_0(\nu)g\rangle.$$

Thus, (6.25) follows. ■

Proof of Theorem 6.3. By Cauchy's integral formula, e^{ith} has the form of

$$e^{ith}=\frac{1}{2\pi i}\operatorname*{st-lim}_{R\nearrow\infty,\varepsilon\searrow 0}\int_{C_{\varepsilon,R}}\frac{e^{itz}}{z-h}dz, \tag{6.28}$$

where st-lim is the strong limit, $R>\|h_{0,0}\|$, $\varepsilon>0$, and $C_{\varepsilon,R}$ is as follows:

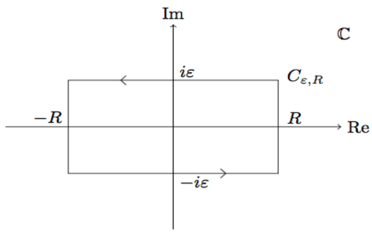

Figure 2: The Contour $C_{\varepsilon,R}$

By Proposition 6.2, we have

$$(z-h)^{-1}\begin{pmatrix} c\\ \psi\end{pmatrix}=\frac{1}{\eta(z)}\begin{pmatrix} c\\ 0\end{pmatrix}+\frac{\lambda}{\eta(z)}\Big\langle g,(z-h_{0,0})^{-1}\psi\Big\rangle\begin{pmatrix} 1\\ 0\end{pmatrix}+\begin{pmatrix} 0\\ (z-h_{0,0})^{-1}\psi\end{pmatrix}$$
$$+\frac{\lambda c}{\eta(z)}\begin{pmatrix} 0\\ (z-h_{0,0})^{-1}g\end{pmatrix}+\frac{\lambda^2\Big\langle g,(z-h_{0,0})^{-1}\psi\Big\rangle}{\eta(z)}\begin{pmatrix} 0\\ (z-h_{0,0})^{-1}g\end{pmatrix}. \tag{6.29}$$

The definition of η, Lemma 6.4, and conditions (A) and (B) imply the following equations:

$$\frac{1}{2\pi i}\lim_{R\nearrow\infty,\varepsilon\searrow 0}\int_{C_{\varepsilon,R}}\frac{e^{itz}}{\eta(z)}dz=\lambda^2\Big\langle g,\frac{e^{ith_{0,0}}}{|\eta_-(h_{0,0})|^2}g\Big\rangle,\tag{6.30}$$

$$\frac{1}{2\pi i}\lim_{R\nearrow\infty,\varepsilon\searrow 0}\int_{C_{\varepsilon,R}}\frac{\lambda e^{itz}}{\eta(z)}\Big\langle g,(z-h_{0,0})^{-1}\psi\Big\rangle dz=\lambda\Big\langle g,\frac{e^{ith_{0,0}}}{\eta_+(h_{0,0})}\psi\Big\rangle+\lambda^3\int_{\sigma_0}\frac{e^{it\nu}\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\psi\Big\rangle}{|\eta_-(\nu)|^2}d\langle g,E_0(\nu)g\rangle,\tag{6.31}$$

$$\begin{aligned}&\frac{1}{2\pi i}\lim_{R\nearrow\infty,\varepsilon\searrow 0}\int_{C_{\varepsilon,R}}\frac{\lambda c e^{itz}}{\eta(z)}\Big\langle \xi,(z-h_{0,0})^{-1}g\Big\rangle dz\\ =&\lambda c\Big\langle \xi,\frac{e^{ith_{0,0}}}{\eta_-(h_{0,0})}g\Big\rangle-\lambda^3 c\int_{\sigma_0}\frac{e^{it\nu}\Big\langle \xi,(h_{0,0}-\nu-i0)^{-1}g\Big\rangle}{|\eta_-(\nu)|^2}d\langle g,E_0(\nu)g\rangle,\end{aligned}\tag{6.32}$$

$$\begin{aligned}&\frac{1}{2\pi i}\lim_{R\nearrow\infty,\varepsilon\searrow 0}\int_{C_{\varepsilon,R}}\frac{\lambda^2 e^{itz}}{\eta(z)}\Big\langle g,(z-h_{0,0})^{-1}\psi\Big\rangle\Big\langle \xi,(z-h_{0,0})^{-1}g\Big\rangle dz\\ =&\lambda^2\int_{\sigma_0}\frac{e^{it\nu}}{\eta_-(\nu)}\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\psi\Big\rangle d\langle \xi,E_0(\nu)g\rangle+\lambda^2\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}\Big\langle \xi,(\nu-h_{0,0}+i0)^{-1}g\Big\rangle d\langle g,E_0(\nu)\psi\rangle\\ &+\lambda^4\int_{\sigma_0}\frac{e^{it\nu}}{|\eta_-(\nu)|^2}\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\psi\Big\rangle\Big\langle \xi,(\nu-h_{0,0}+i0)^{-1}g\Big\rangle d\langle g,E_0(\nu)g\rangle.\end{aligned}\tag{6.33}$$

Hence we conclude that

$$\begin{aligned}c(t)=&\lambda\Big\langle g,\frac{e^{ith_{0,0}}}{\eta_+(h_{0,0})}\psi\Big\rangle+\lambda^2\int_{\sigma_0}\frac{e^{it\nu}}{|\eta_-(\nu)|^2}\Big\{c+\lambda\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\psi\Big\rangle\Big\}d\langle g,E_0(\nu)g\rangle,\\ \langle\xi,\psi(t)\rangle=&\Big\langle \xi,e^{ith_{0,0}}\psi\Big\rangle+\lambda\int_{\sigma_0}\frac{e^{it\nu}}{\eta_-(\nu)}\Big\{c+\lambda\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\psi\Big\rangle\Big\}d\langle \xi,E_0(\nu)g\rangle\\ &-\lambda^2\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}\Big\langle \xi,(h_{0,0}-\nu-i0)^{-1}g\Big\rangle d\langle g,E_0(\nu)\psi\rangle\\ &-\lambda^3\int_{\sigma_0}\frac{e^{it\nu}\Big\langle \xi,(h_{0,0}-\nu-i0)^{-1}g\Big\rangle}{|\eta_-(\nu)|^2}\Big\{c+\lambda\Big\langle g,(\nu-h_{0,0}-i0)^{-1}\psi\Big\rangle\Big\}d\langle g,E_0(\nu)g\rangle\end{aligned}$$

and the theorem is proven.■

7 NESS

In this section, we give the initial state and an explicit formula of NESS. We consider the cases that $h_{0,k}$ is the multiplication operator of $|p|^2/2$, $p\in\mathbb{R}^d$, on $L^2(\mathbb{R}^d)$, $d\geq 3$, or $h_{0,k}=\|A_{G_k}\|\mathbb{1}-A_{G_k}$, where G_k are undirected graphs and A_{G_k} is the adjacency operator of G_k for each $k=1,\dots,N$. If $h_{0,k}$ is the multiplication operator of $|p|^2/2$, then the PF weight v_k is the Dirac delta function δ.

For an operator A on a Hilbert space, we denote the domain of A by $\mathcal{D}(A)$. For each $k=1,\dots,N$, we denote the inverse temperature and the chemical potential of k-th reservoir by $\beta_k>0$ and $\mu_k\leq 0$, respectively. Let v_k be a PF weight for $h_{0,k}$. We set $\mathcal{D}(v)=\bigoplus_{k=1}^N\mathcal{D}(v_k)$ and $(\psi_k)={}^t(0,0,\dots,0,\psi_k,0,\dots,0)$ for $\psi_k\in\mathfrak{K}_k$. Suppose $\psi_k\in\mathcal{D}(v_k)\cap\mathcal{D}((e^{\beta_k(h_{0,k}-\mu_k)}-\mathbb{1})^{-1/2})$. We consider the initial state ω_0 of the k-th reservoir given by

$$\omega_0(W((\psi_k)))=\exp\left(-\frac{1}{2}\langle\psi_k,(\mathcal{N}_k(h_{0,k})+1/2)\psi_k\rangle\right)e^{i\Theta_k(\langle v_k,\psi_k\rangle)},\tag{7.1}$$

where

$$\mathcal{N}_k(x) = (e^{\beta_k(x-\mu_k)} - 1)^{-1} \tag{7.2}$$

and Θ_k is a real valued linear functional on $\mathbb{C}$. Examples of Θ_k are given in Section 9. We assume that $\Theta_k \equiv 0$ whenever $\mu_k < 0$. To obtain an explicit formula of NESS, we assume the following conditions for initial states and form factors:

(C) The initial state ω_0 satisfies

$$\left|\omega_0(a^{\natural_1} a^{\natural_2} \cdots a^{\natural_n})\right| \le n! K_n, \quad n \in \mathbb{N}, \tag{7.3}$$

where $a^{\natural_j} = a({}^t(1,0))$ or $a^\dagger({}^t(1,0))$ and $K_n (> 0)$ satisfies $\lim_{n\to\infty} K_{n+1}/K_n = 0$.

(D) The form factor g_k is in $\mathcal{D}(v_k) \cap \mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2})$ for each $k = 1, \dots, N$.

Lemma 7.1. *Suppose that the form factors g_k and vectors ψ_k belong to $\mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2})$. Then $P_k e^{ith}\, {}^t(c, \psi_1, \dots, \psi_N)$ is in $\mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2})$ for any $t \in \mathbb{R}$ and $c \in \mathbb{C}$, $k = 1, \dots, N$, where P_k is the projection from $\mathcal{K}$ onto $\mathfrak{K}_k$.*

Proof. For simplicity, we assume that $t > 0$. For any $c \in \mathbb{C}$ and $\psi_k \in \mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2})$, $k = 1, \dots, N$, we have that

$$e^{ith}\begin{pmatrix} c \\ \psi \end{pmatrix} = \sum_{n\ge 0} \lambda^n i^n \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \alpha^0_{t_1}(V)\alpha^0_{t_2}(V)\cdots\alpha^0_{t_n}(V) e^{ith_0}\begin{pmatrix} c \\ \psi \end{pmatrix} \tag{7.4}$$

by Dyson series expansion, where $\alpha^0_t(V) = e^{ith_0} V e^{-ith_0}$, V is the operator defined in (6.3), and $\psi = {}^t(\psi_1, \dots, \psi_N)$. For $n \ge 1$, we obtain

$$\begin{aligned}\alpha^0_{t_1}(V)\alpha^0_{t_2}(V)\cdots\alpha^0_{t_n}(V)e^{ith_0}\begin{pmatrix} c \\ \psi \end{pmatrix} =& \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{-it_1h_0}\alpha^0_{t_2}(V)\cdots\alpha^0_{t_n}(V)e^{ith_0}\begin{pmatrix} c \\ \psi \end{pmatrix}\right\rangle e^{it_1h_0}\begin{pmatrix} 0 \\ g \end{pmatrix} \\ &+ \left\langle \begin{pmatrix} 0 \\ g \end{pmatrix}, e^{-it_1h_0}\alpha^0_{t_2}(V)\cdots\alpha^0_{t_n}(V)e^{ith_0}\begin{pmatrix} c \\ \psi \end{pmatrix}\right\rangle e^{it_1h_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned} \tag{7.5}$$

and

$$\left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2} P_k \alpha^0_{t_1}(V)\alpha^0_{t_2}(V)\cdots\alpha^0_{t_n}(V)e^{ith_0}\begin{pmatrix} c \\ \psi \end{pmatrix}\right\| \le \|V\|^{n-1}\left\|\begin{pmatrix} c \\ \psi \end{pmatrix}\right\| \left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2} g_k\right\|. \tag{7.6}$$

It follows that

$$\begin{aligned}\left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2} P_k e^{ith}\begin{pmatrix} c \\ \psi \end{pmatrix}\right\| &\le \left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2}\psi_k\right\| + \left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2} g_k\right\| \left\|\begin{pmatrix} c \\ \psi \end{pmatrix}\right\| \sum_{n\ge 1} \lambda^n \|V\|^{n-1}\frac{t^n}{n!} \\ &= \left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2}\psi_k\right\| + \frac{e^{\lambda t\|V\|} - 1}{\|V\|}\left\|(e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2} g_k\right\| \left\|\begin{pmatrix} c \\ \psi \end{pmatrix}\right\| < \infty.\end{aligned} \tag{7.7}$$

This completes the proof. ■

Remark 7.2. *Note that $\mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2}) = \mathcal{D}((h_{0,k})^{-1/2})$. (cf. Paragraphs before* [13, Theorem 4.5].) *In fact, we consider the continuous function*

$$q(x) = \begin{cases} -\frac{1}{2} & (x = 0), \\ (e^x - 1)^{-1} - x^{-1} & (x > 0). \end{cases} \tag{7.8}$$

The function q is bounded on $[0,\infty)$ and $(e^x-1)^{-1} = q(x)+x^{-1}$, $x\in(0,\infty)$. For any $\varepsilon>0$, the following equation holds:

$$\left\langle \psi, (e^{\beta_k(h_{0,k}+\varepsilon)} - \mathbb{1})^{-1}\psi \right\rangle = \langle \psi, q(\beta_k(h_{0,k}+\varepsilon))\psi\rangle + \left\langle \psi, (\beta_k(h_{0,k}+\varepsilon))^{-1}\psi \right\rangle. \tag{7.9}$$

If $\psi \in \mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2})$, then

$$\lim_{\varepsilon\downarrow 0} \left\langle \psi, (e^{\beta_k(h_{0,k}+\varepsilon)} - \mathbb{1})^{-1}\psi \right\rangle < \infty. \tag{7.10}$$

By the boundedness of q, $\lim_{\varepsilon\downarrow 0} \left\langle \psi, (\beta_k(h_{0,k}+\varepsilon))^{-1}\psi \right\rangle < \infty$. Thus, $\mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2}) \subset \mathcal{D}((h_{0,k})^{-1/2})$. By a similar discussion, we obtain $\mathcal{D}((h_{0,k})^{-1/2}) \subset \mathcal{D}((e^{\beta h_{0,k}} - \mathbb{1})^{-1/2})$. As a consequence, we see that $\mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2}) = \mathcal{D}((h_{0,k})^{-1/2})$.

We define the upper half-plane $\mathbb{C}_+$ on $\mathbb{C}$ by $\mathbb{C}_+ := \{ z\in\mathbb{C} \mid \mathrm{Im} z > 0 \}$ and the Hardy space $\mathbb{H}^\infty(\mathbb{C}_+)$ on $\mathbb{C}_+$ defined by

$$\mathbb{H}^\infty(\mathbb{C}_+) := \left\{ f : \text{ holomorphic on } \mathbb{C}_+ \;\middle|\; \|f\|_\infty := \sup_{z\in\mathbb{C}_+} |f(z)| < \infty \right\}. \tag{7.11}$$

We denote the Hardy space over the lower half-plane $\mathbb{C}_- := \{ z\in\mathbb{C} \mid \mathrm{Im} z < 0 \}$ by $\mathbb{H}^\infty(\mathbb{C}_-)$.

By using Theorem 6.3, we obtain an explicit formula of NESS introduced in (1.6).

Theorem 7.3. *Assume that $h_{0,0}$ is bounded. Under conditions (A) ~ (D), we have that*

$$\lim_{t\to+\infty} \omega_0\circ\alpha_t(W(f)) = \exp\left\{ -\frac{1}{2} S(f) + i\Lambda(f) \right\} \tag{7.12}$$

for $f = {}^t(c,\psi)$ with $c\in\mathbb{C}$ and $\psi \in \mathfrak{k} := \mathfrak{h}(g) \cap \mathcal{D}(v) \cap (\bigoplus_{k=1}^N \mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2}))$, where

$$S(f) = \sum_{l=1}^N \langle \varphi_l(f), (\mathcal{N}_l(h_{0,l}) + 1/2)\varphi_l(f)\rangle, \quad \Lambda(f) = \sum_{l=1}^N \Theta_l(\langle v_l, \varphi_l(f)\rangle), \tag{7.13}$$

and $\langle v_l, \varphi_l(f)\rangle$ is defined by

$$\langle v_l, \varphi_l(f)\rangle := \langle v_l, \psi_l\rangle + \frac{\lambda c\langle v_l, g_l\rangle}{\eta(0)} + \frac{\lambda^2}{\eta(0)}\langle v_l, g_l\rangle \left\langle g, (h_{0,0})^{-1}\psi \right\rangle. \tag{7.14}$$

Remark 7.4. *Theorem 7.3 and the definition of NESS (1.6) imply that NESS exists uniquely. We denote the NESS by ω_+. NESS ω_+ has the form of*

$$\omega_+(\Psi(f)) = \pi^{3/2} \sum_{l=1}^N \Theta_l(\langle v_l, \varphi_l(f)\rangle), \tag{7.15}$$

$$\omega_+(\Psi(f)^2) - \omega_+(\Psi(f))^2 = \sum_{l=1}^N \langle \varphi_l(f), (\mathcal{N}_l(h_{0,l}) + 1/2)\varphi_l(f)\rangle. \tag{7.16}$$

Proof of Theorem 7.3. For a vector $f\in\mathcal{K}$, we denote the scalar part and $\mathfrak{K}_k$-part of f by f_0 and f_k, $k = 1,\ldots,N$, respectively. By (7.1), for $f = {}^t(c,\psi)$ with $c\in\mathbb{C}$ and $\psi\in\mathfrak{k}$, we have that

$$\omega_0\circ\alpha_t(W(f)) = \omega_0(W((e^{ith}f)_0)) \prod_{k=1}^N \omega_0(W((e^{ith}f)_k)). \tag{7.17}$$

First, we consider the limit of $\omega_0(W((e^{ith}f)_0))$. Condition (C) and Theorem 6.3 imply

$$\left|\omega_0(\left\{\Phi((e^{ith}f)_0)\right\}^m)\right| \leq m!(2|c(t)|)^m K_m. \tag{7.18}$$

Since $1/\eta_- \in L^\infty(\mathbb{R})$,

$$\frac{1}{\eta_+(\nu)}\frac{d\langle g, E_0(\nu)\psi\rangle}{d\nu}, \quad \frac{F(\nu; f)}{|\eta_-(\nu)|^2}\frac{d\langle g, E_0(\nu)g\rangle}{d\nu} \in L^1(\mathbb{R}). \tag{7.19}$$

Thus, $\sup_{t\in\mathbb{R}}|c(t)| < \infty$. A theorem of Riemann–Lebesgue (see e.g. [35]) implies that $c(t) \to 0$ as $t \to \infty$. We obtain

$$\begin{aligned}\left|\omega_0(\exp(i\Phi((e^{ith}f)_0))) - 1\right| \leq \sum_{m=1}^{\infty}\frac{1}{m!}\left|\omega_0(\left\{\Phi(e^{ith}f)_0\right\}^m)\right| \leq \sum_{m=1}^{\infty}(2|c(t)|)^m K_m \\ \leq \frac{|c(t)|}{C}\sum_{m=1}^{\infty}(2C)^m K_m \to 0, \quad (t \to \infty),\end{aligned} \tag{7.20}$$

where $C := \sup_{t\in\mathbb{R}}|c(t)|$.

Next, we consider the quadratic part of $\log\omega_0(W((e^{ith}f)_l))$. For $\varepsilon \in (0, \pi/2)$, we put

$$\psi_{\varepsilon,l}(t) := e^{ith_{0,l}}\varphi_l(f) - \lambda^2\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}(h_{0,l} - \nu - i\varepsilon)^{-1}g_l d\langle g, E_0(\nu)\varphi(f)\rangle, \tag{7.21}$$

where the convergence of vector valued integral of (7.21) is in the strong operator topology. Note that $(h_{0,l} - \nu - i\varepsilon)^{-1}$ is bounded. We have that

$$\begin{aligned}&\langle\psi_{\varepsilon,l}(t), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\psi_{\varepsilon,l}(t)\rangle \\ =&\langle\varphi_l(f), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\varphi_l(f)\rangle \\ &-\lambda^2\mathrm{Re}\left\{\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}\left\langle e^{ith_{0,l}}\varphi_l(f), \{\mathcal{N}_l(h_{0,l}) + 1/2\}(h_{0,l} - \nu - i\varepsilon)^{-1}g_l\right\rangle d\langle g, E_0(\nu)\varphi(f)\rangle\right\} \\ &+\lambda^4\int_{\sigma_0}\int_{\sigma_0}\frac{e^{it(\nu-\nu')}}{\eta_-(\nu')\eta_+(\nu)}\left\langle(h_{0,l} - \nu' - i\varepsilon)^{-1}g_l, \{\mathcal{N}_l(h_{0,l}) + 1/2\}(h_{0,l} - \nu - i\varepsilon)^{-1}g_l\right\rangle d\langle g, E_0(\nu)\varphi(f)\rangle d\langle\varphi(f), E_0(\nu')g\rangle.\end{aligned} \tag{7.22}$$

The second term on the right side of (7.22) is

$$\begin{aligned}&\int_{\sigma_0}\frac{e^{it\nu}}{\eta_+(\nu)}\left\langle e^{ith_{0,l}}\varphi_l(f), \{\mathcal{N}_l(h_{0,l}) + 1/2\}(h_{0,l} - \nu - i\varepsilon)^{-1}g_l\right\rangle d\langle g, E_0(\nu)\varphi(f)\rangle \\ =&\int_{\sigma_0}\frac{1}{\eta_+(\nu)}\int_{\sigma_l}\frac{e^{it(\nu-\nu')}}{\nu' - \nu - i\varepsilon}\{\mathcal{N}_l(\nu') + 1/2\}d\langle\varphi_l(f), E_l(\nu')g_l\rangle d\langle g, E_0(\nu)\varphi(f)\rangle \\ =&ie^{t\varepsilon}\int_{\sigma_0}\frac{1}{\eta_+(\nu)}\int_{\sigma_l}\int_0^\infty e^{i(t+s)(\nu-\nu'+i\varepsilon)}ds\{\mathcal{N}_l(\nu') + 1/2\}d\langle\varphi_l(f), E_l(\nu')g_l\rangle d\langle g, E_0(\nu)\varphi(f)\rangle \\ =&ie^{t\varepsilon}\int_t^\infty e^{-s\varepsilon}\int_{\sigma_0}\frac{e^{is\nu}}{\eta_+(\nu)}d\langle g, E_0(\nu)\varphi(f)\rangle\int_{\sigma_l}e^{-is\nu'}\{\mathcal{N}_l(\nu') + 1/2\}d\langle\varphi_l(f), E_l(\nu')g_l\rangle ds.\end{aligned} \tag{7.23}$$

Since $g, \psi \in \bigoplus_{k=1}^N\mathcal{D}((e^{\beta_k h_{0,k}} - \mathbb{1})^{-1/2})$, we have that $g, \psi \in \bigoplus_{k=1}^N\mathcal{D}((h_{0,k})^{-1/2})$ by Remark 7.2. Thus, we obtain $\varphi_l(f) \in \mathcal{D}((h_{0,l})^{-1/2})$ for any $l = 1, \ldots, N$ and

$$\mathcal{N}_l(\nu')\frac{d\langle\varphi_l(f), E_l(\nu')g_l\rangle}{d\nu'} = \nu'\mathcal{N}_l(\nu')\frac{d\left\langle(h_{0,l})^{-1/2}\varphi_l(f), E_l(\nu')(h_{0,l})^{-1/2}g_l\right\rangle}{d\nu'}. \tag{7.24}$$

The above equation (7.24) and conditions (A), (B), and (D) imply that

$$\frac{1}{\eta_+(\nu)}\frac{d\langle g, E_0(\nu)\varphi(f)\rangle}{d\nu}, \{\mathcal{N}_l(\nu') + 1/2\}\frac{d\langle\varphi_l(f), E_l(\nu')g_l\rangle}{d\nu'} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \tag{7.25}$$

Thus, there exist functions $w_1, w_2 \in L^2(\mathbb{R})$ such that

$$\int_t^\infty e^{-s\varepsilon}\int_{\sigma_0}\frac{e^{is\nu}}{\eta_+(\nu)}d\langle g, E_0(\nu)\varphi(f)\rangle\int_{\sigma_l}e^{-is\nu'}\{\mathcal{N}_l(\nu') + 1/2\}d\langle\varphi_l(f), E_l(\nu')g_l\rangle ds = \int_t^\infty e^{-s\varepsilon}\overline{w_1(s)}w_2(s)ds \tag{7.26}$$

by Plancherel theorem.

We consider the third term on the right side of (7.22). Since $\mathcal{N}_l(x) + 1/2$ is in $L^1(\mathbb{R}, d\langle g_l, E_l(x)g_l\rangle)$,

$$\begin{aligned}
&\lambda^4\int_{\sigma_0}\int_{\sigma_0}\frac{e^{it(\nu-\nu')}}{\eta_-(\nu')\eta_+(\nu)}\Big\langle(h_{0,l} - \nu' - i\varepsilon)^{-1}g_l, \{\mathcal{N}_l(h_{0,l}) + 1/2\}(h_{0,l} - \nu - i\varepsilon)^{-1}g_l\Big\rangle d\langle g, E_0(\nu)\varphi(f)\rangle d\langle\varphi(f), E_0(\nu')g\rangle\\
=&\lambda^4\lim_{\delta\searrow 0}\int_{\sigma_0}\int_{\sigma_0}\frac{e^{it(\nu-\nu')}}{\eta_-(\nu')\eta_+(\nu)}\int_{\sigma_l}\frac{\mathcal{N}_l(x+\delta) + 1/2}{(x - \nu' + i\varepsilon)(x - \nu - i\varepsilon)}d\langle g_l, E_l(x)g_l\rangle d\langle g, E_0(\nu)\varphi(f)\rangle d\langle\varphi(f), E_0(\nu')g\rangle.
\end{aligned} \tag{7.27}$$

We define a holomorphic function u_δ on $\mathbb{C}_+$ by

$$u_\delta(z)(= u_\delta(x + iy)) := \frac{1}{\pi}\{\mathcal{N}_l(z + \delta) + 1/2\}\int_{\mathbb{R}}\frac{y}{(x - w)^2 + y^2}\frac{d\langle g_l, E_l(w)g_l\rangle}{dw}dw. \tag{7.28}$$

Note that the support of $d\langle g_l, E_l(w)g_l\rangle/dw$ is contained in $[0, \infty)$, $d\langle g_l, E_0(w)g_l\rangle/dw$ is in $L^\infty(\mathbb{R})$ by condition (A), and $\mathcal{N}_l(z + \delta)$ is analytic and bounded in $D := \{\, z \in \mathbb{C} \mid \mathrm{Im} z > 0, -\delta' < \mathrm{Re} z < \infty \,\}$, where $0 < \delta' < \delta$. Thus, $u_\delta \in \mathbb{H}^\infty(\mathbb{C}_+)$ and $u_\delta(x + iy)$ converges to $\{\mathcal{N}_l(x + \delta) + 1/2\}d\langle g_l, E_0(x)g_l\rangle/dx$ as $y \searrow 0$ in $L^\infty(\mathbb{R})$ by [18, Theorem 3.13.]. For any $R > \|h_{0,0}\|$, we obtain the following equation:

$$\begin{aligned}
&\int_{\sigma_l}\frac{\mathcal{N}_l(x + \delta) + 1/2}{(x - \nu' + i\varepsilon)(x - \nu - i\varepsilon)}d\langle g_l, E_l(x)g_l\rangle = \lim_{R\to\infty,\delta'\searrow 0}\int_{-R}^{R}\frac{u_\delta(x + \delta')}{(x - \nu' + i\varepsilon)(x - \nu - i\varepsilon)}dx\\
&= \lim_{R\to\infty,\delta'\searrow 0}\int_{\gamma_{R,\delta'}}\frac{u_\delta(z + \delta')}{(z - \nu' + i\varepsilon)(z - \nu - i\varepsilon)}dz = 2\pi i\frac{u_\delta(\nu + i\varepsilon)}{\nu - \nu' + 2i\varepsilon},
\end{aligned} \tag{7.29}$$

where $\gamma_{R,\delta'}$ is the contour from $[-\delta', R]$ through $[R, R + iR]$ and $[R + iR, -\delta' + iR]$ to $[-\delta + iR, -\delta']$. By (7.25), (7.27), and $\varepsilon \in (0, \pi/2)$, the last term on the right side of (7.22) has the form of

$$\begin{aligned}
&\lambda^4\int_{\sigma_0}\int_{\sigma_0}\frac{e^{it(\nu-\nu')}}{\eta_-(\nu')\eta_+(\nu)}\Big\langle(h_{0,l} - \nu' - i\varepsilon)^{-1}g_l, (\mathcal{N}_l(h_{0,l}) + 1/2)(h_{0,l} - \nu - i\varepsilon)^{-1}g_l\Big\rangle d\langle g, E_0(\nu)\varphi(f)\rangle d\langle\varphi(f), E_0(\nu')g\rangle\\
=&2\pi i\lambda^4\lim_{\delta\searrow 0}\int_{\sigma_0}\int_{\sigma_0}\frac{e^{it(\nu-\nu')}}{\eta_-(\nu')\eta_+(\nu)}\frac{u_\delta(\nu + i\varepsilon)}{\nu - \nu' + 2i\varepsilon}d\langle g, E_0(\nu)\varphi(f)\rangle d\langle\varphi(f), E_0(\nu')g\rangle\\
=&2\pi\lambda^4 e^{2t\varepsilon}\int_{\sigma_0}\int_{\sigma_0}\frac{u(\nu + i\varepsilon)}{\eta_-(\nu')\eta_+(\nu)}\int_t^\infty e^{is(\nu-\nu'+2i\varepsilon)}ds d\langle g, E_0(\nu)\varphi(f)\rangle d\langle\varphi(f), E_0(\nu')g\rangle\\
=&2\pi\lambda^4 e^{2t\varepsilon}\int_t^\infty e^{-2s\varepsilon}\int_{\sigma_0}\frac{e^{is\nu}u(\nu + i\varepsilon)}{\eta_+(\nu)}d\langle g, E_0(\nu)\varphi(f)\rangle\int_{\sigma_0}\frac{e^{-is\nu'}}{\eta_-(\nu')}d\langle\varphi(f), E_0(\nu')g\rangle ds,
\end{aligned} \tag{7.30}$$

where $u(\nu + i\varepsilon) = \lim_{\delta\searrow 0} u_\delta(\nu + i\varepsilon)$. By (7.25), there exist functions $w_3, w_4 \in L^2(\mathbb{R})$ such that

$$\int_t^\infty e^{-2s\varepsilon}\int_{\sigma_0}\frac{e^{is\nu}u(\nu + i\varepsilon)}{\eta_+(\nu)}d\langle g, E_0(\nu)\varphi(f)\rangle\int_{\sigma_0}\frac{e^{-is\nu'}}{\eta_-(\nu')}d\langle\varphi(f), E_0(\nu')g\rangle ds = \int_t^\infty e^{-2s\varepsilon}\overline{w_3(s)}w_4(s)ds \tag{7.31}$$

by Plancherel theorem. The integral (7.31) is absolutely convergent independent of ε and t. Therefore, we have that

$$\begin{aligned}&\langle \psi_l(t), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\psi_l(t)\rangle = \lim_{\varepsilon\searrow 0} \langle \psi_{\varepsilon,l}(t), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\psi_{\varepsilon,l}(t)\rangle \\ =&\langle \varphi_l(f), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\varphi_l(f)\rangle - \lambda^2 \mathrm{Re}\left\{ i \int_t^\infty \overline{w_1(s)} w_2(s) ds \right\} + 2\pi\lambda^4 \int_t^\infty \overline{w_3(s)} w_4(s) ds. \qquad (7.32)\end{aligned}$$

Thus, we obtain

$$\lim_{t\to\infty} \langle \psi_l(t), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\psi_l(t)\rangle = \langle \varphi_l(f), \{\mathcal{N}_l(h_{0,l}) + 1/2\}\varphi_l(f)\rangle. \qquad (7.33)$$

Finally, we discuss the term $\Theta_l\left(\left\langle v_l, (e^{ith} f)_l\right\rangle\right)$ in (7.1). If $h_{0,k}$ is the multiplication operator of $|p|^2/2$, then we obtain the statement by [34, Theorem 3.1]. Thus, we consider the case where $h_{0,k}$ is the adjacency operator of graphs. For $z \in \mathbb{C}\backslash\sigma_k$ and $\xi \in \mathfrak{k}_k := P_k\mathfrak{k}$, we obtain

$$\left|\left\langle \delta_x, (z - h_{0,k})^{-1}\xi\right\rangle\right| = \left|\int_{\sigma_k} (z - \nu)^{-1} d\langle \delta_x, E_k(\nu)\xi\rangle\right| \le \sup_{\nu\in\sigma_k} |z - \nu|^{-1} \int_{\sigma_k} d|\langle \delta_x, E_0(\nu)\xi\rangle| = \sup_{\nu\in\sigma_k} |z - \nu|^{-1} |\langle \delta_x, \xi\rangle|. \qquad (7.34)$$

Since $\xi \in \mathfrak{k}_k \subset \mathcal{D}(v_k)$, $(z - h_{0,k})^{-1}\xi \in \mathcal{D}(v_k)$ for any $z \in \mathbb{C}\backslash\sigma_k$. It follows that

$$\left\langle v_k, (z - h_{0,k})^{-1}\xi\right\rangle = z^{-1}\left\langle v_k, (z - h_{0,k})(z - h_{0,k})^{-1}\xi\right\rangle = z^{-1}\langle v_k, \xi\rangle. \qquad (7.35)$$

By condition (D) and a theorem of Riemann–Lebesgue, we obtain

$$\begin{aligned}\lim_{t\to\infty} \Lambda_l(e^{ith} f) = \lim_{t\to\infty}\Bigg[&\Theta_l\left(\frac{\langle v, \psi_l\rangle}{2\pi i} \lim_{R\nearrow 0,\varepsilon\searrow 0} \int_{C_{R,\varepsilon}} \frac{e^{itz}}{z} dz\right) + \Theta_l\left(\frac{\lambda c\langle v_l, g_l\rangle}{2\pi i} \lim_{R\nearrow 0,\varepsilon\searrow 0} \int_{C_{R,\varepsilon}} \frac{e^{itz}}{\eta(z)z} dz\right) \\ &+\Theta_l\left(\frac{\lambda^2\langle v_l, g_l\rangle}{2\pi i} \lim_{R\nearrow 0,\varepsilon\searrow 0} \int_{C_{R,\varepsilon}} \frac{e^{itz}\left\langle g, (z - h_{0,0})^{-1}\psi\right\rangle}{\eta(z)z} dz\right)\Bigg] = \Theta_l(\langle v_l, \varphi_l(f)\rangle). \qquad (7.36)\end{aligned}$$

We have completed the proof.■

8 Currents and Entropy Production Rate

We set $(c) = {}^t(c, 0, \ldots, 0)$, $c \in \mathbb{C}$. Following W. Aschbacher, V. Jakšić, Y. Pautrat, and C.-A. Pillet ([4] and [5]), V. Jakšić and C.-A. Pillet [16], for any $l = 1, \ldots, N$, currents J_l and E_l from l-th reservoir to the system is formally defined by

$$J_l = i\lambda a((1))a^\dagger((g_l)) - i\lambda a((g_l))a^\dagger((1)), \qquad (8.1)$$

$$E_l = i\lambda a((1))a^\dagger((h_{0,l}g_l)) - i\lambda a((h_{0,l}g_l))a^\dagger((1)), \qquad (8.2)$$

which are given by the following formal equations:

$$-\frac{d}{dt}\tau_t(d\Gamma(P_0))\Big|_{t=0} = \sum_{l=1}^N J_l, \quad -\frac{d}{dt}\tau_t(d\Gamma(P_0h_0))\Big|_{t=0} = \sum_{l=1}^N E_l, \qquad (8.3)$$

where P_0 is the projection from $\mathcal{K}$ onto $0 \oplus (\bigoplus_{l=1}^n \mathfrak{K}_l)$. Assuming classical thermodynamics, D. Ruelle introduced the entropy production $\mathrm{Ep}(\omega_+)$ in a NESS ω_+ via the following equation:

$$Ep(\omega_+) = \omega_+(\sigma), \qquad (8.4)$$

where

$$\sigma = -\sum_{l=1}^{N} \beta_l (E_l - \mu_l J_l). \tag{8.5}$$

Remark 8.1. *In classical thermodynamics, entropy production rate dS has the form of*

$$dS = \sum_{j=1}^{N} \beta_j (dU_j - \mu_j dN_j), \tag{8.6}$$

where β_j is the inverse temperature, dU_j is the rate of internal energy, μ_j is the chemical potential, and dN_j is the rate of number of particles for each $j \in \{1, \dots, N\}$. V. Jakšić and C.-A. Pilllet proved non-negativity of entropy production rate using the relative modular operators [16, Theorem 1.2]. *Let $(\mathfrak{O}, \alpha)$ be a* C^**-dynamical system and ω_0 be an α-invariant state. We fix a self-adjoint element $V \in \mathfrak{O}$. We denote the perturbed time evolution of α_t by α_t^V. We define the relative entropy* $\mathrm{Ent}(\omega_0 \circ \alpha_t^V | \omega_0)$ *of ω_0 and $\omega_0 \circ \alpha_t^V$ by*

$$\mathrm{Ent}(\omega_0 \circ \alpha_t^V | \omega_0) = \int_0^\infty \log \lambda d \left\langle \Omega_t^V, E(\lambda) \Omega_t^V \right\rangle, \tag{8.7}$$

where Ω_0 and Ω_t^V are the GNS-cyclic vectors with respect to ω_0 and $\omega_0 \circ \alpha_t^V$, respectively, E is the spectral family of the relative modular operator $\Delta_{\Omega_t^V, \Omega_0}$,

$$\Delta_{\Omega_t^V, \Omega_0} = S^*_{\Omega_t^V, \Omega_0} S_{\Omega_t^V, \Omega_0}, \tag{8.8}$$

and $S_{\Omega_t^V, \Omega_0} A \Omega_0 = A^* \Omega_t^V$ for any $A \in \pi_0(\mathfrak{O})''$. Then the entropy production rate is as follows [16, Theorem 1.1]:

$$\mathrm{Ep}(\omega_+) = -\lim_n \frac{1}{T_n} \mathrm{Ent}(\omega_0 \circ \alpha_{T_n}^V | \omega_0) \geq 0. \tag{8.9}$$

Proof of strict positivity is non-trivial.

Corollary 8.2. *Currents J_l and E_l at a NESS ω_+ are given by the following form:*

$$\begin{aligned}
\omega_+(J_l) =& 2\pi\lambda^4 \sum_{k=1}^{N} \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2} (\mathcal{N}_l(\nu) - \mathcal{N}_k(\nu)) \frac{d\langle g_k, E_k(\nu) g_k \rangle}{d\nu} d\langle g_l, E_l(\nu) g_l \rangle \\
&+ \frac{\pi^3 \lambda^2}{\eta(0)} \sum_{k=1}^{N} \{\Theta_k(\langle v_k, g_k \rangle) \Theta_l(i\langle v_l, g_l \rangle) - \Theta_k(i\langle v_k, g_k \rangle) \Theta_l(\langle v_l, g_l \rangle)\},
\end{aligned} \tag{8.10}$$

$$\omega_+(E_l) = 2\pi\lambda^4 \sum_{k=1}^{N} \int_{\sigma_l} \frac{\nu}{|\eta_-(\nu)|^2} (\mathcal{N}_l(\nu) - \mathcal{N}_k(\nu)) \frac{d\langle g_k, E_k(\nu) g_k \rangle}{d\nu} d\langle g_l, E_l(\nu) g_l \rangle. \tag{8.11}$$

Proof. For any $l = 1, \dots, N$, $c \in \mathbb{C}$, and $\xi_l \in \mathfrak{k}_l$, we have

$$a((c)) a^\dagger((\xi_l)) - a^\dagger((c)) a((\xi_l)) = i\Psi((ic)) \Psi((\xi_l)) - i\Psi((c)) \Psi((i\xi_l)). \tag{8.12}$$

Since $[\Psi((c)), \Psi((\xi_l))] = 0$, we obtain

$$\begin{aligned}
&4\omega_+(\Psi((c))\Psi((\xi_l))) = \omega_+(\{\Psi((c) + (\xi_l))\}^2) - \omega_+(\{\Psi((c) - (\xi_l))\}^2) \\
=& \omega_+(\{\Psi((c) + (\xi_l))\}^2) - \omega_+(\Psi((c) + (\xi_l)))^2 - \omega_+(\{\Psi((c) - (\xi_l))\}^2) + \omega_+(\Psi((c) - (\xi_l)))^2 \\
&+ 4\omega_+(\Psi((c)))\omega_+(\Psi((\xi_l))).
\end{aligned} \tag{8.13}$$

From Theorem 7.3 and (8.13), it follows that

$$\begin{aligned}
&\omega_+(\Psi((1))\Psi((i\xi_l))) - \omega_+(\Psi((i))\Psi((\xi_l)))\\
=&2\lambda \mathrm{Im}\langle \xi_l, \{(\mathcal{N}_l(h_{0,l}) + 1/2)/\eta_-(h_{0,l})\}g_l\rangle + 2\lambda^2 \sum_{k=1}^N \mathrm{Im}\Big\langle F(h_{0,k}, (\xi_l))g_k, \Big\{(\mathcal{N}_k(h_{0,k}) + 1/2)/\Big|\eta_-(h_{0,k}\Big|^2)\Big\}g_k\Big\rangle\\
&+\pi^3\left\{\left(\sum_{k=1}^N \Theta_k(\langle v_k, \varphi_k((1))\rangle)\right)\left(\sum_{k=1}^N \Theta_k(\langle v_k, \varphi_k((i\xi_l))\rangle)\right) - \left(\sum_{k=1}^N \Theta_k(\langle v_k, \varphi_k((i))\rangle)\right)\left(\sum_{k=1}^N \Theta_k(\langle v_k, \varphi_k((\xi_l))\rangle)\right)\right\}
\end{aligned} \tag{8.14}$$

by linearity of $\varphi_k(f)$ in f. Note that the element ξ_l is equal to g_l or $h_{0,l}g_l$. Thus the first term of right hand side of (8.14) has the form of

$$\mathrm{Im}\langle \xi_l, \{(\mathcal{N}_l(h_{0,l}) + 1/2)/\eta_-(h_{0,l})\}g_l\rangle = \lambda^2\pi \sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2}\left(\mathcal{N}_l(\nu) + \frac{1}{2}\right)\frac{d\langle g_k, E_k(\nu)g_k\rangle}{d\nu} d\langle \xi_l, E_l(\nu)g_l\rangle. \tag{8.15}$$

If $\xi_l = g_l$ or $h_{0,l}g_l$, then

$$\mathrm{Im}(F(\nu, (\xi_l))) = \lambda\pi \frac{d\langle g_l, E_l(\nu)\xi_l\rangle}{d\nu} = \lambda\pi \frac{d\langle \xi_l, E_l(\nu)g_l\rangle}{d\nu}, \text{ a.e. } \nu \in \mathbb{R}, \tag{8.16}$$

with respect to the Lebesgue measure. The second term of the right hand side of (8.14) has the following form:

$$\begin{aligned}
&\mathrm{Im}\Big\langle F(h_{0,k}, (\xi_l))g_k, \Big\{(\mathcal{N}_k(h_{0,k}) + 1/2)/\Big|\eta_-(h_{0,k})\Big|^2\Big\}g_k\Big\rangle\\
=& -\lambda\pi \int_{\sigma_k} \frac{1}{|\eta_-(\nu)|^2}\frac{d\langle g_l, E_l(\nu)\xi_l\rangle}{d\nu}\left(\mathcal{N}_k(\nu) + \frac{1}{2}\right) d\langle g_k, E_k(\nu)g_k\rangle
\end{aligned} \tag{8.17}$$

By the definition of $\langle v_k, \varphi_k\rangle$, we see that

$$\langle v_k, \varphi((1))\rangle = \frac{\lambda\langle v_k, g_k\rangle}{\eta(0)}, \quad \langle v_k, \varphi_k((\xi_l))\rangle = \delta_{k,l}\langle v_l, \xi_l\rangle + \frac{\lambda^2\langle v_k, g_k\rangle\Big\langle g_l, (h_{0,l})^{-1}\xi_l\Big\rangle}{\eta(0)}. \tag{8.18}$$

Since $g_k \in \mathcal{D}((h_{0,k})^{-1/2})$ for any $k = 1, \ldots, N$, $\eta(0)$ is finite and real. By (8.15), (8.16), (8.17), and (8.18), we obtain

$$
\begin{aligned}
&\omega_+(\Psi((1))\Psi((i\xi_l))) - \omega_+(\Psi((i))\Psi((\xi_l))) \\
=&2\pi\lambda^3 \sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2}(\mathcal{N}_l(\nu) - \mathcal{N}_k(\nu))\frac{d\langle g_k, E_k(\nu)g_k\rangle}{d\nu} d\langle \xi_l, E_l(\nu)g_l\rangle \\
&+\pi^3\left[\left\{\sum_{k=1}^N \Theta_k\left(\frac{\lambda\langle v_k, g_k\rangle}{\eta(0)}\right)\right\}\left\{\sum_{k=1}^N \Theta_k\left(\delta_{k,l} i\langle v_l, \xi_l\rangle + \frac{\lambda^2 i\langle v_k, g_k\rangle\left\langle g_l, (h_{0,l})^{-1}\xi_l\right\rangle}{\eta(0)}\right)\right\}\right. \\
&\left.-\left\{\sum_{k=1}^N \Theta_k\left(\frac{\lambda i\langle v_k, g_k\rangle}{\eta(0)}\right)\right\}\left\{\sum_{k=1}^N \Theta_k\left(\delta_{k,l}\langle v_l, \xi_l\rangle + \frac{\lambda^2\langle v_k, g_k\rangle\left\langle g_l, (h_{0,l})^{-1}\xi_l\right\rangle}{\eta(0)}\right)\right\}\right] \\
=&2\pi\lambda^3 \sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2}(\mathcal{N}_l(\nu) - \mathcal{N}_k(\nu))\frac{d\langle g_k, E_k(\nu)g_k\rangle}{d\nu} d\langle \xi_l, E_l(\nu)g_l\rangle \\
&+\frac{\pi^3\lambda}{\eta(0)}\sum_{k=1}^N \{\Theta_k(\langle v_k, g_k\rangle)\Theta_l(i\langle v_l, \xi_l\rangle) - \Theta_k(i\langle v_k, g_k\rangle)\Theta_l(\langle v_l, \xi_l\rangle)\} \\
&+\frac{\pi^3\lambda^3}{\eta(0)^2}\sum_{j,k=1}^N \left\{\Theta_j\left(\left\langle v_j, g_j\right\rangle\right)\Theta_k(i\langle v_k, g_k\rangle) - \Theta_j\left(i\left\langle v_j, g_j\right\rangle\right)\Theta_k(\langle v_k, g_k\rangle)\right\}\left\langle g_l, (h_{0,l})^{-1}\xi_l\right\rangle,
\end{aligned}
\tag{8.19}
$$

since $\xi_l = g_l$ or $h_{0,l}g_l$ and $\left\langle g_l, (h_{0,l})^{-1}\xi_l\right\rangle$ is real. If $\xi_l = g_l$ (resp. $\xi_l = h_{0,l}g_l$), then we have (8.10) (resp. (8.11)).■

For each $l = 1, \ldots, N$, we define Josephson currents at NESS by

$$
\mathrm{Jos}_l(\omega_+) = \frac{\pi^3\lambda^2}{\eta(0)}\sum_{k=1}^N \{\Theta_k(\langle v_k, g_k\rangle)\Theta_l(i\langle v_l, g_l\rangle) - \Theta_k(i\langle v_k, g_k\rangle)\Theta_l(\langle v_l, g_l\rangle)\}. \tag{8.20}
$$

By Corollary 8.2, we get an explicit form of the mean entropy production rate.

Proposition 8.3. *The entropy production $Ep(\omega_+)$ is given by*

$$
Ep(\omega_+) = \sum_{k,l=1}^N \int_{\sigma_l} \frac{\lambda^4\pi}{|\eta_-(\nu)|^2}\{\beta_l(\nu-\mu_l) - \beta_k(\nu-\mu_k)\}(\mathcal{N}_k(\nu) - \mathcal{N}_l(\nu))\frac{d\langle g_k, E_k(\nu)g_k\rangle}{d\nu} d\langle g_l, E_l(\nu)g_l\rangle. \tag{8.21}
$$

Proof. By Corollary 8.2, we have that

$$
\begin{aligned}
&-\sum_{l=1}^N \beta_l(\omega_+(E_l) - \mu_l\omega_+(J_l)) \\
=&2\lambda^4\pi \sum_{l=1}^N\sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2}(\beta_l\nu - \beta_l\mu_l)(\mathcal{N}_k(\nu) - \mathcal{N}_l(\nu)))\frac{d\langle g_k, E_k(\nu)g_k\rangle}{d\nu} d\langle g_l, E_l(\nu)\xi_l\rangle \\
&+\frac{\pi^3\lambda^2}{\eta(0)}\sum_{l=1}^N \beta_l\mu_l \sum_{k=1}^N \{\Theta_k(\langle v_k, g_k\rangle)\Theta_l(i\langle v_l, g_l\rangle) - \Theta_k(i\langle v_k, g_k\rangle)\Theta_l(\langle v_l, g_l\rangle)\}.
\end{aligned}
\tag{8.22}
$$

If $\mu_l \neq 0$, then $\Theta_l \equiv 0$, and if $\Theta_l \not\equiv 0$, then $\mu_l = 0$. Thus, the last term of (8.22) is equal to zero. It follows that

$$
\begin{aligned}
&-\sum_{l=1}^{N}(\beta_l\omega_+(E_l)-\beta_l\mu_l\omega_+(J_l))\\
&=\sum_{l=1}^{N}\sum_{k=1}^{N}\int_{\sigma_l}\frac{\lambda^4\pi}{|\eta_-(\nu)|^2}(\beta_l\nu-\beta_l\mu_l-\beta_k\nu+\beta_k\mu_k)(\mathcal{N}_k(\nu)-\mathcal{N}_l(\nu)))\frac{d\langle g_k,E_k(\nu)g_k\rangle}{d\nu}d\langle g_l,E_l(\nu)\xi_l\rangle. \quad (8.23)
\end{aligned}
$$

Thus, the proposition follows.■

By Proposition 8.3, the mean entropy production rate is independent of phase terms. Thus, Josephson currents $\mathrm{Jos}_l(\omega_+)$ may occur without entropy production, if the temperatures and the chemical potentials of reservoirs are identical.

For any $k, l \in \{1, \ldots, N\}$, the function

$$
\frac{\lambda^4\pi}{|\eta_-(\nu)|^2}\frac{d\langle g_k,E_k(\nu)g_k\rangle}{d\nu}\frac{d\langle g_l,E_l(\nu)g_l\rangle}{d\nu} \tag{8.24}
$$

corresponds to the “total transmission probability” ([5], [38], and [39]). As in [5], we say that the channel $k \to l$ is open if the set

$$
\left\{ \nu\in\sigma_k\cap\sigma_l \;\middle|\; \frac{1}{|\eta_-(\nu)|^2}\frac{d\langle g_k,E_k(\nu)g_k\rangle}{d\nu}\frac{d\langle g_l,E_l(\nu)g_l\rangle}{d\nu}>0 \right\} \tag{8.25}
$$

has a positive Lebesgue measure. If $\beta_k \neq \beta_l$ or $\mu_k \neq \mu_l$, then the function

$$
\{\beta_l(\nu-\mu_l)-\beta_k(\nu-\mu_k)\}(\mathcal{N}_k(\nu)-\mathcal{N}_l(\nu)) \tag{8.26}
$$

is strictly positive for any finite interval. By the above discussions, we obtain strictly positivity of the entropy production rate.

Theorem 8.4. *If there exists an open channel $k \to l$ such that either $\beta_k \neq \beta_l$ or $\mu_k \neq \mu_l$ for some $k, l \in \{1, \ldots, N\}$, then $Ep(\omega_+) > 0$.*

9 Examples

In this section, we give examples of currents on $\mathbb{R}^d$ and graphs.

9.1 Case of $L^2(\mathbb{R}^d)$, $d \geq 3$

In this subsection, we put $\mathfrak{K}_k = L^2(\mathbb{R}^d)$, $d \geq 3$, for any $k = 1, \ldots, N$. A model consists of a quantum particle and two reservoirs is considered in [33] and [34]. Thus we consider a model consisting grater than two reservoirs and that of phase terms different from [34]. The Hamiltonians $h_{0,k}$ are Fourier transform of positive Laplacian on $L^2(\mathbb{R}^d)$, i.e. the multiplication operator of $|p|^2/2$, $p \in \mathbb{R}^d$. If $g_k \in C_0^\infty(\mathbb{R}^d)$ and g_k are continuous with respect to $|x|$ for any $k = 1, \ldots, N$. Then g_k satisfies condition (A) and we have that

$$
\lim_{\varepsilon\searrow 0}\mathrm{Im}\left\langle g_k,(\nu-h_{0,k}-i\varepsilon)^{-1}g_k\right\rangle=\begin{cases} C(d)\nu^{\frac{d-1}{2}}\left|g_k(\sqrt{2\nu})\right|^2 & (\nu\geq 0),\\ 0 & (\nu<0),\end{cases} \tag{9.1}
$$

where $C(d)$ is a constant depending on the dimension d. Thus, we can find $\lambda > 0$ and $\Omega > 0$ such that the function $\eta(z)$, defined in (6.6), satisfies condition (B). The PF weight for $|x|^2/2$ is the delta function $\delta(x)$. Since $d \geq 3$ and $g_k \in C_0^\infty(\mathbb{R}^d)$, the form factor g_k satisfies condition (D). We fix such g, λ, and Ω.

For any $\psi \in \bigoplus_{k=1}^N C_0^\infty(\mathbb{R}^d)$, we can see that (ψ, g) satisfies condition (Abs) by using Mourre estimate techniques. (See e.g. [38] and [39].) Since $d \geq 3$, $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}((h_{0,k})^{-1/2})$. We put $\mathfrak{h} = \bigoplus_{k=1}^N C_0^\infty(\mathbb{R}^d)$.

Since the form factor g_k is in $C_0^\infty(\mathbb{R}^d)$, there is a compact set $K_k \subset \mathbb{R}^d$ such that $\mathrm{supp} g_k \subset K_k$ for each $k = 1, \dots, N$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the Hilbert spaces defined by

$$\mathcal{K}_1 = \mathbb{C} \oplus \left(\bigoplus_{k=1}^N L^2(K_k) \right), \quad \mathcal{K}_2 = 0 \oplus \left(\bigoplus_{k=1}^N L^2(\mathbb{R}^d \setminus K_k) \right). \tag{9.2}$$

Note that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, $h \restriction_{\mathcal{K}_2} = h_0 \restriction_{\mathcal{K}_2}$, $h\mathcal{K}_1 \subset \mathcal{K}_1$, and $h\mathcal{K}_2 \subset \mathcal{K}_2$. As a consequence, we have that

$$e^{ith} = e^{ith} \restriction_{\mathcal{K}_1} \oplus e^{ith_0} \restriction_{\mathcal{K}_2} \tag{9.3}$$

on $\mathcal{K}_1 \oplus \mathcal{K}_2$. Since $h \restriction_{\mathcal{K}_1}$ and $h_0 \restriction_{\mathcal{K}_1}$ are bounded, we can use Theorems 6.3 and 7.3 and obtain an explicit formula for $e^{ith} \restriction_{\mathcal{K}_1}$.

For $k = 1, \dots, N$, we set $\mu_k = 0$ and

$$\Theta_k^{(1)}(\alpha) = e^{i\tau_k} D_k^{1/2} \alpha + e^{-i\tau_k} D_k^{1/2} \overline{\alpha}, \tag{9.4}$$

$$\Theta_k^{(2)}(\alpha) = s_{1,k} D_k^{1/2} \mathrm{Re}\alpha + s_{2,k} D_k^{1/2} \mathrm{Im}\alpha, \tag{9.5}$$

where $\alpha \in \mathbb{C}$, $\tau_k \in [0, 2\pi)$, $D_k > 0$, $s_1, s_2 \in \mathbb{R}$ and $\overline{\alpha}$ is the complex conjugate for α. The terms $\Theta_k^{(1)}$ and $\Theta_k^{(2)}$ appear in a factor decomposition of quasi-free states with BEC. See [7, Section 5.2.5], [26, (1.18)], and [20, Theorem 4.5]. For $\psi_k \in C_0^\infty(\mathbb{R}^d)$, we define the initial states $\omega_0^{(1)}$ and $\omega_0^{(2)}$ by

$$\omega_0^{(1)}(W((\psi_k))) = \exp\left\{ -\frac{1}{2} \langle \psi_k, (\mathcal{N}_k(h_{0,k}) + 1/2)\psi_k \rangle + i\Theta_k^{(1)}(\langle v_k, \psi_k \rangle) \right\}, \tag{9.6}$$

$$\omega_0^{(2)}(W((\psi_k))) = \exp\left\{ -\frac{1}{2} \langle \psi_k, (\mathcal{N}_k(h_{0,k}) + 1/2)\psi_k \rangle + i\Theta_k^{(2)}(\langle v_k, \psi_k \rangle) \right\}, \tag{9.7}$$

where

$$\mathcal{N}_k(x) = (e^{\beta_k x} - 1)^{-1}. \tag{9.8}$$

We assume that $\omega_0^{(1)}$ and $\omega_0^{(2)}$ satisfy condition (C). Since the vectors $g, \psi \in \mathfrak{h}$ satisfy conditions (Abs), (A), (B), and (D), there exist $\omega_+^{(1)}$ and $\omega_+^{(2)}$ and we have that

$$\begin{aligned} \omega_+^{(1)}(J_l) =& 2\pi\lambda^4 \sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2} (\mathcal{N}_l(\nu) - \mathcal{N}_k(\nu)) \frac{d\langle g_k, E_k(\nu) g_k \rangle}{d\nu} d\langle g_l, E_l(\nu) g_l \rangle \\ &+ \frac{4\pi^3 \lambda^2}{\eta(0)} D_l^{1/2} \sum_{k=1}^N D_k^{1/2} \mathrm{Im}\left(e^{i(\tau_k - \tau_l)} g_k(0) \overline{g_l(0)} \right), \end{aligned} \tag{9.9}$$

$$\begin{aligned} \omega_+^{(2)}(J_l) =& 2\pi\lambda^4 \sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(\nu)|^2} (\mathcal{N}_l(\nu) - \mathcal{N}_k(\nu)) \frac{d\langle g_k, E_k(\nu) g_k \rangle}{d\nu} d\langle g_l, E_l(\nu) g_l \rangle \\ &+ \frac{\pi^3 \lambda^2}{\eta(0)} D_l^{1/2} \sum_{k=1}^N D_k^{1/2} \left\{ (s_{1,k} s_{1,l} + s_{2,k} s_{2,l}) \mathrm{Im}\left(g_k(0) \overline{g_l(0)} \right) + (s_{1,k} s_{2,l} - s_{1,l} s_{2,k}) \mathrm{Re}\left(g_k(0) \overline{g_l(0)} \right) \right\}. \end{aligned} \tag{9.10}$$

9.2 Graphs

In this subsection, we give examples of currents on both periodic graphs and comb graphs. To verify our conditions for the adjacency operators of undirected graphs, we apply results of M. Măntoiu et al. [25].

9.2.1 Adapted Graphs

We recall the definition of adapted graphs introduced in [25]. Let $G = (VG, EG)$ be an undirected graph. For any $x \in VG$, we denote the set of neighbors of x by $N_G(x)$, i.e. $N_G(x) := \{ y \in VG \mid (x, y) \in EG \}$.

Definition 9.1. [25, Definition 3.1.] *A function $\Phi : G \to \mathbb{R}$ is adapted to the graph G if the following conditions hold:*

(i) *There exists $c \geq 0$ such that $|\Phi(x) - \Phi(y)| \leq c$ for any $x, y \in VG$ with $(x, y) \in EG$.*

(ii) *For any $x, y \in VG$, one has*

$$\sum_{z \in N(x) \cap N(y)} \{2\Phi(z) - \Phi(x) - \Phi(y)\} = 0. \tag{9.11}$$

(iii) *For any $x, y \in VG$, one has*

$$\sum_{z \in N(x) \cap N(y)} \{\Phi(z) - \Phi(x)\}\{\Phi(z) - \Phi(y)\}\{2\Phi(z) - \Phi(x) - \Phi(y)\} = 0. \tag{9.12}$$

A pair (G, Φ) is said to be an adapted graph if Φ is adapted to a graph G.

Let (G, Φ) be an adapted graph. We define an unbounded multiplication operator Φ on $\ell^2(VG)$ by $(\Phi f)(x) = \Phi(x) f(x)$, $x \in VG$, where $f \in \ell^2(VG)$ with $\sum_{x \in VG} \Phi(x)^2 |f(x)|^2 < \infty$. We define an operator K on $\ell^2(VG)$ by

$$(K\xi)(x) := i \sum_{y \in N(x)} \{\Phi(y) - \Phi(x)\}\xi(y), \quad \xi \in \ell^2(VG), \quad x \in VG. \tag{9.13}$$

The operator K is self-adjoint and bounded by condition (i) in Definition 9.1. Note that K and A_G are commutative. Since K is self-adjoint, we see the orthogonal decomposition of $\ell^2(VG)$ as

$$\ell^2(VG) = \ker K \oplus \overline{\mathrm{ran} K}, \tag{9.14}$$

where $\mathrm{ran}K$ denotes the range of K. We denote the restriction of A_G onto $\overline{\mathrm{ran}K}$ by $A_{G,0}$.

Theorem 9.2. [25, Theorem 3.3] *Let (G, Φ) be an adapted graph.*

(i) *For any $\xi \in \mathrm{ran}K \cap \mathcal{D}(\Phi)$, there exists a constant $c_\xi > 0$ such that*

$$\sup_{\mu \in \mathbb{R}, \varepsilon > 0} \left| \left\langle \xi, (\mu - A_{G,0} \pm i\varepsilon)^{-1} \xi \right\rangle \right| \leq c_\xi. \tag{9.15}$$

(ii) *The operator $A_{G,0}$ has purely absolutely continuous spectrum.*

9.2.2 Radon–Nikodym Derivative for the Spectral Measure of the Adjacency Operators

In this subsection, we review the Radon–Nikodym derivative of adjacency operators of an adapted graphs (G, Φ). We use the same notation as used in Section 9.2.1. By Theorem 9.2, we have the following lemmas:

Lemma 9.3. *Let (G, Φ) be an adapted graph. Then for any $\xi, \zeta \in \mathrm{ran}K \cap \mathcal{D}(\Phi)$, there exists $c_{\xi,\zeta} > 0$ such that*

$$\sup_{\mu \in \mathbb{R}, \varepsilon > 0} \left| \left\langle \xi, (\mu - A_{G,0} \pm i\varepsilon)^{-1} \zeta \right\rangle \right| < c_{\xi,\zeta}. \tag{9.16}$$

Proof. By polarization identity, we have that

$$4\left\langle \xi, (\mu - A_{G,0} \pm i\varepsilon)^{-1}\zeta \right\rangle = \sum_{k=0}^{3} (-i)^k \left\langle (\xi + i^k\zeta), (\mu - A_{G,0} \pm i\varepsilon)^{-1}(\xi + i^k\zeta) \right\rangle. \tag{9.17}$$

Since $\xi + i^k\zeta \in \operatorname{ran} K \cap \mathcal{D}(\Phi)$, we obtain the statement by Theorem 9.2. ■

By the above lemma, for any $\zeta, \xi \in \operatorname{ran} K \cap \mathcal{D}(\Phi)$, the function $\left\langle \xi, (z - A_{G,0})^{-1}\zeta \right\rangle$ is in $\mathbb{H}^\infty(\mathbb{C}_+)$.

Lemma 9.4. *For any $\xi, \zeta \in \operatorname{ran} K \cap \mathcal{D}(\Phi)$, the function $f_{\xi,\zeta}(z) := \left\langle \xi, (z - A_{G,0})^{-1}\zeta \right\rangle$ is in $\mathbb{H}^\infty(\mathbb{C}_+)$. Moreover, the limit*

$$f^+_{\xi,\zeta}(x) := \lim_{\varepsilon \searrow 0} f_{\xi,\zeta}(x + i\varepsilon) \tag{9.18}$$

exists for a.e. $x \in \mathbb{R}$ with respect to Lebesgue measure and $f^+_{\xi,\zeta} \in L^\infty(\mathbb{R}, dx)$.

Proof. Since $A_{G,0}$ is self-adjoint, $f_{\xi,\zeta}$ is holomorphic in $\mathbb{C}_+$. By [18, Theorem 3.13] and Lemma 9.3, we have the statement. ■

For any $\xi, \zeta \in \operatorname{ran} K \cap \mathcal{D}(\Phi)$, there exists the Radon–Nikodym derivative $d\langle \xi, E(\nu)\zeta \rangle / d\nu$, where E is the spectral measure of $A_{G,0}$.

Lemma 9.5. *For any $\xi, \zeta \in \operatorname{ran} K \cap \mathcal{D}(\Phi)$, the Radon–Nikodym derivative $d\langle \xi, E(\nu)\zeta \rangle / d\nu$ is in $L^p(\mathbb{R})$ for any $p \in \mathbb{N} \cup \{\infty\}$. Moreover, $d\langle \xi, E(\nu)\zeta \rangle / d\nu$ is given by*

$$\frac{d\langle \xi, E(\nu)\zeta \rangle}{d\nu} = \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \left\{ \left\langle \xi, (\nu - A_{G,0} - i\varepsilon)^{-1}\zeta \right\rangle - \left\langle \xi, (\nu - A_{G,0} + i\varepsilon)^{-1}\zeta \right\rangle \right\}, \quad \text{a.e. } x \in \mathbb{R} \tag{9.19}$$

with respect to Lebesgue measure. The support of the function $d\langle \xi, E(\nu)\zeta \rangle / d\nu$ is contained in the spectrum of $A_{G,0}$.

Proof. Since the measure $d\langle \xi, E(\lambda)\zeta \rangle$ is a complex-valued finite measure and absolutely continuous with respect to Lebesgue measure, the Radon–Nikodym derivative $d\langle \xi, E(\nu)\zeta \rangle / d\nu$ is in $L^1(\mathbb{R})$. Note that

$$\frac{d\langle \xi, E(\nu)\zeta \rangle}{d\nu} = \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \left\{ \left\langle \xi, (\nu - A_{G,0} - i\varepsilon)^{-1}\zeta \right\rangle - \left\langle \xi, (\nu - A_{G,0} + i\varepsilon)^{-1}\zeta \right\rangle \right\}, \quad \text{a.e. } x \in \mathbb{R} \tag{9.20}$$

with respect to Lebesgue measure by [18, Theorem 4.15] and polarization identity. Since $\left\langle \xi, (\nu - A_{G,0} \pm i0)^{-1}\zeta \right\rangle$ is in $L^\infty(\mathbb{R})$ by Lemma 9.4, $d\langle \xi, E(\nu)\zeta \rangle / d\nu$ is in $L^p(\mathbb{R})$ for any $p \in \mathbb{N} \cup \{\infty\}$. If $\nu \notin [-\|A_G\|, \|A_G\|]$, then $(\nu - A_{G,0})^{-1}$ is bounded, and

$$\lim_{\varepsilon \to 0} (\nu - A_{G,0} \pm i\varepsilon)^{-1} = (\nu - A_{G,0})^{-1} \tag{9.21}$$

in the operator norm. Thus, the lemma follows. ■

9.2.3 Case of $\mathbb{Z}^d$, $d \geq 3$

In this subsection, we consider $\mathbb{Z}^d$, $d \geq 3$, as graphs. We note that $\mathbb{Z}^d$ has an adapted function Φ defined by

$$\Phi((x_1, \ldots, x_d)) = \sum_{k=1}^{d} x_k, \quad x_k \in \mathbb{Z}. \tag{9.22}$$

Then the operator K defined in (9.13) is of the form

$$K\delta_x = i\sum_{k=1}^{d}(\delta_{x-e_k} - \delta_{x+e_k}), \tag{9.23}$$

where $x, e_k \in \mathbb{Z}^d$, e_k is the element of $\mathbb{Z}^d$, and the k-th component of e_k is 1 and otherwise 0. We put $\mathfrak{K}_k = \ell^2(\mathbb{Z}^d)$, $h_{0,k} = \|A_{\mathbb{Z}^d}\|\mathbb{1} - A_{\mathbb{Z}^d}$, and

$$\mathfrak{h}_k := \operatorname{span}\{\, e^{ith_{0,k}}\delta_x \mid t \in \mathbb{R}, x \in \mathbb{Z}^d \,\} \tag{9.24}$$

for each $k = 1, \ldots, N$. We set

$$g_k = K\delta_{x_k} = i\sum_{j=1}^{d}(\delta_{x_k-e_j} - \delta_{x_k+e_j}) \tag{9.25}$$

for some $x_k \in \mathbb{Z}^d$. By Theorem 9.2, g_k satisfies condition (A). Using the Fourier transformation, we see that

$$\lim_{\varepsilon\searrow 0}\operatorname{Im}\Big\langle g_k, (\nu - h_{0,k} - i\varepsilon)^{-1}g_k\Big\rangle = (2\pi)^{-d/2}\pi\int_{\mathbb{T}^d}\delta(\nu - \sum_{j=1}^{d}\sin^2(\theta_j/2))\left|\sum_{j=1}^{d}\sin\theta_j\right|^2 d\theta. \tag{9.26}$$

Suppose that $\lambda^2 > 0$, $\lambda^2 C_g << 1$, and there exists $\Omega \in (0, \big\|h_{0,0}\big\|)$ such that the right hand side of (9.26) has some strictly positive lower bound for any $\nu \in [\Omega - \lambda^2 C_g, \Omega + \lambda^2 C_g]$. Thus, condition (B) is satisfied. Since $h_{0,k}$ is transient by $d \geq 3$, the form factor g satisfies condition (D). Note that $\bigoplus_{k=1}^{N}(\mathfrak{h}_k \cap \operatorname{ran}K) \subset \mathfrak{h}(g)$ by Lemma 9.3, where $\mathfrak{h}(g)$ is the set defined in (6.9). For initial states $\omega_0^{(1)}$ and $\omega_0^{(2)}$ defined in (9.6) and (9.7) with condition (C), there exist NESS $\omega_+^{(1)}$ and $\omega_+^{(2)}$ which are states on $\mathcal{W}(\mathfrak{k}, \sigma)$, where $\mathfrak{k} = \mathbb{C} \oplus \Big(\bigoplus_{k=1}^{N}(\mathfrak{h}_k \cap \operatorname{ran}K)\Big)$. If the PF weight v is defined by $v(x) = 1$ for any $x \in \mathbb{Z}^d$, then $\langle v_k, g_k\rangle = 0$ for any $k = 1, \ldots, N$. Thus, $\operatorname{Jos}_l(\omega_+) = 0$ for any $l = 1, \ldots, N$.

9.2.4 Regular Admissible Graphs

A graph G is called regular, if for any $x, y \in VG$, $\deg_G(x) = \deg_G(y)$. Recall the definition of admissible graphs (cf. [25]). In this subsection, we assume that G is deduced from a directed graph, i.e., some relation $<$ is given on G such that, for any $x, y \in VG$, $x \sim y$ is equivalent to $x < y$ or $y < x$, and one can not have both $y < x$ and $x < y$. We also write $y > x$ for $x < y$. Then for any $x \in VG$, the neighbor of x, $N_G(x)$, is decomposed into a disjoint union $N_G(x) = N_G^+(x) \cup N_G^-(x)$, where

$$N_G^+(x) := \{\, y \in VG \mid x < y \,\}, \quad N_G^-(x) := \{\, y \in VG \mid y < x \,\}. \tag{9.27}$$

When directions have been fixed, we use the notation $(G, <)$ for the directed graph and say that $(G, <)$ is subjacent to G.

Let $p = x_0x_1\cdots x_n$ be a path. We define the index of path p by

$$\operatorname{ind}(p) := \Big|\{\, j \mid x_{j-1} < x_j \,\}\Big| - \Big|\{\, j \mid x_{j-1} > x_j \,\}\Big|. \tag{9.28}$$

Definition 9.6. [25, Definition 5.1] *A directed graph $(G, <)$ is called admissible if*

(i) *It is univoque, i.e., any closed path in G has index zero.*

(ii) *It is uniform, i.e., for any $x, y \in G$, $\sharp\Big(N_G^-(x) \cap N_G^-(y)\Big) = \sharp\Big(N_G^+(x) \cap N_G^+(y)\Big)$.*

A graph G is called admissible if there exists an admissible directed graph $(G, <)$ subjacent to G.

Definition 9.7. [25, Definition 5.2] *A position function on a directed graph $(G, <)$ is a function $\Phi : G \to \mathbb{Z}$ satisfying $\Phi(x) + 1 = \Phi(y)$ if $x < y$.*

Lemma 9.8. [25, Lemma 5.3]

(i) *A directed graph $(G, <)$ is univoque if and only if it admits a position function.*

(ii) *Any position function on an admissible graph G is surjective.*

(iii) *A position function on a directed graph G is unique up to constant.*

Remark 9.9. *If G is an admissible graph, then there exists a position function Φ. The function Φ satisfies Definition 9.1. Thus, an admissible graph is an adapted graph as well.*

Remark 9.10. *When G is an infinite regular graph, we only consider the PF weight v for the adjacency operator A_G such that $v(x) = 1$ for any $x \in VG$.*

Proposition 9.11. *Let G be an admissible regular graph. Assume that $g \in \mathfrak{h} \cap \operatorname{ran} K$, where $\mathfrak{h}$ is defined by*

$$\mathfrak{h} := \operatorname{span} \{ e^{it(\|A_G\|\mathbb{1} - A_G)} \delta_x \mid x \in VG, t \in \mathbb{R} \}. \tag{9.29}$$

Then $\langle v, g \rangle = 0$.

Proof. Note that $\mathfrak{h} \subset \mathcal{D}(v)$ by [13, Theorem 4.5]. Since $g \in \mathfrak{h} \cap \operatorname{ran} K$, there exists $\zeta \in \ell^2(VG)$ such that $g = K\zeta$, where K is the operator defined in (9.13). Then the vector g is of the form

$$\langle \delta_x, g \rangle = i \sum_{y \in N_G^+(x)} \zeta(y) - i \sum_{y \in N_G^-(x)} \zeta(y), \tag{9.30}$$

where $\zeta(y) = \left\langle \delta_y, \zeta \right\rangle$. Then we have that

$$\langle v, g \rangle = i \left\{ \sum_{x \in VG} \sum_{y \in N_G^+(x)} \zeta(y) - \sum_{x \in VG} \sum_{y \in N_G^-(x)} \zeta(y) \right\} = 0. \tag{9.31}$$

Thus, the proposition is proven.■

By the above proposition, we have the following theorem:

Theorem 9.12. *Let G_k, $k = 1, \ldots, N$, be admissible regular graphs. Fix $g \in \bigoplus_{k=1}^N (\mathfrak{h}_k \cap \operatorname{ran} K_k)$, where $\mathfrak{h}_k$ is defined in (9.29) and K_k is the operator defined in (9.13). For any $k = 1, \ldots, N$, we assume that $h_{0,k} = \left\|A_{G_k}\right\| \mathbb{1} - A_{G_k}$ is transient, there exist $\Omega, \lambda > 0$ such that the function $\eta(z)$ defined in (6.6) satisfies condition (B), and the initial state ω_0 satisfies condition (C). Then there exists NESS ω_+ which is a state on $\mathcal{W}(\mathfrak{k}, \sigma)$, where $\mathfrak{k} = \mathbb{C} \oplus \bigoplus_{k=1}^N (\mathfrak{h}_k \cap \operatorname{ran} K_k)$. Moreover, for any $l = 1, \ldots, N$, we have that*

$$\omega_+(J_l) = 2\pi\lambda^4 \sum_{k=1}^N \int_{\sigma_l} \frac{1}{|\eta_-(v)|^2} (\mathcal{N}_l(v) - \mathcal{N}_k(v)) \frac{d\langle g_k, E_k(v) g_k \rangle}{dv} d\langle g_l, E_l(v) g_l \rangle, \tag{9.32}$$

where J_l is defined in (8.1).

Proof. By assumptions, Theorem 7.3, Corollary 8.2, and Proposition 9.11, we can prove the theorem.■

9.3 Comb Graphs

In this subsection, we consider typical example of non-regular graphs: comb graphs. BEC on comb graphs is studied in [10], [11], and [13]. In [10], R. Burioni, D. Cassi, M. Rasetti, P. Sodano, and A. Vezzani calculated the spectral measure of the adjacency operators of comb graphs $\mathbb{Z}^d \dashv \mathbb{Z}$. Using their results, we calculate currents on comb graphs. First, we recall the definition of comb graphs.

Definition 9.13. *Let G_1 and G_2 be graphs, and let $o \in VG_2$ be a given vertex. Then the comb product $X := G_1 \dashv (G_2, o)$ is a graph with vertex $VX := VG_1 \times VG_2$, and $(g_1, g_2), (g_1', g_2') \in VX$ are adjacent if and only if $g_1 = g_1'$ and $g_2 \sim g_2'$ or $g_2 = g_2' = o$ and $g_1 \sim g_1'$. We call G_1 the base graph, and G_2 the fiber graph.*

We consider the graphs $G_d := \mathbb{Z}^d \dashv (\mathbb{Z}, 0)$, $d \geq 3$. As the case of $\mathbb{Z}^d$, the function Φ defined in (9.22) is adapted to G_{d-1}. Put $h_{0,l} = \left\|A_{G_d}\right\| \mathbb{1} - A_{G_d}$ for any $l = 1, \dots, N$. For $J \in \mathbb{Z}^d$ and $x \in \mathbb{Z}$, the operators K_l, $l = 1, \dots, N$, have the form of

$$K\delta_{J,x} = \begin{cases} i\delta_{J,x-1} - i\delta_{J,x+1} & (x \neq 0) \\ i\sum_{l=1}^{d} \left(\delta_{J-e_l,x} - \delta_{J+e_l,x}\right) + i\delta_{J,-1} - i\delta_{J,1} & (x = 0) \end{cases}. \tag{9.33}$$

Put $g_l = K\delta_{J_l,x_l}$, $l = 1, \dots, N$, where $J_l \in \mathbb{Z}^d$ and $x_l \in \mathbb{Z}$ with $|x_l| >> 1$. Then, by Theorem 9.2 and [11, Theorem 10.14], the form factor g satisfies conditions (A) and (D). By [11, Lemma 9.4], a PF weight v has the following form:

$$v(J, x) = \frac{e^{-|x|\theta_d}}{2\left\|(2\sqrt{d^2+1} - A_{\mathbb{Z}})^{-1}\delta_0\right\| \sinh\theta_d}, \quad J \in \mathbb{Z}^d, \quad x \in \mathbb{Z}, \tag{9.34}$$

with $2\cosh\theta_d = 2\sqrt{d^2+1}$. Another example of v is given in [13]. The form of the spectral measure of A_{G_d} is in [10]. Thus, we can find $\Omega, \lambda > 0$ which satisfy condition (B).

The pairing of g_l and the PF weight $v_l = v$ is given by

$$\begin{aligned} \langle v, g_l \rangle =& i\frac{e^{-|x_l-1|\theta_d}}{2\left\|(2\sqrt{d^2+1} - A_{\mathbb{Z}})^{-1}\delta_0\right\| \sinh\theta_d} - i\frac{e^{-|x_l+1|\theta_d}}{2\left\|(2\sqrt{d^2+1} - A_{\mathbb{Z}})^{-1}\delta_0\right\| \sinh\theta_d} \\ =& i\frac{e^{-|x_l|\theta_d}}{\left\|(2\sqrt{d^2+1} - A_{\mathbb{Z}})^{-1}\delta_0\right\|}. \end{aligned} \tag{9.35}$$

Thus, we define the initial states $\omega_0^{(1)}$ and $\omega_0^{(2)}$ by equations (9.6) and (9.7). Note that $\mathfrak{k} := \bigoplus_{k=1}^{N}(\mathfrak{h}_k \cap \mathrm{ran}K) \subset \mathfrak{h}(g)$ by Lemma 9.3. Thus, there exist NESS $\omega_+^{(1)}$ and $\omega_+^{(2)}$, which are states on $\mathcal{W}(\mathfrak{k}, \sigma)$. If the temperatures are identical, then Josephson currents are given by

$$\omega_+^{(1)}(J_l) = \mathrm{Jos}_l(\omega_+^{(1)}) = \frac{4\pi^3\lambda^2}{\eta(0)} \sum_{k=1}^{N} D_k^{1/2} D_l^{1/2} \sin(\tau_k - \tau_l) \frac{e^{-(|x_k|+|x_l|)\theta_d}}{\left\|(2\sqrt{d^2+1} - A_{\mathbb{Z}})^{-1}\delta_0\right\|^2}, \tag{9.36}$$

$$\omega_+^{(2)}(J_l) = \mathrm{Jos}_l(\omega_+^{(2)}) = \frac{\pi^3\lambda^2}{\eta(0)} \sum_{k=1}^{N} D_k^{1/2} D_l^{1/2} \{s_{1,k}s_{2,l} - s_{1,l}s_{2,k}\} \frac{e^{-(|x_k|+|x_l|)\theta_d}}{\left\|(2\sqrt{d^2+1} - A_{\mathbb{Z}})^{-1}\delta_0\right\|^2}. \tag{9.37}$$

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