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Study on the large time behavior of solutions of the compressible Navier-Stokes equations under the slip boundary condition

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Abstract

The large time behavior of solutions to the compressible Navier-Stokes equations is considered under the slip boundary condition in an infinite cylinder of $\mathbb{R}^3$, $n = 2, 3$. In the case of $n = 2$, it is shown that if the initial data is sufficiently small, then the global solution uniquely exists and the large time behavior of the solution is described by a superposition of one-dimensional nonlinear diffusion waves. In the case of $n = 3$, it is shown that if the initial data is smooth and sufficiently close to the motionless state, then the global solution uniquely exists and the large time behavior of the solution is described by a superposition of one-dimensional nonlinear diffusion waves and a diffusive rigid rotation.

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Results in Chapter 2 were obtained in a joint research with Yoshiyuki Kagei, and published in MI 2018-4, MI Preprint Series, Mathematics for Industry Kyushu University.

Contents

Introduction

This thesis studies the large time behavior of solutions to the compressible Navier-Stokes equations:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{v}) = 0, \tag{0.0.1}$$

$$\rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \mu \mathrm{div} \mathbf{D}(\boldsymbol{v}) - \mu' \nabla \mathrm{div} \boldsymbol{v} + \nabla p(\rho) = \mathbf{0} \tag{0.0.2}$$

in an infinite cylinder $\Omega_\ell = \mathbb{R} \times D_\ell$ of $\mathbb{R}^n$, $n = 2, 3$, where

$$D_\ell = \begin{cases} \{x_2; 0 < x_2 < \ell\} & (n = 2) \\ \{x' = (x_2, x_3); \sqrt{x_2^2 + x_3^2} < \ell\} & (n = 3). \end{cases}$$

Here $\rho = \rho(x,t)$ and $\boldsymbol{v} = {}^\top(v^1(x,t), \cdots, v^n(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \Omega_\ell$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ and satisfies

$$p'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy

$$\mu > 0, \quad \frac{2}{n}\mu + \mu' > 0;$$

div and ∇ denote the usual divergence and gradient with respect to x; $\mathbf{D}(\cdot)$ denotes the deformation tensor whose (j,k)-components $(j, k = 1, \cdots, n)$ are given by

$$\mathbf{D}(\boldsymbol{v})_{jk} = \partial_{x_j} v^k + \partial_{x_k} v^j.$$

Here and in what follows ${}^\top\cdot$ stands for the transposition.

We consider (0.0.1)-(0.0.2) under the slip boundary condition

$$\partial_{x_2} v^1|_{x_2=0,\ell} = 0, \quad v^2|_{x_2=0,\ell} = 0 \;\text{ if }\; n = 2, \tag{0.0.3}$$

$$\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_\ell} = 0, \quad \mathbf{D}(\boldsymbol{v}) \cdot \boldsymbol{n} - \big(\mathbf{D}(\boldsymbol{v})\boldsymbol{n} \cdot \boldsymbol{n}\big)\boldsymbol{n}\big|_{\partial\Omega_\ell} = \mathbf{0} \;\text{ if }\; n = 3. \tag{0.0.4}$$

Here $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega_\ell$, which is given by $\boldsymbol{n} = {}^\top(0, \boldsymbol{n}')$ with $\boldsymbol{n}' = \frac{1}{\ell}x' = \frac{1}{\ell}{}^\top(x_2, x_3)$ being the unit outward normal vector to ∂D_ℓ.

We impose the initial condition

$$\rho|_{t=0} = \rho_0, \quad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0. \tag{0.0.5}$$

Here $\rho_0 = \rho_0(x)$ and $\boldsymbol{v}_0 = \boldsymbol{v}_0(x)$ satisfy $\rho_0(x) \to \rho_*$ and $\boldsymbol{v}_0(x) \to \mathbf{0}$ as $|x| \to \infty$.

In this thesis we will consider the stability of the motionless state $u_s = {}^\top(\rho_*, \mathbf{0})$ and will investigate the large time behavior of solutions around u_s. We thus rewrite (0.0.1)-(0.0.2) into the following equations for the perturbation:

$$\partial_t \phi + \gamma \mathrm{div} \boldsymbol{w} = f^0(\phi, \boldsymbol{w}), \tag{0.0.6}$$

$$\partial_t \boldsymbol{w} - \nu \mathrm{div} \mathbf{D}(\boldsymbol{w}) - \nu' \nabla \mathrm{div} \boldsymbol{w} + \gamma \nabla \phi = \boldsymbol{f}(\phi, \boldsymbol{w}). \tag{0.0.7}$$

Here $u = {}^\top(\phi, \boldsymbol{w})$ with $\phi = \frac{1}{\rho_*}(\rho - \rho_*)$ and $\boldsymbol{w} = \frac{1}{\gamma}\boldsymbol{v}$ denotes the perturbation of $u_s = {}^\top(\rho_*, \mathbf{0})$; ν, ν' and γ are parameters given by

$$\nu = \frac{\mu}{\rho_*}, \quad \nu' = \frac{\mu'}{\rho_*}, \quad \gamma = \sqrt{p'(\rho_*)};$$

and $f(\phi, \boldsymbol{w}) = {}^\top(f^0(\phi, \boldsymbol{w}), \boldsymbol{f}(\phi, \boldsymbol{w}))$ denotes the nonlinear terms:

$$\begin{aligned} f^0(\phi, \boldsymbol{w}) &= -\gamma \mathrm{div}(\phi \boldsymbol{w}), \\ \boldsymbol{f}(\phi, \boldsymbol{w}) &= -\gamma \boldsymbol{w} \cdot \nabla \boldsymbol{w} - \frac{\phi}{1+\phi}\{\nu \mathrm{div} \mathbf{D}(\boldsymbol{w}) + \nu' \nabla \mathrm{div} \boldsymbol{w}\} + \frac{\gamma \phi}{1+\phi} \nabla \phi \\ &\qquad - \frac{\rho_* p''(\rho_*)}{2\gamma(1+\phi)} \nabla(\phi^2) - \frac{\rho_*^2}{2\gamma(1+\phi)} \nabla\big(p^{(3)}(\phi)\phi^3\big), \end{aligned}$$

where

$$p^{(3)}(\phi) = \int_0^1 (1-\theta)^2 p'''\big(\rho_*(1+\theta\phi)\big) \mathrm{d}\theta.$$

The boundary conditions (0.0.3)-(0.0.4) and initial condition (0.0.5) are transformed into

$$\partial_{x_2} w^1|_{x_2=0,\ell} = 0, \quad w^2|_{x_2=0,\ell} = 0 \ \text{ if } \ n = 2, \tag{0.0.8}$$

$$\boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega_\ell} = 0, \quad \mathbf{D}(\boldsymbol{w}) \cdot \boldsymbol{n} - \big(\mathbf{D}(\boldsymbol{w})\boldsymbol{n} \cdot \boldsymbol{n}\big)\boldsymbol{n}\big|_{\partial\Omega_\ell} = \mathbf{0} \ \text{ if } \ n = 3, \tag{0.0.9}$$

and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, \boldsymbol{w}_0). \tag{0.0.10}$$

Here u_0 satisfies $u_0(x) \to 0$ as $|x| \to \infty$.

In this thesis we will show that solutions under the slip boundary condition exhibit the large time behavior completely different from the ones under the non-slip/ Navier-slip boundary conditions.

Large time behavior of solutions of the compressible Navier-Stokes equations in unbounded domains have been studied in detail in various contexts; see, e.g., [5, 6, 7, 10, 13, 15, 18, 20, 21, 25, 26, 27, 30, 33] for the cases of the multi-dimensional whole space, half space and exterior domains. In addition to these domains, problems in infinite layers and cylindrical domains have been also studied, e.g., in [2, 3, 4, 8, 9, 11, 12, 16] under the non-slip boundary condition $\boldsymbol{v}|_{x_2=0,1} = 0$. It was shown in [11, 16] that the large time behavior of perturbations of the motionless state is described by a one-dimensional linear heat equation. This kind of purely diffusive behaviors has been also observed when background flows such as stationary/time-periodic parallel flows and spatially periodic patterns appear, although in these cases the mass of perturbations not only decays diffusively but also is transported by the background flows; see [2, 3, 4, 8, 9, 12]. We also mention the work [23] by H.-L. Li and X. Zhang, where the problem under the Navier-slip boundary condition was considered and an interesting observation on the effect of the slip at the boundary was also made.

In the first part of this thesis, we consider the two-dimensional problem under the slip boundary condition. We will prove that the solution of (0.0.1)-(0.0.2) under the slip boundary condition (0.0.3) with (0.0.5) behaves like a superposition of one-dimensional diffusion waves as $t \to \infty$ as in the case of one-dimensional compressible Navier-Stokes equation, see [19] and [29]. More precisely, consider the problem (0.0.6)-(0.0.10) for u. We prove that, under appropriate conditions for u_0, the solution $u(t)$ satisfies

$$\|\partial_x^k (u - \chi_+ \boldsymbol{a}_+ - \chi_- \boldsymbol{a}_-)(t)\|_{L^2(\Omega_\ell)} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \tag{0.0.11}$$

where $\boldsymbol{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given by

$$\chi_\pm(x_1, t) = z_\pm(x_1 \pm \gamma t, t). \tag{0.0.12}$$

Here $z_\pm = z_\pm(x_1, t)$ are the self-similar solutions of the viscous Burgers equations

$$\partial_t z_\pm - \frac{\nu + \tilde{\nu}}{2} \partial_{x_1}^2 z_\pm \mp c \partial_{x_1}(z_\pm^2) = 0 \tag{0.0.13}$$

satisfying

$$\int_{\mathbb{R}} z_\pm(x_1, t) \mathrm{d}x_1 = \frac{1}{2} \int_{\Omega_\ell} (\phi_0(x) \pm (1 + \phi_0(x)) w_0^1(x)) \mathrm{d}x \tag{0.0.14}$$

for some constant $c \in \mathbb{R}$. In contrast to the case of the non-slip boundary condition, we see that a hyperbolic aspect of (0.0.1)-(0.0.2), i.e., a wave propagation phenomenon, appears in the asymptotic leading part of the solution under the slip boundary condition.

We briefly explain an outline of the proof of the result (0.0.11). To prove (0.0.11), we first establish the decay estimates for $u(t)$. We decompose the solution of (0.0.6)-(0.0.10) into its *low and high frequency* parts. The spectrum of the low-frequency part of the linearized semigroup is different from the one in the case of the non-slip boundary condition; it is the same as that in the case of the one-dimensional compressible Navier-Stokes equation. Therefore, the low-frequency part decays like one-dimensional heat kernel, namely, k-th order derivative decays in the order $O(t^{-\frac{1}{4}-\frac{k}{2}})$ in the L^2 norm. For the high-frequency part (remainder part), we apply the Matsumura-Nishida energy method (see [27]) to see that the high-frequency part decays in the order $O(t^{-\frac{5}{4}})$ in the H^2 norm. Based on the spectral properties of the low-frequency part of the linearized semigroup and the decay estimate for the high-frequency part, we deduce the asymptotic behavior (0.0.11) by applying the argument of [19].

In the second part of this thesis, we consider the three-dimensional problem under the slip boundary condition. We will show that the solution $u(t)$ of (0.0.6)-(0.0.7) under the slip boundary condition (0.0.9) with (0.0.10) behaves like a superposition of one-dimensional nonlinear diffusion waves and a diffusive rigid rotation as $t \to \infty$. More precisely, we prove that, under appropriate conditions for u_0, the solution $u(t)$ satisfies

$$\|\partial_x^k(u - \kappa_+ a_+ - \kappa_- a_- - \kappa_{\mathrm{rig}} a_{\mathrm{rig}})(t)\|_{L^2(\Omega_\ell)} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \tag{0.0.15}$$

where $a_\pm = \frac{1}{2}{}^\top(1, \pm 1, 0, 0)$ and $\kappa_\pm = \kappa_\pm(x_1, t)$ are the nonlinear diffusion waves given by

$$\kappa_\pm(x_1, t) = Z_\pm(x_1 \pm \gamma t, t). \tag{0.0.16}$$

Here $Z_\pm = Z_\pm(x_1, t)$ are the self-similar solutions of the Burgers equations

$$\partial_t Z_\pm - \frac{2\nu + \nu'}{2}\partial_{x_1}^2 Z_\pm \mp c\partial_{x_1}(Z_\pm^2) = 0 \tag{0.0.17}$$

satisfying

$$\int_{\mathbb{R}} Z_\pm(x_1, t)\mathrm{d}x_1 = \frac{1}{2}\int_{\Omega_\ell} \big(\phi_0(x) \pm (1 + \phi_0(x))w_0^1(x)\big)\mathrm{d}x \tag{0.0.18}$$

for some constant $c \in \mathbb{R}$; and

$$a_{\mathrm{rig}} = {}^\top(0, \boldsymbol{a}_{\mathrm{rig}}), \quad \boldsymbol{a}_{\mathrm{rig}} = \frac{1}{\ell^2}\sqrt{\frac{2}{\pi}}\,{}^\top(0, -x_3, x_2), \tag{0.0.19}$$

$$\kappa_{\mathrm{rig}}(x_1, t) = w_{0,\mathrm{rig}}(4\pi\nu t)^{-\frac{1}{2}} e^{-\frac{x_1^2}{4\nu t}} \tag{0.0.20}$$

with $w_{0,\mathrm{rig}} = \int_{\Omega_\ell} \boldsymbol{w}_0 \cdot \boldsymbol{a}_{\mathrm{rig}}\mathrm{d}x$.

We note that, in addition to the wave propagation part $\kappa_+ a_+ + \kappa_- a_-$, the diffusive rigid motion part $\kappa_{\mathrm{rig}} a_{\mathrm{rig}}$ also appears in the asymptotic leading part of the solution in the case of the three-dimensional problem. We also note that the diffusive rigid motion part $\kappa_{\mathrm{rig}} a_{\mathrm{rig}}$ gives the incompressible part of the asymptotic leading part of u since $\mathrm{div}(\kappa_{\mathrm{rig}}\boldsymbol{a}_{\mathrm{rig}}) = 0$.

It should be remarked that the global existence with exponential decay estimate was shown by Shibata and Murata [32] for the problem on a bounded domain under the slip boundary condition (1.1.9), provided that initial data are sufficiently small, and, in addition, orthogonal to rigid motions when the domain is rotationally symmetric. The method in [32] was mainly based on the maximal regularity approach. We also mention the work [22] by Kobayashi and Zajaczkowski, where the global existence on a bounded domain was proved based on the energy method.

We briefly explain a sketch of the proof of the result (0.0.15). As in the case of $n = 2$, we first establish the decay estimates for the solution $u(t)$ of (0.0.6)-(0.0.10). We decompose $u(t)$ into its *low and high frequency* parts. As for the low frequency part, we investigate the spectrum of the low-frequency part of the linearized semigroup and show that the leading part is decomposed into the linear diffusion waves part and the diffusive rigid motion part. As a result, the low frequency part decays like a one-dimensional heat kernel, namely, k-th order derivative decays in the order $O(t^{-\frac{1}{4}-\frac{k}{2}})$ in the L^2 norm. To establish suitable decay estimates for the nonlinear problem, we introduce the momentum formulation for the low frequency part, which makes the equations a conservation form. This enables us to deal with a slowly decaying part caused by the interaction between the diffusion waves and the diffusive rigid motion. For the high frequency part, we apply the Matsumura-Nishida energy method ([27]) and show that the high frequency part decays in the order $O(t^{-\frac{3}{4}})$ in the H^2 norm. To this end, a Korn type inequality plays an important role. Combining the estimates for the low and high frequency parts, we establish the decay estimate of $u(t)$ in H^2 norm. Based on the spectral properties of the low frequency part of the linearized semigroup and the decay estimate for $u(t)$, we deduce the asymptotic behavior (0.0.11) by applying the argument of Kawashima [19].

We finally mention one of motivations of this work. For simplicity we consider the case $n = 2$. The large time behavior of solutions of (0.0.1)-(0.0.2) under the slip boundary condition (0.0.3) would be expected to approximate the behavior of solutions of (0.0.1)-(0.0.2) in a thin fluid layer under the Navier boundary condition. Let us consider (0.0.1)-(0.0.2) in the layer $\Omega_\ell = \{x = (x_1, x_2) \in \mathbb{R}^2;\ x_1 \in \mathbb{R}, 0 < x_2 < \ell\}$ with $0 < \ell \ll 1$ under the boundary condition

$$(\mu \partial_{x_2} v^1 N + k v^1)|_{x_2=0,\ell} = 0, \quad v^2|_{x_2=0,\ell} = 0, \tag{0.0.21}$$

where $k > 0$ is the friction constant and N represents the direction of the outward normal to the boundary $\partial\Omega_\ell$, and hence, $N|_{x_2=0} = -1$ and $N|_{x_2=\ell} = 1$. We introduce the non-dimensional variables $\tilde{x}$, $\tilde{t}$, $\tilde{\rho}$, $\tilde{v}$, $\tilde{p}$ defined by

$$x = \ell\tilde{x}, \quad t = T\tilde{t}, \quad \rho = \rho_*\tilde{\rho}, \quad \boldsymbol{v} = V\tilde{\boldsymbol{v}}, \quad p = \rho_* V^2 \tilde{p}$$

with $T = \frac{\rho_*\ell^2}{\mu}$ and $V = \frac{\ell}{T}$. It follows that $\tilde{\rho}$ and $\tilde{\boldsymbol{v}}$ are governed by the equations

$$\partial_{\tilde{t}}\tilde{\rho} + \mathrm{div}_{\tilde{x}}(\tilde{\rho}\tilde{\boldsymbol{v}}) = 0, \tag{0.0.22}$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{v}} \cdot \nabla_{\tilde{x}}\tilde{\boldsymbol{v}}) - \Delta_{\tilde{x}}\tilde{\boldsymbol{v}} - \frac{\mu + \mu'}{\mu}\nabla_{\tilde{x}}\mathrm{div}_{\tilde{x}}\tilde{\boldsymbol{v}} + \nabla_{\tilde{x}}\tilde{p}(\tilde{\rho}) = 0. \tag{0.0.23}$$

The domain Ω_ℓ is transformed into Ω and the boundary condition (0.0.21) becomes

$$\left.\left(\partial_{\tilde{x}_2}\tilde{v}^1 N + \frac{k\ell}{\mu}\tilde{v}^1\right)\right|_{\tilde{x}_2=0,1} = 0, \quad \tilde{v}^2|_{\tilde{x}_2=0,1} = 0. \tag{0.0.24}$$

Letting $\ell \to 0$ we obtain (0.0.1)-(0.0.2) with μ and $\mu + \mu'$ replaced by 1 and $\frac{\mu+\mu'}{\mu}$, respectively, and the slip boundary condition (0.0.3). Since $\rho(x,t) = \rho_*\tilde{\rho}(\frac{x}{\ell}, \frac{\mu t}{\rho_*\ell^2})$ and $v(x,t) = \frac{\mu}{\rho_*\ell}\tilde{v}(\frac{x}{\ell}, \frac{\mu t}{\rho_*\ell^2})$, we find that the behavior of solutions of (0.0.1)-(0.0.2) in Ω_ℓ under the Navier boundary condition (0.0.21) is expected to be approximated by the large time behavior of solutions of (0.0.1)-(0.0.2) in Ω under the slip boundary condition (0.0.3) when $0 < \ell \ll 1$.

This thesis is organized as follows. In Chapter 1, in the case of $n = 2$, we show that if the initial data is sufficiently small, then the global solution uniquely exists and the large time behavior of the solution is described by a superposition of one-dimensional nonlinear diffusion waves.

In Chapter 2, in the case of $n = 3$, we show that if the initial data is smooth and sufficiently close to the motionless state, then the global solution uniquely exists and the large time behavior of the solution is described by a superposition of one-dimensional nonlinear diffusion waves and a diffusive rigid rotation.

In each section, notation is introduced which is used throughout the chapter and the main results are stated. Continuously, the proofs of the main results are given respectively.

Chapter 1

Large time behavior of solutions to the compressible Navier-Stokes equations in an infinite layer under slip boundary condition

1.1 Formulation of the problem

This chapter studies large time behavior of solutions of the compressible Navier-Stokes equations

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div} v + \nabla P(\rho) = 0 \tag{1.1.2}$$

in an infinite layer Ω of $\mathbb{R}^2$:

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2;\ x_1 \in \mathbb{R}, 0 < x_2 < \ell\}$$

under the slip boundary condition

$$\partial_{x_2} v^1|_{x_2=0,\ell} = 0, \quad v^2|_{x_2=0,\ell} = 0. \tag{1.1.3}$$

Here $\rho = \rho(x,t) > 0$ and $v = {}^\top(v^1(x,t), v^2(x,t))$ denote the unknown density and velocity, respectively, at time $t \geq 0$ and position $x \in \Omega$; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$P'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy

$$\mu > 0, \quad \mu + \mu' \geq 0;$$

div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x. Here and in what follows ${}^\top\cdot$ means the transposition.

We impose the initial condition

$$\rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0. \tag{1.1.4}$$

Here $\rho_0 = \rho_0(x)$ and $v_0 = v_0(x)$ satisfy $\rho_0(x) \to \rho_*$ and $v_0(x) \to 0$ as $|x| \to \infty$.

The aim of this chapter is to investigate the large time behavior of solutions to (1.1.1)-(1.1.4) around the motionless state $\rho = \rho_*$, $v = 0$. We rewrite (1.1.1)-(1.1.2) into the following equations for the perturbation

$$\partial_t \phi + \gamma \mathrm{div} w = f^0(\phi, w), \tag{1.1.5}$$

$$\partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma \nabla \phi = \tilde{f}(\phi, w). \tag{1.1.6}$$

Here $u = {}^\top(\phi, w)$ with $\phi = \frac{1}{\rho_*}(\rho - \rho_*)$ and $w = \frac{1}{\gamma} v$ denotes the perturbation from $u_s = {}^\top(\rho_*, 0)$; ν, $\tilde{\nu}$ and γ are parameters given by

$$\nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_*}, \quad \gamma = \sqrt{P'(\rho_*)};$$

and $f(\phi, w) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$ denote the nonlinear terms:

$$\begin{aligned} f^0(\phi, w) &= -\gamma \mathrm{div}(\phi w), \\ \tilde{f}(\phi, w) &= -\gamma w \cdot \nabla w - \frac{\phi}{1+\phi}\{\nu \Delta w + \tilde{\nu} \nabla \mathrm{div} w\} + \frac{\gamma \phi}{1+\phi} \nabla \phi \\ &\quad - \frac{\rho_*}{\gamma(1+\phi)} \nabla (P^{(2)}(\phi) \phi^2), \end{aligned}$$

where

$$P^{(2)}(\phi) = \int_0^1 (1-\theta) P''(\rho_*(1+\theta\phi)) \mathrm{d}\theta.$$

The boundary condition (1.1.3) and initial condition (1.1.4) are transformed into

$$\partial_{x_2} w^1|_{x_2=0,\ell} = 0, \quad w^2|_{x_2=0,\ell} = 0 \tag{1.1.7}$$

and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{1.1.8}$$

Here u_0 satisfies $u_0(x) \to 0$ as $|x| \to \infty$.

In this chapter we show that the solution of (1.1.1)-(1.1.2) under the slip boundary condition (1.1.3) with (1.1.4) behaves like a superposition of one-dimensional diffusion waves as $t \to \infty$ as in the case of one-dimensional compressible Navier-Stokes equation, see [19] and [29]. More precisely, consider the problem (1.1.5)-(1.1.8) for u. We prove that, under appropriate conditions for u_0, the solution $u(t)$ satisfies

$$\|\partial_x^k (u - \chi_+ \boldsymbol{a}_+ - \chi_- \boldsymbol{a}_-)(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \tag{1.1.9}$$

where $\boldsymbol{a}_{\pm} = {}^{\top}(1, \pm 1, 0)$ and $\chi_{\pm} = \chi_{\pm}(x_1, t)$ are the diffusion waves given by

$$\chi_{\pm}(x_1, t) = z_{\pm}(x_1 \pm \gamma t, t). \tag{1.1.10}$$

Here $z_{\pm} = z_{\pm}(x_1, t)$ are the self-similar solutions of the viscous Burgers equations

$$\partial_t z_{\pm} - \frac{\nu + \tilde{\nu}}{2} \partial_{x_1}^2 z_{\pm} \mp c \partial_{x_1}(z_{\pm}^2) = 0 \tag{1.1.11}$$

satisfying

$$\int_{\mathbb{R}} z_{\pm}(x_1, t) \mathrm{d}x_1 = \frac{1}{2} \int_{\Omega} (\phi_0(x) \pm (1 + \phi_0(x)) w_0^1(x)) \mathrm{d}x \tag{1.1.12}$$

for some constant $c \in \mathbb{R}$. In contrast to the case of the non-slip boundary condition, we see that a hyperbolic aspect of (1.1.1)-(1.1.2) appears in the asymptotic leading part of the solution under the slip boundary condition.

The large time behavior of solutions of (1.1.1)-(1.1.2) under the slip boundary condition (1.1.3) would be expected to approximate the behavior of solutions of (1.1.1)-(1.1.2) in a thin fluid layer under the Navier boundary condition. Let us consider (1.1.1)-(1.1.2) in the layer $\Omega_\ell = \{x = (x_1, x_2) \in \mathbb{R}^2; \, x_1 \in \mathbb{R}, 0 < x_2 < \ell\}$ with $0 < \ell \ll 1$ under the boundary condition

$$(\mu \partial_{x_2} v^1 n + k v^1)|_{x_2 = 0, \ell} = 0, \quad v^2|_{x_2 = 0, \ell} = 0, \tag{1.1.13}$$

where $k > 0$ is the friction constant and n represents the direction of the outward normal to the boundary $\partial\Omega_\ell$, and hence, $n|_{x_2=0} = -1$ and $n|_{x_2=\ell} = 1$. We introduce the non-dimensional variables $\tilde{x}$, $\tilde{t}$, $\tilde{\rho}$, $\tilde{v}$, $\tilde{P}$ defined by

$$x = \ell \tilde{x}, \; t = T\tilde{t}, \; \rho = \rho_* \tilde{\rho}, \; v = V\tilde{v}, \; P = \rho_* V^2 \tilde{P}$$

with $T = \frac{\rho_* \ell^2}{\mu}$ and $V = \frac{\ell}{T}$. It follows that $\tilde{\rho}$ and $\tilde{v}$ are governed by the equations

$$\partial_{\tilde{t}} \tilde{\rho} + \mathrm{div}_{\tilde{x}}(\tilde{\rho}\tilde{v}) = 0, \tag{1.1.14}$$

$$\tilde{\rho}(\partial_{\tilde{t}} \tilde{v} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{v}) - \Delta_{\tilde{x}} \tilde{v} - \frac{\mu + \mu'}{\mu} \nabla_{\tilde{x}} \mathrm{div}_{\tilde{x}} \tilde{v} + \nabla_{\tilde{x}} \tilde{P}(\tilde{\rho}) = 0. \tag{1.1.15}$$

The domain Ω_ℓ is transformed into Ω and the boundary condition (1.1.13) becomes

$$(\partial_{\tilde{x}_2} \tilde{v}^1 n + \tfrac{k\ell}{\mu} \tilde{v}^1)|_{\tilde{x}_2 = 0,1} = 0, \quad \tilde{v}^2|_{\tilde{x}_2 = 0,1} = 0. \tag{1.1.16}$$

Letting $\ell \to 0$ we obtain (1.1.1)-(1.1.2) with μ and $\mu + \mu'$ replaced by 1 and $\frac{\mu+\mu'}{\mu}$, respectively, and the slip boundary condition (1.1.3). Since $\rho(x, t) = \rho_* \tilde{\rho}(\frac{x}{\ell}, \frac{\mu t}{\rho_* \ell^2})$ and $v(x, t) = \frac{\mu}{\rho_* \ell} \tilde{v}(\frac{x}{\ell}, \frac{\mu t}{\rho_* \ell^2})$, we find that the behavior of solutions of (1.1.1)-(1.1.2) in Ω_ℓ under the Navier boundary condition (1.1.13) is expected to be approximated by the large time behavior of solutions of (1.1.1)-(1.1.2) in Ω under the slip boundary condition (1.1.3) when $0 < \ell \ll 1$.

This chapter is organized as follows. In section 1.3 we state the main results of this chapter. In section 1.4 we study the spectral properties of the linearized operator, and in section 1.5 we rewrite (1.1.5)-(1.1.8) into a problem for a system of equations for the low and high frequency parts. Section 1.6 is devoted to estimating the low-frequency part, while the high-frequency part is estimated in section 1.7. In section 1.8 we give the estimates for the nonlinear terms. In section 1.9 we study the asymptotic behavior of the solution of (1.1.5)-(1.1.8).

1.2 Notations

In this section we first introduce some notations which will be used throughout this chapter.

For $1 \leq p \leq \infty$ we denote by $L^p(X)$ the usual Lebesgue space on a domain X and its norm is denoted by $\|\cdot\|_{L^p(X)}$. Let m be a nonnegative integer. The symbol $H^m(X)$ denotes the m-th order L^2-Sobolev space on X with norm $\|\cdot\|_{H^m(X)}$. In particular, we write $\|\cdot\|_{L^2(X)}$ for $H^0(X)$.

We simply denote by $L^p(X)$ (resp., $H^m(X)$) the set of all vector fields $w = {}^\top(w^1, w^2)$ on X with $w^j \in L^p(X)$ (resp, $H^m(X)$), $j = 1, 2$, and its norm is also denoted by $\|\cdot\|_{L^p(X)}$ (resp., $\|\cdot\|_{H^m(X)}$). For $u = {}^\top(\phi, w)$ with $\phi \in H^k(X)$ and $w = {}^\top(w^1, w^2) \in H^m(X)$, we define $\|u\|_{H^k(X)\times H^m(X)}$ by $\|u\|_{H^k(X)\times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^k(X)}$. When $k = m$, we simply write $\|u\|_{H^k(X)\times H^k(X)} = \|u\|_{H^k(X)}$.

Partial derivatives of a function u in x, x_k $(k = 1, 2)$ and t are denoted by $\partial_x u, \partial_{x_k} u$ and $\partial_t u$. We also write the higher order partial derivatives of u in x as $\partial_x^l u = (\partial_x^\alpha u; |\alpha| = l)$.

In the case where $X = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as L^p (resp., H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$. We denote the inner product of $L^2(\Omega)$ by

$$(f, g) = \int_\Omega f(x)g(x)\mathrm{d}x, \qquad f, g \in L^2(\Omega).$$

The average of a function f in x_2 on $(0, 1)$ is denoted by $\langle f\rangle$:

$$\langle f\rangle = \int_0^1 f(x_2)\mathrm{d}x_2.$$

We set

$$H^2_* = \{w = {}^\top(w^1, w^2) \in H^2(\Omega);\ \partial_{x_2} w^1|_{x_2=0,1} = 0, w^2|_{x_2=0,1} = 0\}.$$

For $\alpha \in \mathbb{R}$, we denote by $L^1_\alpha = L^1_\alpha(\Omega)$ the weighted L^1 space with weight $(1 + |x_1|)^\alpha$, and its norm is denoted by

$$\|f\|_{L^1_\alpha} = \int_\Omega (1 + |x_1|)^\alpha |f(x)|\mathrm{d}x.$$

We denote the Fourier transform of $f = f(x_1)$ $(x_1 \in \mathbb{R})$ by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x_1)e^{-i\xi x_1}\mathrm{d}x_1, \quad \xi \in \mathbb{R}.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x_1) = (2\pi)^{-1} \int_{\mathbb{R}} f(\xi) e^{i\xi x_1} \mathrm{d}\xi, \quad x_1 \in \mathbb{R}.$$

For operators A, B, we denote the commutator of A and B by $[A, B]$:

$$[A, B]f = A(Bf) - B(Af).$$

1.3 Main results of Chapter 1

In this section we state the main results of this chapter. We have the following decay estimate of the L^2-norm of the solution u.

Theorem 1.3.1. *There exists a positive number ε_0 such that if $u_0 = {}^\top(\phi_0, w_0) \in (H^2 \times H^2_*) \cap L^1$ with $w_0 = {}^\top(w_0^1, w_0^2)$ satisfies $\|u_0\|_{H^2 \cap L^1} \leq \varepsilon_0$, then problem (1.1.5)-(1.1.8) has a unique global solution*

$$u(t) = {}^\top(\phi(t), w(t)) \in C([0, \infty); H^2 \times H^2_*)$$

and $u(t)$ satisfies

$$\|\partial_x^k u(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1}$$

for $t \geq 0$, $k = 0, 1, 2$.

We next consider the asymptotic behavior of solutions.

Theorem 1.3.2. *In addition to the assumptions of Theorem 1.3.1, if $\phi_0, w_0^1 \in L^1_{1/2}$, then*

$$\|\partial_x^k (u - \chi_+ \boldsymbol{a}_+ - \chi_- \boldsymbol{a}_-)(t)\|_2 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1.$$

Here $\boldsymbol{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given in (1.1.10)-(1.1.12).

The proof of Theorem 1.3.1 will be given in Sections 1.4–1.8, and Theorem 1.3.2 will be proved in Section 1.9.

1.4 Spectral properties of linearized operator

We consider the linearized problem

$$\partial_t u + Lu = F, \quad u|_{t=0} = u_0, \tag{1.4.1}$$

where $u = {}^\top(\phi, w); F = {}^\top(f^0, \tilde{f})$ with $\tilde{f} = {}^\top(f^1, f^2)$ is a given function, and L is an operator of the form

$$L = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla & -\nu \Delta - \tilde{\nu} \nabla \mathrm{div} \end{pmatrix}$$

in $H^1 \times L^2$ with domain $D(L) = H^1 \times H^2_*$.

To investigate (1.4.1), we consider the Fourier transform of (1.4.1) in $x_1 \in \mathbb{R}$:

$$\partial_t \hat{\phi} + i\gamma\xi \hat{w}^1 + \gamma \partial_{x_2} \hat{w}^2 = \hat{f}^0, \tag{1.4.2}$$

$$\partial_t \hat{w}^1 + (\nu + \tilde{\nu})\xi^2 \hat{w}^1 - \nu \partial_{x_2}^2 \hat{w}^1 - i\tilde{\nu}\xi \partial_{x_2} \hat{w}^2 + i\gamma\xi \hat{\phi} = \hat{f}^1, \tag{1.4.3}$$

$$\partial_t \hat{w}^2 + \nu\xi^2 \hat{w}^2 - (\nu + \tilde{\nu})\partial_{x_2}^2 \hat{w}^2 - i\tilde{\nu}\xi \partial_{x_2} \hat{w}^1 + \gamma \partial_{x_2} \hat{\phi} = \hat{f}^2, \tag{1.4.4}$$

$$\partial_{x_2} \hat{w}^1|_{x_2=0,1} = \hat{w}^2|_{x_2=0,1} = 0, \tag{1.4.5}$$

$$\hat{u}|_{t=0} = \hat{u}_0 = {}^\top(\hat{\phi}_0, \hat{w}_0). \tag{1.4.6}$$

We thus arrive at the following problem

$$\partial_t \hat{u} + \hat{L}_\xi \hat{u} = \hat{F}, \quad \hat{u}|_{t=0} = \hat{u}_0, \tag{1.4.7}$$

with a parameter $\xi \in \mathbb{R}$. Here $\hat{u} = \hat{u}(\xi, x_2, t)$; $\hat{L}_\xi$ is the operator

$$\hat{L}_\xi = \begin{pmatrix} 0 & i\gamma\xi & \gamma\partial_{x_2} \\ i\gamma\xi & (\nu+\tilde{\nu})\xi^2 - \nu\partial_{x_2}^2 & -i\tilde{\nu}\xi\partial_{x_2} \\ \gamma\partial_{x_2} & -i\tilde{\nu}\xi\partial_{x_2} & \nu\xi^2 - (\nu+\tilde{\nu})\partial_{x_2}^2 \end{pmatrix}$$

with domain $D(\hat{L}_\xi) = H^1(0,1) \times H^2_*(0,1)$, where $H^2_*(0,1) = \{w = {}^\top(w^1, w^2) \in H^2(0,1);$ $\partial_{x_2} w^1|_{x_2=0,1} = w^2|_{x_2=0,1} = 0\}$. For $-\hat{L}_0$ we have the following result.

Lemma 1.4.1. (i) *$\lambda = 0$ is a semisimple eigenvalue of $-\hat{L}_0$.*
(ii) *The eigenprojection Π for $\lambda = 0$ of $-\hat{L}_0$ is given by*

$$\Pi u = \begin{pmatrix} \langle \phi \rangle \\ \langle w^1 \rangle \\ 0 \end{pmatrix}$$

for $u = {}^\top(\phi, w)$ with $w = {}^\top(w^1, w^2)$.

The proof of Lemma 1.4.1 is straightforward and we omit it.

We next expand $\hat{u}$ and $\hat{F}$ into the Fourier series:

$$\hat{\phi} = \sum_{k=0}^\infty \hat{\phi}_k \cos k\pi x_2, \quad \hat{w}^1 = \sum_{k=0}^\infty \hat{w}^1_k \cos k\pi x_2, \quad \hat{w}^2 = \sum_{k=1}^\infty \hat{w}^2_k \sin k\pi x_2, \tag{1.4.8}$$

$$\hat{f}^0 = \sum_{k=0}^\infty \hat{f}^0_k \cos k\pi x_2, \quad \hat{f}^1 = \sum_{k=0}^\infty \hat{f}^1_k \cos k\pi x_2, \quad \hat{f}^2 = \sum_{k=1}^\infty \hat{f}^2_k \sin k\pi x_2. \tag{1.4.9}$$

It then follows that

$$\partial_t \hat{\phi}_k + i\gamma\xi \hat{w}^1_k + \gamma \hat{w}^2_k k\pi = \hat{f}^0_k, \tag{1.4.10}$$

$$\partial_t \hat{w}^1_k + \nu(\xi^2 + k^2\pi^2)\hat{w}^1_k + \tilde{\nu}\xi^2 \hat{w}^1_k - i\tilde{\nu}k\pi\xi \hat{w}^2_k + i\gamma\xi \hat{\phi}_k = \hat{f}^1_k, \tag{1.4.11}$$

$$\partial_t \hat{w}^2_k + \nu(\xi^2 + k^2\pi^2)\hat{w}^2_k + i\tilde{\nu}k\pi\xi \hat{w}^1_k + \tilde{\nu}k^2\pi^2 \hat{w}^2_k - \gamma k\pi \hat{\phi}_k = \hat{f}^2_k. \tag{1.4.12}$$

We rewrite it in the form

$$\partial_t \hat{u}_k + \hat{L}_{\xi,k}\hat{u}_k = \hat{F}_k, \tag{1.4.13}$$

where $\hat{u}_k = {}^\top(\hat{\phi}_k, \hat{w}_k^1, \hat{w}_k^2)$, $\hat{F}_k = {}^\top(\hat{f}_k^0, \hat{f}_k^1, \hat{f}_k^2)$ and

$$\hat{L}_{\xi,k} = \begin{pmatrix} 0 & i\gamma\xi & \gamma k\pi \\ i\gamma\xi & \nu(\xi^2+k^2\pi^2)+\tilde{\nu}\xi^2 & -i\tilde{\nu}k\pi\xi \\ -\gamma k\pi & i\tilde{\nu}k\pi\xi & \nu(\xi^2+k^2\pi^2)+\tilde{\nu}k^2\pi^2 \end{pmatrix}.$$

As for the the spectrum of $-\hat{L}_{\xi,k}$, we have the following lemma.

Lemma 1.4.2. (i) *The eigenvalues $-\hat{L}_{\xi,k}$ are given by*

$$\begin{aligned} \lambda_{0,k}(\xi) &= -\nu(\xi^2+k^2\pi^2), \\ \lambda_{\pm,k}(\xi) &= -\frac{1}{2}(\nu+\tilde{\nu})(\xi^2+k^2\pi^2) \\ &\quad \pm \frac{1}{2}\sqrt{(\nu+\tilde{\nu})^2(\xi^2+k^2\pi^2)^2 - 4\gamma^2(\xi^2+k^2\pi^2)}. \end{aligned} \tag{1.4.14}$$

(ii) *The eigenprojections for $\lambda_{0,k}$ and $\lambda_{\pm,k}$ are given by the following $P_{0,k}$ and $P_{\pm,k}$, respectively:*

$$P_{0,k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\dfrac{\xi^2}{\xi^2+k^2\pi^2} & \dfrac{ik\pi\xi}{\xi^2+k^2\pi^2} \\ 0 & -\dfrac{ik\pi\xi}{\xi^2+k^2\pi^2} & 1-\dfrac{k^2\pi^2}{\xi^2+k^2\pi^2} \end{pmatrix},$$

$$P_{+,k} = \frac{1}{\lambda_{+,k}-\lambda_{-,k}} \begin{pmatrix} -\lambda_{-,k} & i\gamma\xi & \gamma k\pi \\ i\gamma\xi & \dfrac{\xi^2\lambda_{+,k}}{\xi^2+k^2\pi^2} & -\dfrac{ik\pi\xi\lambda_{+,k}}{\xi^2+k^2\pi^2} \\ -\gamma k\pi & \dfrac{ik\pi\xi\lambda_{+,k}}{\xi^2+k^2\pi^2} & \dfrac{k^2\pi^2\lambda_{+,k}}{\xi^2+k^2\pi^2} \end{pmatrix},$$

$$P_{-,k} = \frac{1}{\lambda_{+,k}-\lambda_{-,k}} \begin{pmatrix} \lambda_{+,k} & -i\gamma\xi & -\gamma k\pi \\ -i\gamma\xi & -\dfrac{\xi^2\lambda_{-,k}}{\xi^2+k^2\pi^2} & \dfrac{ik\pi\xi\lambda_{-,k}}{\xi^2+k^2\pi^2} \\ \gamma k\pi & -\dfrac{ik\pi\xi\lambda_{-,k}}{\xi^2+k^2\pi^2} & -\dfrac{k^2\pi^2\lambda_{-,k}}{\xi^2+k^2\pi^2} \end{pmatrix}.$$

Lemma 1.4.2 can be proved by elementary computations.

1.5 Decay estimate: Proof of Theorem 1.3.1

We consider the nonlinear problem

$$\begin{cases} \partial_t u + Lu = F(u), \\ u|_{t=0} = u_0. \end{cases} \tag{1.5.1}$$

Here $u = {}^\top(\phi, w)$ and $F(u) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$.

One can prove the local solvability for (1.5.1) as in [14].

Proposition 1.5.1. *Assume that $u_0 = {}^\top(\phi_0, w_0) \in H^2 \times H^2_*$ and $\|\phi_0\|_\infty \le \frac{1}{2}$. Then there exists $T_0 > 0$ depending on $\|u_0\|_{H^2}$ such that problem (1.5.1) has a unique solution $u = {}^\top(\phi, w)$ on $[0, T_0]$ satisfying $u \in C([0, T_0]; H^2 \times H^2_*) \cap C^1([0, T_0]; L^2)$ with $w \in L^2(0, T_0; H^3)$ and $\|\phi_0(t)\|_\infty \le \frac{3}{4}$ for $t \in [0, T_0]$. Furthermore, the inequality*

$$\sup_{t\in[0,T_0]} \{\|u(t)\|_{H^2} + \|\partial_t u(t)\|_2\} + \int_0^{T_0} \|w\|^2_{H^3} \mathrm{d}t \le C_0\{1 + \|u_0\|^2_{H^2}\}^a \|u_0\|^2_{H^2} \qquad (1.5.2)$$

holds with some constants $C_0 > 0$ and $a > 0$.

The global existence of $u(t)$ follows in a standard manner from Proposition 1.5.1 and Proposition 1.5.5 below which provides the a priori bound $\|u(t)\|_{H^2} \le C\|u_0\|_{H^2\cap L^1}$ when $\|u_0\|_{H^2\cap L^1}$ is sufficiently small.

We next consider the a priori estimates for $u(t)$. Let r_0 be a number satisfying $0 < r_0 \le 1$. We introduce the cut-off function $\mathbf{1}_{\{|\xi|\le r_0\}}$ defined by

$$\mathbf{1}_{\{|\xi|\le r_0\}} = \begin{cases} 1 & (|\xi| < r_0), \\ 0 & (|\xi| \ge r_0). \end{cases} \qquad (1.5.3)$$

We introduce the projections P_1 and P_∞ defined by

$$P_1 u = \mathcal{F}^{-1}\mathbf{1}_{\{|\xi|\le r_0\}}\Pi\mathcal{F}u, \qquad P_\infty = I - P_1. \qquad (1.5.4)$$

It follows from Lemma 1.4.2 that

$$\begin{aligned} P_1 e^{-tL} u_0 &= \mathcal{F}^{-1}\mathbf{1}_{\{|\xi|\le r_0\}}\left[e^{\lambda_{+,0}t}P_{+,0} + e^{\lambda_{-,0}t}P_{-,0}\right]\Pi\hat{u}_0 \\ &= \mathcal{F}^{-1}\frac{\mathbf{1}_{\{|\xi|\le r_0\}}}{\lambda_{+,0} - \lambda_{-,0}}\left[e^{\lambda_{+,0}t}\begin{pmatrix} -\lambda_{-,0} & i\gamma\xi & 0 \\ i\gamma\xi & \lambda_{+,0} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right. \\ &\qquad \left. + e^{\lambda_{-,0}t}\begin{pmatrix} \lambda_{+,0} & -i\gamma\xi & 0 \\ -i\gamma\xi & -\lambda_{-,0} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\begin{pmatrix} \langle\hat{\phi}_0\rangle \\ \langle\hat{w}_0^1\rangle \\ 0 \end{pmatrix} \end{aligned} \qquad (1.5.5)$$

for $u_0 = {}^\top(\phi_0, w_0^1, w_0^2)$. We also note that $P_1 u$ does not depend on x_2, and so,

$$\partial_{x_2} P_1 u = 0.$$

We decompose $u = {}^\top(\phi, w)$ into

$$u = u_1 + u_\infty,$$

where

$$u_1 = P_1 u = {}^\top(\phi_1, w_1^1, w_1^2), \quad u_\infty = P_\infty u = {}^\top(\phi_\infty, w_\infty^1, w_\infty^2).$$

Remark 1.5.2. *We see from the definition of P_1 that $u_1 = u_1(x_1, t)$ satisfies*

$$\|\partial_{x_1}^{k+l} u_1\|_2 \leq \|\partial_{x_1}^{l} u_1\|_2$$

for arbitrary k and l. We also note that u_∞ satisfies

$$\|u_\infty\|_2 \leq C\|\partial_x u_\infty\|_2.$$

We will frequently make use of these properties in the subsequent arguments.

Proposition 1.5.3. *Let $u(t)$ be a solution of (1.5.1) on $[0, T]$. Assume that $u \in C([0, T]; H^2 \times H^2_*) \cap C^1([0, T]; L^2)$ with $w \in L^2(0, T; H^3)$. Then*

$$u_1 = {}^\top(\phi_1, w_1) \in C^1([0, T]; H^l(\Omega)) \quad (\forall l = 0, 1, 2, \cdots)$$

and

$$u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0, T]; H^2 \times H^2_*) \cap C^1([0, T]; L^2)$$

with $w_\infty \in L^2(0, T; H^3)$.

Furthermore, u_1 and u_∞ satisfy

$$u_1 = P_1 e^{-tL} u_0 + \int_0^t P_1 e^{-(t-\tau)L} F(u(\tau)) \mathrm{d}\tau, \tag{1.5.6}$$

$$\partial_t u_\infty + L u_\infty = F_\infty, \quad u_\infty|_{t=0} = P_\infty u_0, \tag{1.5.7}$$

where $F_\infty = P_\infty F = {}^\top(f^0_\infty,\ \tilde{f}_\infty), \tilde{f}_\infty = (f^1_\infty, f^2_\infty)$.

Proof. Since $P_j L \subset L P_j$ $(j = 1, \infty)$, applying P_j to (1.5.1) we obtain the desired results. □

We define $M(t) \geq 0$ by

$$M(t) = M_1(t) + M_\infty(t) \quad (t \in [0, T]). \tag{1.5.8}$$

Here $M_1(t)$ and $M_\infty(t)$ are defined by

$$M_1(t) = \sup_{0 \leq \tau \leq t} \left\{ \sum_{k=0}^{2} (1+\tau)^{\frac{1}{4}+\frac{k}{2}} \|\partial_{x_1}^k u_1(\tau)\|_2 + (1+\tau)^{\frac{3}{4}} \|\partial_t u_1(\tau)\|_2 \right\},$$

$$M_\infty(t) = \left(\sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{5}{2}} \{\|u_\infty(\tau)\|^2_{H^2} + \|\partial_t u_\infty(\tau)\|_2^2\} \right)^{\frac{1}{2}}.$$

We note that, by the Gagliardo-Nirenberg-Sobolev inequality,

$$\|u_1(t)\|_\infty \leq C\|u_1(t)\|_2^{\frac{1}{2}} \|\partial_{x_1} u_1(t)\|_2^{\frac{1}{2}} \leq C(1+t)^{-\frac{1}{2}} M_1(t),$$
$$\|u_\infty(t)\|_\infty \leq C\|u_\infty(t)\|_{H^2} \leq C(1+t)^{-\frac{5}{4}} M_\infty(t).$$

We introduce the quantities $E_\infty(t)$ and $D_\infty(t)$ for $u_\infty(t) = {}^\top(\phi_\infty(t), w_\infty(t))$:

$$E_\infty(t) = \|u_\infty(t)\|^2_{H^2} + \|\partial_t u_\infty(t)\|^2_2,$$
$$D_\infty(t) = \|\nabla \phi_\infty(t)\|^2_{H^1} + \|\nabla w_\infty(t)\|^2_{H^2} + \|\partial_t u_\infty(t)\|^2_{H^1}.$$

Proposition 1.5.4. *Let $u(t)$ be a solution of (1.5.1) on $[0,T]$. Then there exists a positive constant ε_1 such that if $\|u(t)\|_{H^2} \le \varepsilon_1$ and $M(t) \le 1$ for $t \in [0,T]$, the estimates*

$$M_1(t) \le C\{\|u_0\|_1 + M(t)^2\} \tag{1.5.9}$$

and

$$\begin{aligned} &E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau)\mathrm{d}\tau \\ &\quad \le C\left\{ e^{-at}E_\infty(0) + (1+t)^{-\frac{5}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)}\mathcal{R}(\tau)\mathrm{d}\tau \right\} \end{aligned} \tag{1.5.10}$$

hold uniformly for $t \in [0,T]$ with $C > 0$ independent of T. Here $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant; and $\mathcal{R}(t)$ is a function satisfying the estimate

$$\mathcal{R}(t) \le C\{(1+t)^{-\frac{5}{2}} M(t)^3 + M(t) D_\infty(t)\}. \tag{1.5.11}$$

The estimate (1.5.9) will be proved in Section 1.6, and the estimates (1.5.10) and (1.5.11) will be proved in Sections 1.7 and 1.8.

From Proposition 1.5.4, one can show the following uniform estimate of $M(t)$ as in [12].

Proposition 1.5.5. *If $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then*

$$M(t) \le C\|u_0\|_{H^2\cap L^1}. \tag{1.5.12}$$

Theorem 1.3.1 now follows from Propositions 1.5.1 and 1.5.5.

1.6 Estimates on $P_1 u$

In this section we estimate the low-frequency part $u_1 = P_1 u$ and prove estimate (1.5.9) in Proposition 1.5.4.

Proof of (1.5.9). We see from Lemma 1.4.2 and the definition of Π that

$$\begin{aligned} \|\partial_{x_1}^l e^{-tL} P_1 u_0\|_2 &\le C\left(\int_{\mathbb{R}} |\xi|^{2l} e^{-c_0|\xi|^2 t} \mathbf{1}_{\{|\xi| \le r_0\}} |\Pi \hat{u}_0|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\le C\left(\int_{\mathbb{R}} |\xi|^{2l} e^{-c_0|\xi|^2 t} \mathbf{1}_{\{|\xi| \le r_0\}} \mathrm{d}\xi \right)^{\frac{1}{2}} \|u_0\|_1 \\ &\le C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_1 \end{aligned} \tag{1.6.1}$$

for $l \ge 0$, and hence, by (1.5.6), we have

$$\|\partial_{x_1}^k u_1(t)\|_2 \le \|\partial_{x_1}^k e^{-tL} P_1 u_0\|_2 + \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 F(u(\tau))\|_2 \mathrm{d}\tau$$

$$\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_1 + \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 F(u(\tau))\|_2 \mathrm{d}\tau$$

for $k = 0, 1, 2$.

Let us estimate $\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 F(u(\tau))\|_2 \mathrm{d}\tau$. By the Sobolev inequality: $\|\phi\|_\infty \leq C\|\phi\|_{H^2}$, we see that there exists a positive constant ε_2 such that if $\|u(t)\|_{H^2} \leq \varepsilon_2$ for $t \in [0, T]$, then $\|\phi(t)\|_\infty \leq \frac{1}{2}$ for $t \in [0, T]$, and hence $P_1 F(u)$ is written as

$$P_1 F(u) = P_1 \partial_{x_1} F_0(u) + P_1 \tilde{F}(u),$$

where

$$F_0(u) = \begin{pmatrix} -\gamma \phi_1 w_1^1 \\ -\frac{\gamma}{2}(w_1^1)^2 + (\nu + \tilde{\nu})(\phi_1 \partial_{x_1} w_1^1) + \frac{\gamma}{2}\phi_1^2 - \frac{\rho_*}{\gamma} P^{(2)}(\phi)\phi_1^2 \\ 0 \end{pmatrix}.$$

Here each term in $P_1 \tilde{F}(u)$ includes u_∞, $O((\partial_{x_1} u_1)^2)$, $O(u_1^2 \partial_{x_1} \phi_1)$ or $O(u_1^2 \partial_{x_1}^l w_1)$ $(l = 1, 2)$, and $P_1 \tilde{F}(u)$ is estimated as

$$\|P_1 \tilde{F}(u(\tau))\|_1 \leq C(1+\tau)^{-\frac{5}{4}} M(t)^2.$$

It then follows from (1.6.1) that

$$\begin{aligned} \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 \tilde{F}(u(\tau))\|_2 \mathrm{d}\tau &\leq C \int_0^t (1+t-\tau)^{-\frac{1}{4}-\frac{k}{2}} (1+\tau)^{-\frac{5}{4}} \mathrm{d}\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} M(t)^2. \end{aligned}$$

As for the estimates for $P_1 \partial_{x_1} F_0(u)$ part, we write it as

$$\begin{aligned} &\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 \partial_{x_1} F_0(u(\tau))\|_2 \mathrm{d}\tau \\ &= \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 \partial_{x_1} F_0(u(\tau))\|_2 \mathrm{d}\tau \\ &=: I_1 + I_2. \end{aligned}$$

Since $\partial_{x_1} e^{-tL} P_1 = e^{-tL} P_1 \partial_{x_1}$, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{t}{2}} \|\partial_{x_1}^{k+1} e^{-(t-\tau)L} P_1 F_0(u(\tau))\|_2 \mathrm{d}\tau \\ &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-\frac{1}{2}} \mathrm{d}\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} M(t)^2, \quad k = 0, 1, 2. \end{aligned}$$

As for I_2, we have, for $k = 0, 1$,

$$I_2 \le C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{4}-\frac{k}{2}} (1+\tau)^{-1} \mathrm{d}\tau M(t)^2$$
$$\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}} M(t)^2.$$

For $k = 2$, we have

$$I_2 \le C \int_{\frac{t}{2}}^{t} \|\partial_{x_1} e^{-(t-\tau)L} P_1 \partial_{x_1}^2 F_0(u(\tau))\|_2 \mathrm{d}\tau$$
$$\le C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{3}{2}} \mathrm{d}\tau M(t)^2$$
$$\le C(1+t)^{-\frac{5}{4}} M(t)^2.$$

We thus obtain

$$\|\partial_{x_1}^k u_1(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \{\|u_0\|_1 + M(t)^2\}. \tag{1.6.2}$$

We next estimate the time derivative. We have

$$\|-Lu_1(t)\|_2 \le C\{\|\partial_{x_1}^2 w_1(t)\|_2 + \|\partial_{x_1} u_1(t)\|_2\}$$
$$\le C(1+t)^{-\frac{3}{4}} \{\|u_0\|_1 + M(t)^2\},$$

and

$$\|P_1 F(u(t))\|_2 \le C(1+t)^{-\frac{5}{4}} M(t)^2.$$

Since

$$\partial_t u_1 = -Lu_1 + P_1 F(u),$$

we obtain

$$\begin{aligned} \|\partial_t u_1(t)\|_2 &\le \|Lu_1(t)\|_2 + \|P_1 F(u(t))\|_2 \\ &\le C(1+t)^{-\frac{3}{4}} \{\|u_0\|_1 + M(t)^2\}. \end{aligned} \tag{1.6.3}$$

By (1.6.2) and (1.6.3), we deduce the desired estimate. This completes the proof. □

1.7 Estimates on $P_\infty u$

In this section we estimate the high-frequency part $u_\infty = P_\infty u$ by using the Matsumura-Nishida energy method to prove estimate (1.5.10) in Proposition 1.5.4.

We introduce the quantity $D[w]$ which is defined by

$$D[w] = \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\mathrm{div} w\|_2^2.$$

We also define operators $\tilde{P}_1$ and $\tilde{P}_\infty$ by

$$\tilde{P}_1 \phi = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \le r_0\}} \langle \mathcal{F}\phi \rangle, \quad \tilde{P}_\infty = I - \tilde{P}_1.$$

Note that $P_1 u = {}^\top(\tilde{P}_1 \phi, \tilde{P}_1 w^1, 0)$ and $P_\infty u = {}^\top(\tilde{P}_\infty \phi, \tilde{P}_\infty w^1, w^2)$ for $u = {}^\top(\phi, w^1, w^2)$.

To prove (1.5.10), we prepare some basic estimates.

Proposition 1.7.1. *Let k and j be nonnegative integers satisfying $0 \le 2k + j \le 2$. Then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^k\partial_{x_1}^j u_\infty\|_2^2 + D[\partial_t^k\partial_{x_1}^j w_\infty] + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_t^k\partial_{x_1}^j\dot{\phi}_\infty\|_2^2 \le CR_{j,k}^{(1)}, \tag{1.7.1}$$

where

$$\begin{aligned}
\dot{\phi}_\infty &= \partial_t\phi_\infty + \gamma w\cdot\nabla\phi_\infty,\\
R_{j,k}^{(1)} &= \frac{\gamma}{2}(\mathrm{div}w, |\partial_t^k\partial_{x_1}^j\phi_\infty|^2) - \gamma([\partial_t^k\partial_{x_1}^j, w]\cdot\nabla\phi_\infty, \partial_t^k\partial_{x_1}^j\phi_\infty)\\
&\quad + (\partial_t^k\partial_{x_1}^j\tilde{f}_\infty^0, \partial_t^k\partial_{x_1}^j\phi_\infty) + (\partial_t^k\partial_{x_1}^j\tilde{f}_\infty, \partial_t^k\partial_{x_1}^j w_\infty) + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_t^k\partial_{x_1}^j\tilde{f}_\infty^0\|_2^2.
\end{aligned}$$

Here and in what follows, $\tilde{f}_\infty^0$ denotes

$$\tilde{f}_\infty^0 = \gamma\tilde{P}_1(w\cdot\nabla\phi_\infty) - \gamma\tilde{P}_\infty(w\cdot\nabla\phi_1 + \phi\mathrm{div}w).$$

Proof. Equation (1.5.7) is written as

$$\partial_t\phi_\infty + \gamma w\cdot\nabla\phi_\infty + \gamma\mathrm{div}w_\infty = \tilde{f}_\infty^0, \tag{1.7.2}$$

$$\partial_t w_\infty - \nu\Delta w_\infty - \tilde{\nu}\nabla\mathrm{div}w_\infty + \gamma\nabla\phi_\infty = \tilde{f}_\infty. \tag{1.7.3}$$

We compute $(\partial_{x_1}^j(1.7.2), \partial_{x_1}^j\phi_\infty) + (\partial_{x_1}^j(1.7.3), \partial_{x_1}^j w_\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_1}^j u_\infty\|_2^2 + D[\partial_{x_1}^j w_\infty]\\
&\quad = -\gamma(w\cdot\nabla\partial_{x_1}^j\phi_\infty, \partial_{x_1}^j\phi_\infty) - \gamma([\partial_{x_1}^j, w]\cdot\nabla\phi_\infty, \partial_{x_1}^j\phi_\infty)\\
&\qquad + (\partial_{x_1}^j\tilde{f}_\infty^0, \partial_{x_1}^j\phi_\infty) + (\partial_{x_1}^j\tilde{f}_\infty, \partial_{x_1}^j w_\infty)\\
&\quad = \frac{\gamma}{2}(\mathrm{div}w, |\partial_{x_1}^j\phi_\infty|^2) - \gamma([\partial_{x_1}^j, w]\cdot\nabla\phi_\infty, \partial_{x_1}^j\phi_\infty)\\
&\qquad + (\partial_{x_1}^j\tilde{f}_\infty^0, \partial_{x_1}^j\phi_\infty) + (\partial_{x_1}^j\tilde{f}_\infty, \partial_{x_1}^j w_\infty).
\end{aligned} \tag{1.7.4}$$

We set $\dot{\phi} := \partial_t\phi + \gamma w\cdot\nabla\phi$. From (1.7.2), we have $\partial_{x_1}^j\dot{\phi}_\infty = -\gamma\mathrm{div}\partial_{x_1}^j w_\infty + \partial_{x_1}^j\tilde{f}_\infty^0$, and hence

$$\|\partial_{x_1}^j\dot{\phi}_\infty\|_2^2 \le C(\gamma^2\|\mathrm{div}\partial_{x_1}^j w_\infty\|_2^2 + \|\partial_{x_1}^j\tilde{f}_\infty^0\|_2^2).$$

We thus obtain

$$\begin{aligned}
\frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\dot{\phi}_\infty\|_2^2 &\le C\left\{\nu\|\nabla\partial_{x_1}^j w_\infty\|_2^2 + \tilde{\nu}\|\mathrm{div}\partial_{x_1}^j w_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\tilde{f}_\infty^0\|_2^2\right\}\\
&= C\left\{D[\partial_{x_1}^j w_\infty] + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\tilde{f}_\infty^0\|_2^2\right\}.
\end{aligned} \tag{1.7.5}$$

By (1.7.4) + $\frac{1}{2C}$ × (1.7.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_1}^j u_\infty\|_2^2 + D[\partial_{x_1}^j w_\infty] + \frac{\nu+\tilde{\nu}}{C\gamma^2}\|\partial_{x_1}^j\dot{\phi}_\infty\|_2^2 \le CR_{j,0}^{(1)}. \tag{1.7.6}$$

Replacing $\partial_{x_1}^j$ by ∂_t, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t u_\infty\|_2^2 + D[\partial_t w_\infty] + \frac{\nu+\tilde{\nu}}{C\gamma^2}\|\partial_t \dot{\phi}_\infty\|_2^2 \leq CR_{0,1}^{(1)}. \tag{1.7.7}$$

This completes the proof. □

Proposition 1.7.2. *It holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{D[w_\infty] - 2\gamma(\phi_\infty, \mathrm{div}w_\infty)\} + \frac{1}{2}\|\partial_t u_\infty\|_2^2 \\ &\qquad \leq C\{\gamma^2\|\mathrm{div}w_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2 + \|w\cdot\nabla\phi_\infty\|_2^2\}.\end{aligned} \tag{1.7.8}$$

Proof. We compute $((1.7.2), \partial_t\phi_\infty) + ((1.7.3), \partial_t w_\infty)$ to obtain

$$\begin{aligned}&\|\partial_t u_\infty\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D[w_\infty] + \gamma\{(\mathrm{div}w_\infty, \partial_t\phi_\infty) + (\nabla\phi_\infty, \partial_t w_\infty)\} \\ &\qquad = -\gamma((w\cdot\nabla\phi_\infty), \partial_t\phi_\infty) + (\tilde{f}_\infty^0, \partial_t\phi_\infty) + (\tilde{f}_\infty, \partial_t w_\infty).\end{aligned}$$

Since $(\nabla\phi_\infty, \partial_t w_\infty) = -(\phi_\infty, \mathrm{div}\partial_t w_\infty)$, we have

$$\begin{aligned}&\|\partial_t u_\infty\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D[w_\infty] + \gamma\{(\mathrm{div}w_\infty, \partial_t\phi_\infty) - (\phi_\infty, \mathrm{div}\partial_t w_\infty)\} \\ &\qquad = -\gamma(w\cdot\nabla\phi_\infty, \partial_t\phi_\infty) + (\tilde{f}_\infty^0, \partial_t\phi_\infty) + (\tilde{f}_\infty, \partial_t w_\infty) \\ &\qquad \leq \frac{1}{4}\|\partial_t u_\infty\|_2^2 + C\{\|w\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2\}.\end{aligned} \tag{1.7.9}$$

Adding $-2\gamma(\mathrm{div}w_\infty, \partial_t\phi_\infty)$ to both sides of (1.7.9), we obtain

$$\begin{aligned}&\|\partial_t u_\infty\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D[w_\infty] - \gamma\frac{\mathrm{d}}{\mathrm{d}t}(\phi_\infty, \mathrm{div}w_\infty) \\ &\quad \leq -2\gamma(\mathrm{div}w_\infty, \partial_t\phi_\infty) + \frac{1}{4}\|\partial_t u_\infty\|_2^2 + C\{\|w\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2\} \\ &\quad \leq \frac{1}{2}\|\partial_t u_\infty\|_2^2 + C\{\gamma^2\|\mathrm{div}w_\infty\|_2^2 + \|w\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2\},\end{aligned}$$

which gives the desired estimate. This completes the proof. □

Proposition 1.7.3. *Let j and l be integers satisfying $0 \leq j+l \leq 1$. Then*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 \\ &\quad \leq CR_{j,l}^{(2)} + C\left\{\frac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_{x_1}^j\partial_{x_2}^l w_\infty\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^{j+1}\partial_{x_2}^l\nabla w_\infty\|_2^2\right\},\end{aligned} \tag{1.7.10}$$

where

$$\begin{aligned}R_{j,l}^{(2)} &= \frac{\gamma}{2}(\mathrm{div}w, |\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty|^2) + C\Bigg\{(\nu+\tilde{\nu})\|[\partial_{x_1}^j\partial_{x_2}^{l+1}, w]\cdot\nabla\phi_\infty\|_2^2 \\ &\qquad + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\partial_{x_2}^l h_\infty^0\|_2^2\Bigg\}, \\ h_\infty^0 &= \partial_{x_2}\tilde{f}_\infty^0 + \frac{\gamma}{\nu+\tilde{\nu}}f_\infty^2.\end{aligned}$$

Proof. We compute $\partial_{x_2}(1.7.2) + \frac{\gamma}{\nu+\tilde{\nu}} \times$ (the second component of (1.7.3)) to obtain

$$\begin{aligned}\partial_{x_2}\dot{\phi}_\infty + \gamma\partial_{x_2}\mathrm{div}w_\infty + \frac{\gamma}{\nu+\tilde{\nu}}\partial_t w_\infty^2 - \frac{\nu\gamma}{\nu+\tilde{\nu}}\Delta w_\infty^2 \\ - \frac{\tilde{\nu}\gamma}{\nu+\tilde{\nu}}\partial_{x_2}\mathrm{div}w_\infty + \frac{\gamma^2}{\nu+\tilde{\nu}}\partial_{x_2}\phi_\infty = h_\infty^0.\end{aligned}$$

This gives

$$\partial_{x_2}\dot{\phi}_\infty + \frac{\gamma^2}{\nu+\tilde{\nu}}\partial_{x_2}\phi_\infty = h_\infty^0 - \frac{\gamma}{\nu+\tilde{\nu}}H(w_\infty), \tag{1.7.11}$$

where

$$H(w) = \partial_t w^2 - \nu\partial_{x_1}^2 w^2 + \nu\partial_{x_1}\partial_{x_2}w^1.$$

Applying $\partial_{x_1}^j\partial_{x_2}^l$ to (1.7.11) we have

$$\partial_{x_1}^j\partial_{x_2}^{l+1}\dot{\phi}_\infty + \frac{\gamma^2}{\nu+\tilde{\nu}}\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty = \partial_{x_1}^j\partial_{x_2}^l h_\infty^0 - \frac{\gamma}{\nu+\tilde{\nu}}H(\partial_{x_1}^j\partial_{x_2}^l w_\infty). \tag{1.7.12}$$

We also write (1.7.12) as

$$\begin{aligned}&\partial_t\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty + \frac{\gamma^2}{\nu+\tilde{\nu}}\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty \\ &\quad = -\gamma w\cdot\nabla\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty - \gamma[\partial_{x_1}^j\partial_{x_2}^{l+1}, w]\cdot\nabla\phi_\infty + \partial_{x_1}^j\partial_{x_2}^l h_\infty^0 \\ &\qquad - \frac{\gamma}{\nu+\tilde{\nu}}H(\partial_{x_1}^j\partial_{x_2}^l w_\infty).\end{aligned} \tag{1.7.13}$$

Taking the inner product of (1.7.13) with $\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty$ we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 \\ &\quad = \frac{\gamma}{2}(\mathrm{div}w, |\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty|^2) + (-\gamma[\partial_{x_1}^j\partial_{x_2}^{l+1}, w]\cdot\nabla\phi_\infty \\ &\qquad + \partial_{x_1}^j\partial_{x_2}^l h_\infty^0 - \frac{\gamma}{\nu+\tilde{\nu}}H(\partial_{x_1}^j\partial_{x_2}^l w_\infty), \partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty) \\ &\quad \le \frac{\gamma^2}{2(\nu+\tilde{\nu})}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\gamma}{2}(\mathrm{div}w, |\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty|^2) \\ &\qquad + C\Big\{(\nu+\tilde{\nu})\big\|[\partial_{x_1}^j\partial_{x_2}^{l+1}, w]\cdot\nabla\phi_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\partial_{x_2}^l h_\infty^0\big\|_2^2 \\ &\qquad + \frac{1}{\nu+\tilde{\nu}}\big\|\partial_{x_1}^j\partial_{x_2}^l\partial_t w_\infty\big\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^{j+1}\partial_{x_2}^l\nabla w_\infty\|_2^2\Big\} \\ &\quad = \frac{\gamma^2}{2(\nu+\tilde{\nu})}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + R_{j,l}^{(2)} \\ &\qquad + C\Big\{\frac{1}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^l\partial_t w_\infty\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^{j+1}\partial_{x_2}^l\nabla w_\infty\|_2^2\Big\}.\end{aligned} \tag{1.7.14}$$

The desired estimate follows from (1.7.14). This completes the proof. $\square$

Proposition 1.7.4. *Let j and l be integers satisfying $0 \le j + l \le 1$. Then*

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\dot{\phi}_\infty\|_2^2 \\ \le CR_{j,l}^{(2)} + C\left\{\frac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_{x_1}^j\partial_{x_2}^l w_\infty\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^{j+1}\partial_{x_2}^l\nabla w_\infty\|_2^2\right\}.\end{aligned} \tag{1.7.15}$$

Proof. From (1.7.12), we get

$$\begin{aligned}&\frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\dot{\phi}_\infty\|_2^2 \\ &\quad\le C\bigg\{\frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}}\big\|\partial_{x_1}^j\partial_{x_2}^l\partial_t w_\infty\big\|_2^2 \\ &\qquad + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^{j+1}\partial_{x_2}^l\nabla w_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\partial_{x_2}^l h_\infty^0\|_2^2\bigg\}.\end{aligned} \tag{1.7.16}$$

By (1.7.10) + $\frac{1}{2C}$ × (1.7.16), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\gamma^2}{2(\nu+\tilde{\nu})}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\phi_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{2C\gamma^2}\|\partial_{x_1}^j\partial_{x_2}^{l+1}\dot{\phi}_\infty\|_2^2 \\ \le CR_{j,l}^{(2)} + C\left\{\frac{1}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^l\partial_t w_\infty\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\nabla\partial_{x_1}^{j+1}\partial_{x_2}^l w_\infty\|_2^2\right\}.\end{aligned} \tag{1.7.17}$$

This completes the proof. □

Proposition 1.7.5. *Let j and l be integers satisfying $0 \le j + l \le 1$. Then*

$$\begin{aligned}&\frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_x^{l+1}\phi_\infty\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_x^{l+2}w_\infty\|_2^2 \\ &\quad\le C\bigg\{\frac{1}{\nu+\tilde{\nu}}\|\partial_t\partial_{x_1}^j w_\infty\|_{H^l}^2 + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\dot{\phi}\|_{H^{l+1}}^2 \\ &\qquad + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\partial_{x_1}^j\tilde{f}_\infty^0\|_{H^{l+1}}^2 + \frac{1}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\tilde{f}_\infty\|_{H^l}^2\bigg\}.\end{aligned} \tag{1.7.18}$$

To prove Proposition 1.7.5 we will apply the following lemma.

Lemma 1.7.6. *Let $u = {}^\top(p, v)$ be a solution of the Stokes system*

$$\begin{cases}\mathrm{div}\ v = f, \\ -\Delta v + \nabla p = g, \\ \partial_{x_2}v^1|_{x_2=0,1} = v^2|_{x_2=0,1} = 0.\end{cases} \tag{1.7.19}$$

Then there exists a constant $C > 0$ such that, for $0 \le j + l \le 1$,

$$\|\partial_{x_1}^j\partial_x^{l+2}v\|_2 + \|\partial_{x_1}^j\partial_x^{l+1}p\|_2 \le C\{\|\partial_{x_1}^j\partial_x^{l+1}f\|_2 + \|\partial_{x_1}^j\partial_x^l g\|_2\}.$$

It is not difficult to prove Lemma 1.7.6 by using the Fourier transform in x_1 and the Fourier series expansion in x_2 as in (1.4.8) and (1.4.9). More precisely, one can prove Lemma 1.7.6 for $l = 0$ by using Lemma 1.4.2 with $\nu = \gamma = 1, \tilde{\nu} = 0$; and for $l = 1$, in addition to Lemma 1.4.2, we also use the equations (1.7.19) to estimate $\|\partial_{x_2}^2 p\|_2$ and $\|\partial_{x_2}^3 v\|_2$. We omit the detail.

Proof of Proposition 1.7.5. We rewrite equation (1.7.2)-(1.7.3) in the following form:

$$\begin{cases} \text{div } w_\infty = \dfrac{1}{\gamma}(\tilde{f}^0_\infty - \dot{\phi}_\infty), \\ -\Delta w_\infty + \nabla\left(\dfrac{\gamma}{\nu}\phi_\infty\right) = \dfrac{1}{\nu}\left\{\tilde{f}_\infty - \left(\partial_t w_\infty - \dfrac{\tilde{\nu}}{\gamma}\nabla(\tilde{f}^0_\infty - \dot{\phi}_\infty)\right)\right\}. \end{cases} \tag{1.7.20}$$

It then follows from Lemma 1.7.6 that

$$\begin{aligned} &\frac{\gamma^2}{\nu^2}\|\partial_{x_1}^j \partial_x^{l+1}\phi_\infty\|_2^2 + \|\partial_{x_1}^j \partial_x^{l+2} w_\infty\|_2^2 \\ &\quad \le C\left\{\left\|\frac{1}{\gamma}\partial_{x_1}^j(\tilde{f}^0_\infty - \dot{\phi}_\infty)\right\|_{H^{l+1}}^2 \right. \\ &\qquad \left. + \left\|\frac{1}{\nu}\partial_{x_1}^j(\tilde{f}_\infty - \partial_t w_\infty + \frac{\tilde{\nu}}{\gamma}\nabla(\tilde{f}^0_\infty - \dot{\phi}_\infty))\right\|_{H^l}^2\right\} \\ &\quad \le C\left\{\frac{1}{\nu^2}\|\partial_t\partial_{x_1}^j w_\infty\|_{H^l}^2 + \frac{\nu^2+\tilde{\nu}^2}{\gamma^2\nu^2}\|\partial_{x_1}^j\dot{\phi}\|_{H^{l+1}}^2 \right. \\ &\qquad \left. + \frac{\nu^2+\tilde{\nu}^2}{\gamma^2\nu^2}\|\partial_{x_1}^j\tilde{f}^0_\infty\|_{H^{l+1}}^2 + \frac{1}{\nu^2}\|\partial_{x_1}^j\tilde{f}_\infty\|_{H^l}^2\right\}. \end{aligned} \tag{1.7.21}$$

By $\frac{\nu^2}{\nu+\tilde{\nu}} \times$ (1.7.21) we obtain the desired result. □

We are now in a position to prove (1.5.10).

Proof of (1.5.10). We compute $b_1 \times \{(1.7.1)_{j=k=0} + (1.7.1)_{j=1,k=0}\} + (1.7.8)$ with a positive number b_1. Taking b_1 suitably large, we see that

$$E_0(t) = b_1\|u_\infty(t)\|_2^2 + D[w_\infty(t)] - 2\gamma(\phi_\infty(t), \text{div} w_\infty(t))$$

is equivalent to

$$\|u_\infty(t)\|_2^2 + D[w_\infty(t)],$$

and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + \frac{1}{2}D_1(t) \le CN_1(t), \tag{1.7.22}$$

where

$$E_1(t) = E_0(t) + b_1\|\partial_{x_1}u_\infty(t)\|_2^2,$$

$$
\begin{aligned}
D_1(t) &= b_1 \sum_{j=0}^{1} \left(D[\partial_{x_1}^j w_\infty(t)] + \frac{\nu+\tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \dot{\phi}_\infty(t)\|_2^2 \right) + \|\partial_t u_\infty(t)\|_2^2, \\
N_1(t) &= \sum_{j=0}^{1} \left| R_{j,0}^{(1)} \right| + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2 + \|w \cdot \nabla \phi_\infty\|_2^2.
\end{aligned}
$$

We next consider $b_2 \times (1.7.22) + (1.7.15)_{j=l=0}$. Then, with a suitably large $b_2 > 0$, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E_2(t) + \frac{1}{2} D_2(t) \le C N_2(t), \tag{1.7.23}
$$

where

$$
\begin{aligned}
E_2(t) &= b_2 E_1(t) + \|\partial_{x_2} \phi_\infty(t)\|_2^2, \\
D_2(t) &= \frac{b_2}{2} D_1(t) + \frac{\gamma^2}{\nu+\tilde{\nu}} \|\partial_{x_2} \phi_\infty(t)\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2} \|\dot{\phi}_\infty(t)\|_{H^1}^2, \\
N_2(t) &= b_2 N_1(t) + \left| R_{0,0}^{(2)} \right|.
\end{aligned}
$$

It then follows from $b_3 \times (1.7.23) + (1.7.18)_{j=l=0}$ with a suitably large $b_3 > 0$ that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E_3(t) + \frac{1}{2} D_3(t) \le C N_3(t), \tag{1.7.24}
$$

where

$$
\begin{aligned}
E_3(t) &= b_3 E_2(t), \\
D_3(t) &= \frac{b_3}{2} D_2(t) + \frac{\gamma^2}{\nu+\tilde{\nu}} \|\partial_x \phi_\infty(t)\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}} \|\partial_x^2 w_\infty(t)\|_2^2, \\
N_3(t) &= b_3 N_2(t) + \frac{\nu+\tilde{\nu}}{\gamma^2} \|\tilde{f}_\infty^0\|_{H^1}^2.
\end{aligned}
$$

We next compute $(1.7.24) + b_4 \times \{(1.7.1)_{j=2,k=0} + (1.7.1)_{j=0,k=1}\} + (1.7.15)_{j=1,l=0}$. Taking $b_4 > 0$ suitably large, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E_4(t) + \frac{1}{2} D_4(t) \le C N_4(t), \tag{1.7.25}
$$

where

$$
\begin{aligned}
E_4(t) &= E_3(t) + b_4 \{ \|\partial_{x_1}^2 u_\infty(t)\|_2^2 + \|\partial_t u_\infty(t)\|_2^2 \} + \|\partial_{x_1}\partial_{x_2} \phi_\infty(t)\|_2^2, \\
D_4(t) &= D_3(t) + b_4 \{ D[\partial_{x_1}^2 w_\infty(t)] + D[\partial_t w_\infty(t)] \} + \frac{\gamma^2}{\nu+\tilde{\nu}} \|\partial_{x_1}\partial_{x_2} \phi_\infty(t)\|_2^2 \\
&\quad + \frac{\nu+\tilde{\nu}}{\gamma^2} \{ \|\partial_{x_1} \dot{\phi}_\infty(t)\|_{H^1}^2 + \|\partial_t \dot{\phi}_\infty(t)\|_2^2 \}, \\
N_4(t) &= N_3(t) + \left| R_{2,0}^{(1)}(t) \right| + \left| R_{0,1}^{(1)}(t) \right| + \left| R_{1,0}^{(2)}(t) \right|.
\end{aligned}
$$

It then follows from $b_5 \times (1.7.25) + (1.7.18)_{j=1,l=0}$ with a suitably large $b_5 > 0$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_5(t) + \frac{1}{2}D_5(t) \leq CN_5(t), \tag{1.7.26}$$

where

$$\begin{aligned}
E_5(t) &= b_5 E_4(t),\\
D_5(t) &= \frac{b_5}{2}D_4(t) + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}\partial_x\phi_\infty(t)\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_{x_1}\partial_x^2 w_\infty(t)\|_2^2,\\
N_5(t) &= N_4(t) + \|\partial_{x_1}\tilde{f}^0_\infty(t)\|_{H^1}^2 + \|\partial_{x_1}\tilde{f}_\infty(t)\|_2^2.
\end{aligned}$$

We next consider $b_6 \times (1.7.26) + (1.7.15)_{j=0,l=1}$. Then, with $b_6 > 0$ suitably large, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_6(t) + \frac{1}{2}D_6(t) \leq CN_6(t), \tag{1.7.27}$$

where

$$\begin{aligned}
E_6(t) &= b_6 E_5(t) + \|\partial_{x_2}^2\phi_\infty(t)\|_2^2,\\
D_6(t) &= \frac{b_6}{2}D_5(t) + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_2}^2\phi_\infty(t)\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2}\|\dot{\phi}_\infty(t)\|_{H^2}^2,\\
N_6(t) &= N_5(t) + \left|R^{(2)}_{0,1}(t)\right|.
\end{aligned}$$

We then deduce from $b_7 \times (1.7.27) + (1.7.18)_{j=0,l=1}$ with a suitably large $b_7 > 0$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_7(t) + \frac{1}{2}D_7(t) \leq CN_7(t), \tag{1.7.28}$$

where

$$\begin{aligned}
E_7(t) &= b_7 E_6(t),\\
D_7(t) &= \frac{b_7}{2}D_6(t) + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_x^2\phi_\infty(t)\|_2^2 + \frac{\nu^2}{\nu+\tilde{\nu}}\|\partial_x^3 w_\infty(t)\|_2^2,\\
N_7(t) &= N_6(t) + \|\tilde{f}^0_\infty(t)\|_{H^2}^2 + \|\tilde{f}_\infty(t)\|_{H^1}^2.
\end{aligned}$$

By (1.7.2) we have

$$\|\partial_t\phi_\infty\|_{H^1}^2 \leq C\{\gamma^2\|\nabla w_\infty\|_{H^1}^2 + \gamma^2\|w\cdot\nabla\phi_\infty\|_{H^1}^2 + \|\tilde{f}^0_\infty\|_{H^1}^2\}. \tag{1.7.29}$$

We then deduce from $b_8 \times (1.7.28) + (1.7.29)$ with a suitably large $b_8 > 0$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_8(t) + \frac{1}{2}D_8(t) \leq CN_8(t), \tag{1.7.30}$$

where

$$E_8(t) = b_8 E_7(t),$$

$$D_8(t) = \frac{b_8}{2} D_7(t) + \|\partial_t \phi_\infty(t)\|_{H^1}^2,$$
$$N_8(t) = N_7(t) + \|(w \cdot \nabla \phi_\infty)(t)\|_{H^1}^2.$$

Note that $D_8(t)$ is equivalent to $D_\infty(t)$. By Remark 1.5.2, we have $D_8(t) \geq c_1 E_8(t)$ for some constant $c_1 > 0$, and hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_8(t) + c_1 E_8(t) + D_8(t) \leq 2C N_8(t).$$

This implies

$$E_8(t) + \int_0^t e^{-c_1(t-\tau)} D_8(\tau)\mathrm{d}\tau \leq e^{-c_1 t} E_8(0) + 2C \int_0^t e^{-c_1(t-\tau)} N_8(\tau)\mathrm{d}\tau.$$

Since, by (1.7.3),

$$\nu \partial_{x_2}^2 w_\infty^1 = \partial_t w_\infty^1 - \nu \partial_{x_1}^2 w_\infty^1 - \tilde{\nu} \partial_{x_1}(\partial_{x_1} w_\infty^1 + \partial_{x_2} w_\infty^2) + \gamma \partial_{x_1} \phi_\infty - f_\infty^1,$$
$$(\nu + \tilde{\nu}) \partial_{x_2}^2 w_\infty^2 = \partial_t w_\infty^2 - \nu \partial_{x_1}^2 w_\infty^2 - \tilde{\nu} \partial_{x_2} \partial_{x_1} w_\infty^1 + \gamma \partial_{x_2} \phi_\infty - f_\infty^2,$$

and

$$\|\partial_{x_1} \partial_{x_2} w_\infty(t)\|_2^2 = (\partial_{x_1}^2 w_\infty(t), \partial_{x_2}^2 w_\infty(t)) \leq \|\partial_{x_1}^2 w_\infty(t)\|_2 \|\partial_{x_2}^2 w_\infty(t)\|_2,$$

we have

$$\begin{aligned} \|\partial_{x_1} \partial_{x_2} w_\infty(t)\|_2^2 + \|\partial_{x_2}^2 w(t)\|_2^2 &\leq C\{E_8(t) + \|\tilde{f}_\infty(t)\|_2^2\} \\ &\leq C\{E_8(t) + (1+t)^{-\frac{5}{2}} M(t)^4\}. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} &E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau)\mathrm{d}\tau \\ &\quad \leq C\left\{ e^{-at} E_\infty(0) + (1+t)^{-\frac{5}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} N_8(\tau)\mathrm{d}\tau \right\} \end{aligned}$$

holds uniformly for $t \in [0, T]$ with $C > 0$ independent of T. Here $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant. We will see in section 1.8 that N_8 satisfies the estimate

$$N_8(t) \leq C\{(1+t)^{-\frac{5}{2}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty(t)\}.$$

We thus obtain estimate (1.5.10) with $\mathcal{R}(t) = N_8(t)$ that satisfies (1.5.11). This completes the proof. □

1.8 Estimates on nonlinearities

In this section we estimate the nonlinearities to establish (1.5.11) for $\mathcal{R}(t) = N_8(t)$.

By using the Gagliardo-Nirenberg-Sobolev inequality and Remark 1.5.2 we have the following estimates on $f^0 = -\gamma \mathrm{div}(\phi w)$.

Proposition 1.8.1. *The following estimates hold uniformly for $t \in [0, T]$ with $C > 0$ independent of T :*

$$\|\phi \text{div } w\|_{H^2} \le C\big\{(1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t)\sqrt{D_\infty(t)}\big\}, \tag{1.8.1}$$

$$\|w \cdot \nabla\phi\|_{H^1} + \|w \cdot \nabla\phi_1\|_{H^2} \le C(1+t)^{-\frac{5}{4}} M(t)^2, \tag{1.8.2}$$

$$|(\text{div} w, |\partial_t^k \partial_{x_1}^j \phi_\infty|^2)| \le C\big\{(1+t)^{-\frac{5}{2}} M(t)^3 + (1+t)^{-\frac{5}{2}} M(t) D_\infty(t)\big\}, \tag{1.8.3}$$

$$\|[\partial_t^k \partial_x^j, w] \cdot \nabla\phi_\infty\|_2 \le C\big\{(1+t)^{-2} M(t)^2 + (1+t)^{-\frac{5}{4}} M(t)\sqrt{D_\infty(t)}\big\} \tag{1.8.4}$$

for $2k + j \le 2$ and

$$\|\tilde{f}_\infty^0\|_{H^2} \le C\big\{(1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t)\sqrt{D_\infty(t)}\big\}, \tag{1.8.5}$$

$$\|\partial_t \tilde{f}_\infty^0\|_2 \le C\big\{(1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t)\sqrt{D_\infty(t)}\big\}. \tag{1.8.6}$$

We next consider the estimates for $\tilde{f}_\infty$. We recall that $\|\phi(t)\|_\infty \le \frac{1}{2}$ whenever $\|u(t)\|_{H^2} \le \varepsilon_2$, which follows from the Sobolev inequality $\|u(t)\|_\infty \le C\|u(t)\|_{H^2}$.

Proposition 1.8.2. *If $\|u(t)\|_{H^2} \le \varepsilon_2$ and $M(t) \le 1$ for $t \in [0, T]$, then*

$$\|\tilde{f}_\infty\|_2 \le C(1+t)^{-\frac{5}{4}} M(t)^2, \tag{1.8.7}$$

$$\|\tilde{f}_\infty\|_{H^1} \le C\big\{(1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t)\sqrt{D_\infty(t)}\big\}, \tag{1.8.8}$$

$$|(\partial_t \tilde{f}_\infty, \partial_t w_\infty)| \le C\{(1+t)^{-\frac{5}{2}} M(t)^3 + M(t) D_\infty(t)\}, \tag{1.8.9}$$

where $C > 0$ is a constant independent of T.

Proof. The estimates (1.8.7)-(1.8.9) can be proved by using the Gagliardo-Nirenberg-Sobolev inequality and Remark 1.5.2. We here estimate the term $(\partial_t(\frac{\phi}{1+\phi}\Delta w), \partial_t w_\infty)$ only, which appears in (1.8.9). We set $g(\phi) = \frac{\phi}{1+\phi}$. Then

$$\begin{aligned}
&|(\partial_t(g(\phi)\Delta w), \partial_t w_\infty)| \\
&\quad = |-(\partial_t(\nabla(g(\phi))\nabla w), \partial_t w_\infty) - (\partial_t(g(\phi)\nabla w), \partial_t \nabla w_\infty)| \\
&\quad \le |(\partial_t(g'(\phi)\nabla\phi\nabla w), \partial_t w_\infty)| + |(\partial_t(g(\phi)\nabla w), \partial_t \nabla w_\infty)| \\
&\quad =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

The first term on the right is estimated as

$$\begin{aligned}
\mathrm{I} &\le |(g''(\phi)\partial_t\phi\nabla\phi\nabla w, \partial_t w_\infty)| + |(g'(\phi)\partial_t\nabla\phi\nabla w, \partial_t w_\infty)| \\
&\quad + |(g'(\phi)\nabla\phi\partial_t\nabla w, \partial_t w_\infty)| \\
&\le C\{\|\partial_t\phi\|_4\|\nabla\phi\|_4\|\nabla w\|_\infty\|\partial_t w_\infty\|_2 + \|\partial_t\nabla\phi\|_2\|\nabla w\|_\infty\|\partial_t w_\infty\|_2 \\
&\quad + \|\nabla\phi\|_4\|\partial_t\nabla w\|_2\|\partial_t w_\infty\|_4\} \\
&\le C\{\|\partial_t\phi\|_{H^1}\|\nabla\phi\|_{H^1}\|\nabla w\|_{H^2}\|\partial_t w_\infty\|_2 + \|\partial_t\nabla\phi\|_2\|\nabla w\|_{H^2}\|\partial_t w_\infty\|_2 \\
&\quad + \|\nabla\phi\|_{H^1}\|\partial_t\nabla w\|_2\|\partial_t w_\infty\|_{H^1}\} \\
&\le C\{(1+t)^{-\frac{5}{2}} M(t)^3 + (1+t)^{-\frac{1}{2}} M(t) D_\infty(t)\}.
\end{aligned}$$

As for II, we have

$$
\begin{aligned}
\mathrm{II} &\leq C\{\|g'(\phi)\partial_t\phi\nabla w\|_2 + \|g(\phi)\partial_t\nabla w\|_2\}\|\partial_t\nabla w_\infty\|_2 \\
&\leq C\{\|\partial_t\phi\|_4\|\nabla w\|_4 + \|g(\phi)\|_\infty\|\partial_t\nabla w\|_2\}\|\partial_t\nabla w_\infty\|_2 \\
&\leq C\{\|\partial_t\phi\|_{H^1}\|\nabla w\|_{H^1} + \|\phi\|_\infty\|\partial_t\nabla w\|_2\}\|\partial_t\nabla w_\infty\|_2 \\
&\leq C\{(1+t)^{-\frac{5}{2}}M(t)^3 + M(t)D_\infty(t)\}.
\end{aligned}
$$

The other terms can be estimated similarly. This completes the proof. □

The desired estimate for $\mathcal{R}(t) = N_8(t)$ follows from Propositions 1.8.1 and 1.8.2.

1.9 Asymptotic behavior: Proof of Theorem 1.3.2

In this section we prove Theorem 1.3.2. To this end we rewrite (1.1.1)-(1.1.2) in the form of conservation laws.

We set

$$m = \rho v = \rho_*(1+\phi)v.$$

Then (1.1.1)-(1.1.2) is written as

$$\begin{cases} \partial_t\rho + \mathrm{div}\, m = 0, \\ \partial_t m - \mu\Delta\left(\dfrac{m}{\rho}\right) - (\mu+\mu')\nabla\mathrm{div}\left(\dfrac{m}{\rho}\right) + \nabla P(\rho) + \mathrm{div}\left(\dfrac{m\otimes m}{\rho}\right) = 0, \end{cases} \tag{1.9.1}$$

and the boundary condition (1.1.3) is transformed into

$$\partial_{x_2}\left(\frac{m^1}{\rho}\right)\bigg|_{x_2=0,1} = 0, \quad m^2\big|_{x_2=0,1} = 0. \tag{1.9.2}$$

We note that, from the proof of Theorem 1.3.1,

$$\|m^2(t)\|_2 = \|\gamma\rho(t)w^2(t)\|_2 = O(t^{-\frac{5}{4}}) \ \text{ as } \ t\to\infty.$$

Therefore, to prove Theorem 1.3.2, it suffices to investigate the asymptotic behavior of ${}^\top(\phi, m^1)$.

We decompose ${}^\top(\phi, m^1)$ as

$$
\begin{gathered}
\phi = \Phi + \Phi_\infty, \quad \Phi = \phi_1 = \tilde{P}_1\phi, \quad \Phi_\infty = \phi_\infty = \tilde{P}_\infty\phi, \\
m^1 = \rho_*\gamma(M + M_\infty), \quad M = \frac{1}{\rho_*\gamma}\tilde{P}_1 m^1, \quad M_\infty = \frac{1}{\rho_*\gamma}\tilde{P}_\infty m^1.
\end{gathered}
$$

Note that $w^1 = \frac{M+M_\infty}{1+\phi}$.

Applying P_1 to (1.9.1) and using (1.9.2), we have

$$\begin{cases} \partial_t\Phi + \gamma\partial_{x_1}M = 0, \\ \partial_t M - (\nu+\tilde{\nu})\partial_{x_1}^2 M + \gamma\partial_{x_1}\Phi = \partial_{x_1}\tilde{P}_1 g(U) + \partial_{x_1}\tilde{P}_1\tilde{g}. \end{cases} \tag{1.9.3}$$

Here $U = {}^\top(\Phi, M)$,

$$g(U) = -\frac{\rho_* P''(\rho_*)}{2\gamma}\Phi^2 - \gamma M^2,$$
$$\tilde{g} = \tilde{g}(x,t) = -(\nu+\tilde{\nu})\partial_{x_1}(\phi w^1) - \frac{\rho_* P''(\rho_*)}{2\gamma}(2\Phi\Phi_\infty + \Phi_\infty^2)$$
$$- \gamma(2MM_\infty + M_\infty^2) + \gamma(\phi w^1(M + M_\infty)),$$

where $\phi = \Phi + \Phi_\infty,\ w^1 = \frac{M+M_\infty}{1+\phi}$.

We write (1.9.3) in the form

$$\begin{cases} \partial_t U + L_0 U = \partial_{x_1} P_0 G(U) + \partial_{x_1} P_0 \tilde{G}, \quad U = P_0 U, \\ U|_{t=0} = P_0 U_0, \end{cases} \tag{1.9.4}$$

where $U_0 = {}^\top(\phi_0, \frac{1}{\rho_*\gamma} m_0^1) = {}^\top(\phi_0, (1+\phi_0)w_0^1)$,

$$L_0 = \begin{pmatrix} 0 & \gamma\partial_{x_1} \\ \gamma\partial_{x_1} & -(\nu+\tilde{\nu})\partial_{x_1}^2 \end{pmatrix},$$
$$G(U) = \begin{pmatrix} 0 \\ g(U) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix},$$

and P_0 denotes the projection defined by

$$P_0(U) = \begin{pmatrix} \tilde{P}_1 \Phi \\ \tilde{P}_1 M \end{pmatrix}$$

for $U = {}^\top(\Phi, M)$.

We see from Lemma 1.4.2 that

$$e^{-tL_0} = \mathcal{F}^{-1}(e^{\lambda_+ t} P_+ + e^{\lambda_- t} P_-)\mathcal{F},$$

where

$$\lambda_\pm = \lambda_{\pm,0} = -\frac{1}{2}(\nu+\tilde{\nu})\xi^2 \pm \frac{1}{2}\sqrt{(\nu+\tilde{\nu})^2\xi^4 - 4\gamma^2\xi^2},$$
$$P_\pm = \pm\frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} -\lambda_\mp & i\gamma\xi \\ i\gamma\xi & \lambda_\pm \end{pmatrix}.$$

We observe that, for $|\xi| \ll 1$,

$$\lambda_\pm = -\frac{\nu+\tilde{\nu}}{2}\xi^2 \pm i\gamma\xi + O(\xi^3),$$
$$P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}(1 + O(\xi)).$$

We define $S(t)$ and $S_\pm(t)$ by

$$S(t) = S_+(t) + S_-(t),$$

$$S_\pm(t) = \mathcal{F}^{-1}\hat{S}_\pm(t)\mathcal{F},$$
$$\hat{S}_\pm(t) = \frac{1}{2}e^{-\frac{\nu+\tilde{\nu}}{2}\xi^2 t \pm i\gamma\xi t}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

Clearly, $e^{-tL_0}P_0$ has the same estimate as that for $e^{-tL}P_1$ such as (1.6.1). Furthermore, $e^{-tL_0}P_0$ is approximated by $S(t)$ in the following way. We define Π_0 by

$$\Pi_0 U_0 = {}^\top(\langle\phi_0\rangle, \langle M_0\rangle) \ \text{ for } \ U_0 = {}^\top(\phi_0, M_0).$$

Note that $\Pi_0 P_0 = P_0\Pi_0 = P_0$.

Lemma 1.9.1. *The following estimates hold uniformly for $t > 0$:*

(i) $\|\partial_{x_1}^k e^{-tL_0}P_0U_0\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_1,$

(ii) $\|\partial_{x_1}^k S_\pm(t)P_0U_0\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_1,$
$\|\partial_{x_1}^k S_\pm(t)\Pi_0(I-P_0)U_0\|_2 \le Ct^{-\frac{k}{2}}e^{-c_0t}\|U_0\|_2, \quad c_0 = \frac{1}{2}(\nu+\tilde{\nu})r_0^2,$

(iii) $\|\partial_{x_1}^k\big(e^{-tL_0} - S(t)\big)P_0U_0\|_2 \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_1.$

Proof. The estimates in (i) and (ii) can be obtained by the same computation as in (1.6.1). As for (iii), since

$$\begin{aligned}
&|e^{\lambda_\pm t}P_\pm - \hat{S}_\pm(t)| \\
&\quad = C\big\{|e^{-\frac{\nu+\tilde{\nu}}{2}\xi^2 t\pm i\gamma\xi t}(e^{\lambda_\pm t + \frac{\nu+\tilde{\nu}}{2}\xi^2 t \mp i\gamma\xi t} - 1)| + C|\xi|e^{Re\lambda_\pm t}\big\} \\
&\quad \le C(|\xi|^3 t + |\xi|)e^{-\frac{\nu+\tilde{\nu}}{4}\xi^2 t},
\end{aligned}$$

we have the desired estimate. □

We denote by $U^{(0)}(t) = {}^\top(\phi^{(0)}(x_1,t), M^{(0),1}(x_1,t))$ the solution of the following integral equation:

$$U^{(0)}(t) = S(t)\Pi_0 U_0 + \int_0^t S(t-\tau)\partial_{x_1}G(U^{(0)}(\tau))\mathrm{d}\tau. \tag{1.9.5}$$

We see from (1.9.4) that $U(t)$ is written as

$$U(t) = e^{-tL_0}P_0U_0 + \int_0^t e^{-(t-\tau)L_0}P_0\partial_{x_1}(G(U) + \tilde{G})(\tau)\mathrm{d}\tau. \tag{1.9.6}$$

We will show that $U(t)$ is approximated by $U^0(t)$ as $t \to \infty$.

By Lemma 1.9.1, we have the following estimates for $U^{(0)}(t)$.

Proposition 1.9.2. *If $\|U_0\|_{H^2\cap L^1} \ll 1$, then (1.9.5) has a unique solution $U^{(0)}(t)$ that satisfies*

$$\|\partial_{x_1}^k U^{(0)}(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{H^2\cap L^1}, \ k = 0,1,2, \tag{1.9.7}$$
$$\|\partial_{x_1}^k U^{(0)}(t)\|_\infty \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|U_0\|_{H^2\cap L^1}, \ k = 0,1. \tag{1.9.8}$$

We have the following estimate for $U(t)-U^{(0)}(t)$.

Theorem 1.9.3. *If* $\|U_0\|_{H^2\cap L^1}\ll 1$, *then*

$$\|\partial_{x_1}^k(U(t)-U^{(0)}(t))\|_2\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}\|U_0\|_{H^2\cap L^1},\ k=0,1,$$

for any $\delta>0$.

Proof. We introduce $N(t)$ defined by

$$N(t)=\sup_{0\le\tau\le t}\left\{\sum_{k=0}^{1}(1+\tau)^{\frac{3}{4}+\frac{k}{2}-\delta}\|\partial_{x_1}^k(U(\tau)-U^{(0)}(\tau))\|_2\right\}.$$

It follows form (1.9.5)-(1.9.6) that $U(t)-U^{(0)}(t)$ is written as

$$U(t)-U^{(0)}(t)=\sum_{j=0}^{4}I_j(t),$$

where

$$\begin{aligned}
I_0(t)&=(e^{-tL_0}P_0-S(t)\Pi_0)U_0,\\
I_1(t)&=\int_0^t S(t-\tau)P_0\partial_{x_1}(G(U(\tau))-G(U^{(0)}(\tau)))\mathrm{d}\tau,\\
I_2(t)&=\int_0^t (e^{-(t-\tau)L_0}-S(t-\tau))P_0\partial_{x_1}G(U(\tau))\mathrm{d}\tau,\\
I_3(t)&=-\int_0^t S(t-\tau)(I-P_0)\partial_{x_1}G(U^{(0)}(\tau))\mathrm{d}\tau,\\
I_4(t)&=\int_0^t e^{-(t-\tau)L_0}P_0\partial_{x_1}\tilde{G}(\tau)\mathrm{d}\tau.
\end{aligned}$$

As for $I_0(t)$, we see from Lemma 1.9.1 (ii), (iii) that

$$\begin{aligned}
\|\partial_{x_1}^k I_0(t)\|_2&\le\|\partial_{x_1}^k(e^{-tL_0}-S(t))P_0U_0\|_2+\|S(t)\Pi_0(I-P_0)\partial_{x_1}^k U_0\|_2\\
&\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{H^1\cap L^1}.
\end{aligned}$$

As for I_1, by Theorem 1.3.1 and Proposition 1.9.2, we have

$$\begin{aligned}
\|G(U)-G(U^{(0)})\|_1&\le C\{\|\Phi^2-(\phi^{(0)})^2\|_1+\|M^2-(M^{(0),1})^2\|_1\}\\
&\le C\|U+U^{(0)}\|_2\|U-U^{(0)}\|_2\\
&\le C(1+t)^{-1+\delta}N(t)\|U_0\|_{H^2\cap L^1},
\end{aligned}$$

and similarly,

$$\|\partial_{x_1}\big(G(U)-G(U^{(0)})\big)\|_1\le C\{\|\partial_{x_1}(U+U^{(0)})\|_2\|U-U^{(0)}\|_2$$

$$
\begin{aligned}
&\qquad + \|U+U^{(0)}\|_2\|\partial_{x_1}(U-U^{(0)})\|_2\} \\
&\le C(1+t)^{-\frac{3}{2}+\delta}N(t)\|U_0\|_{H^2\cap L^1}.
\end{aligned}
$$

It then follows from Lemma 1.9.1 (ii) that

$$
\begin{aligned}
\|\partial_{x_1}^k I_1(t)\|_2 &\le C\left\{\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1+\delta}\mathrm{d}\tau\right. \\
&\qquad \left. + \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1-\frac{k}{2}+\delta}\mathrm{d}\tau\right\} N(t)\|U_0\|_{H^2\cap L^1} \\
&\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}N(t)\|U_0\|_{H^2\cap L^1}.
\end{aligned}
$$

We next estimate $\partial_{x_1}^k I_2(t)$. Since

$$
\begin{aligned}
&\|\partial_{x_1}G(U)\|_1 \le C\|U\|_2\|\partial_{x_1}U\|_2 \le C(1+t)^{-1}M(t)^2, \\
&\|\partial_{x_1}^2 G(U)\|_1 \le C\{\|U\|_2\|\partial_{x_1}^2 U\|_2 + \|\partial_{x_1}U\|_2^2\} \le C(1+t)^{-\frac{3}{2}}M(t)^2.
\end{aligned}
$$

We see from Lemma 1.9.1 (iii) that

$$
\begin{aligned}
\|\partial_{x_1}^k I_2(t)\|_2 &\le C\left\{\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1}\mathrm{d}\tau\right. \\
&\qquad \left. + \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1-\frac{k}{2}}\mathrm{d}\tau\right\} M(t)^2 \\
&\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}\|U_0\|_{H^2\cap L^1}.
\end{aligned}
$$

Concerning $I_3(t)$, we first observe that $\partial_{x_1}G(U^{(0)}) = \Pi_0\partial_{x_1}G(U^{(0)})$ since $\partial_{x_1}G(U^{(0)})$ depends only on x_1 and t. Furthermore, we have

$$
\begin{aligned}
\|\partial_{x_1}G(U^{(0)})\|_2 &\le C\|U^{(0)}\|_\infty\|\partial_{x_1}U^{(0)}\|_2 \le C(1+t)^{-\frac{5}{4}}\|U_0\|_{H^2\cap L^1}^2, \\
\|\partial_{x_1}^2 G(U^{(0)})\|_2 &\le C\{\|U^{(0)}\|_\infty\|\partial_{x_1}^2 U^{(0)}\|_2 + \|\partial_{x_1}U^{(0)}\|_\infty\|\partial_{x_1}U^{(0)}\|_2\} \\
&\le C(1+t)^{-\frac{7}{4}}\|U_0\|_{H^2\cap L^1}^2.
\end{aligned}
$$

It then follows from Lemma 1.9.1 (ii) that

$$
\begin{aligned}
\|\partial_{x_1}^k I_3(t)\|_2 &\le C\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}\mathrm{d}\tau\|U_0\|_{H^2\cap L^1}^2 \\
&\le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}\|U_0\|_{H^2\cap L^1}.
\end{aligned}
$$

As for $I_4(t)$, we have

$$
\begin{aligned}
\|\tilde{G}\|_1 &\le C(1+t)^{-1}M(t)^2, \\
\|\partial_{x_1}\tilde{G}\|_1 &\le C(1+t)^{-\frac{3}{2}}M(t)^2,
\end{aligned}
$$

and hence, similarly to the estimate for $\partial_{x_1}^k I_2(t)$,

$$\|\partial_{x_1}^k I_4(t)\|_2 \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}\|U_0\|_{H^2\cap L^1}.$$

This completes the proof. $\square$

Proof of Theorem 1.3.2. It suffices to show that $\|\partial_{x_1}^k(U^{(0)}-\chi_+\boldsymbol{b}_+-\chi_-\boldsymbol{b}_-)(t)\|_2$ for $k=0,1$, where $\boldsymbol{b}_\pm = {}^\top(1,\pm 1)\in\mathbb{R}^2$. Here $\chi_\pm=\chi_\pm(x_1,t)$ is the diffusion waves given in (1.1.10)-(1.1.12) with $c=\frac{1}{2}(a+b), a=-\frac{\rho_* P''(\rho_*)}{2\gamma}, b=-\gamma$. We follow the arguments in [19] and [17]. We write U_0 as

$$U_0 = U_{0+}+U_{0-},$$

where

$$U_{0\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1\\ \pm 1 & 1\end{pmatrix}\Pi_0 U_0 = \frac{1}{2}\left\langle \phi_0 \pm \frac{1}{\rho_*\gamma}m_0^1\right\rangle \boldsymbol{b}_\pm.$$

It then follows that

$$U^{(0)}(t) = S_+(t)U_{0+}+S_-(t)U_{0-}+I_{1,+}(t)+I_{1,-}(t),$$

where

$$I_{1,\pm}(t) = \int_0^t S_\pm(t-\tau)\partial_{x_1}\begin{pmatrix} 0\\ a(\phi^{(0)})^2+b(M^{(0),1})^2\end{pmatrix}\mathrm{d}t.$$

We write $I_{1,\pm}(t)$ as

$$I_{1,\pm} = \pm\frac{1}{2}\int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}\big(a(\phi^{(0)})^2+b(M^{(0),1})^2\big)\mathrm{d}\tau\boldsymbol{b}_\pm,$$

where

$$e^{-tL_\pm}u_0 = \mathcal{F}^{-1}\big[e^{(-\frac{\nu+\tilde{\nu}}{2}\xi^2\pm i\gamma\xi)t}\hat{u}_0\big].$$

We note that $e^{-tL_\pm}$ satisfies the same estimates as those for $S_\pm(t)$ in Lemma 1.9.1 (ii).

We define $V(t)={}^\top(\eta(t),\zeta(t))$ by

$$\begin{aligned} U^{(0)}(t) &= \chi_+(t)\boldsymbol{b}_+ + \chi_-(t)\boldsymbol{b}_- + V(t)\\ &= \begin{pmatrix}\chi_+ + \chi_- + \eta\\ \chi_+ - \chi_- + \zeta\end{pmatrix},\end{aligned}$$

and introduce

$$Y(t) = \sup_{0\le\tau\le t}\{(1+\tau)^{\frac{1}{2}}\|V(\tau)\|_2 + (1+\tau)\|\partial_{x_1}V(\tau)\|_2\}.$$

We write

$$\begin{aligned}(\phi^{(0)})^2 &= (\chi_+ + \chi_- + \eta)(\chi_+ + \chi_- + \eta)\\ &= \chi_+^2+\chi_-^2+2\chi_+\chi_- + (\chi_+ + \chi_+)\eta + \eta(\chi_+ + \chi_- + \eta)\end{aligned}$$

$$
\begin{aligned}
&= \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + (\chi_+ + \chi_- + \phi^{(0)})\eta \\
&= \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + \sigma_1\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
(M^{(0),1})^2 &= \chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + (\chi_+ - \chi_- + M^{(0),1})\zeta \\
&= \chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + \sigma_2\zeta,
\end{aligned}
$$

where $\sigma_1 = \chi_+ + \chi_- + \phi^{(0)}$ and $\sigma_2 = \chi_+ - \chi_- + M^{(0),1}$. It then follows that $I_{1,\pm}(t)$ is written in the following forms

$$
\begin{aligned}
I_{1,\pm}(t) = \pm\frac{1}{2}\int_0^t &e^{-(t-\tau)L_\pm}\partial_{x_1}((a+b)(\chi_+^2 + \chi_-^2) \\
&+ 2(a-b)\chi_+\chi_- + a\sigma_1\eta + b\sigma_2\zeta)\mathrm{d}\tau\boldsymbol{b}_\pm.
\end{aligned}
$$

Since $\chi_\pm$ satisfies

$$
\chi_\pm(t) = e^{-tL_\pm}\chi_{0\pm} \pm \frac{a+b}{2}\int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}(\chi_\pm^2)(\tau)\mathrm{d}\tau,
$$

where $\chi_{0\pm} = \chi_\pm(0)$, we see that

$$
\begin{aligned}
V(t) &= U^{(0)}(t) - \chi_+(t)\boldsymbol{b}_+ - \chi_-(t)\boldsymbol{b}_- \\
&= S_+(t)(U_{0+} - \chi_{0+}\boldsymbol{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\boldsymbol{b}_-) + I_{1,+} + I_{1,-} \\
&\quad - \frac{a+b}{2}\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\chi_+^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad + \frac{a+b}{2}\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\chi_-^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&= S_+(t)(U_{0+} - \chi_{0+}\boldsymbol{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\boldsymbol{b}_-) \\
&\quad + \frac{1}{2}(a+b)\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\chi_-^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - \frac{1}{2}(a+b)\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\chi_+^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&\quad + (a-b)\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\chi_+\chi_-)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - (a-b)\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\chi_+\chi_-)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&\quad + \frac{1}{2}a\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_1\eta)(\tau)\mathrm{d}\tau\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - \frac{1}{2}a\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_1\eta)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&\quad + \frac{1}{2}b\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_2\zeta)(\tau)\mathrm{d}\tau\boldsymbol{b}_+
\end{aligned}
$$

$$-\frac{1}{2}b\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_2\zeta)(\tau)\mathrm{d}\tau\boldsymbol{b}_-.$$

It then follows that

$$\begin{aligned}
\|\partial_{x_1}^k V(t)\|_2 \le& \sum_{j=\pm}\|\partial_{x_1}^k S_j(t)(U_{0j}-\chi_{0j}\boldsymbol{b}_j)\|_2 \\
&+ C_1\big(\|\partial_{x_1}^k w_+(t)\|_2+\|\partial_{x_1}^k w_-(t)\|_2\big) \\
&+ C_2\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+}\partial_{x_1}(\chi_+\chi_-)(\tau)\|_2\mathrm{d}\tau \\
&+ C_3\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-}\partial_{x_1}(\chi_+\chi_-)(\tau)\|_2\mathrm{d}\tau \\
&+ C_4\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_1\eta)(\tau)\|_2\mathrm{d}\tau \\
&+ C_5\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_1\eta)(\tau)\|_2\mathrm{d}\tau \\
&+ C_6\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_2\zeta)(\tau)\|_2\mathrm{d}\tau \\
&+ C_7\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_2\zeta)(\tau)\|_2\mathrm{d}\tau \\
=:& \sum_{j=\pm}\|\partial_{x_1}^k S_j(t)(U_{0j}-\chi_{0j}\boldsymbol{b}_j)\|_2+\sum_{j=1}^7 I_j,
\end{aligned}$$

where

$$\begin{aligned}
w_\pm(t) &= \int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}(\chi_\mp^2)(\tau)\mathrm{d}\tau, \\
C_1 &= \frac{1}{2}|a+b|,\quad C_2=C_3=|a-b|,\quad C_4=C_5=\frac{1}{2}|a|,\quad C_6=C_7=\frac{1}{2}|b|.
\end{aligned}$$

Since

$$\int_{\mathbb{R}}(U_{0\pm}-\chi_{0\pm}\boldsymbol{b}_\pm)\mathrm{d}x_1=\left[\frac{1}{2}\int_\Omega\left(\phi^{(0)}\pm\frac{1}{\rho_*\gamma}m_0^1\right)\mathrm{d}x-\int_{\mathbb{R}}\chi_{0\pm}\mathrm{d}x_1\right]\boldsymbol{b}_\pm=0,$$

we have

$$\|\partial_{x_1}^k S_\pm(t)(U_{0\pm}-\chi_{0\pm}\boldsymbol{b}_\pm)\|_2\le Ct^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1_{1/2}}.$$

As for I_1, we apply the estimates for $w_\pm$ by [24] (see also [17, Lemma 4.2]) to obtain

$$I_1\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2.$$

We next estimate I_2. For $1\le p\le\infty$ and $l\ge 0$, we have

$$\|\partial_x^l(\chi_+\chi_-)(t)\|_1\le Ce^{-ct}\|u_0\|_{H^2\cap L^1}^2. \tag{1.9.9}$$

See [19] and [17] for estimate (1.9.9). It then follows from (1.9.9) and Lemma 1.9.1 (ii) that

$$
\begin{aligned}
I_2 &\leq C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(\|\chi_+\chi_-(\tau)\|_1 + \|\partial_{x_1}^{k+1}(\chi_+\chi_-)(\tau)\|_2)\mathrm{d}\tau \\
&\leq C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} e^{-c\tau}\mathrm{d}\tau \|u_0\|_{H^2\cap L^1}^2 \\
&\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2.
\end{aligned}
$$

Similarly, we have $I_3 \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2$.

We next estimate I_4. By Lemma 1.9.1, we have

$$
\begin{aligned}
I_4 &\leq C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|\sigma_1\eta(\tau)\|_1\mathrm{d}\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}}\|\partial_{x_1}^k(\sigma_1\eta)(\tau)\|_2\mathrm{d}\tau \\
&\quad + C\int_0^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_{x_1}^k(\sigma_1\eta)(\tau)\|_2\mathrm{d}\tau \\
&=: I_{41} + I_{42} + I_{43}.
\end{aligned}
$$

By applying Proposition 1.9.2 and the following estimate

$$
\|\partial_{x_1}^k\chi_\pm(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_1, \tag{1.9.10}
$$

we see that $\|\sigma_1(\tau)\|_2 \leq C(1+\tau)^{-\frac{1}{4}}\|u_0\|_{H^2\cap L^1}$. Since $\|\sigma_1\eta\|_1 \leq \|\sigma_1\|_2\|\eta\|_2$, we have

$$
\begin{aligned}
I_{41} &= CY(t)\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\|u_0\|_{H^2\cap L^1} \\
&\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{42} &= CY(t)\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}\mathrm{d}\tau\|u_0\|_{H^2\cap L^1} \\
&\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t),
\end{aligned}
$$

and

$$
\begin{aligned}
I_{43} &= CY(t)\int_0^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1-\frac{k}{2}}\mathrm{d}\tau\|u_0\|_{H^2\cap L^1} \\
&\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t).
\end{aligned}
$$

We thus obtain $I_4 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t)$. We can obtain the estimates for I_5, I_6, I_7 in a similar manner. It then follows that if $\|u_0\|_{H^2\cap L^1} \ll 1$, we have

$$
\|\partial_x^k V(t)\|_2 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1} \tag{1.9.11}
$$

for $k = 0, 1$.

Since $w^1 = \frac{1}{\rho_* \gamma} m^1 - \phi w^1$, we have $w_1^1 = M - \tilde{P}_1(\phi w^1)$, and so,

$$u = \begin{pmatrix} \phi_1 \\ w_1^1 \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_\infty \\ w_\infty^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} \Phi \\ M \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_\infty \\ -\tilde{P}_1(\phi w^1) + w_\infty^1 \\ w^2 \end{pmatrix}.$$

The desired estimate in Theorem 1.3.2 thus follows from Theorem 9.3 and (1.9.11). This completes the proof. □

Chapter 2

Asymptotic behavior of solutions of the compressible Navier-Stokes equations in a cylinder under the slip boundary condition

In this chapter, the large time behavior of solutions to the compressible Navier-Stokes equations around the motionless state is considered in a cylinder under the slip boundary condition. It is shown that if the initial data is sufficiently small, the global solution uniquely exists and the large time behavior of the solution is described by a superposition of one-dimensional nonlinear diffusion waves and a diffusive rigid rotation.

2.1 Formulation of the problem

This chapter studies the large time behavior of solutions of the compressible Navier-Stokes equations

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{v}) = 0, \tag{2.1.1}$$

$$\rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \mu \mathrm{div} \mathbf{D}(\boldsymbol{v}) - \mu' \nabla \mathrm{div} \boldsymbol{v} + \nabla p(\rho) = \mathbf{0} \tag{2.1.2}$$

in an infinite cylinder $\Omega_\ell = \mathbb{R} \times D_\ell$:

$$\Omega_\ell = \{x = (x_1, x');\ x_1 \in \mathbb{R},\ x' = (x_2, x_3) \in D_\ell\},$$
$$D_\ell = \{x' = (x_2, x_3);\ x_2^2 + x_3^2 \leq \ell^2\},$$

under the slip boundary condition

$$\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_\ell} = 0, \quad \mathbf{D}(\boldsymbol{v}) \cdot \boldsymbol{n} - \big(\mathbf{D}(\boldsymbol{v})\boldsymbol{n} \cdot \boldsymbol{n}\big)\boldsymbol{n}\big|_{\partial\Omega_\ell} = \mathbf{0}. \tag{2.1.3}$$

Here $\rho = \rho(x, t)$ and $\boldsymbol{v} = {}^\top (v^1(x,t), v^2(x,t), v^3(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \Omega_\ell$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ and satisfies

$$p'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy

$$\mu > 0, \quad \frac{2}{3}\mu + \mu' > 0;$$

div and ∇ denote the usual divergence and gradient with respect to x; $\mathbf{D}(\cdot)$ denotes the deformation tensor whose (j,k)-components $(j,k = 1,2,3)$ are given by

$$\mathbf{D}(\boldsymbol{v})_{jk} = \partial_{x_j} v^k + \partial_{x_k} v^j;$$

$\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega_\ell$, which is given by $\boldsymbol{n} = {}^\top(0, \boldsymbol{n}')$ with $\boldsymbol{n}' = \frac{1}{\ell}x' = \frac{1}{\ell}{}^\top(x_2, x_3)$ being the unit outward normal vector to ∂D_ℓ. Here and in what follows ${}^\top\cdot$ stands for the transposition.

We impose the initial condition

$$\rho|_{t=0} = \rho_0, \quad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0. \tag{2.1.4}$$

Here $\rho_0 = \rho_0(x)$ and $\boldsymbol{v}_0 = \boldsymbol{v}_0(x)$ satisfy $\rho_0(x) \to \rho_*$ and $\boldsymbol{v}_0(x) \to \mathbf{0}$ as $|x| \to \infty$.

In this chapter we will consider the stability of the motionless state $u_s = {}^\top(\rho_*, \mathbf{0})$ and will investigate the large time behavior of solutions around u_s. We thus rewrite (2.1.1)-(2.1.2) into the following equations for the perturbation

$$\partial_t\phi + \gamma \mathrm{div}\boldsymbol{w} = f^0(\phi, \boldsymbol{w}), \tag{2.1.5}$$

$$\partial_t\boldsymbol{w} - \nu \mathrm{div}\mathbf{D}(\boldsymbol{w}) - \nu'\nabla\mathrm{div}\boldsymbol{w} + \gamma\nabla\phi = \boldsymbol{f}(\phi, \boldsymbol{w}). \tag{2.1.6}$$

Here $u = {}^\top(\phi, \boldsymbol{w})$ with $\phi = \frac{1}{\rho_*}(\rho - \rho_*)$ and $\boldsymbol{w} = \frac{1}{\gamma}\boldsymbol{v}$ denotes the perturbation of $u_s = {}^\top(\rho_*, \mathbf{0})$; ν, ν', γ are parameters given by

$$\nu = \frac{\mu}{\rho_*}, \quad \nu' = \frac{\mu'}{\rho_*}, \quad \gamma = \sqrt{p'(\rho_*)};$$

and $f(\phi, \boldsymbol{w}) = {}^\top(f^0(\phi, \boldsymbol{w}), \boldsymbol{f}(\phi, \boldsymbol{w}))$ denotes the nonlinear terms:

$$\begin{aligned} f^0(\phi, \boldsymbol{w}) &= -\gamma \mathrm{div}(\phi\boldsymbol{w}), \\ \boldsymbol{f}(\phi, \boldsymbol{w}) &= -\gamma\boldsymbol{w}\cdot\nabla\boldsymbol{w} - \frac{\phi}{1+\phi}\{\nu\mathrm{div}\mathbf{D}(\boldsymbol{w}) + \nu'\nabla\mathrm{div}\boldsymbol{w}\} + \frac{\gamma\phi}{1+\phi}\nabla\phi \\ &\quad - \frac{\rho_* p''(\rho_*)}{2\gamma(1+\phi)}\nabla(\phi^2) - \frac{\rho_*^2}{2\gamma(1+\phi)}\nabla\big(p^{(3)}(\phi)\phi^3\big), \end{aligned}$$

where

$$p^{(3)}(\phi) = \int_0^1 (1-\theta)^2 p'''\big(\rho_*(1+\theta\phi)\big)\mathrm{d}\theta.$$

The boundary condition (2.1.3) and initial condition (2.1.4) are transformed into

$$\boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega_\ell} = 0, \quad \mathbf{D}(\boldsymbol{w})\cdot\boldsymbol{n} - (\mathbf{D}(\boldsymbol{w})\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n}\big|_{\partial\Omega_\ell} = \mathbf{0}, \tag{2.1.7}$$

and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, \boldsymbol{w}_0). \tag{2.1.8}$$

Here u_0 satisfies $u_0(x) \to 0$ as $|x| \to \infty$.

In this chapter we show that the solution $u(t)$ of (1.1.7)-(1.1.8) under the slip boundary condition (1.1.9) with (1.1.10) behaves like a superposition of one-dimensional nonlinear diffusion waves and a diffusive rigid rotation as $t \to \infty$. More precisely, we prove that, under appropriate conditions for u_0, the solution $u(t)$ satisfies

$$\|\partial_x^k(u - \kappa_+ a_+ - \kappa_- a_- - \kappa_{\mathrm{rig}} a_{\mathrm{rig}})(t)\|_{L^2(\Omega_\ell)} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \tag{2.1.9}$$

where $a_\pm = \frac{1}{2}{}^\top(1, \pm 1, 0, 0)$ and $\kappa_\pm = \kappa_\pm(x_1, t)$ are the nonlinear diffusion waves given by

$$\kappa_\pm(x_1, t) = Z_\pm(x_1 \pm \gamma t, t). \tag{2.1.10}$$

Here $Z_\pm = Z_\pm(x_1, t)$ are the self-similar solutions of the Burgers equations

$$\partial_t Z_\pm - \frac{2\nu + \nu'}{2}\partial_{x_1}^2 Z_\pm \mp c\partial_{x_1}(Z_\pm^2) = 0 \tag{2.1.11}$$

satisfying

$$\int_{\mathbb{R}} Z_\pm(x_1, t)\mathrm{d}x_1 = \frac{1}{2}\int_{\Omega_\ell} \big(\phi_0(x) \pm (1 + \phi_0(x))w_0^1(x)\big)\mathrm{d}x \tag{2.1.12}$$

for some constant $c \in \mathbb{R}$; and

$$a_{\mathrm{rig}} = {}^\top(0, \boldsymbol{a}_{\mathrm{rig}}), \quad \boldsymbol{a}_{\mathrm{rig}} = \frac{1}{\ell^2}\sqrt{\frac{2}{\pi}}\ {}^\top(0, -x_3, x_2), \tag{2.1.13}$$

$$\kappa_{\mathrm{rig}}(x_1, t) = w_{0,\mathrm{rig}}(4\pi\nu t)^{-\frac{1}{2}} e^{-\frac{x_1^2}{4\nu t}} \tag{2.1.14}$$

with $w_{0,\mathrm{rig}} = \int_{\Omega_\ell} \boldsymbol{w}_0 \cdot \boldsymbol{a}_{\mathrm{rig}}\mathrm{d}x$. We note that, in addition to the wave propagation part $\kappa_+ a_+ + \kappa_- a_-$, the diffusive rigid motion part $\kappa_{\mathrm{rig}} a_{\mathrm{rig}}$ also appears in the asymptotic leading part of the solution in the case of the cylinder. We also note that the diffusive rigid motion part $\kappa_{\mathrm{rig}} a_{\mathrm{rig}}$ gives the incompressible part of the asymptotic leading part of u since $\mathrm{div}(\kappa_{\mathrm{rig}}\boldsymbol{a}_{\mathrm{rig}}) = 0$. It should be remarked that the global existence with exponential decay estimate was shown by Shibata and Murata [32] for the problem on a bounded domain under the slip boundary condition (1.1.9), provided that initial data are sufficiently small, and, in addition, orthogonal to rigid motions when the domain is rotationally symmetric. The method in [32] was mainly based on the maximal regularity approach. We also mention the work [22] by Kobayashi and Zajaczkowski, where the global existence on a bounded domain was proved based on the energy method.

This chapter is organized as follows. In Section 2 we introduce notations and rewrite the problem in a non-dimensional form. In Section 3 we state the main results of this chapter. In Section 4 we study the spectral properties of the linearized operator and derive a Korn type inequality. In Section 5 we rewrite the problem (1.1.7)-(1.1.10) into a problem for a system of equations for the low and high frequency parts and introduce the momentum formulation for the low frequency part. Section 6 is devoted to estimating the low frequency part, while the high frequency part is estimated in Section 7. In Section 8, we give necessary estimates for the nonlinearities. In Section 9 we study the asymptotic behavior of the solution of (1.1.7)-(1.1.10).

2.2 Preliminaries

In this section we first introduce notations which will be used throughout the chapter. We then rewrite the problem into a non-dimensional form.

2.2.1 Notation

For $1 \le p \le \infty$ we denote by $L^p(E)$ the usual Lebesgue space on a domain E of $\mathbb{R}^n$ and its norm is denoted by $\|\cdot\|_{L^p(E)}$. Let m be a nonnegative integer. The symbol $H^m(E)$ denotes the m-th order L^2-Sobolev space on E with norm $\|\cdot\|_{H^m(E)}$.

We simply denote by $L^p(E)$ (resp., $H^m(E)$) the set of all vector fields $\boldsymbol{w} = {}^\top(w^1, \cdots, w^n)$ on E with $w^j \in L^p(E)$ (resp, $H^m(E)$), $j = 1, \cdots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(E)}$ (resp., $\|\cdot\|_{H^m(E)}$). For $u = {}^\top(\phi, \boldsymbol{w})$ with $\phi \in H^k(E)$ and $\boldsymbol{w} = {}^\top(w^1, \cdots, w^n) \in H^m(E)$, we define $\|u\|_{H^k(E)\times H^m(E)}$ by $\|u\|_{H^k(E)\times H^m(E)} = \|\phi\|_{H^k(E)} + \|\boldsymbol{w}\|_{H^k(E)}$. When $k = m$, we simply write $\|u\|_{H^k(E)\times H^k(E)} = \|u\|_{H^k(E)}$.

Partial derivatives of a function v in x, x_1, x' and t are denoted by $\partial_x v, \partial_{x_1} v, \partial_{x'} v$ and $\partial_t v$. We also write the higher order partial derivatives of v in x as $\partial_x^l v = (\partial_x^\alpha v; |\alpha| = l)$.

In Section 4 we will consider the complex-valued functions. We set $\Omega := \Omega_1$. In the case where $E = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as L^p (resp., H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$. We denote the inner product of $L^2(\Omega)$ by

$$(f, g) = \int_\Omega f(x)\overline{g(x)}\mathrm{d}x, \quad f, g \in L^2(\Omega).$$

Here $\bar{g}$ denotes the complex conjugate of g.

We set $D := D_1$. The inner product of $L^2(D)$ is denoted by

$$(f, g)_{L^2(D)} = \int_D f(x')\overline{g(x')}\mathrm{d}x', \quad f, g \in L^2(D).$$

We also define a weighted inner product $\langle f, g\rangle$ by

$$\langle f, g\rangle = \frac{1}{|D|}(f, g)_{L^2(D)} = \frac{1}{|D|}\int_D f(x')\overline{g(x')}\mathrm{d}x',$$

where $|D| = \int_D \mathrm{d}x'$. For $f \in L^1(D)$ we denote the mean value of f over D by $\langle f\rangle$:

$$\langle f\rangle = \langle f, 1\rangle = \frac{1}{|D|}\int_D f\mathrm{d}x'.$$

For $\alpha \in \mathbb{R}$, we denote by $L^1_\alpha = L^1_\alpha(\Omega)$ the weighted L^1 space with weight $(1 + |x_1|)^\alpha$, and its norm is denoted by

$$\|f\|_{L^1_\alpha} = \int_\Omega (1 + |x_1|)^\alpha |f(x)|\mathrm{d}x.$$

We denote the Fourier transform of $f = f(x_1)$ ($x_1 \in \mathbb{R}$) by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x_1)e^{-i\xi x_1}\mathrm{d}x_1, \quad \xi \in \mathbb{R}.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f](x_1) = (2\pi)^{-1} \int_{\mathbb{R}} f(\xi) e^{i\xi x_1} \mathrm{d}\xi, \quad x_1 \in \mathbb{R}.$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$.

For operators A, B, we denote the commutator of A and B by $[A, B]$:

$$[A, B]f = A(Bf) - B(Af).$$

2.2.2 Non-dimensionalization

In this subsection we rewrite the problem into the one in a non-dimensional form. We introduce the following non-dimensional variables:

$$\tilde{x} = \frac{1}{\ell} x, \quad \tilde{t} = \frac{\gamma}{\ell} t, \quad \tilde{\rho} = \frac{\rho}{\rho_*}, \quad \tilde{\boldsymbol{v}} = \frac{1}{\gamma} \boldsymbol{v}, \quad \tilde{p} = \frac{1}{\rho_* \gamma^2} p,$$

where

$$\gamma = \sqrt{p'(\rho_*)}.$$

The problem (1.1.1)-(1.1.2) is then transformed into the following non-dimensional problem on $\Omega = \mathbb{R} \times D$:

$$\partial_{\tilde{t}} \tilde{\rho} + \mathrm{div}_{\tilde{x}} (\tilde{\rho} \tilde{\boldsymbol{v}}) = 0, \tag{2.2.1}$$

$$\tilde{\rho}(\partial_{\tilde{t}} \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{v}} \cdot \nabla_{\tilde{x}} \tilde{\boldsymbol{v}}) - \nu \mathrm{div}_{\tilde{x}} \big(\mathbf{D}_{\tilde{x}} (\tilde{\boldsymbol{v}}) \big) - \nu' \nabla_{\tilde{x}} \mathrm{div}_{\tilde{x}} \tilde{\boldsymbol{v}} + \nabla_{\tilde{x}} \tilde{p}(\tilde{\rho}) = \mathbf{0}. \tag{2.2.2}$$

Here ν and ν' are non-dimensional parameters given by

$$\nu = \frac{\mu}{\rho_* \gamma \ell}, \quad \nu' = \frac{\mu'}{\rho_* \gamma \ell}.$$

The boundary condition (1.1.5) and initial condition (1.1.6) are transformed into

$$\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{n}}|_{\partial\Omega} = 0, \quad \mathbf{D}(\tilde{\boldsymbol{v}}) \cdot \tilde{\boldsymbol{n}} - (\mathbf{D}(\tilde{\boldsymbol{v}}) \tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{n}}) \tilde{\boldsymbol{n}}|_{\partial\Omega} = \mathbf{0}, \tag{2.2.3}$$

and

$$(\tilde{\rho}, \tilde{\boldsymbol{v}})|_{t=0} = (\tilde{\rho}_0, \tilde{\boldsymbol{v}}_0). \tag{2.2.4}$$

In what follows, for simplicity, we omit the tildes of $\tilde{x}, \tilde{t}, \tilde{\rho}, \tilde{\boldsymbol{v}}$ and $\tilde{p}$ and write them as $x, t, \rho, \boldsymbol{v}$ and p, respectively. Observe that, due to the non-dimensionalization, we have

$$D = \{x' = (x_2, x_3); x_2^2 + x_3^2 \leq 1\}, \quad |D| = \int_D \mathrm{d}x' = \pi,$$

and thus

$$\langle f \rangle = \frac{1}{\pi} \int_D f(x') \mathrm{d}x'.$$

The perturbation equations for the non-dimensionalized problem (2.2.1)-(2.2.2) is given by

$$\partial_t \phi + \mathrm{div} \boldsymbol{w} = f^0(\phi, \boldsymbol{w}), \tag{2.2.5}$$
$$\partial_t \boldsymbol{w} - \nu \mathrm{div} \mathbf{D}(\boldsymbol{w}) - \nu' \nabla \mathrm{div} \boldsymbol{w} + \nabla \phi = \boldsymbol{f}(\phi, \boldsymbol{w}). \tag{2.2.6}$$

Here $u = {}^\top(\phi, \boldsymbol{w})$ with $\phi = \rho - 1$ and $\boldsymbol{w} = \boldsymbol{v}$ denotes the non-dimensional perturbation, and $f(\phi, \boldsymbol{w}) = {}^\top(f^0(\phi, \boldsymbol{w}), \boldsymbol{f}(\phi, \boldsymbol{w}))$ denotes the non-dimensional nonlinear terms:

$$\begin{aligned} f^0(\phi, \boldsymbol{w}) &= -\mathrm{div}(\phi \boldsymbol{w}), \\ \boldsymbol{f}(\phi, \boldsymbol{w}) &= -\boldsymbol{w} \cdot \nabla \boldsymbol{w} - \frac{\phi}{1+\phi} \{\nu \mathrm{div} \mathbf{D}(\boldsymbol{w}) + \nu' \nabla \mathrm{div} \boldsymbol{w}\} + \frac{\phi}{1+\phi} \nabla \phi \\ &\quad - \frac{p''(1)}{2(1+\phi)} \nabla(\phi^2) - \frac{1}{2(1+\phi)} \nabla \big(p^{(3)}(\phi) \phi^3\big), \end{aligned}$$

where

$$p^{(3)}(\phi) = \int_0^1 (1-\theta)^2 p'''(1+\theta\phi) \mathrm{d}\theta.$$

The non-dimensional form of the boundary and initial conditions are

$$\boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathbf{D}(\boldsymbol{w}) \cdot \boldsymbol{n} - (\mathbf{D}(\boldsymbol{w}) \boldsymbol{n} \cdot \boldsymbol{n}) \boldsymbol{n}\big|_{\partial\Omega} = \mathbf{0}, \tag{2.2.7}$$

with $\boldsymbol{n} = {}^\top(0, \boldsymbol{n}')$, $\boldsymbol{n}' = {}^\top(x_2, x_3)$, and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, \boldsymbol{w}_0). \tag{2.2.8}$$

Here u_0 satisfies $u_0(x) \to 0$ as $|x| \to \infty$.

2.3 Main results of Chapter 2

In this section we state the main results of this chapter. We begin with the global existence result. We set

$$H^2_* = H^2_*(\Omega) = \{\boldsymbol{w} \in H^2(\Omega)^3; \boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \mathbf{D}(\boldsymbol{w}) \cdot \boldsymbol{n} - (\mathbf{D}(\boldsymbol{w}) \boldsymbol{n} \cdot \boldsymbol{n}) \boldsymbol{n}\big|_{\partial\Omega} = \mathbf{0}\}.$$

Theorem 2.3.1. *There exists a positive constant ε_0 such that if $u_0 = {}^\top(\phi_0, \boldsymbol{w}_0) \in (H^2 \times H^2_*) \cap (L^1 \times L^1)$ and $\|u_0\|_{H^2 \cap L^1} \le \varepsilon_0$, then problem (1.1.7)-(1.1.10) has a unique global solution*

$$u(t) = {}^\top(\phi(t), \boldsymbol{w}(t)) \in C\left([0, \infty); H^2 \times H^2_*\right)$$

and $u(t)$ satisfies

$$\|\partial_{x_1}^k u(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} \tag{2.3.1}$$

for $t \ge 0$, $k = 0, 1$.

We next consider the asymptotic behavior of solutions.

Theorem 2.3.2. *In addition to the assumption of Theorem 2.3.1, assume that* $u_0 \in L^1_{1/2} \times L^1_{1/2}$. *Then*

$$\|\partial^k_{x_1}(u - \chi_+ b_+ - \chi_- b_- - \chi_{\rm rig} b_{\rm rig})(t)\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1. \tag{2.3.2}$$

Here $b_\pm = \frac{1}{2}{}^\top(1, \pm 1, 0, 0)$; $\chi_\pm = \chi_\pm(x_1, t)$ *are the diffusion waves given by* $\chi_\pm(x_1, t) = z_\pm(x_1 \pm t, t)$ *with* $z_\pm(x_1, t) = Z(\ell x_1, \frac{\ell}{\gamma} t)$ *and* $\chi_{\rm rig}$ *is the diffusive rigid rotation given by* $\chi_{\rm rig}(x_1, t) = \kappa_{\rm rig}(\ell x_1, \frac{\ell}{\gamma} t)$, *where* $Z_\pm$ *and* $\kappa_{\rm rig}$ *are defined in (2.1.10)-(2.1.12) and (2.1.13)-(2.1.14), respectively.*

The proof of Theorem 2.3.1 will be given in Sections 2.4–2.8, and Theorem 2.3.2 will be proved in Section 2.9.

2.4 Spectral properties of the linearized operator

In this section we investigate the spectral properties of the linearized operators and give a Korn type inequality, which will be used in the proof Theorem 2.3.1.

We consider the linearized equation

$$\partial_t u + Lu = 0, \quad u\big|_{t=0} = u_0 = {}^\top(\phi_0, \boldsymbol{w}_0), \tag{2.4.1}$$

where $u = {}^\top(\phi, \boldsymbol{w})$ and L is an operator on $L^2 \times L^2$ of the form

$$L = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & -\nu \mathrm{div}\mathbf{D}(\cdot) - \nu' \nabla \mathrm{div} \end{pmatrix}$$

with domain $D(L) = H^1 \times H^2_*$. It was shown in [28] that $-L$ generates an analytic semigroup e^{-tL} on $L^2 \times L^2$.

To investigate the spectrum of L, we consider the Fourier transform of (2.4.1) in x_1 variable, which take the form

$$\partial_t \hat{\phi} + i\xi \hat{w}^1 + \nabla' \cdot \hat{\boldsymbol{w}}' = 0, \tag{2.4.2}$$

$$\partial_t \hat{w}^1 + \nu(\xi^2 - \Delta')\hat{w}^1 - i(\nu + \nu')\xi(i\xi \hat{w}^1 + \nabla' \cdot \hat{\boldsymbol{w}}') + i\xi\hat{\phi} = 0, \tag{2.4.3}$$

$$\partial_t \hat{\boldsymbol{w}}' + \nu\big(\xi^2 \hat{\boldsymbol{w}}' - \nabla' \cdot \mathbf{D}'(\hat{\boldsymbol{w}}')\big) - \nu'\nabla'\nabla' \cdot \hat{\boldsymbol{w}}' - i(\nu + \nu')\xi \nabla' \hat{w}^1 + \nabla' \hat{\phi} = \mathbf{0}'. \tag{2.4.4}$$

Here $\nabla' = {}^\top(\partial_{x_2}, \partial_{x_3})$, $\Delta' = \partial^2_{x_2} + \partial^2_{x_3}$; and $\mathbf{D}'(\boldsymbol{w}')$ is the 2×2 matrix with (j,k)-components $\mathbf{D}'(\boldsymbol{w}')_{j,k} = \partial_{x_j} w^k + \partial_{x_k} w^j$, $j, k = 2, 3$, and $\mathbf{0}' = {}^\top(0, 0)$.

We thus arrive at the following problem

$$\partial_t \hat{u} + \hat{L}_\xi \hat{u} = 0, \quad \hat{u}|_{t=0} = \hat{u}_0 \tag{2.4.5}$$

with a parameter $\xi \in \mathbb{R}$. Here $\hat{L}_\xi$ is an operator on $X := H^1(D) \times L^2(D)^3$ of the form

$$\hat{L}_\xi = \begin{pmatrix} 0 & i\xi & {}^\top\nabla' \\ i\xi & (2\nu + \nu')\xi^2 - \nu\Delta' & -i(\nu + \nu')\xi {}^\top\nabla' \\ \nabla' & -i\xi(\nu + \nu')\nabla' & \nu\big(\xi^2 - \nabla' \cdot \mathbf{D}'(\cdot)\big) - \nu\nabla'{}^\top\nabla' \end{pmatrix}$$

with domain $D(\hat{L}_\xi) = H^1(D) \times H^2_*(D)$, where

$$H^2_*(D) = \Big\{ \boldsymbol{w} = {}^\top(w^1, \boldsymbol{w}') \in H^2(D)^3;\ \frac{\partial w^1}{\partial \boldsymbol{n}'}\Big|_{\partial D} = 0,\ \boldsymbol{w}' \cdot \boldsymbol{n}'|_{\partial D} = 0,$$
$$\mathbf{D}'(\boldsymbol{w}') \cdot \boldsymbol{n}' - (\mathbf{D}'(\boldsymbol{w}')\boldsymbol{n}' \cdot \boldsymbol{n}')\boldsymbol{n}'\big|_{\partial D} = \mathbf{0}' \Big\}.$$

It was shown in [28] that $-\hat{L}_0 = -\hat{L}_\xi|_{\xi=0}$ generates an analytic semigroup. Since

$$\|(\hat{L}_\xi - \hat{L}_0)u\|_{H^1(D)\times L^2(D)} \le C|\xi|\big(1 + |\xi|\big)\|u\|_{L^2(D)\times H^1(D)},$$

one can see from a standard perturbation argument that $-\hat{L}_\xi$ generates an analytic semigroup for any fixed $\xi \in \mathbb{R}$. It follows that $e^{-tL} = \mathcal{F}^{-1}e^{-t\hat{L}_\xi}\mathcal{F}$.

To investigate the spectral properties of $-\hat{L}_\xi$, we introduce the following notations. For $u = {}^\top(\phi, \boldsymbol{w}),\ \boldsymbol{w} = {}^\top(w^1, \boldsymbol{w}')$, we define $\Pi_0 u,\ \Pi_1 u$ and $\Pi_{\mathrm{rig}} u$ by

$$\Pi_0 u = \begin{pmatrix} \langle \phi \rangle \\ \mathbf{0} \end{pmatrix},$$
$$\Pi_1 u = \begin{pmatrix} 0 \\ \boldsymbol{\Pi}_1 \boldsymbol{w} \end{pmatrix}, \quad \boldsymbol{\Pi}_1 \boldsymbol{w} = \begin{pmatrix} \langle w^1 \rangle \\ \mathbf{0}' \end{pmatrix},$$
$$\Pi_{\mathrm{rig}} u = \begin{pmatrix} 0 \\ \boldsymbol{\Pi}_{\mathrm{rig}} \boldsymbol{w} \end{pmatrix}, \quad \boldsymbol{\Pi}_{\mathrm{rig}} \boldsymbol{w} = \langle \boldsymbol{w}, \boldsymbol{b}_{\mathrm{rig}} \rangle \boldsymbol{b}_{\mathrm{rig}},$$

where

$$\boldsymbol{b}_{\mathrm{rig}} = \begin{pmatrix} 0 \\ \boldsymbol{b}'_{\mathrm{rig}} \end{pmatrix}, \quad \boldsymbol{b}'_{\mathrm{rig}} = \sqrt{2}\begin{pmatrix} -x_3 \\ x_2 \end{pmatrix}.$$

Note that $\langle \boldsymbol{b}_{\mathrm{rig}}, \boldsymbol{b}_{\mathrm{rig}} \rangle = \langle \boldsymbol{b}'_{\mathrm{rig}}, \boldsymbol{b}'_{\mathrm{rig}} \rangle = 1$ and that Π_j $(j = 0, 1, \mathrm{rig})$ are projections satisfying $\Pi_j \Pi_k = 0$ $(j \ne k)$. We also define the projection $\boldsymbol{\Pi}'_{\mathrm{rig}}$ by

$$\boldsymbol{\Pi}'_{\mathrm{rig}} \boldsymbol{w}' = \langle \boldsymbol{w}', \boldsymbol{b}'_{\mathrm{rig}} \rangle \boldsymbol{b}'_{\mathrm{rig}}.$$

By a direct computation, one can verify the following lemma.

Lemma 2.4.1. (i) *For any $\xi \in \mathbb{R}$, $-\hat{L}_\xi$ has the following eigenvalues*

$$\lambda_\pm(\xi) = -\frac{2\nu + \nu'}{2}\xi^2 \pm \frac{1}{2}\sqrt{(2\nu + \nu')^2\xi^4 - 4\xi^2},$$
$$\lambda_{\mathrm{rig}}(\xi) = -\nu\xi^2.$$

(ii) *Let $\hat{\Pi}_\pm(\xi)$ be the projection operator given by*

$$\hat{\Pi}_\pm(\xi) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} \begin{pmatrix} \mp\lambda_\mp(\xi) & \pm i\xi & {}^\top\mathbf{0}' \\ \pm i\xi & \pm\lambda_\pm(\xi) & {}^\top\mathbf{0}' \\ \mathbf{0}' & \mathbf{0}' & O' \end{pmatrix} (\Pi_0 + \Pi_1)$$

for $\xi \ne \pm\frac{2}{2\nu+\nu'}$, where

$$O' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and let $\hat{\Pi}_{\rm rig}(\xi)$ *be the projection operator given by*

$$\hat{\Pi}_{\rm rig}(\xi) = \Pi_{\rm rig}$$

for $\xi \in \mathbb{R}$. *Then for* $j, k = \pm, {\rm rig}$,

$$\hat{\Pi}_j(\xi)^2 = \hat{\Pi}_j(\xi), \quad \hat{\Pi}_j(\xi)\hat{\Pi}_k(\xi) = 0 \quad (j \neq k)$$

and

$$-\hat{\Pi}_j(\xi)\hat{L}_\xi \subset -\hat{L}_\xi\hat{\Pi}_j(\xi) = \lambda_j(\xi)\hat{\Pi}_j(\xi).$$

(iii) *If* $|\xi| < \frac{2}{2\nu+\nu'}$, *then*

$$\lambda_+(\xi) = \overline{\lambda_-(\xi)}, \quad {\rm Re}\ \lambda_\pm(\xi) = -\frac{2\nu+\nu'}{2}\xi^2.$$

Furthermore there exists a positive constant R_0 *such that*

$$\begin{aligned} &{\rm Im}\ \lambda_\pm(\xi) = \pm i(\xi + \tilde{\lambda}(\xi)), \\ &\hat{\Pi}_\pm(\xi) = \Pi_{\pm,0} + \tilde{\Pi}_\pm(\xi) \end{aligned}$$

for $|\xi| \le R_0$, *where*

$$\Pi_{\pm,0} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 & {}^\top\mathbf{0}' \\ \pm 1 & 1 & {}^\top\mathbf{0}' \\ \mathbf{0}' & \mathbf{0}' & O' \end{pmatrix}(\Pi_0 + \Pi_1);$$

$\tilde{\lambda}(\xi)$ *and* $\tilde{\Pi}_\pm(\xi)$ *satisfy*

$$|\tilde{\lambda}(\xi)| \le C|\xi|^3, \quad \|\tilde{\Pi}_\pm(\xi)\| \le C|\xi|$$

uniformly for $|\xi| \le R_0$. *Here* $\|\tilde{\Pi}_\pm(\xi)\|$ *denotes the matrix norm of* $\tilde{\Pi}_\pm(\xi)$.

Let F be a smooth function on $[0, T]$ with values in X, and let u_0 be in X. Then $u(t)$ defined by

$$u(t) = e^{-t\hat{L}_\xi}u_0 + \int_0^t e^{-(t-s)\hat{L}_\xi}F(s){\rm d}s$$

is a unique solution of

$$\partial_t u + \hat{L}_\xi u = F, \quad u|_{t=0} = u_0. \tag{2.4.6}$$

We decompose (2.4.6) into $\Pi_j(\xi)$ ($j = \pm, {\rm rig}$) and $I - (\Pi_+ + \Pi_- + \Pi_{\rm rig})(\xi)$ parts; applying these projections to (2.4.6), we have

$$\partial_t u_j - \lambda_j(\xi)u_j = F_j, \quad u_j|_{t=0} = u_{0,j}, \quad j = \pm, {\rm rig}, \tag{2.4.7}$$

$$\partial_t \tilde{u} + \hat{L}_\xi \tilde{u} = \tilde{F}, \quad \tilde{u}|_{t=0} = \tilde{u}_0. \tag{2.4.8}$$

Here

$$u_j(t) = \hat{\Pi}_j(\xi)u(t), \quad F_j(t) = \hat{\Pi}_j(\xi)F(t), \quad u_{0,j} = \hat{\Pi}_j(\xi)u_0, \ j = \pm, {\rm rig};$$

and

$$\tilde{u}(t) = \big(I - \hat{\Pi}_+(\xi) - \hat{\Pi}_-(\xi) - \hat{\Pi}_{\rm rig}(\xi)\big)u(t),$$

and so on. It then follows from (2.4.7) that

$$u_j(t) = e^{\lambda_j(\xi)t}u_{0,j} + \int_0^t e^{\lambda_j(\xi)(t-s)}F_j(s)\mathrm{d}s,$$
$$\hat{\Pi}_j(\xi)e^{-t\hat{L}_\xi}u_0 = e^{\lambda_j(\xi)t}\hat{\Pi}_j(\xi)u_0.$$

By Lemma 2.4.1, we deduce that for any nonnegative integers k and l,

$$\|\xi^k\hat{\Pi}_\pm(\xi)e^{-t\hat{L}_\xi}u_0\|_{L^2(D)} \le C_k(1+t)^{-\frac{k}{2}}e^{-\frac{2\nu+\nu'}{k}\xi^2 t}\|u_0\|_{L^1(D)} \tag{2.4.9}$$

uniformly in $|\xi| \le R_0$ and $t \ge 0$, and

$$\|\xi^k\partial_{x'}^l\Pi_{\mathrm{rig}}e^{-t\hat{L}_\xi}u_0\|_{L^2(D)} \le C_{k,l}(1+t)^{-\frac{k}{2}}e^{-\frac{\nu}{k}\xi^2 t}\|u_0\|_{L^1(D)}$$

uniformly in $|\xi| \le R_1$ and $t \ge 0$, where R_1 is an arbitrarily fixed positive constant.

We next introduce a projection on *a low frequency* (*slowly decaying*) part. For a given positive number R, let $\mathbf{1}_R$ be a function on $\mathbb{R}$ given by

$$\mathbf{1}_R(\xi) = \begin{cases} 1 & (|\xi| \le R), \\ 0 & (|\xi| > R). \end{cases}$$

We define P_1 by

$$P_1 = \mathcal{F}^{-1}\big[\mathbf{1}_{R_0}(\xi)\big(\hat{\Pi}_+(\xi) + \hat{\Pi}_-(\xi)\big) + \mathbf{1}_{R_1}(\xi)\Pi_{\mathrm{rig}}\big]\mathcal{F}, \tag{2.4.10}$$

where R_0 is the positive constant given in Lemma 2.4.1 and R_1 is a positive constant to be determined later. We also set

$$P_\infty = I - P_1. \tag{2.4.11}$$

It then follows that P_1 is a bounded projection on $H^k \times H^k$ for all k, and that P_1 satisfies $\partial_{x_1}P_1 = P_1\partial_{x_1}$,

$$\|P_1 u\|_{H^k} \le C_k\|u\|_2, \tag{2.4.12}$$

and

$$\begin{aligned} P_1 e^{-tL} &= e^{-tL}P_1 \\ &= \mathcal{F}^{-1}\big[\mathbf{1}_{R_0}(\xi)\big(e^{\lambda_+(\xi)t}\hat{\Pi}_+(\xi) + e^{\lambda_-(\xi)t}\hat{\Pi}_-(\xi)\big) + \mathbf{1}_{R_1}(\xi)e^{\lambda_{\mathrm{rig}}(\xi)t}\Pi_{\mathrm{rig}}\big]\mathcal{F}. \end{aligned}$$

By (2.4.9), a standard argument shows that the P_1-part of e^{-tL} has the following decay properties.

Lemma 2.4.2. *For any nonnegative integers k and l, $P_1e^{-tL} = e^{-tL}P_1$ satisfies the estimate*

$$\|\partial_{x_1}^k\partial_{x'}^l P_1 e^{-tL}u_0\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_1 \tag{2.4.13}$$

uniformly for $t \ge 0$ and $u_0 \in (L^2 \times L^2) \cap (L^1 \times L^1)$.

We next investigate the asymptotic behavior of $e^{-tL}P_1$ as $t\to\infty$. We define $S(t), S_{\pm}(t)$ and $S_{\rm rig}(t)$ by

$$
\begin{aligned}
S(t) &= S_+(t)+S_-(t)+S_{\rm rig}(t),\\
S_j(t) &= \mathcal{F}^{-1}\hat{S}_j(t)\mathcal{F}, \quad j=\pm,\ \text{rig},\\
\hat{S}_{\pm}(t) &= \frac{1}{2}e^{-\frac{2\nu+\nu'}{2}\xi^2 t\pm i\xi t}\Pi_{\pm,0},\\
\hat{S}_{\rm rig}(t) &= e^{-\nu\xi^2 t}\Pi_{\rm rig}.
\end{aligned}
$$

We then see that $S(t)$ gives the asymptotic leading part of $e^{-tL}P_1$ in the following sense.

Lemma 2.4.3. *Let $k=0,1$. The following estimates hold uniformly for $t>0$:*

(i) $\|\partial_{x_1}^k S_{\pm}(t)P_1u_0\|_2 \le C(1+t)^{-\frac14-\frac{k}{2}}\|u_0\|_1$,
$\|\partial_{x_1}^k S_{\pm}(t)(I-P_1)u_0\|_2 \le C\big\{t^{-\frac{k}{2}}e^{-\frac{2\nu+\nu'}{4}R_0^2 t}\|u_0\|_2+(1+t)^{-\frac34-\frac{k}{2}}\|u_0\|_1\big\}$,

(ii) $\|\partial_{x_1}^k S_{\rm rig}(t)P_1u_0\|_2 \le C(1+t)^{-\frac14-\frac{k}{2}}\|u_0\|_1$,
$\|\partial_{x_1}^k S_{\rm rig}(t)(I-P_1)u_0\|_2 \le Ct^{-\frac{k}{2}}e^{-\frac{\nu}{2}R_1^2 t}\|u_0\|_2$,

(iii) $\|\partial_{x_1}^k \big(e^{-tL}-S(t)\big)P_1u_0\|_2 \le C(1+t)^{-\frac34-\frac{k}{2}}\|u_0\|_1$.

Proof. The first inequality of (i) can be obtained in a standard manner. As for the second one, since $\hat{\Pi}_{\pm}(\xi)=\Pi_{\pm,0}+\tilde{\Pi}_{\pm}(\xi)$, we see that

$$
\begin{aligned}
\Pi_{\pm,0}(I-P_1) &= \mathcal{F}^{-1}\big[\Pi_{\pm,0}(I-\mathbf{1}_{R_0})+\mathbf{1}_{R_0}\big(I-\hat{\Pi}_+(\xi)-\hat{\Pi}_-(\xi)\big)\big]\\
&= \mathcal{F}^{-1}\big[(1-\mathbf{1}_{R_0})\Pi_{\pm,0}-\mathbf{1}_{R_0}\Pi_{\pm,0}\big(\tilde{\Pi}_+(\xi)+\tilde{\Pi}_-(\xi)\big)\big].
\end{aligned}
$$

It then follows from Lemma 2.4.1 (iii) that

$$
\begin{aligned}
&\|\partial_{x_1}^k S_{\pm}(t)(I-P_1)u_0\|_2^2\\
&\quad\le C\left\{\int_{|\xi|\ge R_0}|\xi|^{2k}e^{-(2\nu+\nu')\xi^2 t}\|\hat{u}_0\|_{L^2(D)}^2\mathrm{d}\xi\right.\\
&\qquad\qquad\left.+\int_{|\xi|\le R_0}|\xi|^{2(k+1)}e^{-(2\nu+\nu')\xi^2 t}\|\hat{u}_0\|_{L^1(D)}^2\mathrm{d}\xi\right\}\\
&\quad\le C\{t^{-k}e^{-\frac{2\nu+\nu'}{2}R_0^2 t}\|u_0\|_2^2+(1+t)^{-\frac32-k}\|u_0\|_1^2\},
\end{aligned}
$$

which is the desired inequality.

The inequalities in (ii) can be obtained similarly to those in (i). As for (iii), we see from Lemma 2.4.1 (iii) that

$$
\begin{aligned}
\big|e^{\lambda_{\pm}(\xi)t}-e^{(-\frac{2\nu+\nu'}{2}\xi^2\pm i\xi)t}\big| &= \big|e^{(-\frac{2\nu+\nu'}{2}\xi^2\pm i\xi)t}(e^{\pm i\tilde{\lambda}_{\pm}(\xi)t}-1)\big|\\
&\le C|\xi|^3te^{-\frac{2\nu+\nu'}{2}\xi^2 t}
\end{aligned}
$$

for $|\xi| \le R_0$. We thus obtain

$$\|\partial_{x_1}^k \big(e^{-tL} - S(t)\big)P_1 u_0\|_2 \le C \left(\int_{|\xi|\le R_0} |\xi|^{2(k+1)} e^{-(2\nu+\nu')\xi^2 t} \mathrm{d}\xi \right)^{\frac{1}{2}} \|u_0\|_1$$
$$\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|u_0\|_1.$$

This completes the proof. □

Remark 2.4.4. *Let us consider the inhomogeneous problem*

$$\partial_t u + Lu = F, \quad u|_{t=0} = u_0, \tag{2.4.14}$$

where $F = {}^\top(f^0, \boldsymbol{f})$, $\boldsymbol{f} = {}^\top(f^1, \boldsymbol{f}')$. *Let* $u_1(t) = P_1 u(t)$. *Applying* P_1 *to (2.4.14), we have*

$$\partial_t u_1 + Lu_1 = P_1 F, \quad u_1|_{t=0} = P_1 u_0. \tag{2.4.15}$$

Since $\hat{\Pi}_+(\xi) + \hat{\Pi}_-(\xi) = \Pi_0 + \Pi_1$, *we see that* P_1 *is written as*

$$P_1 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}(\xi)\Pi_0 + \mathbf{1}_{R_0}(\xi)\Pi_1 + \mathbf{1}_{R_1}(\xi)\Pi_{\mathrm{rig}}]\mathcal{F}.$$

It then follows that

$$u_1(t) = \sigma^0(t) b_0 + \sigma^1(t) b_1 + \sigma^{\mathrm{rig}}(t) b_{\mathrm{rig}},$$

where

$$b_0 = {}^\top(1, 0, \mathbf{0}'), \quad b_1 = {}^\top(0, 1, \mathbf{0}'),$$
$$\sigma^0 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{\phi}\rangle], \quad \sigma^1 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{w}^1\rangle], \quad \sigma^{\mathrm{rig}} = \mathcal{F}^{-1}[\mathbf{1}_{R_1}\langle\hat{\boldsymbol{w}}', \boldsymbol{b}'_{\mathrm{rig}}\rangle].$$

The problem (2.4.15) is then written as

$$\begin{cases} \partial_t \sigma^0 + \partial_{x_1}\sigma^1 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{f}^0\rangle], \\ \partial_t \sigma^1 - (2\nu+\nu')\partial_{x_1}^2 \sigma^1 + \partial_{x_1}\sigma^0 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{f}^1\rangle], \\ \partial_t \sigma^{\mathrm{rig}} - \nu\partial_{x_1}^2\sigma^{\mathrm{rig}} = \mathcal{F}^{-1}[\mathbf{1}_{R_1}\langle\hat{\boldsymbol{f}}', \boldsymbol{b}'_{\mathrm{rig}}\rangle], \\ \sigma^0|_{t=0} = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{\phi}_0\rangle],\ \sigma^1|_{t=0} = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{w}_0^1\rangle],\ \sigma^{\mathrm{rig}}|_{t=0} = \mathcal{F}^{-1}[\mathbf{1}_{R_1}\langle\hat{\boldsymbol{w}}'_0, \boldsymbol{b}'_{\mathrm{rig}}\rangle], \end{cases} \tag{2.4.16}$$

where $u_0 = {}^\top(\phi_0, \boldsymbol{w}_0)$, $\boldsymbol{w}_0 = {}^\top(w_0^1, \boldsymbol{w}'_0)$. *Therefore,* ${}^\top(\sigma^0, \sigma^1)$ *is a solution of linearized one-dimensional compressible Navier-Stokes equations on* $\mathbb{R}$, *and* σ^{rig} *is a solution of a linear one-dimensional heat equation on* $\mathbb{R}$.

As for the P_∞-part we will use the following Poincaré's and Korn type inequalities.

Lemma 2.4.5. *If the constant* R_1 *in the definition of* P_1 *is taken suitably large, then the following estimates*

$$\|\phi\|_2 \le C\|\nabla\phi\|_2,$$
$$\|\boldsymbol{w}\|_2 \le C_1\|\nabla\boldsymbol{w}\|_2 \le C_2\|\mathbf{D}(\boldsymbol{w})\|_2$$

hold for $u = {}^\top(\phi, \boldsymbol{w}) = P_\infty u$ *with* $\boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$.

To prove Lemma 2.4.5, we prepare the following Korn's inequality.

Lemma 2.4.6. *There holds the estimate*

$$\|\nabla \boldsymbol{w}\|_2 \leq C\{\|\mathbf{D}(\boldsymbol{w})\|_2 + \|\boldsymbol{\Pi}'_{\mathrm{rig}}\boldsymbol{w}'\|_2\} \tag{2.4.17}$$

for $\boldsymbol{w} = {}^\top(w^1, \boldsymbol{w}') \in H^1(\Omega)$ *with* $\boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$.

Proof. We first assume that $\boldsymbol{\Pi}'_{\mathrm{rig}}\boldsymbol{w}'(x_1,\cdot) = \mathbf{0}$ for a.e. x_1. By the definition of $\mathbf{D}(\boldsymbol{w})$, we have

$$\begin{aligned}\|\mathbf{D}(\boldsymbol{w})\|_2^2 &= (\mathbf{D}(\boldsymbol{w}), \mathbf{D}(\boldsymbol{w}))\\ &= 2\sum_{j,k=1}^{3}\int_\Omega |\partial_{x_j} w^k|^2 \mathrm{d}x + 2\mathrm{Re}\sum_{j,k=1}^{3}\int_\Omega \partial_{x_j} w^k \overline{\partial_{x_k} w^j}\mathrm{d}x.\end{aligned} \tag{2.4.18}$$

As for the second term on the right-hand side, since $\boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, we see, by integration by parts, that

$$\begin{aligned}&\sum_{j,k=1}^{3}\int_\Omega \partial_{x_j} w^k \overline{\partial_{x_k} w^j}\mathrm{d}x\\ &= \sum_{k=1}^{3}\int_\Omega \partial_{x_1} w^k \overline{\partial_{x_k} w^1}\mathrm{d}x + \sum_{j=2,3}\int_\Omega \partial_{x_j} w^1 \overline{\partial_{x_1} w^j}\mathrm{d}x\\ &\quad + \sum_{j=2,3}\sum_{k=2,3}\int_\Omega \partial_{x_j} w^k \overline{\partial_{x_k} w^j}\mathrm{d}x\\ &= \int_\Omega \mathrm{div}\boldsymbol{w}\ \overline{\partial_{x_1} w^1}\mathrm{d}x + \int_\Omega \partial_{x_1} w^1 \overline{\nabla'\cdot\boldsymbol{w}'}\mathrm{d}x + \int_{\mathbb{R}} I(\boldsymbol{w}',\boldsymbol{w}')(x_1)\mathrm{d}x_1,\end{aligned} \tag{2.4.19}$$

where

$$I(\boldsymbol{w}'_1,\boldsymbol{w}'_2) = \sum_{j=2,3}\sum_{k=2,3}\int_D \partial_{x_j} w_1^k \overline{\partial_{x_k} w_2^j}\mathrm{d}x', \quad \boldsymbol{w}'_m = {}^\top(w_m^2, w_m^3), \quad m = 1, 2.$$

Since, $\boldsymbol{\Pi}'_{\mathrm{rig}}\boldsymbol{w}'(x_1,\cdot) = \mathbf{0}$, according to the proof of [34, Lemma 4], one can show that for any $\delta > 0$ there exists a positive constant C_δ such that

$$\begin{aligned}\mathrm{Re}\ I(\boldsymbol{w}',\boldsymbol{w}')(x_1) \geq \|\nabla'\cdot\boldsymbol{w}'(x_1,\cdot)\|_{L^2(D)}^2 - \delta\|\nabla'\boldsymbol{w}(x_1,\cdot)\|_{L^2(D)}^2\\ - C_\delta\|\mathbf{D}'(\boldsymbol{w}'(x_1,\cdot))\|_{L^2(D)}^2.\end{aligned} \tag{2.4.20}$$

This, together with (2.4.19), gives

$$\mathrm{Re}\sum_{j,k=1}^{3}\int_\Omega \partial_{x_j} w^k \overline{\partial_{x_k} w^j}\mathrm{d}x \geq \|\mathrm{div}\boldsymbol{w}\|_2^2 - \delta\|\nabla'\boldsymbol{w}'\|_2^2 - C_\delta\|\mathbf{D}'(\boldsymbol{w}')\|_2^2. \tag{2.4.21}$$

It then follows from (2.4.18) and (2.4.21) that

$$\|\mathbf{D}(\boldsymbol{w})\|_2^2 \geq 2\|\nabla\boldsymbol{w}\|_2^2 + 2\|\mathrm{div}\boldsymbol{w}\|_2^2 - 2\delta\|\nabla'\boldsymbol{w}'\|_2^2 - 2C_\delta\|\mathbf{D}'(\boldsymbol{w}')\|_2^2. \tag{2.4.22}$$

Taking $\delta = \frac{1}{2}$, we have

$$\|\nabla \boldsymbol{w}\|_2^2 + \|\text{div}\boldsymbol{w}\|_2^2 \le \|\mathbf{D}(\boldsymbol{w})\|_2^2 + 2C\|\mathbf{D}'(\boldsymbol{w}')\|_2^2 \le C\|\mathbf{D}(\boldsymbol{w})\|_2^2. \tag{2.4.23}$$

This shows the desired inequality for $\boldsymbol{w}$ satisfying $\boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$ and $\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'(x_1,\cdot) = \mathbf{0}$ for a.e. x_1.

For general $\boldsymbol{w}$ with $\boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, we decompose $\boldsymbol{w}'$ in (2.4.20) as

$$\boldsymbol{w}' = \boldsymbol{v}' + \mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}', \tag{2.4.24}$$

where $\boldsymbol{v}' = (I - \mathbf{\Pi}'_{\text{rig}})\boldsymbol{w}'$. Then $\mathbf{\Pi}'_{\text{rig}}\boldsymbol{v}' = 0$ and

$$I(\boldsymbol{w}',\boldsymbol{w}') = I(\boldsymbol{v}',\boldsymbol{v}') + I(\boldsymbol{v}',\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}') + I(\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}',\boldsymbol{v}') + I(\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}',\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}').$$

Since $\nabla'\cdot(\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}') = 0$ and $\mathbf{D}'(\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}') = O'$, we have $\nabla'\cdot\boldsymbol{v}' = \nabla'\cdot\boldsymbol{w}'$ and $\mathbf{D}'(\boldsymbol{v}') = \mathbf{D}'(\boldsymbol{w}')$. Then, applying (2.4.20) with $\boldsymbol{w}'$ replaced by $\boldsymbol{v}'$, we have

$$\begin{aligned}\text{Re } I(\boldsymbol{v}',\boldsymbol{v}') &\ge \|\nabla'\cdot\boldsymbol{v}'\|_{L^2(D)}^2 - \delta\|\nabla'\boldsymbol{v}'\|_{L^2(D)}^2 - C_\delta\|\mathbf{D}'(\boldsymbol{v}')\|_{L^2(D)}^2 \\ &= \|\nabla'\cdot\boldsymbol{w}'\|_{L^2(D)}^2 - \delta\|\nabla'\boldsymbol{v}'\|_{L^2(D)}^2 - C_\delta\|\mathbf{D}'(\boldsymbol{w}')\|_{L^2(D)}^2 \\ &\ge \|\nabla'\cdot\boldsymbol{w}'\|_{L^2(D)}^2 - \delta\|\nabla'\boldsymbol{w}'\|_{L^2(D)}^2 - C_\delta\|\mathbf{\Pi}_{\text{rig}}\boldsymbol{w}'\|_{L^2(D)}^2 \\ &\quad - C_\delta\|\mathbf{D}'(\boldsymbol{w}')\|_{L^2(D)}^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}|I(\boldsymbol{v}',\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}') + I(\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}',\boldsymbol{v}')| &\le \delta\|\nabla'\boldsymbol{v}'\|_{L^2(D)}^2 + C_\delta\|\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'\|_{L^2(D)}^2 \\ &\le \delta\|\nabla'\boldsymbol{w}'\|_{L^2(D)}^2 + C_\delta\|\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'\|_{L^2(D)}^2, \\ |I(\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}',\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}')| &\le C\|\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'\|_{L^2(D)}^2.\end{aligned}$$

Therefore, instead of (2.4.20), we obtain

$$\begin{aligned}\text{Re } I(\boldsymbol{w}',\boldsymbol{w}')(x_1) &\ge \|\nabla'\cdot\boldsymbol{w}'(x_1,\cdot)\|_{L^2(D)}^2 - 2\delta\|\nabla'\boldsymbol{w}'(x_1,\cdot)\|_{L^2(D)}^2 \\ &\quad - C_\delta\|\mathbf{D}'(\boldsymbol{w}'(x_1,\cdot))\|_{L^2(D)}^2 - C_\delta\|\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'(x_1,\cdot)\|_{L^2(D)}^2.\end{aligned}$$

This, together with (2.4.19), then gives

$$\begin{aligned}&\text{Re } \sum_{j,k=1}^{3}\int_\Omega \partial_{x_j}w^k\overline{\partial_{x_k}w^j}\mathrm{d}x \\ &\quad \ge \|\text{div}\boldsymbol{w}\|_2^2 - 2\delta\|\nabla'\boldsymbol{w}'\|_2^2 - C_\delta\|\mathbf{D}'(\boldsymbol{w}')\|_2^2 - C_\delta\|\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'\|_2^2,\end{aligned}$$

and hence

$$\begin{aligned}\|\mathbf{D}(\boldsymbol{w})\|_2^2 &\ge 2\|\nabla\boldsymbol{w}\|_2^2 + 2\|\text{div}\boldsymbol{w}\|_2^2 - 4\delta\|\nabla'\boldsymbol{w}'\|_2^2 \\ &\quad - 2C_\delta\|\mathbf{D}'(\boldsymbol{w}')\|_2^2 - 2C_\delta\|\mathbf{\Pi}'_{\text{rig}}\boldsymbol{w}'\|_2^2.\end{aligned}$$

Taking $\delta = \frac{1}{4}$, we obtain the desired inequality. This completes the proof. □

We now give a proof of Lemma 2.4.5.

Proof of Lemma 2.4.5. Let $u = {}^\top(\phi, \boldsymbol{w})$ satisfy $u = P_\infty u$. Since $\tilde{\Pi}_+(\xi) + \tilde{\Pi}_-(\xi) = \Pi_0 + \Pi_1$, we see that

$$\phi = \mathcal{F}^{-1}[\hat{\phi} - \mathbf{1}_{R_0}\langle\hat{\phi}\rangle] = \mathcal{F}^{-1}\big[(1-\mathbf{1}_{R_0})\hat{\phi} + \mathbf{1}_{R_0}(\hat{\phi} - \langle\hat{\phi}\rangle)\big], \tag{2.4.25}$$

$$w^1 = \mathcal{F}^{-1}[\hat{w}^1 - \mathbf{1}_{R_0}\langle\hat{w}^1\rangle] = \mathcal{F}^{-1}\big[(1-\mathbf{1}_{R_0})\hat{w}^1 + \mathbf{1}_{R_0}(\hat{w}^1 - \langle\hat{w}^1\rangle)\big]. \tag{2.4.26}$$

By the Poincaré's inequality, we have

$$\|\mathcal{F}^{-1}\big[\mathbf{1}_{R_0}(\hat{\phi} - \langle\hat{\phi}\rangle)\big]\|_2 \le C\|\nabla'\phi\|_2 \le C\|\nabla\phi\|_2. \tag{2.4.27}$$

Since

$$1 - \mathbf{1}_{R_0}(\xi) = \begin{cases} 0 & (|\xi| \le R_0), \\ 1 & (|\xi| > R_0), \end{cases}$$

we have

$$\|\mathcal{F}^{-1}[(1-\mathbf{1}_{R_0})\hat{\phi}]\|_2 \le \frac{C}{R_0}\|\xi\hat{\phi}\|_2 = \frac{\sqrt{2\pi}C}{R_0}\|\partial_{x_1}\phi\|_2 \le \frac{C}{R_0}\|\nabla\phi\|_2. \tag{2.4.28}$$

We thus obtain

$$\|\phi\|_2 \le C\|\nabla\phi\|_2. \tag{2.4.29}$$

Similarly we have

$$\|w^1\|_2 \le C\|\nabla w^1\|_2. \tag{2.4.30}$$

We next show that $\|\nabla\boldsymbol{w}\|_2 \le C\|\mathbf{D}(\boldsymbol{w})\|_2$ if $R_1 > 0$ is taken suitably large. We first observe that

$$\begin{aligned} \boldsymbol{w} &= \mathcal{F}^{-1}[(I - \mathbf{1}_{R_0}\boldsymbol{\Pi}_1 - \mathbf{1}_{R_1}\boldsymbol{\Pi}_{\mathrm{rig}})\hat{\boldsymbol{w}}] \\ &= \mathcal{F}^{-1}[(\mathbf{1}_{R_0}(I - (\boldsymbol{\Pi}_1 + \boldsymbol{\Pi}_{\mathrm{rig}})) + (I - \mathbf{1}_{R_1}) + (\mathbf{1}_{R_1} - \mathbf{1}_{R_0})(I - \boldsymbol{\Pi}_{\mathrm{rig}}))\hat{\boldsymbol{w}}] \\ &=: \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3. \end{aligned}$$

As for $\boldsymbol{w}_1$, since $\boldsymbol{\Pi}'_{\mathrm{rig}}\boldsymbol{w}'_j = \mathbf{0}'$ $(j = 1, 3)$, we see from Lemma 2.4.6 that

$$\|\nabla\boldsymbol{w}_j\|_2 \le C\|\mathbf{D}(\boldsymbol{w}_j)\|_2, \quad j = 1, 3. \tag{2.4.31}$$

Lemma 2.4.6 also shows that

$$\begin{aligned} \|\nabla\boldsymbol{w}_2\|_2 &\le C\{\|\mathbf{D}(\boldsymbol{w}_2)\|_2 + \|\boldsymbol{\Pi}'_{\mathrm{rig}}\boldsymbol{w}'_2\|_2\} \\ &\le C\{\|\mathbf{D}(\boldsymbol{w}_2)\|_2 + \|\boldsymbol{w}'_2\|_2\} \\ &\le C\{\|\mathbf{D}(\boldsymbol{w}_2)\|_2 + \frac{1}{R_1}\|\partial_{x_1}\boldsymbol{w}'_2\|_2\}. \end{aligned}$$

Set $R_2 = \max\{R_0, \frac{1}{2C}\}$. Then

$$\|\nabla\boldsymbol{w}_2\|_2 \le 2C\|\mathbf{D}(\boldsymbol{w}_2)\|_2.$$

We thus obtain

$$\begin{aligned}\|\nabla \boldsymbol{w}\|_2 &\le \|\nabla \boldsymbol{w}_1\|_2 + \|\nabla \boldsymbol{w}_2\|_2 + \|\nabla \boldsymbol{w}_3\|_2\\ &\le C\{\|\mathbf{D}(\boldsymbol{w}_1)\|_2 + \|\mathbf{D}(\boldsymbol{w}_2)\|_2 + \|\mathbf{D}(\boldsymbol{w}_3)\|_2\}.\end{aligned}$$

Noting that supp $\hat{\boldsymbol{w}}_1 \subset \{|\xi| \le R_0\}$, supp $\hat{\boldsymbol{w}}_2 \subset \{|\xi| \ge R_1\}$ and supp $\hat{\boldsymbol{w}}_3 \subset \{R_0 \le |\xi| \le R_1\}$, we see that

$$\begin{aligned}2\pi\|\mathbf{D}(\boldsymbol{w})\|_2^2 &= \sum_{j=1}^{3} \|\hat{\mathbf{D}}(\hat{\boldsymbol{w}}_j)\|_2^2 + 2\mathrm{Re}\ (\hat{\mathbf{D}}(\hat{\boldsymbol{w}}_1), \hat{\mathbf{D}}(\hat{\boldsymbol{w}}_2))\\ &\quad + 2\mathrm{Re}\ (\hat{\mathbf{D}}(\hat{\boldsymbol{w}}_2), \hat{\mathbf{D}}(\hat{\boldsymbol{w}}_3)) + 2\mathrm{Re}\ (\hat{\mathbf{D}}(\hat{\boldsymbol{w}}_3), \hat{\mathbf{D}}(\hat{\boldsymbol{w}}_1))\\ &= \sum_{j=1}^{3} \|\hat{\mathbf{D}}(\hat{\boldsymbol{w}}_j)\|_2^2\\ &= 2\pi \sum_{j=1}^{3} \|\mathbf{D}(\boldsymbol{w}_j)\|_2^2.\end{aligned}$$

Therefore, we have

$$\|\nabla \boldsymbol{w}\|_2 \le C\|\mathbf{D}(\boldsymbol{w})\|_2. \tag{2.4.32}$$

We finally prove that $\|\boldsymbol{w}\|_2 \le C\|\nabla \boldsymbol{w}\|_2$. In view of (2.4.30), it suffices to prove that $\|\boldsymbol{w}'\|_2 \le C\|\nabla' \boldsymbol{w}'\|_2$. But, since $\boldsymbol{w}' \cdot \boldsymbol{n}'|_{\partial D} = 0$, we have $\|\boldsymbol{w}'(x_1, \cdot)\|_{L^2(D)} \le C\|\nabla' \boldsymbol{w}'(x_1, \cdot)\|_{L^2(D)}$, which gives the desired estimate. This completes the proof. □

Hereafter we fix the constant R_1 in the definition of P_1 so that Lemma 2.4.5 holds true.

2.5 Reformulation of problem

In this section we reformulate the problem. The problem (2.2.5)-(2.2.8) is written as

$$\begin{cases}\partial_t u + Lu = F(u),\\ u|_{t=0} = u_0.\end{cases} \tag{2.5.1}$$

Here $u = {}^\top(\phi, \boldsymbol{w}) \in D(L)$ and $F(u) = {}^\top(f^0(\phi, \boldsymbol{w}), \boldsymbol{f}(\phi, \boldsymbol{w}))$.

One can prove the local solvability for (2.5.1) as in [14]. See also [28].

Proposition 2.5.1. *Assume that $u_0 = {}^\top(\phi_0, \boldsymbol{w}_0) \in H^2 \times H^2_*$ and $\|\phi_0\|_\infty \le \frac{1}{2}$. Then there exists a positive number T_0 depending on $\|u_0\|_{H^2}$ such that the problem (2.5.1) has a unique solution $u = {}^\top(\phi, \boldsymbol{w})$ on $[0, T_0]$ satisfying $u \in C([0, T_0]; H^2 \times H^2_*) \cap C^1([0, T_0]; L^2 \times L^2)$ with $\boldsymbol{w} \in \cap_{j=0}^{1} H^j(0, T_0; H^{3-2j})$ and $\|\phi_0(t)\|_\infty \le \frac{3}{4}$ for $t \in [0, T_0]$. Furthermore, the inequality*

$$\sup_{t\in[0,T_0]} \{\|u(t)\|_{H^2} + \|\partial_t u(t)\|_2\} + \int_0^{T_0} \|\boldsymbol{w}\|_{H^3}^2 \mathrm{d}t \le C\{1 + \|u_0\|_{H^2}^2\}^\beta \|u_0\|_{H^2}^2 \tag{2.5.2}$$

holds with some positive constants C and β.

The global existence of $u(t)$ follows in a standard manner from Proposition 2.5.1 and the *a priori* bound $\|u(t)\|_{H^2} \leq C\|u_0\|_{H^2\cap L^1}$ for u_0 sufficiently small in $H^2 \cap L^1$. The *a priori* bound will be given in Proposition 2.5.7 below.

To solve the problem (2.5.1), we decompose (2.5.1) into a problem for a low frequency part $u_1(t)$ of $u(t)$ and a one for a high frequency part $u_\infty(t)$ of $u(t)$.

We decompose $u = {}^\top(\phi, \boldsymbol{w})$ into

$$u = u_1 + u_\infty,$$

where

$$\begin{aligned} u_1 &= P_1 u = {}^\top(\phi_1, \boldsymbol{w}_1), & \boldsymbol{w}_1 &= {}^\top(w_1^1, \boldsymbol{w}_1'), \\ u_\infty &= P_\infty u = {}^\top(\phi_\infty, \boldsymbol{w}_\infty), & \boldsymbol{w}_\infty &= {}^\top(w_\infty^1, \boldsymbol{w}_\infty'). \end{aligned}$$

By applying the operators P_1 and P_∞ to (2.5.1), we obtain the following proposition.

Proposition 2.5.2. *Let T be a given positive number and let $u(t)$ be a solution of (2.5.1) on $[0,T]$. Assume that $u \in C([0,T]; H^2 \times H_*^2) \cap C^1([0,T]; L^2 \times L^2)$ with $\boldsymbol{w} \in \cap_{j=0}^1 H^j(0,T; H^{3-2j})$. Then*

$$u_1 = {}^\top(\phi_1, \boldsymbol{w}_1) \in C^1\big([0,T]; H^l \times [H^l \cap H_*^2]\big)$$

for $l = 0, 1, 2, \cdots$, and

$$u_\infty = {}^\top(\phi_\infty, \boldsymbol{w}_\infty) \in C([0,T]; H^2 \times H_*^2) \cap C^1([0,T]; L^2 \times L^2)$$

with $\boldsymbol{w}_\infty \in \cap_{j=0}^1 H^j(0,T; H^{3-2j})$.

Furthermore, u_1 and u_∞ satisfy

$$u_1 = e^{-tL_0} P_1 u_0 + \int_0^t P_1 e^{-(t-\tau)L_0} F_1(u(\tau)) \mathrm{d}\tau, \tag{2.5.3}$$

$$\partial_t u_\infty + L u_\infty = F_\infty(u), \quad u_\infty|_{t=0} = P_\infty u_0. \tag{2.5.4}$$

Here $F_1(u) = P_1 F(u_1 + u_\infty)$, $F_\infty(u) = P_\infty F(u_1 + u_\infty)$.

We define $M(t) \geq 0$ by

$$M(t) = M_1(t) + M_\infty(t) \quad (t \in [0,T]), \tag{2.5.5}$$

where $M_1(t)$ and $M_\infty(t)$ are defined by

$$M_1(t) = \sup_{0 \leq \tau \leq t} \left\{ \sum_{k=0}^{1} (1+\tau)^{\frac{1}{4}+\frac{k}{2}} \|\partial_{x_1}^k u_1(\tau)\|_2 + (1+\tau)^{\frac{3}{4}} \|\partial_t u_1(\tau)\|_2 \right\},$$

$$M_\infty(t) = \left(\sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{3}{2}} \{ \|u_\infty(\tau)\|_{H^2}^2 + \|\partial_t u_\infty(\tau)\|_2^2 \} \right)^{\frac{1}{2}}.$$

By the definition of P_1, we see that

$$P_1 u = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{\phi}\rangle]b_0 + \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{w}^1\rangle]b_1 + \mathcal{F}^{-1}[\mathbf{1}_{R_1}\langle\hat{\boldsymbol{w}}', \boldsymbol{b}'_{\mathrm{rig}}\rangle]b_{\mathrm{rig}},$$

and $\mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{\phi}\rangle]$, $\mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{w}^1\rangle]$ and $\mathcal{F}^{-1}[\mathbf{1}_{R_1}\langle\hat{\boldsymbol{w}}', \boldsymbol{b}'_{\mathrm{rig}}\rangle]$ are functions of x_1 only. Therefore, by the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\|u_1(t)\|_\infty \le C\|u_1(t)\|_2^{\frac{1}{2}}\|\partial_{x_1}u_1(t)\|_2^{\frac{1}{2}} \le C(1+t)^{-\frac{1}{2}}M_1(t).$$

As for $u_\infty(t)$, we have

$$\|u_\infty(t)\|_\infty \le C\|u_\infty(t)\|_{H^2} \le C(1+t)^{-\frac{3}{4}}M_\infty(t).$$

We also introduce the quantities $E_\infty(t)$ and $D_\infty(t)$ for $u_\infty(t) = {}^\top(\phi_\infty(t), \boldsymbol{w}_\infty(t))$:

$$\begin{aligned} E_\infty(t) &= \|u_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_2^2, \\ D_\infty(t) &= \|\nabla\phi_\infty(t)\|_{H^1}^2 + \|\nabla\boldsymbol{w}_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_{H^1}^2. \end{aligned}$$

Theorem 2.3.1 is an immediate consequence of the following proposition.

Proposition 2.5.3. *If $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then*

$$M(t) \le C\|u_0\|_{H^2\cap L^1}. \tag{2.5.6}$$

To prove Proposition 2.5.3, we reformulate (2.5.3)-(2.5.4) as follows. We will make use of a momentum formulation for the low frequency part which is useful to derive the decay estimate. Let $u = u_1 + u_\infty$, where u_1 and u_∞ satisfy (2.5.3)-(2.5.4). Then, u satisfies (2.2.5)-(2.2.8). If we write u as $u = {}^\top(\phi, \boldsymbol{w})$, then ϕ and $\boldsymbol{w}$ satisfy

$$\partial_t\phi + \mathrm{div}\boldsymbol{w} = g^0(\phi, \boldsymbol{w}), \tag{2.5.7}$$

$$\partial_t\boldsymbol{w} - \nu\mathrm{div}\mathbf{D}(\boldsymbol{w}) - \nu'\nabla\mathrm{div}\boldsymbol{w} + \nabla\phi = \boldsymbol{g}(\phi, \boldsymbol{w}), \tag{2.5.8}$$

where

$$\begin{aligned} g^0(\phi, \boldsymbol{w}) &= -\mathrm{div}(\phi\boldsymbol{w}), \\ \boldsymbol{g}(\phi, \boldsymbol{w}) &= -\phi\partial_t\boldsymbol{w} - (1+\phi)\boldsymbol{w}\cdot\nabla\boldsymbol{w} - \big(\nabla p(\rho) - \nabla\phi\big) \\ &= -\phi\partial_t\boldsymbol{w} - (1+\phi)\boldsymbol{w}\cdot\nabla\boldsymbol{w} - \frac{p''(1)}{2}\nabla(\phi^2) - \frac{1}{2}\nabla\big(p^{(3)}(\phi)\phi^3\big). \end{aligned}$$

Here

$$p^{(3)}(\phi) = \int_0^1 (1-\theta)^2 p'''(1+\theta\phi)\mathrm{d}\theta.$$

From the system (2.5.7)-(2.5.8) we derive a momentum formulation for the low frequency part. As in [36], we introduce a dimensionless momentum $\boldsymbol{m}$:

$$\boldsymbol{m} = (1+\phi)\boldsymbol{w}, \tag{2.5.9}$$

and define its low-frequency part $\boldsymbol{m}_1$ by

$$\boldsymbol{m}_1 = \boldsymbol{w}_1 + \boldsymbol{P}_1(\phi \boldsymbol{w}), \tag{2.5.10}$$

where $\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_\infty$ and the operator $\boldsymbol{P}_1$ is defined by

$$\boldsymbol{P}_1 \boldsymbol{w} = \mathcal{F}^{-1}\big(\mathbf{1}_{R_0}\boldsymbol{\Pi}_1 \hat{\boldsymbol{w}} + \mathbf{1}_{R_1}\boldsymbol{\Pi}_{\mathrm{rig}}\hat{\boldsymbol{w}}\big).$$

Note that $\boldsymbol{P}_1$ is a bounded projection from L^2 to H^k for any nonnegative integer k and it holds that $\partial_{x_1}\boldsymbol{P}_1 = \boldsymbol{P}_1\partial_{x_1}$ and

$$\|\boldsymbol{P}_1 \boldsymbol{w}\|_{H^k} \le C_k \|\boldsymbol{w}\|_2. \tag{2.5.11}$$

Before proceeding further, we show that $\boldsymbol{w}_1$ is uniquely determined by $\boldsymbol{m}_1, \phi$ and $\boldsymbol{w}_\infty$ through the relation (2.5.10), i.e.,

$$\boldsymbol{w}_1 = \boldsymbol{m}_1 - \boldsymbol{P}_1\big(\phi(\boldsymbol{w}_1 + \boldsymbol{w}_\infty)\big). \tag{2.5.12}$$

Lemma 2.5.4. *There exists a positive constant δ_0 such that the following assertion holds true. Let*

$$\boldsymbol{m}_1 \in C^1([0,T]; L^2),\ \ \boldsymbol{w}_\infty \in \cap_{j=0}^{1} C^j([0,T]; H^{2-2j}),\ \ \phi \in \cap_{j=0}^{1} C^j([0,T]; H^{2-2j}).$$

If $\|\phi\|_{C([0,T];H^2)} + \|\partial_t \phi\|_{C([0,T];L^2)} \le \delta_0$, then there uniquely exists $\boldsymbol{w}_1 \in C^1([0,T]; L^2)$ that satisfies (2.5.12).

Furthermore, there hold the estimates

$$\|\boldsymbol{w}_1\|_{C([0,T];L^2)} \le C\left(\|\boldsymbol{m}_1\|_{C([0,T];H^2)} + \|\phi\|_{C([0,T];H^2)}\|\boldsymbol{w}_\infty\|_{C([0,T];H^2)}\right), \tag{2.5.13}$$

$$\begin{aligned} &\|\partial_t \boldsymbol{w}_1\|_{C([0,T];L^2)} \\ &\quad \le C\big(\|\partial_t \boldsymbol{m}_1\|_{C([0,T];H^2)} + \|\partial_t \phi\|_{C([0,T];L^2)}\|\boldsymbol{w}_\infty\|_{C([0,T];H^2)} \\ &\qquad\quad + \|\phi\|_{C([0,T];H^2)}\|\partial_t \boldsymbol{w}_\infty\|_{C([0,T];L^2)}\big), \end{aligned} \tag{2.5.14}$$

where C is a positive constant independent of T.

Proof. We first observe that, by the Sobolev inequality and (2.5.11),

$$\|\boldsymbol{w}_1\|_\infty \le C\|\boldsymbol{w}_1\|_{H^2} = C\|\boldsymbol{P}_1 \boldsymbol{w}_1\|_{H^2} \le C\|\boldsymbol{w}_1\|_2.$$

Let $\|\phi\|_{C([0,T];H^2)} + \|\partial_t \phi\|_{C([0,T];L^2)} \le \delta_0$. We set $\Gamma(\boldsymbol{w}_1) = \boldsymbol{m}_1 - \boldsymbol{P}_1\big(\phi(\boldsymbol{w}_1 + \boldsymbol{w}_\infty)\big)$. We claim that Γ is a map on $C^1([0,T]; L^2)$. In fact, by the Sobolev inequality and (2.5.11), we have

$$\begin{aligned} &\|\Gamma(\boldsymbol{w}_1)\|_{C([0,T];L^2)} \\ &\quad \le \|\boldsymbol{m}_1\|_{C([0,T];L^2)} + C\|\phi(\boldsymbol{w}_1 + \boldsymbol{w}_\infty)\|_{C([0,T];L^2)} \\ &\quad \le C\left\{\|\boldsymbol{m}_1\|_{C([0,T];L^2)} + \|\phi\|_{C([0,T];L^\infty)}\|\boldsymbol{w}_1 + \boldsymbol{w}_\infty\|_{C([0,T];L^2)}\right\} \\ &\quad \le C\left\{\|\boldsymbol{m}_1\|_{C([0,T];L^2)} + \|\phi\|_{C([0,T];H^2)}\big(\|\boldsymbol{w}_1\|_{C([0,T];L^2)} + \|\boldsymbol{w}_\infty\|_{C([0,T];L^2)}\big)\right\}, \end{aligned} \tag{2.5.15}$$

and

$$
\begin{aligned}
&\|\partial_t \Gamma(\boldsymbol{w}_1)\|_{C([0,T];L^2)} \\
&\quad \le \|\partial_t \boldsymbol{m}_1\|_{C([0,T];L^2)} + C\|\partial_t \phi(\boldsymbol{w}_1 + \boldsymbol{w}_\infty)\|_{C([0,T];L^2)} \\
&\qquad + C\|\phi(\partial_t \boldsymbol{w}_1 + \partial_t \boldsymbol{w}_\infty)\|_{C([0,T];L^2)} \\
&\quad \le C\Big\{\|\partial_t \boldsymbol{m}_1\|_{C([0,T];L^2)} + C\|\partial_t \phi\|_{C([0,T];L^2)}\big(\|\boldsymbol{w}_1\|_{C([0,T];L^\infty)} + \|\boldsymbol{w}_\infty\|_{C([0,T];L^\infty)}\big) \\
&\qquad + C\|\phi\|_{C([0,T];L^\infty)}\big(\|\partial_t \boldsymbol{w}_1\|_{C([0,T];L^2)} + \|\partial_t \boldsymbol{w}_\infty\|_{C([0,T];L^2)}\big)\Big\} \qquad (2.5.16) \\
&\quad \le C\Big\{\|\partial_t \boldsymbol{m}_1\|_{C([0,T];L^2)} + \|\partial_t \phi\|_{C([0,T];L^2)}\big(\|\boldsymbol{w}_1\|_{C([0,T];L^2)} + \|\boldsymbol{w}_\infty\|_{C([0,T];H^2)}\big) \\
&\qquad + \|\phi\|_{C([0,T];H^2)}\big(\|\partial_t \boldsymbol{w}_1\|_{C([0,T];L^2)} + \|\partial_t \boldsymbol{w}_\infty\|_{C([0,T];L^2)}\big)\Big\}.
\end{aligned}
$$

Therefore, Γ is a map on $C^1([0,T];L^2)$.

We next show that Γ is a contraction map. By the Sobolev inequality and (2.5.11), we have

$$
\begin{aligned}
\|\Gamma(\boldsymbol{w}_1^{(1)}) - \Gamma(\boldsymbol{w}_1^{(2)})\|_{C([0,T];L^2)} &\le C\|\phi\|_{C([0,T];L^\infty)}\|\boldsymbol{w}_1^{(1)} - \boldsymbol{w}_1^{(2)}\|_{C([0,T];L^2)} \\
&\le C\|\phi\|_{C([0,T];H^2)}\|\boldsymbol{w}_1^{(1)} - \boldsymbol{w}_1^{(2)}\|_{C([0,T];L^2)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\partial_t\big[\Gamma(\boldsymbol{w}_1^{(1)}) - \Gamma(\boldsymbol{w}_1^{(2)})\big]\big\|_{C([0,T];L^2)} \\
&\quad \le C\Big\{\|\phi\|_{C([0,T];L^\infty)}\|\partial_t \boldsymbol{w}_1^{(1)} - \partial_t \boldsymbol{w}_1^{(2)}\|_{C([0,T];L^2)} \\
&\qquad + \|\partial_t \phi\|_{C([0,T];L^2)}\|\boldsymbol{w}_1^{(1)} - \boldsymbol{w}_1^{(2)}\|_{C([0,T];L^\infty)}\Big\} \\
&\quad \le C\left\{\|\phi\|_{C([0,T];H^2)} + \|\partial_t \phi\|_{C([0,T];L^2)}\right\}\|\boldsymbol{w}_1^{(1)} - \boldsymbol{w}_1^{(2)}\|_{C^1([0,T];L^2)}.
\end{aligned}
$$

We thus obtain

$$
\|\Gamma(\boldsymbol{w}_1^{(1)}) - \Gamma(\boldsymbol{w}_1^{(2)})\|_{C^1([0,T];L^2)} \le C\delta_0\|\boldsymbol{w}_1^{(1)} - \boldsymbol{w}_1^{(2)}\|_{C^1([0,T];L^2)}.
$$

Therefore, if $\delta_0 > 0$ is sufficiently small, then Γ is a contraction. By the contraction mapping principle, we thus see that there exists a unique $\boldsymbol{w}_1 \in C^1([0,T];L^2)$ such that $\boldsymbol{w}_1 = \Gamma(\boldsymbol{w}_1)$. This shows the unique existence of $\boldsymbol{w}_1$ satisfying (2.5.12). The estimates (2.5.13) and (2.5.14) now follow from (2.5.15) and (2.5.16). This completes the proof. □

Remark 2.5.5. *We see from (2.5.11) that $\boldsymbol{w}_1$ in Lemma 2.5.4 satisfies $\boldsymbol{w}_1 \in C^1([0,T];H^k)$ for any k and*

$$
\|\boldsymbol{w}_1\|_{C^1([0,T];H^k)} \le C_k\|\boldsymbol{w}_1\|_{C^1([0,T];L^2)}. \qquad (2.5.17)
$$

By (2.5.7), we have

$$
\begin{aligned}
(1+\phi)\boldsymbol{w}\cdot\nabla\boldsymbol{w} &= \mathrm{div}\big((1+\phi)\boldsymbol{w}\otimes\boldsymbol{w}\big) - \boldsymbol{w}\mathrm{div}\big((1+\phi)\boldsymbol{w}\big) \\
&= \mathrm{div}\big((1+\phi)\boldsymbol{w}\otimes\boldsymbol{w}\big) + \boldsymbol{w}\partial_t\phi.
\end{aligned} \qquad (2.5.18)
$$

Therefore, applying the operator P_1 to (2.5.7)-(2.5.8), we obtain

$$\partial_t \phi_1 + \mathrm{div}\ \boldsymbol{m}_1 = 0,$$

and

$$\begin{aligned}
&\boldsymbol{P}_1(\partial_t \boldsymbol{w} - \nu \mathrm{div} \mathbf{D}(\boldsymbol{w}) - \nu' \nabla \mathrm{div} \boldsymbol{w}) + \nabla \phi_1 \\
&\quad = \boldsymbol{P}_1\big(- \phi \partial_t \boldsymbol{w} - (1+\phi) \boldsymbol{w} \cdot \nabla \boldsymbol{w} - \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3)\big) \\
&\quad = \boldsymbol{P}_1\big(- \phi \partial_t \boldsymbol{w} - \mathrm{div}((1+\phi) \boldsymbol{w} \otimes \boldsymbol{w}) + \boldsymbol{w} \mathrm{div}((1+\phi)\boldsymbol{w}) \\
&\qquad\quad - \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3)\big) \\
&\quad = \boldsymbol{P}_1\big(- \phi \partial_t \boldsymbol{w} - \mathrm{div}((1+\phi) \boldsymbol{w} \otimes \boldsymbol{w}) - \boldsymbol{w} \partial_t \phi - \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3)\big) \\
&\quad = \boldsymbol{P}_1\big(- \partial_t (\phi \boldsymbol{w}) - \mathrm{div}((1+\phi) \boldsymbol{w} \otimes \boldsymbol{w})\big) + \boldsymbol{P}_1\big(- \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3)\big).
\end{aligned}$$

The latter equation is written as

$$\begin{aligned}
&\boldsymbol{P}_1\big(\partial_t ((1+\phi)\boldsymbol{w}) - \nu \mathrm{div} \mathbf{D}(\boldsymbol{w}) - \nu' \nabla \mathrm{div} \boldsymbol{w}\big) + \nabla \phi_1 \\
&\quad = -\boldsymbol{P}_1\big(\mathrm{div}((1+\phi) \boldsymbol{w} \otimes \boldsymbol{w})\big) + \boldsymbol{P}_1\big(- \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3)\big).
\end{aligned}$$

We thus obtain

$$\partial_t \phi_1 + \mathrm{div}\ \boldsymbol{m}_1 = 0, \tag{2.5.19}$$

$$\begin{aligned}
&\partial_t \boldsymbol{m}_1 - \nu \mathrm{div}\ \mathbf{D}(\boldsymbol{m}_1) - \nu' \nabla \mathrm{div}\ \boldsymbol{m}_1 + \nabla \phi_1 \\
&\quad = -\nu \mathrm{div} \mathbf{D}(\boldsymbol{P}_1(\phi \boldsymbol{w})) - \nu' \nabla \mathrm{div} \boldsymbol{P}_1(\phi \boldsymbol{w}) \\
&\qquad - \boldsymbol{P}_1\big(\mathrm{div}((1+\phi) \boldsymbol{w} \otimes \boldsymbol{w}) - \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3)\big).
\end{aligned} \tag{2.5.20}$$

Setting

$$u_{1*} = {}^\top(\phi_1, \boldsymbol{m}_1), \tag{2.5.21}$$

we arrive at

$$\begin{cases} \partial_t u_{1*} + L u_{1*} = P_1 G_1(u) + P_1 G_2(u), \\ \boldsymbol{m}_1 = \boldsymbol{w}_1 + \boldsymbol{P}_1(\phi \boldsymbol{w}), \\ u_{1*}|_{t=0} = P_1 u_{*0}, \end{cases} \tag{2.5.22}$$

where $u = {}^\top(\phi, \boldsymbol{w})$, $\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_\infty$, and

$$\begin{aligned}
&u_{*0} = {}^\top(\phi_0, \boldsymbol{m}_0), \quad \boldsymbol{m}_0 = \boldsymbol{w}_0 + \phi_0 \boldsymbol{w}_0, \\
&G_1(u) = \begin{pmatrix} -\mathrm{div}(\phi \boldsymbol{w}) \\ -\mathrm{div}((1+\phi) \boldsymbol{w} \otimes \boldsymbol{w}) - \tfrac{p''(1)}{2} \nabla(\phi^2) - \tfrac{1}{2} \nabla (p^{(3)}(\phi)\phi^3) \end{pmatrix}, \\
&G_2(u) = L \begin{pmatrix} 0 \\ \boldsymbol{P}_1(\phi \boldsymbol{w}) \end{pmatrix}.
\end{aligned}$$

Since $\hat{\Pi}_+(\xi) + \hat{\Pi}_-(\xi) = \Pi_0 + \Pi_1$, we see that

$$P_1\big(G_1(u) + G_2(u)\big) = P_1 \partial_{x_1} G(u) + P_1 \partial_{x_1} \tilde{G}(u), \tag{2.5.23}$$

where

$$G(u) = -\begin{pmatrix} 0 \\ (w^1)^2 + \frac{p''(1)}{2}\phi^2 \\ w^1\boldsymbol{w}' \end{pmatrix}, \tag{2.5.24}$$

$$\tilde{G}(u) = -\begin{pmatrix} 0 \\ \phi(w^1)^2 + \frac{1}{2}p^{(3)}(\phi)\phi^3 + (2\nu+\nu')\partial_{x_1}(\phi w^1) \\ \phi w^1\boldsymbol{w}' + \nu\partial_{x_1}(\phi\boldsymbol{w}') \end{pmatrix}. \tag{2.5.25}$$

In terms of $u_{1*}(t)$ and $u_\infty(t)$, Proposition 2.5.2 is restated as follows.

Proposition 2.5.6. *Let $u(t)$ be a solution of (1.5.1) on $[0,T]$. Assume that $u \in C([0,T]; H^2 \times H^2_*) \cap C^1([0,T]; L^2 \times L^2)$ with $\boldsymbol{w} \in \cap_{j=0}^1 H^j(0,T; H^{3-2j})$. Then*

$$u_{1*} = {}^\top(\phi_1, \boldsymbol{m}_1) \in C^1([0,T]; H^l \times H^l)$$

for $l = 0, 1, 2, \cdots$, and

$$u_\infty = {}^\top(\phi_\infty, \boldsymbol{w}_\infty) \in C([0,T]; H^2 \times H^2_*) \cap C^1([0,T]; L^2 \times L^2)$$

with $\boldsymbol{w}_\infty \in \cap_{j=0}^1 H^j(0,T; H^{3-2j})$.

Furthermore, u_{1*} and u_∞ satisfy

$$u_{1*}(t) = e^{-tL}P_1 u_{*0} + \int_0^t e^{-(t-\tau)L}P_1\partial_{x_1}\big(G(u) + \tilde{G}(u)\big)(\tau)\mathrm{d}\tau, \tag{2.5.26}$$

$$\boldsymbol{m}_1 = \boldsymbol{w}_1 + \boldsymbol{P}_1(\phi\boldsymbol{w}), \tag{2.5.27}$$

$$\partial_t u_\infty + Lu_\infty = F_\infty(u), \quad u_\infty|_{t=0} = P_\infty u_0, \tag{2.5.28}$$

where $u = {}^\top(\phi, \boldsymbol{w})$, $\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_\infty$; $u_{*0} = {}^\top(\phi_0, \boldsymbol{m}_0)$, $\boldsymbol{m}_0 = \boldsymbol{w}_0 + \phi_0\boldsymbol{w}_0$; $G(u)$ and $\tilde{G}(u)$ are the ones given by (2.5.24) and (2.5.25); $F_\infty(u) = P_\infty F(u) =: {}^\top(f^0_\infty(u), \boldsymbol{f}_\infty(u))$, $\boldsymbol{f}_\infty(u) =: {}^\top(f^1_\infty(u), \boldsymbol{f}'_\infty(u))$.

We will consider $u_{1*}(t)$ and $u_\infty(t)$ to establish suitable *a priori* estimates.

We define $M_*(t) \geq 0$ by

$$M_*(t) = M_{1*}(t) + M_\infty(t) \quad (t \in [0,T]). \tag{2.5.29}$$

Here $M_{1*}(t)$ is defined by

$$M_{1*}(t) = \sup_{0\leq\tau\leq t}\left\{\sum_{k=0}^1 (1+\tau)^{\frac{1}{4}+\frac{k}{2}}\|\partial_{x_1}^k u_{1*}(\tau)\|_2 + (1+\tau)^{\frac{3}{4}}\|\partial_t u_{1*}(\tau)\|_2\right\},$$

and $M_\infty(t)$ is defined as before.

We see from Lemma 5.4 (and its proof) that $M(t)$ is equivalent to $M_*(t)$ if $\|\phi\|_{C([0,t];H^2)} + \|\partial_t\phi\|_{C([0,t];L^2)} \leq \delta_0$. We also note that, by the Gagliardo-Nirenberg-Sobolev inequality,

$$\|u_{1*}(t)\|_\infty \leq C\|u_{1*}(t)\|_2^{\frac{1}{2}}\|\partial_{x_1}u_{1*}(t)\|_2^{\frac{1}{2}} \leq C(1+t)^{-\frac{1}{2}}M_{1*}(t).$$

As for $M_{1*}(t)$ and $M_\infty(t)$, we will prove the following estimates.

Proposition 2.5.7. *Let $u(t)$ be a solution of (1.5.1) on $[0,T]$. Then there exists a positive constant ε_1 such that if $\|u(t)\|_{H^2} \le \varepsilon_1$ and $M_*(t) \le 1$ for $t \in [0,T]$, the estimates*

$$M_{1*}(t) \le C\{\|u_{*0}\|_1 + M_*(t)^2\} \tag{2.5.30}$$

and

$$\begin{aligned} &E_\infty(t) + \int_0^t e^{-d(t-\tau)} D_\infty(\tau)\mathrm{d}\tau \\ &\le C\left\{e^{-dt}E_\infty(0) + (1+t)^{-\frac{3}{2}}M_*(t)^4 + \int_0^t e^{-d(t-\tau)}\mathcal{R}(\tau)\mathrm{d}\tau\right\} \end{aligned} \tag{2.5.31}$$

hold uniformly for $t \in [0,T]$ with a positive constant C independent of T. Here $d = d(\nu,\nu')$ is a positive constant, and $\mathcal{R}(t)$ is a quantity that satisfies

$$\mathcal{R}(t) \le C\{(1+t)^{-\frac{3}{2}}M_*(t)^3 + M_*(t)D_\infty(t)\}. \tag{2.5.32}$$

The estimate (2.5.30) will be proved in Section 6, and the estimates (2.5.31) and (2.5.32) will be proved in Sections 7 and 8, respectively.

From Propositions 1.5.1 and 2.5.7, one can show the following uniform estimate of $M_*(t)$ as in [12].

Proposition 2.5.8. *If $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then*

$$M_*(t) \le C\|u_0\|_{H^2\cap L^1}. \tag{2.5.33}$$

Proposition 2.5.3 immediately follows from Proposition 2.5.8.

2.6 Estimates for $u_{1*}(t)$

In this section, we estimate the low-frequency part $u_{1*} = {}^\top(\phi_1, \boldsymbol{m}_1)$ and prove the estimate (2.5.30) in Proposition 2.5.7.

As for the nonlinearities, by using (2.4.12), it is not difficult to show the following estimates.

Lemma 2.6.1. *Let $k = 0,1$. Then*

$$\left\|\partial_{x_1}^k P_1\big(G(u) + \tilde{G}(u)\big)\right\|_1 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}M_*(t)^2(1+M_*(t)), \tag{2.6.1}$$

$$\left\|P_1\partial_{x_1}\big(G(u) + \tilde{G}(u)\big)\right\|_2 \le C(1+t)^{-\frac{3}{4}}M_*(t)^2(1+M_*(t)). \tag{2.6.2}$$

We now give a proof of (2.5.30).

Proof of (2.5.30). By (2.5.26) and Lemma 2.4.2, we have

$$\begin{aligned} &\|\partial_{x_1}^k u_{1*}(t)\|_2 \\ &\le \|\partial_{x_1}^k e^{-tL}P_1 u_{*0}\|_2 + \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L}P_1\partial_{x_1}(G(u)+\tilde{G}(u))(\tau)\|_2 \mathrm{d}\tau \\ &\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_{*0}\|_1 + \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L}P_1\partial_{x_1}(G(u)+\tilde{G}(u))(\tau)\|_2 \mathrm{d}\tau \end{aligned}$$

for $k=0,1$. As for the second term on the right-hand side, we write it as

$$\begin{aligned}&\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L}P_1\partial_{x_1}(G(u)+\tilde{G}(u))(\tau)\|_2 \mathrm{d}\tau\\&=\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right)\|\partial_{x_1}^k e^{-(t-\tau)L}P_1\partial_{x_1}(G(u)+\tilde{G}(u))(\tau)\|_2 \mathrm{d}\tau\\&=: I_1(t)+I_2(t).\end{aligned}$$

As for $I_1(t)$, since $\partial_{x_1}P = P\partial_{x_1}$, we write $\partial_{x_1}^k e^{-(t-\tau)L}P_1\partial_{x_1} = \partial_{x_1}^{k+1}e^{-(t-\tau)L}P_1$, and applying Lemmas 2.4.2 and 2.6.1 to obtain

$$I_1(t)\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{1}{2}}\mathrm{d}\tau M_*(t)^2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}M_*(t)^2.$$

As for $I_2(t)$, applying Lemmas 2.4.2 and 2.6.1, we have

$$I_2(t)\le C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{1}{4}-\frac{k}{2}}(1+\tau)^{-1}\mathrm{d}\tau M_*(t)^2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}M_*(t)^2.$$

We thus obtain

$$\|\partial_{x_1}^k u_{1*}(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\{\|u_{*0}\|_1+M_*(t)^2\} \tag{2.6.3}$$

for $k=0,1$.

Let us estimate the time derivative. By Remark 2.4.4, we have

$$Lu_{1*}=\begin{pmatrix}0 & \partial_{x_1} & {}^\top\mathbf{0}'\\ \partial_{x_1} & -(2\nu+\nu')\partial_{x_1}^2 & {}^\top\mathbf{0}'\\ \mathbf{0}' & \mathbf{0}' & -\nu\partial_{x_1}^2 I'\end{pmatrix}u_{1*},\quad I'=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

This, together with (2.4.12), implies that

$$\begin{aligned}\|-Lu_{1*}(t)\|_2 &\le C\{\|\partial_{x_1}^2\boldsymbol{m}_1(t)\|_2+\|\partial_{x_1}u_{1*}(t)\|_2\}\\&\le C(1+t)^{-\frac{3}{4}}\{\|u_{*0}\|_1+M_*(t)^2\}.\end{aligned}$$

Since $\partial_t u_{1*} = -Lu_{1*}+P_1\partial_{x_1}\big(G(u)+\tilde{G}(u)\big)$, applying Lemma 2.6.1, we have

$$\begin{aligned}\|\partial_t u_{1*}(t)\|_2 &\le C\{\|Lu_{1*}(t)\|_2+\|P_1(G(u)+\tilde{G}(u))(t)\|_2\}\\&\le C(1+t)^{-\frac{3}{4}}\{\|u_{*0}\|_1+M_*(t)^2\}.\end{aligned}\tag{2.6.4}$$

By (2.6.3) and (2.6.4), we deduce the desired estimate (2.5.30). This completes the proof.
□

2.7 Estimates for $u_\infty(t)$

In this section, we estimate the high-frequency part $u_\infty = P_\infty u$ by using the Matsumura-Nishida energy method ([27]) and prove estimate (2.5.31) with $\mathcal{R}(t)$ satisfying (2.5.32) in Proposition 2.5.7.

We define the operator $\boldsymbol{P}_\infty$ by

$$\boldsymbol{P}_\infty = I - \boldsymbol{P}_1.$$

Throughout this section we set $\nu_0 = \min\left\{\frac{2}{3}\nu, \frac{2}{3}\nu + \nu'\right\}$. As was shown in [32], since

$$\|\mathrm{div}\ \boldsymbol{w}_\infty\|_2^2 \leq 3\sum_{j=1}^{3} \|\partial_{x_j} w_\infty^j\|_2^2 \leq \frac{3}{4}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2, \tag{2.7.1}$$

we see that

$$\frac{\nu}{2}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \nu'\|\mathrm{div}\ \boldsymbol{w}_\infty\|_2^2 \geq \frac{3}{4}\nu_0\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2. \tag{2.7.2}$$

In fact, (2.7.2) clearly holds if $\nu' \geq 0$. If $\nu' < 0$, by (2.7.1)

$$\begin{aligned}\frac{\nu}{2}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \nu'\|\mathrm{div}\ \boldsymbol{w}_\infty\|_2^2 &\geq \left(\frac{\nu}{2} + \frac{3}{4}\nu'\right)\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 \\ &= \frac{3}{4}\left(\frac{2}{3}\nu + \nu'\right)\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2.\end{aligned}$$

We thus obtain (2.7.2).

To prove (2.5.31), we prepare some basic estimates.

Proposition 2.7.1. *Let k and j be nonnegative integers satisfying $0 \leq 2k + j \leq 2$. Then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^k\partial_{x_1}^j u_\infty\|_2^2 + \frac{3}{8}c_K\nu_0\|\nabla\partial_t^k\partial_{x_1}^j \boldsymbol{w}_\infty\|_2^2 + \frac{3\nu_0}{4}\|\partial_t^k\partial_{x_1}^j \dot{\phi}_\infty\|_2^2 \leq CR_{j,k}^{(1)}, \tag{2.7.3}$$

where $c_K = \frac{C_1^2}{C_2^2}$ with C_1 and C_2 being the constants in Lemma 2.4.5,

$$\begin{aligned}\dot{\phi}_\infty &= \partial_t\phi_\infty + \boldsymbol{w}\cdot\nabla\phi_\infty, \\ R_{j,k}^{(1)} &= \frac{1}{2}(\mathrm{div}\boldsymbol{w}, |\partial_t^k\partial_{x_1}^j\phi_\infty|^2) - ([\partial_t^k\partial_{x_1}^j, \boldsymbol{w}]\cdot\nabla\phi_\infty, \partial_t^k\partial_{x_1}^j\phi_\infty) \\ &\quad + (\partial_t^k\partial_{x_1}^j\tilde{f}_\infty^0, \partial_t^k\partial_{x_1}^j\phi_\infty) + (\partial_t^k\partial_{x_1}^j\boldsymbol{f}_\infty, \partial_t^k\partial_{x_1}^j\boldsymbol{w}_\infty) + \frac{3\nu_0}{2}\|\partial_t^k\partial_{x_1}^j\tilde{f}_\infty^0\|_2^2, \\ \tilde{f}_\infty^0 &= \tilde{f}_\infty^0(u) = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\widehat{\boldsymbol{w}\cdot\nabla\phi_\infty}\rangle] - (\boldsymbol{w}\cdot\nabla\phi_1 + \phi\ \mathrm{div}\boldsymbol{w}) \\ &\qquad + \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\widehat{\boldsymbol{w}\cdot\nabla\phi_1 + \phi\ \mathrm{div}\boldsymbol{w}}\rangle].\end{aligned}$$

Here and in what follows we abbreviate $f_\infty^0(u)$, $\tilde{f}_\infty^0(u)$, $\boldsymbol{f}_\infty(u)$ and $F_\infty(u)$ as f_∞^0, $\tilde{f}_\infty^0$, $\boldsymbol{f}_\infty$ and F_∞, respectively.

Proof. Equation (2.5.28) is written as

$$\partial_t\phi_\infty + \boldsymbol{w}\cdot\nabla\phi_\infty + \mathrm{div}\boldsymbol{w}_\infty = \tilde{f}^0_\infty, \tag{2.7.4}$$

$$\partial_t\boldsymbol{w}_\infty - \nu\mathrm{div}\mathbf{D}(\boldsymbol{w}_\infty) - \nu'\nabla\mathrm{div}\boldsymbol{w}_\infty + \nabla\phi_\infty = \boldsymbol{f}_\infty. \tag{2.7.5}$$

We compute $(\partial^j_{x_1}(2.7.4), \partial^j_{x_1}\phi_\infty) + (\partial^j_{x_1}(2.7.5), \partial^j_{x_1}\boldsymbol{w}_\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^j_{x_1}u_\infty\|_2^2 + \frac{\nu}{2}\|\partial^j_{x_1}\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \nu'\|\partial^j_{x_1}\mathrm{div}\boldsymbol{w}_\infty\|_2^2 \\
&\quad = -(\partial^j_{x_1}(\boldsymbol{w}\cdot\nabla\phi_\infty), \partial^j_{x_1}\phi_\infty) + (\partial^j_{x_1}\tilde{f}^0_\infty, \partial^j_{x_1}\phi_\infty) + (\partial^j_{x_1}\boldsymbol{f}_\infty, \partial^j_{x_1}\boldsymbol{w}_\infty) \\
&\quad = -(\boldsymbol{w}\cdot\nabla\partial^j_{x_1}\phi_\infty, \partial^j_{x_1}\phi_\infty) - ([\partial^j_{x_1}, \boldsymbol{w}]\cdot\nabla\phi_\infty, \partial^j_{x_1}\phi_\infty) \\
&\qquad + (\partial^j_{x_1}\tilde{f}^0_\infty, \partial^j_{x_1}\phi_\infty) + (\partial^j_{x_1}\boldsymbol{f}_\infty, \partial^j_{x_1}\boldsymbol{w}_\infty) \\
&\quad = \frac{1}{2}(\mathrm{div}\boldsymbol{w}, |\partial^j_{x_1}\phi_\infty|^2) - ([\partial^j_{x_1}, \boldsymbol{w}]\cdot\nabla\phi_\infty, \partial^j_{x_1}\phi_\infty) + (\partial^j_{x_1}\tilde{f}^0_\infty, \partial^j_{x_1}\phi_\infty) \\
&\qquad + (\partial^j_{x_1}\boldsymbol{f}_\infty, \partial^j_{x_1}\boldsymbol{w}_\infty).
\end{aligned} \tag{2.7.6}$$

We set $\dot{\phi} := \partial_t\phi + \boldsymbol{w}\cdot\nabla\phi_\infty$. From (2.7.4), we have $\partial^j_{x_1}\dot{\phi}_\infty = -\mathrm{div}\partial^j_{x_1}\boldsymbol{w}_\infty + \partial^j_{x_1}\tilde{f}^0_\infty$, and hence, by (2.7.1)

$$\begin{aligned}
\|\partial^j_{x_1}\dot{\phi}_\infty\|_2^2 &\le 2(\|\mathrm{div}\partial^j_{x_1}\boldsymbol{w}_\infty\|_2^2 + \|\partial^j_{x_1}\tilde{f}^0_\infty\|_2^2) \\
&\le \frac{3}{2}\|\mathbf{D}(\partial^j_{x_1}\boldsymbol{w}_\infty)\|_2^2 + 2\|\partial^j_{x_1}\tilde{f}^0_\infty\|_2^2.
\end{aligned}$$

We thus obtain

$$\frac{3}{4}\nu_0\|\partial^j_{x_1}\dot{\phi}_\infty\|_2^2 \le \frac{3}{8}\nu_0\|\mathbf{D}(\partial^j_{x_1}\boldsymbol{w}_\infty)\|_2^2 + \frac{3}{2}\nu_0\|\partial^j_{x_1}\tilde{f}^0_\infty\|_2^2. \tag{2.7.7}$$

It then follows from (2.4.17), (2.7.6) and (2.7.7) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^j_{x_1}u_\infty\|_2^2 + \frac{3}{8}c_K\nu_0\|\nabla\partial^j_{x_1}\boldsymbol{w}_\infty\|_2^2 + \frac{3\nu_0}{4}\|\partial^j_{x_1}\dot{\phi}_\infty\|_2^2 \le CR^{(1)}_{j,0}. \tag{2.7.8}$$

Replacing $\partial^j_{x_1}$ by ∂_t, we also have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t u_\infty\|_2^2 + \frac{3}{8}c_K\nu_0\|\nabla\partial_t\boldsymbol{w}_\infty\|_2^2 + \frac{3\nu_0}{4}\|\partial_t\dot{\phi}_\infty\|_2^2 \le CR^{(1)}_{0,1}. \tag{2.7.9}$$

This completes the proof. □

Proposition 2.7.2. *It holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\nu}{2}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \nu'\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 - 2(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty)\right\} + \frac{1}{2}\|\partial_t\boldsymbol{w}_\infty\|_2^2 \\
&\quad \le 4\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 + \|\boldsymbol{w}\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}^0_\infty\|_2^2 + \|\boldsymbol{f}_\infty\|_2^2.
\end{aligned} \tag{2.7.10}$$

Proof. We compute $((2.7.4), \partial_t \phi_\infty) + ((2.7.5), \partial_t \boldsymbol{w}_\infty)$ to obtain

$$\begin{aligned}
&\|\partial_t u_\infty\|_2^2 + \frac{\nu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \frac{\nu'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 \\
&\quad + \{(\mathrm{div}\boldsymbol{w}_\infty, \partial_t\phi_\infty) + (\nabla\phi_\infty, \partial_t\boldsymbol{w}_\infty)\} \\
&= (\tilde{f}^0_\infty, \partial_t\phi_\infty) + (\boldsymbol{f}_\infty, \partial_t\boldsymbol{w}_\infty) - (\boldsymbol{w}\cdot\nabla\phi_\infty, \partial_t\phi_\infty).
\end{aligned}$$

Since

$$\begin{aligned}
(\nabla\phi_\infty, \partial_t\boldsymbol{w}_\infty) &= -(\phi_\infty, \partial_t\mathrm{div}\boldsymbol{w}_\infty) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty) + (\partial_t\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty),
\end{aligned}$$

we have

$$\begin{aligned}
&\|\partial_t u_\infty\|_2^2 + \frac{\nu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \frac{\nu'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 - \frac{\mathrm{d}}{\mathrm{d}t}(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty) \\
&= (\tilde{f}^0_\infty, \partial_t\phi_\infty) + (\boldsymbol{f}_\infty, \partial_t\boldsymbol{w}_\infty) - 2(\partial_t\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty) - (\boldsymbol{w}\cdot\nabla\phi_\infty, \partial_t\phi_\infty) \qquad (2.7.11) \\
&\le \frac{1}{4}\|\partial_t u_\infty\|_2^2 + C\{\|\boldsymbol{w}\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}^0_\infty\|_2^2 + \|\boldsymbol{f}_\infty\|_2^2\}.
\end{aligned}$$

Adding $-2(\mathrm{div}\boldsymbol{w}_\infty, \partial_t\phi_\infty)$ to both sides of (2.7.11), we obtain

$$\begin{aligned}
&\|\partial_t u_\infty\|_2^2 + \frac{\nu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \frac{\nu'}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 - \frac{\mathrm{d}}{\mathrm{d}t}(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty) \\
&\le -2(\mathrm{div}\boldsymbol{w}_\infty, \partial_t\phi_\infty) + \frac{1}{4}\|\partial_t u_\infty\|_2^2 + C\{\|\boldsymbol{w}\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}^0_\infty\|_2^2 + \|\boldsymbol{f}_\infty\|_2^2\} \\
&\le \frac{1}{2}\|\partial_t u_\infty\|_2^2 + C\{\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 + \|\boldsymbol{w}\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}^0_\infty\|_2^2 + \|\boldsymbol{f}_\infty\|_2^2\},
\end{aligned}$$

which gives the desired estimate. This completes the proof. □

We next establish the interior estimates. Let $\zeta \in C^\infty(D)$. It follows from (2.7.4) and (2.7.5) that

$$\partial_t(\zeta\phi_\infty) + \boldsymbol{w}\cdot\nabla(\zeta\phi_\infty) + \mathrm{div}(\zeta\boldsymbol{w}_\infty) = g^0_\infty + \boldsymbol{w}_\infty\cdot\nabla\zeta, \qquad (2.7.12)$$

$$\partial_t(\zeta\boldsymbol{w}_\infty) - \nu\mathrm{div}\mathbf{D}(\zeta\boldsymbol{w}_\infty) - \nu'\nabla\mathrm{div}(\zeta\boldsymbol{w}_\infty) + \nabla(\zeta\phi_\infty) = \boldsymbol{g}_\infty, \qquad (2.7.13)$$

where

$$\begin{aligned}
g^0_\infty &= \zeta\tilde{f}^0_\infty + (\boldsymbol{w}\cdot\nabla\zeta)\phi_\infty, \\
\boldsymbol{g}_\infty &= \zeta\boldsymbol{f}_\infty + \nu[\zeta, \mathrm{div}\mathbf{D}]\boldsymbol{w}_\infty + \nu'[\zeta, \nabla\mathrm{div}]\boldsymbol{w}_\infty + \phi_\infty\nabla\zeta.
\end{aligned}$$

We introduce $\zeta_{in}(x') = \zeta_{in}(|x'|) \in C_c^\infty(D)$ satisfying

$$\zeta_{in}(x') = 1 \ \text{ for } \ |x'| \le \frac{3}{4}, \quad \zeta_{in}(x') = 0 \ \text{ for } \ |x'| \ge \frac{7}{8}.$$

Setting $\zeta = \zeta_{in}$ in (2.7.12)-(2.7.13), we obtain the following interior estimates.

Proposition 2.7.3. *Let* $1 \le |\alpha| \le 2$. *Then*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha(\zeta_{in}u_\infty)\|_2^2 + \frac{3}{8}c_K\nu_0\|\nabla\partial_x^\alpha(\zeta_{in}\boldsymbol{w}_\infty)\|_2^2 + \frac{3\nu_0}{4}\|\partial_x^\alpha(\zeta_{in}\dot{\phi}_\infty)\|_2^2 \\ &\quad \le C\{R_\alpha^{(2)} + \nu_0\|\boldsymbol{w}_\infty\|_{H^{|\alpha|}}^2\} + \varepsilon\|\phi_\infty\|_{H^{|\alpha|}}^2 + \varepsilon\|\boldsymbol{w}_\infty\|_{H^{|\alpha|+1}}^2 + \frac{C}{\varepsilon}\|\boldsymbol{w}_\infty\|_{H^{|\alpha|}}^2,\end{aligned} \tag{2.7.14}$$

where

$$\begin{aligned}R_\alpha^{(2)} &= \frac{1}{2}(\mathrm{div}\boldsymbol{w}, |\partial_x^\alpha(\zeta_{in}\phi_\infty)|^2) - ([\partial_x^\alpha, \boldsymbol{w}]\cdot\nabla(\zeta_{in}\phi_\infty), \partial_x^\alpha(\zeta_{in}\phi_\infty)) \\ &\quad + (\partial_x^\alpha g^0_{\infty,in}, \partial_x^\alpha(\zeta_{in}\phi_\infty)) + (\partial_x^\alpha(\zeta_{in}\boldsymbol{f}_\infty), \partial_x^\alpha(\zeta_{in}\boldsymbol{w}_\infty)) + \frac{3\nu_0}{2}\|\partial_x^\alpha(\zeta_{in}g^0_\infty)\|_2^2.\end{aligned}$$

Here $g^0_{\infty,in}$ *and* $\boldsymbol{g}_{\infty,in}$ *are the functions obtained by replacing* ζ *in* g^0_∞ *and* $\boldsymbol{g}_\infty$ *with* ζ_{in}*, respectively.*

Proof. Applying ∂_x^α to (2.7.12) and (2.7.13), we see that

$$\begin{aligned}&\partial_t\partial_x^\alpha(\zeta\phi_\infty) + \boldsymbol{w}\cdot\nabla\partial_x^\alpha(\zeta\phi_\infty) + \mathrm{div}\partial_x^\alpha(\zeta\boldsymbol{w}_\infty) \\ &\quad = \partial_x^\alpha g^0_\infty + \partial_x^\alpha(\boldsymbol{w}_\infty\cdot\nabla\zeta) - [\partial_x^\alpha, \boldsymbol{w}]\cdot\nabla(\zeta\phi_\infty),\end{aligned} \tag{2.7.15}$$

$$\begin{aligned}&\partial_t\partial_x^\alpha(\zeta\boldsymbol{w}_\infty) - \nu\mathrm{div}\mathbf{D}(\partial_x^\alpha(\zeta\boldsymbol{w}_\infty)) - \nu'\nabla\mathrm{div}\partial_x^\alpha(\zeta\boldsymbol{w}_\infty) + \nabla\partial_x^\alpha(\zeta\phi_\infty) \\ &\quad = \partial_x^\alpha\boldsymbol{g}_\infty.\end{aligned} \tag{2.7.16}$$

Since $\partial_x^\alpha(\zeta_{in}\boldsymbol{w}_\infty) = 0$ for $|x'| \ge \frac{7}{8}$, one can obtain the desired estimates as in the proof of Proposition 2.7.1. This completes the proof. $\square$

We next consider the estimates near the boundary ∂D. For this purpose we rewrite (2.7.12)-(2.7.13) in the cylindrical coordinates (x_1, r, θ) with $x_2 = r\cos\theta, x_3 = r\sin\theta$. The velocity field $\boldsymbol{w}$ is written as

$$\boldsymbol{w} = E\tilde{\boldsymbol{w}}, \quad \tilde{\boldsymbol{w}} = {}^\top(\tilde{w}^1, \tilde{w}^2, \tilde{w}^3) = {}^\top(w^1, w^r, w^\theta),$$

where

$$E = (\boldsymbol{e}_1, \boldsymbol{e}_r, \boldsymbol{e}_\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The gradient $\nabla_x = {}^\top(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ is rewritten as

$$\nabla_x = \tilde{E}\nabla_y, \quad \nabla_y = {}^\top(\partial_{y_1}, \partial_{y_2}, \partial_{y_3}) = {}^\top(\partial_{x_1}, \partial_r, \partial_\theta),$$

where

$$\tilde{E} = (\boldsymbol{e}_1, \boldsymbol{e}_r, \frac{1}{r}\boldsymbol{e}_\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\frac{1}{r}\sin\theta \\ 0 & \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}.$$

For an integer m satisfying $m \ge 1$, we define $\boldsymbol{w}_m$ by

$$\boldsymbol{w}_m = E\tilde{\boldsymbol{w}}_m,$$

$$\tilde{\boldsymbol{w}}_m = {}^\top(\partial_\theta^m \tilde{w}_m^1, \partial_\theta^m \tilde{w}_m^2, \partial_\theta^m \tilde{w}_m^3) = {}^\top(\partial_\theta^m w_m^1, \partial_\theta^m w_m^r, \partial_\theta^m w_m^\theta),$$

and

$$\boldsymbol{w}_m = {}^\top(w_m^1, w_m^2, w_m^3), \quad \tilde{\boldsymbol{w}}_m = {}^\top(\tilde{w}_m^1, \tilde{w}_m^2, \tilde{w}_m^3).$$

Straightforward computations yield the following relations. We take $\zeta_b(x') = \zeta_b(|x'|) \in C^\infty(\overline{D})$ such that $\zeta_b(x') = 1$ for $|x'| \geq \frac{5}{8}$ and $\zeta_b(x') = 0$ for $|x'| \leq \frac{1}{2}$.

Lemma 2.7.4. *Let $\boldsymbol{w}_m$ be defined as above. Then*

$$\|\partial_x^\alpha \partial_{x_1}^j (\zeta_b \boldsymbol{w}_m)\|_2 \leq \|\partial_x^\alpha \partial_{x_1}^j \partial_\theta^m (\zeta_b \boldsymbol{w})\|_2 + C \sum_{p=0}^{m-1} \|\partial_{x_1}^j \partial_\theta^p \boldsymbol{w}\|_{H^{|\alpha|}}, \tag{2.7.17}$$

$$\|\partial_x^\alpha \partial_{x_1}^j \partial_\theta^m (\zeta_b \boldsymbol{w})\|_2 \leq \|\partial_x^\alpha \partial_{x_1}^j (\zeta_b \boldsymbol{w}_m)\|_2 + C \sum_{p=0}^{m-1} \|\partial_{x_1}^j \partial_\theta^p \boldsymbol{w}\|_{H^{|\alpha|}}. \tag{2.7.18}$$

Proof. Since, by $\boldsymbol{w} = E\tilde{\boldsymbol{w}}$,

$$\begin{aligned}
\boldsymbol{w}_m = E \begin{pmatrix} \partial_\theta^m \tilde{w}^1 \\ \partial_\theta^m \tilde{w}^2 \\ \partial_\theta^m \tilde{w}^3 \end{pmatrix} &= E\partial_\theta^m (E^{-1}\boldsymbol{w}) \\
&= E(E^{-1}\partial_\theta^m \boldsymbol{w} + [\partial_\theta^m, E^{-1}]\boldsymbol{w}) \\
&= \partial_\theta^m \boldsymbol{w} + E[\partial_\theta^m, E^{-1}]\boldsymbol{w}.
\end{aligned}$$

we have

$$\begin{aligned}
\|\partial_x^\alpha \boldsymbol{w}_m\|_2 &\leq \|\partial_x^\alpha \partial_\theta^m \boldsymbol{w} + \partial_x^\alpha (E[\partial_\theta^m, E^{-1}]\boldsymbol{w})\|_2 \\
&\leq \|\partial_x^\alpha \partial_\theta^m \boldsymbol{w}\|_2 + C \sum_{p=0}^{m-1} \|\partial_\theta^p \boldsymbol{w}\|_{H^{|\alpha|}}, \\
\|\partial_x^\alpha \partial_\theta^m \boldsymbol{w}\|_2 &\leq \|\partial_x^\alpha \boldsymbol{w}_m\|_2 + C \sum_{p=0}^{m-1} \|\partial_\theta^p \boldsymbol{w}\|_{H^{|\alpha|}}.
\end{aligned}$$

This completes the proof. □

We see from (2.7.12) and (2.7.13) with $\zeta = \zeta_b$ that

$$\partial_t(\zeta_b \phi_\infty) + \boldsymbol{w} \cdot \nabla(\zeta_b \phi_\infty) + \mathrm{div}(\zeta_b \boldsymbol{w}_\infty) = g_{\infty,b}^0 + \boldsymbol{w}_\infty \cdot \nabla(\zeta_b \phi_\infty), \tag{2.7.19}$$

$$\begin{aligned}
&\partial_t(\zeta_b w_\infty^1) - \nu\Big(\partial_{x_1}^2(\zeta_b w_\infty^1) + \frac{1}{r}\partial_r(r\partial_r(\zeta_b w_\infty^1)) + \frac{1}{r^2}\partial_\theta^2(\zeta_b w_\infty^1)\Big) \\
&\quad - (\nu + \nu')\partial_{x_1}\mathrm{div}(\zeta_b \boldsymbol{w}_\infty) + \partial_{x_1}(\zeta_b \phi_\infty) = g_{\infty,b}^1,
\end{aligned} \tag{2.7.20}$$

$$\begin{aligned}
&\partial_t(\zeta_b w_\infty^r) - \nu\Big(\partial_{x_1}^2(\zeta_b w_\infty^r) + \frac{1}{r}\partial_r(r\partial_r(\zeta_b w_\infty^r)) + \frac{1}{r^2}\partial_\theta^2(\zeta_b w_\infty^r) \\
&\quad - \frac{1}{r^2}(\zeta_b w_\infty^r) - \frac{2}{r^2}\partial_\theta(\zeta_b w_\infty^\theta)\Big) - (\nu + \nu')\partial_r \mathrm{div}(\zeta_b \boldsymbol{w}_\infty) + \partial_r(\zeta_b \phi_\infty) = g_{\infty,b}^r,
\end{aligned} \tag{2.7.21}$$

$$\partial_t(\zeta_b w_\infty^\theta) - \nu\Big(\partial_{x_1}^2(\zeta_b w_\infty^\theta) + \frac{1}{r}\partial_r(r\partial_r(\zeta_b w_\infty^\theta)) + \frac{1}{r^2}\partial_\theta^2(\zeta_b w_\infty^\theta) \\ - \frac{1}{r^2}(\zeta_b w_\infty^\theta) + \frac{2}{r^2}\partial_\theta(\zeta_b w_\infty^r)\Big) - \frac{\nu+\nu'}{r}\partial_\theta \mathrm{div}(\zeta_b \boldsymbol{w}_\infty) + \frac{1}{r}\partial_\theta(\zeta_b\phi_\infty) = g_{\infty,b}^\theta, \tag{2.7.22}$$

Here $g_{\infty,b}^j = \boldsymbol{g}_{\infty,b}\cdot\boldsymbol{e}_j$ $(j=1,r,\theta)$; $g_{\infty,b}^0$ and $\boldsymbol{g}_{\infty,b}$ are the functions obtained by replacing ζ in g_∞^0 and $\boldsymbol{g}_\infty$ with ζ_b; and div $\boldsymbol{w} = \partial_{x_1}w^1 + \frac{1}{r}\partial_r(rw^r) + \frac{1}{r}\partial_\theta(w^\theta)$.

Note that ∂_θ^mdiv $\boldsymbol{w}$ = div $\boldsymbol{w}_m$. Furthermore the boundary condition (1.1.9) is transformed into

$$\partial_r w^1|_{\partial D} = 0, \quad w^r|_{\partial D} = 0, \quad \partial_r w^\theta - \frac{1}{r}w^\theta|_{\partial\Omega} = 0. \tag{2.7.23}$$

Since $\boldsymbol{w}_\infty$ satisfies the boundary condition (1.1.9), $\tilde{\boldsymbol{w}}_\infty = {}^\top(w_\infty^1, w_\infty^r, w_\infty^\theta) = E^{-1}\boldsymbol{w}_\infty$ satisfies (2.7.23). We thus obtain

$$\partial_{x_1}^j\partial_\theta^k\partial_r w_\infty^1|_{\partial\Omega} = 0,\ \partial_{x_1}^j\partial_\theta^k w_\infty^r|_{\partial\Omega} = 0,\ \partial_{x_1}^j\partial_\theta^k\partial_r w_\infty^\theta - \frac{1}{r}\partial_{x_1}^j\partial_\theta^k w_\infty^\theta|_{\partial\Omega} = 0. \tag{2.7.24}$$

Applying $\partial_{x_1}^j\partial_\theta^m$ to (2.7.19)-(2.7.23), we see that $\zeta_b\phi_{\infty,m} = \zeta_b\partial_\theta^m\phi_\infty$ and $\zeta_b\boldsymbol{w}_{\infty,m} = E\zeta_b\tilde{\boldsymbol{w}}_{\infty,m}$ satisfy

$$\partial_t\partial_{x_1}^j(\zeta_b\phi_{\infty,m}) + \boldsymbol{w}\cdot\nabla\partial_{x_1}^j(\zeta_b\phi_{\infty,m}) + \mathrm{div}(\partial_{x_1}^j(\zeta_b\boldsymbol{w}_m)) \\ = \partial_{x_1}^j\partial_\theta^m g_{\infty,b}^0 + \partial_{x_1}^j\partial_\theta^m(\boldsymbol{w}_\infty\cdot\nabla\zeta_b) - [\partial_{x_1}^j\partial_\theta^m, \boldsymbol{w}]\nabla(\zeta_b\phi_\infty), \tag{2.7.25}$$

$$\partial_t\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m}) - \nu\mathrm{div}\mathbf{D}(\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})) - \nu'\nabla\mathrm{div}(\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})) \\ + \nabla\partial_{x_1}^j(\zeta_b\phi_{\infty,m}) = \partial_{x_1}^j\boldsymbol{g}_{\infty,b,m}, \tag{2.7.26}$$

and

$$\partial_{x_1}^j\boldsymbol{w}_{\infty,m}\cdot\boldsymbol{n}|_{\partial D} = 0, \quad \big[\mathbf{D}(\partial_{x_1}^j\boldsymbol{w}_{\infty,m})\cdot\boldsymbol{n} - (\mathbf{D}(\partial_{x_1}^j\boldsymbol{w}_{\infty,m})\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n}\big]\big|_{\partial D} = \mathbf{0}. \tag{2.7.27}$$

Here $\boldsymbol{g}_{\infty,b,m} = E\tilde{\boldsymbol{g}}_{\infty,b,m}$ with $\tilde{\boldsymbol{g}}_{\infty,b,m} = {}^\top(\partial_\theta^m g_{\infty,b}^1, \partial_\theta^m g_{\infty,b}^r, \partial_\theta^m g_{\infty,b}^\theta)$.

As in the proof of Proposition 2.7.1, we now obtain the following estimates for the tangential derivatives.

Proposition 2.7.5. *Let j and m be nonnegative integers satisfying $1 \le j+m \le 2$. Then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\phi_\infty)\|_2^2 + \|\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})\|_2^2\right\} + \frac{3}{8}c_K\nu_0\|\nabla\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})\|_2^2 \\ + \frac{3\nu_0}{4}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\dot{\phi}_\infty)\|_2^2 \\ \le C\{R_{j,m}^{(3)} + \nu_0\|\boldsymbol{w}_\infty\|_{H^{j+m}}^2\} + \varepsilon\|\phi_\infty\|_{H^{j+m}}^2 + \varepsilon\|\boldsymbol{w}_\infty\|_{H^{j+m+1}}^2 + \frac{C}{\varepsilon}\|\boldsymbol{w}_\infty\|_{H^{j+m}}^2, \tag{2.7.28}$$

where

$$R_{j,m}^{(3)} = \frac{1}{2}(\mathrm{div}\boldsymbol{w}, |\partial_{x_1}^j\partial_\theta^m(\zeta_b\phi_\infty)|^2) - ([\partial_{x_1}^j\partial_\theta^m, \boldsymbol{w}]\cdot\nabla(\zeta_b\phi_\infty), \partial_{x_1}^j\partial_\theta^m(\zeta_b\phi_\infty)) \\ + (\partial_{x_1}^j\partial_\theta^m g_{\infty,b}^0, \partial_{x_1}^j\partial_\theta^m(\zeta_b\phi_\infty)) + (\partial_{x_1}^j\boldsymbol{f}_{\infty,b,m}, \partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})) \\ + \frac{3\nu_0}{2}\|\partial_{x_1}^j\partial_\theta^m g_{\infty,b}^0\|_2^2.$$

Here $\boldsymbol{f}_{\infty,b,m} = E\tilde{\boldsymbol{f}}_{\infty,b,m}$ with $\tilde{\boldsymbol{f}}_{\infty,b,m} = {}^\top(\partial_\theta^m(\zeta_b f^1), \partial_\theta^m(\zeta_b f^r), \partial_\theta^m(\zeta_b f^\theta))$.

Proof. The only difference from the proof of Proposition 2.7.1 appears in the Korn's inequality:

$$
\begin{aligned}
\frac{3}{4}c_K\nu_0\|\nabla(\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m}))\|_2^2 \le \frac{3}{4}\nu_0\|\mathbf{D}(\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m}))\|_2^2 \\
+ C\nu_0\|\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})\|_2^2.
\end{aligned} \tag{2.7.29}
$$

This completes the proof. □

We next estimate the normal derivatives of ϕ_∞. For this purpose, we rewrite the equations (2.5.28) by using the cylindrical coordinates in the following way:

$$
\begin{cases}
\partial_t\phi_\infty + \mathrm{div}\boldsymbol{w}_\infty = \tilde{f}^0_\infty - \boldsymbol{w}\cdot\nabla\phi_\infty, & (2.7.30)\\
\partial_t w^1_\infty - \nu\left(\partial_{x_1}^2 w^1_\infty + \frac{1}{r}\partial_r(r\partial_r w^1_\infty) + \frac{1}{r^2}\partial_\theta^2 w^1_\infty\right) - (\nu+\nu')\partial_{x_1}\mathrm{div}\boldsymbol{w}_\infty & (2.7.31)\\
\qquad + \partial_{x_1}\phi_\infty = f^1_\infty, & \\
\partial_t w^r_\infty - \nu\left(\partial_{x_1}^2 w^r_\infty + \frac{1}{r}\partial_r(r\partial_r w^r_\infty) + \frac{1}{r^2}\partial_\theta^2 w^r_\infty - \frac{1}{r^2}w^r_\infty - \frac{2}{r^2}\partial_\theta w^\theta_\infty\right) & (2.7.32)\\
\qquad - (\nu+\nu')\partial_r\mathrm{div}\boldsymbol{w}_\infty + \partial_r\phi_\infty = f^r_\infty, & \\
\partial_t w^\theta_\infty - \nu\left(\partial_{x_1}^2 w^\theta_\infty + \frac{1}{r}\partial_r(r\partial_r w^\theta_\infty) + \frac{1}{r^2}\partial_\theta^2 w^\theta_\infty - \frac{1}{r^2}w^\theta_\infty + \frac{2}{r^2}\partial_\theta w^r_\infty\right) & (2.7.33)\\
\qquad - \frac{\nu+\nu'}{r}\partial_\theta\mathrm{div}\boldsymbol{w}_\infty + \frac{1}{r}\partial_\theta\phi_\infty = f^\theta_\infty, &
\end{cases}
$$

where

$$
f^j_\infty = \boldsymbol{f}_\infty\cdot\boldsymbol{e}_j, \quad j\in\{1,r,\theta\}.
$$

Proposition 2.7.6. *Let j,l and m be nonnegative integers satisfying $0\le j+l+m\le 1$. Then*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}\phi_\infty\|_2^2 + \frac{1}{2(2\nu+\nu')}\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}\phi_\infty\|_2^2 + (2\nu+\nu')\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}\dot{\phi}_\infty\|_2^2\\
&\quad\le C\{R^{(4)}_{j,m,l} + (2\nu+\nu')\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^l M(\boldsymbol{w}_\infty)\|_2^2 + \frac{1}{2\nu+\nu'}\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^l\partial_t(w^r_\infty)\|_2^2\},
\end{aligned} \tag{2.7.34}
$$

where

$$
\begin{aligned}
R^{(4)}_{j,m,l}(t) &= \frac{1}{2}|(\mathrm{div}\boldsymbol{w}, |\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}\phi_\infty|^2)| + \frac{1}{2}|(\boldsymbol{w}\cdot\nabla|\zeta_b|^2, |\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}\phi_\infty|^2)|\\
&\quad + (2\nu+\nu')\|\zeta_b[\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}, \boldsymbol{w}]\cdot\nabla\phi_\infty\|_2^2 + (2\nu+\nu')\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^{l+1}\tilde{f}^0_\infty\|_2^2\\
&\quad + \frac{1}{2\nu+\nu'}\|\zeta_b\partial_{x_1}^j\partial_\theta^m\partial_r^l f^r_\infty\|_2^2,\\
M(\boldsymbol{w}_\infty) &= \frac{\nu}{2\nu+\nu'}\left\{\partial_{x_1}^2 w^r_\infty + \frac{1}{r^2}\partial_\theta^2 w^r_\infty - \frac{2}{r^2}\partial_\theta w^\theta_\infty - \partial_r\partial_{x_1}w^1_\infty - \partial_r(\tfrac{1}{r}\partial_\theta w^\theta_\infty)\right\}.
\end{aligned}
$$

Proof. By applying ∂_r to (2.7.30) to obtain

$$\partial_t\partial_r\phi_\infty + \partial_r(\boldsymbol{w}\cdot\nabla\phi_\infty) + \partial_r \mathrm{div}\boldsymbol{w}_\infty = \partial_r\tilde{f}^0_\infty, \tag{2.7.35}$$

where $\tilde{f}^0_\infty = -\phi\mathrm{div}\boldsymbol{w}_\infty$.

We can rewrite (2.7.32) as following

$$\begin{aligned}&\partial_t w^r_\infty - \nu\left(\partial^2_{x_1}w^r_\infty + \frac{1}{r^2}\partial^2_\theta w^r_\infty - \frac{2}{r^2}\partial_\theta w^\theta_\infty - \partial_r\partial_{x_1}w^1_\infty - \partial_r\left(\frac{1}{r}\partial_\theta w^\theta_\infty\right)\right)\\ &\quad - (2\nu+\nu')\partial_r\mathrm{div}\boldsymbol{w}_\infty + \partial_r\phi_\infty = f^r_\infty.\end{aligned} \tag{2.7.36}$$

We next compute (2.7.35) + $\frac{1}{2\nu+\nu'}\times$ (2.7.36) to obtain

$$\partial_t\partial_r\phi_\infty + \partial_r(\boldsymbol{w}\cdot\nabla\phi_\infty) + \frac{1}{2\nu+\nu'}\partial_r\phi_\infty = h, \tag{2.7.37}$$

where

$$\begin{aligned}h &= \partial_r\tilde{f}^0_\infty + \frac{1}{2\nu+\nu'}f^r_\infty - \frac{1}{2\nu+\nu'}\partial_t w^r_\infty + M(\boldsymbol{w}_\infty),\\ M(\boldsymbol{w}_\infty) &= \frac{\nu}{2\nu+\nu'}\left\{\partial^2_{x_1}w^r_\infty + \frac{1}{r^2}\partial^2_\theta w^r_\infty - \frac{2}{r^2}\partial_\theta w^\theta_\infty - \partial_r\partial_{x_1}w^1_\infty - \partial_r(\tfrac{1}{r}\partial_\theta w^\theta_\infty)\right\}.\end{aligned}$$

We write (2.7.37) in following form:

$$\partial_t\partial_r\phi_\infty + \frac{1}{2\nu+\nu'}\partial_r\phi_\infty = -\partial_r(\boldsymbol{w}\cdot\nabla\phi_\infty) + h. \tag{2.7.38}$$

Applying $\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^l_r$ to (2.7.38), we have

$$\begin{aligned}&\partial_t\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r(\zeta_b\phi_\infty) + \frac{1}{2\nu+\nu'}\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r(\zeta_b\phi_\infty)\\ &\quad = -\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r(\boldsymbol{w}\cdot\nabla\phi_\infty) + \partial^j_{x_1}\partial^m_\theta\partial^l_r h\\ &\quad = -\boldsymbol{w}\cdot\nabla(\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty) - \zeta_b[\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r, \boldsymbol{w}]\cdot\nabla\phi_\infty + \zeta_b\partial^j_{x_1}\partial^m_\theta\partial^l_r h.\end{aligned} \tag{2.7.39}$$

By taking the inner product of (2.7.39) with $\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty\|^2_2 + \frac{1}{2\nu+\nu'}\|\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty\|^2_2\\ &\quad = (-\boldsymbol{w}\cdot\nabla(\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty), \zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty)\\ &\qquad + (-\zeta_b[\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r, \boldsymbol{w}]\cdot\nabla\phi_\infty + \zeta_b\partial^j_{x_1}\partial^m_\theta\partial^l_r h, \zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty)\\ &\quad = \frac{1}{2}(\mathrm{div}\boldsymbol{w}_\infty, |\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty|^2)\\ &\qquad + (-\zeta_b[\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r, \boldsymbol{w}]\cdot\nabla\phi_\infty + \zeta_b\partial^j_{x_1}\partial^m_\theta\partial^l_r h, \zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty)\\ &\quad \le \frac{1}{2(2\nu+\nu')}\|\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty\|^2_2 + \frac{1}{2}(\mathrm{div}\boldsymbol{w}_\infty, |\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r\phi_\infty|^2)\\ &\qquad + C\{(2\nu+\nu')\|\zeta_b[\partial^j_{x_1}\partial^m_\theta\partial^{l+1}_r, \boldsymbol{w}]\cdot\nabla\phi_\infty\|^2_2 + (2\nu+\nu')\|\zeta_b\partial^j_{x_1}\partial^m_\theta\partial^l_r h\|^2_2\}.\end{aligned} \tag{2.7.40}$$

We also write (2.7.37) as following form

$$\partial_r \dot{\phi}_\infty + \frac{1}{2\nu+\nu'} \partial_r \phi_\infty = h, \tag{2.7.41}$$

Applying $\zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r^l$ to (2.7.41), we have

$$\partial_{x_1}^j \partial_\theta^m \partial_r^{l+1} \dot{\phi}_\infty + \frac{1}{2\nu+\nu'} \zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r^{l+1} \phi_\infty = \zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r^l h. \tag{2.7.42}$$

It then follows that

$$\begin{aligned} &(2\nu+\nu')\|\zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r^{l+1} \dot{\phi}_\infty\|_2^2 \\ &\le C\left\{ \frac{1}{2\nu+\nu'} \|\zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r^{l+1} \phi_\infty\|_2^2 + (2\nu+\nu')\|\zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r^l h\|_2^2 \right\}. \end{aligned} \tag{2.7.43}$$

By (2.7.40) + $\frac{1}{2C}$ × (2.7.43), we obtain the desired result. This completes the proof. □

We next estimate higher order derivatives of ϕ_∞ and $\boldsymbol{w}_\infty$.

Proposition 2.7.7. *Let $k = 2, 3$. Then*

$$\begin{aligned} &\frac{1}{\nu^2}\|\phi_\infty\|_{H^{k-1}}^2 + \|\boldsymbol{w}_\infty\|_{H^k}^2 \\ &\le C\left\{ \frac{1}{\nu^2}\|\partial_t \boldsymbol{w}_\infty\|_{H^{k-2}}^2 + \frac{1}{\nu^2}\|\boldsymbol{f}_\infty\|_{H^{k-2}}^2 + \frac{\nu^2+\nu'^2}{\nu^2}\|\tilde{f}_\infty^0\|_{H^{k-1}}^2 \right. \\ &\qquad \left. + \|\partial_x \boldsymbol{w}_\infty\|_2^2 + \frac{\nu^2+\nu'^2}{\nu^2}\|\dot{\phi}_\infty\|_{H^{k-1}}^2 \right\}. \end{aligned} \tag{2.7.44}$$

Proposition 2.7.8. *Let j and m be nonnegative integers satisfying $j + m = 1$. Then*

$$\begin{aligned} &\frac{1}{\nu^2}\|\partial_{x_1}^j \partial_\theta^m (\zeta_b \phi_\infty)\|_{H^1}^2 + \|\partial_{x_1}^j (\zeta_b \boldsymbol{w}_{\infty,m})\|_{H^2}^2 \\ &\le C\left\{ \frac{1}{\nu^2}\|\partial_t \partial_{x_1}^j (\zeta_b \boldsymbol{w}_{\infty,m})\|_2^2 + \frac{1}{\nu^2}\|\partial_{x_1}^j \boldsymbol{g}_{\infty,b,m}\|_2^2 + \frac{\nu^2+\nu'^2}{\nu^2}\|\partial_{x_1}^j \partial_\theta^m g_{\infty,b}^0\|_{H^1}^2 \right. \\ &\qquad + \frac{\nu^2+\nu'^2}{\nu^2}\|\partial_{x_1}^j \partial_\theta^m (\boldsymbol{w}_\infty \cdot \nabla \zeta_b)\|_2^2 + \|\partial_{x_1}^j \partial_x (\zeta_b \boldsymbol{w}_{\infty,m})\|_2^2 \\ &\qquad \left. + (\nu^2+\nu'^2)\|\partial_{x_1}^j \partial_\theta^m (\zeta_b \dot{\phi}_\infty)\|_{H^1}^2 \right\}. \end{aligned} \tag{2.7.45}$$

Propositions 2.7.7 and 2.7.8 are obtained by applying the following lemma.

Lemma 2.7.9. *Let $u = {}^\top(\phi, \boldsymbol{w})$ be a solution of the Stokes system*

$$\begin{cases} \mathrm{div}\ \boldsymbol{w} = f, \\ -\mathrm{div}\ \mathbf{D}(\boldsymbol{w}) + \nabla \phi = \boldsymbol{g}, \\ \boldsymbol{w} \cdot \boldsymbol{n}|_{\partial D} = 0, \quad \left[\mathbf{D}(\boldsymbol{w}) \cdot \boldsymbol{n} - (\mathbf{D}(\boldsymbol{w})\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{n}\right]\big|_{\partial D} = \mathbf{0}. \end{cases} \tag{2.7.46}$$

Then there exists a positive constant C such that

$$\|\boldsymbol{w}\|_{H^k} + \|\phi\|_{H^{k-1}} \le C\{\|f\|_{H^{k-1}} + \|\boldsymbol{g}\|_{H^{k-2}} + \|\partial_x \boldsymbol{w}\|_2\}. \tag{2.7.47}$$

Lemma 2.7.9 can be proved by using a similar argument to those in [35, Theorem II I.1.5.1] and [11, Appendix] based on the following estimates obtained in [31]:

$$\begin{aligned}&\|\boldsymbol{w}\|_{H^{k+2}(\tilde{D})}+\|\phi\|_{H^{k+1}(\tilde{D})}\\ &\quad\le C\{\|f\|_{H^{k+1}(\tilde{D})}+\|\boldsymbol{g}\|_{H^{k}(\tilde{D})}+\|\boldsymbol{h}\|_{H^{k+1}(\tilde{D})}\}\quad (k=0,1)\end{aligned}$$

for a solution of the problem

$$\begin{cases}\mathrm{div}\ \boldsymbol{w}=f,\\ -\mathrm{div}\ \mathbf{D}(\boldsymbol{w})+\nabla\phi=\boldsymbol{g},\\ \boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\tilde{D}}=0,\quad \big[\mathbf{D}(\boldsymbol{w})\cdot\boldsymbol{n}-(\mathbf{D}(\boldsymbol{w})\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n}\big]\big|_{\partial\tilde{D}}=\boldsymbol{h},\end{cases}$$

where $\tilde{D}$ is a bounded domain with smooth boundary. In [31], the above estimate was proved for $k=0$, but one can also prove it for $k=1$ according to the proof in [31].

Proof of Proposition 2.7.7. We rewrite equations (1.7.5)-(2.7.5) in the following form:

$$\begin{cases}\mathrm{div}\boldsymbol{w}_\infty=\tilde{f}^0_\infty-\dot{\phi}_\infty,\\ -\mathrm{div}\mathbf{D}(\boldsymbol{w}_\infty)+\nabla\left(\dfrac{1}{\nu}\phi_\infty\right)=\dfrac{1}{\nu}\left\{\boldsymbol{f}_\infty-\Big(\partial_t\boldsymbol{w}_\infty-\nu'\nabla(\boldsymbol{f}^0_\infty-\dot{\phi}_\infty)\Big)\right\}.\end{cases}\tag{2.7.48}$$

It then follows from Lemma 2.7.9 that

$$\begin{aligned}&\frac{1}{\nu^2}\|\phi_\infty\|^2_{H^{k-1}}+\|\boldsymbol{w}_\infty\|^2_{H^k}\\ &\le C\left\{\|\boldsymbol{f}^0_\infty-\dot{\phi}_\infty\|^2_{H^{k-1}}+\left\|\frac{1}{\nu}\left\{\boldsymbol{f}_\infty-\Big(\partial_t\boldsymbol{w}_\infty-\nu'\nabla(\boldsymbol{f}^0_\infty-\dot{\phi}_\infty)\Big)\right\}\right\|^2_{H^{k-2}}+\|\partial_x\boldsymbol{w}_\infty\|_2^2\right\}\\ &\le C\bigg\{\frac{1}{\nu^2}\|\partial_t\boldsymbol{w}_\infty\|^2_{H^{k-2}}+\frac{1}{\nu^2}\|\boldsymbol{f}_\infty\|^2_{H^{k-2}}+\frac{\nu^2+\nu'^2}{\nu^2}\|\boldsymbol{f}^0_\infty\|^2_{H^{k-1}}+\|\partial_x\boldsymbol{w}_\infty\|_2^2\\ &\qquad+(\nu^2+\nu'^2)\|\dot{\phi}_\infty\|^2_{H^{k-1}}\bigg\}.\end{aligned}\tag{2.7.49}$$

This completes the proof. □

Proof of Proposition 2.7.8. We rewrite equations (2.7.25)-(2.7.26) in the following form:

$$\begin{cases}\mathrm{div}(\partial^j_{x_1}(\zeta_b\boldsymbol{w}_m))\\ \quad=\partial^j_{x_1}\partial^m_\theta g^0_{\infty,b}+\partial^j_{x_1}\partial^m_\theta(\boldsymbol{w}_\infty\cdot\nabla\zeta_b)-[\partial^j_{x_1}\partial^m_\theta,\boldsymbol{w}]\nabla(\zeta_b\phi_\infty)-\partial^j_{x_1}\partial^m_\theta(\zeta_b\dot{\phi}_\infty),\\ -\mathrm{div}\mathbf{D}(\partial^j_{x_1}(\zeta_b\boldsymbol{w}_{\infty,m}))+\nabla\left(\dfrac{1}{\nu}\partial^j_{x_1}(\zeta_b\phi_{\infty,m})\right)\\ \quad=\dfrac{1}{\nu}\Big\{\partial^j_{x_1}\boldsymbol{g}_{\infty,b,m}-\partial_t\partial^j_{x_1}(\zeta_b\boldsymbol{w}_{\infty,m})-\nu'\nabla\big(\partial^j_{x_1}\partial^m_\theta g^0_{\infty,b}\\ \qquad+\partial^j_{x_1}\partial^m_\theta(\boldsymbol{w}_\infty\cdot\nabla\zeta_b)-[\partial^j_{x_1}\partial^m_\theta,\boldsymbol{w}]\nabla(\zeta_b\phi_\infty)-\partial^j_{x_1}\partial^m_\theta(\zeta_b\dot{\phi}_\infty)\big)\Big\},\end{cases}\tag{2.7.50}$$

The desired result can be obtained by applying Lemma 2.7.9 in a similar manner to the proof of Proposition 2.7.7. This completes the proof. □

Proposition 2.7.10. *Let $k = 0, 1$. Then*

$$\|\partial_t \phi_\infty\|_{H^k} \leq C\{\|\tilde{f}^0_\infty\|_{H^k} + \|\boldsymbol{w}\cdot\nabla\phi_\infty\|_{H^k} + \|\mathrm{div}\ \boldsymbol{w}_\infty\|_{H^k}\}. \tag{2.7.51}$$

Proof. The estimate (2.7.51) immediately follows from the relation

$$\partial_t \phi_\infty = \tilde{f}^0_\infty - \boldsymbol{w}\cdot\nabla\phi_\infty - \mathrm{div}\ \boldsymbol{w}_\infty.$$

This completes the proof. □

We are now in a position to prove (2.5.31) with $\mathcal{R}(t)$ satisfying (2.5.32).

Proof of (2.5.31) with $\mathcal{R}(t)$ satisfying (2.5.32). The proof is given by a suitable linear combination of the inequalities obtained in Propositions 1.7.1–2.7.10 as in [27]. (See also [15, 12].) We first observe that

$$\begin{aligned}|2(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty)| \leq 2\|\phi_\infty\|_2\|\mathrm{div}\boldsymbol{w}_\infty\|_2 &\leq \frac{\nu_0}{2}\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 + \frac{1}{\nu_0}\|\phi_\infty\|_2^2 \\ &\leq \frac{3}{8}\nu_0\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \frac{1}{\nu_0}\|\phi_\infty\|_2^2.\end{aligned}$$

It then follows that

$$\begin{aligned}&\frac{\nu}{2}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \nu'\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 - 2(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty) \\ &\quad\geq \frac{3}{4}\nu_0\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 - \frac{3}{8}\nu_0\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 - \frac{1}{\nu_0}\|\phi_\infty\|_2^2 \\ &\quad= \frac{3}{8}\nu_0\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 - \frac{1}{\nu_0}\|\phi_\infty\|_2^2 \\ &\quad\geq \frac{3}{8}c_K\nu_0\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 - \frac{1}{\nu_0}\|\phi_\infty\|_2^2.\end{aligned}$$

We also have

$$4\|\mathrm{div}(\boldsymbol{w}_\infty)\|_2^2 \leq 12\|\nabla\boldsymbol{w}_\infty\|_2^2.$$

Therefore, setting $c_1 = \max\{\frac{2}{\nu_0}, \frac{64}{c_K\nu_0}\}$, we see from $(1 + c_1)\times(1.7.4)_{k=j=0} + (2.7.10)$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + \frac{1}{2}D_1(t) \leq CN_1(t), \tag{2.7.52}$$

where

$$\begin{aligned}E_1(t) &= (1+c_1)\|u_\infty\|_2^2 + \frac{\nu}{2}\|\mathbf{D}(\boldsymbol{w}_\infty)\|_2^2 + \nu'\|\mathrm{div}\boldsymbol{w}_\infty\|_2^2 - 2(\phi_\infty, \mathrm{div}\boldsymbol{w}_\infty), \\ D_1(t) &= \frac{3}{4}\left(1 + \frac{c_1}{2}\right)c_K\nu_0\|\nabla\boldsymbol{w}_\infty\|_2^2 + \frac{3}{2}(1+c_1)\nu\|\dot{\phi}_\infty\|_2^2 + \|\partial_t\boldsymbol{w}_\infty\|_2^2, \\ N_1(t) &= \left|R^{(1)}_{0,0}(t)\right| + \|\boldsymbol{w}\cdot\nabla\phi_\infty\|_2^2 + \|\tilde{f}^0_\infty\|_2^2 + \|\boldsymbol{f}_\infty\|_2^2.\end{aligned}$$

Note that there exists a positive constant $C > 0$ such that

$$C^{-1}\{\|u_\infty(t)\|_2^2 + \|\nabla \boldsymbol{w}_\infty(t)\|_2^2\} \le E_1(t) \le C\{\|u_\infty(t)\|_2^2 + \|\nabla \boldsymbol{w}_\infty(t)\|_2^2\}.$$

We next consider $c_2 \times (2.7.52) + (2.7.14)_{|\alpha|=1} + (2.7.28)_{j+m=1}$ with $c_2 = c_2'(1+\frac{1}{\varepsilon})$. Then, with a suitably large positive constant c_2', we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) + \frac{1}{2}D_2(t) \le CN_2(t) + 5\varepsilon(\|\phi_\infty\|_{H^1}^2 + \|\boldsymbol{w}_\infty\|_{H^2}^2), \tag{2.7.53}$$

where

$$\begin{aligned}
E_2(t) &= c_2E_1(t) + \|\nabla(\zeta_{in}u_\infty)\|_2^2 + \sum_{j+m=1}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\phi_\infty)\|_2^2 + \|\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})\|_2^2,\\
E_2(t) &= c_2D_1(t) + \frac{3}{4}c_K\nu_0\{\|\partial_x^2(\zeta_{in}\boldsymbol{w}_\infty)\|_2^2 + \sum_{j+m=1}\|\nabla\partial_{x_1}^j(\zeta_b\boldsymbol{w}_m)\|_2^2\}\\
&\quad + \frac{3\nu_0}{2}\{\|\nabla(\zeta_{in}\dot{\phi}_\infty)\|_2^2 + \sum_{j+m=1}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\dot{\phi})\|_2^2\},\\
N_2(t) &= c_2N_1(t) + \sum_{|\alpha|=1}R_\alpha^{(2)}(t) + \sum_{j+m=1}\left|R_{j,m}^{(3)}(t)\right|.
\end{aligned}$$

We next consider $c_3 \times (2.7.53) + (2.7.34)_{j=m=l=0}$ with a positive constant c_3. Then, taking c_3 suitably large and apply Lemma 2.7.4, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_3(t) + \frac{1}{2}D_3(t) \le CN_3(t) + 5c_3\varepsilon(\|\phi_\infty\|_{H^1}^2 + \|\boldsymbol{w}_\infty\|_{H^2}^2), \tag{2.7.54}$$

where

$$\begin{aligned}
E_3(t) &= c_3E_2(t) + \|\zeta_b\partial_r\phi_\infty(t)\|_2^2,\\
D_3(t) &= c_3D_2(t) + \frac{1}{2\nu+\nu'}\|\zeta_b\partial_r\phi_\infty(t)\|_2^2 + (2\nu+\nu')\|\zeta_b\partial_r\dot{\phi}_\infty(t)\|_2^2,\\
N_3(t) &= c_3N_2(t) + R_{0,0,0}^{(4)}.
\end{aligned}$$

We are now in a position to derive the H^1-energy estimate. We consider $c_4 \times (2.7.54) + \frac{\nu^2}{2\nu+\nu'} \times (2.7.44)_{k=2}$ with a positive constant c_4. By taking c_4 suitably large and ε suitably small, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_4(t) + \frac{1}{2}D_4(t) \le CN_4(t), \tag{2.7.55}$$

where

$$\begin{aligned}
E_4(t) &= c_4E_3(t),\\
D_4(t) &= c_4D_3(t) + \frac{1}{2\nu+\nu'}\|\phi_\infty\|_{H^1}^2 + \frac{\nu^2}{2\nu+\nu'}\|\boldsymbol{w}_\infty\|_{H^2}^2,\\
N_4(t) &= c_4N_3(t) + \frac{1}{2\nu+\nu'}\|\boldsymbol{f}_\infty\|_2^2 + (2\nu+\nu')\|\tilde{f}_\infty^0\|_{H^1}^2.
\end{aligned}$$

Let us proceed to establish the H^2-energy estimate. With a suitably large positive constant c_5' we see from $(1.7.4)_{k=1,j=0}, (2.7.14)_{|\alpha|=2}, (2.7.28)_{j+m=2}$ and $c_5 \times (2.7.55), c_5 = c_5'(1+\frac{1}{\varepsilon})$, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_5(t) + \frac{1}{2}D_5(t) \le CN_5(t) + 9\varepsilon(\|\phi_\infty\|_{H^2}^2 + \|\boldsymbol{w}_\infty\|_{H^3}^2), \tag{2.7.56}$$

where

$$\begin{aligned}
E_5(t) &= c_5E_4(t) + \|\partial_t u_\infty(t)\|_2^2 + \|\partial_x^2(\zeta_b u_\infty)\|_2^2 + \sum_{j+m=2}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\phi_\infty)\|_2^2 \\
&\quad + \|\partial_{x_1}^j(\zeta_b\boldsymbol{w}_{\infty,m})\|_2^2, \\
D_5(t) &= c_5D_4(t) + \frac{3}{4}c_K\nu_0\|\nabla\partial_t\boldsymbol{w}_\infty(t)\|_2^2 + \frac{3\nu_0}{2}\|\partial_t\dot{\phi}_\infty\|_2^2 \\
&\quad + \frac{3}{4}c_K\nu_0\left(\|\partial_x^3(\zeta_{in}\boldsymbol{w}_\infty)\|_2^2 + \sum_{j+m=2}\|\nabla\partial_{x_1}^j\zeta_b(\boldsymbol{w}_{\infty,m})\|_2^2\right) \\
&\quad + \frac{3\nu_0}{2}\left(\|\partial_x^2(\zeta_{in}\dot{\phi}_\infty)\|_2^2 + \sum_{j+m=2}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\dot{\phi})\|_2^2\right), \\
N_5(t) &= c_5N_4(t) + R_{0,1}^{(1)}(t) + \sum_{|\alpha|=2}R_\alpha^{(2)}(t) + \sum_{j+m=2}R_{j,m}^{(3)}(t).
\end{aligned}$$

We note that, by Lemma 2.7.4, $E_5(t)$ and $D_5(t)$ are equivalent to

$$c_5E_4(t) + \|\partial_t u_\infty(t)\|_2^2 + \|\partial_x^2(\zeta_b u_\infty)\|_2^2 + \sum_{j+m=2}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b u_\infty)\|_2^2$$

and

$$\begin{aligned}
&c_5D_4(t) + \frac{3}{4}c_K\nu_0\left(\|\nabla\partial_t\boldsymbol{w}_\infty(t)\|_2^2 + \|\partial_x^3(\zeta_{in}\boldsymbol{w}_\infty)\|_2^2 + \sum_{j+m=2}\|\nabla\partial_{x_1}^j\partial_\theta^m(\zeta_b\boldsymbol{w}_\infty)\|_2^2\right) \\
&\quad + \frac{3\nu_0}{2}\left(\|\partial_t\dot{\phi}_\infty\|_2^2 + \|\partial_x^2(\zeta_{in}\dot{\phi}_\infty)\|_2^2 + \sum_{j+m=2}\|\partial_{x_1}^j\partial_\theta^m(\zeta_b\dot{\phi}_\infty)\|_2^2\right),
\end{aligned}$$

respectively, by taking c_5' suitably large.

We next consider $c_6 \times (2.7.56) + (2.7.34)_{j+m=1,l=0}$ with a positive constant c_6. Since

$$\sum_{j+m=1}\|\partial_{x_1}^j\partial_\theta^m\partial_t(\zeta_b w^r)\|_2^2 + \|\partial_{x_1}^j\partial_\theta^m M(\zeta_b\boldsymbol{w}_\infty)\|_2^2 \le CD_5(t),$$

taking c_6 suitably large, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_6(t) + \frac{1}{2}D_6(t) \le CN_6(t) + 9c_6\varepsilon(\|\phi_\infty\|_{H^2}^2 + \|\boldsymbol{w}_\infty\|_{H^3}^2), \tag{2.7.57}$$

where

$$E_6(t) = c_6 E_5(t) + \sum_{j+m=1} \|\zeta_b \partial_{x_1}^j \partial_\theta^m \phi_\infty(t)\|_2^2,$$

$$D_6(t) = c_6 D_5(t) + \sum_{j+m=1} \left\{ \frac{1}{2\nu+\nu'} \|\zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r \phi_\infty(t)\|_2^2 + (2\nu+\nu') \|\zeta_b \partial_{x_1}^j \partial_\theta^m \partial_r \dot{\phi}_\infty(t)\|_2^2 \right\},$$

$$N_6(t) = c_6 N_5(t) + \sum_{j+m=1} R_{j,m,0}^{(4)}(t).$$

It then follows from $c_7 \times (2.7.57) + \frac{\nu^2}{2\nu+\nu'} \times (2.7.45)$ with a suitably large positive constant c_7 that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E_7(t) + \frac{1}{2} D_7(t) \le C N_7(t) + 9 c_6 c_7 \varepsilon (\|\phi_\infty\|_{H^2}^2 + \|\boldsymbol{w}_\infty\|_{H^3}^2), \tag{2.7.58}$$

where

$$E_7(t) = c_7 E_6(t),$$

$$D_7(t) = c_7 D_6(t) + \sum_{j+m=1} \left\{ \frac{1}{2\nu+\nu'} \|\partial_{x_1}^j \partial_\theta^m (\zeta_b \phi_\infty)\|_{H^1}^2 + \frac{\nu^2}{2\nu+\nu'} \|\partial_{x_1}^j (\zeta_b \boldsymbol{w}_{\infty,m})\|_{H^2}^2 \right\},$$

$$N_7(t) = c_7 N_6(t) + \sum_{j+m=1} \left\{ \frac{1}{2\nu+\nu'} \|\partial_{x_1}^j \boldsymbol{g}_{\infty,b,m}\|_2^2 + (2\nu+\nu') \|\partial_{x_1}^j \partial_\theta^m g_{\infty,b}^0\|_2^2 \right\}.$$

We note that, by Lemma 2.7.4, $D_7(t)$ is equivalent to

$$c_7 D_6(t) + \sum_{j+m=1} \left\{ \frac{1}{2\nu+\nu'} \|\partial_{x_1}^j \partial_\theta^m (\zeta_b \phi_\infty)\|_{H^1}^2 + \frac{\nu^2}{2\nu+\nu'} \|\partial_{x_1}^j \partial_\theta^m (\zeta_b \boldsymbol{w}_\infty)\|_{H^2}^2 \right\}.$$

We next consider $c_8 \times (2.7.58) + (2.7.45)_{j=m=0,l=1}$ with a positive constant c_8. Since

$$\|\partial_r \partial_t (\zeta_b w_\infty^r)\|_2^2 + \|\partial_r M(\zeta_b \boldsymbol{w}_\infty)\|_2^2 \le C D_7(t),$$

by taking c_8 suitably large, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E_8(t) + \frac{1}{2} D_8(t) \le C N_8(t) + 9 c_6 c_7 c_8 \varepsilon (\|\phi_\infty\|_{H^2}^2 + \|\boldsymbol{w}_\infty\|_{H^3}^2), \tag{2.7.59}$$

where

$$E_8(t) = c_8 E_7(t) + \|\zeta_b \partial_r^2 \phi_\infty(t)\|_2^2,$$

$$D_8(t) = c_8 D_7(t) + \frac{1}{2\nu+\nu'} \|\zeta_b \partial_r^2 \phi_\infty(t)\|_2^2 + (2\nu+\nu') \|\zeta_b \partial_r^2 \dot{\phi}_\infty(t)\|_2^2,$$

$$N_8(t) = c_8 N_7(t) + R_{0,0,1}^{(4)}.$$

It then follows from $c_9 \times (2.7.59) + \frac{\nu^2}{2\nu+\nu'} \times (2.7.44)_{k=3}$ with a suitably large c_9 and a suitably small ε that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E_9(t) + \frac{1}{2} D_9(t) \le C N_9(t), \tag{2.7.60}$$

where

$$\begin{aligned}
E_9(t) &= c_9 E_8(t),\\
D_9(t) &= c_9 D_8(t) + \frac{1}{2\nu+\nu'}\|\phi_\infty(t)\|_{H^2}^2 + \frac{\nu^2}{2\nu+\nu'}\|\boldsymbol{w}_\infty(t)\|_{H^3}^2,\\
N_9(t) &= c_9 N_8(t) + \frac{1}{2\nu+\nu'}\|\boldsymbol{f}_\infty\|_{H^1}^2 + (2\nu+\nu')\|\tilde{f}^0_\infty\|_{H^2}^2.
\end{aligned}$$

Since $D_9(t) \geq d_1 E_9(t)$ for some positive constant d_1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_9(t) + \frac{d_1}{2}E_9(t) + \frac{1}{2}D_9(t) \leq 2CN_9(t), \tag{2.7.61}$$

$$E_9(t) + \frac{1}{2}\int_0^t e^{-\frac{d_1}{2}(t-s)}D_9(s)\mathrm{d}s \leq e^{-\frac{d_1}{2}t}E_9(0) + 2C\int_0^t e^{-\frac{d_1}{2}(t-s)}N_9(s)\mathrm{d}s. \tag{2.7.62}$$

Since $\boldsymbol{w}_\infty$ satisfies

$$\begin{cases}
-\nu \mathrm{div}\mathbf{D}(\boldsymbol{w}_\infty) - \nu'\nabla\mathrm{div}\boldsymbol{w}_\infty = \boldsymbol{f}_\infty - (\partial_t \boldsymbol{w}_\infty + \nabla\phi_\infty),\\
\boldsymbol{w}_\infty \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathbf{D}(\boldsymbol{w}_\infty)\cdot\boldsymbol{n} - \big(\mathbf{D}(\boldsymbol{w}_\infty)\boldsymbol{n}\cdot\boldsymbol{n}\big)\boldsymbol{n}\big|_{\partial\Omega} = \mathbf{0},
\end{cases}$$

by the elliptic estimate, we have

$$\begin{aligned}
\|\boldsymbol{w}_\infty(t)\|_{H^2}^2 &\leq C\{\|\boldsymbol{f}_\infty(t)\|_2^2 + \|\partial_t\boldsymbol{w}_\infty(t)\|_2^2 + \|\nabla\phi_\infty(t)\|_2^2\}\\
&\leq C\{E_9(t) + \|\boldsymbol{f}_\infty(t)\|_2^2\}.
\end{aligned} \tag{2.7.63}$$

It then follows from $(2.7.51)_{k=1}$, (2.7.62) and (2.7.63) that

$$\begin{aligned}
&E_{10}(t) + \frac{1}{2}\int_0^t e^{-\frac{d_1}{2}(t-s)}D_{10}(s)\mathrm{d}s\\
&\leq C\left\{e^{-\frac{d_1}{2}t}E_{10}(0) + \|\boldsymbol{f}_\infty(t)\|_2^2 + \int_0^t e^{-\frac{d_1}{2}(t-s)}N_{10}(s)\mathrm{d}s\right\}.
\end{aligned} \tag{2.7.64}$$

Here

$$\begin{aligned}
E_{10}(t) &= c_{10}E_9(t) + \|\boldsymbol{w}_\infty(t)\|_{H^2}^2,\\
D_{10}(t) &= c_{10}D_9(t) + \|\partial_t\phi_\infty(t)\|_{H^1}^2,\\
N_{10}(t) &= c_{10}N_9(t) + \|\tilde{f}^0_\infty(t)\|_{H^1}^2 + \|(\boldsymbol{w}\cdot\nabla\phi_\infty)(t)\|_{H^1}^2,
\end{aligned}$$

with some suitably large positive constant c_{10}.

By Propositions 2.8.1 and 2.8.2 below, we have

$$\|\boldsymbol{f}_\infty\|_2^2 \leq C(1+t)^{-\frac{3}{2}}M(t)^4 \leq C(1+t)^{-\frac{3}{2}}M_*(t)^4$$

and

$$\begin{aligned}
N_{10}(t) &\leq C\{(1+t)^{-\frac{3}{2}}M(t)^3 + M(t)D_\infty(t)\}\\
&\leq C\{(1+t)^{-\frac{3}{2}}M_*(t)^3 + M_*(t)D_\infty(t)\}.
\end{aligned}$$

Combing these estimates with (2.7.64), we obtain (2.5.31) with $\mathcal{R}(t)$ satisfying (2.5.32), by setting $\mathcal{R}(t) = N_{10}(t)$ in (2.7.64). This completes the proof. □

2.8 Estimates for the nonlinearities

In this section we estimate the nonlinearities. Throughout this section we assume that $u = {}^\top(\phi, \boldsymbol{w}) \in C([0,T]; H^2 \times H^2_*)$ is a solution of (1.5.1) with $\boldsymbol{w} \in L^2(0,T;H^3)$ for a given positive number T. Observe that u is written as $u = u_1 + u_\infty,\ u_j = {}^\top(\phi_j, \boldsymbol{w}_j),\ (j = 1, \infty)$; and $\phi_1 = \phi_1(x_1)$ and $\boldsymbol{w}_1 = {}^\top(w_1^1, w_1^{\mathrm{rig}}\boldsymbol{b}'_{\mathrm{rig}})$ with $w_1^1 = w_1^1(x_1)$ and $w_1^{\mathrm{rig}} = w_1^{\mathrm{rig}}(x_1)$.

Proposition 2.8.1. *The following estimates hold uniformly for $t \in [0,T]$ with $C > 0$ independent of T :*

$$\|\phi \mathrm{div}\ \boldsymbol{w}\|_{H^2} \le C\big\{(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)\sqrt{D_\infty(t)}\big\}, \tag{2.8.1}$$

$$\|\boldsymbol{w}\cdot\nabla\phi_\infty\|_{H^1} + \|\boldsymbol{w}\cdot\nabla\phi_1\|_{H^2} \le C(1+t)^{-\frac{5}{4}}M(t)^2, \tag{2.8.2}$$

$$|(\mathrm{div}\boldsymbol{w}, |\partial_t^k\partial_{x_1}^j\phi_\infty|^2)| \le C\big\{(1+t)^{-\frac{5}{2}}M(t)^3 + (1+t)^{-\frac{5}{2}}M(t)D_\infty(t)\big\}, \tag{2.8.3}$$

$$\|[\partial_t^k\partial_x^j, \boldsymbol{w}]\cdot\nabla\phi_\infty\|_2 \le C\big\{(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{3}{4}}M(t)\sqrt{D_\infty(t)}\big\} \tag{2.8.4}$$

for $1 \le 2k + j \le 2$ and

$$\|\tilde{f}^0_\infty\|_{H^2} \le C\big\{(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)\sqrt{D_\infty(t)}\big\}, \tag{2.8.5}$$

$$\|\partial_t\tilde{f}^0_\infty\|_2 \le C\big\{(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)\sqrt{D_\infty(t)}\big\}. \tag{2.8.6}$$

Noting that $\mathrm{div}\boldsymbol{w}_1 = \partial_{x_1}w_1^1(x_1)$ and $\phi_1 = \phi_1(x_1)$, one can prove Proposition 2.8.1 by straightforward applications of the Gagliardo-Nirenberg-Sobolev inequality.

We next consider the estimates for $\boldsymbol{f}_\infty$. We observe that there exists a positive constant ε_2 such that $\|\phi(t)\|_\infty \le \frac{1}{2}$ whenever $\|u(t)\|_{H^2} \le \varepsilon_2$, which follows from the Sobolev inequality $\|u(t)\|_\infty \le C\|u(t)\|_{H^2}$.

Proposition 2.8.2. *If $\|u(t)\|_{H^2} \le \varepsilon_2$ and $M(t) \le 1$ for $t \in [0,T]$, then*

$$\|\boldsymbol{f}_\infty\|_2 \le C(1+t)^{-\frac{3}{4}}M(t)^2, \tag{2.8.7}$$

$$\|\boldsymbol{f}_\infty\|_{H^1} \le C\big\{(1+t)^{-\frac{3}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)\sqrt{D_\infty(t)}\big\}, \tag{2.8.8}$$

$$|(\partial_t\boldsymbol{f}_\infty, \partial_t\boldsymbol{w}_\infty)| \le C\{(1+t)^{-2}M(t)^3 + M(t)D_\infty(t)\}, \tag{2.8.9}$$

where C is a positive constant independent of T.

Noting that $\phi_1 = \phi_1(x_1)$ and

$$\nu\mathrm{div}\mathbf{D}(\boldsymbol{w}_1) + \nu'\nabla\mathrm{div}\boldsymbol{w}_1 = \begin{pmatrix} (2\nu+\nu')\partial_{x_1}^2 w_1^1(x_1) \\ \nu\partial_{x_1}^2 w_1^{\mathrm{rig}}(x_1)\boldsymbol{b}'_{\mathrm{rig}} \end{pmatrix},$$

one can prove Proposition 2.8.2 by using the Gagliardo-Nirenberg-Sobolev inequality. We also note that the most slowly decaying term arises from $\boldsymbol{w}_1\cdot\nabla\boldsymbol{w}_1$ which contains $-2(w_1^{\mathrm{rig}})^2\boldsymbol{n}$. Cf., [1, Section 7].

The estimate (2.5.32) for $\mathcal{R}(t) = N_{10}(t)$ follows from Propositions 2.8.1 and 2.8.2.

2.9 Asymptotic behavior

In this section we give a proof of Theorem 2.3.2. To this end, we first show that the low frequency part $u_{1*}(t)$ is approximated by the solution

$$\sigma_*(t) = \sigma_*^0(t)b_0 + \sigma_*^1(t)b_1 + \sigma_*^{\mathrm{rig}}(t)b_{\mathrm{rig}}, \quad \sigma_*^j = \sigma_*^j(x_1,t) \ (j=0,1,\mathrm{rig}) \tag{2.9.1}$$

of the following integral equation as $t \to \infty$:

$$\sigma_*(t) = S(t)P_1 u_{*0} + \int_0^t S(t-\tau)\partial_{x_1} G(\sigma_*)(\tau)\mathrm{d}\tau. \tag{2.9.2}$$

We note that, since P_1 is written as $P_1 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\Pi_0 + \mathbf{1}_{R_0}\Pi_1 + \mathbf{1}_{R_1}\Pi_{\mathrm{rig}}]$, $P_1 u_{*0}$ takes the form

$$P_1 u_{*0} = \tilde{\sigma}_{*0}^0 b_0 + \tilde{\sigma}_{*0}^1 b_1 + \tilde{\sigma}_{*0}^{\mathrm{rig}} b_{\mathrm{rig}}, \tag{2.9.3}$$

where $\tilde{\sigma}_{*0}^0 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{\phi}_0\rangle]$, $\tilde{\sigma}_{*0}^1 = \mathcal{F}^{-1}[\mathbf{1}_{R_0}\langle\hat{m}_0^1\rangle]$ and $\tilde{\sigma}_{*0}^{\mathrm{rig}} = \mathcal{F}^{-1}[\mathbf{1}_{R_1}\langle \boldsymbol{m}_0', \boldsymbol{b}_{\mathrm{rig}}'\rangle]$ with $u_{*0} = {}^\top(\phi_0, \boldsymbol{m}_0)$, $\boldsymbol{m}_0 = {}^\top(m_0^1, \boldsymbol{m}_0')$. We also note that

$$\begin{aligned} G(\sigma_*) &= -\Big((\sigma_*^1)^2 + \frac{p''(1)}{2}(\sigma_*^0)^2\Big)b_1 - \sigma_*^1\sigma_*^{\mathrm{rig}} b_{\mathrm{rig}} \\ &= -\Big((\sigma_*^1)^2 + \frac{p''(1)}{2}(\sigma_*^0)^2\Big)b_+ + \Big((\sigma_*^1)^2 + \frac{p''(1)}{2}(\sigma_*^0)^2\Big)b_- - \sigma_*^1\sigma_*^{\mathrm{rig}} b_{\mathrm{rig}}. \end{aligned} \tag{2.9.4}$$

By a standard way, one can show the following estimates for $\sigma_*(t)$ by using Lemma 2.4.3.

Lemma 2.9.1. *If $\|u_0\|_{H^1\cap L^1}$ is sufficiently small, then (2.9.2) has a unique solution $\sigma_*(t)$ that satisfies*

$$\|\partial_{x_1}^k \sigma_*(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_{H^1\cap L^1} \quad (k=0,1), \tag{2.9.5}$$

$$\|\partial_{x_1}^k \sigma_*(t)\|_\infty \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^1\cap L^1} \quad (k=0,1). \tag{2.9.6}$$

We have the following estimate for $u_{1*}(t) - \sigma_*(t)$ which shows that the asymptotic leading part of $u_{1*}(t)$ is given by $\sigma_*(t)$ for large t.

Theorem 2.9.2. *If $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then*

$$\|u_{1*}(t) - \sigma_*(t)\|_2 \le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^2\cap L^1},$$

for any $\delta > 0$.

Proof. We introduce $N(t)$ defined by

$$N(t) = \sup_{0\le\tau\le t}\Big\{(1+\tau)^{\frac{3}{4}-\delta}\|u_{1*}(\tau) - \sigma_*(\tau)\|_{H^1}\Big\}.$$

It follows from (2.5.26) and (2.9.2) that $u_{1*}(t) - \sigma_*(t)$ is written as

$$u_{1*}(t) - \sigma_*(t) = \sum_{j=0}^{4} I_j(t),$$

where

$$
\begin{aligned}
I_0(t) &= \big(e^{-tL} - S(t)\big)P_1 u_{*0},\\
I_1(t) &= \int_0^t S(t-\tau)P_1\partial_{x_1}\big(G(u(\tau)) - G(\sigma_*(\tau))\big)\mathrm{d}\tau,\\
I_2(t) &= \int_0^t \big(e^{-(t-\tau)L} - S(t-\tau)\big)\, P_1\partial_{x_1}G\big(u(\tau)\big)\mathrm{d}\tau,\\
I_3(t) &= -\int_0^t S(t-\tau)(I-P_1)\partial_{x_1}G\big(\sigma_*(\tau)\big)\mathrm{d}\tau,\\
I_4(t) &= \int_0^t e^{-(t-\tau)L}P_1\partial_{x_1}\tilde{G}\big(u(\tau)\big)\mathrm{d}\tau.
\end{aligned}
$$

As for $I_0(t)$, Lemma 2.4.3 (iii) shows that

$$
\|\partial_{x_1}^k I_0(t)\|_2 \le \|\partial_{x_1}^k (e^{-tL} - S(t))P_1 u_{*0}\|_2 \le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^1\cap L^1}
$$

for $k = 0, 1$.

As for $I_1(t)$, we write it as

$$
I_1(t) = \int_0^t \partial_{x_1} S(t-\tau)P_1\big(G(u(\tau)) - G(\sigma_*(\tau))\big)\mathrm{d}\tau.
$$

Since

$$
\begin{aligned}
&\|G(u) - G(\sigma_*)\|_1\\
&\quad \le C\{\|(w^1)^2 - (\sigma_*^1)^2\|_1 + \|\phi^2 - (\sigma_*^0)^2\|_1 + \|w^1\boldsymbol{w}' - \sigma_*^1\sigma_*^{\mathrm{rig}}\boldsymbol{b}'_{\mathrm{rig}}\|_1\}\\
&\quad \le C(1+t)^{-1+\delta}N(t)\|u_0\|_{H^2\cap L^1},
\end{aligned}
$$

it follows from (2.4.12) and Lemma 2.4.3 (i), (ii) that

$$
\begin{aligned}
\|\partial_{x_1}^k I_1(t)\|_2 &\le C\left\{\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1+\delta}\mathrm{d}\tau\right\} N(t)\|u_0\|_{H^2\cap L^1}\\
&\le C(1+t)^{-\frac{3}{4}+\delta}N(t)\|u_0\|_{H^2\cap L^1}
\end{aligned}
$$

for $k = 0, 1$.

We next estimate $I_2(t)$. Since

$$
\|\partial_{x_1}G(u)\|_1 \le C\|u\|_2\|\partial_{x_1}u\|_2 \le C(1+t)^{-1}M(t)^2,
$$

we have

$$
\begin{aligned}
\|\partial_{x_1}^k I_2(t)\|_2 &\le C\left\{\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1}\mathrm{d}\tau\right\} M(t)^2\\
&\le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^2\cap L^1}
\end{aligned}
$$

for $k=0,1$.

As for $I_3(t)$, we see from Lemma 2.9.1 that

$$\|\partial_{x_1} G(\sigma_*)\|_2 \le C\|\sigma_*\|_\infty \|\partial_{x_1}\sigma_*\|_2 \le C(1+t)^{-\frac{5}{4}}\|u_0\|^2_{H^1\cap L^1},$$
$$\|\partial_{x_1} G(\sigma_*)\|_1 \le C\|\sigma_*\|_2 \|\partial_{x_1}\sigma_*\|_2 \le C(1+t)^{-1}\|u_0\|^2_{H^1\cap L^1}.$$

It follows from Lemma 2.4.3 (i), (ii) that

$$\begin{aligned}
&\|\partial^k_{x_1} I_3(t)\|_2 \\
&\le C\int_0^t \left\{(t-\tau)^{-\frac{k}{2}} e^{-d_2(t-\tau)}(1+\tau)^{-\frac{5}{4}} + (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1}\right\} \mathrm{d}\tau \|u_0\|^2_{H^2\cap L^1} \\
&\le C\left\{(1+t)^{-\frac{5}{4}} + (1+t)^{-\frac{3}{4}+\delta}\right\}\|u_0\|_{H^2\cap L^1}
\end{aligned}$$

for $k=0,1$.

As for $I_4(t)$, we write it as

$$I_4(t) = \int_0^t \partial_{x_1} e^{-(t-\tau)L} P_1 \tilde{G}\big(u(\tau)\big)\mathrm{d}\tau.$$

Since

$$\|\tilde{G}\|_1 \le C(1+t)^{-1}M(t)^2,$$

similarly to the estimate for $\partial^k_{x_1} I_2(t)$, we obtain

$$\|\partial^k_{x_1} I_4(t)\|_2 \le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^2\cap L^1}$$

for $k=0,1$. This completes the proof. □

Proof of Theorem 2.3.2. By Theorem 9.2, it suffices to show that

$$\left\|\partial^k_{x_1}\left(\sigma_* - \chi_+ b_+ - \chi_- b_- - \chi_{\mathrm{rig}} b_{\mathrm{rig}}\right)(t)\right\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^1\cap L^1}$$

for $k=0,1$. Here $\chi_\pm = \chi_\pm(x_1,t)$ is the diffusion waves given in (2.1.10)-(2.1.12) with $c = \frac{1}{4}\big(1+\frac{p''(1)}{2}\big)$, and χ_{rig} is given in (2.1.14). We write $\sigma_*(t)$ as

$$\sigma_*(t) = \sigma_*^+(t)b_+ + \sigma_*^-(t)b_- + \sigma_*^{\mathrm{rig}}(t)b_{\mathrm{rig}},$$

where

$$\sigma_*^\pm(t) = \sigma_*^0(t) \pm \sigma_*^1(t).$$

We also write $P_1 u_{*0}$ as

$$\begin{aligned}
P_1 u_{*0} &= \sigma^0_{*0} b_0 + \sigma^1_{*0} b_1 + \sigma^{\mathrm{rig}}_{*0} b_{\mathrm{rig}} + \tilde{\sigma}_{*0} \\
&= \sigma^+_{*0} b_+ + \sigma^-_{*0} b_- + \sigma^{\mathrm{rig}}_{*0} b_{\mathrm{rig}} + \tilde{\sigma}_{*0},
\end{aligned} \tag{2.9.7}$$

where

$$\sigma^0_{*0} = \langle \phi_0 \rangle, \quad \sigma^1_{*0} = \langle m^1_0 \rangle, \quad \sigma^{\mathrm{rig}}_{*0} = \langle \boldsymbol{m}'_0, \boldsymbol{b}'_{\mathrm{rig}} \rangle,$$

$$\tilde{\sigma}_{*0} = \mathcal{F}^{-1}\big[(1-\mathbf{1}_{R_0})(\langle\hat{\phi}_0\rangle b_0 + \langle\hat{m}_0^1\rangle b_1) + (1-\mathbf{1}_{R_1})\langle \boldsymbol{m}_0', \boldsymbol{b}_{\mathrm{rig}}'\rangle \boldsymbol{b}_{\mathrm{rig}}'\big]$$

and

$$\sigma_{*0}^{\pm} = \sigma_{*0}^0 \pm \sigma_{*0}^1.$$

We then obtain

$$\begin{aligned}\sigma_*(t) = {} & S_+(t)(\sigma_{*0}^+ b_+) + S_-(t)(\sigma_{*0}^- b_-) + S_{\mathrm{rig}}(t)(\sigma_{*0}^{\mathrm{rig}} b_{\mathrm{rig}}) + S(t)\tilde{\sigma}_{*0} \\ & + I_+(t) + I_-(t) + I_{\mathrm{rig}}(t),\end{aligned}$$

where

$$\begin{aligned}I_\pm(t) &= \mp\int_0^t S_\pm(t-\tau)\left[\partial_{x_1}\left((\sigma_*^1)^2 + \tfrac{p''(1)}{2}(\sigma_*^0)^2\right)(\tau)b_\pm\right]\mathrm{d}\tau, \\ I_{\mathrm{rig}}(t) &= -\int_0^t S_{\mathrm{rig}}(t-\tau)\left[\partial_{x_1}\big(\sigma_*^1\sigma_*^{\mathrm{rig}}\big)(\tau)b_{\mathrm{rig}}\right]\mathrm{d}\tau.\end{aligned}$$

Note that

$$\sigma_*^0(\tau) = \frac{1}{2}\big(\sigma_*^+(\tau) + \sigma_*^-(\tau)\big), \quad \sigma_*^1(\tau) = \frac{1}{2}\big(\sigma_*^+(\tau) - \sigma_*^-(\tau)\big).$$

We define $V(t) = \eta_+(t)b_+ + \eta_-(t)b_- + \eta_{\mathrm{rig}}(t)b_{\mathrm{rig}}$ by

$$\eta_\pm(t) = \sigma_*^\pm(t) - \chi_\pm(t), \quad \eta_{\mathrm{rig}}(t) = \sigma_*^{\mathrm{rig}}(t) - \chi_{\mathrm{rig}}(t),$$

and introduce

$$Y(t) = \sup_{0\le\tau\le t}\left\{(1+\tau)^{\frac{1}{2}}\|V(\tau)\|_2 + (1+\tau)\|\partial_{x_1}V(\tau)\|_2\right\}.$$

We write

$$\begin{aligned}\big(\sigma_*^1\big)^2 = \frac{1}{4}\big(\sigma_*^+ - \sigma_*^-\big)^2 &= \frac{1}{4}(\eta_+ + \chi_+ - \eta_- - \chi_-)^2 \\ &= \frac{1}{4}\big\{\chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + \zeta_-(\eta_+ - \eta_-)\big\},\end{aligned}$$

$$\begin{aligned}\big(\sigma_*^0\big)^2 = \frac{1}{4}\big(\sigma_*^+ + \sigma_*^-\big)^2 &= \frac{1}{4}(\eta_+ + \chi_+ + \eta_- + \chi_-)^2 \\ &= \frac{1}{4}\{\chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + \zeta_+(\eta_+ + \eta_-)\},\end{aligned}$$

and

$$\begin{aligned}\sigma_*^1\sigma_*^{\mathrm{rig}} &= \frac{1}{2}\big(\sigma_*^+ - \sigma_*^-\big)\sigma_*^{\mathrm{rig}} \\ &= \frac{1}{2}(\eta_+ + \chi_+ - \eta_- - \chi_-)(\eta_{\mathrm{rig}} + \chi_{\mathrm{rig}}) \\ &= \frac{1}{2}\{\chi_+\chi_{\mathrm{rig}} - \chi_-\chi_{\mathrm{rig}} + (\eta_+ + \chi_+ - \eta_- - \chi_-)\eta_{\mathrm{rig}} + (\eta_+ - \eta_-)\chi_{\mathrm{rig}}\},\end{aligned}$$

where $\zeta_\pm = 2(\chi_+ \pm \chi_-) + \eta_+ \pm \eta_-$. It then follows that $I_\pm(t)$ and $I_{\rm rig}(t)$ are written as

$$
\begin{aligned}
I_\pm(t) = \mp \int_0^t S_\pm(t-\tau)\big[\partial_{x_1}\big\{\tfrac14\big(1+\tfrac{p''(1)}{2}\big)(\chi_+^2+\chi_-^2) - \tfrac12\big(1-\tfrac{p''(1)}{2}\big)\chi_+\chi_- \\
+ \tfrac14\zeta_-(\eta_+-\eta_-) + \tfrac{p''(1)}{8}\zeta_+(\eta_++\eta_-)\big\}b_\pm\big]\mathrm{d}\tau, \\
I_{\rm rig}(t) = -\frac12\int_0^t S_{\rm rig}(t-\tau)\big[\partial_{x_1}\big\{\chi_+\chi_{\rm rig} - \chi_-\chi_{\rm rig} + (\eta_+ + \chi_+ - \eta_- - \chi_-)\eta_{\rm rig} \\
+ (\eta_+ - \eta_-)\chi_{\rm rig}\big\}b_{\rm rig}\big]\mathrm{d}\tau.
\end{aligned}
$$

We observe that $\chi_\pm = \chi_\pm(x_1,t)$ and $\chi_{\rm rig} = \chi_{\rm rig}(x_1,t)$ satisfy

$$
\partial_t\chi_\pm - \frac{2\nu+\nu'}{2}\partial_{x_1}^2\chi_\pm \mp \partial_{x_1}\chi_\pm \pm \frac14\big(1+\tfrac{p''(1)}{2}\big)\partial_{x_1}(\chi_\pm)^2 = 0
$$

and

$$
\partial_t\chi_{\rm rig} - \nu\partial_{x_1}^2\chi_{\rm rig} = 0,
$$

respectively. Therefore, we have

$$
\chi_\pm(t)b_\pm = S_\pm(t)[\chi_{\pm0}b_\pm] \mp \frac14\big(1+\tfrac{p''(1)}{2}\big)\int_0^t S_\pm(t-\tau)[\partial_{x_1}(\chi_\pm)^2 b_\pm](\tau)\mathrm{d}\tau
$$

and

$$
\chi_{\rm rig}(t)b_{\rm rig} = S_{\rm rig}(t)[\chi_{\rm rig0}b_{\rm rig}],
$$

where $\chi_{\pm0} = \chi_\pm(0)$ and $\chi_{\rm rig0} = \chi_{\rm rig}(0)$. It then follows that

$$
\begin{aligned}
V(t) &= \sigma_*(t) - \chi_+(t)b_+ - \chi_-(t)b_- - \chi_{\rm rig}(t)b_{\rm rig} \\
&= S_+(t)\big[(\sigma_{*0}^+ - \chi_{0+})b_+\big] + S_-(t)\big[(\sigma_{*0}^- - \chi_{0-})b_-\big] \\
&\quad + S_{\rm rig}(t)\big[(\sigma_{*0}^{\rm rig} - \chi_{\rm rig0})b_{\rm rig}\big] + S(t)\tilde\sigma_{*0} + I_+ + I_- + I_{\rm rig} \\
&\quad + \frac14\big(1+\tfrac{p''(1)}{2}\big)\int_0^t S_+(t-\tau)\big[\partial_{x_1}(\chi_+^2)(\tau)b_+\big]\mathrm{d}\tau \\
&\quad - \frac14\big(1+\tfrac{p''(1)}{2}\big)\int_0^t S_-(t-\tau)\big[\partial_{x_1}(\chi_-^2)(\tau)b_-\big]\mathrm{d}\tau \\
&= S_+(t)\big[(\sigma_{*0}^+ - \chi_{0+})b_+\big] + S_-(t)\big[(\sigma_{*0}^- - \chi_{0-})b_-\big] \\
&\quad + S_{\rm rig}(t)\big[(\sigma_{*0}^{\rm rig} - \chi_{*0}^{\rm rig})b_{\rm rig}\big] + \sum_{k=1}^5 J_k,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 = &-\frac14\big(1+\tfrac{p''(1)}{2}\big)\int_0^t S_+(t-\tau)\big[\partial_{x_1}(\chi_-^2)(\tau)b_+\big]\mathrm{d}\tau \\
&+\frac14\big(1+\tfrac{p''(1)}{2}\big)\int_0^t S_-(t-\tau)\big[\partial_{x_1}(\chi_+^2)(\tau)b_-\big]\mathrm{d}\tau,
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \frac{1}{2}\big(1-\tfrac{p''(1)}{2}\big)\int_0^t S_+(t-\tau)\big[\partial_{x_1}(\chi_+\chi_-)(\tau)b_+\big]\mathrm{d}\tau \\
&\quad -\frac{1}{2}\big(1-\tfrac{p''(1)}{2}\big)\int_0^t S_-(t-\tau)\big[\partial_{x_1}(\chi_+\chi_-)(\tau)b_-\big]\mathrm{d}\tau, \\
J_3 &= -\frac{1}{4}\int_0^t S_+(t-\tau)\big[\partial_{x_1}\big(\zeta_-(\eta_+-\eta_-)\big)(\tau)b_+\big]\mathrm{d}\tau \\
&\quad -\frac{p''(1)}{8}\int_0^t S_+(t-\tau)\big[\partial_{x_1}\big(\zeta_+(\eta_++\eta_-)\big)(\tau)b_+\big]\mathrm{d}\tau \\
&\quad +\frac{1}{4}\int_0^t S_-(t-\tau)\big[\partial_{x_1}\big(\zeta_-(\eta_+-\eta_-)\big)(\tau)b_-\big]\mathrm{d}\tau \\
&\quad +\frac{p''(1)}{8}\int_0^t S_-(t-\tau)\big[\partial_{x_1}\big(\zeta_+(\eta_++\eta_-)\big)(\tau)b_-\big]\mathrm{d}\tau, \\
J_4 &= -\frac{1}{2}\int_0^t S_{\mathrm{rig}}(t-\tau)\big[\partial_{x_1}(\chi_+\chi_{\mathrm{rig}}-\chi_-\chi_{\mathrm{rig}})b_{\mathrm{rig}}\big]\mathrm{d}\tau, \\
J_5 &= -\frac{1}{2}\int_0^t S_{\mathrm{rig}}(t-\tau)\big[\partial_{x_1}\{(\eta_++\chi_+-\eta_--\chi_-)\eta_{\mathrm{rig}}\}b_{\mathrm{rig}}\big]\mathrm{d}\tau \\
&\quad -\frac{1}{2}\int_0^t S_{\mathrm{rig}}(t-\tau)\big[\partial_{x_1}\{(\eta_+-\eta_-)\chi_{\mathrm{rig}}\}b_{\mathrm{rig}}\big]\mathrm{d}\tau.
\end{aligned}
$$

Since

$$\int_{\mathbb{R}}(\sigma_{*0}^{\pm}-\chi_{\pm 0})\mathrm{d}x_1=0,$$

we have

$$\big\|\partial_{x_1}^k S_\pm(t)\big[(\sigma_{*0}^{\pm}-\chi_{\pm 0})b_\pm\big]\big\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1_{1/2}}$$

for $k=0,1$. Similarly,

$$\big\|\partial_{x_1}^k S_{\mathrm{rig}}(t)\big[(\sigma_{*0}^{\mathrm{rig}}-\chi_{\mathrm{rig}0})b_{\mathrm{rig}}\big]\big\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1_{1/2}}$$

for $k=0,1$. We also have

$$\|\partial_{x_1}^k S(t)\tilde{\sigma}_{*0}\|_2 \le Ce^{-C_1 t}\|u_0\|_{H^2}, \quad k=0,1,$$

for some positive constant C.

As for J_1, we apply the estimates for the interaction between two diffusion waves in different fields by T.-P. Liu [24] (see also [17, Lemma 4.2]) to obtain

$$\|\partial_{x_1}^k J_1(t)\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2.$$

We next estimate J_2. For $l\ge 0$, we have

$$\|\partial_{x_1}^l(\chi_+\chi_-)(t)\|_1 \le Ce^{-d_3 t}\|u_0\|_{H^2\cap L^1}^2 \tag{2.9.8}$$

with some positive constant d_3, which was obtained in [19]. It then follows that

$$\begin{aligned}\|\partial_{x_1}^k J_2(t)\|_2 &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\big(\|\chi_+\chi_-(\tau)\|_1 + \|\partial_{x_1}^{k+1}(\chi_+\chi_-)(\tau)\|_2\big)\mathrm{d}\tau\\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} e^{-d_3\tau}\mathrm{d}\tau\|u_0\|^2_{H^2\cap L^1}\\ &\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|u_0\|^2_{H^2\cap L^1}.\end{aligned}$$

We next estimate J_3. We have

$$\begin{aligned}&\left\|\partial_{x_1}^k\int_0^t S_+(t-\tau)\big[\partial_{x_1}\big(\zeta_-(\eta_+-\eta_-)\big)(\tau)b_+\big]\mathrm{d}\tau\right\|_2\\ &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|\zeta_-(\eta_+-\eta_-)(\tau)\|_1\mathrm{d}\tau\\ &\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}}\|\partial_{x_1}^k\big(\zeta_-(\eta_+-\eta_-)\big)(\tau)\|_2\mathrm{d}\tau\\ &\quad + C\int_0^t e^{-d_2(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_{x_1}^k\big(\zeta_-(\eta_+-\eta_-)\big)(\tau)\|_2\mathrm{d}\tau\\ &=: J_{31}+J_{32}+J_{33}\end{aligned}$$

for $k=0,1$. Since

$$\|\partial_{x_1}^k\chi_\pm(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}, \tag{2.9.9}$$

we see that $\|\zeta_-(\tau)\|_2 \le C(1+\tau)^{-\frac{1}{4}}\|u_0\|_{H^2\cap L^1}$. This, together with

$$\|\zeta_-(\eta_+-\eta_-)\|_1 \le \|\zeta_-\|_2\|\eta_+-\eta_-\|_2 \le C\|\zeta_-\|_2\|V\|_2,$$

implies that

$$\begin{aligned}J_{31} &= CY(t)\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\|u_0\|_{H^2\cap L^1}\\ &\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t).\end{aligned}$$

Similarly,

$$\begin{aligned}J_{32} &= CY(t)\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}\mathrm{d}\tau\|u_0\|_{H^2\cap L^1}\\ &\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t),\end{aligned}$$

and

$$\begin{aligned}J_{33} &= CY(t)\int_0^t e^{-d_2(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1-\frac{k}{2}}\mathrm{d}\tau\|u_0\|_{H^2\cap L^1}\\ &\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t).\end{aligned}$$

We thus obtain

$$\left\| \partial_{x_1}^k \int_0^t S_+(t-\tau)\big[\partial_{x_1}\big(\zeta_-(\eta_+ - \eta_-)\big)(\tau) b_+\big] \mathrm{d}\tau \right\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2\cap L^1} Y(t).$$

The other terms in J_3 can be estimated in a similar manner. We thus obtain

$$\|\partial_{x_1}^k J_3(t)\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2\cap L^1} Y(t).$$

To estimate J_4, as in the case of J_2, we make use of the estimate

$$\|\partial_x^l (\chi_\pm \chi_{\mathrm{rig}})(t)\|_1 \le C e^{-d_3 t} \|u_0\|_{H^2\cap L^1}^2 \tag{2.9.10}$$

for $l \ge 0$, which was obtained in [19]. We can obtain the estimate for J_4 in a similar manner to that for J_2. The estimate for J_5 can be obtained in a similar manner to that for J_3. It then follows that if $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, we have

$$\|\partial_x^k V(t)\|_2 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2\cap L^1_{1/2}} \tag{2.9.11}$$

for $k = 0, 1$. This completes the proof. □

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