Iwasawa theory for representations of knot groups

丹下, 稜斗

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Ryoto TANGE

Graduate School of Mathematics
Kyushu University

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Abstract

In the spirit of arithmetic topology, we study a topological analogue of Iwasawa theory for representations of Galois groups. First, we present a generalization of the Fox formula for twisted Alexander invariants associated to representations of knot groups over rings of S-integers of F, where S is a finite set of finite primes of a number field F. Second, we study the twisted knot module for the universal deformation of an SL_2-representation of a knot group, and introduce an associated L-function, which may be seen as an analogue of the algebraic p-adic L-function associated to the Selmer module for the universal deformation of a Galois representation.

This thesis is based on [Tan18] and [KMTT18].

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Notation

For an integral domain A, we denote by $\mathrm{char}(A)$ the characteristic of A and we denote by $Q(A)$ the field of fractions of A.

For a, b in a commutative ring A, $a \doteq b$ means $a = bu$ for some unit $u \in A^{\times}$.

For a field F, we denote by $\overline{F}$ a fixed algebraic closure of F.

For positive integers m, n, and for a finite set of finite primes $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ of a number field F, $m =_S n$ means $m = np_1^{e_1} \cdots p_r^{e_r}$ for some integers $e_1, \dots, e_r \in \mathbb{Z}$, where $(p_i) = \mathfrak{p}_i \cap \mathbb{Z}$. Note that $m =_S n$ if and only if $|m|_p = |n|_p$ for all $(p) \notin \{(p_1), \dots, (p_r)\}$, where $|\cdot|_p$ is the p-adic absolute value normalized by $|p|_p = p^{-1}$.

For a local ring R, we denote by $\mathfrak{m}_R$ the maximal ideal of R.

Contents

0 Introduction

0.1 Historical background

In the late of 1950's, Iwasawa introduced the theory of $\mathbb{Z}_p$-extensions, p being a prime number, and studied the asymptotic formula for p-ideal class groups in a $\mathbb{Z}_p$-extension. Starting from 1980's, several generalizations of Iwasawa theory has been considered and has been recently intensively developed in number theory. One of the main generalization of Iwasawa theory was initiated by Mazur, Kato, Greenberg, Coates, etc., which considers the p-adic Galois representations. Meanwhile, motivated by the study of Hida on p-adic Hecke algebras, Mazur introduced the deformation of p-adic Galois representations. As an application, Kato and Ochiai showed that the dual Selmer module of the universal ordinary modular GL_2-deformation is finitely generated and torsion over the p-adic Hecke algebra.

On the other hand in knot theory, in the early 1950's, Fox proved that the order of the first homology group of coverings branched along the knot complement may be computed by using the Alexander polynomial of a knot. In the middle of 1990's, using representations of knot groups, Lin and Wada independently introduced the generalization of the Alexander polynomial, which is called the twisted Alexander invariants. This is one of the reasons why studies of knots from the viewpoint of representations of knot groups, such as character varieties, are still intriguing.

Comparing these two theories, one might notice that there are some mysterious similarities. The first mathematician who had an insight into the analogies between knots and primes was Mazur ([Maz64]). After a long silence, the dictionaries between topology of 3-manifolds and arithmetic of number rings were started to be investigated systematically by Kapranov, Reznikov, and Morishita in the latter half of 1990s. Shortly thereafter, significant progresses have been made and this area - now called *arithmetic topology* - is becoming a driving force to obtain parallel results between 3-dimensional topology and number theory (cf. [Kap95], [Mor10], [Mor12], [Rez97], [Rez00]). In particular, it is known that there is an analogy between Iwasawa theory and Alexander theory, where the 1st homology group corresponds to the ideal class group. One of the most interesting problems in arithmetic topology is to study a topological analogue of Iwasawa theory for Galois group representations. In this thesis, we focus on this open problem illustrated with some concrete examples.

0.2 Basic analogies

It has been known that there are intriguing analogies between knot theory and number theory (cf. [Mor12]). Here is the dictionary of basic analogies.

Number theory	**Knot theory**
prime ideal (p) $\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$	knot K $S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$
p-adic integers $\mathrm{Spec}(\mathbb{Z}_p)$	tubular neighborhood V_K
p-adic numbers $\mathrm{Spec}(\mathbb{Q}_p) = \mathrm{Spec}(\mathbb{Z}_p) \setminus \mathrm{Spec}(\mathbb{F}_p)$	boundary torus $\partial V_K = V_K \setminus K$
decomposition group over p $D_p = \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}_p))$	peripheral group of K $D_K = \pi_1(\partial V_K)$
monodromy $[\gamma, \mathbb{Q}_p]$	meridian m
Frobenius $[p, \mathbb{Q}_p]$	longitude l
prime complement $X_p = \mathrm{Spec}(\mathbb{Z}) \setminus (p)$	knot complement $E_K = S^3 \setminus \mathrm{int}(V_K)$
prime group $G_p = \pi_1^{\text{ét}}(X_p)$	knot group $G_K = \pi_1(E_K)$

Based on the above guiding principals, there are close parallels between Alexander-Fox theory and Iwasawa theory ([Maz64], [Mor12, Ch.9–13]). From the viewpoint of deformations of group representations ([Maz89]), they are concerned with abelian deformations of representations of knot and Galois groups and the associated topological and arithmetic invariants such as the Alexander and Iwasawa polynomials, respectively.

Iwasawa theory	**Alexander theory**
$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty / X_p)$ X_p^∞: cyclotomic p-cover of X_p	$G_K^{\mathrm{ab}} = \mathrm{Gal}(E_K^\infty / E_K)$ E_K^∞: infinite cyclic cover of E_K
1-dim. universal representation $\chi_p : G_p \to \mathbb{Z}_p[[G_p^{\mathrm{ab}}]]^\times \risingdotseq \mathbb{Z}_p[[T]]^\times$	1-dim. universal representation $\chi_K : G_K \to \mathbb{Z}[G_K^{\mathrm{ab}}]^\times = \mathbb{Z}[t^{\pm 1}]^\times$
Iwasawa module $H_1(X_p^\infty, \mathbb{Z}_p) = H_1(X_p, \chi_p)$	knot module $H_1(E_K^\infty, \mathbb{Z}) = H_1(E_K, \chi_K)$
Iwasawa polynomial (algebraic p-adic L-function) $\Delta_0(H_1(X_p, \chi_p))$	Alexander polynomial $\Delta_0(H_1(E_K, \chi_K))$

Here $\Delta_0(H_1(X_K, \chi_K))$ (resp. $\Delta_0(H_1(X_p, \chi_p))$) means the greatest common divisor of generators of the initial Fitting ideal of $H_1(X_K, \chi_K)$ (resp. $H_1(X_p, \chi_p)$) over the ring $\mathbb{Z}[t^{\pm 1}]$ (resp. $\mathbb{Z}_p[[T]]$).

0.3 The results of this thesis

Here are the contents of this thesis. In Chapter 1, we present a generalization of the Fox formula for twisted Alexander invariants associated to irreducible representations of knot groups over rings of S-integers of F, where S is a finite set of finite primes of a number field F. The Fox formula is one of the important results in knot theory which expresses the order of the first integral homology group of the n-fold cyclic cover branched over a knot in terms of the Alexander polynomial. Let K be a knot in the 3-sphere S^3, M_n the n-fold cyclic cover branched over K, and $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ the Alexander polynomial of K. If $\Delta_K(t)$ and $t^n - 1$ have no common roots in $\mathbb{C}$, then the Fox formula is given by

$$\#H_1(M_n; \mathbb{Z}) = \left| \prod_{i=1}^{n} \Delta_K(\zeta_n^i) \right|,$$

where $\#G$ denotes the order of a group G and ζ_n is a primitive n-th root of unity ([Fox56]). As an application, it follows immediately from the Fox formula that the asymptotic growth of integral homology groups holds:

$$\lim_{n \to \infty} \frac{1}{n} \log(\#H_1(M_n, \mathbb{Z})) = \log \mathbb{M}(\Delta_K(t)),$$

where $\mathbb{M}(\Delta_K(t))$ is the Mahler measure of $\Delta_K(t)$ ([Mah62]). We remark that this asymptotic growth may be seen as an analogue of the Iwasawa asymptotic formula for p-ideal class groups in a $\mathbb{Z}_p$-extension, p being a prime number ([Iwa59]). The analogies with number theory are the motivation of our study ([Mor12]).

Recently, a generalization of the Alexander polynomial, called a *twisted Alexander invariant*, which was introduced by Lin ([Lin01]) and Wada ([Wad94]), is playing an important role in knot theory ([FV11]). It is known ([KL99], [SW09]) that the twisted Alexander invariant relates to the twisted homology group of a knot complement associated to a $\mathrm{GL}_m(R)$-representation of a knot group G_K, where R is a Noetherian UFD.

The purpose of Chapter 1 is to consider a generalization of the Fox formula for twisted Alexander invariants. The following Theorem is our main result (see Notation for symbols $\doteq$ and $=_S$. In particular, when S is the empty set, $=_S$ means the usual equality):

Theorem A *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ be a representation. Assume that $H_0(X_\infty; V_\rho) = 0$, where X_∞ is the infinite cyclic cover, and V_ρ is a representation space. Let $\Delta_{K,\rho}(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ be the twisted Alexander invariant of K associated to ρ. If $\Delta_{K,\rho}(t) \neq 0$, and $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$, then we have*

$$\#H_1(X_n; V_\rho) =_S \left| \mathrm{N}_{F/\mathbb{Q}} \left(\prod_{i=1}^{n} \Delta_{K,\rho}(\zeta_n^i) \right) \right|,$$

where X_n is the n-fold cyclic cover of the knot complement, $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$ is a norm map, and ζ_n is a primitive n-th root of unity.

In particular, when ρ is irreducible over $\mathcal{O}_{F,S}$, namely if the composite of ρ with the natural map $\mathrm{GL}_m(\mathcal{O}_{F,S}) \to \mathrm{GL}_m(\mathbb{F}(\mathfrak{p}))$ is irreducible over the residue field $\mathbb{F}(\mathfrak{p}) = (\mathcal{O}_{F,S})_{\mathfrak{p}}/\mathfrak{p}(\mathcal{O}_{F,S})_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_{F,S}$, where $(\mathcal{O}_{F,S})_{\mathfrak{p}}$ is the localization of $\mathcal{O}_{F,S}$ at $\mathfrak{p}$, we have $H_0(X_\infty; V_\rho) = 0$ and so the conditions of Theorem hold.

The case when $n = 1$ for a certain integral representation of a link group was already proved by Silver–Williams ([SW09]) using the dynamical method ([Sch95]). Our proof is elementary and uses some number theoretic arguments. The idea of the proof is mainly generalizations of [Web79] and [Cro63]. By the Mostow rigidity theorem, any holonomy representations attached to the complete hyperbolic structure on the interior of the knot complement can be lifted to a representation over S-integers. Therefore, our result is applicable to those holonomy representations. As an application, it follows immediately from Theorem that the asymptotic growth formulas of twisted homology groups hold:

$$\lim_{n\in\mathbb{N};\gcd(n,p)=1} \frac{1}{n} \log |\# H_1(X_n, V_\rho)|_p = \log \mathbb{M}_p(\mathrm{N}_{F/\mathbb{Q}}(\Delta_{K,\rho}(t))),$$

where p is a prime number which is not lying below S, $|\cdot|_p$ is the p-adic absolute value, and $\mathbb{M}_p$ is the Ueki p-adic Mahler measure of $\Delta_{K,\rho}(t)$ ([Uek17]). In particular, when S is the empty set, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\# H_1(X_n, V_\rho)) = \log \mathbb{M}(\mathrm{N}_{F/\mathbb{Q}}(\Delta_{K,\rho}(t))).$$

We remark that these asymptotic growth formulas may be seen as analogues of the asymptotic formula for the Tate–Shafarevich groups or the Selmer groups of p-adic Galois representations in a $\mathbb{Z}_p$-extension, which was firstly studied by Mazur ([Maz72]).

Number theory	**Knot theory**
Iwasawa asymptotic formula for the p-ideal class group	asymptotic formula for the knot module
asymptotic formula for the Tate-Shafarevich/Selmer group	asymptotic formula for the twisted knot module

In Chapter 2, we study the twisted knot module for the universal deformation of an SL_2-representation of a knot group, and introduce an associated L-function, which may be seen as an analogue of the algebraic p-adic L-function associated to the Selmer module for the universal deformation of a Galois representation. It has been known that there are intriguing analogies between knot theory and number theory (cf. [Mor12]). In particular, it may be noteworthy that there are close parallels between Alexander-Fox theory and Iwasawa theory ([Maz64], [Mor12, Ch.9–13]). From the viewpoint of deformations of group representations ([Maz89]), they are concerned with abelian deformations of representations of knot and Galois groups and the associated topological and arithmetic invariants such as the Alexander and Iwasawa polynomials, respectively. In [Maz00], Mazur proposed a number of problems in pursuing these analogies for non-abelian deformations of higher dimensional representations. To carry out Mazur's perspective, as a first step, we developed a deformation theory for SL_2-representations of knot groups in [MTTU17]. In this paper, we continue our study and introduce a certain L-function associated to the twisted knot module for the universal deformation of a knot group representation, which may be seen as an analogue of the algebraic p-adic L-function associated to the Selmer module for the universal deformation of a Galois representation ([Gre94]).

Let K be a knot in the 3-sphere S^3 and $G_K := \pi_1(S^3 \setminus K)$ the knot group. Fix a field k whose characteristic is not 2 and a complete discrete valuation ring $\mathcal{O}$ whose residue field is k. Let $\overline{\rho} : G_K \to \mathrm{SL}_2(k)$ be a given absolutely irreducible representation. It was shown in [MTTU17] that there exists the universal deformation $\boldsymbol{\rho} : G_K \to \mathrm{SL}_2(\boldsymbol{R}_{\overline{\rho}})$ of $\overline{\rho}$, where $\boldsymbol{R}_{\overline{\rho}}$ is a complete local $\mathcal{O}$-algebra whose residue field is k. Assume that $\boldsymbol{R}_{\overline{\rho}}$ is a Noetherian factorial domain. In this paper, we study the twisted knot module $H_1(\boldsymbol{\rho}) := H_1(S^3 \setminus K; \boldsymbol{\rho})$ with coefficients in the universal deformation $\boldsymbol{\rho}$, and introduce the associated L-function $L_K(\boldsymbol{\rho})$ defined on the universal deformation space $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$ as $\Delta_0(H_1(\boldsymbol{\rho}))$, the greatest common divisor of generators of the initial Fitting ideal of $H_1(\boldsymbol{\rho})$ over the universal deformation ring $\boldsymbol{R}_{\overline{\rho}}$. In terms of our $H_1(\boldsymbol{\rho})$ and $L_K(\boldsymbol{\rho})$, we then formulate the problems proposed by Mazur (questions 1 and 2 of [Maz00, page 440]) as follows.

(1) Is $H_1(\boldsymbol{\rho})$ finitely generated and torsion as an $\boldsymbol{R}_{\overline{\rho}}$-module ?

(2) Investigate the order of the zeroes of $L_K(\boldsymbol{\rho})$ at prime divisors of $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$.

The corresponding problems of (1) and (2) in the arithmetic counterpart, say $(1)_{\mathrm{arith}}$ and $(2)_{\mathrm{arith}}$ respectively, are important issues in number theory (cf. questions 1 and 2 of [Maz00, page 454]). In fact, $(1)_{\mathrm{arith}}$ is a part of the so-called main conjecture for p-adic deformations of a Galois representation. For the cyclotomic deformation of a Dirichlet character, the affirmative answer to $(1)_{\mathrm{arith}}$ is a basic result in Iwasawa theory ([Iwa73]), which asserts that the classical Iwasawa module is finitely generated and torsion over the Iwasawa algebra. For the Hida deformation (universal ordinary modular GL_2-deformation) ([Hid86a], [Hid86b]), the affirmative answer to $(1)_{\mathrm{arith}}$ has been shown by Kato and Ochiai ([Kat04], [Och01], [Och06]), which asserts that the dual Selmer module of the Hida deformation is finitely generated and torsion over the universal ordinary modular deformation ring (p-adic Hecke algebra). The problem $(2)_{\mathrm{arith}}$ remains an interesting problem to be explored and it is related to deep arithmetic issues (cf. question 3 of [Maz00, page 454], and Ribet's theorem on Herbrand's theorem for example [MW84]).

So it may be interesting to study the above problems (1) and (2) in the knot theoretic situation. Our results concerning these are as follows. For (1), we give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated and torsion over $\boldsymbol{R}_{\overline{\rho}}$ under certain conditions using a twisted Alexander invariant of K (cf. Theorem 2.3.2.2). For

(2), we give some concrete examples for 2-bridge knots K such that $L_K(\boldsymbol{\rho})$ has only one zero of order 0 or 2 (cf. Subsection 2.4.3).

Number theory	**Knot theory**
Selmer module	twisted knot module
L-function associated to Selmer module for universal deformations of representation of Galois group	L-function associated to twisted knot module for universal deformations of representation of knot group

1 Fox formulas for twisted homology groups associated to representations of knot groups

In this Chapter, we present a generalization of the Fox formula for twisted Alexander invariants associated to irreducible representations of knot groups over rings of S-integers of F, where S is a finite set of finite primes of a number field F. As an application, we give the asymptotic growth of twisted homology groups.

The contents of this Chapter are organized as follows. In Sections 1.1-1.3, we review some basic materials in knot theory. In Section 1.4, we review an algebraic tool, which is called a resultant. In Section 1.5, we prepare some number theoretic lemmas, which will be used in the sequel. In Section 1.6, we prove the main result. In Section 1.7, we give an application which determines the asymptotic growth of the twisted homology groups using the Mahler measure. In the last Section, we give some concrete examples.

1.1 Twisted chain complexes

In this Section, we define twisted chain complexes, which are based on [KL99] and [SW09].

Let K be a knot in the 3-sphere S^3 and let $X := S^3 \setminus K$ denote the knot complement of K. Let $G_K := \pi_1(X)$ denote the knot group of K.

Let $\widetilde{X} \to X$ be the universal cover of X. Let R be a Noetherian UFD. Fix a cellular chain complex $C_*(X) = C_*(X; R)$ of X with coefficients in R and let $C_*(\widetilde{X}) = C_*(\widetilde{X}; R)$ be the cellular chain complex of $\widetilde{X}$ induced from $C_*(X)$. Since G_K acts on $\widetilde{X} \to X$ as the covering transformation group, $C_*(\widetilde{X})$ is a free left $R[G_K]$-module.

Let $\rho : G_K \to \mathrm{GL}_m(R)$ be a representation and $V_\rho = R^m$ the representation space of ρ. Note that G_K acts naturally from right on V_ρ. We define the *ρ-twisted chain complex $C_*(X; V_\rho)$ of X* by

$$C_*(X; V_\rho) := V_\rho \otimes_{R[G_K]} C_*(\widetilde{X}).$$

We define the *i-th ρ-twisted knot module $H_i(X; V_\rho)$* by

$$H_i(X; V_\rho) := H_i(C_*(X; V_\rho)).$$

Next, we define the chain complex associated to ρ for the cyclic cover of X. Let $Y \to X$ be the infinite cyclic cover (resp. the n-fold cyclic cover) of X. Let $\alpha : G_K \to G_K^{\mathrm{ab}} \simeq \mathbb{Z} = \langle t \rangle$ (resp. $\alpha_n : G_K \to \mathbb{Z}/n\mathbb{Z} = \langle t_n \rangle$) be the abelianization homomorphism. Then $V_\rho[t^{\pm 1}] = R[t^{\pm 1}] \otimes_R V_\rho$ (resp. $V_\rho[\langle t_n \rangle] = R[\langle t_n \rangle] \otimes_R V_\rho$) is a right $R[G_K]$-module via

(1.1.1)

$$(r(t) \otimes \boldsymbol{v}).g := r(t) \cdot \alpha(g) \otimes \boldsymbol{v}\rho(g), \ \text{(resp. } (r(t_n) \otimes \boldsymbol{v}).g := r(t_n) \cdot \alpha_n(g) \otimes \boldsymbol{v}\rho(g), \text{)}$$

where $r(t) \in R[t^{\pm 1}]$, $r(t_n) \in R[\langle t_n \rangle]$, $\boldsymbol{v} \in V_\rho$ and $g \in G_K$.

We define the ρ*-twisted chain complex* $C_*(Y; V_\rho)$ *of* Y by

$$C_*(Y; V_\rho) := V_\rho[t^{\pm 1}] \otimes_{R[G_K]} C_*(\widetilde{X}),\ (\text{resp. } C_*(Y; V_\rho) := V_\rho[\langle t_n \rangle] \otimes_{R[G_K]} C_*(\widetilde{X}),)$$

and the ρ-twisted homology $H_i(Y; V_\rho)$ by

$$H_i(Y; V_\rho) := H_i(C_*(Y; V_\rho)).$$

When Y is the infinite cyclic cover $X_\infty \to X$, we call $H_i(X_\infty; V_\rho)$ the *i-th ρ-twisted Alexander module*. Note that the covering transformation $t : X_\infty \to X_\infty$ induces the action of $\langle t \rangle \simeq \mathbb{Z}$ on $C_*(X_\infty; V_\rho)$ defined by the following:

$$\begin{array}{rcl} t_\# : C_*(X_\infty; V_\rho) & \to & C_*(X_\infty; V_\rho); \\ (r(t) \otimes \boldsymbol{v}) \otimes z & \mapsto & (r(t) \cdot t \otimes \boldsymbol{v}) \otimes z. \end{array}$$

Since G_K is a finitely presented group and V_ρ is a free R-module of finite rank, $C_*(X_\infty; V_\rho)$ is a finitely generated $R[t^{\pm 1}]$-module, and since $R[t^{\pm 1}]$ is a Noetherian ring, $H_1(X_\infty; V_\rho)$ is a finitely generated $R[t^{\pm 1}]$-module.

1.2 Twisted Alexander invariants

The twisted Alexander invariant $\Delta_{K,\rho}(t)$ is defined as follows ([GKM06], [Wad94]). We keep the same notations as before. Recall that R is a Noetherian UFD. Note that the knot group G_K has a Wirtinger presentation

$$\langle g_1,\ \ldots, g_q \mid r_1 = \cdots = r_{q-1} = 1 \rangle. \tag{1.2.1}$$

Let F_0 be the free group on the words $g_1, \ldots, g_q$ and $\pi : R[F_0] \to R[G_K]$ denote the natural surjective homomorphism of group rings. We write the same g_i for the image of g_i in G_K. We denote by the same α for the R-algebra homomorphism $R[G_K] \to R[t^{\pm 1}]$, which is induced by α. Let us denote by the same ρ for the R-algebra homomorphism $R[G_K] \to \mathrm{M}_m(R)$, which is induced by ρ. Then we have the tensor product representation

$$\rho \otimes \alpha : R[G_K] \longrightarrow \mathrm{M}_m(R[t^{\pm 1}]),$$

and the R-algebra homomorphism

$$\Phi := (\rho \otimes \alpha) \circ \pi : R[F_0] \longrightarrow \mathrm{M}_m(R[t^{\pm 1}]).$$

Let us consider the (big) $(q-1) \times q$ matrix P, whose (i, j) component is defined by the $m \times m$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial g_j}\right),$$

where $\frac{\partial}{\partial g_j} : R[F_0] \to R[F_0]$ denotes the Fox derivative ([Fox53]) over R extended from $\mathbb{Z}$. For $1 \leq j \leq q$, let P_j denote the matrix obtained by deleting the j-th column from P and we regard P_j as an $(q-1)m \times (q-1)m$ matrix over $R[t^{\pm 1}]$.

It is known ([Wad94]) that there is k ($1 \le k \le q$) such that $\det(\Phi(g_k - 1)) \neq 0$ and that the ratio

$$\Delta_{K,\rho}(t) := \frac{\det(P_k)}{\det \Phi(g_k - 1)} \in Q(R)(t)$$

is independent of such k's. We call $\Delta_{K,\rho}(t)$ the *twisted Alexander invariant* of K associated to ρ.

Similarly to the classical case, there is a relation between the twisted Alexander invariant and the order ideals of ρ-twisted Alexander module. Let us recall the definition of the order ideal. Let M be a finitely generated R-module. Let us take a finite presentation of M over R:

$$R^m \xrightarrow{\partial} R^n \longrightarrow M \longrightarrow 0,$$

where ∂ is an $m \times n$ matrix over R. We define the *order ideal* $E_0(M)$ of M to be the ideal generated by n-minors of ∂. The order ideal depends only on M and independent of the choice of a presentation. Let $\Delta_0(M)$ be the greatest common divisor of generators of $E_0(M)$, which is well-defined up to multiplication by a unit of R.

Proposition 1.2.0.1 ([KL99], [SW09]) *Let $X_\infty \to X$ be the infinite cyclic cover of X. For any representation $\rho : G_K \to \mathrm{GL}_m(R)$, we have*

$$\Delta_{K,\rho}(t) = \frac{\Delta_0(H_1(X_\infty; V_\rho))}{\Delta_0(H_0(X_\infty; V_\rho))}.$$

In particular, when $\rho : G_K \to \mathrm{GL}_m(R)$ is irreducible over a commutative UFD R, we have the following Corollary. We say that $\rho : G_K \to \mathrm{GL}_m(R)$ is *irreducible* over a commutative UFD R if the composite of ρ with the natural map $\mathrm{GL}_m(R) \to \mathrm{GL}_m(\mathbb{F}(\mathfrak{p}))$ is irreducible over the residue field $\mathbb{F}(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of R, where $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$.

Corollary 1.2.0.2 *Let $\rho : G_K \to \mathrm{GL}_m(A)$ be an irreducible representation over a PID A. Then we have*

$$\Delta_0(H_0(X_\infty; V_\rho)) \doteq 1 \in A[t^{\pm 1}],$$

and hence

$$\Delta_{K,\rho}(t) \doteq \Delta_0(H_1(X_\infty; V_\rho)).$$

In particular, $\Delta_{K,\rho}(t)$ is a Laurent polynomial over A.

In order to prove Corollary 1.2.0.2, we use the following Lemma.

Lemma 1.2.0.3 ([DFJ12, Lemma 2.5]) *Let $\rho_\mathbb{F} : G_K \to \mathrm{GL}_m(\mathbb{F})$ be an irreducible representation over a field $\mathbb{F}$. Then we have*

$$\Delta_0(H_0(X_\infty; V_{\rho_\mathbb{F}})) \doteq 1 \in \mathbb{F}[t^{\pm 1}].$$

First, it is easy to see that $\Delta_0(H_0(X_\infty; V_\rho)) \bmod (\mathfrak{p}) = \Delta_0(H_0(X_\infty; V_{\rho_{\mathbb{F}(\mathfrak{p})}}))$ for any prime ideal $\mathfrak{p}$ of A, where $(\mathfrak{p})$ denotes the ideal of $A[t^{\pm 1}]$ generated by $\mathfrak{p}$. Suppose $\Delta_0(H_0(X_\infty; V_\rho))$ is not a unit in $A[t^{\pm 1}]$. Then there exists a maximal ideal $\mathfrak{m}$ of $A[t^{\pm 1}]$ containing $\Delta_0(H_0(X_\infty; V_\rho))$. Since A is a PID, $\mathfrak{m}$ is generated by a prime element q of A and $f(t) \in A[t^{\pm 1}]$ such that $f(t) \bmod (\mathfrak{q})$ is irreducible over $\mathbb{F}(\mathfrak{q})$, where $\mathfrak{q}$ is a prime ideal of A generated by q. Hence, there exist $g(t)$, $h(t) \in A[t^{\pm 1}]$ such that $\Delta_0(H_0(X_\infty; V_\rho)) = qg(t) + f(t)h(t)$. Therefore, we have

$$\Delta_0(H_0(X_\infty; V_{\rho_{\mathbb{F}(\mathfrak{q})}})) = \Delta_0(H_0(X_\infty; V_\rho)) \bmod (\mathfrak{q}) = f(t)h(t) \bmod (\mathfrak{q}).$$

On the other hand, by Lemma 1.2.0.3, we have $\Delta_0(H_0(X_\infty; V_{\rho_{\mathbb{F}(\mathfrak{q})}})) \doteq 1 \in \mathbb{F}(\mathfrak{q})[t^{\pm 1}]$. This contradicts that $f(t) \bmod (\mathfrak{q})$ is irreducible over $\mathbb{F}(\mathfrak{q})$. $\square$

1.3 Twisted Wang sequences

In this Section, we formulate an analogue of Wang sequence for twisted homology groups. We keep the same notations as before. Recall that R is a Noetherian UFD. Note that for the n-fold cyclic cover $X_n \to X$, the covering $p_n : X_\infty \to X_n$ induces the following map:

$$\begin{array}{rcl} p_{n\#} : C_*(X_\infty; V_\rho) & \to & C_*(X_n; V_\rho); \\ (r(t) \otimes \boldsymbol{v}) \otimes z & \mapsto & (r(t_n) \otimes \boldsymbol{v}) \otimes z. \end{array}$$

Lemma 1.3.0.1 *Let $\rho : G_K \to \mathrm{GL}_m(R)$ be a representation. Then we have an exact sequence*

$$0 \to C_*(X_\infty; V_\rho) \overset{t_\#^n - 1}{\to} C_*(X_\infty; V_\rho) \overset{p_{n\#}}{\to} C_*(X_n; V_\rho) \to 0.$$

proof By tensoring V_ρ over R with the exact sequence

$$0 \to R[t^{\pm 1}] \overset{t^n - 1}{\to} R[t^{\pm 1}] \to R[\langle t_n \rangle] \to 0,$$

we have the exact sequence

$$0 \to V_\rho[t^{\pm 1}] \overset{t^n - 1}{\to} V_\rho[t^{\pm 1}] \to V_\rho[\langle t_n \rangle] \to 0. \tag{1.3.1}$$

By the right $R[G_K]$-module structures of $V_\rho[t^{\pm 1}]$ and $V_\rho[\langle t_n \rangle]$ defined in (1.1.1), we can see (1.3.1) is an exact sequence of the right $R[G_K]$-modules. Since $C_*(\widetilde{X})$ is a free left $R[G_K]$-module of finite rank, by tensoring $C_*(\widetilde{X})$ from the right over $R[G_K]$ with (1.3.1), we obtain the assertion. $\square$

Note that Lemma 1.3.0.1 induces the following long exact sequence, which is called the *twisted Wang sequence*:

$$\cdots \to H_1(X_\infty; V_\rho) \overset{t_*^n - 1}{\to} H_1(X_\infty; V_\rho) \overset{p_{n*}}{\to} H_1(X_n; V_\rho) \overset{\partial_{1*}}{\to} H_0(X_\infty; V_\rho) \to \cdots .$$

Hence, when $H_0(X_\infty; V_\rho) = 0$, we have the following relation between $H_1(X_n; V_\rho)$ and $H_1(X_\infty; V_\rho)$:

Proposition 1.3.0.2 *Let $\rho : G_K \to \mathrm{GL}_m(R)$ be a representation. Assume that $H_0(X_\infty; V_\rho) = 0$. Then we have*

$$H_1(X_n; V_\rho) \simeq H_1(X_\infty; V_\rho)/(t^n - 1)H_1(X_\infty; V_\rho).$$

As we discussed in Section 1.2, when $\rho : G_K \to \mathrm{GL}_m(A)$ is an irreducible representation over a PID A, we have $\Delta_0(H_0(X_\infty; V_\rho)) \doteq 1$, and so by [Hil12, Theorem 3.1], we have $H_0(X_\infty; V_\rho) = 0$. Therefore, we have the following Corollary.

Corollary 1.3.0.3 *Let $\rho : G_K \to \mathrm{GL}_m(A)$ be an irreducible representation over a PID A. Then we have*

$$H_1(X_n; V_\rho) \simeq H_1(X_\infty; V_\rho)/(t^n - 1)H_1(X_\infty; V_\rho).$$

1.4 Resultants

In this Section, we recall the definition of the resultant and state the relation between the resultant and the order ideal.

Let A be an integral domain. Consider the following two non-zero polynomials in $A[t]$ factor in $\overline{Q(A)}$:

$$f = f(t) = a\prod_{i=1}^{m}(t - \xi_i),\ \ g = g(t) = b\prod_{j=1}^{n}(t - \zeta_j).$$

Then we define the *resultant* $\mathrm{Res}(f, g)$ for polynomials f and g by

$$\mathrm{Res}(f, g) := a^m b^n \prod_{i,j}(\xi_i - \zeta_j) = a^m \prod_i g(\xi_i).$$

For polynomials f, $g \in A[t]$, it is easy to see that $\mathrm{Res}(f, g) = 0$ if and only if f and g have a common root in $\overline{Q(A)}$. In addition, the resultant is symmetric up to the sign and is multiplicative ([Lan02, IV.8, IX.3]):

$$\mathrm{Res}(f, g) = (-1)^{\deg(f\cdot g)}\mathrm{Res}(g, f),$$

$$\mathrm{Res}(f, g \cdot h) = \mathrm{Res}(f, g) \cdot \mathrm{Res}(f, h),$$

where f, g, $h \in A[t^{\pm1}]$. The resultant can be generalized for Laurent polynomials since it is insensitive to units ut^i with $u \in A^\times$ and $i \in \mathbb{Z}$.

The following Lemmas claim that when R is a Noetherian UFD, the greatest common divisor of generators of the order ideal of finitely generated torsion $R[t^{\pm1}]$-module is computable by using the resultant. Note that we say the Laurent polynomial in $R[t^{\pm1}]$ is *doubly monic* if the highest and lowest coefficients are units in R.

Lemma 1.4.0.1 ([Hil12, Theorem 3.13]) *Let R be a Noetherian UFD and N a finitely generated torsion $R[t^{\pm1}]$-module. Let $f(t) \in R[t^{\pm1}]$ be a doubly monic polynomial. Then $N/f(t)N$ is a torsion R-module if and only if $\Delta_0(N)|_{t=\zeta} \neq 0$ for all non-zero roots ζ of $f(t)$ in $\overline{Q(R)}$.*

Lemma 1.4.0.2 ([Hil12, Corollary 3.13.1]) *Let R be a Noetherian UFD and let $f(t),\ g(t) \in R[t^{\pm 1}]$ having no common roots in $\overline{Q(R)}$. If $f(t)$ or $g(t)$ is doubly monic, then*

$$\Delta_0(R[t^{\pm 1}]/(f(t), g(t))) \doteq \mathrm{Res}(f(t), g(t)).$$

1.5 Number theoretic lemmas

Let us recall some facts and notions in number theory, which we shall use in the following. We refer to [MR03, 6.1] and [Ono90, 2.8-2.10] for these materials. Let F be a number field. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be a finite set of finite primes of F and $\mathcal{O}_{F,S}$ the ring of S-integers, namely

$$\mathcal{O}_{F,S} := \{a \in F \mid v_{\mathfrak{p}}(a) \geq 0 \text{ for all } \mathfrak{p} \in S_F \setminus S\},$$

where $v_{\mathfrak{p}}$ is an additive valuation of F at $\mathfrak{p}$, and S_F is the set of all finite primes of F. It is known that $\mathcal{O}_{F,S}$ is always Noetherian and if we take a sufficiently large finite set T of finite primes of F containing S, then $\mathcal{O}_{F,T}$ turns out to be a PID. Therefore, we may always take S so that $\mathcal{O}_{F,S}$ is a PID. For $a \in F$, we define the *norm* $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$ *of* a by

$$\mathrm{N}_{F/\mathbb{Q}}(a) := \prod_{\sigma} \sigma(a),$$

where σ runs over all embeddings of F in $\mathbb{C}$. For an integral ideal I of $\mathcal{O}_F$, we define the *norm* $\mathrm{N}I$ by $\#\mathcal{O}_F/I$. It is extended multiplicatively for a fractional ideal of $\mathcal{O}_F$. For $a \in F^{\times}$, we have $|\mathrm{N}_{F/\mathbb{Q}}(a)| = \mathrm{N}(a)$, where $(a) = a\mathcal{O}_F$ is the principal ideal generated by a. We say that a fractional ideal J of $\mathcal{O}_F$ is *prime to* S if any prime factor of J is not in S.

Lemma 1.5.0.1 *For $a \in \mathcal{O}_{F,S} \setminus \{0\}$, we have*

$$\#\mathcal{O}_{F,S}/a\mathcal{O}_{F,S} =_S |\mathrm{N}_{F/\mathbb{Q}}(a)|.$$

proof For $a \in \mathcal{O}_{F,S} \setminus \{0\}$, we can write $(a) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \mathfrak{a}$, where $e_i \in \mathbb{Z}$ and $\mathfrak{a}$ is an integral ideal prime to S. Then we have

$$\begin{array}{rcl} |\mathrm{N}_{F/\mathbb{Q}}(a)| & = & \mathrm{N}(a) \\ & = & \mathrm{N}\mathfrak{p}_1^{e_1} \cdots \mathrm{N}\mathfrak{p}_r^{e_r} \mathrm{N}\mathfrak{a} \\ & =_S & \mathrm{N}\mathfrak{a} \\ & = & \#\mathcal{O}_F/\mathfrak{a} \\ & =_S & \#\mathcal{O}_{F,S}/a\mathcal{O}_{F,S}. \quad \square \end{array}$$

Using these norms and Lemma 1.5.0.1, we have the following Lemmas. The proof of Lemma 1.5.0.3 is a generalization of [Web79].

Lemma 1.5.0.2 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $f(t),\ g(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ and* *assume that either $f(t)$ or $g(t)$ is doubly monic. Then $\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t))$ is a torsion $\mathcal{O}_{F,S}$-module if and only if $f(t)$ and $g(t)$ have no common roots in $\overline{F}$. When $f(t)$ and $g(t)$ have no common roots in $\overline{F}$, we have*

$$\#\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)) =_S |\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(f(t), g(t)))|.$$

proof Suppose $f(t)$ is doubly monic. Since $\Delta_0(\mathcal{O}_{F,S}[t^{\pm 1}]/(g(t))) = g(t)$, and $g(\zeta) \neq 0$ for all non-zero roots ζ of $f(t)$ in $\overline{F}$, by Lemma 1.4.0.1, $\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t))$ is a torsion $\mathcal{O}_{F,S}$-module. A similar argument holds when we suppose $g(t)$ is doubly monic.

Next, let us investigate the order of $\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t))$. Since we suppose that $\mathcal{O}_{F,S}$ is a PID, by regarding $\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t))$ as an $\mathcal{O}_{F,S}$-module, using the structure theorem for modules over PIDs, we have

$$\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)) \simeq \mathcal{O}_{F,S}/a_1\mathcal{O}_{F,S} \oplus \cdots \oplus \mathcal{O}_{F,S}/a_s\mathcal{O}_{F,S}, \tag{1.5.1}$$

where $a_1, \ldots, a_s \in \mathcal{O}_{F,S}$. So we have

$$\Delta_0(\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t))) = u \cdot a_1 \cdot \cdots \cdot a_s,$$

where $u \in \mathcal{O}_{F,S}^{\times}$, and hence we have

$$\left|\mathrm{N}_{F/\mathbb{Q}}\left(\prod_{i=1}^{s} a_i\right)\right| =_S \left|\mathrm{N}_{F/\mathbb{Q}}\big(\Delta_0(\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)))\big)\right|. \tag{1.5.2}$$

Therefore, we have

$$\begin{aligned}
\#\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)) = & \prod_{i=1}^{s} \#\mathcal{O}_{F,S}/a_i\mathcal{O}_{F,S} && \text{(by (1.5.1))} \\
&=_S \prod_{i=1}^{s} |\mathrm{N}_{F/\mathbb{Q}}(a_i)| && \text{(by Lemma 1.5.0.1)} \\
&=_S \left|\mathrm{N}_{F/\mathbb{Q}}\big(\Delta_0(\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)))\big)\right| && \text{(by (1.5.2))}
\end{aligned}$$

and hence by Lemma 1.4.0.2, we have

$$\#\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)) =_S |\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(f(t), g(t)))|. \quad \square$$

Lemma 1.5.0.3 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let N be a finitely generated $\mathcal{O}_{F,S}[t^{\pm 1}]$-module with $\mathrm{rank}_F(N \otimes_{\mathcal{O}_{F,S}} F) < \infty$ and having no submodule of finite length. Then $N/(t^n - 1)N$ is a torsion $\mathcal{O}_{F,S}$-module if and only if $\Delta_0(N)$ and $t^n - 1$ have no common roots in $\overline{F}$. When $\Delta_0(N)$ and $t^n - 1$ have no common roots in $\overline{F}$, we have*

$$\#N/(t^n - 1)N =_S |\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(t^n - 1,\ \Delta_0(N)))|.$$

proof By the right exactness of the tensor product, we have

$$(N/(t^n-1)N)\otimes_{\mathcal{O}_{F,S}} F \simeq (N\otimes_{\mathcal{O}_{F,S}} F)/(t^n-1)(N\otimes_{\mathcal{O}_{F,S}} F),$$

and hence $N/(t^n-1)N$ is a torsion $\mathcal{O}_{F,S}$-module if and only if

$$(N\otimes_{\mathcal{O}_{F,S}} F)/(t^n-1)(N\otimes_{\mathcal{O}_{F,S}} F)=0,$$

which is the same as the multiplication map $t^n-1 : N\otimes_{\mathcal{O}_{F,S}} F \to N\otimes_{\mathcal{O}_{F,S}} F$ is surjective. Since $\mathrm{rank}_F(N\otimes_{\mathcal{O}_{F,S}} F)$ is finite, the surjectivity of the map $t^n-1 : N\otimes_{\mathcal{O}_{F,S}} F \to N\otimes_{\mathcal{O}_{F,S}} F$ is equivalent to its injectivity. Suppose $N\otimes_{\mathcal{O}_{F,S}} F \simeq \oplus_{i=1}^{s} F[t^{\pm 1}]/(h_i(t))$, where $h_i(t)\in \mathcal{O}_{F,S}[t^{\pm 1}]$. Since N has no non-zero $\mathcal{O}_{F,S}$-torsion, the map $t^n-1 : N\otimes_{\mathcal{O}_{F,S}} F \to N\otimes_{\mathcal{O}_{F,S}} F$ is injective if and only if $\prod_{i=1}^{s} h_i(t)=\Delta_0(N)$ and t^n-1 have no common roots in $\overline{F}$.

Next, let us investigate the order of $N/(t^n-1)N$ when $\Delta_0(N)$ and t^n-1 have no common roots in $\overline{F}$. Let M be an $\mathcal{O}_{F,S}[t^{\pm 1}]$-submodule of N such that $M\simeq \oplus_{i=1}^{s}\mathcal{O}_{F,S}[t^{\pm 1}]/(h_i(t))$. Then we have $M\otimes_{\mathcal{O}_{F,S}} F = N\otimes_{\mathcal{O}_{F,S}} F \simeq \oplus_{i=1}^{s} F[t^{\pm 1}]/(h_i(t))$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Ker}(\Phi_2) \arrow[r] \arrow[d] & \mathrm{Ker}(\Phi_3) \arrow[d] \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-6ex]\tikztotarget.west) |- (\tikztotarget)}] & \\
M \arrow[r] \arrow[d, "\Phi_1"] & N \arrow[r] \arrow[d, "\Phi_2"] & N/M \arrow[r] \arrow[d, "\Phi_3"] & 0 \\
M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N/M \arrow[r] \arrow[d] & 0 \\
\mathrm{Coker}(\Phi_1) \arrow[r] & \mathrm{Coker}(\Phi_2) \arrow[r] & \mathrm{Coker}(\Phi_3) \arrow[r] & 0,
\end{tikzcd}$$

where the homomorphisms Φ_1, Φ_2 and Φ_3 are the multiplication maps t^n-1. Since the map $t^n-1 : N\otimes_{\mathcal{O}_{F,S}} F \to N\otimes_{\mathcal{O}_{F,S}} F$ is injective and N has no submodule of finite length, we have $\mathrm{Ker}(\Phi_2)=0$. Hence, we have

$$0\to \mathrm{Ker}(\Phi_3)\to \mathrm{Coker}(\Phi_1)\to \mathrm{Coker}(\Phi_2)\to \mathrm{Coker}(\Phi_3)\to 0. \tag{1.5.3}$$

Since $N\otimes_{\mathcal{O}_{F,S}} F = M\otimes_{\mathcal{O}_{F,S}} F$, by the right exactness of the tensor product, we have

$$(N/M)\otimes_{\mathcal{O}_{F,S}} F \simeq (N\otimes_{\mathcal{O}_{F,S}} F)/(M\otimes_{\mathcal{O}_{F,S}} F)=0,$$

and hence N/M is torsion $\mathcal{O}_{F,S}$-module. Therefore, we have

$$\#\mathrm{Ker}(\Phi_3)=\#\mathrm{Coker}(\Phi_3). \tag{1.5.4}$$

On the other hand, since $M\simeq \oplus_{i=1}^{s}\mathcal{O}_{F,S}[t^{\pm 1}]/(h_i(t))$, by Lemma 1.5.0.2, we have

$$\begin{aligned}
\#\mathrm{Coker}(\Phi_1) &= \#M/(t^n-1)M \\
&=_S \left|\mathrm{N}_{F/\mathbb{Q}}\left(\mathrm{Res}\left(t^n-1,\ \prod_{i=1}^{s} h_i(t)\right)\right)\right| \\
&=_S \left|\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(t^n-1,\ \Delta_0(N)))\right|.
\end{aligned}$$

Therefore by (1.5.3) and (1.5.4), we have

$$\#N/(t^n-1)N = \#\mathrm{Coker}(\Phi_2) = \#\mathrm{Coker}(\Phi_1) =_S \left|\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(t^n-1,\ \Delta_0(N)))\right|.$$

□

1.6 Fox formulas for twisted Alexander invariants

In this Section, we formulate an analogue of the Fox formula for twisted Alexander invariants associated to $\mathrm{GL}_m(\mathcal{O}_{F,S})$-representations of knot groups under the assumption $H_0(X_\infty; V_\rho) = 0$. The proof is a generalization of [Cro63].

Theorem 1.6.0.1 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ be a representation. Assume that $H_0(X_\infty; V_\rho) = 0$ and let $\Delta_{K,\rho}(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ be the twisted Alexander invariant of K associated to ρ. If $\Delta_{K,\rho}(t) \neq 0$, and $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$, then we have*

$$\#H_1(X_n; V_\rho) =_S \left|\mathrm{N}_{F/\mathbb{Q}}\left(\prod_{i=1}^{n} \Delta_{K,\rho}(\zeta_n^i)\right)\right|,$$

where ζ_n is a primitive n-th root of unity.

proof We will verify that all conditions of Lemma 1.5.0.3 hold for $N := H_1(X_\infty; V_\rho)$.

(N has no submodule of finite length) The knot complement X is homotopy equivalent to a finite 2-complex W with one 0-cell, q 1-cells, and $q-1$ 2-cells [Lic97, Chapter 11]. Since $H_0(X_\infty; V_\rho) = 0$, the cellular chain complex gives a resolution

$$C_2(W_\infty; V_\rho) \to C_1(W_\infty; V_\rho)' \to N = H_1(X_\infty; V_\rho) \to 0,$$

where $C_2(W_\infty; V_\rho)$ and $C_1(W_\infty; V_\rho)'$ are free $\mathcal{O}_{F,S}[t^{\pm 1}]$-modules of rank $q-1$. Since N is a torsion R-module, this is a short free resolution, so we may apply [Hil12, Theorem 3.8]. Hence, N has no submodule of finite length.

($\mathrm{rank}_F(N \otimes_{\mathcal{O}_{F,S}} F) < \infty$) Since N is a finitely generated R-torsion module, $N \otimes_R F[t^{\pm 1}]$ is a finitely generated $F[t^{\pm 1}]$-torsion module. Hence, it has finite rank as an F-vector space, by the structure theorem for modules over PIDs.

Hence, by applying Corollary 1.2.0.2, Proposition 1.3.0.2 and Lemma 1.5.0.3, we have

$$\begin{aligned}
\#H_1(X_n;V_\rho) &= \#H_1(X_\infty;V_\rho)/(t^n-1)H_1(X_\infty;V_\rho) \\
&=_S \left|\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(t^n-1,\ \Delta_{K,\rho}(t)))\right| \\
&=_S \left|\mathrm{N}_{F/\mathbb{Q}}\left(\prod_{i=1}^{n}\Delta_{K,\rho}(\zeta_n^i)\right)\right|. \qquad \square
\end{aligned}$$

In particular, when $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ is irreducible, we have the following Corollary.

Corollary 1.6.0.2 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ be an irreducible representation, and let $\Delta_{K,\rho}(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ be the twisted Alexander invariant of K associated to ρ. If $\Delta_{K,\rho}(t)$ and t^n-1 have no common roots in $\overline{F}$, then we have*

$$\#H_1(X_n;V_\rho) =_S \left|\mathrm{N}_{F/\mathbb{Q}}\left(\prod_{i=1}^{n}\Delta_{K,\rho}(\zeta_n^i)\right)\right|,$$

where ζ_n is a primitive n-th root of unity.

1.7 Asymptotic growth of twisted homology groups

We keep the notation as in Section 1.6. Let $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Assume that $H_0(X_\infty;V_\rho) = 0$, and $\Delta_{K,\rho}(t)$ and t^n-1 have no common roots in $\overline{F}$ for all positive integers n. Set $\overline{\Delta_{K,\rho}}(t) := \mathrm{N}_{F/\mathbb{Q}}(\Delta_{K,\rho}(t)) \in \mathbb{Z}_{S_0}[t^{\pm 1}]$, where $S_0 = \{\mathfrak{p}_1 \cap \mathbb{Z}, \dots, \mathfrak{p}_r \cap \mathbb{Z}\}$, and $\mathbb{Z}_{S_0}$ is the ring of S_0-integers of $\mathbb{Q}$. Then by Theorem 1.6.0.1, we have

$$\#H_1(X_n, V_\rho) =_S \prod_{i=1}^{n}\left|\overline{\Delta_{K,\rho}}(\zeta_n^i)\right|. \tag{1.7.1}$$

As we remarked in Notation, when $(p) \notin S_0$, (1.7.1) is equivalent to

$$|\#H_1(X_n, V_\rho)|_p = \prod_{i=1}^{n}\left|\overline{\Delta_{K,\rho}}(\zeta_n^i)\right|_p. \tag{1.7.2}$$

Here, $|\cdot|_p$ is the p-adic absolute value on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$, where $\mathbb{C}_p$ is the p-adic completion of an algebraic closure of the p-adic number field.

For $f(t) \in \mathbb{Z}_{S_0}[t^{\pm 1}]$, we define the *Mahler measure* $\mathbb{M}(f(t))$ of $f(t)$ ([Mah62]) by

$$\mathbb{M}(f(t)) := \exp\left(\int_0^1 \log|f(e^{2\pi\sqrt{-1}x})|dx\right).$$

If $f(t)$ factors as $f(t) = at^e \prod_{j=1}^{d}(t - \xi_j)$ in $\mathbb{C}$, then by Jensen's formula, we have $\mathbb{M}(f(t)) = a \prod_{j=1}^{d} \max(|\xi_j|, 1)$. For $f(t) \in \mathbb{C}_p[t^{\pm 1}] \setminus \{0\}$ with no root on $|z|_p = 1$, we define the *Ueki p-adic Mahler measure* $\mathbb{M}_p(f(t))$ of $f(t)$ ([Uek17]) by

$$\mathbb{M}_p(f(t)) := \exp\left(\lim_{n\in\mathbb{N};\gcd(n,p)=1} \frac{1}{n} \sum_{i=1}^{n} \log |f(e^{\frac{2\pi\sqrt{-1}}{n}i})|_p \right).$$

Now we are ready to state our Theorem.

Theorem 1.7.0.1 *When $(p) \notin S_0$, we have*

$$\lim_{n\in\mathbb{N};\gcd(n,p)=1} \frac{1}{n} \log |\#H_1(X_n, V_\rho)|_p = \log \mathbb{M}_p(\overline{\Delta_{K,\rho}}(t)).$$

In particular, when S is the empty set, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\#H_1(X_n, V_\rho)) = \log \mathbb{M}(\overline{\Delta_{K,\rho}}(t)).$$

proof By (1.7.2), we have

$$\begin{aligned} \lim_{n\in\mathbb{N};\gcd(n,p)=1} \frac{1}{n} \log |\#H_1(X_n, V_\rho)|_p &= \lim_{n\in\mathbb{N};\gcd(n,p)=1} \frac{1}{n} \sum_{i=1}^{n} \log |\overline{\Delta_{K,\rho}}(e^{\frac{2\pi\sqrt{-1}}{n}i})|_p \\ &= \log \mathbb{M}_p(\overline{\Delta_{K,\rho}}(t)). \end{aligned}$$

In particular, when S is the empty set, by (1.7.1), we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log(\#H_1(X_n, V_\rho)) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |\overline{\Delta_{K,\rho}}(e^{\frac{2\pi\sqrt{-1}}{n}i})| \\ &= \int_0^1 \log |(\overline{\Delta_{K,\rho}}(e^{2\pi\sqrt{-1}x})| dx \\ &= \log \mathbb{M}(\overline{\Delta_{K,\rho}}(t)). \end{aligned}$$

$\square$

Remark 1.7.0.2 Theorem 1.7.0.1 is a generalization of the result by González-Acuña and Short ([GAnS91]), and by Noguchi ([Nog07]), where the case ρ is a trivial representation over $\mathbb{Z}$ was studied.

1.8 Examples

In this Section, we discuss some concrete examples, where K will be a 2-bridge knot and ρ will be an irreducible $\mathrm{SL}_2(\mathcal{O}_F)$-representation of a knot group G_K with $\mathcal{O}_F$ a PID. The computation is based on Mathematica.

Example 1.8.0.1 Let K be the trefoil knot, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle.$$

Consider the following representation:

$$\rho : G_K \to \mathrm{SL}_2(\mathbb{Z})\,; \quad \rho(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(g_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then we have $\Delta_{K,\rho}(t) = t^2 + 1$ and hence by applying Corollary 1.6.0.2, we have the following:

$$\#H_1(X_n; V_\rho) = \begin{cases} 2 \text{ (when } n \equiv 1,\ 3 \bmod 4), \\ 4 \text{ (when } n \equiv 2 \bmod 4). \end{cases}$$

Example 1.8.0.2 Let K be the knot 5_1, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 g_2 g_1 = g_2 g_1 g_2 g_1 g_2 \rangle.$$

Consider the following representation:

$$\rho : G_K \to \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]\right); \quad \rho(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(g_2) = \begin{pmatrix} 1 & 0 \\ -\frac{3+\sqrt{5}}{2} & 1 \end{pmatrix},$$

where $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ is the ring of integers of $\mathbb{Q}(\sqrt{5})$. Then we have $\Delta_{K,\rho}(t) = (t^2+1)(t^4 - \frac{1+\sqrt{5}}{2}t^2 + 1)$ and hence by applying Corollary 1.6.0.2, we have the following:

$$\begin{aligned} \#H_1(X; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(3 - \sqrt{5}) = 4, \\ \#H_1(X_2; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(2(7 - 3\sqrt{5})) = 16, \\ \#H_1(X_3; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(-2) = 4, \\ \#H_1(X_5; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(8) = 64. \end{aligned}$$

Example 1.8.0.3 Let K be the figure-eight knot, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2^{-1} g_1^{-1} g_2 g_1 = g_2 g_1 g_2^{-1} g_1^{-1} g_2 \rangle.$$

Consider the following representation:

$$\rho : G_K \to \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right); \quad \rho(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(g_2) = \begin{pmatrix} 1 & 0 \\ \frac{1+\sqrt{-3}}{2} & 1 \end{pmatrix},$$

where $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is the ring of integers of $\mathbb{Q}(\sqrt{-3})$. Then we have $\Delta_{K,\rho}(t) = \frac{1}{t^2}(t^2 - 4t + 1) \doteq t^2 - 4t + 1$ and hence by applying Corollary 1.6.0.2, we have the following:

$$\begin{aligned} \#H_1(X; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-2) = 4, \\ \#H_1(X_2; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-12) = 144, \\ \#H_1(X_3; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-50) = 2500, \\ \#H_1(X_4; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-192) = 36864. \end{aligned}$$

Since $\overline{\Delta_{K,\rho}}(t) = (t^2 - 4t + 1)^2$, by Theorem 1.7.0.1, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\#H_1(X_n, V_\rho)) = 2\log(2+\sqrt{3}).$$

2 L-functions for twisted homology groups associated to deformations of representations of knot groups

In this Chapter, we study the twisted knot module for the universal deformation of an SL_2-representation of a knot group, and introduce an associated L-function, which may be seen as an analogue of the algebraic p-adic L-function associated to the Selmer module for the universal deformation of a Galois representation. We then investigate two problems proposed by Mazur: Firstly we show the torsion property of the twisted knot module over the universal deformation ring under certain conditions. Secondly we compute the L-function by some concrete examples for 2-bridge knots.

Here are contents of this Chapter. In Section 2.1, we recall the deformation theory for SL_2-representations of a group, which was developed in [MTTU17]. In Section 2.2, we show the relation between the universal deformation ring and the character scheme over $\mathbb{Z}$ of SL_2-representations. In Section 2.3, we study the twisted knot module with coefficients in the universal deformation of an SL_2-representation of a knot group, and introduce an associated L-function. In Section 2.4, we discuss some examples for some 2-bridge knots, for which we study Mazur's problems.

2.1 The universal deformation

In this section, we present a summary of the deformation theory for SL_2-representations of a group, which was developed in [MTTU17]. We also discuss the obstruction to the deformation problem for a group representation. Throughout this section, let G denote a group.

2.1.1 Pseudo-representations and their deformations

Let A be a commutative ring with identity. A map $T : G \to A$ is called a *pseudo*-SL_2*-representation* over A if the following four conditions are satisfied:

(P1) $T(e) = 2$ ($e :=$ the identity element of G),
(P2) $T(g_1g_2) = T(g_2g_1)$ for any $g_1, g_2 \in G$,
(P3) $T(g_1)T(g_2)T(g_3) + T(g_1g_2g_3) + T(g_1g_3g_2) - T(g_1g_2)T(g_3) - T(g_2g_3)T(g_1) - T(g_1g_3)T(g_2) = 0$ for any $g_1, g_2, g_3 \in G$,
(P4) $T(g)^2 - T(g^2) = 2$ for any $g \in G$.

Note that the conditions (P1) $\sim$ (P3) are nothing but Taylor's conditions for a pseudo-representation of degree 2 ([Tay91]) and that (P4) is the condition for determinant 1. In the following, we say simply a *pseudo-representation* for a pseudo-SL_2-representation. The trace $\mathrm{tr}(\rho)$ of a representation $\rho : G \to \mathrm{SL}_2(A)$ satisfies the conditions (P1) $\sim$ (P4) ([Pro76, Theorem 4.3]), and, conversely, a pseudo-SL_2-representation is shown to be obtained as the trace of a representation under certain conditions (See Theorem 2.1.2.1 below).

Let k be a perfect field and let $\mathcal{O}$ be a complete discrete valuation ring with the residue field $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$. There is a unique subgroup V of $\mathcal{O}^\times$ such that

$k^\times \simeq V$ and $\mathcal{O}^\times = V \times (1+\mathfrak{m}_\mathcal{O})$. The composition map $\lambda : k^\times \simeq V \hookrightarrow \mathcal{O}^\times$ is called the *Teichmüller lift* which satisfies $\lambda(\alpha) \bmod \mathfrak{m}_\mathcal{O} = \alpha$ for $\alpha \in k$. It is extended to $\lambda : k \hookrightarrow \mathcal{O}$ by $\lambda(0) := 0$. Let $\mathfrak{CL}_\mathcal{O}$ be the category of complete local $\mathcal{O}$-algebras with residue field k. A morphism in $\mathfrak{CL}_\mathcal{O}$ is an $\mathcal{O}$-algebra homomorphism inducing the identity on residue fields.

Let $\overline{T} : G \to k$ be a pseudo-representation over k. A couple (R, T) is called an SL_2*-deformation* of $\overline{T}$ if $R \in \mathfrak{CL}_\mathcal{O}$ and $T : G \to R$ is a pseudo-SL_2-representation over R such that $T \bmod \mathfrak{m}_R = \overline{T}$. In the following, we say simply a *deformation* of $\overline{T}$ for an SL_2-deformation. A deformation $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ of $\overline{T}$ is called a *universal deformation* if the following universal property is satisfied: "For any deformation (R, T) of $\overline{T}$ there exists a unique morphism $\psi : \boldsymbol{R}_{\overline{T}} \to R$ in $\mathfrak{CL}_\mathcal{O}$ such that $\psi \circ \boldsymbol{T} = T$." Namely the correspondence $\psi \mapsto \psi \circ \boldsymbol{T}$ gives the bijection

$$\mathrm{Hom}_{\mathfrak{CL}_\mathcal{O}}(\boldsymbol{R}_{\overline{T}}, R) \simeq \{(R, T) \,|\, \text{deformation of } \overline{T}\}.$$

By the universal property, a universal deformation $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ of $\overline{T}$ is unique (if it exists) up to isomorphism. The $\mathcal{O}$-algebra $\boldsymbol{R}_{\overline{T}}$ is called the *universal deformation ring* of $\overline{T}$.

Theorem 2.1.1.1 [MTTU17, Theorem 1.2.1]. *For a pseudo-representation $\overline{T} : G \to k$, there exists a universal deformation $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ of $\overline{T}$.*

We recall the construction of $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$. Let X_g denote a variable indexed by $g \in G$. Then the universal deformation ring $\boldsymbol{R}_{\overline{T}}$ is given by

$$\boldsymbol{R}_{\overline{T}} = \mathcal{O}[[X_g \ (g \in G)]]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal of the formal power series ring $\mathcal{O}[[X_g \ (g \in G)]]$ generated by the elements of following type: Setting $T_g := X_g + \lambda(\overline{T}(g))$,

(1) $T_e - 2 = X_e + \lambda(\overline{T}(e)) - 2$,
(2) $T_{g_1 g_2} - T_{g_2 g_1} = X_{g_1 g_2} - X_{g_2 g_1}$,
(3) $T_{g_1} T_{g_2} T_{g_3} + T_{g_1 g_2 g_3} + T_{g_1 g_3 g_2} - T_{g_1 g_2} T_{g_3} - T_{g_2 g_3} T_{g_1} - T_{g_1 g_3} T_{g_2}$,
(4) $T_g^2 - T_{g^2} - 2$,

for $g, g_1, g_2, g_3 \in G$. The universal deformation $\boldsymbol{T} : G \to \boldsymbol{R}_{\overline{T}}$ is given by

$$\boldsymbol{T}(g) := T_g \bmod \mathcal{I}.$$

Then, for any deformation (R, T) of $\overline{T}$, the morphism $\psi : \boldsymbol{R}_{\overline{T}} \to R$ in $\mathfrak{CL}_\mathcal{O}$ defined by $\psi(X_g) := T(g) - \lambda(\overline{T}(g))$ satisfies $\psi \circ \boldsymbol{T} = T$.

We note that $\boldsymbol{R}_{\overline{T}}$ constructed above is a complete Noetherian local $\mathcal{O}$-algebra if G is a finitely generated group.

2.1.2 Deformations of an SL_2-representation

We keep the same notations as in 1.1. In this subsection we assume that $\mathrm{char}(k) \neq 2$, so that 2 is invertible in $\mathcal{O}$ and hence in any $R \in \mathfrak{CL}_\mathcal{O}$. Let

$\overline{\rho} : G \to \mathrm{SL}_2(k)$ be a given representation. We call a couple (R, ρ) an SL_2-*deformation* of $\overline{\rho}$ if $R \in \mathfrak{CL}_{\mathcal{O}}$ and $\rho : G \to \mathrm{SL}_2(R)$ is a representation such that $\rho \bmod \mathfrak{m}_R = \overline{\rho}$. In the following, we say simply a *deformation* of $\overline{\rho}$ for an SL_2-deformation. A deformation $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ of $\overline{\rho}$ is called a *universal deformation* of $\overline{\rho}$ if the following universal property is satisfied: "For any deformation (R, ρ) of $\overline{\rho}$ there exists a unique morphism $\psi : \boldsymbol{R}_{\overline{\rho}} \to R$ in $\mathfrak{CL}_{\mathcal{O}}$ such that $\psi \circ \boldsymbol{\rho} \approx \rho$". Here two representations ρ_1, ρ_2 of degree 2 over a local ring A are said to be *strictly equivalent*, denoted by $\rho_1 \approx \rho_2$, if there is $\gamma \in I_2 + \mathrm{M}_2(\mathfrak{m}_A)$ such that $\rho_2(g) = \gamma^{-1}\rho_1(g)\gamma$ for all $g \in G$. Namely the correspondence $\psi \mapsto \psi \circ \boldsymbol{\rho}$ gives the bijection

$$\mathrm{Hom}_{\mathfrak{CL}_{\mathcal{O}}}(\boldsymbol{R}_{\overline{\rho}}, R) \simeq \{(R, \rho) \,|\, \text{deformation of } \overline{\rho}\}/\approx .$$

By the universal property, a universal deformation $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ of $\overline{\rho}$ is unique (if it exists) up to strict equivalence. The $\mathcal{O}$-algebra $\boldsymbol{R}_{\overline{\rho}}$ is called the *universal deformation ring* of $\overline{\rho}$.

A deformation (R, ρ) of $\overline{\rho}$ gives rise to a deformation $(R, \mathrm{tr}(\rho))$ of the pseudo-representation $\mathrm{tr}(\overline{\rho}) : G \to k$. Assume that $\overline{\rho}$ is absolutely irreducible, namely, the composite of $\overline{\rho}$ with an inclusion $\mathrm{SL}_2(k) \hookrightarrow \mathrm{SL}_2(\overline{k})$ is irreducible for an algebraic closure $\overline{k}$ of k. Then, by using theorems of Caryaol [Car94, Theorem 1] and Nyssen [Nys96, Theorem 1], it can be shown that this correspondence by the trace is indeed bijective. It is here that the condition $\mathrm{char}(k) \neq 2$ is used.

Theorem 2.1.2.1 [MTTU17, Theorem 2.1.2]. *Let $\overline{\rho} : G \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation and let $R \in \mathfrak{CL}_{\mathcal{O}}$. Then the correspondence $\rho \mapsto \mathrm{tr}(\rho)$ gives the following bijection:*

$$\{\rho : G \to \mathrm{SL}_2(R) \,|\, \textit{deformation of } \overline{\rho} \textit{ over } R\}/\approx \\ \longrightarrow \{T : G \to R \,|\, \textit{deformation of } \mathrm{tr}(\overline{\rho}) \textit{ over } R\}.$$

Now, by Theorem 2.1.1.1, there exists the universal deformation $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ of a pseudo-representation $\overline{T} = \mathrm{tr}(\overline{\rho})$. By Theorem 2.1.2.1, we have a deformation $\boldsymbol{\rho} : G \to \mathrm{SL}_2(\boldsymbol{R}_{\overline{T}})$ of $\overline{\rho}$ such that $\mathrm{tr}(\boldsymbol{\rho}) = \boldsymbol{T}$. Then we can verify that $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{\rho})$ satisfies the desired property of the universal deformation of $\overline{\rho}$.

Theorem 2.1.2.2 [MTTU17, Theorem 2.2.2]. *Let $\overline{\rho} : G \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation. Then there exists the universal deformation $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ of $\overline{\rho}$, where $\boldsymbol{R}_{\overline{\rho}}$ is given as $\boldsymbol{R}_{\overline{T}}$ for $\overline{T} := \mathrm{tr}(\overline{\rho})$ in Theorem 2.2.1.1.*

2.1.3 Obstructions

We recall basic facts on a presentation of a complete local $\mathcal{O}$-algebra and the obstruction for the deformation problem. For $R \in \mathfrak{CL}_{\mathcal{O}}$, we define the *relative cotangent space* $\mathfrak{t}^*_{R/\mathcal{O}}$ of R by the k-vector space $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}}R)$ and the *relative tangent space* $\mathfrak{t}_{R/\mathcal{O}}$ of R by the dual k-vector space of $\mathfrak{t}^*_{R/\mathcal{O}}$. We note that they are same as the cotangent and tangent spaces of $R/\mathfrak{m}_{\mathcal{O}}R = R \otimes_{\mathcal{O}} k$, respectively. The following lemma is a well-known fact which can be proved using Nakayama's lemma (cf. [Til96, Lemma 5.1]).

Lemma 2.1.3.1 *Let $d := \dim_k \mathfrak{t}_{R/\mathcal{O}}$ and assume $d < \infty$. Let $x_1, \ldots, x_d$ be elements of R whose images in $R \otimes_{\mathcal{O}} k$ form a system of parameters of the local k-algebra $R \otimes_{\mathcal{O}} k$. Then there is a surjective $\mathcal{O}$-algebra homomorphism*

$$\eta : \mathcal{O}[[X_1, \ldots, X_d]] \longrightarrow R$$

in $\mathfrak{CL}_{\mathcal{O}}$ such that $\eta(X_i) = x_i$ for $1 \leq i \leq d$.

Let $\mathrm{Ad}(\overline{\rho})$ be the k-vector space $\mathrm{sl}_2(k) := \{X \in \mathrm{M}_2(k) \,|\, \mathrm{tr}(X) = 0\}$ on which G acts by $g.X := \overline{\rho}(g)X\overline{\rho}(g)^{-1}$ for $g \in G$ and $X \in \mathrm{sl}_2(k)$. It is well-known ([Maz89, 1.6]) that there is a canonical isomorphism between the relative cotangent space $\mathfrak{t}^*_{\boldsymbol{R}_{\overline{\rho}}/\mathcal{O}}$ and the 1st group cohomology $H^1(G, \mathrm{Ad}(\overline{\rho}))$. We say that the deformation problem for $\overline{\rho}$ is *unobstructed* if the 2nd cohomology $H^2(G, \mathrm{Ad}(\overline{\rho}))$ vanishes. The following proposition is also well-known.

Proposition 2.1.3.2 [Maz89, 1.6, Proposition 2]). *Suppose that the deformation problem for $\overline{\rho}$ is unobstructed and $\dim_k H^1(G, \mathrm{Ad}(\overline{\rho})) < \infty$. Then the map η in Lemma 2.1.3.1 with $R = \boldsymbol{R}_{\overline{\rho}}$ is isomorphic*

$$\eta : \mathcal{O}[[X_1, \ldots, X_d]] \xrightarrow{\sim} \boldsymbol{R}_{\overline{\rho}}.$$

In this paper, we are interested in the case that G is a knot group, namely, the fundamental group of the complement of a knot in the 3-sphere S^3. We note that the deformation problem is not unobstructed in general for a knot group representation $\overline{\rho}$, as shown in Subsection 2.2.3.

2.2 Character schemes

In this section, we show the relation between the universal deformation ring in Section 1 and the character scheme of SL_2-representations.

In Subsection 2.2.1, we recall the constructions and some facts concerning the character scheme and the skein algebra over $\mathbb{Z}$, and then describe their relation. For the details on the materials, we consult [CS83], [LM85, Chapter 1], [Nak00] and [Sai96]. In Subsection 2.2.2, via the skein algebra, we show that the universal deformation ring may be seen as an infinitesimal deformation of the character algebra. In Subsection 2.2.3, we show that the deformation problem is not unobstructed for a knot group in general, using Thurston's result on the character variety.

2.2.1 Character schemes and skein algebras over $\mathbb{Z}$

Let G be a group. Let $\mathcal{F}$ be the functor from the category $\mathfrak{Com.Ring}$ of commutative rings with identity to the category of sets defined by

$$\mathcal{F}(A) := \{G \to \mathrm{SL}_2(A) \,|\, \text{representation}\}$$

for $A \in \mathfrak{Com.Ring}$. The functor $\mathcal{F}$ is represented by a pair $(\mathcal{A}(G), \sigma_G)$, where $\mathcal{A}(G) \in \mathfrak{Com.Ring}$ and $\sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G))$ is a representation, which satisfies the following universal property: "For any $A \in \mathfrak{Com.Ring}$ and a representation $\rho : G \to \mathrm{SL}_2(A)$, there is a unique morphism $\psi : \mathcal{A}(G) \to A$ in $\mathfrak{Com.Ring}$ such that $\psi \circ \sigma_G = \rho$." Thus the correspondence $\psi \mapsto \psi \circ \sigma_G$ gives the bijection

$$\mathrm{Hom}_{\mathfrak{Com.Ring}}(\mathcal{A}(G), A) \simeq \{G \to \mathrm{SL}_2(A) \,|\, \text{representation}\}.$$

By the universal property, the pair $(\mathcal{A}(G), \sigma_G)$ is unique (if exists) up to isomorphism. We call $\mathcal{A}(G)$ the *universal representation algebra* over $\mathbb{Z}$ and $\sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G))$ the *universal representation*. The pair $(\mathcal{A}(G), \sigma_G)$ is constructed as follows. Let $X(g) = (X_{ij}(g))_{1 \le i,j \le 2}$ be 2×2 matrix whose entries $X_{ij}(g)$'s are variables indexed by $1 \le i, j \le 2$ and $g \in G$. Then $\mathcal{A}(G)$ is given as

$$\mathcal{A}(G) = \mathbb{Z}[X_{ij}(g)\ (1 \le i, j \le 2; g \in G)]/J,$$

where J is the ideal of the polynomial ring $\mathbb{Z}[X_{ij}(g)\ (1 \le i, j \le 2; g \in G)]$ generated by

$$X_{ij}(e) - \delta_{ij},\ X_{ij}(g_1 g_2) - \sum_{k=1}^{2} X_{ik}(g_1) X_{kj}(g_2),\ \det(X(g)) - 1$$

for $1 \le i, j \le 2$ and $g \in G$, and the representation $\sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G))$ is given by

$$\sigma_G(g) := X(g) \bmod J \quad (g \in G).$$

We note that when G is presented by finitely many generators $g_1, \dots, g_n$ subject to the relations $r_l = 1$ $(l \in L)$, $\mathcal{A}(G)$ is given by

$$\mathcal{A}(G) = \mathbb{Z}[X_{ij}(g_h)\ (1 \le h \le n, 1 \le i, j \le 2)]/J'$$

for the ideal J' generated by

$$r_l(X(g_1), \dots, X(g_n))_{ij} - \delta_{ij},\ \ \det(X(g_h)) - 1,$$

where $1 \le i, j \le 2, l \in L, 1 \le h \le n$ and $r_l(X(g_1), \dots, X(g_n))_{ij}$ denotes the (i, j)-entry of $r_l(X(g_1), \dots, X(g_n))$. The universal representation σ_G is given by

$$\sigma_G(g_h) = X(g_h) \bmod J' \ \ (1 \le h \le n).$$

So $\mathcal{A}(G)$ is a finitely generated algebra over $\mathbb{Z}$ if G is a finitely generated group. We denote by $\mathcal{R}(G)$ the affine scheme $\mathrm{Spec}(\mathcal{A}(G))$ and call it the *representation scheme* of G over $\mathbb{Z}$. So A-rational points of $\mathcal{R}(G)$ corresponds bijectively to representations $G \to \mathrm{SL}_2(A)$ for any $A \in \mathfrak{Com.Ring}$. For $\mathfrak{p} \in \mathcal{R}(G)$, we let $\rho_{\mathfrak{p}} := \psi_{\mathfrak{p}} \circ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)/\mathfrak{p})$ be the corresponding representation, where $\psi_{\mathfrak{p}} : \mathcal{A}(G) \to \mathcal{A}(G)/\mathfrak{p}$ is the natural homomorphism.

We say that a representation $\rho : G \to \mathrm{SL}_2(A)$ with $A \in \mathfrak{Com.Ring}$ is *absolutely irreducible* if the composite of ρ with the natural map $\mathrm{SL}_2(A) \to \mathrm{SL}_2(k(\mathfrak{p}))$ is absolutely irreducible over the residue field $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$.

Let PGL_2 be the group scheme over $\mathbb{Z}$ whose coordinate ring $A(\mathrm{PGL}_2)$ is the subring of the graded ring $\mathbb{Z}[Y_{ij}\ (1 \le i,j \le 2)]_{\det(Y)}$ consisting of homogeneous elements of degree 0, where the degree of Y_{ij} is 1. The adjoint action $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \to \mathcal{R}(G)$ is given by the dual action

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2);\quad X_{ij}(g) \mapsto (YX(g)Y^{-1})_{ij} \otimes Y_{kl},$$

where $Y = (Y_{ij})_{1 \le i,j \le 2}$ and $(YX(g)Y^{-1})_{ij}$ denotes the (i,j)-entry of $YX(g)Y^{-1}$. Let $\mathcal{B}(G)$ be the invariant subalgebra of $\mathcal{A}(G)$ under this action of PGL_2

$$\begin{aligned} \mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \mid \mathrm{Ad}^*(x) = x \otimes 1\}. \end{aligned}$$

We call $\mathcal{B}(G)$ the *character algebra* of G over $\mathbb{Z}$. We denote by $\mathcal{X}(G)$ the affine scheme $\mathrm{Spec}(\mathcal{B}(G))$ and call it the *character scheme* of G over $\mathbb{Z}$. The natural inclusion

$$\iota : \mathcal{B}(G) \hookrightarrow \mathcal{A}(G)$$

induces a morphism of schemes

$$\iota^{\#} : \mathcal{R}(G) \longrightarrow \mathcal{X}(G).$$

We denote the image of $\mathfrak{p}(= \rho_{\mathfrak{p}}) \in \mathcal{R}(G)$ in $\mathcal{X}(G)$ under $\iota^{\#}$ by $[\mathfrak{p}](= [\rho_{\mathfrak{p}}])$.

According to [PS00, Definition 2.5] and [Sai96, 3.1], we define the *skein algebra* $\mathcal{C}(G)$ over $\mathbb{Z}$ by

$$\mathcal{C}(G) := \mathbb{Z}[t_g\ (g \in G)]/I,$$

where I is the ideal of the polynomial ring $\mathbb{Z}[t_g\ (g \in \Pi)]$ generated by the polynomials of the form

$$t_e - 2,\ \ t_{g_1} t_{g_2} - t_{g_1 g_2} - t_{g_1^{-1} g_2} \quad (g_1, g_2 \in G).$$

We note that $\mathcal{C}(G)$ is a finitely generated algebra over $\mathbb{Z}$ if G is a finitely generated group ([Sai96, 3.2]). We denote by $\mathcal{S}(G)$ the affine scheme $\mathrm{Spec}(\mathcal{C}(G))$ and call it the *skein scheme* of G over $\mathbb{Z}$.

Since $\mathrm{tr}(\sigma_G(g))$ $(g \in G)$ is invariant under the adjoint action of PGL_2 and we have the formula

$$\mathrm{tr}(\sigma_G(g_1))\mathrm{tr}(\sigma_G(g_2)) - \mathrm{tr}(\sigma_G(g_1 g_2)) - \mathrm{tr}(\sigma_G(g_1^{-1} g_2)) = 0$$

for $g_1, g_2 \in G$, which is derived from the Cayley-Hamilton relation, we obtain a $\mathbb{Z}$-algebra homomorphism

$$\tau : \mathcal{C}(G) \longrightarrow \mathcal{B}(G)$$

defined by

$$\tau(t_g) := \mathrm{tr}(\sigma_G(g)) \qquad (g \in G).$$

It induces the morphism of schemes

$$\tau^{\#} : \mathcal{X}(G) \longrightarrow \mathcal{S}(G).$$

We set

$$\varphi := \iota \circ \tau : \mathcal{C}(G) \longrightarrow \mathcal{A}(G)$$

so that we have the morphism of schemes

$$\varphi^{\#} = \tau^{\#} \circ \iota^{\#} : \mathcal{R}(G) \longrightarrow \mathcal{S}(G).$$

Now we define the *discriminant ideal* $\Delta(G)$ of $\mathcal{C}(G)$ by the ideal generated by the images of the elements in $\mathbb{Z}[t_g\ (g \in \pi)]$ of the form

$$\Delta(g_1, g_2) := t_{g_1 g_2 g_1^{-1} g_2^{-1}} - 2 = t_{g_1}^2 + t_{g_2}^2 + t_{g_1 g_2}^2 - t_{g_1} t_{g_2} t_{g_1 g_2} - 4 \ \ (g_1, g_2 \in G),$$

and the *discriminant subscheme* by $V(\Delta(G)) = \mathrm{Spec}(\mathcal{C}(G)/\Delta(G))$. We define the open subschemes $\mathcal{S}(G)_{\mathrm{a.i}}$, $\mathcal{X}(G)_{\mathrm{a.i}}$ and $\mathcal{R}(G)_{\mathrm{a.i}}$ of $\mathcal{S}(G)$, $\mathcal{X}(G)$ and $\mathcal{R}(G)$, respectively, by

$$\begin{aligned}
\mathcal{S}(G)_{\mathrm{a.i}} &:= \mathcal{S}(G) \setminus V(\Delta(G)), \\
\mathcal{X}(G)_{\mathrm{a.i}} &:= \mathcal{X}(G) \setminus (\tau^{\#})^{-1}(V(\Delta(G))), \\
\mathcal{R}(G)_{\mathrm{a.i}} &:= \mathcal{R}(G) \setminus (\varphi^{\#})^{-1}(V(\Delta(G))).
\end{aligned}$$

The following theorem, due to Kyoji Saito, is fundamental for our purpose.

Theorem 2.2.1.1 ([Sai96, 4.2, 4.3], [Nak00, Corollary 6.8]).
(1) *For* $\mathfrak{p} \in \mathcal{R}(G)$, $\rho_{\mathfrak{p}}$ *is absolutely irreducible if and only if* $\mathfrak{p} \in \mathcal{R}(G)_{\mathrm{a.i}}$.
(2) *The restriction of* $\varphi^{\#}$ *to* $\mathcal{R}(G)_{\mathrm{a.i}}$

$$\varphi^{\#}_{\mathrm{a.i}} : \mathcal{R}(G)_{\mathrm{a.i}} \longrightarrow \mathcal{S}(G)_{\mathrm{a.i}}$$

is a principal PGL_2*-bundle.*
(3) *The restriction of* $\tau^{\#}$ *to* $\mathcal{X}(G)_{\mathrm{a.i}}$ *is an isomorphism*

$$\tau^{\#}_{\mathrm{a.i}} : \mathcal{X}(G)_{\mathrm{a.i}} \xrightarrow{\sim} \mathcal{S}(G)_{\mathrm{a.i}}.$$

By virtue of Theorem 2.2.1.1 (1), we call $\mathcal{S}(G)_{\mathrm{a.i}}$, $\mathcal{X}(G)_{\mathrm{a.i}}$ and $\mathcal{R}(G)_{\mathrm{a.i}}$ the *absolutely irreducible part* of $\mathcal{S}(G)$, $\mathcal{X}(G)$ and $\mathcal{R}(G)$, respectively. We note that $\mathcal{X}(G)_{\mathrm{a.i}} (\simeq \mathcal{S}(G)_{\mathrm{a.i}})$ represents the functor $\overline{\mathcal{F}}$ from the category $\mathfrak{Sch}$ of schemes to the category of sets, which associates to a scheme X the set of isomorphism classes of absolutely irreducible representations $G \to \mathrm{SL}_2(\Gamma(X, \mathcal{O}_X))$:

$$\overline{\mathcal{F}}(X) := \{G \to \mathrm{SL}_2(\Gamma(X, \mathcal{O}_X)) \,|\, \text{absolutely irreducible representation}\}/\sim.$$

Since $\varphi^{\#}_{\mathrm{a.i}}$ and $\tau^{\#}_{\mathrm{a.i}}$ are defined over $\mathbb{Z}$, they induces maps on A-rational points for $A \in \mathfrak{Com.Ring}$:

$$\begin{aligned}
\varphi^{\#}_{\mathrm{a.i}}(A) &: \mathcal{R}(G)_{\mathrm{a.i}}(A) \longrightarrow \mathcal{S}(G)_{\mathrm{a.i}}(A), \\
\tau^{\#}_{\mathrm{a.i}}(A) &: \mathcal{X}(G)_{\mathrm{a.i}}(A) \xrightarrow{\sim} \mathcal{S}(G)_{\mathrm{a.i}}(A).
\end{aligned}$$

By Theorem 2.2.1.1 (3), we have the following

Corollary 2.2.1.2 *Let $\overline{\rho} : G \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation over a field k so that $\overline{\rho} \in \mathcal{R}(G)_{\mathrm{a.i}}(k)$. Let $[\overline{\rho}] \in \mathcal{X}(G)_{\mathrm{a.i}}(k)$ also denote the corresponding prime ideal of $\mathcal{B}(G)$. Then the morphism τ induces an isomorphism of local rings:*

$$\mathcal{C}(G)_{\tau^{\#}([\overline{\rho}])} \simeq \mathcal{B}(G)_{[\overline{\rho}]}.$$

The following proposition can be proved by using the vanishing of the Galois cohomology $H^1(k, \mathrm{PGL}_2(\overline{k})) = 1$ for a field k whose Brauer group $\mathrm{Br}(k) = 0$ ([Ser73, III, 2.2]) and Skolem-Noether theorem. For example, when k is a finite field or an algebraically closed field, $\mathrm{Br}(k) = 0$.

Proposition 2.2.1.3 ([Fuk98, Lemma 3.3.1], [Har08, Proposition 2.2.27]) *Let k be a field whose Brauer group $\mathrm{Br}(k) = 0$. Then $\varphi^{\#}_{\mathrm{a.i}}$ induces the following bijection on k-rational points:*

$$\varphi^{\#}_{\mathrm{a.i}}(k) : \ \mathcal{R}(G)_{\mathrm{a.i}}(k)/\mathrm{PGL}_2(k) \xrightarrow{\sim} \mathcal{S}(G)_{\mathrm{a.i}}(k).$$

2.2.2 The relation between the universal deformation ring and the character scheme

Let k be a perfect field with $\mathrm{char}(k) \neq 2$ and let $\mathcal{O}$ be a discrete valuation ring with residue field k. Let $\overline{\rho} : G \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation and let $\overline{T} : G \to k$ be a pseudo-SL_2-representation over k given by the character $\mathrm{tr}(\overline{\rho})$. Let $\boldsymbol{R}_{\overline{\rho}} (= \boldsymbol{R}_{\overline{T}})$ be the universal deformation ring of $\overline{\rho}$ (or $\overline{T}$) as in Theorem 2.1.2.2. Recall that $\boldsymbol{R}_{\overline{T}}$ is a complete local $\mathcal{O}$-algebra whose residue field is k. On the other hand, let $\mathcal{B}(G)$ and $\mathcal{S}(G)$ be the character algebra and skein algebra of G over $\mathbb{Z}$, respectively. We set

$$\begin{aligned} \mathcal{B}(G)_k &:= \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \quad \mathcal{X}(G)_k := \mathrm{Spec}(\mathcal{B}(G)_k) = \mathcal{X}(G) \otimes_{\mathbb{Z}} k, \\ \mathcal{C}(G)_k &:= \mathcal{C}(G) \otimes_{\mathbb{Z}} k, \quad \mathcal{S}(G)_k := \mathrm{Spec}(\mathcal{C}(G)_k) = \mathcal{S}(G) \otimes_{\mathbb{Z}} k. \end{aligned}$$

We also denote by $\mathcal{X}(G)_k^{\mathrm{a.i}}$ and $\mathcal{S}(G)_k^{\mathrm{a.i}}$ the absolutely irreducible part of $\mathcal{X}(G)_k$ and $\mathcal{S}(G)_k$, respectively. By Theorem 2.2.1.1 (3), we have $\mathcal{X}(G)_k^{\mathrm{a.i}} \simeq \mathcal{S}(G)_k^{\mathrm{a.i}}$. The following theorem tells us that the universal deformation ring $\boldsymbol{R}_{\overline{\rho}}$ may be seen as an infinitesimal deformation of the character k-algebra $\mathcal{B}(G)_k$ at $[\overline{\rho}]$.

Theorem 2.2.2.1 *Let $[\overline{\rho}]$ denote the maximal ideal of $\mathcal{B}(G)_k$ corresponding to the representation $\overline{\rho}$. We then have an isomorphism of k-algebras*

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]},$$

where $(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$ denotes the $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Proof. By the construction of $\boldsymbol{R}_{\overline{T}}$ in Theorem 2.2.1.1, we have

$$\boldsymbol{R}_{\overline{\rho}} = \mathcal{O}[[X_g \ (g \in G)]]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal of the power series ring $\mathcal{O}[[X_g \ \ (g \in G)]]$ generated by elements of the form: setting $T_g := X_g + \lambda(\overline{T}(g))$ (λ : the Teichmüller lift),

(1) $T_e - 2$,
(2) $T_{g_1 g_2} - T_{g_2 g_1}$,
(3) $T_{g_1} T_{g_2} T_{g_3} + T_{g_1 g_2 g_3} + T_{g_1 g_3 g_2} - T_{g_1 g_2} T_{g_3} - T_{g_2 g_3} T_{g_1} - T_{g_1 g_3} T_{g_2}$,
(4) $T_g^2 - T_{g^2} - 2$,

where $g, g_1, g_2, g_3 \in G$.

On the other hand, let $\psi : \mathcal{B}(G)_k \to k$ be the morphism in $\mathfrak{Com}.\mathfrak{Ring}$ corresponding to $[\overline{\rho}] \in \mathcal{X}(G)_{\mathrm{a.i}}(k)$. Since $\psi(\mathrm{tr}(\sigma_G(g))) = \mathrm{tr}(\overline{\rho}(g)) = \overline{T}(g)$ for $g \in G$, the maximal ideal $[\overline{\rho}] = \mathrm{Ker}(\psi)$ of $\mathcal{B}(G)_k$ corresponds to the maximal ideal $(t_g - \overline{T}(g)\ (g \in G))$ of $\mathcal{C}(G)_k$. Therefore Corollary 2.2.1.2 yields

$$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]} \simeq k[[x_g\ (g \in G)]]/I^{\wedge},$$

where $x_g := t_g - \overline{T}(g)$ $(g \in G)$ and $I^{\wedge}$ is the ideal of the power series ring $k[[x_g\ (g \in G)]]$ generated by elements of the form

$$t_e - 2,\ t_{g_1} t_{g_2} - t_{g_1 g_2} - t_{g_1^{-1} g_2}\ (g_1, g_2 \in G).$$

So, in order to show that the correspondence $x_g \mapsto X_g \otimes 1$ gives the desired isomorphism $(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]} \simeq \boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k$, it suffices to show the following

Lemma 2.2.2.2 *Let T be a function on G with values in an integral domain whose characteristic is not 2. Let* (P) *be the relations given by* (P1) $T(1) = 2$,
(P2) $T(g_1 g_2) = T(g_2 g_1)$,
(P3) $T(g_1)T(g_2)T(g_3) + T(g_1 g_2 g_3) + T(g_1 g_3 g_2) - T(g_1 g_2)T(g_3) - T(g_2 g_3)T(g_1) - T(g_1 g_3)T(g_2) = 0$,
(P4) $T(g)^2 - T(g^2) = 2$, *and let* (C) *be the relations given by* (C1) $T(1) = 2$,
(C2) $T(g_1)T(g_2) = T(g_1 g_2) + T(g_1^{-1} g_2)$,
where g, g_1, g_2, g_3 are any elements in G.

Then (P) *and* (C) *are equivalent.*

Proof of Lemma 2.2.2.2. (P) $\Rightarrow$ (C): Letting $g_2 = g_1$ in (P3), we have

$$T(g_1)^2 T(g_3) - T(g_1^2)T(g_3) + T(g_1^2 g_3) + T(g_1 g_3 g_1) - 2T(g_1 g_3)T(g_1) = 0.$$

Using (P2) and (P4), we have

$$2(T(g_3) + T(g_1^2 g_3) - T(g_1 g_3)T(g_1)) = 0.$$

Letting g_3 be replaced by $g_1^{-1} g_2$ in the above equation and noting T has the value in an integral domain whose characteristic is not 2, we obtain (C2).

(C) $\Rightarrow$ (P). Letting $g_2 = 1$ in (C2) and using (C1), we have

$$T(g) = T(g^{-1}) \text{ for any } g \in G.$$

Exchanging g_1 and g_2 in (C2) each other and using the above relation, we have

$$T(g_2)T(g_1) = T(g_2 g_1) + T(g_2^{-1} g_1) = T(g_2 g_1) + T(g_1^{-1} g_2)$$

and hence we obtain (P2). Next letting g_1 be replaced by g_1g_3 in (C2), we have

$$-T(g_1g_3)T(g_2)+T(g_1g_3g_2)+T(g_3^{-1}g_1^{-1}g_2)=0, \tag{2.2.2.1}$$

and letting g_2 be replaced by g_2g_3 in (C2), we have

$$-T(g_1)T(g_2g_3)+T(g_1g_2g_3)+T(g_1^{-1}g_2g_3)=0. \tag{2.2.2.2}$$

By (C2), we have

$$\begin{aligned} T(g_3^{-1}g_1^{-1}g_2) &= T(g_3)T(g_1^{-1}g_2)-T(g_3g_1^{-1}g_2) \\ &= T(g_3)T(g_1)T(g_2)-T(g_1g_2)T(g_3)-T(g_3g_1^{-1}g_2). \end{aligned}$$

Hence, using (P2) proved already, we have

$$\begin{aligned} T(g_3^{-1}g_1^{-1}g_2)+T(g_1^{-1}g_2g_3) &= T(g_1)T(g_2)T(g_3)-T(g_1g_2)T(g_3) \\ &\quad -T(g_3g_1^{-1}g_2)+T(g_1^{-1}g_2g_3) \\ &= T(g_1)T(g_2)T(g_3)-T(g_1g_2)T(g_3). \end{aligned} \tag{2.2.2.3}$$

Summing up (2.2.2.1) and (2.2.2.2) together with (2.2.2.3), we obtain (P3). Finally putting $g_1=g_2$ in (C2) and using (C1), we obtain (P4). □

By Lemma 2.1.3.1 and Theorem 2.2.2.1, we have the following

Corollary 2.2.2.3 *Let $[\overline{\rho}]$ be a regular k-rational point of $\mathcal{X}(G)_k^{\mathrm{a.i}}$. Let d be the dimension of the irreducible component of $\mathcal{X}(G)_k^{\mathrm{a.i}}$ containing $[\overline{\rho}]$ so that $(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$ is a power series ring over k on a regular system of parameters $z_1,\ldots,z_d$. Let $x_1,\ldots,x_d$ be elements of $\boldsymbol{R}_{\overline{\rho}}$ such that the image of x_i in $\boldsymbol{R}_{\overline{\rho}}\otimes_{\mathcal{O}}k\simeq(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$ is z_i for $1\le i\le d$. Then there is a surjective $\mathcal{O}$-algebra homomorphism*

$$\eta:\mathcal{O}[[X_1,\ldots,X_d]]\longrightarrow\boldsymbol{R}_{\overline{T}}$$

in $\mathfrak{CL}_{\mathcal{O}}$ such that $\eta(X_i)=x_i$ for $1\le i\le d$.

By Corollary 2.2.2.3, we obtain the following criterion which determines the universal deformations for many examples. See Section 4.

Theorem 2.2.2.4 *Let notations and assumptions be as in Corollary 2.2.2.3. We suppose that there are $g_1,\ldots,g_d\in G$ such that $z_i=t_i-\mathrm{tr}(\overline{\rho}(g_i))$ for $1\le i\le d$, where t_i denotes a variable corresponding to the regular function $\mathrm{tr}(\sigma_G(g_i))$. Choose $\alpha_i\in\mathcal{O}$ such that α_i mod $\mathfrak{m}_{\mathcal{O}}=\mathrm{tr}(\overline{\rho}(g_i))$ for $1\le i\le d$ and suppose that $\rho:G\to\mathrm{SL}_2(\mathcal{O}[[t_1-\alpha_1,\ldots,t_d-\alpha_d]])$ is a deformation of $\overline{\rho}$ satisfying*

$$\mathrm{tr}(\rho(g_i))=t_i\quad(1\le i\le d).$$

Then $(\mathcal{O}[[t_1-\alpha_1,\ldots,t_d-\alpha_d]],\rho)$ is the universal deformation of $\overline{\rho}$.

Proof. By the universal property of $(\boldsymbol{R}_{\overline{\rho}},\boldsymbol{\rho})$, there is a morphism

$$\psi:\boldsymbol{R}_{\overline{\rho}}\longrightarrow\mathcal{O}[[t_1-\alpha_1,\ldots,t_d-\alpha_d]]$$

in $\mathfrak{CL}_{\mathcal{O}}$ such that $\psi \circ \boldsymbol{\rho} \approx \rho$. Hence we have

$$\psi(\operatorname{tr}(\boldsymbol{\rho}(g_i))) = \operatorname{tr}(\rho(g_i)) = t_i, \ \ 1 \le i \le d. \tag{2.2.2.4}$$

By Corollary 2.2.2.3, there is a surjective morphism

$$\eta : \mathcal{O}[[X_1, \ldots, X_d]] \longrightarrow \boldsymbol{R}_{\overline{\rho}}$$

in $\mathfrak{CL}_{\mathcal{O}}$ such that $\eta(X_i) = \operatorname{tr}(\boldsymbol{\rho}(g_i)) - \alpha_i$ for $1 \le i \le d$. Since $\psi \circ \eta : \mathcal{O}[[X_1, \ldots, X_d]] \to \mathcal{O}[[t_1 - \alpha_1, \ldots, t_d - \alpha_d]]$ is a morphism in $\mathfrak{CL}_{\mathcal{O}}$ and satisfies, by (2.2.2.4),

$$\psi \circ \eta(X_i) = t_i - \alpha_i \quad (1 \le i \le d),$$

$\psi \circ \eta$ is an isomorphism in $\mathfrak{CL}_{\mathcal{O}}$. Since η is surjective, η must be isomorphic and so is ψ. $\Box$

2.2.3 The case of a knot group

Let K be a knot in the 3-sphere S^3 and let E_K denote the knot complement $S^3 \setminus K$. Let G_K denote the knot group of K, $G_K := \pi_1(E_K)$. It is well known that G_K has the following presentation of deficiency one (for example, the Wirtinger presentation):

$$G_K = \langle g_1, \ldots, g_n \mid r_1 = \cdots = r_{n-1} = 1 \rangle . \tag{2.2.3.1}$$

Let k be a field with $\operatorname{char}(k) \neq 2$. Let $\overline{\rho} : G_K \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation and let $\boldsymbol{R}_{\overline{\rho}}$ be the universal deformation ring as in Theorem 2.1.2.2. Since the character variety $\mathcal{X}(G_K)_k$ of a knot group G_K over a field k has been extensively studied (see [CS83], [Le93], [Har08] etc), we can determine $\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k$ by Theorem 2.2.2.1 and even $\boldsymbol{R}_{\overline{\rho}}$ by Theorem 2.2.2.4 for some knots K. In fact, in [MTTU17], Morishita, Takakura, Terashima and Ueki determined $\boldsymbol{R}_{\overline{\rho}}$ for a certain Riley-type representations $\overline{\rho}$ for a 2-bridge knots K. See also Section 4 for other examples.

It is a delicate problem, however, to determine the universal deformation ring $\boldsymbol{R}_{\overline{\rho}}$ for a knot group representation $\overline{\rho}$ in general, since the deformation problem for $\overline{\rho}$ is not unobstructed in general for a knot group G_K, as the following theorem shows.

Theorem 2.2.3.1 *We suppose that $\rho : G_K \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation and that there is a subring A of a finite algebraic number field F and a finite prime $\mathfrak{p}$ of F such that A is $\mathfrak{p}$-integral and the image of ρ is contained in $\mathrm{SL}_2(A)$. Set $k := A/\mathfrak{p}$ and $\overline{\rho} := \rho \bmod \mathfrak{p} : G_K \to \mathrm{SL}_2(k)$. Then we have*

$$H^2(G_K, \operatorname{Ad}(\overline{\rho})) \neq 0.$$

We note that the assumption in Theorem 2.2.3.1 is satisfied, for instance, when K is a hyperbolic knot and ρ is the holonomy representation attached to a hyperbolic structure on E_K such that the completion is a closed or a cone 3-manifold. For the proof of Theorem 2.2.3.1, we recall the following lemma, which is a special case of a more general result, due to Thurston, for 3-manifolds.

Lemma 2.2.3.2 ([CS83, Proposition 3.2.1] *For an irreducible representation $\rho : G_K \to \mathrm{SL}_2(\mathbb{C})$, the irreducible component of $\mathcal{X}(G_K)_{\mathbb{C}}$ containing $[\rho]$ has the dimension greater than* 0.

Proof of Theorem 2.2.3.1. Let W be the CW complex attached to the presentation (2.2.3.1). We recall herewith the construction of W:
• We prepare 0-cell b^*, 1-cells $g_1^*, \ldots, g_n^*$, where each g_i^* corresponds to the generator g_i, 2-cells $r_1^*, \ldots, r_{n-1}^*$, where each r_j^* corresponds to the relator r_j.
• We attach each 1-cell g_i^* to the 0-cell b^* so that we obtain a bouquet.
• We attach the boundary of each 2-cell r_j^* to 1-cells of the bouquet, according to words in r_j.
We note that the knot complement E_K and the CW complex W are homotopically equivalent by Whitehead's theorem, because they are both the Eilenberg-MacLane space $K(G_K, 1)$.

Let $\mathrm{Ad}(\rho)$ be the A-module $\mathrm{sl}_2(A)$ on which G_K acts by $g.X := \rho(g)X\rho(g)^{-1}$ for $g \in G_K$ and $X \in \mathrm{sl}_2(A)$. We let $\mathrm{Ad}(\rho)_{\mathbb{C}} := \mathrm{Ad}(\rho) \otimes_A \mathbb{C} = \mathrm{sl}_2(\mathbb{C})$ on which G_K acts as $g \otimes \mathrm{id}_{\mathbb{C}}$ for $g \in G_K$. Since the Euler characteristic of W is zero, we have

$$
\begin{aligned}
\sum_{i=0}^{2}(-1)^i \dim_{\mathbb{C}} H^i(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}}) &= \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{C}} C^i(W; \mathrm{Ad}(\rho)_{\mathbb{C}}) \\
&= 3\sum_{i=0}^{2}(-1)^i \dim_{\mathbb{C}} C^i(W; \mathbb{C}) \\
&= 0.
\end{aligned} \tag{2.2.3.2}
$$

Since ρ is irreducible, we have $H^0(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}}) = 0$ by Schur's lemma. So, by (2.2.3.2), we have

$$\dim_{\mathbb{C}} H^2(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}}) = \dim_{\mathbb{C}} H^1(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}}). \tag{2.2.3.3}$$

Since $H^1(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}})$ contains the tangent space of the character variety $\mathcal{X}(G_K)_{\mathbb{C}}$ at $[\rho]$ ([Por97, Proposition 3.5]), Lemma 2.2.3.2 implies $H^1(G_K, \mathrm{Ad}(\rho))_{\mathbb{C}}) \neq 0$. So, by (2.2.3.3), we have $H^2(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}}) \neq 0$. Since $H^2(G_K, \mathrm{Ad}(\rho)_{\mathbb{C}}) = H^2(G_K, \mathrm{Ad}(\rho)_A) \otimes_A \mathbb{C}$, we have

$$H^2(G_K, \mathrm{Ad}(\rho)) \neq 0. \tag{2.2.3.4}$$

Let $\mathrm{Ad}(\overline{\rho}) := \mathrm{Ad}(\rho) \otimes_A k = \mathrm{sl}_2(k)$ on which G_K acts as $g \otimes \mathrm{id}_k$ for $g \in G_K$. Let us consider the differentials of cochains

$$
\begin{aligned}
&d : C^1(W; \mathrm{Ad}(\rho)) \longrightarrow C^2(W; \mathrm{Ad}(\rho)), \\
&\overline{d} := d \otimes (\mathrm{mod}\ \mathfrak{p}) : C^1(W; \mathrm{Ad}(\overline{\rho})) \longrightarrow C^2(W; \mathrm{Ad}(\overline{\rho})).
\end{aligned}
$$

By (2.2.3.4), all $3n$-minors of d are zero. Therefore all $3n$-minors of $\overline{d}$ are zero and hence $H^2(G_K, \mathrm{Ad}(\overline{\rho})) \neq 0$. $\square$.

2.3 L-functions associated to the universal deformations

In this section, we study the twisted knot module $H_1(\boldsymbol{\rho}) = H_1(E_K; \boldsymbol{\rho})$ with coefficients in the universal deformation $\boldsymbol{\rho}$ of an SL_2- representation of a knot group G_K, and introduce the associated L-function $L_K(\boldsymbol{\rho})$. We then formulate two problems proposed by Mazur ([Maz00]): the torsion property of $H_1(\boldsymbol{\rho})$ over the universal deformation ring $\boldsymbol{R}_{\overline{\rho}}$ (Problem 2.3.2.1) and the generic simplicity of the zeroes of $L_K(\boldsymbol{\rho})$ (Problem 2.3.2.7). Our main theorem in this section (Theorem 2.3.2.2) gives a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated and torsion over $\boldsymbol{R}_{\overline{\rho}}$ using a twisted Alexander invariant of K.

2.3.1 Fitting ideals and twisted Alexander invariants

Let A be a Noetherian integrally closed domain. Let M, M' be finitely generated A-modules. We say that a homomorphism $\varphi : M \to M'$ is a *pseudo-isomorphism* if the annihilators of $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are not contained in height 1 prime ideals of A.

Lemma 2.3.1.1 (cf. [Ser95, Lemma 5]). *For any finitely generated torsion A-module M, there are positive integers $e_1, \ldots, e_s$, height 1 prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of A for some $s \geq 1$, and a pseudo-isomorphism*

$$\varphi : M \longrightarrow \bigoplus_{i=1}^{s} A/\mathfrak{p}_i^{e_i}.$$

Here the set $\{(\mathfrak{p}_i, e_i)\}$ is uniquely determined by M. If A is a Noetherian factorial domain further, each prime ideal $\mathfrak{p}_i$ of height 1 is a principal ideal $\mathfrak{p}_i = (f_i)$ for a prime element f_i of A.

We note that a regular local ring is a Noetherian factorial local domain (Auslander-Buchsbaum). For example, the Iwasawa algebra $\mathcal{O}[[X]]$ is a 2-dimensional regular local ring, where $\mathcal{O}$ is a complete discrete valuation ring with $\mathrm{char}(\mathcal{O}) = 0$ and finite residue field. Then it is known in Iwasawa theory ([Iwa73]) that a height 1 prime ideal of $\mathcal{O}[[X]]$ is (ϖ) for a prime element ϖ of $\mathcal{O}$ or (f) for an irreducible distinguished polynomial $f \in \mathcal{O}[X]$, and a pseudo-isomorphism means a homomorphism with finite kernel and cokernel ([Was97, §13.2]).

Let A be a Noetherian factorial domain and let M be a finitely generated A-module. Let us take a finite presentation of M over A

$$A^m \xrightarrow{\partial} A^n \longrightarrow M \longrightarrow 0,$$

where ∂ is an $n \times m$ matrix over A. For a non-negative integer d, we define the d-th *Fitting ideal* (*elementary ideal*) $E_d(M)$ of M to be the ideal generated by $(n-d)$ minors of ∂. If $d \geq n$, we let $E_d(M) := A$, and if $n - d > m$, we let $E_d(M) := 0$. These ideals depend only on M and independent of the choice of a presentation. The initial Fitting ideal $E_0(M)$ is called the *order ideal* of M. Let $\Delta_d(M)$ be the greatest common divisor of generators of $E_d(M)$, which is well defined up to multiplication by a unit of A. The *rank* of M over A is defined by the dimension of $M \otimes_A Q(A)$ over $Q(A)$. The following facts are well known ([Hil12, Ch.3], [Kaw96, 7.2]).

Lemma 2.3.1.2 *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of finitely generated A-modules. Then we have the followings.*
(1) $\Delta_0(M_2) \doteq \Delta_0(M_1)\Delta_0(M_3)$.
(2) *If the A-torsion subgroup of M_3 is zero and r is the rank of M_3 over A, then* $\Delta_d(M_2) \doteq \Delta_{d-r}(M_1)$.

For example, suppose A is a principal ideal domain and M is a finitely generated torsion A-module. Then we have $M \simeq \bigoplus_{i=1}^{s} A/(a_i)$ with $(a_1) \supset \cdots \supset (a_s)$, and $E_d(M) = (a_1 \cdots a_{s-d})$, $\Delta_d(M) \doteq a_1 \cdots a_{s-d}$ for $d < s$. As another example, let A be the Iwasawa algebra $\mathcal{O}[[X]]$ and M a finitely generated torsion A-module. Then there is a pseudo-isomorphism $\varphi : M \to \bigoplus_{i=1}^{s} A/(f_i^{e_i})$, where f_i is a prime element of $\mathcal{O}$ or an irreducible distinguished polynomial in $\mathcal{O}[X]$. If φ is injective, in particular, if M has no non-trivial finite A-submodule, we have $E_0(M) = (f)$, $\Delta_0(M) \doteq f$, where f is the *Iwasawa polynomial* $\prod_{i=1}^{s} f_i^{e_i}$ ([MW84, Appendix]). For higher Fitting ideals $E_d(M)$ for $d > 0$ in Iwasawa theory, we refer to [Kur03].

Next, let us define the twisted Alexander invariant for a finite connected CW complex (see 1.2 for the case of a knot complement). Let C be a finite connected CW complex. Let $G := \pi_1(C)$ be the fundamental group of C which is supposed to have the finite presentation

$$G = \langle g_1, \ldots, g_n \mid r_1 = \cdots = r_m = 1 \rangle ,$$

where relators $r_1, \ldots, r_m$ are words of the letters $g_1, \ldots, g_n$. We suppose that there is a surjective homomorphism

$$\alpha : G \longrightarrow \langle t \rangle \simeq \mathbb{Z}.$$

Let A be a Noetherian factorial domain. We denote by the same α the group A-algebra homomorphism $A[G] \to A[t^{\pm 1}]$, which is induced by α. Let

$$\rho : G \longrightarrow \mathrm{GL}_N(A)$$

be a representation of G of degree N over A and let us denote by the same ρ the A-algebra homomorphism $A[G] \to \mathrm{M}_N(A)$ induced by ρ. Then we have the tensor product representation

$$\rho \otimes \alpha : A[G] \longrightarrow \mathrm{M}_N(A[t^{\pm 1}]).$$

The twisted Alexander invariant $\Delta(C, \rho; t) \in A[t^{\pm 1}]$ is defined as follows. Let F be the free group on $g_1, \ldots, g_n$ and let $\pi : F \to G$ be the natural homomorphism. We denote by the same π the A-algebra homomorphism $A[F] \to A[G]$ induced by π. Then we have the A-algebra homomorphism

$$\Phi := (\rho \otimes \alpha) \circ \pi : A[F] \longrightarrow \mathrm{M}_N(A[t^{\pm 1}]).$$

Let $\frac{\partial}{\partial g_i} : A[F] \to A[F]$ be the Fox derivative over A, extended from $\mathbb{Z}$ ([Fox53]). Let us consider the (big) $n \times m$ matrix P, called the *twisted Alexander matrix*, whose (i, j) component is defined by the $N \times N$ matrix

$$\Phi\left(\frac{\partial r_j}{\partial g_i}\right).$$

For $1 \leq i \leq n$, let P_i denote the matrix obtained by deleting the i-th row from P and we regard P_i as an $(n-1)N \times mN$ matrix over $A[t^{\pm 1}]$. We note that $A[t^{\pm 1}]$ is also a Noetherian factorial domain. Let D_i be the greatest common divisor of all $(n-1)N$-minors of P_i. Then it is known that there is i $(1 \leq i \leq n)$ such that $\det(\Phi(g_i - 1)) \neq 0$ and that the ratio

$$\Delta(C, \rho; t) := \frac{D_i}{\det \Phi(g_i - 1)} \ (\in Q(A)(t)) \tag{2.3.1.1}$$

is independent of such i's and is called the *twisted Alexander invariant* of C associated to ρ ([Wad94]).

2.3.2 L-functions associated to the universal deformations

Let K be a knot in the 3-sphere S^3 and let E_K denote the knot complement $S^3 \setminus K$. Let G_K denote the knot group $\pi_1(E_K)$ of K, which has the following presentation:

$$G_K = \langle g_1, \ldots, g_n \mid r_1 = \cdots = r_{n-1} = 1 \rangle. \tag{2.3.2.1}$$

Let F be the free group on the words $g_1, \ldots, g_n$ and let $\pi : \mathbb{Z}[F] \to \mathbb{Z}[G]$ be the natural homomorphism of group rings. We write the same g_i for the image of g_i in G_K.

Let $\overline{\rho} : G_K \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation of G_K over a perfect field k with $\mathrm{char}(k) \neq 2$. Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k and let $\mathfrak{CL}_{\mathcal{O}}$ be the category of complete local $\mathcal{O}$-algebras with residue field k. Let $\boldsymbol{\rho} : G_K \to \mathrm{SL}_2(\boldsymbol{R}_{\overline{\rho}})$ be the universal deformation of $\overline{\rho}$ (Theorem 2.1.2.2). We denote by the same $\boldsymbol{\rho}$ the induced algebra homomorphism $\mathbb{Z}[G_K] \to \mathrm{M}_2(\boldsymbol{R}_{\overline{\rho}})$. Let $V_{\boldsymbol{\rho}}$ be the representation space (of column vectors) $(\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$ of $\boldsymbol{\rho}$ on which G_K acts from the left via $\boldsymbol{\rho}$. We will compute the *twisted knot module*

$$H_*(\boldsymbol{\rho}) := H_*(E_K; V_{\boldsymbol{\rho}})$$

with coefficients in $V_{\boldsymbol{\rho}}$ as the homology of the chain complex $C_*(W; V_{\boldsymbol{\rho}})$ of the CW complex W attached to the presentation (3.2.1). The CW complex W was given in Subsection 2.2.3. Since $H_1(W; \mathbb{Z}) = H_1(E_K; \mathbb{Z}) = \langle t \rangle \simeq \mathbb{Z}$, we take $\alpha : \pi_1(W) \to \langle t \rangle$ to be the abelianization map.

For a representation $\rho : G_K \to \mathrm{SL}_2(A)$, where A is a Noetherian factorial domain, we define the *twisted Alexander invariant* $\Delta_K(\rho; t)$ of K associated to ρ by (cf. 1.2)

$$\Delta_K(\rho; t) := \Delta(W, \rho; t).$$

We note that $\Delta_K(\rho; t)$ coincides with the Reidemeister torsion of E_K (or W) associated to the representation $\rho \otimes \alpha$ over $Q(A)(t)$ [FV11, Proposition 2.2] By Proposition 1.2.0.1, the relation between the twisted Alexander invariant $\Delta_K(\rho; t)$ and the initial Fitting ideals of $H_i(\rho \otimes \alpha) := H_i(E_K; \rho \otimes \alpha)$ $(i = 0, 1)$ is given by

$$\Delta_K(\rho; t) \doteq \frac{\Delta_0(H_1(\rho \otimes \alpha))}{\Delta_0(H_0(\rho \otimes \alpha))}. \tag{2.3.2.2}$$

Following Mazur's question 1 of [Maz00, page 440], we may ask the following

Problem 2.3.2.1 *Is $H_1(\boldsymbol{\rho})$ a finitely generated and torsion $\boldsymbol{R}_{\overline{\rho}}$-module ?*

Here is our main theorem, which gives an affirmative answer to Problem 2.3.2.1 under some conditions using a twisted Alexander invariant of K.

Theorem 2.3.2.2 *Notations being as above, suppose that the following two conditions are satisfied*
(1) *$\boldsymbol{R}_{\overline{\rho}}$ is a Noetherian integral domain.*
(2) *There is a deformation $\rho : G_K \to \mathrm{SL}_2(R)$ of $\overline{\rho}$, where $R \in \mathfrak{CL}_{\mathcal{O}}$ is a Noetherian factorial domain, and $g \in G_K$ such that*
(2-1) $\det(\rho(g) - I) \neq 0$ and
(2-2) $\Delta_K(\rho; 1) \neq 0$.
Then we have $H_1(\boldsymbol{\rho}) = 0$.

Proof. We may assume that $g = g_n$ in the presentation (2.3.2.1) of G_K. We consider the following chain complex $C_*(\boldsymbol{\rho}) := C_*(W; V_{\boldsymbol{\rho}})$ ([Kaw96, 7.1]):

$$0 \longrightarrow C_2(\boldsymbol{\rho}) \xrightarrow{\partial_2} C_1(\boldsymbol{\rho}) \xrightarrow{\partial_1} C_0(\boldsymbol{\rho}) \longrightarrow 0,$$

defined by

$$\left\{ \begin{array}{l} C_0(\boldsymbol{\rho}) := V_{\boldsymbol{\rho}}, \\ C_1(\boldsymbol{\rho}) := (V_{\boldsymbol{\rho}})^{\oplus n}, \\ C_2(\boldsymbol{\rho}) := (V_{\boldsymbol{\rho}})^{\oplus (n-1)}, \end{array} \right. \quad \left\{ \begin{array}{l} \partial_1 := (\boldsymbol{\rho}(g_1) - I, \dots, \boldsymbol{\rho}(g_n) - I), \\ \partial_2 := \left(\boldsymbol{\rho} \circ \pi \left(\dfrac{\partial r_j}{\partial g_i} \right) \right), \end{array} \right.$$

where $\frac{\partial}{\partial g_i} : \mathbb{Z}[F] \to \mathbb{Z}[F]$ denotes the Fox derivative ([Fox53]), and ∂_2 is regarded as a (big) $n \times (n-1)$ matrix whose (i,j)-entry is the 2×2 matrix $\boldsymbol{\rho} \circ \pi \left(\frac{\partial r_j}{\partial g_i} \right)$.

By the condition (2), let $\psi : \boldsymbol{R}_{\overline{\rho}} \to R$ be a morphism in $\mathfrak{CL}_{\mathcal{O}}$ such that $\psi \circ \boldsymbol{\rho} \approx \rho$. Since $\psi(\det(\boldsymbol{\rho}(g_n) - I)) = \det(\rho(g_n) - I) \neq 0$ by (2-1), we have $\det(\boldsymbol{\rho}(g_n) - I) \in Q(\boldsymbol{R}_{\overline{\rho}})^\times$ by the condition (1). Hence we have

$$H_0(\boldsymbol{\rho}) \otimes_{\boldsymbol{R}_{\overline{\rho}}} Q(\boldsymbol{R}_{\overline{\rho}}) = 0. \tag{2.3.2.3}$$

Let $C_1'(\boldsymbol{\rho})$ be the $\boldsymbol{R}_{\overline{\rho}}$-submodule of $C_1(\boldsymbol{\rho})$ consisting of the first $(n-1)$ components so that $C_1(\boldsymbol{\rho}) = C_1'(\boldsymbol{\rho}) \oplus V_{\boldsymbol{\rho}}$ and let ∂_2' be the $(n-1) \times (n-1)$ matrix obtained deleting the n-th row from ∂_2. Consider the $\boldsymbol{R}_{\overline{\rho}}$-homomorphism

$$\partial_2' : C_2(\boldsymbol{\rho}) \longrightarrow C_1'(\boldsymbol{\rho}).$$

Then, by the definition (2.3.1.1) of the twisted Alexander invariant, we have

$$\Delta_K(\rho;1) = \frac{\psi(\det(\partial_2'))}{\psi(\det(\boldsymbol{\rho}(g_n) - I))}. \tag{2.3.2.4}$$

By the conditions (2-1), (2-2) and (2.3.2.4), we have $\det(\partial_2') \in Q(\boldsymbol{R}_{\overline{\rho}})^\times$. Hence we have

$$H_2(\boldsymbol{\rho}) \otimes_{\boldsymbol{R}_{\overline{\rho}}} Q(\boldsymbol{R}_{\overline{\rho}}) = 0. \tag{2.3.2.5}$$

Since the Euler characteristic of W is zero, we have

(2.3.2.6)

$$\begin{aligned}\sum_{i=0}^{3}(-1)^i \dim_{Q(\boldsymbol{R}_{\overline{\rho}})} H_i(\boldsymbol{\rho}) \otimes_{\boldsymbol{R}_{\overline{\rho}}} Q(\boldsymbol{R}_{\overline{\rho}}) \quad &= \sum_{i=0}^{3}(-1)^i \dim_{Q(\boldsymbol{R}_{\overline{\rho}})} C_i(\boldsymbol{\rho}) \otimes_{\boldsymbol{R}_{\overline{\rho}}} Q(\boldsymbol{R}_{\overline{\rho}}) \\ &= (\mathrm{rank}_{\boldsymbol{R}_{\overline{\rho}}} V_{\boldsymbol{\rho}}) \sum_{i=0}^{3}(-1)^i \mathrm{rank}_{\mathbb{Z}}\, C_i(W) \\ &= 0.\end{aligned}$$

Therefore, by (2.3.2.3), (2.3.2.5) and (2.3.2.6), we have

$$\mathrm{rank}_{\boldsymbol{R}_{\overline{\rho}}} H_1(\boldsymbol{\rho}) = \dim_{Q(\boldsymbol{R}_{\overline{\rho}})} H_1(\boldsymbol{\rho}) \otimes_{\boldsymbol{R}_{\overline{\rho}}} Q(\boldsymbol{R}_{\overline{\rho}}) = 0$$

and hence $H_1(\boldsymbol{\rho})$ is torsion over $\boldsymbol{R}_{\overline{\rho}}$. Since $\boldsymbol{R}_{\overline{\rho}}$ is Noetherian and $H_1(\boldsymbol{\rho})$ is a quotient of a submodule of $(V_{\boldsymbol{\rho}})^{\oplus n} = (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2n}$, $H_1(\boldsymbol{\rho})$ is Noetherian, in particular, finitely generated over $\boldsymbol{R}_{\overline{\rho}}$. $\square$

It may be interesting to note that the condition (2-2) in Theorem 2.3.2.2 on a twisted Alexander polynomial is reminiscent of Kato's result in number theoretic situation ([Kat04]), which asserts that the non-vanishing of the L-function at 1 of a modular form implies the finiteness of the Selmer module of the associated p-adic Galois representation.
As a special case of Theorem 2.3.2.2, the above proof shows the following.

Corollary 2.3.2.3 *Notations being as above, suppose that the following two conditions are satisfied:*
(1) $\boldsymbol{R}_{\overline{\rho}}$ is a Noetherian integral domain.
(2) There is $g \in G_K$ such that $\det(\overline{\rho}(g) - I) \neq 0$ and $\Delta_K(\overline{\rho};1) \neq 0$.
Then we have $H_1(\boldsymbol{\rho}) = 0$.

Proof. By the assumptions, we have $\det(\boldsymbol{\rho}(g_n) - I), \det(\partial_2') \in (\boldsymbol{R}_{\overline{\rho}})^\times$, from which we easily see that $\mathrm{Ker}(\partial_1) = \mathrm{Im}(\partial_2)$ and hence $H_1(\boldsymbol{\rho}) = 0$. $\square$

Assume that $\boldsymbol{R}_{\overline{\rho}}$ is a Noetherian factorial domain and the condition (2) of Theorem 2.3.2.2. When $H_1(\boldsymbol{\rho})$ is a torsion $\boldsymbol{R}_{\overline{\rho}}$-module, we are interested in the invariant

$$L_K(\boldsymbol{\rho}) := \Delta_0(H_1(\boldsymbol{\rho})), \tag{2.3.2.7}$$

which we call the *L-function* of the knot K associated to $\boldsymbol{\rho}$ (cf. Remark 2.3.2.5 (3) below). We note that it is a computable invariant by the following

Proposition 2.3.2.4 *Notations being as above, we have*

$$L_K(\boldsymbol{\rho}) \doteq \Delta_2(\mathrm{Coker}(\partial_2))$$

Proof. This follows from the exact sequence of $\boldsymbol{R}_{\overline{\rho}}$-modules

$$0 \longrightarrow H_1(\boldsymbol{\rho}) \longrightarrow \mathrm{Coker}(\partial_2) \longrightarrow V_{\boldsymbol{\rho}} = (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2} \longrightarrow 0$$

and Lemma 2.3.1.2 (2). $\square$

Remark 2.3.2.5 (1) The L-function $L_K(\boldsymbol{\rho})$ is determined up to multiplication by a unit of $\boldsymbol{R}_{\overline{\rho}}$.
(2) When $H_*(\boldsymbol{\rho}) \otimes_{\boldsymbol{R}_{\overline{\rho}}} Q(\boldsymbol{R}_{\overline{\rho}}) = 0$, we have the Reidemeister torsion $\Delta_K(\boldsymbol{\rho};1) \in Q(\boldsymbol{R}_{\overline{\rho}})$ of E_K associated to $\boldsymbol{\rho}$, which is an invariant defined without indeterminacy. It may be non-trivial, even when $H_*(\boldsymbol{\rho}) = 0$.
(3) Our L-function $L_K(\boldsymbol{\rho})$ may be seen as an analogue in knot theory of the algebraic p-adic L-function for the universal Galois deformation in number theory ([Gre94]). In terms of [Maz00], the $\boldsymbol{R}_{\overline{\rho}}$-module $H_1(\boldsymbol{\rho})$ gives a coherent torsion sheaf $\mathcal{H}_1(\boldsymbol{\rho})$ on the universal deformation space $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$ and $L_K(\boldsymbol{\rho})$ gives a non-zero section of $\mathcal{H}_1(\boldsymbol{\rho})$.

We find the following necessary condition for the L-function $L_K(\boldsymbol{\rho})$ to be non-trivial under a mild condition.

Proposition 2.3.2.6 *Assume that $\Delta_0(H_0(\boldsymbol{\rho})) \doteq 1$. If $L_K(\boldsymbol{\rho}) \not\doteq 1$, we have $\Delta_K(\overline{\rho};1) = 0$.*

Proof. By (2.3.2.2), (2.3.2.7) and our assumption, we have

$$\Delta_K(\boldsymbol{\rho};1) \doteq L_K(\boldsymbol{\rho}). \tag{2.3.2.8}$$

Suppose $L_K(\boldsymbol{\rho}) \not\doteq 1$, which means $L_K(\boldsymbol{\rho}) \in \mathfrak{m}_{\boldsymbol{R}_{\overline{\rho}}}$. Let $\varphi : \boldsymbol{R}_{\overline{\rho}} \to k$ be the homomorphism taking mod $\mathfrak{m}_{\boldsymbol{R}_{\overline{\rho}}}$. Then, by the functorial property of the twisted Alexander invariant and (2.3.2.9) , we have

$$\Delta_K(\overline{\rho};1) = \varphi(\Delta_K(\boldsymbol{\rho};1)) = \varphi(L_K(\boldsymbol{\rho})) = 0. \ \square$$

Following Mazur's question 2 of [Maz00, page 440], we may ask the following

Problem 2.3.2.7 *Investigate the order of the zeroes of $L_K(\boldsymbol{\rho})$ on $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$ at prime divisors.*

In the next section, we verify Problem 2.3.2.7 affirmatively by some examples.

Remark 2.3.2.8 In [Maz00], Mazur works over a field k (in fact, the field of complex numbers) and so the L-function discussed there is, in our terms, given by

$$L_K(\boldsymbol{\rho}_k) := \Delta_0(H_1(\boldsymbol{\rho}_k)),$$

where $\boldsymbol{\rho}_k : G_K \to \mathrm{SL}_2(\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k)$ is the representation obtained by taking mod $\mathfrak{m}_{\mathcal{O}}$ of $\boldsymbol{\rho}$. Therefore our L-function $L_K(\boldsymbol{\rho})$ in (3.2.6) is a finer object than $L_K(\boldsymbol{\rho}_k)$.

2.4 Examples

In this section, we discuss concrete examples of the universal deformations of some representations of 2-bridge knot groups over finite fields and the associated L-functions.

Let K be a 2-bridge knot in the 3-sphere S^3, given as the Schubert form $B(m,n)$ where m and n are odd integers with $m > 0, -m < n < m$ and g.c.d$(m,n) = 1$. The knot group G_K is known to have a presentation of the form

$$G_K = \langle g_1, g_2 \mid wg_1 = g_2 w \rangle,$$

where w is a word $w(g_1, g_2)$ of g_1 and g_2 which has the following symmetric form

$$w = w(g_1, g_2) = g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_1^{\epsilon_{m-2}} g_2^{\epsilon_{m-1}},$$
$$\epsilon_i = (-1)^{[in/m]} = \epsilon_{m-i} \quad ([\cdot] = \text{Gauss symbol}).$$

We write the same g_i for the image of the word g_i in G_K.

Let A be a commutative ring with identity. For $a \in A^\times$ and $b \in A$, we consider two matrices $C(a)$ and $D(a,b)$ in $\mathrm{SL}_2(A)$ defined by

$$C(a) := \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix}, \quad D(a,b) := \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$$

and we set

$$W(a,b) := C(a)^{\epsilon_1} D(a,b)^{\epsilon_2} \cdots C(a)^{\epsilon_{m-2}} D(a,b)^{\epsilon_{m-1}}.$$

It is easy to see that there are (Laurent) polynomials $w_{ij}(t,u) \in \mathbb{Z}[t^{\pm}, u]$ $(1 \leq i,j \leq 2)$ such that $W(a,b) = (w_{ij}(a,b))$. Let $\varphi(t,u) := w_{11}(t,u) + (t^{-1} - t)w_{12}(t,u) \in \mathbb{Z}[t^{\pm}, u]$. Then it is shown ([Ril84]) that there is a unique polynomial $\Phi(x,u) \in \mathbb{Z}[x,u]$ such that

$$\Phi(t + t^{-1}, u) = t^l \varphi(t,u)$$

for an integer l.

Let k be a field with $\mathrm{char}(k) \neq 2$ and let $\mathcal{O}$ be a complete discrete valuation ring with residue field k. Let $\mathcal{X}(G_K)_k$ denote the character variety of G_K over k. The proof of Proposition 1.4.1 of [CS83] tells us that any $\mathrm{tr}(\sigma_{G_K}(g))$ $(g \in G_K)$ is given as a polynomial of $\mathrm{tr}(\sigma_{G_K}(g_1))(= \mathrm{tr}(\sigma_{G_K}(g_2)))$

and $\mathrm{tr}(\sigma_{G_K}(g_1g_2))$ with coefficients in $\mathbb{Z}$. In particular, the character algebra $\mathcal{B}(G_K)_k$ is generated by $\mathrm{tr}(\sigma_{G_K}(g_1))$ and $\mathrm{tr}(\sigma_{G_K}(g_1g_2))$ over k. Let x and y denote the variables corresponding, respectively, to the coordinate functions $\mathrm{tr}(\sigma_{G_K}(g_1))$ and $\mathrm{tr}(\sigma_{G_K}(g_1g_2))$ on $\mathcal{X}(G_K)$. This variable x is consistent with the variable x of $\Phi(x,u)$ (and so causes no confusion). Since $\mathrm{tr}(C(a)) = a + a^{-1}$ and $\mathrm{tr}(C(a)D(a,b)) = a^2 + a^{-2} + b$, the coordinate variables x and y are related with t and u by

$$x = t + t^{-1}, \ \ y = t^2 + t^{-2} + u = x^2 + u - 2.$$

The following theorem is due to Le.

Theorem 2.4.0.1 ([Le93, Theorem 3.3.1]) *We have*

$$\mathcal{X}(G_K)_k = \mathrm{Spec}(k[x,y]/((y - x^2 + 2)\Phi(x, y - x^2 + 2))).$$

Here, for a k-algebra A, the A-rational points on $\Phi(x, y-x^2+2) = 0$ correspond bijectively to isomorphism classes of absolutely irreducible representation $G_K \to \mathrm{SL}_2(A)$ except the finitely many intersection points with $y - x^2 + 2 = 0$.

Example 2.4.0.2 (1) *When K is the trefoil knot $B(3,1)$, we see $\Phi(x, y - x^2 + 2) = y - 1$.*
(2) *When K is the figure eight knot $B(5,3)$, we have $\Phi(x, y - x^2 + 2) = y^2 - (1 + x^2)y + 2x^2 - 1$.*
(3) *When $K := B(7,3)$, the knot 5_2, we have $\Phi(x, y - x^2 + 2) = y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1$.*

By Theorem 2.4.0.1, we have the following

Corollary 2.4.0.3 *Let $\overline{\rho} : G_K \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation so that $[\overline{\rho}]$ is a regular k-rational point of $\mathcal{X}(G_K)_k^{\mathrm{a.i}}$. Then we have*

$$(\mathcal{B}(G_K)_k)^{\wedge}_{[\overline{\rho}]} \simeq k[[x - \mathrm{tr}(\overline{\rho}(g_1))]].$$

So, by Theorem 2.2.2.4, we have

Corollary 2.4.0.4 *Let $\overline{\rho}$ be as in Corollary 2.4.0.3. Suppose that $\rho : G_K \to \mathrm{SL}_2(\mathcal{O}[[x - \alpha]])$, where α is an element of $\mathcal{O}$ such that α mod $\mathfrak{m}_{\mathcal{O}} = \mathrm{tr}(\overline{\rho}(g_1))$, is a deformation of $\overline{\rho}$ satisfying*

$$\mathrm{tr}(\rho(g_1)) = x. \tag{2.4.1}$$

Then the pair $(\mathcal{O}[[x - \alpha]], \rho)$ is the universal deformation of $\overline{\rho}$.

In the following, we discuss some concrete examples, where k will be a finite field $\mathbb{F}_p$ for some odd prime number p.

Convention. Let R be a complete local ring with residue field $R/\mathfrak{m}_R = \mathbb{F}_p$. When the equation $X^2 = a$ for $a \in R$ has two simple roots in R, we denote by $\sqrt{a}$ for the "positive" solution, namely, $(\sqrt{a})^2 = a$ and $\sqrt{a}$ mod $\mathfrak{m}_R \in \{1, \dots, \frac{p-1}{2}\}$.

2.4.1 Riley representations

For each 2-bridge knot K, there is a representation $\rho_{\mathrm{Riley},\omega} : G_K \to \mathrm{SL}_2(\mathbb{C})$ called Riley representation which is defined by the following:

$$g_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; g_2 \mapsto \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix},$$

where ω is a non-zero root of $\Phi(2,\omega) = 0$, and $\Phi(2,u) \in \mathbb{Z}[u]$ is a monic polynomial defined before. By taking an odd prime p satisfying $\omega \bmod p \in \mathbb{F}_p$, we can consider the mod p representation $\overline{\rho}_{\mathrm{Riley},\omega} : G_K \to \mathrm{SL}_2(\mathbb{F}_p)$. It is known, due to J. Ueki, that the universal deformation $\boldsymbol{\rho}_{\mathrm{Riley},\omega} : G_K \to \mathrm{SL}_2(\mathbb{Z}_p[[x-2]])$ of $\overline{\rho}_{\mathrm{Riley},\omega}$ is given by the following [MTTU17, Theorem 4.3.3]:

$$\boldsymbol{\rho}_{\mathrm{Riley},\omega}(g_1) := \begin{pmatrix} \frac{x}{2} & 1 \\ \frac{x^2-4}{4} & \frac{x}{2} \end{pmatrix}, \; \boldsymbol{\rho}_{\mathrm{Riley},\omega}(g_2) := \begin{pmatrix} \frac{x}{2} & \frac{(1-v(x))^2 s(x)}{x^2-4} \\ \frac{(1+v(x))^2 s(x)}{4} & \frac{x}{2} \end{pmatrix},$$

where $s(x) \in \mathbb{Z}_p[[x-2]]^\times$ depends on K, and $v(x) := \sqrt{1 + \frac{x^2-4}{s(x)}} \in \mathbb{Z}_p[[x-2]]^\times$.

Let us consider the L-function associated to $\boldsymbol{\rho}_{\mathrm{Riley},\omega}$. Having Corollary 2.3.2.3 in our mind, since we have $\det(\overline{\rho}_{\mathrm{Riley},\omega}(g_1 g_2) - I_2) = -\omega \bmod p \neq 0 \in \mathbb{F}_p$, we check the value of $\Delta_K(\overline{\rho}_{\mathrm{Riley},\omega}; 1)$. In order to do this, we consider the total representation $\rho_\Phi : G_K \to \mathrm{SL}_{2N}(\mathbb{Z})$ associated to $\rho_{\mathrm{Riley},\omega}$, where $N := \deg(\Phi(2,u))$.

Recall that the total representation $\rho_\Phi : G_K \to \mathrm{SL}_{2N}(\mathbb{Z})$ is a representation induced by $\rho_{\mathrm{Riley},\omega}$, which is given by the following:

$$g_1 \mapsto \begin{pmatrix} I_N & I_N \\ 0 & I_N \end{pmatrix}, \; g_2 \mapsto \begin{pmatrix} I_N & 0 \\ C & I_N \end{pmatrix},$$

where C is the companion matrix of $\Phi(2,u)$, namely when $\Phi(2,u)$ is given by $u^N + c_{N-1}u^{N-1} + \cdots + c_0$, then

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{N-1} \end{pmatrix}.$$

It is known, due to Silver and Williams, that the absolute value of $\Delta_{K,\rho_\Phi}(1)$ is given by the following.

Theorem 2.4.1.1 ([SW09, Theorem 5.1]) Let K be a 2-bridge knot and $\rho_\Phi : G_K \to \mathrm{SL}_{2N}(\mathbb{Z})$ be a total representation associated to $\rho_{\mathrm{Riley},\omega}$. Then we have $|\Delta_{K,\rho_\Phi}(1)| = 2^N$.

Also, note that the relation between $\Delta_{K,\rho_\Phi}(t)$ and $\Delta_{K,\rho_{\mathrm{Riley},\omega}}(t)$ is given by the following:

Lemma 2.4.1.2 (cf. [KSW99]) *We have*

$$\Delta_{K,\rho_\Phi}(t) = \prod_{\omega} \Delta_{K,\rho_{\mathrm{Riley},\omega}}(t),$$

where ω range over the roots of $\Phi(2,u) = 0$.

Now, let us consider the value of $\Delta_K(\overline{\rho}_{\mathrm{Riley},\omega};1)$. Since p is odd, by taking mod p in the equality of Theorem 2.4.1.1, we have $\Delta_{K,\rho_\Phi}(1) \bmod p \neq 0 \in \mathbb{F}_p$. Then by using Lemma 2.4.1.2, we have $\prod_\omega \Delta_{K,\rho_{\mathrm{Riley},\omega}}(1) \bmod p \neq 0 \in \mathbb{F}_p$, and so $\Delta_{K,\rho_{\mathrm{Riley},\omega}}(1) \bmod p \neq 0 \in \mathbb{F}_p$, namely $\Delta_{K,\overline{\rho}_{\mathrm{Riley},\omega}}(1) \neq 0$.

Therefore, by Corollary 2.3.2.3, we have

$$H_1(\boldsymbol{\rho}_{\mathrm{Riley},\omega}) = 0,\ L_K(\boldsymbol{\rho}_{\mathrm{Riley},\omega}) \doteq 1.$$

2.4.2 Holonomy representations

For each hyperbolic knot K, there is a representation $\rho_{\mathrm{hol}} : G_K \to \mathrm{PSL}_2(\mathbb{C})$ called holonomy representation, which is determined by the complete hyperbolic structure of the complement of K. It is known that we can lift ρ_{hol} to $\mathrm{SL}_2(\mathbb{C})$-representation, which we also denote by $\rho_{\mathrm{hol}} : G_K \to \mathrm{SL}_2(\mathbb{C})$. When K is a 2-bridge knot, we have the following expression for ρ_{hol}:

Lemma 2.4.2.1 ([DHY09]) *Let K be a 2-bridge hyperbolic knot and $\rho_{\mathrm{hol}} : G_K \to \mathrm{SL}_2(\mathbb{C})$ a lift of the holonomy representation. Then ρ_{hol} is given, up to conjugation, by*

$$g_1 \mapsto \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ g_2 \mapsto \pm \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix},$$

where ω is a root of $\Phi(2,\omega) = 0$.

Corresponding to the sign, denote the above representations by $\rho_{\mathrm{hol},\pm} : G_K \to \mathrm{SL}_2(\mathbb{C})$ respectively. As before, by taking odd prime p satisfying $\omega \in \mathbb{F}_p$, we can consider the mod p representations $\overline{\rho}_{\mathrm{hol},\pm} : G_K \to \mathrm{SL}_2(\mathbb{F}_p)$ and those universal deformations $\boldsymbol{\rho}_{\mathrm{hol},\pm} : G_K \to \mathrm{SL}_2(\mathbb{Z}_p[[x \pm 2]])$ respectively.

Now, let us consider the L-function associated to $\boldsymbol{\rho}_{\mathrm{hol},\pm}$. For the case of $\boldsymbol{\rho}_{\mathrm{hol},+}$, we have the same discussion as 2.4.1, so we have

$$H_1(\boldsymbol{\rho}_{\mathrm{hol},+}) = 0,\ L_K(\boldsymbol{\rho}_{\mathrm{hol},+}) \doteq 1.$$

For the case of $\boldsymbol{\rho}_{\mathrm{hol},-}$, namely when the trace of $\overline{\rho}_{\mathrm{hol},-}(g_1)$ is $p-2 \in \mathbb{F}_p$, we will see in 2.4.3 (3) that it depends on primes whether $H_1(\boldsymbol{\rho}_{\mathrm{hol},-})$ and $L_K(\boldsymbol{\rho}_{\mathrm{hol},-})$ are trivial or not.

2.4.3 Other examples

(1) Let $K := B(3,1)$, the trefoil knot, whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle.$$

We have $\mathcal{X}(G_K)_{\mathrm{a.i}}(k) = \{(x,y) \in k^2 \mid y = 1\}$.

Let $k = \mathbb{F}_3$ and $\mathcal{O} = \mathbb{Z}_3$, and consider the following absolutely irreducible representation whose $\mathrm{PGL}_2(\mathbb{F}_3)$-conjugacy class corresponds to the regular $\mathbb{F}_3$-rational point $(x,y) = (2,1)$ of $\mathcal{X}(G_K)_{\mathrm{a.i}}$ (Proposition 2.2.1.3):

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

Let $\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$ be the representation defined by

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$
$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix}.$$

We see by the straightforward computation that $\boldsymbol{\rho}_1$ is indeed a representation of G_K and a deformation of $\overline{\rho}_1$ (see our convention). Moreover, we have $\mathrm{tr}(\boldsymbol{\rho}_1(g_1)) = x$, hence $\boldsymbol{\rho}_1$ satisfies the condition (2.4.1). Therefore $(\boldsymbol{R}_{\overline{\rho}_1} = \mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is the universal deformation of $\overline{\rho}_1$.

We easily see that $\Delta_0(H_0(\boldsymbol{\rho}_1)) \doteq 1$ and $\Delta_K(\overline{\rho}_1; t) = 1+t^2$, hence, $\Delta_K(\overline{\rho}_1; 1) = 2 \neq 0$. Therefore, by Proposition 2.3.2.6, we have

$$H_1(\boldsymbol{\rho}_1) = 0,\ L_K(\boldsymbol{\rho}_1) \doteq 1.$$

(2) Let $K := B(5,3)$, the figure eight knot, whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1g_2^{-1}g_1^{-1}g_2g_1 = g_2g_1g_2^{-1}g_1^{-1}g_2 \rangle.$$

We have $\mathcal{X}(G_K)_{\mathrm{a.i}}(k) = \{(x,y) \in k^2 \mid y^2 - (1+x^2)y + 2x^2 - 1 = 0\} \setminus \{(\pm\sqrt{5}, 3)\}$.

Let $k = \mathbb{F}_7$ and $\mathcal{O} = \mathbb{Z}_7$, and consider the following absolutely irreducible representation whose $\mathrm{PGL}_2(\mathbb{F}_7)$-conjugacy class corresponds to the regular $\mathbb{F}_7$-rational points $(x,y) = (5,5)$ of $\mathcal{X}(G_K)_{\mathrm{a.i}}$:

$$\overline{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{F}_7);\ \overline{\rho}_2(g_1) = \begin{pmatrix} 0 & 6 \\ 1 & 5 \end{pmatrix}, \overline{\rho}_2(g_2) = \begin{pmatrix} 5 & 6 \\ 1 & 0 \end{pmatrix}.$$

Let $\boldsymbol{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{Z}_7[[x+2]])$ be the representation defined by

$$\boldsymbol{\rho}_2(g_1) = \begin{pmatrix} \frac{x+\sqrt{\frac{x^2-5+u(x)}{2}}}{2} & -1 \\ -\frac{x^2-3-u(x)}{8} & \frac{x-\sqrt{\frac{x^2-5+u(x)}{2}}}{2} \end{pmatrix},$$
$$\boldsymbol{\rho}_2(g_2) = \begin{pmatrix} \frac{x-\sqrt{\frac{x^2-5+u(x)}{2}}}{2} & -1 \\ -\frac{x^2-3-u(x)}{8} & \frac{x+\sqrt{\frac{x^2-5+u(x)}{2}}}{2} \end{pmatrix},$$

where $u(x) := \sqrt{(x^2-1)(x^2-5)}$. We see by the straightforward computation that $\boldsymbol{\rho}_2$ is indeed a representation of G_K and a deformation of $\overline{\rho}_2$. Moreover, we have $\mathrm{tr}(\boldsymbol{\rho}_2(g_1)) = x$, hence $\boldsymbol{\rho}_2$ satisfies the condition (2.4.1). Therefore $(\boldsymbol{R}_{\overline{\rho}_2} = \mathbb{Z}_7[[x+2]], \boldsymbol{\rho}_2)$ is the universal deformation of $\overline{\rho}_2$.

We easily see that $\det(\overline{\rho}_2(g_2)-I) = 4 \neq 0$ and that $\Delta_K(\overline{\rho}_2; t) = t^{-2}+4t^{-1}+1$, hence, $\Delta_K(\overline{\rho}_2; 1) = 6 \neq 0$. Therefore, by Corollary 2.3.2.3, we have

$$H_1(\boldsymbol{\rho}_2) = 0,\ L_K(\boldsymbol{\rho}_2) \doteq 1.$$

(3) Let $K := B(7,3)$, the knot 5_2, whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1g_2g_1^{-1}g_2^{-1}g_1g_2g_1 = g_2g_1g_2g_1^{-1}g_2^{-1}g_1g_2\rangle.$$

We have $\mathcal{X}(G_K)_{\mathrm{a.i}}(k) = \{(x,y) \in k^2 \mid y^3-(x^2+1)y^2+(3x^2-2)y-2x^2+1 = 0\} \setminus \{(\pm\sqrt{\frac{7}{2}}, \frac{3}{2})\}$.

Firstly, let $k = \mathbb{F}_{11}$ and $\mathcal{O} = \mathbb{Z}_{11}$, and consider the following absolutely irreducible representation whose $\mathrm{PGL}_2(\mathbb{F}_{11})$-conjugacy class corresponds to the regular $\mathbb{F}_{11}$-rational point $(x,y) = (5,5)$ of $\mathcal{X}(G_K)_{\mathrm{a.i}}$:

$$\overline{\rho}_3 : G_K \to \mathrm{SL}_2(\mathbb{F}_{11});\ \overline{\rho}_3(g_1) = \begin{pmatrix} 5 & 10 \\ 1 & 0 \end{pmatrix},\ \overline{\rho}_3(g_2) = \begin{pmatrix} 5 & 1 \\ 10 & 0 \end{pmatrix}.$$

Let $\alpha := \frac{3-\sqrt{5}}{2}, \xi := \frac{4-\sqrt{5}}{4} \in \mathbb{Z}_{11}$ so that $\alpha \bmod 11 = 5, \xi \bmod 11 = 0 \in \mathbb{F}_{11}$. Let $s = s(x)$ be the unique solution in $\mathbb{Z}_{11}[[x-\alpha]]$ satisfying the equation

$$64s^3 - 16(2x^2+5)s^2 + 4(x^4+9x^2+6)s - 4x^4 - 6x^2 - 1 = 0 \tag{2.4.3.1}$$

and

$$s(\alpha) = \xi. \tag{2.4.3.2}$$

Such an $s(x)$ is proved, by Hensel's lemma ([Ser68, §4, Proposition 7]) to exist uniquely. Now, let $\boldsymbol{\rho}_3 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{11}[[x-\alpha]])$ be the representation defined by

$$\boldsymbol{\rho}_3(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-4s(x)}}{2} & -1 \\ -s(x)+1 & \frac{x-\sqrt{x^2-4s(x)}}{2} \end{pmatrix},$$
$$\boldsymbol{\rho}_3(g_2) = \begin{pmatrix} \frac{x+\sqrt{x^2-4s(x)}}{2} & 1 \\ s(x)-1 & \frac{x-\sqrt{x^2-4s(x)}}{2} \end{pmatrix}.$$

We can verify by (2.4.3.1) that $\boldsymbol{\rho}_3$ is indeed a representation of G_K and by (2.4.3.2) that $\boldsymbol{\rho}_3$ is a deformation of $\overline{\rho}_3$. Moreover, we have $\mathrm{tr}(\boldsymbol{\rho}_3(g_1)) = x$, hence $\boldsymbol{\rho}_3$ satisfies the condition (2.4.1). Therefore $(\boldsymbol{R}_{\overline{\rho}_3} = \mathbb{Z}_{11}[[x-\alpha]], \boldsymbol{\rho}_3)$ is the universal deformation of $\overline{\rho}_3$.

Consider the 11-adic lifting $\rho_3 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{11})$ of $\overline{\rho}_3$ defined by $\boldsymbol{\rho}_3|_{x=5}$:

$$\rho_3(g_1) = \begin{pmatrix} \frac{5+\sqrt{25-4\mu}}{2} & -1 \\ -\mu+1 & \frac{5-\sqrt{25-4\mu}}{2} \end{pmatrix},$$

$$\rho_3(g_2) = \begin{pmatrix} \frac{5+\sqrt{25-4\mu}}{2} & 1 \\ \mu - 1 & \frac{5-\sqrt{25-4\mu}}{2} \end{pmatrix},$$

where μ is the unique solution in $\mathbb{Z}_{11}$ satisfying (4.5.1) with $x = 5$ and $\mu \bmod 11 = 0$. Then we easily see that $\det(\rho_3(g_2) - I) = -3 \neq 0$, and that $\Delta_K(\rho_3; t) = -2\{-8\mu^2+58\mu-52+5t+(-8\mu^2+58\mu-52)t^2\}$, hence, $\Delta_K(\rho_3; 1) = -2(-16\mu^2+116\mu-99) \neq 0$. Therefore, by Theorem 2.3.2.2, $H_1(\boldsymbol{\rho}_3)$ is a finitely generated torsion $\mathbb{Z}_{11}[[x-\alpha]]$-module.

We let $r := g_1g_2g_1^{-1}g_2^{-1}g_1g_2g_1g_2^{-1}g_1^{-1}g_2g_1g_2^{-1}g_1^{-1}g_2^{-1}$ and set

$$\partial_2 = \left(\boldsymbol{\rho}_3\left(\frac{\partial r}{\partial g_1}\right), \boldsymbol{\rho}_3\left(\frac{\partial r}{\partial g_2}\right)\right) = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4).$$

By the computer calculation, we find that all 2-minors of ∂_2 are given by

$$\begin{aligned}
\det(\boldsymbol{a}_1, \boldsymbol{a}_2) &= 2(x-2)\{4(s-1)x^2 + x - 4(2s-1)^2\}, \\
\det(\boldsymbol{a}_1, \boldsymbol{a}_3) &= -\tfrac{1}{2}\{4(s-1)x^4 - 2(8s^2-2s-5)x^2 + 4(s-1)x \\
&\quad +(4s-3)(12s-5)\}(x-2-\sqrt{x^2-4s}), \\
\det(\boldsymbol{a}_1, \boldsymbol{a}_4) &= 4(s-1)x^4 - 8(s-1)x^3 - 4(4s^2-5s+2)x^2 \\
&\quad +4(8s^2-7s+2)x - (4s-1)^2, \\
\det(\boldsymbol{a}_2, \boldsymbol{a}_3) &= -\{4(s-1)x^4 - 8(s-1)x^3 - 4(4s^2-5s+2)x^2 \\
&\quad +4(8s^2-7s+2)x - (4s-1)^2\}, \\
\det(\boldsymbol{a}_2, \boldsymbol{a}_4) &= 2\{4(s-1)x^2 + x - 4(2s-1)^2\}(x-2+\sqrt{x^2-4s}), \\
\det(\boldsymbol{a}_3, \boldsymbol{a}_4) &= 2(x-2)\{4(s-1)x^2 + x - 4(2s-1)^2\}.
\end{aligned} \tag{2.4.3.3}$$

By (2.4.3.3) and the computer calculation, we find that $x = \alpha$ ($s(\alpha) = \frac{4-\sqrt{5}}{4}$) gives a common zero of all 2-minors of ∂_2 and and their derivatives and is not a common zero of the third order derivatives of all 2-minors. Hence the greatest common divisor of all 2-minors is $(x-\alpha)^2$. Therefore, by Proposition 2.3.2.4, we have

$$H_1(\boldsymbol{\rho}_3) \simeq \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}, \ L_K(\boldsymbol{\rho}_3) \doteq (x-\alpha)^2.$$

Secondly, let $k = \mathbb{F}_{19}$ and $\mathcal{O} = \mathbb{Z}_{19}$, and consider the following absolutely irreducible representation whose $\mathrm{PGL}_2(\mathbb{F}_{19})$-conjugacy class corresponds to the regular $\mathbb{F}_{19}$-rational point $(x, y) = (6, 6)$ of $\mathcal{X}(G_K)_{\mathrm{a.i}}$:

$$\overline{\rho}_4 : G_K \to \mathrm{SL}_2(\mathbb{F}_{19}); \ \overline{\rho}_4(g_1) = \begin{pmatrix} 14 & 1 \\ 1 & 11 \end{pmatrix}, \ \overline{\rho}_4(g_2) = \begin{pmatrix} 11 & 1 \\ 1 & 14 \end{pmatrix}.$$

Let $\beta := \frac{3+\sqrt{5}}{2}, \zeta := \frac{7+\sqrt{5}}{8} \in \mathbb{Z}_{19}$ so that $\beta \bmod 19 = 6, \zeta \bmod 19 = 2 \in \mathbb{F}_{19}$. Let $v = v(x)$ be the unique solution in $\mathbb{Z}_{19}[[x-\beta]]$ satisfying the equation

$$64v^3 - 16(x^2+7)v^2 + 28(x^2+2)v - 12x^2 - 7 = 0 \tag{2.4.3.4}$$

and

$$v(\beta) = \zeta. \tag{2.4.3.5}$$

Such a $v(x)$ is proved, by Hensel's lemma ([Ser68, §4, Proposition 7]) to exist uniquely. Now, let $\boldsymbol{\rho}_4 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19}[[x-\beta]])$ be the representation defined by

$$\begin{aligned}\boldsymbol{\rho}_4(g_1) &= \begin{pmatrix} \frac{x+\sqrt{x^2-4v(x)}}{2} & 1 \\ v(x)-1 & \frac{x-\sqrt{x^2-4v(x)}}{2} \end{pmatrix}, \\ \boldsymbol{\rho}_4(g_2) &= \begin{pmatrix} \frac{x-\sqrt{x^2-4v(x)}}{2} & 1 \\ v(x)-1 & \frac{x+\sqrt{x^2-4v(x)}}{2} \end{pmatrix}.\end{aligned}$$

We can verify by (2.4.3.4) that $\boldsymbol{\rho}_4$ is indeed a representation of G_K and by (2.4.3.5) that $\boldsymbol{\rho}_4$ is a deformation of $\overline{\rho}_4$. Moreover, we have $\mathrm{tr}(\boldsymbol{\rho}_4(g_1)) = x$, hence $\boldsymbol{\rho}_4$ satisfies the condition (2.4.1). Therefore $(\boldsymbol{R}_{\overline{\rho}_4} = \mathbb{Z}_{19}[[x-\beta]], \boldsymbol{\rho}_4)$ is the universal deformation of $\overline{\rho}_4$.

Consider the 19-adic lifting $\rho_4 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19})$ of $\overline{\rho}_4$ defined by $\boldsymbol{\rho}_4|_{x=6}$:

$$\rho_4(g_1) = \begin{pmatrix} \frac{6+\sqrt{36-4\nu}}{2} & 1 \\ \nu-1 & \frac{6-\sqrt{36-4\nu}}{2} \end{pmatrix},$$

$$\rho_4(g_2) = \begin{pmatrix} \frac{6-\sqrt{36-4\nu}}{2} & 1 \\ \nu-1 & \frac{6+\sqrt{36-4\nu}}{2} \end{pmatrix},$$

where ν is the unique solution in $\mathbb{Z}_{19}$ satisfying (4.5.4) with $x = 6$ and $\nu \bmod 19 = 2$. Then we easily see that $\det(\rho_4(g_2) - I) = -4 \neq 0$, and that $\Delta_K(\rho_4; t) = -2\{-8\nu^2+80\nu-74+6t+(-8\nu^2+80\nu-74)t^2\}$, hence, $\Delta_K(\rho_4; 1) = -2(-16\nu^2+160\nu-142) \neq 0$. Therefore, by Theorem 2.3.2.2, $H_1(\boldsymbol{\rho}_4)$ is a finitely generated torsion $\mathbb{Z}_{19}[[x-\beta]]$-module.

We set

$$\partial_2 = \left(\boldsymbol{\rho}_4\left(\frac{\partial r}{\partial g_1}\right), \boldsymbol{\rho}_4\left(\frac{\partial r}{\partial g_2}\right)\right) = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4).$$

By the computer calculation, we find that all 2-minors of ∂_2 are given by

$$\begin{aligned}
\det(\boldsymbol{b}_1, \boldsymbol{b}_2) &= 2(x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}, \\
\det(\boldsymbol{b}_1, \boldsymbol{b}_3) &= -\tfrac{1}{2}\{4(v-1)x^2 - 4(v-1)x - (4v-3)^2\}\sqrt{x^2-4v} \\
\det(\boldsymbol{b}_1, \boldsymbol{b}_4) &= 4(v-1)x^4 - (8v-9)x^3 - 2(8v^2-10v+5)x^2 \\
&\quad +4(8v^2-9v+3)x - (4v-3)^2 \\
&\quad -(x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}\sqrt{x^2-4v}, \\
\det(\boldsymbol{b}_2, \boldsymbol{b}_3) &= -\{4(v-1)x^4 - (8v-9)x^3 - 2(8v^2-10v+5)x^2 \\
&\quad +4(8v^2-9v+3)x - (4v-3)^2\} \\
&\quad -(x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}\sqrt{x^2-4v}, \\
\det(\boldsymbol{b}_2, \boldsymbol{b}_4) &= 2\{4(v-1)x^2 + x - 4(2v-1)^2\}\sqrt{x^2-4v}, \\
\det(\boldsymbol{b}_3, \boldsymbol{b}_4) &= 2(x-2)\{4(v-1)x^2 + x - 4(2v-1)^2\}.
\end{aligned} \tag{2.4.3.6}$$

By (2.4.3.6) and the computer calculation, we find that $x = \beta$ ($v(\beta) = \frac{7+\sqrt{5}}{8}$) is a common zero of all 2-minors of ∂_2 and their derivatives and is not a common zero of the third order derivatives of all 2-minors. Hence the greatest common divisor of all 2-minors is $(x-\beta)^2$. Therefore, by Proposition 2.3.2.4, we have

$$H_1(\boldsymbol{\rho}_4) \simeq \mathbb{Z}_{19} \oplus \mathbb{Z}_{19},\ L_K(\boldsymbol{\rho}_4) \doteq (x-\beta)^2.$$

We see that all examples above answer Problems 2.3.2.1 affirmatively and answer Problem 2.3.2.7 concretely.

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