Construction of a regularization-independent supercurrent in terms of the gradient flow

笠井, 彩

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Construction of a regularization-independent supercurrent in terms of the gradient flow

笠井 彩 (Aya Kasai)
Department of physics, Graduate School of Science, Kyushu University

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Abstract

We construct a regularization-independent representation of the supercurrent—the Noether current associated with supersymmetry—in the four-dimensional $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric Yang–Mills theories. For this, we employ the so-called gradient flow. The gradient flow is the evolution of quantum fields along a fictitious time according to diffusion-type equations. A salient feature of the gradient flow is that composite operators (i.e., local products) of flowed fields automatically become renormalized ultraviolet finite operators under the ordinary parameter renormalization of the original field theory. This implies that any operator represented by flowed fields (and renormalized parameters) is independent of the regularization. We obtain such a representation for the supercurrent in the above supersymmetric gauge theories by combining a detailed one-loop level analysis of supersymmetric Ward–Takahashi identities in the Wess–Zumino gauge and the small flow-time expansion. We believe that our representation of the properly-normalized conserved supercurrent will be very useful, for instance, in the parameter tuning toward the supersymmetric point in the future lattice numerical simulations of the above mentioned supersymmetric gauge theories.

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Chapter 1

Introduction

So far, the so-called Standard Model (SM) perfectly describes the dynamics of elementary particles to the energy scale of $\mathcal{O}(10^2)$GeV. The SM is a renormalizable Quantum Field Theory (QFT) based on the gauge principle and it subsumes three forces in nature; the electromagnetic interaction, the weak interaction, and the strong interaction. It is quite conceivable, however, that the SM is not the ultimate theory; we expect that physics Beyond the Standard Model (BSM) emerges in higher energy scales. One of the reasons for this belief is that the gravitational interaction, which is characterized by the Planck energy scale $M_P \sim \mathcal{O}(10^{18})$GeV, is not contained in the above list; we do not know a renormalizable form of the quantized gravity [1, 2]. Another reason is that, assuming the Planck scale provides ultraviolet (UV) cutoff (this implies that the SM holds up to very high energy scale), the mass of the Higgs scalar $m \sim 125$GeV is unnaturally small [6]: Through radiative corrections, the observed Higgs scalar's mass m and its bare mass m_0 will be related roughly as

$$m^2 - m_0^2 \sim \mathcal{O}(M_P^2), \tag{1.1}$$

Since $M_P \sim \mathcal{O}(10^{18})$GeV and $m \sim 125$GeV, this relation requires extreme fine tuning of m_0^2 in $\sim 10^{-30}$%. Another strong suggestion for the BSM physics is the existence of dark matter [3, 4, 5], the matter which interacts with the particles in the SM very weakly.

An attractive possibility that would answer the above three questions is *supersymmetry* (SUSY). First, the superstring theory, the only known consistent theory of quantized gravity, (usually) requires SUSY for its self-consistency. Second, because SUSY is a symmetry between bosons and fermions (see below), there occurs cancellation of radiative corrections between bosons and fermions. This weakens the cutoff dependence M_P^2 in (1.1) roughly to $\ln(M_P^2/\mu^2)$ and may solve the naturalness problem. Lastly, supersymmetric theories contain bosons and fermions as pairs (SUSY partners). Some of (many) unobserved particles in those pairs are candidates of dark matter.

Let us illustrate the idea of SUSY, by taking the simplest SUSY model, a system of a free complex scalar field $\phi(x)$ and a free Weyl spinor field $\psi_\alpha(x)$, $\alpha = 1, 2$ in the four-dimensional Minkowski spacetime. The Lagrangian density is given by

$$\mathcal{L} = -\partial^\mu \phi^\dagger \partial_\mu \phi + i\partial^\mu \bar{\psi}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \psi_\alpha, \tag{1.2}$$

where $\sigma_\mu^{\dot{\alpha}\alpha}$ are the Pauli matrices ($\sigma_{\mu=0}^{\dot{\alpha}\alpha} = 1$). This Lagrangian is invariant under the following

SUSY transformation up to the total divergence:

$$\phi \to \phi + \sqrt{2}\xi^{\alpha}\psi_{\alpha} \tag{1.3}$$

$$\psi \to \psi + i\sqrt{2}\sigma^{\mu}\bar{\xi}\partial_{\mu}\phi. \tag{1.4}$$

(The parameter ξ_α of SUSY transformation is a Grassmann-odd Weyl spinor.) The above SUSY transformation "mixes" the bosonic field $\phi(x)$ with the fermionic field $\psi_\alpha(x)$.

More generally, SUSY is characterized by the algebra formed by the generators of SUSY transformations, Q^A_α, $\bar{Q}_{\dot{\beta}^B}$, $A, B = 1, 2, \dots \mathcal{N}$ ($\mathcal{N}$ can be 1,2,3,4 in the four-dimensional spacetime). These form a closed super Lie algebra with the generators p_μ, $M_{\mu\nu}$ of the Poincaré transformations as [24].

$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\beta}}\} = 2\delta^{AB} p_\mu \sigma^\mu_{\alpha\dot{\beta}}, \tag{1.5}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} = 0, \tag{1.6}$$

$$[p_\mu, Q^A_\alpha] = [p_\mu, \bar{Q}^B_{\dot{\beta}}] = 0, \tag{1.7}$$

$$[M_{\mu\nu}, Q^A_\alpha] = \frac{1}{2}[\sigma_\mu, \sigma_\nu]^\beta_\alpha Q^A_\beta. \tag{1.8}$$

Note that since the momentum p_μ is the generator of the spacetime translation, SUSY should be regarded as a spacetime symmetry (although it is fermionic).

Now, we are interested in non-perturbative aspects of QFT. We know there exist important non-perturbative quantum effects in nature, such as the spontaneous chiral symmetry breaking and the quark confinement. In the context of SUSY theories, we are interested in a non-perturbative spontaneous SUSY breaking (SUSY must be spontaneously broken in our physical world in some way to explain non-degeneracy of bosons and fermions), the spectrum of bound states and so on. The Lattice field theory [7] is the most well-developed method which enables to study non-perturbative phenomena in QFT. In this framework, the continuous spacetime is approximated by a discrete set of points (lattice). Then the functional integral that defines QFT is carried out by applying the Monte Carlo simulation method [7, 8, 9]. This lattice structure, however, explicitly breaks symmetries associated with the continuous spacetime, such as translational invariance. As we have seen in Eq. (1.5), SUSY is a spacetime symmetry related to the translation. Thus SUSY is explicitly broken in lattice field theory. If a preferred symmetry is broken by the regularization (in the present case, by the lattice regularization), one has to generally tune parameters in the Lagrangian so that the symmetry is restored in the continuum limit. More definitely, one has to tune parameters so that the Ward–Takahashi (WT) identities associated with the symmetry broken by regularization is restored. For example, when the Wilson lattice fermion action [10] is employed to describe the four-dimensional (4D) $\mathcal{N} = 1$ super Yang–Mills theory (SYM), one has to tune the gaugino (the SUSY partner of the gauge boson) mass parameter. However, since the WT identities contains the Noether current associated with the symmetry broken by regularization, one has to find a correct expression of the Noether current at the same time as the parameter tuning. Note that since the Noether current is a composite operator whose finiteness is guaranteed by the associated WT identities, the construction of the Noether current is quite non-trivial when the regularization breaks the relevant symmetry.

In this paper, we construct a regularization-independent representation of the properly-normalized conserved supercurrent—the Noether current associated with SUSY—in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM by employing the so-called gradient flow. Since this representation, which may be used with the lattice regularization, a precise form determined a priori of the supercurrent will be useful, first of all, in the parameter tuning toward the supersymmetric continuum limit in future lattice numerical simulations of the above mentioned supersymmetric gauge theories.

Relating to our analysis, there exists another complication in SUSY gauge theories. The full linear SUSY multiplet of the gauge boson contains many non-dynamical auxiliary fields; the sole role of these fields is to close the SUSY algebra without using the equations of motion. However, some of these non-dynamical field has the mass dimension zero. If the regularization does not preserve SUSY, we have to add counterterms to the action to restore SUSY, there exists a mass dimension zero field, any function of this field may be a counterterm; it is not clear one can control such a counterterm of an arbitrary functional form.

A natural way out of this complication is to take the so-called Wess–Zumino (WZ) gauge [30], in which the SUSY multiplet contains only dynamical (ordinary) fields, the gauge field A_μ^a and the gaugino fields λ_α^a, $\bar{\lambda}_{\dot{\alpha}}^a$. In our analyses in Chap. 3 and Chap. 4, we always take this WZ gauge. The drawback of the WZ gauge is that with the WZ gauge SUSY transformations become non-linear in fields; this non-linearity produces many composite operators in the SUSY WZ identities and makes the renormalization of the supercurrent complicated as we will see in Chap. 3 and in Chap. 4.

Let us describe the gradient flow [11]–[15] in the pure Yang–Mills (YM) theory in the $D = 4$ Euclidean spacetime, whose action is given by

$$S_{\mathrm{YM}} = -\frac{1}{2g_0^2}\int d^4x\,\mathrm{tr}\Big[F_{\mu\nu}(x)F_{\mu\nu}(x)\Big], \tag{1.9}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{1.10}$$

where A_μ is the YM gauge field. The gradient flow is the evolution of the gauge field along a fictitious time $t \geq 0$. The evolution along the flow time t is defined by the flow equation

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu, \quad B_\mu(t=0,x) = A_\mu(x), \tag{1.11}$$

where the flowed field strength $G_{\mu\nu}$ and the flowed covariant derivative D_μ are defined as

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot]. \tag{1.12}$$

Here α_0 is a constant and does not affect gauge-invariant observables (see Chap. 2 for details). When $\alpha_0 = 0$, the flow equation is expressed as

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)}. \tag{1.13}$$

The right-hand side is the gradient of the action, so is the name of the gradient flow.

Lüscher and Weisz [14] studied the renormalization property of correlation functions of the flowed gauge field $B_\mu(x)$. They showed in all order of perturbation theory that any correlation functions and composite operators (i.e., local product) of the flowed gauge field are UV finite without any multiplicative renormalization, once the parameters in the Yang–Mills theory are renormalized in the usual way (for the proof of this renormalizability of the gradient flow, see Refs. [14, 15, 16]).[1] The proof of this renormalizability in Ref. [14] is briefly sketched in Chap. 2. Since composite

[1] Precisely speaking, for matter fields such as the fermion field and the scalar field, even the flowed fields require the wave function renormalization. We will encounter this situation in Chap. 3 and Chap. 4.

operators of the flowed gauge field are UV finite, they are independent of the way of regularization; they are thus *regularization-independent*.

One can further expand a composite operator of the flowed gauge field in terms of composite operators of the un-flowed gauge field in the small flow-time limit $t \to 0$ [14]. This is the *small flow-time expansion* and its general form reads

$$\mathcal{O}(t,x) = \zeta_1(t)\mathcal{O}_1(x) + \zeta_2(t)\mathcal{O}_2(x) + \ldots + \mathcal{O}(t). \tag{1.14}$$

Since the flow time has the mass dimension -2, this series is an expansion in terms of local composite operators with increasing mass dimensions. This expansion can be used, by inverting this relation, to represent a composite operator of the un-flowed field by the composite operator of the flowed field. Since the latter is independent of the regularization as already noted, using this expansion, we can express any finite operator of the un-flowed field in a regularization independent way.

This sort of representation of a finite operator in terms of the gradient flow was first considered for the energy-momentum tensor (EMT), the Noether current associated with the translational invariance, in Ref. [18]. In the pure YM theory, the EMT is

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x)\right]. \tag{1.15}$$

This expression in quantum theory assumes the dimensional regularization. On the other hand, for the following dimension 4 gauge invariant composite operators,

$$U_{\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x), \tag{1.16}$$

$$E(t,x) \equiv \frac{1}{4}G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x), \tag{1.17}$$

the small flow-time expansion reads

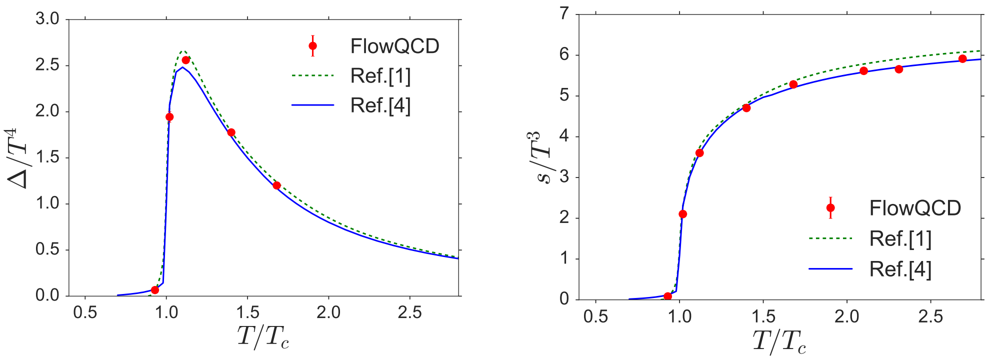
$$U_{\mu\nu}(t,x) = c_T(t)T_{\mu\nu}(x) + \frac{1}{4}c_S(t)\delta_{\mu\nu}\{F^a_{\rho\sigma}F^a_{\rho\sigma}\}_R(x) + \mathcal{O}(t), \tag{1.18}$$

$$E(t,x) = \frac{1}{4}c_E(t)\{F^a_{\rho\sigma}F^a_{\rho\sigma}\}_R(x) + \mathcal{O}(t), \tag{1.19}$$

The coefficients $c_T(t)$, $c_S(t)$, and $c_E(t)$ for $t \to 0$ can be determined by perturbation theory. Then inverting these relations with respect to the EMT, we have

$$T_{\mu\nu}(x) = \frac{1}{c_T(t)}U_{\mu\nu}(t,x) - \frac{c_S(t)}{c_T(t)c_E(t)}\delta_{\mu\nu}E(t,x) + \mathcal{O}(t). \tag{1.20}$$

Finally, we take the small flow-time limit $t \to 0$; this limit justifies the perturbative computation of the expansion coefficients and also removes the last $O(t)$ term in the representation. The resulting representation is independent of the regularization and we can use it in lattice numerical simulations. For example, we can use it for the computation of thermodynamic quantities of the YM theory at finite temperature. Figure 1.1 is the result of Ref. [21] on two thermodynamic quantities, the trace anomaly and the entropy density, in the $SU(3)$ pure YM theory. The red symbols are obtained by the finite temperature expectation value of the gradient representation of the EMT (1.20); it is clear that the representation has the correct normalization.



(a) (b)

Figure 1.1: These two figures show the results of Ref. [21] on two thermodynamic quantities, the trace anomaly (the left panel) and the entropy density (the right panel), in the $SU(3)$ pure YM theory. The horizontal axial is the temperature T in units of the confinement/deconfinement critical temperature T_c. The curves are results of preceding studies, Refs. [22] and [23], which uses completely different method to ours.

In this paper, we carry out a similar construction of the supercurrent.

This thesis is organized as follows: In Chap. 2, we briefly sketch the proof of the renormalizability of the gradient flow in Ref. [14] in the four-dimensional (4D) pure Yang–Mills (YM) theory. The reader who are mainly interested in the practical application of the gradient flow to the supercurrent may skip this chapter. In Chap. 3 and in Chap. 4, we consider the 4D supersymmetric Yang–Mills theory (SYM) with the $\mathcal{N}=1$ supersymmetry (SUSY) and the $\mathcal{N}=2$ SUSY, respectively. The Lagrangian densities for both theories are constructed in the Wess–Zumino gauge under dimensional regularization. As already mentioned, this setup explicitly breaks SUSY and the construction of the Noether current associated with SUSY, the supercurrent, becomes quite non-trivial. First, we derive Ward–Takahashi (WT) identities associated with SUSY in terms of bare quantities. These identities contain many SUSY breaking composite operators. Next, through one-loop level computations, we determine the renormalization of various composite operators appearing in the SUSY WT identities. Then, after reorganizing various terms in the SUSY WT relations by using Schwinger–Dyson equations, we find that the expression of the renormalized supercurrent in the one-loop level whose total divergence generates correct SUSY transformations on renormalized fields. Then, finally, by using the small flow-time expansion, we re-express the renormalized supercurrent in terms of composite operators of flowed fields. The resulting expression of the supercurrent is independent on the adopted regularization. In Chap. 3, the small flow-time expansion is calculated by using the background field method [17] and, in Chap. 4, the small flow time expansion is calculated diagrammatically (the flow Feynman rules are summarized in Appendix B). Chap. 5 is devoted to Conclusion. There are five appendices that contain various elements needed in the main text.

Our expressions for the correctly-normalized supercurrents in terms of the gradient flow will be useful in lattice simulations of those supersymmetric gauge theories. For example, the result in Chap. 3 should be useful for the parameter tuning in lattice simulations of the 4D $\mathcal{N}=1$ SYM (see Refs. [31]-[53]) toward the SUSY point. The result in Chap. 4 should also be useful for the parameter tuning toward the SUSY point in lattice simulations of the 4D $\mathcal{N} = 2$ SYM [28, 54, 55]. It will be interesting to further generalize our results to more realistic supersymmetric systems that contain matter fields—our study here may be regarded as the first step toward such an enterprise.

Chapter 2

Proof of the renormalizability of the gradient flow

As already noted in Introduction, our construction relies on a renormalization property of the gradient flow; local products of flowed fields become UV finite once bare parameters in the original action are renormalized. For matter fields such as fermion, wave function renormalization of flowed elementary fields is also necessary. Such UV finite composite operators are independent of the way of regularization (when the cutoff is sent to infinity). In this section, we briefly sketch the proof of this renormalizability of the gradient flow in the case of the 4D pure Yang–Mills (YM) theory [14].

2.1 Action of the YM theory and the flow equation

We consider the pure Yang–Mills theory in the D-dimensional euclidean spacetime. The action is

$$S_{\mathrm{YM}} = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right], \tag{2.1}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]. \tag{2.2}$$

For this system, the flow of the gauge field $A_\mu(x)$ is defined by

$$A_\mu(x) \to B_\mu(t,x), \quad \text{where} \quad B_\mu(t=0,x) = A_\mu(x), \tag{2.3}$$

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu, \tag{2.4}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot], \tag{2.5}$$

where α_0 is a constant.

First, we show that the value of α_0 does not affect any gauge-invariant observables. Consider the following $(D+1)$-dimensional gauge transformation in the flow equation (2.4),

$$\delta B_\mu(t,x) = D_\mu \omega(t,x). \tag{2.6}$$

If we assume that $\omega(t,x)$ obeys

$$\partial_t \omega - \alpha_0 D_\mu \partial_\mu \omega = -\delta\alpha_0 \partial_\nu B_\nu, \tag{2.7}$$

then the flow equation (2.4) is modified so that

$$\alpha_0 \to \alpha_0 + \delta\alpha_0. \tag{2.8}$$

This means that a shift of α_0 can be induced by the $(D+1)$-dimensional gauge transformation that leaves any gauge invariant quantities intact. Thus, any gauge invariant quantities are independent of the parameter α_0. In Chap. 3 and in Chap. 4, we choose $\alpha_0 = 1$.

Now, we can decompose the flow equation (2.4) into the linear part and the non-linear part:

$$\partial_t B^a_\mu = \partial_\nu \partial_\nu B^a_\mu + (\alpha_0 - 1)\partial_\mu \partial_\nu B^a_\nu + R^a_\mu, \tag{2.9}$$

$$R^a_\mu = 2f^{abc} B^b_\nu \partial_\nu B^c_\mu - f^{abc} B^b_\nu \partial_\mu B^c_\nu + (\alpha_0 - 1) f^{abc} B^b_\mu \partial_\nu B^c_\nu + f^{abc} f^{cde} B_\nu B^d_\nu B^e_\mu. \tag{2.10}$$

The solution for the linear part (obtained by setting $R^a_\mu = 0$) is given by

$$B^{0a}_\mu(t,x) = \int d^D y \, K_t(x-y)_{\mu\nu} A^a_\nu(y), \tag{2.11}$$

$$K_t(x)_{\mu\nu} = \int_p \frac{e^{ipx}}{p^2} \left\{ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 t p^2} \right\}, \tag{2.12}$$

where we used the abbreviation $\int_p \equiv \int \frac{d^D p}{(2\pi)^D}$; we use this notation throughout this paper.

The solution to the full flow equation can be formally written down as

$$B^a_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A^a_\nu(y) + \int_0^t ds \, K_{t-s}(x-y)_{\mu\nu} R^a_\nu(s,y) \right]. \tag{2.13}$$

One can easily see that this is actually the solution of the flow equation (2.4). In this formal solution, the non-linear term R_μ contains B_μ itself. Thus, by substituting the zeroth order solution $B^{0a}_\mu = \int d^D y \, K_t(x-y)_{\mu\nu} A^a_\nu(y)$ back into R^a_μ, as the solution in the first order of R^a_μ, we have

$$\begin{aligned} B^{1a}_\mu(t,x) = &\int d^D y \, K_t(x-y)_{\mu\nu} A^a_\nu(y) \\ &+ \int d^D y \int_0^t ds \, K_{t-s}(x-y)_{\mu\nu} \Big[2 f^{abc} B^{0b}_\nu(s,y) \partial_\nu B^{0c}_\mu(s,y) - f^{abc} B^{0b}_\nu(s,y) \partial_\mu B^{0c}_\nu(s,y) \\ &\quad + (\alpha_0 - 1) f^{abc} B^{0b}_\mu(s,y) \partial_\nu B^{0c}_\nu(s,y) \\ &\quad + f^{abc} f^{cde} B^{0b}_\nu(s,y) B^{0d}_\nu(s,y) B^{0e}_\mu(s,y) \Big]. \end{aligned}$$

Repeating this kind of iteration, we can obtain the perturbative solution to any order.

The quantum correlation functions of the flowed gauge field $B^a_\mu(t,x)$ are defined by the functional integral over the original gauge field (the non-flow field) $A^a_\mu(x)$. For example, perturbatively, the two-point function (the propagator) $\langle B_\mu^a(t,x)B_\rho^b(u,z)\rangle$ is computed as

$$\begin{aligned}
\langle B_\mu^a(t,x)B_\rho^b(u,z)\rangle =&\langle B_\mu^{0a}(t,x)B_\rho^{0b}(u,z)\rangle \\
&+\int d^Dy\int_0^t ds\, K_{t-s}(x-y)_{\mu\nu}\Big[\\
&\times\langle 2f^{acd}B_\nu^c(s,y)\partial_\nu B_\mu^d(s,y)B_\rho^{0b}(u,z)\rangle-\langle f^{acd}B_\nu^c(s,y)\partial_\mu B_\nu^d(s,y)B_\rho^{0b}(u,z)\rangle \\
&\quad+(\alpha_0-1)\langle f^{acd}B_\mu^c(s,y)\partial_\nu B_\nu^d(s,y)B_\rho^{0b}(u,z)\rangle \\
&\quad+\langle f^{acd}f^{def}B_\nu^c(s,y)B_\nu^e(s,y)B_\mu^f(s,y)B_\rho^{0b}(u,z)\rangle\Big]+\Big[(a,\mu,t,x)\leftrightarrow(b,\rho,u,z)\Big] \\
&=\langle B_\mu^{0a}(t,x)B_\rho^{0b}(u,z)\rangle \\
&+\int d^Dy\int_0^t ds\, K_{t-s}(x-y)_{\mu\nu}\Big[\\
&\times\langle 2f^{acd}B_\nu^{0c}(s,y)\partial_\nu B_\mu^{0d}(s,y)B_\rho^{0b}(u,z)\rangle-\langle f^{acd}B_\nu^{0c}(s,y)\partial_\mu B_\nu^{0d}(s,y)B_\rho^{0b}(u,z)\rangle \\
&\quad+(\alpha_0-1)\langle f^{acd}B_\mu^{0c}(s,y)\partial_\nu B_\nu^{0d}(s,y)B_\rho^{0b}(u,z)\rangle,
\end{aligned}\tag{2.14}$$

where we have noted that the quadratic and the cubic terms in R_μ^a (2.10) are respectively given by

$$X_\mu^{(2)a}\equiv 2f^{abc}B_\nu^b\partial_\nu B_\mu^c-f^{abc}B_\nu^b\partial_\mu B_\nu^c+(\alpha_0-1)f^{abc}B_\mu^b\partial_\nu B_\nu^c,\tag{2.15}$$
$$X_\mu^{(3)a}\equiv f^{abc}f^{cde}B_\nu^bB_\nu^dB_\mu^e.\tag{2.16}$$

For the first term in the above expression, for example, we have

$$\begin{aligned}
\langle B_\mu^{0a}(t,x)B_\nu^{0b}(s,y)\rangle^{\rm tree}=&\int_0^t dt_1\int_0^s ds_1\int d^Dx_1\int d^Dy_1 \\
&\times K_{t-t_1}(x-x_1)_{\mu\rho}K_{s-s_1}(y-y_1)_\rho\langle A_\rho^a(x_1)A_\sigma^b(y_1)\rangle^{\rm tree} \\
=&\,g_0^2\delta^{ab}\int_p e^{ip(x-y)}\frac{1}{(p^2)^2}\Big\{(\delta_{\mu\nu}p^2-p_\mu p_\nu)e^{-(t+s)p^2} \\
&+\frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\Big\},
\end{aligned}\tag{2.17}$$

where we have used the tree-level propagator of the original gauge field.

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle^{\rm tree}=g_0^2\delta^{ab}\int_p e^{ip(x-y)}\frac{1}{(p^2)^2}\Big[(\delta_{\mu\nu}p^2-p_\mu p_\nu)+\frac{1}{\lambda_0}p_\mu p_\nu\Big].\tag{2.18}$$

For this, we have assumed the following gauge-fixing term and the Faddeev–Popov ghost term:

$$S_{\rm gf}=-\frac{\lambda_0}{g_0^2}\int d^Dx\,{\rm tr}\Big[\partial_\mu A_\mu(x)\partial_\nu A_\nu(x)\Big],\tag{2.19}$$
$$S_{c\bar c}=-\frac{2}{g_0^2}\int d^Dx\,{\rm tr}\Big[\partial_\mu\bar c(x)D_\mu c(x)\Big].\tag{2.20}$$

The perturbative expansion such as the one in Eq. (2.14) can be represented diagrammatically. First, we represent the function $K_{t-s}(x-y)_{\mu\nu}$ (called the heat kernel) as an arrowed line (called the flow line) in Fig. 2.1. This appears only in the combination such as $\int d^D y \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu}$ and so the flow time t is always greater than the flow time s; the arrow in the flow line indicates this direction in the flow time. Another line is the flow propagator in Fig. 2.2. This simply represents

Figure 2.1: The flow line with arrow represents the heat kernel. The arrow from s to t shows that $t > s$.

the Wick contraction of B-fields in Eq. (2.17).

Figure 2.2: The wavy line (flow propagator) represents $\langle B^{0a}_\mu(t,x) B^{0b}_\nu(s,y)\rangle^{\text{tree}}$ in Eq. (2.17).

Beside these two types of lines, there are two types of "flow vertices" in Fig. 2.3 and Fig. 2.4 indicated by white blobs. These flow vertices arise from $X^{(2)a}_\mu$ and $X^{(3)a}_\mu$ in Eqs. (2.15) and (2.16), respectively. There also exist "ordinary" vertices that arise from the action of the original gauge theory $S = S_{\text{YM}} + S_{\text{gf}} + S_{c\bar{c}}$. In the next section, we construct the action of gauge theory in the $(D+1)$-dimensional spacetime, introducing a new field $L_\mu(t,x)$, that reproduces the flow line, the flow propagator, the flow vertices, and the ordinary Yang–Mills vertices.

2.2 $(D+1)$-dimensional gauge theory that reproduces the flow Feynman rules

For later discussion, we construct the $(D+1)$-dimensional action that reproduces the above Feynman rules for the flow Feynman diagrams. We then argue the property of needed counterterms for this $(D+1)$-dimensional theory that removes UV divergences.

First, we consider the flowed gauge field $B_\mu(t,x)$ as if it is an independent degrees of freedom in a $(D+1)$ dimensional gauge theory. The action consists of two parts, S_D and S_{D+1}:

$$S = \underbrace{S_{\text{YM}} + S_{\text{gf}} + S_{c\bar{c}}}_{S_D} + \underbrace{S_{\text{fl}} + S_{d\bar{d}}}_{S_{D+1}}, \tag{2.21}$$

$$S_{\text{YM}} = -\frac{1}{2g_0^2} \int d^D x\, \text{tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right], \tag{2.22}$$

$$S_{\text{gf}} = -\frac{\lambda_0}{g_0^2} \int d^D x\, \text{tr}\Big[\partial_\mu A_\mu(x) \partial_\nu A_\nu(x)\Big], \tag{2.23}$$

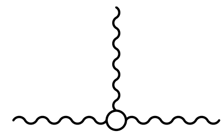


Figure 2.3: One of flow vertices arises from $X_\mu^{(2)a}$ in Eq. (2.15).

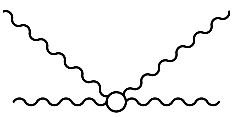


Figure 2.4: One of flow vertices arises from $X_\mu^{(3)a}$ in Eq. (2.16).

$$S_{c\bar{c}} = -\frac{2}{g_0^2}\int d^D x\,\mathrm{tr}\Big[\partial_\mu \bar{c}(x) D_\mu c(x)\Big], \tag{2.24}$$

$$S_{\mathrm{fl}} = -2\int_0^\infty dt \int d^D x\,\mathrm{tr}\Big[L_\mu(t,x)(\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu)(t,x)\Big], \tag{2.25}$$

$$S_{d\bar{d}} = -2\int_0^\infty dt \int d^D x\,\mathrm{tr}\Big[\bar{d}(t,x)(\partial_t d - \alpha_0 D_\mu \partial_\mu d)(t,x)\Big]. \tag{2.26}$$

The first part S_D is just the D dimensional Yang–Mills theory with the gauge fixing term in Eqs. (2.19) and (2.20). S_{fl} contains the $(D+1)$-dimensional freedom $L_\mu(t,x)$ and $B_\mu(t,x)$, where $L_\mu(t,x)$ is a Lagrange multiplier. After integrating over $L_\mu(t,x)$, the functional integral over $B_\mu(t,x)$ is restricted to the solution of the flow equation 2.4; the boundary condition $B_\mu(t=0,x) = A_\mu(x)$ has to be assumed later. $S_{d\bar{d}}$ is on the other hand composed from $\bar{d}(t,x)$ and $d(t,x)$, where $d(t,x)$ is a $(D+1)$-dimensional ghost and $\bar{d}(t,x)$ is a Lagrange multiplier that imposes the flow equation,

$$\partial_t d = \alpha_0 D_\mu \partial_\mu d(t,x), \qquad d(t=0,x) = c(x). \tag{2.27}$$

We can easily show that this $(D+1)$-dimensional action S reproduces the Feynman rules for flow Feynman diagrams in the previous section. The rules of ordinary vertices (denoted by black blobs) are read off from S_D. The flow vertices come from S_{fl}, especially from LB^2, LB^3 terms:

- LBB vertex
  $\int_0^\infty dt \int d^D x\, L_\mu^a X_\mu^{(2)a}$
- $LBBB$ vertex
  $\int_0^\infty dt \int d^D x\, L_\mu^a X_\mu^{(3)a}$

These flow vertices are denoted by white blobs as noted already. The flow line (the heat kernel) now corresponds to the LB propagator, the inverse of the coefficient of LB term in S_{fl}.

$$(\delta_{\mu\rho}\partial_t - \delta_{\mu\rho}\partial^x_\sigma\partial^x_\sigma - (\alpha_0 - 1)\partial^x_\mu\partial^x_\rho)\langle B^a_\rho(t,x)L^b_\nu(s,y)\rangle^{\text{tree}} = \delta^{ab}\delta_{\mu\nu}\delta(t-s)\delta(x-y)$$
$$\langle B^a_\mu(t,x)L^b_\nu(s,y)\rangle^{\text{tree}} = \delta^{ab}\theta(t-s)K_{t-s}(x-y)_{\mu\nu} \qquad (2.28)$$

The LB propagator is denoted by an arrow that goes from L to B. One can see that any flow Feynman diagram drawn with these vertices and lines has a same value as the flow Feynman diagram drawn with the rules in the previous section. From $S_{d\bar{d}}$, we also have the following additional vertex and propagators for the flowed ghost fields,

- the flow line for the ghost field= $\bar{d}d$ propagator
  $\langle \bar{d}^a(t,x)d^b(s,y)\rangle^{\text{tree}} = \delta^{ab}\delta(t-s)\int_p e^{-\alpha_0(t-s)p^2}$
- the flow propagator for the ghost field
  $\langle \bar{d}^a(t,x)d^b(s,y)\rangle^{\text{tree}} = \delta^{ab}\int_p e^{-\alpha_0(t+s)p^2}e^{ip(x-y)}\frac{1}{p^2}$
- $\bar{d}dB$ flow-vertex
  $\int_0^\infty dt\int d^Dx\,\alpha_0 f^{abc}\bar{d}^a B^b_\mu \partial_\mu d^c$

These are necessary only to keep the BRS symmetry in the $(D+1)$-dimensional theory.

It is possible to show that counterterms that are needed for the cancellation of UV divergences arise in S_D only and S_{D+1} does not need any renormalization. As the example, let us consider the two-point function of B^a_μ in the one-loop level. The flow Feynman diagrams that are relevant for $\langle B^a_\mu(t,x)B^b_\nu(s,y)\rangle$ are depicted in Fig. 2.5. After calculating these diagrams according

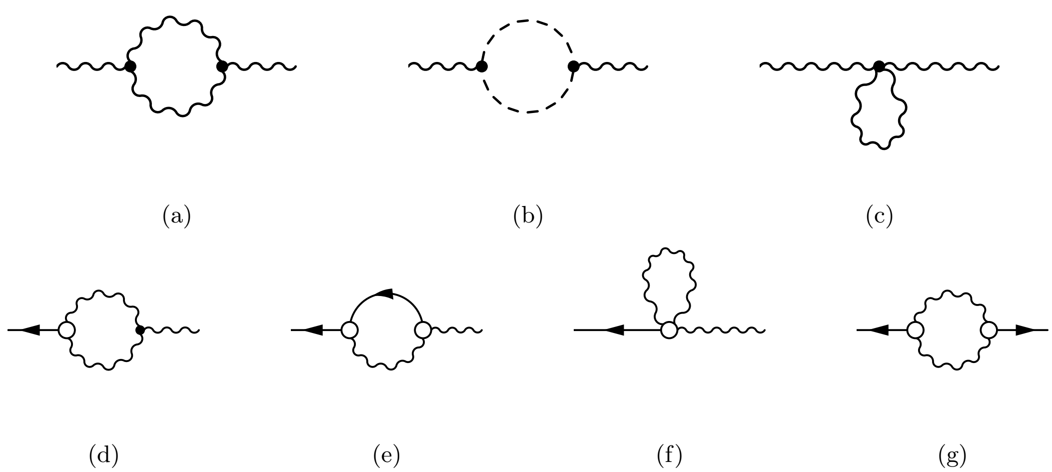


Figure 2.5

to the Feynman rules, $\langle B^a_\mu(t,x)B^b_\nu(s,y)\rangle$ to the one-loop level turns out to be in the dimensional regularization $D = 4 - 2\epsilon$ (this is the expression for the gauge group $G = SU(N)$),

$$\begin{aligned}\langle B^a_\mu(t,x)B^b_\nu(s,y)\rangle^{1\,\text{loop}} &= g^2\delta^{ab}\int_p e^{ip(x-y)}\frac{1}{(p^2)^2}\\ &\quad\times\left\{(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1+\frac{b_0}{\epsilon}g^2)e^{-(t+s)p^2}\right.\\ &\quad\left.+\lambda^{-1}(1-\frac{c_0-b_0}{\epsilon}g^2)p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right\}, \end{aligned} \tag{2.29}$$

$$b_0 = \frac{N}{16\pi^2}\frac{11}{3}, \tag{2.30}$$

$$c_0 = \frac{N}{16\pi^2}\left(\frac{13}{6} - \frac{1}{2\lambda}\right). \tag{2.31}$$

In addition to this, we also have the counterterm that arises from the substitutions (the one-loop level renormalization),

$$g_0^2 = \mu^{2\epsilon}g^2\left(1 - \frac{b_0}{\epsilon}g^2\right), \tag{2.32}$$

$$\lambda_0 = \left(1 - \frac{c_0}{\epsilon}g^2\right)\lambda, \tag{2.33}$$

in the tree-level propagator,

$$\langle B^a_\mu(t,x)B^b_\nu(s,y)\rangle^{\text{tree}} = g_0^2\delta^{ab}\int_p e^{ip(x-y)}\frac{1}{(p^2)^2}\left\{(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \lambda_0^{-1}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right\}. \tag{2.34}$$

We see that UV divergences in $\langle B^a_\mu(t,x)B^b_\nu(s,y)\rangle^{1\,\text{loop}}$ are precisely cancelled by the counterterm. This shows that in the one-loop level the parameter renormalization make $\langle B^a_\mu(t,x)B^b_\nu(s,y)\rangle^{1\,\text{loop}}$ finite without any wave function renormalization.

We want to show that the above UV finiteness of the correlation functions of the flowed gauge field without the wave function renormalization persists, not only in the one-loop level, in all orders of perturbation theory.

First, we note that possible UV divergences occur only at the "boundary" of the $(D+1)$-dimensional spacetime, i.e., at the zero flow time $t = 0$. This follows from the observation that in any loop in a flow Feynman diagram, if at least one of the flow times of flow vertices is non-zero, then the loop integral absolutely convergent, because of the Gaussian damping factor in the heat kernel or the flow propagator. For this, an important fact is that there is no loop consisting only of the heat kernel, $\langle B^a_\mu(t,x)L^b_\nu(s,y)\rangle^{\text{tree}} = \delta^{ab}\theta(t-s)K_{t-s}(x-y)_{\mu\nu}$; because of the step function $\theta(t-s)$, such a loop is measured-zero and can be neglected. All UV divergences in the $(D+1)$-dimensional field theory occurs at the boundary at $t = 0$.

Another crucial observation is that the 1PI diagram that contains the flowed gauge field B should always accompany L at least at one of external vertices. This follows from the above Feynman rule (the vertices containing B always accompany one L) and again from the fact that there is no loop consisting only of the heat kernel $\langle BL\rangle$. The only way to make a loop without any external L vertex is a loop consisting only of $\langle BL\rangle$ lines, but this is impossible.

A similar statement holds for the $(D+1)$-dimensional ghost d; possible UV diverging 1PI vertices are at the zero flow time and it must accompany at least one Lagrange multiplier $\bar{d}$ at least at one of external vertices.

From these considerations and from the fact that the divergent part must be a local polynomial of fields of the mass dimension 4 and the ghost number 0, we see that the most general form of the divergent part which contains $(D+1)$-dimensional fields is (in the l-th loop level),

$$2g^{2l}\int d^Dx\,\mathrm{tr}\Big[z_1L_\mu(0,x)A_{\mu R}(x)+z_2\bar{d}(0,x)c_R(x)\Big], \tag{2.35}$$

where we have noted the boundary conditions, $B(t=0,x)=A_{R\mu}(x)$ and $d(t=0,x)=c_R(x)$.

On the other hand, it turns out that the above $(D+1)$-dimensional gauge theory $S=S_D+S_{D+1}$ is invariant under the BRS transformations of the form,

$$\delta_{BRS}\,B_\mu=D_\mu d, \tag{2.36}$$
$$\delta_{BRS}\,L_\mu=[L_\mu,d], \tag{2.37}$$
$$\delta_{BRS}\,d=-d^ad^bT^aT^b, \tag{2.38}$$
$$\delta_{BRS}\,\bar{d}=D_\mu L_\mu-\{d,d\}. \tag{2.39}$$

We now show that the counterterm (2.35) cannot exist by the restriction implies by the BRS invariance. From the BRS invariance, we have following Ward–Takahashi (WT) relations:

$$\lambda_0\langle B^a_\mu(t,x)\partial_\nu A^b_\nu(y)\partial_\rho A^c_\rho(z)\rangle=-\langle(D_\mu d)^a(t,x)\bar{c}^b(y)\partial_\rho A^c_\rho(z)\rangle, \tag{2.40}$$

because the action and the functional integration measure are BRS invariant. Under the standard renormalization in the original Yang–Mills theory,

$$\lambda\langle B^a_\mu(t,x)\partial_\nu A^b_{\nu R}(y)\partial_\rho A^c_{\rho R}(z)\rangle=-\langle(D_\mu d)^a(t,x)\bar{c}^b_R(y)\partial_\rho A^c_{\rho R}(z)\rangle. \tag{2.41}$$

The counterterm (2.35) further contributes to both sides of the WT identity (2.40) through the tree-level diagrams in Fig. 2.6. From Fig. 2.6, we see that the counterterm (2.35) contributes to the left-side by $3z_1g^{2l}$ while to the right-side by $[(z_1+z_2)+(z_1+z_2)+z_2]g^{2l}=(2z_1+3z_2)g^{2l}$. Thus the WT identity for the BRS symmetry in the l-loop order implies $z_1=z_2=0$. There is no need of the counterterm of the form (2.35). Only counterterms needed is the counterterms in the original ordinary YM theory. This shows in particular that there is no need of the wave function renormalization of the flowed gauge field. It can be seen this UV finiteness persists even for composite operators (i.e., the local products) of the flowed gauge field. The crucial point is again that there is no closed loop being consisting the heat kernels.

In Ref. [14], it is claimed that for $t\to0$, where t is the flow time, any composite operator (i.e., the local product) of the flowed field can be expressed by an asymptotic series of composite operators of the un-flowed field with increasing mass dimensions. This *small flow-time expansion* works, for example, for the following gauge invariant dimension 4 operator,

$$E(t,x)=\frac{1}{4}G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x). \tag{2.42}$$

as

$$E(t,x)=\langle E(t,x)\rangle+c_E(t)\{\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\}_R(x)+\mathcal{O}(t). \tag{2.43}$$

The first term is the unit operator and the second term is a renormalized operator of the dimension 4 at the zero flow time. The last $\mathcal{O}(t)$ represents the contribution of operators of the dimension 6

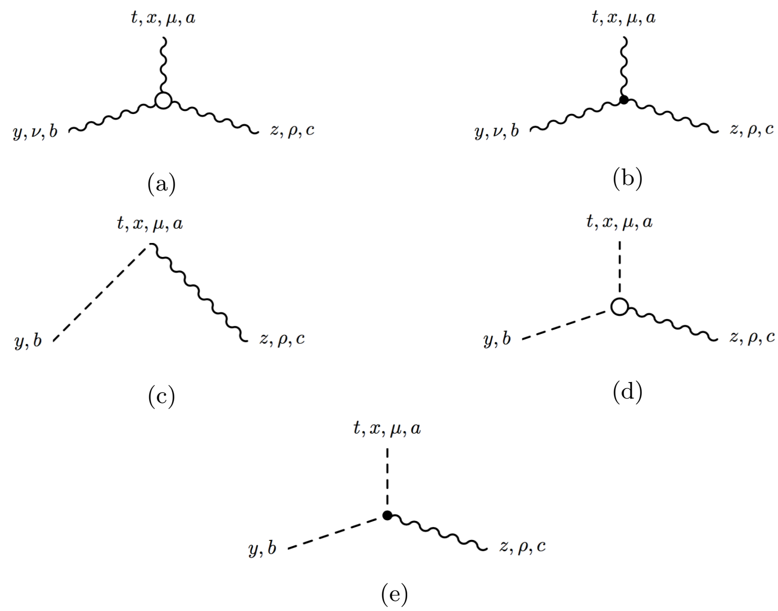


Figure 2.6: The tree diagrams that are relevant to the contributions of the counterterm (2.35) to the both sides of Eq. (2.40). The diagrams (a) and (b) show the contributions to the left-hand side, while diagrams (c)–(e) show the contributions to the right-hand side. If the diagram has a $B_\mu(t \neq 0, x)A_\nu(y)$ line, it can be decomposed into $B_\mu(t \neq 0, x)L_\rho(0, z)$ line and $A_\rho(z)A_\nu(y)$ line by using the two-point vertex $g^{2l}z_1 L_\mu(t = 0, x)A_\nu(x)$ in the first term of Eq. (2.35). The diagram with this decomposed BA line has the contribution of $g^{2l}z_1$ times the value of the original diagram. For example, the diagram (a) has two BA lines and gives the contribution $2g^{2l}z_1\lambda_0\langle B^a_\mu(t, x)\partial_\nu A^b_\nu(y)\partial_\rho A^c_\rho(z)\rangle^{\text{tree}}$ to the left-hand side of Eq. (2.40). Similarly, if the diagram has a $d(t \neq 0, z)\bar{c}(y)$ line, it can be decomposed to $d(t \neq 0, x)\bar{d}(0, z)$ and $c(z)\bar{c}(y)$. The diagram with this decomposed $d\bar{c}$ line has the contribution of $g^{2l}z_2$ times the value of the original diagram.

or higher. This small flow-time expansion provides a possible way to represent an operator in the original gauge theory in terms of composite operators of the flowed field that does not require the wave function renormalization. In Chap. 3 and Chap. 4, this technique will be fully utilized to find the expression of the supercurrent in terms of flowed fields.

Chapter 3

The 4D $\mathcal{N} = 1$ super Yang–Mills theory

In Introduction, we mentioned the application of the gradient flow to the constructing of a regularization-independent expression for the energy–momentum tensor (EMT). In this section, we consider the construction of a regularization-independent supercurrent in terms of flowed fields in the simplest 4D supersymmetric gauge theory, the $\mathcal{N} = 1$ SYM.

3.1 Action, super transformations, and BRS-invariance

We consider the $\mathcal{N} = 1$ SYM in the D dimensional Euclidean spacetime. The action is given by

$$S = \frac{1}{4g_0^2} \int d^D x \, F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \frac{1}{2} \int d^D x \, \bar{\psi}^a(x) \not{\mathcal{D}}^{ab} \psi^b(x), \tag{3.1}$$

where

$$\mathcal{D}_\mu^{ab} = \delta^{ab} \partial_\mu + \mathcal{A}_\mu^{ab}(x), \quad \mathcal{A}_\mu^{ab} = f^{acb} A_\mu^c, \quad \not{\mathcal{D}}^{ab} = \mathcal{D}_\mu^{ab} \gamma_\mu, \tag{3.2}$$

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + f^{abc} A_\mu^b(x) A_\nu^c(x). \tag{3.3}$$

As explained in Introduction, this is the expression in the Wess–Zumino (WZ) gauge and only the gauge field and the gaugino field exist. The SUSY transformation in the WZ gauge is non-linear and its explicit form is given by

$$\delta_\xi A_\mu^a(x) = g_0 \bar{\xi} \gamma_\mu \psi^a(x), \tag{3.4}$$

$$\delta_\xi \psi^a(x) = -\frac{1}{2g_0} \sigma_{\mu\nu} \xi F_{\mu\nu}^a(x), \tag{3.5}$$

$$\delta_\xi \bar{\psi}^a(x) = \frac{1}{2g_0} \bar{\xi} \sigma_{\mu\nu} F_{\mu\nu}^a(x). \tag{3.6}$$

When $D = 4$, noting the fact that both the gaugino field and the SUSY transformation parameter ξ are the Majorana spinor $\bar{\psi} = \psi^T(-C^{-1})$, one can see that the action is invariant under the above

SUSY transformation up to the surface term. Thus, if we make the transformation parameter local $\xi \to \xi(x)$, the variation of the action takes the form of $\int d^D x\, \partial_\mu \xi(x) S_\mu$. The supercurrent in the classical level is then given by

$$S_\mu = -\frac{1}{2g_0}\sigma_{\rho\sigma}\gamma_\mu \psi^a(x) F^a_{\rho\sigma}(x). \tag{3.7}$$

For the following perturbative calculations, we need to introduce the gauge fixing term and the Faddeev–Popov ghost term. We set

$$S_{\text{gf}} = \frac{\lambda_0}{2g_0^2}\int d^D x\, \partial_\mu A^a_\mu(x) \partial_\nu A^a_\nu(x), \tag{3.8}$$

$$S_{c\bar{c}} = \frac{1}{g_0^2}\int d^D x\, \partial_\mu \bar{c}^a(x) \mathcal{D}^{ab}_\mu c^b(x). \tag{3.9}$$

We note that $S + S_{\text{gf}} + S_{c\bar{c}}$ is invariant under BRS transformation

$$\delta_B A^a_\mu(x) = \mathcal{D}^{ab}_\mu c^b(x), \tag{3.10}$$

$$\delta_B \psi^a(x) = -f^{abc} c^b(x) \psi^c(x), \tag{3.11}$$

$$\delta_B \bar{\psi}^a(x) = -f^{abc} c^b(x) \bar{\psi}^c(x), \tag{3.12}$$

$$\delta_B c^a(x) = -\frac{1}{2} f^{abc} c^b(x) c^c(x), \tag{3.13}$$

$$\delta_B \bar{c}^a(x) = \lambda_0 \partial_\mu A^a_\mu(x). \tag{3.14}$$

This means that the expectation value of a BRS-exact operator $\delta_B \mathcal{O}$ is zero since $\langle \delta_B \mathcal{O}\rangle = \langle -\delta_B(S + S_{\text{gf}} + S_{c\bar{c}})\mathcal{O}\rangle = 0$ under the functional integration.

The above gauge fixing and the Faddeev–Popov ghost terms are however not invariant under SUSY:

$$\begin{aligned}\delta_\xi S_{\text{gf}} &= -\frac{\lambda_0}{g_0}\int d^D x\, \partial_\mu \partial_\nu A^a_\nu(x) \bar{\xi}\gamma_\mu \psi^a(x) \\ &= -\int d^D x\, \bar{\xi} X_{\text{gf}}(x),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\delta_\xi S_{c\bar{c}} &= -\frac{1}{g_0}\int d^D x\, f^{abc} \partial_\mu \bar{c}^a(x) c^b(x) \bar{\xi}\gamma_\mu \psi^c(x) \\ &= -\int d^D x\, \bar{\xi} X_{c\bar{c}}(x),\end{aligned} \tag{3.16}$$

where

$$X_{\text{gf}}(x) \equiv \frac{\lambda_0}{g_0}\partial_\mu \partial_\nu A^a_\nu(x) \gamma_\mu \psi^a(x), \tag{3.17}$$

$$X_{c\bar{c}}(x) \equiv \frac{1}{g_0} f^{abc} \partial_\mu \bar{c}^a(x) c^b(x) \gamma_\mu \psi^c(x). \tag{3.18}$$

We note that the combination $X_{\text{gf}} + X_{c\bar{c}}$ is BRS-exact

$$X_{\text{gf}} + X_{c\bar{c}} = \delta_B\left(\frac{\lambda_0}{g_0}\partial_\mu \bar{c}^a(x) \gamma_\mu \psi^a(x)\right). \tag{3.19}$$

This combination thus gives no contribution in correlation functions of gauge-invariant operators.

When $D \neq 4$ as assumed in dimensional regularization, also the classical action S is not invariant under SUSY. This occurs because the Fierz identity in $D = 4$ does not hold for $D = 4-2\epsilon$ with $\epsilon \neq 0$. The explicit SUSY breaking term is

$$\begin{aligned}\delta_\xi S &= -\frac{1}{2}g_0 \int d^D x\, f^{abc}\bar{\xi}\gamma_\mu \psi^a(x)\bar{\psi}^b(x)\gamma_\mu\psi^c(x) \\ &= -\int d^D x\, \bar{\xi} X_{\text{Fierz}}(x),\end{aligned} \tag{3.20}$$

where

$$X_{\text{Fierz}}(x) \equiv \frac{1}{2}g_0 f^{abc}\gamma_\mu\psi^a(x)\bar{\psi}^b(x)\gamma_\mu\psi^c(x). \tag{3.21}$$

Thus, for $D \neq 4$, considering the SUSY transformation with a localized parameter $\xi(x)$ in the functional integration containing $A^b_\alpha(y)\bar{\psi}^c(z)$, we have the following SUSY WT identity,

$$\begin{aligned}&\left\langle\left[\partial_\mu S_\mu(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x)\right] A^b_\alpha(y)\bar{\psi}^c(z)\right\rangle \\ &= -\delta(x-y)\left\langle g_0\gamma_\alpha\psi^b(y)\bar{\psi}^c(z)\right\rangle - \delta(x-z)\left\langle A^b_\alpha(y)\frac{1}{2g_0}\sigma_{\beta\gamma}F^c_{\beta\gamma}(z)\right\rangle\end{aligned} \tag{3.22}$$

and similarly, the functional integration containing $\bar{\psi}^b(y)c^c(z)\bar{c}^d(w)$,

$$\begin{aligned}&\left\langle\left[\partial_\mu S_\mu(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x)\right]\bar{\psi}^b(y)c^c(z)\bar{c}^d(w)\right\rangle \\ &= -\delta(x-y)\left\langle\frac{1}{2g_0}\sigma_{\beta\gamma}F^b_{\beta\gamma}(y)c^c(z)\bar{c}^d(w)\right\rangle.\end{aligned} \tag{3.23}$$

These are our basic relations.

Now, we first note that the effect of $X_{\text{Fierz}}(x)$ can be canceled by the counterterm

$$S' \equiv -\frac{1}{(4\pi)^2}C_2(G)\frac{1}{6}\int d^D x\, F^a_{\mu\nu}(x)F^a_{\mu\nu}(x). \tag{3.24}$$

This can be seen by calculating the following one-loop diagram. The calculation shows that inserting

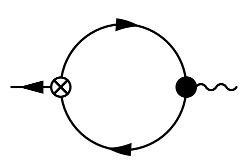


Figure 3.1: diagram X-fierz

$X_{\text{Fierz}}(x)$ into the first WT identity amounts to inserting the following operator in the tree-level expectation value

$$\langle X_{\text{Fierz}}(x)A^b_\alpha(y)\bar{\psi}^c(z)\rangle^{1\,\text{loop}} = \langle\frac{g_0}{(4\pi)^2}C_2(G)\frac{2}{3}\partial_\mu F^a_{\mu\nu}(x)\gamma_\nu\psi^a(x)A^b_\alpha(y)\bar{\psi}^c(z)\rangle^{\text{tree}} + \mathcal{O}(A^2_\mu). \tag{3.25}$$

The same operator appears also in the second WT identity in one-loop level. The last abbreviated term $\mathcal{O}(A^2)$ stands for higher order terms in the gauge field that cannot be determined in the present analysis. The variation of the above counterterm is

$$\langle X'(x)A^b_\alpha(y)\bar{\psi}^c(z)\rangle^{\rm tree} = -\langle X_{\rm Fierz}(x)A^b_\alpha(y)\bar{\psi}^c(z)\rangle^{1\,\rm loop} + \mathcal{O}(A^2_\mu), \tag{3.26}$$

where

$$\delta S' = -\int d^Dx\,\bar{\xi}(x)X'(x). \tag{3.27}$$

Thus the counterterm S' cancels the effect of $X_{\rm Fierz}(x)$ up to $\mathcal{O}(A^2_\mu)$. We will discuss the form of $\mathcal{O}(A^2_\mu)$ later.

Thus, under the presence of the above counterterm, we have the following SUSY WT identities,

$$\begin{aligned}&\left\langle [\partial_\mu S_\mu(x) + X_{\rm gf}(x) + X_{c\bar{c}}(x)]\,A^b_\alpha(y)\bar{\psi}^c(z)\right\rangle' \\ &= -\delta(x-y)\left\langle g_0\gamma_\alpha\psi^b(y)\bar{\psi}^c(z)\right\rangle' - \delta(x-z)\left\langle A^b_\alpha(y)\frac{1}{2g_0}\sigma_{\beta\gamma}F^c_{\beta\gamma}(z)\right\rangle',\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}&\left\langle [\partial_\mu S_\mu(x) + X_{\rm gf}(x) + X_{c\bar{c}}(x)]\,\bar{\psi}^b(y)c^c(z)\bar{c}^d(w)\right\rangle' \\ &= -\delta(x-y)\left\langle \frac{1}{2g_0}\sigma_{\beta\gamma}F^b_{\beta\gamma}(y)c^c(z)\bar{c}^d(w)\right\rangle',\end{aligned} \tag{3.29}$$

where the prime $'$ indicates that we use the action with the above counterterm $S + S_{\rm gf} + S_{c\bar{c}} + S'$.

3.2 Renormalization of the supercurrent in the one-loop level

Next, we determine the renormalization of $S_\mu(x)$ in the one-loop level. We first re-express all bare quantities in the SUSY WT identities by renormalized ones. We define

$$\Delta \equiv \frac{g^2}{(4\pi)^2}C_2(G)\frac{1}{\epsilon} \tag{3.30}$$

for notational convenience. In the minimal subtraction (MS) scheme, we have following relations between the bare and the renormalized quantities:

$$g_0 = \mu^\epsilon\left(1-\frac{3}{2}\Delta\right)g(\mu), \tag{3.31}$$

$$\lambda_0 = (1-\Delta)\,\lambda, \tag{3.32}$$

$$A^a_\mu(x) = (1-\Delta)\,A^a_{\mu R}(x), \tag{3.33}$$

$$\psi^a(x) = \left(1-\frac{1}{2}\Delta\right)\psi^a_R(x), \tag{3.34}$$

$$c^a(x) = \left(1-\frac{5}{4}\Delta\right)c^a_R(x), \tag{3.35}$$

$$F^a_{\mu\nu}(x) = \left(1-\frac{5}{2}\Delta\right)\left(\partial_\mu A^a_{\nu R}(x) - \partial_\nu A^a_{\mu R}(x)\right) + \left(1-\frac{11}{4}\Delta\right)f^{abc}\left\{A^b_\mu(x)A^c_\nu(x)\right\}_R. \tag{3.36}$$

We further consider the renormalization of composite operators $X_{\rm gf}$ and $X_{c\bar{c}}$. Re-expressing bare quantities in $X_{\rm gf}$ and $X_{c\bar{c}}$ by renormalized quantities, we obtain

$$X_{\rm gf}(x) = (1-\Delta)\frac{\lambda}{g}\partial_\mu\partial_\nu A^a_{\nu R}(x)\gamma_\mu\psi^a_R(x), \tag{3.37}$$

$$X_{c\bar{c}}(x) = \left(1-\frac{3}{2}\Delta\right)\frac{1}{g}f^{abc}\partial_\mu\bar{c}^a_R(x)c^b_R(x)\gamma_\mu\psi^c_R(x). \tag{3.38}$$

Compared to a simple product of elementary fields, the composite operator (i.e., the local product) such as $\partial_\mu\partial_\nu A^a_{\nu R}(x)\gamma_\mu\psi^a_R(x)$ and $f^{abc}\partial_\mu\bar{c}^a_R(x)c^b_R(x)\gamma_\mu\psi^c_R(x)$ produces additional UV divergences which require further renormalization. The UV divergences are determined by evaluating following one-loop diagrams. The sum of these diagrams tells that

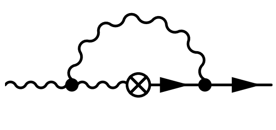


Figure 3.2: a

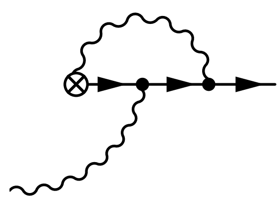


Figure 3.3: b

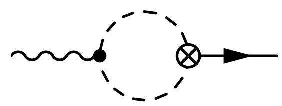


Figure 3.4: c

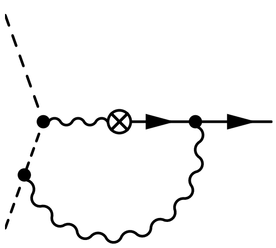


Figure 3.5: d

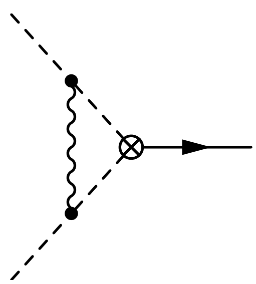


Figure 3.6: e

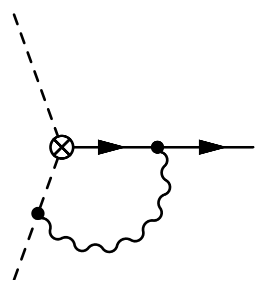


Figure 3.7: f

$$\begin{aligned}
&\left[\frac{\lambda}{g}\partial_\mu\partial_\nu A^a_{\nu R}(x)\gamma_\mu\psi^a_R(x) + \frac{1}{g}f^{abc}\partial_\mu\bar{c}^a_R(x)c^b_R(x)\gamma_\mu\psi^c_R(x)\right]\Bigg|_{\text{1PI, divergent part}} \\
&= 2\Delta\frac{\lambda}{g}\partial_\mu\partial_\nu A^a_{\nu R}(x)\gamma_\mu\psi^a_R(x) + \frac{1}{2}\Delta\frac{1}{g}f^{abc}\partial_\mu\bar{c}^a_R(x)c^b_R(x)\gamma_\mu\psi^c_R(x) \\
&\quad + \Delta\partial_\mu\left\{-\frac{1}{2g}\sigma_{\rho\sigma}\gamma_\mu\psi^a_R(x)[\partial_\rho A^a_{\sigma R}(x) - \partial_\sigma A^a_{\rho R}(x)]\right\} \\
&\quad\quad + 2\Delta\left(-\frac{1}{g^2}\right)\partial_\mu\partial_\mu A^a_{\nu R}(x)g\gamma_\nu\psi^a_R(x)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{2}\Delta\frac{1}{2g}[\partial_\mu A^a_{\nu R}(x)-\partial_\nu A^a_{\mu R}(x)]\sigma_{\mu\nu}\partial\!\!\!/\psi^a_R(x)\\
&\quad+\frac{1}{4g}\partial_\mu\{[A^a_{\nu R}(x)\gamma_\nu\gamma_\mu+2A^a_{\mu R}(x)]\partial\!\!\!/\psi^a_R(x)\}+\Delta\mathcal{O}(A^2_{\mu R}).
\end{aligned}\tag{3.39}
$$

From this, we see that the completely renormalized composite operators $X_{\mathrm{gf}R}(x)$ and $X_{c\bar{c}R}(x)$ are given by

$$
\begin{aligned}
&X_{\mathrm{gf}}(x)+X_{c\bar{c}}(x)\\
&=(1+\Delta)X_{\mathrm{gf}R}(x)+(1-\Delta)X_{c\bar{c}R}(x)\\
&\quad+\Delta\partial_\mu\left\{-\frac{1}{2g}\sigma_{\rho\sigma}\gamma_\mu\psi^a_R(x)[\partial_\rho A^a_{\sigma R}(x)-\partial_\sigma A^a_{\rho R}(x)]\right\}\\
&\quad\quad+2\Delta\left(-\frac{1}{g^2}\right)\partial_\mu\partial_\mu A^a_{\nu R}(x)g\gamma_\nu\psi^a_R(x)\\
&\quad\quad\quad+\frac{3}{2}\Delta\frac{1}{2g}[\partial_\mu A^a_{\nu R}(x)-\partial_\nu A^a_{\mu R}(x)]\sigma_{\mu\nu}\partial\!\!\!/\psi^a_R(x)\\
&\quad\quad\quad\quad+\frac{1}{4g}\partial_\mu\{[A^a_{\nu R}(x)\gamma_\nu\gamma_\mu+2A^a_{\mu R}(x)]\partial\!\!\!/\psi^a_R(x)\}+\Delta\mathcal{O}(A^2_{\mu R}).
\end{aligned}\tag{3.40}
$$

For the later argument, it is convenient to express this in the following form using the equations of motion

$$
\begin{aligned}
&X_{\mathrm{gf}}(x)+X_{c\bar{c}}(x)\\
&=(1+\Delta)(X_{\mathrm{gf}R}(x)+X_{c\bar{c}R}(x))+2\Delta g\gamma_\mu\psi^a_R\frac{\delta S_{c\bar{c}}}{\delta A^a_\mu(x)}\\
&\quad+\Delta\partial_\mu S_\mu(x)\\
&\quad\quad+2\Delta g\gamma_\mu\psi^a_R\frac{\delta(S+S_{\mathrm{gf}})}{\delta A^a_\mu(x)}\\
&\quad\quad\quad+\frac{3}{2}\Delta\frac{1}{2g}[\partial_\mu A^a_{\nu R}(x)-\partial_\nu A^a_{\mu R}(x)]\sigma_{\mu\nu}\frac{\delta S^t}{\delta\bar{\psi}^a(x)}\\
&\quad\quad\quad\quad+\frac{1}{4g}\partial_\mu\left[(A^a_{\nu R}(x)\gamma_\nu\gamma_\mu+2A^a_{\mu R}(x))\frac{\delta S^t}{\delta\bar{\psi}^a(x)}\right]+\Delta\mathcal{O}(A^2_{\mu R})\\
&=(1+\Delta)(X_{\mathrm{gf}R}(x)+X_{c\bar{c}R}(x))\\
&\quad+\Delta\partial_\mu S_\mu(x)\\
&\quad\quad+2\Delta g\gamma_\mu\psi^a_R\frac{\delta S^t}{\delta A^a_\mu(x)}\\
&\quad\quad\quad+\frac{3}{2}\Delta\frac{1}{2g}[\partial_\mu A^a_{\nu R}(x)-\partial_\nu A^a_{\mu R}(x)]\sigma_{\mu\nu}\frac{\delta S^t}{\delta\bar{\psi}^a(x)}\\
&\quad\quad\quad\quad+\frac{1}{4g}\partial_\mu\left[(A^a_{\nu R}(x)\gamma_\nu\gamma_\mu+2A^a_{\mu R}(x))\frac{\delta S^t}{\delta\bar{\psi}^a(x)}\right]+\Delta\mathcal{O}(A^2_{\mu R}),
\end{aligned}\tag{3.41}
$$

where S^t denotes the total action, $S^t=S+S_{\mathrm{gf}}+S_{c\bar{c}}$. To derive this, we have noted the relation such as $\Delta g\gamma_\mu\psi^a_R\delta S_{c\bar{c}}/(\delta A^a_\mu)=-\Delta X_{c\bar{c}R}$. The explicit form of the $\mathcal{O}(A^2_R)$ term in this expression is different from that in Eq. (3.40).

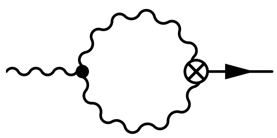


Figure 3.8: g

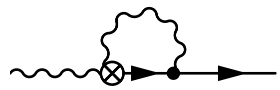


Figure 3.9: h

Next, we determine the renormalization of the supercurrent $S_\mu(x)$ in a similar way. First by substituting the relations (3.31), (3.33), (3.34), (3.35), and (3.36) into $S_\mu(x)$, we have

$$S_\mu(x) = -\frac{1}{2g}\sigma_{\rho\sigma}\gamma_\mu\psi^a_R(x)\left[\partial_\rho A^a_{\sigma R}(x) - \partial_\sigma A^a_{\rho R}(x) + f^{abc}\{A^b_\rho(x)A^c_\sigma(x)\}_R\right] + \Delta\mathcal{O}(A^2_{\mu R}). \tag{3.42}$$

The composite operators in $S_\mu(x)$, such as $\psi^a_R(x)\partial_\rho A^a_{\sigma R}(x)$ and $\psi^a_R(x)f^{abc}\{A^b_\rho(x)A^c_\sigma(x)\}_R$ produce further UV divergences. These can be determined by the computation of the diagrams 3.2, 3.3, 3.5 and the following new diagrams. Sum of these diagrams shows that

$$\begin{aligned}
&-\frac{1}{2g}\sigma_{\rho\sigma}\gamma_\mu\psi^a_R(x)\left[\partial_\rho A^a_{\sigma R}(x) - \partial_\sigma A^a_{\rho R}(x) + f^{abc}\{A^b_\rho(x)A^c_\sigma(x)\}_R\right]\Bigg|_{\text{1PI, divergent part}} \\
&= -\Delta\frac{1}{4g}[A^a_{\nu R}(x)\gamma_\nu\gamma_\mu + 2A^a_{\mu R}(x)]\partial\!\!\!/\psi^a_R(x) + \Delta\mathcal{O}(A^2_{\mu R}).
\end{aligned} \tag{3.43}$$

From this we see that the renormalized supercurrent $S_{\mu R}(x)$ is given from $S_\mu(x)$ as

$$S_\mu(x) = S_{\mu R}(x) - \Delta\frac{1}{4g}[A^a_{\nu R}(x)\gamma_\nu\gamma_\mu + 2A^a_{\mu R}(x)]\partial\!\!\!/\psi^a_R(x) + \Delta\mathcal{O}(A^2_{\mu R}). \tag{3.44}$$

This can be rewritten as

$$S_\mu(x) = S_{\mu R}(x) - \Delta\frac{1}{4g}[A^a_{\nu R}(x)\gamma_\nu\gamma_\mu + 2A^a_{\mu R}(x)]\frac{\delta S^t}{\delta\bar\psi^a(x)} + \Delta\mathcal{O}(A^2_{\mu R}). \tag{3.45}$$

Thus we have obtained the renormalized supercurrent $S_{\mu R}(x)$ in dimensional regularization in the one-loop level.

Using all the above relations between the bare quantities and the renormalized ones, we obtain the SUSY WT identities among renormalized quantities:

$$\begin{aligned}
&\left\langle\left[\partial_\mu S_{\mu R}(x) + X_{\text{gf}R}(x) + X_{c\bar{c}R}(x) + \mathcal{O}(A^2_{\mu R})\right] A^b_{\alpha R}(y)\bar\psi^c_R(z)\right\rangle' \\
&= -\delta(x-y)\left\langle g\gamma_\alpha\psi^b_R(y)\bar\psi^c_R(z)\right\rangle' \\
&\quad -\delta(x-z)\left\langle A^b_{\alpha R}(y)\frac{1}{2g}\sigma_{\beta\gamma}\left[\partial_\beta A^c_{\gamma R}(z) - \partial_\gamma A^c_{\beta R}(z) + f^{cde}\{A^d_\beta(z)A^e_\gamma(z)\}_R\right]\right\rangle'
\end{aligned} \tag{3.46}$$

and

$$\left\langle\left[\partial_\mu S_{\mu R}(x)+X_{\mathrm{gf}R}(x)+X_{c\bar{c}R}(x)+\Delta\mathcal{O}(A_{\mu R}^2)\right]\bar{\psi}_R^b(y)c_R^c(z)\bar{c}_R^d(w)\right\rangle' \\ =-\delta(x-y)\left\langle\frac{1}{2g}\sigma_{\beta\gamma}\left[\partial_\beta A_{\gamma R}^b(y)-\partial_\gamma A_{\beta R}^b(y)+\{f^{bef}A_\beta^e A_\gamma^f\}_R(y)\right]c_R^c(z)\bar{c}_R^d(w)\right\rangle'. \quad (3.47)$$

In deriving these, we have used the following relations holding in the tree-level approximation,

$$\left\langle\frac{\delta S^t}{\delta A_{\mu R}^a(x)}A_{\rho R}^b(y)\right\rangle'=\delta^{ab}\delta_{\nu\rho}\delta(x-y), \quad (3.48)$$

$$\left\langle\frac{\delta S^t}{\delta\bar{\psi}_R^a(x)}\bar{\psi}_R^b(y)\right\rangle'=\delta^{ab}\delta(x-y), \quad (3.49)$$

$$\left\langle X_{c\bar{c}R}(x)A_\alpha^b(y)\bar{\psi}_R^c(z)\right\rangle'=0, \quad (3.50)$$

$$\left\langle\left(-\frac{1}{g^2}\right)\partial_\mu\partial_\mu A_{\nu R}^a(x)g\gamma_\nu\psi_R^a(x)\bar{\psi}_R^b(y)c_R^c(z)\bar{c}_R^d(w)\right\rangle'=\left\langle X_{c\bar{c}}(x)\bar{\psi}_R^b(y)c_R^c(z)\bar{c}_R^d(w)\right\rangle'. \quad (3.51)$$

Equations (3.46) and (3.47) show that the combination $X_{current}(x)\equiv\partial_\mu S_{\mu R}(x)+X_{\mathrm{gf}R}(x)+X_{c\bar{c}R}(x)$ generates the properly normalized SUSY transformations on renormalized elementary fields. For renormalized *composite operators*, however, whether $X_{current}(x)$ generates the correct SUSY transformation or not is not obvious. If we focus on "on mass-shell" correlation functions, where all composite operators and $X_{current}(x)$ are separated to each other in position space, we can still argue that $X_{current}(x)$ has the correct normalization because any new UV divergences arise from the equal point limit between $X_{current}(x)$ and the composite operators.

If we consider only such on mass-shell correlation functions containing only gauge invariant operators, we can further simplify $X_{current}$ as follows. We first note that in on mass-shell correlation functions we may use the equations of motion. Thus, Eq. (3.41) yields

$$X_{\mathrm{gf}}+X_{c\bar{c}}=X_{\mathrm{gf}R}+X_{c\bar{c}R}+\Delta\left(\partial_\mu S_\mu+X_{\mathrm{gf}}+X_{c\bar{c}}\right)+\mathcal{O}(A_{\mu R}^2). \quad (3.52)$$

Furthermore, since in the one-loop approximation, we can use the tree-level relation, $\partial_\mu S_\mu+X_{\mathrm{gf}}+X_{c\bar{c}}=0$, under Δ, we can set

$$X_{\mathrm{gf}}+X_{c\bar{c}}=X_{\mathrm{gf}R}+X_{c\bar{c}R}+\mathcal{O}(A_{\mu R}^2) \quad (3.53)$$

within on mass-shell correlation functions. From the discussion around Eq.(3.19), however, $X_{\mathrm{gf}}+X_{c\bar{c}}$ is BRS-exact and thus can be neglected in correlation functions with gauge invariant operators.

This shows that now $X_{current}(x)$ is reduced to $\partial_\mu S_{\mu R}(x)$. However, the usage of the tree level equations of motion in Eq. (3.45) tells

$$S_{\mu R}(x)\to S_\mu(x)+\Delta\mathcal{O}(A_{\mu R}^2), \quad (3.54)$$

we see that $X_{current}(x)$ is further reduced to $\partial_\mu S_{\mu R}(x)$ with

$$S_{\mu R}(x)\to S_\mu(x)=-\frac{1}{2g_0}\sigma_{\rho\sigma}\gamma_\mu\psi^a(x)F_{\rho\sigma}^a(x), \quad (3.55)$$

where we have required that the supercurrent is gauge invariant to fix the ambiguity of the $\Delta\mathcal{O}(A_{\mu R}^2)$ term.

The above one-loop analysis thus shows that, within on mass-shell correlation functions on gauge invariant operators, the combination,

$$S_{\mu R} = -\frac{1}{2g_0}\sigma_{\rho\sigma}\gamma_\mu\psi^a(x)F^a_{\rho\sigma}(x) + \mathcal{O}(g_0^3). \tag{3.56}$$

can be regarded as a properly normalized supercurrent.

3.3 Small flow-time expansion of the supercurrent

Our next (and last) task is to re-express the renormalized supercurrent (3.56) in terms of flowed fields. To obtain the relation between composite operators of flowed fields and composite operators of un-flowed fields, we employ the small flow-time expansion.

The flow equations for gauge and gaugino fields are

$$\partial_t B^a_\mu(t,x) = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}(t,x) + \alpha_0\mathcal{D}^{ab}_\mu\partial_\nu B^b_\nu, \qquad B^a_\mu(t=0,x) = A^a_\mu(x), \tag{3.57}$$
$$\partial_t\chi^a(t,x) = (\mathcal{D}_\mu\mathcal{D}_\mu - \alpha_0\partial_\mu\mathcal{B}_\mu)^{ab}\chi^b(t,x), \qquad \chi^a(t=0,x) = \psi^a(x), \tag{3.58}$$
$$\partial_t\bar{\chi}^a(t,x) = \bar{\chi}^b(t,x)(\overleftarrow{\mathcal{D}}_\mu\overleftarrow{\mathcal{D}}_\mu + \alpha_0\partial_\mu\mathcal{B}_\mu)^{ba}, \qquad \bar{\chi}^a(t=0,x) = \bar{\psi}^a(x), \tag{3.59}$$

where

$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu + \mathcal{B}^{ab}_\mu, \tag{3.60}$$
$$\overleftarrow{\mathcal{D}}_\mu{}^{ab} = \delta^{ab}\overleftarrow{\partial}_\mu - \mathcal{B}^{ab}_\mu, \tag{3.61}$$
$$\mathcal{B}^{ab}_\mu = f^{acb}B^c_\mu. \tag{3.62}$$

The value of α_0 is arbitrary as long as we focus on gauge-invariant operators as explained in Chap. 2.

To obtain a small flow-time expansion for the supercurrent (3.56), we start from the small flow-time expansion of the gauge invariant operator $\chi^a(t,x)G^a_{\rho\sigma}(t,x)$. From the consideration of the Lorenz covariance, the gauge invariance, and the mass dimension (recall that the flow time possess the mass dimension -2), the most general form of the small flow-time expansion is

$$\begin{aligned}\chi^a(t,x)G^a_{\rho\sigma}(t,x) =&\zeta_1(t)\psi^a(x)F^a_{\rho\sigma}(x)\\ &+ \zeta_2(t)\left[\gamma_\rho\gamma_\alpha\psi^a(x)F^a_{\alpha\sigma}(x) - \gamma_\sigma\gamma_\alpha\psi^a(x)F^a_{\alpha\rho}(x)\right]\\ &+ \zeta_3(t)\sigma_{\alpha\beta}\sigma_{\rho\sigma}\psi^a(x)F^a_{\alpha\beta}(x) + \mathcal{O}(t).\end{aligned} \tag{3.63}$$

The coefficients $\zeta_i(t)$, $i = 1, 2, 3$ are expanded in the loop order,

$$\zeta_1(t) = 1 + \zeta_1^{(1)}(t) + \dots, \quad \zeta_2(t) = \zeta_2^{(1)}(t) + \dots, \quad \zeta_3(t) = \zeta_3^{(1)}(t) + \dots. \tag{3.64}$$

Noting that Eq. (3.63) can be inverted as

$$\begin{aligned}\psi^a(x)F^a_{\rho\sigma}(x) = &\left[1 - \zeta_1^{(1)}(t)\right]\chi^a(t,x)G^a_{\rho\sigma}(t,x)\\ &- \zeta_2^{(1)}(t)\left[\gamma_\rho\gamma_\alpha\chi^a(t,x)G^a_{\alpha\sigma}(t,x) - \gamma_\sigma\gamma_\alpha\chi^a(t,x)G^a_{\alpha\rho}(t,x)\right]\\ &- \zeta_3(t)\sigma_{\alpha\beta}\sigma_{\rho\sigma}\chi^a(t,x)G^a_{\alpha\beta}(t,x) + \mathcal{O}(t)\end{aligned} \tag{3.65}$$

to the one-loop order, we can obtain the expression for the renormalized supercurrent

$$\begin{aligned}S_{\mu R}(x) = &-\frac{1}{2g_0}\left[1-\zeta_1^{(1)}(t)-2(D-3)\zeta_2^{(1)}(t)+(D-9)(D-4)\zeta_3^{(1)}(t)\right]\sigma_{\rho\sigma}\gamma_\mu\chi^a(t,x)G^a_{\rho\sigma}(t,x)\\&-\frac{1}{2g_0}\left[4(D-4)\zeta_2^{(1)}(t)-4(D-5)(D-4)\zeta_3^{(1)}(t)\right]\gamma_\rho\chi^a(t,x)G^a_{\rho\mu}(t,x)\\&+\mathcal{O}(t)+\mathcal{O}(g_0^3).\end{aligned} \tag{3.66}$$

Thus, once one-loop coefficients $\zeta_i^{(1)}$, $i=1,2,3$ are obtained we have the regularization-independent expression of the supercurrent in terms of flowed fields (for this we should finally take $t\to 0$).

Here we use the background field method [17] to compute $\zeta_i^{(1)}$, $i=1,2,3$. In this method, all the fields are decomposed into the classical background fields and the quantum fields:

$$A_\mu(x)=\hat{A}_\mu(x)+a_\mu(x), \qquad B_\mu(t,x)=\hat{B}_\mu(t,x)+b_\mu(t,x), \tag{3.67}$$

$$\psi(x)=\hat{\psi}(x)+p(x), \qquad \chi(t,x)=\hat{\chi}(t,x)+k(t,x), \tag{3.68}$$

$$\bar{\psi}(x)=\hat{\bar{\psi}}(x)+\bar{p}(x), \qquad \bar{\chi}(t,x)=\hat{\bar{\chi}}(t,x)+\bar{k}(t,x). \tag{3.69}$$

Background fields are denoted with the hat $\hat{}$. In the background field method, we modify the "gauge fixing term" in the flow-equations as

$$\partial_t B^a_\mu(t,x)+\mathcal{D}^{ab}_\nu G^b_{\nu\mu}(t,x)+\alpha_0(\mathcal{D}_\mu\hat{\mathcal{D}}_\nu)^{ab}b^b_\nu(t,x), \qquad B^a_\mu(t=0,x)=A^a_\mu(x), \tag{3.70}$$

$$\partial_t\chi^a(t,x)=\left\{(\mathcal{D}^2)^{ab}-\alpha_0 f^{acb}[\hat{\mathcal{D}}_\mu b_\mu(t,x)]^c\right\}\chi^b(t,x), \qquad \chi^a(t=0,x)=\psi^a(x), \tag{3.71}$$

$$\partial_t\bar{\chi}^a(t,x)=\bar{\chi}^b(t,x)\left\{(\overleftarrow{\mathcal{D}}^2)^{ba}+\alpha_0 f^{bca}[\hat{\mathcal{D}}_\mu b_\mu(t,x)]^c\right\}, \qquad \bar{\chi}^a(t=0,x)=\bar{\psi}^a(x). \tag{3.72}$$

It can be shown that any gauge invariant quantity are not affected by the value of α_0 [17].

The flow of the background fields is taken as [17]

$$\partial_t\hat{B}^a_\mu(t,x)=\hat{\mathcal{D}}^{ab}_\nu\hat{G}^b_{\nu\mu}(t,x), \qquad \hat{B}^a_\mu(t=0,x)=\hat{A}^a_\mu(x), \tag{3.73}$$

$$\partial_t\hat{\chi}^a(t,x)=(\hat{\mathcal{D}}^2)^{ab}\hat{\chi}^b(t,x), \qquad \hat{\chi}^a(t=0,x)=\hat{\psi}^a(x), \tag{3.74}$$

$$\partial_t\hat{\bar{\chi}}^a(t,x)=\hat{\bar{\chi}}^b(t,x)(\overleftarrow{\hat{\mathcal{D}}}^2)^{ba}, \qquad \hat{\bar{\chi}}^a(t=0,x)=\hat{\bar{\psi}}^a(x). \tag{3.75}$$

We further assume that the background fields satisfy the equations of motion

$$\hat{\mathcal{D}}^{ab}_\mu\hat{F}^b_{\mu\nu}=0, \tag{3.76}$$

$$\hat{\mathcal{D}}^{ab}_\mu\hat{\psi}^b=0, \tag{3.77}$$

$$\hat{\bar{\psi}}^b\overleftarrow{\hat{\mathcal{D}}}^{ba}_\mu=0. \tag{3.78}$$

With this assumption, the background gauge field does not flow $\hat{B}^a_\mu(t,x)=\hat{A}^a_\mu(x)$. Also, for the tree-level fermion tadpoles, $\langle p^a\rangle^{(0)}=\langle\bar{p}^a\rangle^{(0)}=0$, where the superscript $^{(0)}$ stands for the tree-level approximation. These assumptions also imply $\langle a^a\rangle^{(0)}=\mathcal{O}(\hat{\psi}^2)$ and by the flow equation for quantum fields, we see that $\langle b^a(t,x)-a^a(x)\rangle^{(0)}=\mathcal{O}(t,\,\hat{\psi}^2)$, and $\langle k^a\rangle^{(0)}=\langle\bar{k}^a\rangle^{(0)}=\mathcal{O}(t,\hat{\psi}^3)$.

In the background field method, the gauge fixing term is chosen as

$$S_{\text{bcgf}} = \frac{\lambda_0}{2g_0^2}\int d^D x\, \hat{\mathcal{D}}_\mu a_\mu^a(x)\hat{\mathcal{D}}_\nu a_\nu^a(x). \tag{3.79}$$

In this background gauge, the total action is invariant under the following background gauge transformation,

$$\delta \hat{A}_\mu^a(x) = \hat{\mathcal{D}}_\mu^{ab}\theta^b(x). \tag{3.80}$$

Substituting the decomposition (3.67)–(3.69) into Eq. (3.65), we have

$$\begin{aligned}
&[\hat{\chi}^a(t,x) + k^a(t,x)]\left[\hat{F}_{\mu\nu}^a(x) + \hat{\mathcal{D}}_\mu^{ab} b_\nu^b(t,x) - \hat{\mathcal{D}}_\nu^{ab} b_\mu^b(t,x) + f^{abc} b_\mu^b(t,x) b_\nu^c(t,x)\right]\\
&\quad - \left[\hat{\psi}^a(x) + p^a(x)\right]\left[\hat{F}_{\mu\nu}^a(x) + \hat{\mathcal{D}}_\mu^{ab} a_\nu^b(x) - \hat{\mathcal{D}}_\nu^{ab} a_\mu^b(x) + f^{abc} a_\mu^b(x) a_\nu^c(x)\right]\\
&= \zeta_1^{(1)}(t)\hat{\psi}^a(x)\hat{F}_{\mu\nu}^a(x) + \zeta_2^{(1)}(t)\left[\gamma_\mu\gamma_\rho\hat{\psi}^a(x)\hat{F}_{\rho\nu}^a(x) - \gamma_\nu\gamma_\rho\hat{\psi}^a(x)\hat{F}_{\rho\mu}^a(x)\right]\\
&\quad + \zeta_3^{(1)}(t)\sigma_{\rho\sigma}\sigma_{\mu\nu}\hat{\psi}^a(x)\hat{F}_{\rho\sigma}^a(x) + \mathcal{O}(t).
\end{aligned} \tag{3.81}$$

Thus the coefficients $\zeta_i^{(1)}(t)$ in Eq. (3.81) can be determined by calculating the expectation value of the left-hand side in the presence of the background fields and comparing it with the right-hand side. Since the right-hand side of Eq. (3.81) does not have the terms $\mathcal{O}(\hat{\psi}^2)$ or $\mathcal{O}(\hat{\mathcal{D}}\hat{\psi})$, such contributions in the left-hand side can be neglected. Noting that there is no one-loop diagram that contributes to $f^{abc}\langle k^a(t,x)b_\mu^b(t,x)b_\nu^c(t,x)\rangle$ and $f^{abc}\langle p^a(x)a_\mu^b(x)a_\nu^c(x)\rangle$, all the expectation values are now linear (i.e., the tadpole) or quadratic in quantum fields.

The covariance of the expectation values under the background gauge transformation tells that the tadpole $\langle b_\mu^a(t,x)\rangle$ is $\mathcal{O}(t)$ and can be neglected; note that $\langle b_\mu^a\rangle$ behaves as the adjoint representation under the background gauge transformation. Then the lowest dimensional candidate is $t\hat{\mathcal{D}}_\nu^{ab}\hat{F}_{\nu\mu}^b$ and this is already $\mathcal{O}(t)$ and can be neglected. These arguments considerably simplify the calculation.

Using the results in [17] on the tree-level propagators in the background fields,

$$\langle b_\mu^a(t,x) b_\nu^b(s,y)\rangle = g_0^2\int_{t+s}^\infty d\xi\,(e^{\xi\hat{\Delta}_x})_{\mu\nu}^{ab}\delta(x-y) + \mathcal{O}(\hat{\psi}^2), \tag{3.82}$$

$$\langle k^a(t,x) b_\mu^b(s,y)\rangle = -g_0^2\left(e^{t\hat{\not{\mathcal{D}}}_x^2}\frac{1}{\hat{\not{\mathcal{D}}}_x}\right)^{ac} f^{cde}\gamma_\nu\hat{\psi}^e\int_s^\infty du\,(e^{u\hat{\Delta}_x})_{\nu\mu}^{db}\delta(x-y) + \mathcal{O}(\hat{\mathcal{D}}\hat{\psi},\hat{\psi}^3), \tag{3.83}$$

and the formal solution to the flow equation (3.71),

$$\begin{aligned}
k^a(t,x) =& (e^{t\hat{\mathcal{D}}^2})^{ab} p^b(x)\\
&+ \int_0^t ds\,[e^{(t-s)\hat{\mathcal{D}}^2}]^{ab}\left[2f^{bcd} b_\mu^c(s,x)\hat{\mathcal{D}}_\mu^{de} + f^{bcd}f^{dfe} b_\mu^c(s,x) b_\mu^f(s,x)\right]\\
&\quad\times\left\{[e^{s\hat{\mathcal{D}}^2}]^{eg}\hat{\psi}^g(x) + k^e(s,x)\right\},
\end{aligned} \tag{3.84}$$

where

$$\hat{\Delta}_{\mu\nu}^{ab} = \delta_{\mu\nu}(\hat{\mathcal{D}}^2)^{ab} + 2\hat{\mathcal{F}}_{\mu\nu}^{ab}, \tag{3.85}$$

$$\hat{\mathcal{F}}_{\mu\nu}^{ab} = f^{acb}\hat{F}_{\mu\nu}^c, \tag{3.86}$$

we can calculate the expectation values of flowed fields as (the superscript $^{(1)}$ stands for the one-loop level calculation),

$$\langle k(t,x)\hat{F}_{\mu\nu}(x)\rangle_{\rm 1PI}^{(1)} = \langle k(t,x)\rangle_{\rm 1PI}^{(1)}\hat{F}_{\mu\nu}(x) = \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{2}{D-4}(8\pi t)^{2-D/2}\hat{\psi}(x)\hat{F}_{\mu\nu}(x), \tag{3.87}$$

$$\langle \hat{\psi}(x)[b_\mu(t,x),b_\nu(t,x)] - \hat{\psi}(x)[a_\mu(x),a_\nu(x)]\rangle_{\rm 1PI}^{(1)} = \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{-4}{D-4}(8\pi t)^{2-D/2}\hat{\psi}(x)\hat{F}_{\mu\nu}(x), \tag{3.88}$$

$$\begin{aligned}
&\langle k(t,x)\left(\hat{D}_\mu b_\nu(t,x) - \hat{D}_\nu b_\mu(t,x)\right) - p(x)\left(\hat{D}_\mu a_\nu(x) - \hat{D}_\nu a_\mu(x)\right)\rangle_{\rm 1PI}^{(1)} \\
&= \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{2}{D(D-2)(D-4)}(8\pi t)^{2-D/2} \\
&\quad\times\left\{D\left(\gamma_\mu\gamma_\rho\hat{\psi}(x)\hat{F}_{\rho\nu}(x) - \gamma_\nu\gamma_\rho\hat{\psi}(x)\hat{F}_{\rho\mu}(x)\right) + 2\sigma_{\rho\sigma}\sigma_{\mu\nu}\hat{\psi}(x)\hat{F}_{\rho\sigma}(x)\right\}.
\end{aligned} \tag{3.89}$$

Substituting these into Eq. (3.81), we obtain

$$\zeta_1^{(1)}(t) = \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{-2}{D-4}(8\pi t)^{2-D/2}, \tag{3.90}$$

$$\zeta_2^{(1)}(t) = \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{2}{(D-2)(D-4)}(8\pi t)^{2-D/2}, \tag{3.91}$$

$$\zeta_3^{(1)}(t) = \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{4}{D(D-2)(D-4)}(8\pi t)^{2-D/2}. \tag{3.92}$$

Recalling Eq. (3.66), we obtain the supercurrent in terms of the flowed fields

$$\begin{aligned}
S_{\mu R}(x) = &-\frac{1}{2g_0}\left[1 + \frac{g_0^2}{(4\pi)^2}C_2(G)\frac{2(D-18)}{D(D-2)}(8\pi t)^{2-D/2}\right]\sigma_{\rho\sigma}\gamma_\mu\chi^a(t,x)G_{\rho\sigma}^a(t,x) \\
&-\frac{1}{2g_0}\frac{g_0^2}{(4\pi)^2}C_2(G)\frac{8(D-10)}{D(D-2)}(8\pi t)^{2-D/2}\gamma_\nu\chi^a(t,x)G_{\nu\mu}^a(t,x) + \mathcal{O}(t) + \mathcal{O}(g_0^3).
\end{aligned} \tag{3.93}$$

Finally, since the flowed gaugino field $\chi^a(t,x)$ itself requires the wave functional renormalization, we replace it by the following "ringed variable" [19] (see also Appendix C),

$$\begin{aligned}
\mathring{\chi}(t,x) &= \sqrt{\frac{-\dim(G)}{(4\pi)^2t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}}\,\chi(t,x) \\
&= \frac{1}{(8\pi t)^{\epsilon/2}}\left\{1 + \frac{g^2}{(4\pi)^2}C_2(G)\left[\frac{3}{2}\frac{1}{\epsilon} + \frac{3}{2}\ln(8\pi\mu^2 t) - \frac{1}{2}\ln(432)\right] + \mathcal{O}(g^4)\right\}\chi(t,x).
\end{aligned} \tag{3.94}$$

This variable is convenient because it does not require the wave function renormalization. The coupling constant g_0 also requires the renormalization as Eq. (3.31),

$$g_0^2 = \mu^{2\epsilon}g^2\left[1 + \frac{g^2}{(4\pi)^2}C_2(G)\frac{1}{\epsilon}(-3) + \mathcal{O}(g^4)\right]. \tag{3.95}$$

We thus rewrite the supercurrent in terms of these renormalized quantities as

$$S_{\mu R}(x) = -\frac{1}{2g}\left\{1+\frac{g^2}{(4\pi)^2}C_2(G)\left[-\frac{7}{2}-\frac{3}{2}\ln(8\pi\mu^2 t)+\frac{1}{2}\ln(432)\right]\right\}\sigma_{\rho\sigma}\gamma_\mu \mathring{\chi}^a(t,x)G^a_{\rho\sigma}(t,x)$$
$$-\frac{g^2}{(4\pi)^2}C_2(G)3\gamma_\nu\mathring{\chi}^a(t,x)G^a_{\nu\mu}(t,x)+\mathcal{O}(t)+\mathcal{O}(g^3). \tag{3.96}$$

Since this is manifestly UV-finite, this is independent of the regularization. By the fact that the supercurrent in Eq. (3.44) is completely written by bare quantities, we can replace the coupling constant by the running coupling consonant with an arbitrary mass scale μ. We may take $\mu=\frac{1}{\sqrt{8t}}$. Then,

$$S_{\mu R}(x) = -\frac{1}{2g(1/\sqrt{8t})}\left\{1+\frac{g(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left[-\frac{7}{2}-\frac{3}{2}\ln\pi+\frac{1}{2}\ln(432)\right]\right\}\sigma_{\rho\sigma}\gamma_\mu \mathring{\chi}^a(t,x)G^a_{\rho\sigma}(t,x)$$
$$-\frac{g(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)3\gamma_\nu\mathring{\chi}^a(t,x)G^a_{\nu\mu}(t,x)+\mathcal{O}(t)+\mathcal{O}(g(1/\sqrt{8t})^3). \tag{3.97}$$

Since $g(1/\sqrt{8t})$ goes to zero in the $t\to 0$ limit thanks to the asymptotic freedom, the $t\to 0$ limit can eliminate both $\mathcal{O}(t)$ and $\mathcal{O}(g(1/\sqrt{8t})^3)$ errors in the expression. In this way, we have

$$S_{\mu R}(x) = \lim_{t\to 0}\left(-\frac{1}{2g(1/\sqrt{8t})}\left\{1+\frac{g(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left[-\frac{7}{2}-\frac{3}{2}\ln\pi+\frac{1}{2}\ln(432)\right]\right\}\right.$$
$$\times\sigma_{\rho\sigma}\gamma_\mu \mathring{\chi}^a(t,x)G^a_{\rho\sigma}(t,x)$$
$$\left.-\frac{g(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)3\gamma_\nu\mathring{\chi}^a(t,x)G^a_{\nu\mu}(t,x)\right). \tag{3.98}$$

This is a regularization-independent expression for the supercurrent we were looking for. If one prefers the $\overline{\text{MS}}$ scheme instead of the MS scheme, it is sufficient to make the following replacement in Eq. (3.98)

$$\ln\pi \to \gamma - 2\ln 2, \tag{3.99}$$

where γ denotes Euler's constant.

Chapter 4

The 4D $\mathcal{N} = 2$ super Yang–Mills theory

In this chapter, we consider the Yang–Mills theory with the extended $\mathcal{N} = 2$ supersymmetry as our second example for which a regularization-independent expression of the supercurrent is constructed. Following almost the same line of arguments as in the previous chapter, we find the renormalized supercurrent in dimensional regularization in the one-loop level and then express it in terms of the flowed fields through the small flow-time expansion.

4.1 Action, transformation, and symmetry

The action of the $\mathcal{N} = 2$ SYM in the Wess–Zumino gauge is

$$\begin{aligned}\mathcal{L} =& \frac{1}{4g_0^2} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \bar{\psi}^a(x) \not{\mathcal{D}}^{ab} \psi^b(x) \\ &+ \mathcal{D}_\mu \varphi^{\dagger a}(x) \mathcal{D}_\mu \varphi^a(x) - \frac{1}{2} g_0^2 f^{abc} f^{ade} \varphi^{\dagger b}(x) \varphi^c(x) \varphi^{\dagger d}(x) \varphi^e(x) \\ &+ \sqrt{2} g_0 f^{abc} \bar{\psi}^a(x) \left(P_+ \varphi^b(x) - P_- \varphi^{\dagger b}(x) \right) \psi^c(x),\end{aligned} \tag{4.1}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\mu^c, \tag{4.2}$$

$$\mathcal{D}_\mu^{ab} = \delta^{ab} \partial_\mu + \mathcal{A}_\mu^{ab}, \quad \not{\mathcal{D}}^{ab} = \gamma_\mu \mathcal{D}_\mu^{ab}, \tag{4.3}$$

$$\mathcal{A}_\mu^{ab} = f^{acb} A_\mu^c. \tag{4.4}$$

As the name $\mathcal{N} = 2$ implies, there are two Weyl (or Majorana in four-dimensional spacetime) supercharges in this theory and these can be combined into a single Dirac supercharge. Similarly, the gaugino field $\psi(x)$ is a Dirac fermion instead of Majorana. Thus the $\mathcal{N} = 2$ theory requires further boson fields to balance the degrees of freedom between bosons and fermions. The complex scalar field $\varphi(x)$ does this job.

The above action is invariant under the following $\mathcal{N} = 2$ SUSY transformation:

$$\delta_\chi A^a_\mu = \frac{1}{2} g_0 \left(\bar{\chi}\gamma_\mu \psi^a - \bar{\psi}^a \gamma_\mu \chi \right), \tag{4.5}$$

$$\delta_\chi \varphi^a = -\frac{1}{\sqrt{2}} \left[\bar{\chi} P_- \psi^a - \bar{\psi}^a P_- \chi \right], \tag{4.6}$$

$$\delta_\chi \varphi^{a\dagger} = \frac{1}{\sqrt{2}} \left[\bar{\chi} P_+ \psi^a - \bar{\psi}^a P_+ \chi \right], \tag{4.7}$$

$$\begin{aligned}\delta_\chi \psi^a = &-\frac{1}{4g_0}\sigma_{\mu\nu}\chi F^a_{\mu\nu} \\ &- \frac{1}{\sqrt{2}} \left[\gamma_\mu P_+ \chi \mathcal{D}_\mu \varphi^a - \gamma_\mu P_- \chi \mathcal{D}_\mu \varphi^{a\dagger} \right] \\ &- \frac{1}{2} g_0 \gamma_5 \chi f^{abc} \varphi^{b\dagger} \varphi^c, \end{aligned} \tag{4.8}$$

$$\begin{aligned}\delta_\chi \bar{\psi}^a = &\frac{1}{4g_0}\bar{\chi}\sigma_{\mu\nu} F^a_{\mu\nu} \\ &- \frac{1}{\sqrt{2}} \left[\bar{\chi}\gamma_\mu P_- \mathcal{D}_\mu \varphi^a - \bar{\chi}\gamma_\mu P_+ \mathcal{D}_\mu \varphi^{a\dagger} \right] \\ &- \frac{1}{2} g_0 \bar{\chi} \gamma_5 f^{abc} \varphi^{b\dagger} \varphi^c. \end{aligned} \tag{4.9}$$

Considering the variation of the action under the localized SUSY transformation $\xi \to \xi(x)$ and $\bar{\xi} \to \bar{\xi}(x)$, we have the classical supercurrents,

$$\begin{aligned} S_\mu = &-\frac{1}{4g_0}\sigma_{\rho\sigma}\gamma_\mu \psi^a F^a_{\rho\sigma} \\ &+ \frac{1}{\sqrt{2}}\gamma_\nu\gamma_\mu P_+ \psi^a \mathcal{D}_\nu \varphi^a - \frac{1}{\sqrt{2}}\gamma_\nu\gamma_\mu P_- \psi^a \mathcal{D}_\nu \varphi^{\dagger a} + \frac{1}{2} g_0 f^{abc}\gamma_5\gamma_\mu \psi^a \varphi^{\dagger b}\varphi^c, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \bar{S}_\mu = &-\frac{1}{4g_0}\bar{\psi}^a\gamma_\mu \sigma_{\rho\sigma} F^a_{\rho\sigma} \\ &- \frac{1}{\sqrt{2}}\bar{\psi}^a\gamma_\mu\gamma_\nu P_+ \mathcal{D}_\nu \varphi^a + \frac{1}{\sqrt{2}}\bar{\psi}^a\gamma_\mu\gamma_\nu P_- \mathcal{D}_\nu \varphi^{\dagger a} - \frac{1}{2} g_0 f^{abc}\bar{\psi}^a\gamma_\mu\gamma_5 \varphi^{\dagger b}\varphi^c. \end{aligned} \tag{4.11}$$

These are the "canonical" supercurrents. From the perspective of the theory that has the classical scale-invariance, it is natural to use the following "improved" supercurrents

$$S^{\mathrm{imp}}_\mu \equiv S_\mu + \frac{\sqrt{2}}{3}\sigma_{\mu\nu}\partial_\nu (P_+ \psi^a \varphi^a - P_- \psi^a \varphi^{\dagger a}), \tag{4.12}$$

$$\bar{S}^{\mathrm{imp}}_\mu \equiv \bar{S}_\mu - \frac{\sqrt{2}}{3}\partial_\nu (\bar{\psi}^a P_+ \varphi^a - \bar{\psi}^a P_- \varphi^{\dagger a})\sigma_{\nu\mu}. \tag{4.13}$$

The added terms do not affect the conservation law $\partial_\mu S^{\mathrm{imp}}_\mu = \partial_\mu \bar{S}^{\mathrm{imp}}_\mu = 0$ because $\partial_\mu\partial_\nu\sigma_{\mu\nu} = 0$. Using the identity $\gamma_\mu\sigma_{\rho\sigma}\gamma_\mu = 0$ in $D = 4$, we see these currents are γ-traceless:

$$\gamma_\mu S^{\mathrm{imp}}_\mu = \bar{S}^{\mathrm{imp}}_\mu \gamma_\mu = 0, \tag{4.14}$$

under the equations of motion. In the following oe-loop calculation, however, we will find that these currents are not finite and require the renormalization, unlike the canonical supercurrent in the $\mathcal{N}=1$ SYM. These can be made UV finite by further adding terms being proportional to the equations of motion:

$$\tilde{S}_\mu^{\rm imp} \equiv S_\mu^{\rm imp} - \frac{1}{2\sqrt{2}}\gamma_\mu(P_-\not{\mathcal{D}}\psi^a\varphi^a - P_+\not{\mathcal{D}}\psi^a\varphi^{\dagger a} - \sqrt{2}g_0 f^{abc}\gamma_5\psi^a\varphi^{\dagger b}\varphi^c), \tag{4.15}$$

$$\tilde{\bar{S}}_\mu^{\rm imp} \equiv \bar{S}_\mu^{\rm imp} + \frac{1}{2\sqrt{2}}(\bar{\psi}^a\overleftarrow{\not{\mathcal{D}}}P_-\varphi^a - \bar{\psi}^a\overleftarrow{\not{\mathcal{D}}}P_+\varphi^{\dagger a} - \sqrt{2}g_0 f^{abc}\bar{\psi}^a\gamma_5\varphi^{\dagger b}\varphi^c)\gamma_\mu. \tag{4.16}$$

Some calculation shows that these can be expressed in simpler forms,

$$\begin{aligned}
\tilde{S}_\mu^{\rm imp} = &-\frac{1}{4g_0}\sigma_{\rho\sigma}\gamma_\mu\psi^a F_{\rho\sigma}^a \\
&+\frac{1}{2\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu}-\delta_{\mu\nu}\right)(P_+\mathcal{D}_\nu\psi^a\varphi^a - P_-\mathcal{D}_\nu\psi^a\varphi^{\dagger a}) \\
&-\frac{1}{\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu}-\delta_{\mu\nu}\right)(P_+\psi^a\mathcal{D}_\nu\varphi^a - P_-\psi^a\mathcal{D}_\nu\varphi^{\dagger a}),
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\tilde{\bar{S}}_\mu^{\rm imp} = &-\frac{1}{4g_0}\bar{\psi}^a\gamma_\mu\sigma_{\rho\sigma}F_{\rho\sigma}^a \\
&-\frac{1}{2\sqrt{2}}(\mathcal{D}_\nu\bar{\psi}^a P_+\varphi^a - \mathcal{D}_\nu\bar{\psi}^a P_-\varphi^{\dagger a})\left(\frac{1}{3}\sigma_{\nu\mu}-\delta_{\nu\mu}\right) \\
&+\frac{1}{\sqrt{2}}(\bar{\psi}^a P_+\mathcal{D}_\nu\varphi^a - \bar{\psi}^a P_-\mathcal{D}_\nu\varphi^{\dagger a})\left(\frac{1}{3}\sigma_{\nu\mu}-\delta_{\nu\mu}\right).
\end{aligned} \tag{4.18}$$

In these forms, it is clear that these are γ-traceless for $D=4$ even without using the equations of motion, because $\gamma_\mu\sigma_{\rho\sigma}\gamma_\mu = 0$ and $\gamma_\mu[(1/3)\sigma_{\mu\nu}-\delta_{\mu\nu}] = 0$ for $D=4$. We take these classical supercurrents as our starting point for our argument in the one-loop level.

For perturbative calculations, we introduce the gauge fixing and the Faddeev–Popov ghost terms as before,

$$S_{\rm gf} = \frac{\lambda_0}{2g_0^2}\int d^Dx\,\partial_\mu A_\mu^a(x)\partial_\nu A_\nu^a(x), \tag{4.19}$$

$$S_{c\bar{c}} = -\frac{1}{g_0^2}\int d^Dx\,\bar{c}^a(x)\partial_\mu\mathcal{D}_\mu c^a(x), \tag{4.20}$$

These terms breaks SUSY and give rise to the following SUSY breaking terms:

$$\delta_\xi S_{\rm gf} = -\int d^Dx\,(\bar{\xi}X_{\rm gf} + \bar{X}_{\rm gf}\xi), \tag{4.21}$$

$$\delta_\xi S_{c\bar{c}} = -\int d^Dx\,(\bar{\xi}X_{c\bar{c}} + \bar{X}_{c\bar{c}}\xi), \tag{4.22}$$

where

$$X_{\rm gf} = \frac{\lambda_0}{2g_0}\gamma_\mu\psi^a\partial_\mu\partial_\nu A_\nu^a, \tag{4.23}$$

$$\bar{X}_{\rm gf} = -\frac{\lambda_0}{2g_0}\bar{\psi}^a\gamma_\mu\partial_\mu\partial_\nu A_\nu^a, \tag{4.24}$$

and

$$X_{c\bar{c}} = \frac{1}{2g_0} f^{abc} \partial_\mu \bar{c}^a c^b \gamma_\mu \psi^c, \tag{4.25}$$

$$\bar{X}_{c\bar{c}} = -\frac{1}{2g_0} f^{abc} \partial_\mu \bar{c}^a c^b \bar{\psi}^c \gamma_\mu. \tag{4.26}$$

We now derive SUSY WT identities starting from

$$\left\langle \delta_\xi \left[\begin{Bmatrix} A_\nu^b(y) \\ \varphi^b(y) \\ \varphi^{\dagger b}(y) \end{Bmatrix} \bar{\psi}^c(z) \right] \right\rangle = 0, \tag{4.27}$$

and

$$\left\langle \delta_\xi \left[\bar{\psi}^b(y) c^c(z) \bar{c}^d(w) \right] \right\rangle = 0. \tag{4.28}$$

Under the dimensional regularization $D = 4 - 2\epsilon$, there is an additional explicit SUSY breakings $X_{\text{Fierz}}(x)$ and $\bar{X}_{\text{Fierz}}(x)$ arise from $\delta_\xi S = -\int d^D x\, [\bar{\xi}(\partial_\mu S_\mu + X_{\text{Fierz}}) + (\partial_\mu \bar{S}_\mu + \bar{X}_{\text{Fierz}})\xi]$ for the same reason as in the $\mathcal{N}=1$ SYM. The SUSY WT identities thus become

$$\begin{aligned}
&\left\langle \left[\partial_\mu \tilde{S}_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x) \right] A_\nu^b(y) \bar{\psi}^c(z) \right\rangle \\
&= -\delta(x-y) \frac{1}{2} g_0 \left\langle \gamma_\nu \psi^b(y) \bar{\psi}^c(z) \right\rangle \\
&\quad - \delta(x-z) \frac{1}{4g_0} \left\langle A_\nu^b(y) \sigma_{\rho\sigma} F_{\rho\sigma}^c(z) \right\rangle + \delta(x-z) \frac{1}{2} g_0 \left\langle A_\nu^b(y) \gamma_5 f^{cde} \varphi^{\dagger d}(z) \varphi^e(z) \right\rangle \\
&\quad + \delta(x-z) \frac{1}{\sqrt{2}} \left\langle A_\nu^b(y) \gamma_\rho \left[P_- \mathcal{D}_\rho \varphi^c(z) - P_+ \mathcal{D}_\rho \varphi^{\dagger c}(z) \right] \right\rangle \\
&\quad - \partial_\mu^x \delta(x-z) \frac{1}{2\sqrt{2}} \left\langle A_\nu^b(y) \gamma_\mu \left[P_- \varphi^c(z) - P_+ \varphi^{\dagger c}(z) \right] \right\rangle,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&\left\langle \left[\partial_\mu \tilde{S}_\mu^{\text{imp}}(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{c\bar{c}}(x) \right] \varphi^b(y) \bar{\psi}^c(z) \right\rangle \\
&= \delta(x-y) \frac{1}{\sqrt{2}} \left\langle P_- \psi^b(y) \bar{\psi}^c(z) \right\rangle \\
&\quad - \delta(x-z) \frac{1}{4g_0} \left\langle \varphi^b(y) \sigma_{\rho\sigma} F_{\rho\sigma}^c(z) \right\rangle + \delta(x-z) \frac{1}{2} g_0 \left\langle \varphi^b(y) \gamma_5 f^{cde} \varphi^{\dagger d}(z) \varphi^e(z) \right\rangle \\
&\quad + \delta(x-z) \frac{1}{\sqrt{2}} \left\langle \varphi^b(y) \gamma_\rho \left[P_- \mathcal{D}_\rho \varphi^c(z) - P_+ \mathcal{D}_\rho \varphi^{\dagger c}(z) \right] \right\rangle \\
&\quad - \partial_\mu^x \delta(x-z) \frac{1}{2\sqrt{2}} \left\langle \varphi^b(y) \gamma_\mu \left[P_- \varphi^c(z) - P_+ \varphi^{\dagger c}(z) \right] \right\rangle,
\end{aligned} \tag{4.30}$$

$$
\begin{aligned}
&\left\langle\left[\partial_\mu \tilde{S}_\mu^{\mathrm{imp}}(x)+X_{\mathrm{Fierz}}(x)+X_{\mathrm{gf}}(x)+X_{c\bar{c}}(x)\right] \varphi^{\dagger b}(y)\bar{\psi}^c(z)\right\rangle \\
&=-\delta(x-y)\frac{1}{\sqrt{2}}\left\langle P_+\psi^b(y)\bar{\psi}^c(z)\right\rangle \\
&\quad-\delta(x-z)\frac{1}{4g_0}\left\langle \varphi^{\dagger b}(y)\sigma_{\rho\sigma}F^c_{\rho\sigma}(z)\right\rangle+\delta(x-z)\frac{1}{2}g_0\left\langle\varphi^{\dagger b}(y)\gamma_5 f^{cde}\varphi^{\dagger d}(z)\varphi^e(z)\right\rangle \\
&\quad+\delta(x-z)\frac{1}{\sqrt{2}}\left\langle\varphi^{\dagger b}(y)\gamma_\rho\left[P_-\mathcal{D}_\rho\varphi^c(z)-P_+\mathcal{D}_\rho\varphi^{\dagger c}(z)\right]\right\rangle \\
&\quad-\partial^x_\mu\delta(x-z)\frac{1}{2\sqrt{2}}\left\langle\varphi^{\dagger b}(y)\gamma_\mu\left[P_-\varphi^c(z)-P_+\varphi^{\dagger c}(z)\right]\right\rangle, \qquad (4.31)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\langle\left[\partial_\mu \tilde{S}_\mu^{\mathrm{imp}}(x)+X_{\mathrm{Fierz}}(x)+X_{\mathrm{gf}}(x)+X_{c\bar{c}}(x)\right] \bar{\psi}^b(y)c^c(z)\bar{c}^d(w)\right\rangle \\
&=-\delta(x-y)\frac{1}{4g_0}\left\langle\sigma_{\rho\sigma}F^b_{\rho\sigma}(y)c^c(z)\bar{c}^d(w)\right\rangle+\delta(x-y)\frac{1}{2}g_0\left\langle\gamma_5 f^{bef}\varphi^{\dagger e}(y)\varphi^f(y)c^c(z)\bar{c}^d(w)\right\rangle \\
&\quad+\delta(x-y)\frac{1}{\sqrt{2}}\left\langle\gamma_\rho\left[P_-\mathcal{D}_\rho\varphi^c(z)-P_+\mathcal{D}_\rho\varphi^{\dagger c}(z)\right]c^c(z)\bar{c}^d(w)\right\rangle \\
&\quad-\partial^x_\mu\delta(x-y)\frac{1}{2\sqrt{2}}\left\langle\gamma_\mu\left[P_-\varphi^b(y)-P_+\varphi^{\dagger b}(y)\right]c^c(z)\bar{c}^d(w)\right\rangle. \qquad (4.32)
\end{aligned}
$$

We will rewrite these identities in terms of renormalized quantities at the one-loop level. Before going to this, let us study the effect of X_{Fierz} and $\bar{X}_{\mathrm{Fierz}}$.

4.2 Effect of $X_{\mathrm{Fierz}}(x)$

As for the $\mathcal{N}=1$ case, the effect of X_{Fierz} can be absorbed into an appropriate counterterm. The one-loop level contribution to $\langle X_{\mathrm{Fierz}}(x)A^b_\alpha(y)\bar{\psi}^c(z)\rangle$, $\langle X_{\mathrm{Fierz}}(x)\varphi^b(y)\bar{\psi}^c(z)\rangle$, $\langle X_{\mathrm{Fierz}}(x)\varphi^{\dagger b}(y)\bar{\psi}^c(z)\rangle$ arises from the diagrams in Fig. 4.1. These yield

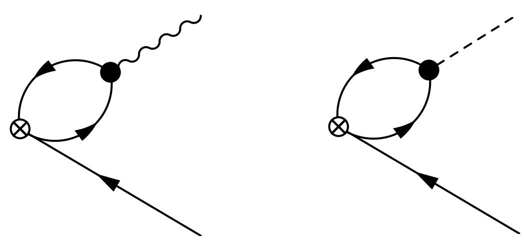


Figure 4.1

$$
\begin{aligned}
&\left\langle X_{\mathrm{Fierz}}(x)\begin{Bmatrix}A^b_\alpha(y)\\ \varphi^b(y)\\ \varphi^{\dagger b}(y)\end{Bmatrix}\bar{\psi}^c(z)\right\rangle \\
&=\frac{g_0^2}{(4\pi)^2}C_2(G)\delta^{bc}\frac{\Gamma(D/2)^2}{\Gamma(D)}\Gamma(2-D/2)(-1)(D-4)\int_p e^{ip(x-y)}\int_q e^{iq(x-z)}
\end{aligned}
$$

$$\times\left(\frac{p^2}{4\pi}\right)^{D/2-2}\begin{Bmatrix} g_0(-p^2\gamma_\alpha+\not{p}p_\alpha) \\ \frac{1}{\sqrt{2}}\frac{1}{D-2}[(D-1)+3\gamma_5]p^2 \\ -\frac{1}{\sqrt{2}}\frac{1}{D-2}[(D-1)-3\gamma_5]p^2 \end{Bmatrix}\frac{1}{p^2}\frac{1}{i\not{q}}. \tag{4.33}$$

This shows that the one-loop effect of X_{Fierz} can be represented as

$$X_{\text{Fierz}}\overset{D\to 4}{\to}\frac{g_0^2}{(4\pi)^2}C_2(G)\left(\frac{1}{3g_0}\gamma_\nu\psi^a\partial_\mu F^a_{\mu\nu}-\frac{1}{\sqrt{2}}P_+\psi^a\partial_\mu\partial_\mu\varphi^{\dagger a}+\frac{1}{\sqrt{2}}P_-\psi^a\partial_\mu\partial_\mu\varphi^a\right). \tag{4.34}$$

This effect thus can be removed by adding the following counterterm to the action,

$$\mathcal{L}'\equiv\frac{g_0^2}{(4\pi)^2}C_2(G)\left(-\frac{1}{6g_0^2}F^a_{\mu\nu}F^a_{\mu\nu}+\frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a+\partial_\mu\varphi^{\dagger a}\partial_\mu\varphi^{\dagger a}\right). \tag{4.35}$$

In what follows, the symbol $\langle\,\rangle$ stands for the expectation value with respect to the action with the above counterterm, $\mathcal{L}+\mathcal{L}'$.

4.3 Renormalized supercurrent in the $\mathcal{N}=2$ SYM

In this section, we will write Eqs. (4.29), (4.30), (4.31), and (4.29) in terms of renormalized quantities.

First, we summarize the renormalization of parameters and basic fields:

$$g_0=\mu^\epsilon(1-\Delta)g, \tag{4.36}$$

$$\lambda_0=\lambda, \tag{4.37}$$

$$A^a_\mu=(1-\Delta)A^a_{\mu R}, \tag{4.38}$$

$$\begin{Bmatrix}\psi^a\\ \bar\psi^a\end{Bmatrix}=(1-\Delta)\begin{Bmatrix}\psi^a_R\\ \bar\psi^a_R\end{Bmatrix}, \tag{4.39}$$

$$\begin{Bmatrix}\varphi^a\\ \varphi^{a\dagger}\end{Bmatrix}=\begin{Bmatrix}\varphi^a_R\\ \varphi^{a\dagger}_R\end{Bmatrix}, \tag{4.40}$$

$$c^a=\left(1-\frac{1}{2}\Delta\right)c^a_R, \tag{4.41}$$

$$\bar c^a=(1-\Delta)\bar c^a_R \tag{4.42}$$

$$F^a_{\mu\nu}=\left(1-\frac{5}{2}\Delta\right)(\partial_\mu A^a_{\nu R}-\partial_\nu A^a_{\mu R})+\left(1-\frac{11}{4}\Delta\right)\{f^{abc}A^b_\mu A^c_\nu\}_R. \tag{4.43}$$

These renormalization factors can be determined from the renormalization in the $\mathcal{N}=1$ SYM by doubling the contribution of the fermion field and further computing the contribution of the scalar field; the Feynman diagrams required in the latter computation are summarized in Appendix D.

We also need the renormalization factors for the following gauge covariant composite operators appearing the SUSY WT identities:

$$f^{abc}\varphi^{\dagger b}\varphi^c=(1-\Delta)\{f^{abc}\varphi^{\dagger b}\varphi^c\}_R, \tag{4.44}$$

$$\mathcal{D}_\mu\varphi^a=\left(1-\frac{3}{2}\Delta\right)\partial_\mu\varphi^a_R+\left(1-\frac{15}{8}\Delta\right)\{f^{abc}A^b_\mu\varphi^c\}_R, \tag{4.45}$$

$$\mathcal{D}_\mu\varphi^{\dagger a}=\left(1-\frac{3}{2}\Delta\right)\partial_\mu\varphi^{\dagger a}_R+\left(1-\frac{15}{8}\Delta\right)\{f^{abc}A^b_\mu\varphi^{\dagger c}\}_R. \tag{4.46}$$

Next, we consider the renormalization the composite operators $X_{\rm gf}$ and $X_{c\bar{c}}$. Substituting the above renormalizations and computing the UV divergences arise from the 1PI diagrams A01–A06 in Appendix D, we find

$$\begin{aligned}
X_{\rm gf} + X_{c\bar{c}} = (1+\Delta)X_{{\rm gf}R} &+ (1-\Delta)X_{c\bar{c}R} \\
&+ \Delta\partial_\mu \left[-\frac{1}{4g}\sigma_{\rho\sigma}\gamma_\mu\psi_R^a(\partial_\rho A_{\sigma R}^a - \partial_\sigma A_{\rho R}^a) \right] \\
&+ \Delta\left(-\frac{1}{g}\right)\gamma_\nu\psi_R^a\partial_\mu\partial_\mu A_{\nu R}^a \\
&+ \Delta\frac{3}{8g}\sigma_{\mu\nu}\partial\!\!\!/\psi_R^a(\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) \\
&+ \Delta\frac{1}{8g}\partial_\mu\left[(A_{\nu R}^a\gamma_\nu\gamma_\mu + 2A_{\mu R}^a)\partial\!\!\!/\psi_R^a\right] \\
&+ \Delta\left(-\frac{1}{\sqrt{2}}\right)\partial_\mu\left[(P_+\partial_\nu\varphi_R^a - P_-\partial_\nu\varphi_R^{\dagger a})\gamma_\nu\gamma_\mu\psi_R^a\right] \\
&+ \Delta\left(-\frac{1}{2\sqrt{2}}\right)(P_+\varphi_R^a - P_-\varphi_R^{\dagger a})\partial_\mu\partial_\mu\psi_R^a \\
&+ \Delta\frac{1}{4}gf^{abc}\gamma_\mu\psi_R^a(\varphi_R^{\dagger b}\overleftrightarrow{\partial}_\mu\varphi_R^c) \\
&+ \Delta\left(-\frac{3}{4}\right)gf^{abc}\gamma_5\gamma_\mu\psi_R^a\partial_\mu(\varphi_R^{\dagger b}\varphi_R^c) \\
&+ \Delta(-1)gf^{abc}\gamma_5\partial\!\!\!/\psi_R^a\varphi_R^{\dagger b}\varphi_R^c \\
&+ \mathcal{H}_1,
\end{aligned} \tag{4.47}$$

where $\mathcal{H}_1$ denotes the possible "higher order terms" of the form,

$$\Delta\left[O(\psi_R A_R^2) + O(\psi_R A_R\varphi_R) + O(\psi_R\varphi_R^3) + O(\psi_R^3)\right], \tag{4.48}$$

which cannot be determined from the analysis of the present WT identities; later we will fix these ambiguities.

For later convenience, we re-express the above relation by using the equations of motion:

$$\begin{aligned}
&X_{\rm gf} + X_{c\bar{c}} \\
&= (1+\Delta)\left(X_{{\rm gf}R} + X_{c\bar{c}R}\right) \\
&\quad + \Delta\partial_\mu\tilde{S}_\mu^{\rm imp} + \Delta\frac{1}{8g}\partial_\mu\left[(A_{\nu R}^a\gamma_\nu\gamma_\mu + 2A_{\mu R}^a)\frac{\delta S^t}{\delta\bar{\psi}^a}\right] \\
&\quad + \Delta g\gamma_\mu\psi_R^a\frac{\delta S^t}{\delta A_\mu^a} \\
&\quad + \Delta\left[\frac{3}{8g}\sigma_{\mu\nu}(\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) - g\gamma_5\{f^{abc}\varphi^{\dagger b}\varphi^c\}_R \right. \\
&\qquad\qquad \left. - \frac{3}{2\sqrt{2}}\gamma_\mu(\partial_\mu\varphi_R^a P_- - \partial_\mu\varphi_R^{\dagger a}P_+)\right]\frac{\delta S^t}{\delta\bar{\psi}^a} \\
&\quad + \Delta(-\sqrt{2})P_-\psi_R^a\frac{\delta S^t}{\delta\varphi^a} + \Delta\sqrt{2}P_+\psi_R^a\frac{\delta S^t}{\delta\varphi^{\dagger a}}
\end{aligned}$$

$$+ \mathcal{H}_2, \tag{4.49}$$

where

$$S^t \equiv S + S_{\rm gf} + S_{c\bar{c}} \tag{4.50}$$

is the total action.[1] Here, $\mathcal{H}_2$ denotes the possible higher order terms again. In deriving the above expression, we have noted the relation $\Delta g \gamma_\mu \psi_R^a \frac{\delta S_{c\bar{c}}}{\delta A_\mu^a} = -2\Delta X_{c\bar{c}R}$.

We next consider the renormalize of the supercurrent $\tilde{S}_\mu^{\rm imp}$ itself. Combining the results in the $\mathcal{N} = 1$ case, Eqs. (4.36)–(4.43), and calculating the UV divergences arising from the 1PI diagrams, A01, A02, B02, B03, B04, C01, C02, and C03 in Appendix D, we have

$$\begin{aligned}
\tilde{S}_\mu^{\rm imp} &= \tilde{S}_{\mu R}^{\rm imp} + \Delta \frac{1}{4g} \sigma_{\rho\sigma} \gamma_\mu \psi_R^a (\partial_\rho A_{\sigma R}^a - \partial_\sigma A_{\rho R}^a) \\
&\quad + \Delta \frac{3}{4g} \left(\frac{1}{3} \sigma_{\mu\sigma} - \delta_{\mu\sigma} \right) \gamma_\rho \psi_R^a (\partial_\rho A_{\sigma R}^a - \partial_\nu A_{\sigma R}^a) \\
&\quad + \Delta \left(-\frac{1}{8g} \right) (A_{\nu R}^a \gamma_\nu \gamma_\mu + 2A_{\mu R}^a) \frac{\delta S^t}{\delta \bar{\psi}^a} \\
&\quad + \mathcal{H}_{3\mu} \\
&= \tilde{S}_{\mu R}^{\rm imp} + \Delta \left(-\frac{1}{8g} \right) (A_{\nu R}^a \gamma_\nu \gamma_\mu + 2A_{\mu R}^a) \frac{\delta S^t}{\delta \bar{\psi}^a} + \mathcal{H}_{3\mu},
\end{aligned} \tag{4.51}$$

where we have noted the identity for $D = 4$:

$$\left(\frac{1}{3} \sigma_{\mu\sigma} - \delta_{\mu\sigma} \right) \gamma_\rho \mathcal{A}_{\rho\sigma} = -\frac{1}{3} \sigma_{\rho\sigma} \gamma_\mu \mathcal{A}_{\rho\sigma}. \tag{4.52}$$

Then, Eqs. (4.49) and (4.51) tell that

$$\begin{aligned}
&\partial_\mu \tilde{S}_\mu^{\rm imp} + X_{\rm gf} + X_{c\bar{c}} \\
&= (1 + \Delta) \left(\partial_\mu \tilde{S}_{\mu R}^{\rm imp} + X_{{\rm gf}R} + X_{c\bar{c}R} \right) \\
&\quad + \Delta g \gamma_\mu \psi_R^a \frac{\delta S^t}{\delta A_\mu^a} \\
&\quad + \Delta \left[\frac{3}{8g} \sigma_{\mu\nu} (\partial_\mu A_{\nu R}^a - \partial_\nu A_{\mu R}^a) - g\gamma_5 \{ f^{abc} \varphi^{\dagger b} \varphi^c \}_R \right. \\
&\qquad\qquad \left. - \frac{3}{2\sqrt{2}} \gamma_\mu (\partial_\mu \varphi_R^a P_- - \partial_\mu \varphi_R^{\dagger a} P_+) \right] \frac{\delta S^t}{\delta \bar{\psi}^a} \\
&\quad + \Delta(-\sqrt{2}) P_- \psi_R^a \frac{\delta S^t}{\delta \varphi^a} + \Delta \sqrt{2} P_+ \psi_R^a \frac{\delta S^t}{\delta \varphi^{\dagger a}} \\
&\quad + \underbrace{\partial_\mu \mathcal{H}_{3\mu} + \mathcal{H}_2}_{\mathcal{H}_4}.
\end{aligned} \tag{4.53}$$

Substituting Eq. (4.53) and Eqs. (4.36)–(4.46) into the SUSY WT identities, Eqs. (4.29)–(4.32), we obtain the SUSY WT identities in terms of renormalized quantities.

[1] We should add Eq. (4.35) to S^t, but its effect in Eq. (4.49) is higher order and is negligible.

For illustration, we explain the detailed calculation for Eq. (4.29). After substituting Eq. (4.53) in Eq. (4.29), the left-hand side of Eq. (4.29) becomes

$$\begin{aligned}
&(1-\Delta)\left\langle\left[\partial_\mu \tilde{S}^{\text{imp}}_{\mu R}(x)+X_{\text{gf}R}(x)+X_{c\bar{c}R}(x)+\mathcal{H}_4(x)\right]A^b_{\nu R}(y)\bar{\psi}^c_R(z)\right\rangle\\
&-2\Delta\delta(x-y)\left(-\frac{1}{2}\right)g\left\langle\gamma_\nu\psi^b_R(y)\bar{\psi}^c_R(z)\right\rangle\\
&-\frac{3}{2}\Delta\delta(x-z)\left(-\frac{1}{4g}\right)\left\langle A^b_{\nu R}(y)\sigma_{\rho\sigma}\left[\partial_\rho A^c_{\sigma R}(z)-\partial_\sigma A^c_{\rho R}(z)\right]\right\rangle\\
&-2\Delta\delta(x-z)\frac{1}{2}g\left\langle A^b_{\nu R}(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle\\
&-\frac{3}{2}\Delta\delta(x-z)\frac{1}{\sqrt{2}}\left\langle A^b_{\nu R}(y)\gamma_\rho\left[P_-\partial_\rho\varphi^c_R(z)-P_+\partial_\rho\varphi^{\dagger c}_R(z)\right]\right\rangle,
\end{aligned} \tag{4.54}$$

where we have noted Schwinger–Dyson equations

$$\left\langle\mathcal{F}^a_\mu(x)\frac{\delta S^t}{\delta A^a_\mu(x)}A^b_\nu(y)\bar{\psi}^c(z)\right\rangle=\delta(x-y)\left\langle\mathcal{F}^b_\nu(y)\bar{\psi}^c(z)\right\rangle, \tag{4.55}$$

$$\left\langle\mathcal{F}^a(x)\frac{\delta S^t}{\delta\bar{\psi}^a(x)}\begin{Bmatrix}A^b_\alpha(y)\\ \varphi^b(y)\\ \varphi^{\dagger b}(y)\end{Bmatrix}\bar{\psi}^c(z)\right\rangle=\delta(x-z)\left\langle\begin{Bmatrix}A^b_\alpha(y)\\ \varphi^b(y)\\ \varphi^{\dagger b}(y)\end{Bmatrix}\mathcal{F}^c(z)\right\rangle, \tag{4.56}$$

$$\left\langle\mathcal{F}^a(x)\frac{\delta S^t}{\delta\varphi^a(x)}\varphi^b(y)\bar{\psi}^c(z)\right\rangle=\delta(x-y)\left\langle\mathcal{F}^b(y)\bar{\psi}^c(z)\right\rangle, \tag{4.57}$$

$$\left\langle\mathcal{F}^a(x)\frac{\delta S^t}{\delta\varphi^{\dagger a}(x)}\varphi^{\dagger b}(y)\bar{\psi}^c(z)\right\rangle=\delta(x-y)\left\langle\mathcal{F}^b(y)\bar{\psi}^c(z)\right\rangle. \tag{4.58}$$

After using (4.36)–(4.46), the right-hand side of Eq. (4.29) becomes

$$\begin{aligned}
&(1-3\Delta)\delta(x-y)\left(-\frac{1}{2}\right)g\left\langle\gamma_\nu\psi^b_R(y)\bar{\psi}^c_R(z)\right\rangle\\
&+\left(1-\frac{5}{2}\Delta\right)\delta(x-z)\left(-\frac{1}{4g}\right)\left\langle A^b_{\nu R}(y)\sigma_{\rho\sigma}\left[\partial_\rho A^c_{\sigma R}(z)-\partial_\sigma A^c_{\rho R}(z)+\mathcal{H}'(z)\right]\right\rangle\\
&+(1-3\Delta)\delta(x-z)\frac{1}{2}g\left\langle A^b_{\nu R}(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle\\
&+\left(1-\frac{5}{2}\Delta\right)\delta(x-z)\frac{1}{\sqrt{2}}\left\langle A^b_{\nu R}(y)\gamma_\rho\left[P_-\partial_\rho\varphi^c_R(z)-P_+\partial_\rho\varphi^{\dagger c}_R(z)+\mathcal{H}'(z)\right]\right\rangle\\
&+(1-\Delta)(-1)\partial^x_\mu\delta(x-z)\frac{1}{2\sqrt{2}}\left\langle A^b_{\nu R}(y)\gamma_\mu\left[P_-\varphi^c_R(z)-P_+\varphi^{\dagger c}_R(z)\right]\right\rangle.
\end{aligned} \tag{4.59}$$

Then after we transfer the last four lines of the left-hand side to the right-hand side, we find that every terms have the common factor $1-\Delta$. In this way, we have

$$\begin{aligned}
&\left\langle\left[\partial_\mu \tilde{S}^{\text{imp}}_{\mu R}(x)+X_{\text{gf}R}(x)+X_{c\bar{c}R}(x)+\mathcal{H}_4(x)\right]A^b_{\nu R}(y)\bar{\psi}^c_R(z)\right\rangle\\
&=\delta(x-y)\left(-\frac{1}{2}\right)g\left\langle\gamma_\nu\psi^b_R(y)\bar{\psi}^c_R(z)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \delta(x-z)\left(-\frac{1}{4g}\right)\left\langle A^b_{\nu R}(y)\sigma_{\rho\sigma}\left[\partial_\rho A^c_{\sigma R}(z) - \partial_\sigma A^c_{\rho R}(z) + \mathcal{H}'(z)\right]\right\rangle \\
&+ \delta(x-z)\frac{1}{2}g\left\langle A^b_{\nu R}(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle \\
&+ \delta(x-z)\frac{1}{\sqrt{2}}\left\langle A^b_{\nu R}(y)\gamma_\rho\left[P_-\partial_\rho\varphi^c_R(z) - P_+\partial_\rho\varphi^{\dagger c}_R(z) + \mathcal{H}'(z)\right]\right\rangle \\
&- \partial^x_\mu\delta(x-z)\frac{1}{2\sqrt{2}}\left\langle A^b_{\nu R}(y)\gamma_\mu\left[P_-\varphi^c_R(z) - P_+\varphi^{\dagger c}_R(z)\right]\right\rangle . \qquad (4.60)
\end{aligned}
$$

In a similar way, from Eqs. (4.30)–(4.32), we have

$$
\begin{aligned}
&\left\langle\left[\partial_\mu\tilde{S}^{\mathrm{imp}}_{\mu R}(x) + X_{\mathrm{gf}R}(x) + X_{c\bar{c}R}(x) + \mathcal{H}_4(x)\right]\varphi^b_R(y)\bar{\psi}^c_R(z)\right\rangle \\
&= \delta(x-y)\frac{1}{\sqrt{2}}\left\langle P_-\psi^b_R(y)\bar{\psi}^c_R(z)\right\rangle \\
&\quad - \delta(x-z)\frac{1}{4g}\left\langle\varphi^b_R(y)\sigma_{\rho\sigma}\left[\partial_\rho A^c_{\sigma R}(z) - \partial_\sigma A^c_{\rho R}(z) + \mathcal{H}'(z)\right]\right\rangle \\
&\quad + \delta(x-z)\frac{1}{2}g\left\langle\varphi^b_R(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle \\
&\quad + \delta(x-z)\frac{1}{\sqrt{2}}\left\langle\varphi^b_R(y)\gamma_\rho\left[P_-\partial_\rho\varphi^c_R(z) - P_+\partial_\rho\varphi^{\dagger c}_R(z) + \mathcal{H}'(z)\right]\right\rangle \\
&\quad - \partial^x_\mu\delta(x-z)\frac{1}{2\sqrt{2}}\left\langle\varphi^b_R(y)\gamma_\mu\left[P_-\varphi^c_R(z) - P_+\varphi^{\dagger c}_R(z)\right]\right\rangle , \qquad (4.61)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\langle\left[\partial_\mu\tilde{S}^{\mathrm{imp}}_{\mu R}(x) + X_{\mathrm{gf}R}(x) + X_{c\bar{c}R}(x) + \mathcal{H}_4(x)\right]\varphi^{\dagger b}_R(y)\bar{\psi}^c_R(z)\right\rangle \\
&= -\delta(x-y)\frac{1}{\sqrt{2}}\left\langle P_+\psi^b_R(y)\bar{\psi}^c_R(z)\right\rangle \\
&\quad - \delta(x-z)\frac{1}{4g}\left\langle\varphi^{\dagger b}_R(y)\sigma_{\rho\sigma}\left[\partial_\rho A^c_{\sigma R}(z) - \partial_\sigma A^c_{\rho R}(z) + \mathcal{H}'(z)\right]\right\rangle \\
&\quad + \delta(x-z)\frac{1}{2}g\left\langle\varphi^{\dagger b}_R(y)\gamma_5\{f^{cde}\varphi^{\dagger d}\varphi^e\}_R(z)\right\rangle \\
&\quad + \delta(x-z)\frac{1}{\sqrt{2}}\left\langle\varphi^{\dagger b}_R(y)\gamma_\rho\left[P_-\partial_\rho\varphi^c_R(z) - P_+\partial_\rho\varphi^{\dagger c}_R(z) + \mathcal{H}'(z)\right]\right\rangle \\
&\quad - \partial^x_\mu\delta(x-z)\frac{1}{2\sqrt{2}}\left\langle\varphi^{\dagger b}_R(y)\gamma_\mu\left[P_-\varphi^c_R(z) - P_+\varphi^{\dagger c}_R(z)\right]\right\rangle , \qquad (4.62)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\langle\left[\partial_\mu\tilde{S}^{\mathrm{imp}}_{\mu R}(x) + X_{\mathrm{gf}R}(x) + X_{c\bar{c}R}(x) + \mathcal{H}_4(x)\right]\bar{\psi}^b_R(y)c^c_R(z)\bar{c}^d_R(w)\right\rangle \\
&= -\delta(x-y)\frac{1}{4g}\left\langle\sigma_{\rho\sigma}\left[\partial_\rho A^b_{\sigma R}(y) - \partial_\sigma A^b_{\rho R}(y) + \mathcal{H}'(y)\right]c^c_R(z)\bar{c}^d_R(w)\right\rangle \\
&\quad + \delta(x-y)\frac{1}{2}g\left\langle\gamma_5\{f^{bef}\varphi^{\dagger e}\varphi^f\}_R(y)c^c_R(z)\bar{c}^d_R(w)\right\rangle \\
&\quad + \delta(x-y)\frac{1}{\sqrt{2}}\left\langle\gamma_\rho\left[P_-\partial_\rho\varphi^c_R(y) - P_+\partial_\rho\varphi^{\dagger c}_R(y) + \mathcal{H}'(y)\right]c^c_R(z)\bar{c}^d_R(w)\right\rangle
\end{aligned}
$$

$$- \partial_\mu^x \delta(x-y) \frac{1}{2\sqrt{2}} \left\langle \gamma_\mu \left[P_- \varphi_R^b(y) - P_+ \varphi_R^{\dagger b}(y) \right] c_R^c(z) \bar{c}_R^d(w) \right\rangle . \tag{4.63}$$

From the above SUSY WT identities, we can infer that the combination

$$\partial_\mu \tilde{S}_{\mu R}^{\mathrm{imp}} + X_{\mathrm{gf}R} + X_{c\bar{c}R} + \mathcal{H}_4 \tag{4.64}$$

generates the correct SUSY transformations on renormalized elementary fields. Also in the on mass-shell correlation functions with composite operators, this combination can be regarded to have a proper normalization because no new UV divergences associated with the equal-point limit arises in on mass-shell correlation functions.

Now we simplify the expression (4.64) by considering its insertion into the on mass-shell correlation functions with gauge-invariant operators.

First, since the equations of motion hold within the on mass-shell correlation functions, Eq. (4.49) reduces to

$$X_{\mathrm{gf}} + X_{c\bar{c}} = X_{\mathrm{gf}R} + X_{c\bar{c}R} + \Delta \left(\partial_\mu \tilde{S}_\mu^{\mathrm{imp}} + X_{\mathrm{gf}} + X_{c\bar{c}} \right) + \mathcal{H}_2. \tag{4.65}$$

We further note that $\partial_\mu \tilde{S}_\mu^{\mathrm{imp}} + X_{\mathrm{gf}} + X_{c\bar{c}} = 0$ under tree-level equations of motion. Thus, to the one-loop level,

$$X_{\mathrm{gf}} + X_{c\bar{c}} = X_{\mathrm{gf}R} + X_{c\bar{c}R} + \mathcal{H}_2. \tag{4.66}$$

This combination, however, vanishes inside correlation functions with gauge-invariant operators, because $X_{\mathrm{gf}} + X_{c\bar{c}}$ is BRS-exact according to Eq. (3.19). Then Eq. (4.64) can be replaced by

$$\begin{aligned} &\partial_\mu \tilde{S}_{\mu R}^{\mathrm{imp}} + \mathcal{H}_4 - \mathcal{H}_2 \\ &= \partial_\mu (\tilde{S}_{\mu R}^{\mathrm{imp}} + \mathcal{H}_{3\mu}). \end{aligned} \tag{4.67}$$

Since Eq. (4.51) shows that $\tilde{S}_\mu^{\mathrm{imp}} = \tilde{S}_{\mu R}^{\mathrm{imp}} + \mathcal{H}_{3\mu}$, the combination (4.64), when inserted in the on mass-shell correlation functions of gauge-invariant operators, can be replaced by $\partial_\mu \tilde{S}_\mu^{\mathrm{imp}}$.

The bottom line of the above analyses is that as far as the insertion in the on mass-shell correlation functions of gauge-invariant operators is concerned, the bare supercurrents

$$\tilde{S}_\mu^{\mathrm{imp}}, \quad \bar{\tilde{S}}_\mu^{\mathrm{imp}} \tag{4.68}$$

are properly normalized.

4.4 Small flow-time expansion of the supercurrent

In the previous section, we have found that the correctly normalized supercurrent in the $\mathcal{N}=2$ SYM theory in the WZ gauge (under the dimensional regularization) is the bare supercurrent $\tilde{S}_\mu^{\mathrm{imp}}$ itself. Our next (and final) task is to express this supercurrent in terms of the flowed fields. For this, we again calculate the small flow-time expansion of relevant flowed operators. Unlike the case of the $\mathcal{N}=1$ SYM theory in Sec 3.3, we do this without using the background field method, because the tree-level propagators in the presence of the background scalar field are rather cumbersome; we do calculations using flow Feynman diagrams.

The flow equations in the present system are defined by

$$\partial_t B^a_\mu(t,x) = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}(t,x) + \alpha_0 \mathcal{D}^{ab}_\mu \partial_\nu B^b_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x), \tag{4.69}$$
$$\partial_t \chi^a(t,x) = (\mathcal{D}_\mu \mathcal{D}_\mu - \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ab} \chi^b(t,x), \qquad \chi^a(t=0,x) = \psi^a(x), \tag{4.70}$$
$$\partial_t \bar{\chi}^a(t,x) = \bar{\chi}^b(t,x)(\overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}_\mu + \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ba}, \qquad \bar{\chi}^a(t=0,x) = \bar{\psi}(x), \tag{4.71}$$
$$\partial_t \phi^a(t,x) = (\mathcal{D}_\mu \mathcal{D}_\mu - \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ab} \phi^b(t,x), \qquad \phi^a(t=0,x) = \varphi^a(x), \tag{4.72}$$
$$\partial_t \phi^{a\dagger}(t,x) = \phi^{b\dagger}(t,x)(\overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}_\mu + \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ba}, \qquad \phi^{a\dagger}(t=0,x) = \varphi^{a\dagger}(x), \tag{4.73}$$
$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu + \mathcal{B}^{ab}_\mu, \tag{4.74}$$
$$\overleftarrow{\mathcal{D}}^{ab}_\mu = \delta^{ab}\overleftarrow{\partial}_\mu - \mathcal{B}^{ab}_\mu, \tag{4.75}$$
$$\mathcal{B}^{ab}_\mu = f^{acb} B^c_\mu. \tag{4.76}$$

The supercurrent $\tilde{S}^{\rm imp}_\mu$ contains the following composite operators:

$$\frac{1}{g_0}\psi^a F^a_{\mu\nu}, \tag{4.77}$$
$$\psi^a \mathcal{D}_\mu \varphi^a, \tag{4.78}$$
$$\psi^a \varphi^a, \tag{4.79}$$
$$\psi^a \varphi^{\dagger a}, \tag{4.80}$$
$$g_0 f^{abc} \psi^a \phi^{b\dagger} \phi^c. \tag{4.81}$$

Similar to Chap. 3, we have to calculate the small flow-time expansion of the flowed version of these operators

$$\mathcal{O}_1(t,x) = \frac{1}{g_0}\chi^a(t,x) G^a_{\mu\nu}(t,x), \tag{4.82}$$
$$\mathcal{O}_2(t,x) = \chi^a(t,x)\mathcal{D}_\mu \phi^a(t,x), \tag{4.83}$$
$$\mathcal{O}_3(t,x) = \chi^a(t,x)\phi^a(t,x) \tag{4.84}$$
$$\mathcal{O}_4(t,x) = \mathcal{P}\mathcal{O}_3(t,x) = \chi^a(t,x)\phi^{\dagger a}(t,x), \tag{4.85}$$
$$\mathcal{O}_5(t,x) = g_0 f^{abc}\chi^a(t,x)\phi^{\dagger b}(t,x)\phi^c(t,x), \tag{4.86}$$

where $\mathcal{P}$ is the parity transformation defined in Appendix A. For this, we compute the expectation values,

$$\langle \mathcal{O}_i(t,x)\mathcal{O}_{\rm ex1}(y_1)\mathcal{O}_{\rm ex2}(y_2)\ldots\rangle, (i=1,2,3), \tag{4.87}$$

with all the possible un-flowed external fields $\mathcal{O}_{\rm ex}(y)$ in one-loop level. We do this by diagrammatically. The rules for the flow Feynman diagram are summarized in Appendix B.

Taking $\mathcal{O}_1(t,x) = \frac{1}{g_0}\chi^a G^a_{\mu\nu}$ as the example, we illustrate the calculation of the small flow-time expansion. We first consider the cases of the external fields, $\mathcal{O}_{\rm ex1}\mathcal{O}_{\rm ex2} = \mathcal{O}(A\psi)$, $\mathcal{O}(AA\psi)$. These correlation functions in the one-loop level are already calculated in the $\mathcal{N}=1$ case and because there is no one-loop diagram with a scalar propagator loop, the expansion coefficients in Eqs. (3.90)-(3.92) for the $\mathcal{N}=1$ SYM can be used without change.

Next, we have to consider the cases of the external fields containing the scalar field. The relevant diagrams in the one-loop level are A01–A06 in Appendix D. Two diagrams, A01 and A02 contribute to the correlation functions of the form, $\langle \mathcal{O}_1(t,x)\mathcal{O}_{\mathrm{ex1}}(y)\mathcal{O}_{\mathrm{ex2}}(z)\rangle$, where $\mathcal{O}_{\mathrm{ex1}}(y)\mathcal{O}_{\mathrm{ex2}}(z) = \mathcal{O}(\psi\varphi)$. The remaining four diagrams A03–A06 give rise to $\langle \mathcal{O}_1(t,x)\mathcal{O}_{\mathrm{ex1}}(y_1)\mathcal{O}_{\mathrm{ex2}}(y_2)\mathcal{O}_{\mathrm{ex3}}(y_3)\rangle$, where $\mathcal{O}_{\mathrm{ex1}}(y_1)\mathcal{O}_{\mathrm{ex2}}(y_2)\mathcal{O}_{\mathrm{ex3}}(y_3) = \mathcal{O}(A\psi\varphi)$, $\mathcal{O}(\psi\varphi\varphi^\dagger)$. According to the Feynman rules in Appendix B, we can explicitly calculate the small flow-time expansion of diagrams A01–A06. Combining Eqs. (3.90)-(3.92) and somewhat lengthy calculation of A01–A06, we obtain the small flow-time expansion of $\chi^a G^a_{\mu\nu}$,

$$
\begin{aligned}
&\frac{1}{g_0}\chi^a(t,x)G^a_{\mu\nu}(t,x) \\
&= \left[1+\frac{-2}{D-4}\xi(t)\right]\frac{1}{g_0}\psi^a(x)F^a_{\mu\nu}(x) \\
&\quad + \xi(t)\left\{\frac{2}{(D-4)(D-2)}\frac{1}{g_0}\left[\gamma_\mu\gamma_\rho\psi^a(x)F^a_{\rho\nu}(x) - \gamma_\nu\gamma_\rho\psi^a(x)F^a_{\rho\mu}(x)\right]\right. \\
&\qquad\qquad \left. + \frac{4}{(D-4)(D-2)D}\frac{1}{g_0}\sigma_{\rho\sigma}\sigma_{\mu\nu}\psi^a(x)F^a_{\rho\sigma}(x)\right\} \\
&\quad + \xi(t)\sqrt{2}\left\{\frac{4}{(D-4)(D-2)D}\gamma_\rho\gamma_\mu\gamma_\nu\left[P_+\psi^a(x)\mathcal{D}_\rho\varphi^a(x) - P_-\psi^a(x)\mathcal{D}_\rho\varphi^{\dagger a}(x)\right]\right. \\
&\qquad\qquad + \frac{-2}{(D-2)D}\gamma_\nu\left[P_+\psi^a(x)\mathcal{D}_\mu\varphi^a(x) - P_-\psi^a(x)\mathcal{D}_\mu\varphi^{\dagger a}(x)\right] \\
&\qquad\qquad + \frac{-2}{(D-4)(D-2)}\gamma_\nu\left[P_+\mathcal{D}_\mu\psi^a(x)\varphi^a(x) - P_-\mathcal{D}_\mu\psi^a(x)\varphi^{\dagger a}(x)\right] \\
&\qquad\qquad + \frac{2(D+4)}{(D-2)D(D+2)}\gamma_\nu\gamma_5\mathcal{D}_\mu\psi^a(x)\left[\varphi^a(x)+\varphi^{\dagger a}(x)\right] \\
&\qquad\qquad \left. + \frac{2}{(D-2)(D+2)}\gamma_\nu\gamma_5\psi^a(x)\mathcal{D}_\mu\left[\varphi^a(x)+\varphi^{\dagger a}(x)\right]\right\} - (\mu\leftrightarrow\nu) \\
&\quad + \xi(t)\frac{8}{(D-4)(D-2)D}g_0 f^{abc}\sigma_{\mu\nu}\gamma_5\psi^a(x)\varphi^{\dagger b}(x)\varphi^c(x) + O(t),
\end{aligned}
\tag{4.88}
$$

where $\xi(t) \equiv \frac{g_0^2}{(4\pi)^2}C_2(G)(8\pi t)^{2-D/2}$.

A similar calculation on $\chi^a\mathcal{D}_\mu\phi^a$ yields

$$
\begin{aligned}
&\chi^a(t,x)\mathcal{D}_\mu\phi^a(t,x) \\
&= \left[1+\frac{2(D-1)}{(D-4)(D-2)}\xi(t)\right]\psi^a(x)\mathcal{D}_\mu\varphi^a(x) \\
&\quad + \xi(t)\left\{\frac{2}{(D-4)(D-2)}\sigma_{\mu\nu}\psi^a(x)\mathcal{D}_\nu\varphi^a(x)\right. \\
&\qquad\qquad + \frac{2(D-1)}{(D-4)D}\mathcal{D}_\mu\psi^a(x)\varphi^a(x) \\
&\qquad\qquad \left. + \frac{-2}{(D-4)D}\sigma_{\mu\nu}\mathcal{D}_\nu\psi^a(x)\varphi^a(x)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\xi(t)\Bigg\{\frac{4}{(D-4)D}P_-\psi^a(x)\mathcal{D}_\mu\varphi^a(x)\\
&\qquad+\frac{8}{(D-4)(D-2)D}\sigma_{\mu\nu}P_-\psi^a(x)\mathcal{D}_\nu\varphi^a(x)\\
&\qquad+\frac{4}{(D-4)(D-2)}P_-\mathcal{D}_\mu\psi^a(x)\varphi^a(x)\\
&\qquad+\frac{-4}{(D-2)(D+2)}P_-\psi^a(x)\mathcal{D}_\mu\left[\varphi^a(x)+\varphi^{\dagger a}(x)\right]\\
&\qquad+\frac{-4(D+4)}{(D-2)D(D+2)}P_-\mathcal{D}_\mu\psi^a(x)\left[\varphi^a(x)+\varphi^{\dagger a}(x)\right]\Bigg\}\\
&+\xi(t)\sqrt{2}\Bigg\{\frac{-2}{(D-4)(D-2)D}\frac{1}{g_0}\gamma_\mu\sigma_{\rho\sigma}P_-\psi^a(x)F^a_{\rho\sigma}(x)\\
&\qquad+\frac{8}{(D-4)(D-2)D}\frac{1}{g_0}\gamma_\nu P_-\psi^a(x)F^a_{\mu\nu}(x)\\
&\qquad+\frac{-2(D+4)}{(D-4)(D-2)D}g_0f^{abc}\gamma_\mu P_-\psi^a(x)\varphi^{\dagger b}(x)\varphi^c(x)\\
&\qquad+\frac{-2}{(D-2)D}g_0f^{abc}\gamma_\mu\gamma_5\psi^a(x)\varphi^{\dagger b}(x)\varphi^c(x)\Bigg\}+O(t),
\end{aligned}
\tag{4.89}
$$

where diagrams B01–B20 and C01–C07 in Appendix D are relevant.

The flow Feynman diagrams, B01, B04, B06, B10, and B12, give rise to

$$
\begin{aligned}
\chi^a(t,x)\phi^a(t,x)&=\left[1+\frac{4(D-1)}{(D-4)(D-2)}\xi(t)\right]\psi^a(x)\varphi^a(x)\\
&\quad+\xi(t)\Bigg\{\frac{8}{(D-4)(D-2)}P_-\psi^a(x)\varphi^a(x)\\
&\qquad+\frac{-8}{(D-2)D}P_-\psi^a(x)\left[\varphi^a(x)+\varphi^{\dagger a}(x)\right]\Bigg\}+O(t).
\end{aligned}
\tag{4.90}
$$

Applying the parity transformations in Appendix A, we have

$$
\begin{aligned}
\chi^a(t,x)\phi^{\dagger a}(t,x)&=\left[1+\frac{4(D-1)}{(D-4)(D-2)}\xi(t)\right]\psi^a(x)\varphi^{\dagger a}(x)\\
&\quad+\xi(t)\Bigg\{\frac{8}{(D-4)(D-2)}P_+\psi^a(x)\varphi^{\dagger a}(x)\\
&\qquad+\frac{-8}{(D-2)D}P_+\psi^a(x)\left[\varphi^a(x)+\varphi^{\dagger a}(x)\right]\Bigg\}+O(t).
\end{aligned}
\tag{4.91}
$$

Finally, for $g_0f^{abc}\chi^a\phi^{\dagger b}\phi^c$, from the diagrams D01–D11,

$$
\begin{aligned}
&g_0f^{abc}\chi^a(t,x)\phi^{\dagger b}(t,x)\phi^c(t,x)\\
&=\left[1+\frac{2(3D^2-6D-8)}{(D-4)(D-2)D}\xi(t)\right]g_0f^{abc}\psi^a(x)\varphi^{\dagger b}(x)\varphi^c(x)\\
&\quad+\xi(t)\sqrt{2}\frac{2}{(D-4)(D-2)}\gamma_\mu\left[P_+\mathcal{D}_\mu\psi^a(x)\varphi^a(x)+P_-\mathcal{D}_\mu\psi^a(x)\varphi^{\dagger a}(x)\right]+O(t).
\end{aligned}
\tag{4.92}
$$

We now have the small flow-time expansion for all the flowed operators relevant to the representation of the supercurrent.

By inverting the above relations on the un-flowed composite operators, we obtain the operators in the supercurrent in terms of the flowed fields. For example, Eq. (4.90) gives

$$\begin{aligned}\psi^a(x)\varphi^a(x) = &\left[1 + \frac{-4(D-1)}{(D-4)(D-2)}\xi(t)\right]\chi^a(t,x)\phi^a(t,x)\\ &+ \xi(t)\left\{\frac{-8}{(D-4)(D-2)}P_-\chi^a(t,x)\phi^a(t,x)\right.\\ &\left.+ \frac{8}{(D-2)D}P_-\chi^a(t,x)\left[\phi^a(t,x)+\phi^{\dagger a}(t,x)\right]\right\} + O(t). \end{aligned} \tag{4.93}$$

The flowed gaugino field and the flowed scalar field in these expressions, however, require the wave function renormalization [15]). We thus express these field by the UV-finite ringed gaugino field and the scalar field. The relations between the original flowed fields and the ringed fields are shown in Appendix C.

Finally, by substituting the composite operators in the supercurrent by flowed operators and re-express it in terms of the ringed flowed fields and the renormalized gauge coupling, we have

$$\begin{aligned}&\tilde{S}_\mu^{\text{imp}}\\ &= \left\{1 + \frac{g^2}{(4\pi)^2}C_2(G)\left[-\ln(8\pi\mu^2 t) - \frac{9}{4} + \frac{1}{2}\ln(432)\right]\right\}\left(-\frac{1}{4g}\right)\sigma_{\rho\sigma}\gamma_\mu\mathring{\chi}^a G^a_{\rho\sigma}\\ &\quad - \frac{g}{(4\pi)^2}C_2(G)\gamma_\nu\mathring{\chi}^a G^a_{\nu\mu}\\ &\quad + \left\{1 + \frac{g^2}{(4\pi)^2}C_2(G)\left[-\frac{19}{4} + 4\ln 2 + \frac{1}{2}\ln(432)\right]\right\}\\ &\qquad \times \frac{1}{2\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)(P_+\mathcal{D}_\nu\mathring{\chi}^a\mathring{\phi}^a - P_-\mathcal{D}_\nu\mathring{\chi}^a\mathring{\phi}^{\dagger a})\\ &\quad - \frac{3}{\sqrt{2}}\frac{g^2}{(4\pi)^2}C_2(G)(P_+\mathcal{D}_\mu\mathring{\chi}^a\mathring{\phi}^a - P_-\mathcal{D}_\mu\mathring{\chi}^a\mathring{\phi}^{\dagger a})\\ &\quad + \left\{1 + \frac{g^2}{(4\pi)^2}C_2(G)\left[\frac{1}{2} + 4\ln 2 + \frac{1}{2}\ln(432)\right]\right\}\\ &\qquad \times\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)(P_+\mathring{\chi}^a\mathcal{D}_\nu\mathring{\phi}^a - P_-\mathring{\chi}^a\mathcal{D}_\nu\mathring{\phi}^{\dagger a})\\ &\quad + \frac{1}{\sqrt{2}}\frac{g^2}{(4\pi)^2}C_2(G)\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)\gamma_5\mathcal{D}_\nu\mathring{\chi}^a(\mathring{\phi}^a + \mathring{\phi}^{\dagger a})\\ &\quad + \frac{1}{2\sqrt{2}}\frac{g^2}{(4\pi)^2}C_2(G)\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)\gamma_5\mathring{\chi}^a\mathcal{D}_\nu(\mathring{\phi}^a + \mathring{\phi}^{\dagger a})\\ &\quad - \frac{1}{4}\frac{g^3}{(4\pi)^2}C_2(G)f^{abc}\gamma_5\gamma_\mu\mathring{\chi}^a\mathring{\phi}^{\dagger b}\mathring{\phi}^c + O(t). \end{aligned} \tag{4.94}$$

The conjugate of the supercurrent $\tilde{\bar{S}}_\mu^{\text{imp}}$ can be obtained from the charge conjugation $\tilde{S}_\mu^{\text{imp}} \to$

$C(\tilde{\bar{S}}_\mu^{\mathrm{imp}})^T$ as

$$
\begin{aligned}
&\tilde{\bar{S}}_\mu^{\mathrm{imp}} \\
&= \left\{1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\ln(8\pi\mu^2 t) - \frac{9}{4} + \frac{1}{2}\ln(432)\right]\right\} \left(-\frac{1}{4g}\right) \mathring{\bar{\chi}}^a \gamma_\mu \sigma_{\rho\sigma} G^a_{\rho\sigma} \\
&\quad + \frac{g}{(4\pi)^2} C_2(G) \mathring{\bar{\chi}}^a \gamma_\nu G^a_{\nu\mu} \\
&\quad + \left\{1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\frac{19}{4} + 4\ln 2 + \frac{1}{2}\ln(432)\right]\right\} \\
&\qquad\qquad \times \left(-\frac{1}{2\sqrt{2}}\right) (\mathcal{D}_\nu \mathring{\bar{\chi}}^a P_+ \mathring{\phi}^a - \mathcal{D}_\nu \mathring{\bar{\chi}}^a P_- \mathring{\phi}^{\dagger a}) \left(\frac{1}{3}\sigma_{\nu\mu} - \delta_{\nu\mu}\right) \\
&\quad + \frac{3}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) (\mathcal{D}_\mu \mathring{\bar{\chi}}^a P_+ \mathring{\phi}^a - \mathcal{D}_\mu \mathring{\bar{\chi}}^a P_- \mathring{\phi}^{\dagger a}) \\
&\quad + \left\{1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[\frac{1}{2} + 4\ln 2 + \frac{1}{2}\ln(432)\right]\right\} \\
&\qquad\qquad \times \frac{1}{\sqrt{2}} (P_+ \mathring{\bar{\chi}}^a \mathcal{D}_\nu \mathring{\phi}^a - P_- \mathring{\bar{\chi}}^a \mathcal{D}_\nu \mathring{\phi}^{\dagger a}) \left(\frac{1}{3}\sigma_{\nu\mu} - \delta_{\nu\mu}\right) \\
&\quad - \frac{1}{\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) \mathcal{D}_\nu \mathring{\bar{\chi}}^a \gamma_5 (\mathring{\phi}^a + \mathring{\phi}^{\dagger a}) \left(\frac{1}{3}\sigma_{\nu\mu} - \delta_{\nu\mu}\right) \\
&\quad - \frac{1}{2\sqrt{2}} \frac{g^2}{(4\pi)^2} C_2(G) \mathring{\bar{\chi}}^a \gamma_5 \mathcal{D}_\nu (\mathring{\phi}^a + \mathring{\phi}^{\dagger a}) \left(\frac{1}{3}\sigma_{\nu\mu} - \delta_{\nu\mu}\right) \\
&\quad + \frac{1}{4} \frac{g^3}{(4\pi)^2} C_2(G) f^{abc} \mathring{\bar{\chi}}^a \gamma_\mu \gamma_5 \mathring{\phi}^{\dagger b} \mathring{\phi}^c + O(t).
\end{aligned}
\tag{4.95}
$$

These are our main results on the supercurrents in the 4D $\mathcal{N} = 2$ SYM. Expressed only in (ringed) flow fields and the renormalized coupling, these are manifestly UV finite as they should be (as the Noether current operators). Thus these expressions are regularization independent. Since both sides of Eqs. (4.94) and (4.95) are independent of the renormalization scale μ, we can set it arbitrary. Taking $\mu = 1/\sqrt{8t}$, both the higher loop corrections and the last $\mathcal{O}(t)$ terms can be neglected in the limit $t \to 0$ since the theory is asymptotic free (the beta function in $\mathcal{N} = 2$ SYM to all orders in perturbation theory $\beta(g) \equiv \frac{1}{\mu}\frac{\partial}{\partial\mu} g(\mu) = -2g^3 C_2(G)/(4\pi)^2$ (Refs. [56, 57, 58, 59, 60])).

Chapter 5

Conclusion

In this thesis, we constructed a regularization-independent expression for the supercurrent (the Noether current associated with supersymmetry) in the four-dimensional $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Yang–Mills theories by employing the gradient flow. Our primary motivation for this study is possible non-perturbative analyses of supersymmetric gauge theories by lattice numerical simulations in the future. For numerical simulations the field contents in the so-called Wess–Zumino (WZ) gauge should be advantageous. So we adopted this WZ gauge. With this WZ gauge, however, the SUSY transformation becomes non-linear. Elements in our (perturbative) analysis, the dimensional regularization, the gauge fixing and the Faddeev–Popov ghost terms, break supersymmetry. For this reason, first of all, we had to find a correct expression of the supercurrent (under the dimensional regularization). Through a rather lengthy analysis at the one-loop level, we found the expression of a properly-normalized supercurrent at the one-loop level that works within on-mass-shell correlation functions with gauge invariant operators. We then express this in terms of field variables obtained by flow equations by using the small flow-time expansion. The resulting expressions are manifestly UV finite as should be for Noether current operators. In the small flow-time limit, the expression is expected to be exact, providing a regularization-independent representation of the supercurrent. Since this representation is regularization independent, this can also be used with lattice regularization. We believe that a priori knowledge on the properly-normalized supercurrent will be quite useful in future lattice numerical simulations of supersymmetric gauge theories because the conservation of this current can be used the parameter tuning toward the supersymmetric point. Also, it must be interesting to generalize our construction to more general supersymmetric models which include matter multiplets. It must be also interesting to give a further understanding on the mechanism behind the UV finiteness of the gradient flow. A consideration on the possible relationship between the gradient low and the Wilsonian renormalization group [61, 62] may give a clue on this issue.

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Appendix A

Notation

Throughout the thesis, we adopt the following notational conventions.

We always assume the natural system of units in which $c = \hbar = 1$.

Repeated indices are always summed over with $\mu = 0, 1, 2, 3$. When we are considering the Euclidean spacetime, the upper and lower Lorentz indices are not distinguished.

The generators of the algebra of the gauge group G are all *anti*-Hermitian:

$$\left[T^a, T^b\right] = f^{abc} T^c, \tag{A.1}$$

and the Dynkin index $T(R)$ and the Casimir $C_2(R)$ are defined by

$$\operatorname{tr}(T^a T^b) = -T(R)\delta^{ab}, \tag{A.2}$$

$$T^a T^a = -C_2(R)\mathbf{1}. \tag{A.3}$$

In particular, for the adjoint representation A, the generator is $(T_A^a)_{bc} = -f^{abc}$ and for $G = SU(N)$,

$$T(A) = C_2(A) = C_2(G) = N, \tag{A.4}$$

i.e., $f^{abc} f^{dbc} = C_2(G)\delta^{ad}$. We note the identity,

$$f^{cXa} f^{aYb} f^{bZc} = -\frac{1}{2} C_2(G) f^{XYZ}. \tag{A.5}$$

This follows from a consideration of $\operatorname{tr}(T_A^a T_A^b T_A^c)$.

The gamma matrices obey $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and all the gamma matrices are Hermitian. The trace over the spinor indices is set $\operatorname{tr}(1) = 4$ even under the dimensional regularization $D = 4 - 2\epsilon$.

For fields in the adjoint representation, $\phi^a(x)$, we also use the notation $\phi(x) = \phi^a T^a$. The covariant derivative for ϕ and for ϕ^a are thus defined respectively by

$$D_\mu = \partial_\mu + [A_\mu, \cdot], \tag{A.6}$$

$$\begin{aligned} \mathcal{D}_\mu^{ab} &= \delta^{ab}\partial_\mu + A_\mu^c f^{acb} \\ &= \delta^{ab}\partial_\mu + \mathcal{A}_\mu^{ab}. \end{aligned} \tag{A.7}$$

The abbreviation $\mathcal{D}_\mu \phi^a = \mathcal{D}_\mu^{ab} \phi^b$ is also used.

We define the chiral matrix and the chiral projections for any $D = 4 - 2\epsilon$ by,

$$\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3, \qquad P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5). \tag{A.8}$$

Then we have

$$\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \begin{cases} 4\epsilon_{\mu\nu\rho\sigma}, & \mu,\nu,\rho,\sigma \in \{0,1,2,3\}, \\ 0, & \text{otherwise}, \end{cases} \tag{A.9}$$

where the totally anti-symmetric tensor is normalized as $\epsilon_{0123} = 1$. We also use the definition

$$\sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \tag{A.10}$$

The charge conjugation matrix C satisfies

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \tag{A.11}$$

and thus

$$C^{-1}\sigma_{\mu\nu}C = -\sigma_{\mu\nu}^T, \qquad C^{-1}\gamma_5 C = \gamma_5^T. \tag{A.12}$$

The charge conjugation transformation for fields is defined by

$$\psi(x) \to C\bar{\psi}^T(x), \qquad \bar{\psi}(x) \to -\psi^T(x)C^{-1}, \tag{A.13}$$
$$A_\mu(x) \to A_\mu(x), \tag{A.14}$$
$$\varphi(x) \to -\varphi(x), \qquad \varphi^\dagger(x) \to -\varphi^\dagger(x), \tag{A.15}$$
$$c(x) \to c(x), \qquad \bar{c}(x) \to \bar{c}(x). \tag{A.16}$$

The charge conjugation on the flowed fields is defined similarly.

The parity conjugations for the fields are, on the other hand, defined by

$$\psi(x) \to \gamma_0\psi(\tilde{x}), \qquad \bar{\psi}(x) \to \bar{\psi}(\tilde{x})\gamma_0, \tag{A.17}$$
$$A_0(x) \to A_0(\tilde{x}), \qquad A_i(x) \to -A_i(\tilde{x}), \tag{A.18}$$
$$\varphi(x) \to -\varphi^\dagger(\tilde{x}), \qquad \varphi^\dagger(x) \to -\varphi(\tilde{x}), \tag{A.19}$$
$$c(x) \to c(\tilde{x}), \qquad \bar{c}(x) \to \bar{c}(\tilde{x}), \tag{A.20}$$

where the i denotes the spatial directions and $\tilde{x} \equiv (x_0, -x_i)$. The parity transformation on the flowed fields is defined similarly.

Appendix B

Flow Feynman rules in the $\mathcal{N} = 2$ SYM

In Chap. 4, we consider the calculation of the flow Feynman diagrams in Appendix D. Here, we summarize the required flow Feynman rules.

The flow equations for the fields are defined by

$$\partial_t B_\mu^a(t,x) = \mathcal{D}_\nu G_{\nu\mu}(t,x) + \alpha_0 \mathcal{D}_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x), \tag{B.1}$$

$$\partial_t \chi^a(t,x) = (\mathcal{D}_\mu \mathcal{D}_\mu - \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ab} \chi^b(t,x), \qquad \chi^a(t=0,x) = \psi^a(x), \tag{B.2}$$

$$\partial_t \bar{\chi}^a(t,x) = \bar{\chi}^b(t,x)(\overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}_\mu + \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ba}, \qquad \bar{\chi}^a(t=0,x) = \bar{\psi}(x), \tag{B.3}$$

$$\partial_t \phi^a(t,x) = (\mathcal{D}_\mu \mathcal{D}_\mu - \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ab} \phi^b(t,x), \qquad \phi^a(t=0,x) = \varphi^a(x), \tag{B.4}$$

$$\partial_t \phi^{a\dagger}(t,x) = \phi^{b\dagger}(t,x)(\overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}_\mu + \alpha_0 \partial_\mu \mathcal{B}_\mu)^{ba}, \qquad \phi^{a\dagger}(t=0,x) = \varphi^{a\dagger}(x), \tag{B.5}$$

$$(\mathcal{D}_\mu)^{ab} \equiv \delta^{ab}\partial_\mu + B_\mu^c f^{acb} = \delta^{ab}\partial_\mu + \mathcal{B}_\mu^{ab} \tag{B.6}$$

$$(\overleftarrow{\mathcal{D}}_\mu)^{ba} \equiv \delta^{ba}\overleftarrow{\partial}_\mu + B_\mu^c f^{bac} = \delta^{ba}\overleftarrow{\partial}_\mu - \mathcal{B}_\mu^c f^{bca}, \tag{B.7}$$

where α_0 is a constant that can be chosen arbitrarily as far as gauge-invariant observables are concerned (see Chap. 2).

With the choice $\alpha_0 = 1$, the exact solutions to the flow equations are

$$B_\mu^a(t,x) = \int d^D y \left[K_t(x-y) A_\mu^a(y) + \int_0^t ds\, K_{t-s}(x-y) R_\mu^a(s,y) \right] \tag{B.8}$$

$$\chi^a(t,x) = \int d^D y \left[K_t(x-y) \psi^a(y) + \int_0^t ds\, K_{t-s}(x-y) \Delta'^{ac}(s,y) \chi^c(s,y) \right], \tag{B.9}$$

$$\bar{\chi}^a(t,x) = \int d^D y \left[K_t(x-y) \bar{\psi}^a(y) + \int_0^t ds\, K_{t-s}(x-y) \bar{\chi}^c(s,y) \bar{\Delta}'^{ca}(s,y) \right], \tag{B.10}$$

$$\phi^a(t,x) = \int d^D y \left[K_t(x-y) \varphi^a(y) + \int_0^t ds\, K_{t-s}(x-y) \Delta'^{ac}(s,y) \phi^c(s,y) \right], \tag{B.11}$$

$$\phi^{a\dagger}(t,x) = \int d^D y \left[K_t(x-y) \varphi^{a\dagger}(y) + \int_0^t ds\, K_{t-s}(x-y) \Delta'^{ac}(s,y) \phi^{c\dagger}(s,y) \right], \tag{B.12}$$

where the heat kernel $K_t(x)$ and the non-linear terms of the flow equations $R^a_\mu(t,x)$, $\Delta'^{ac}(t,x)$, $\bar{\Delta}'^{ac}(t,x)$ are defined by

$$K_t(x) = \int \frac{d^D p}{(2\pi)^D}\, e^{ipx} e^{-tp^2}, \tag{B.13}$$

$$\begin{aligned} R^a_\mu(t,x) \equiv\, & 2f^{abc} B^b_\nu(t,x)\partial_\nu B^c_\mu(t,x) - f^{abc} B^b_\nu(t,x)\partial_\mu B^c_\nu(t,x), \\ & + (\alpha_0 - 1) f^{abc} B^b_\mu(t,x)\partial_\nu B^c_\nu(t,x) + f^{abc} f^{cde} B^b_\nu(t,x) B^d_\nu(t,x) B^e_\mu(t,x) \end{aligned} \tag{B.14}$$

$$\Delta'^{ac}(t,x) \equiv 2f^{abc} B^b_\mu(t,x)\partial_\mu + f^{abe} f^{edc} B^b_\mu(t,x) B^d_\mu(t,x), \tag{B.15}$$

$$\bar{\Delta}'^{ca}(t,x) \equiv -2f^{cba}\overleftarrow{\partial_\mu} B^b_\mu + f^{cdb} f^{bea} B^d_\mu B^e_\mu. \tag{B.16}$$

The non-linear terms $R^a_\mu(t,x)$, $\Delta'^{ab}(t,x)$, $\bar{\Delta}'^{ab}(t,x)$, are represented by following flow vertices:

- $B_\mu B_\nu \chi$ three-point vertex
  $\int d^D y \int_0^t ds\, K_{t-s}(x-y) 2f^{abc} B^b_\mu(s,y)\chi^c(s,y)$,
- $B_\mu B_\nu B_\rho \chi$ four-point vertex
  $\int d^D y \int_0^t ds\, K_{t-s}(x-y) f^{abc} f^{edc} B^b_\mu(s,y) B^d_\mu(s,y)\chi^c(s,y)$
- $B_\mu B_\nu \phi$ three-point vertex
  $\int d^D y \int_0^t ds\, K_{t-s}(x-y) 2f^{abc} B^b_\mu(s,y)\phi^c(s,y)$,
- $B_\mu B_\nu B_\rho \phi$ four-point vertex
  $\int d^D y \int_0^t ds\, K_{t-s}(x-y) f^{abc} f^{edc} B^b_\mu(s,y) B^d_\mu(s,y)\phi^c(s,y)$
- $B_\mu B_\nu B_\rho$ three-point vertex
  $\int d^D y \int_0^t ds\, K_{t-s}(x-y)\Big(2f^{abc} B^b_\nu(s,y)\partial_\nu B^c_\mu(s,y) - f^{abc} B^b_\nu(s,y)\partial_\mu B^c_\nu(s,y)\Big)$
- $B_\mu B_\nu B_\rho B_\sigma$ four-point vertex
  $\int d^D y \int_0^t ds\, K_{t-s}(x-y) f^{abc} f^{cde} B^b_\nu(s,y) B^d_\nu(s,y) B^e_\mu(s,y)$

These flow vertices are denoted by white blobs in figures in Appendix D.

For the flow lines (i.e., the heat kernels), we use doubled lines in figures in Appendix D; this convention differs from that in Chap. 2. The flow propagators are denoted by single lines

Besides flow vertices, ordinary vertices come from the original $\mathcal{N}=2$ SYM action with the gauge fixing and the ghost terms

$$S = S_{\mathcal{N}=2\,\mathrm{SYM}} + S_{\mathrm{gf}} + S_{c\bar{c}}, \tag{B.17}$$

$$\begin{aligned} S_{\mathcal{N}=2\,\mathrm{SYM}} = \int d^D x \biggl[&\frac{1}{4g_0^2} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) + \bar{\psi}^a(x) \not{D}^{ab} \psi^b(x) \\ &+ D_\mu \varphi^{\dagger a}(x) D_\mu \varphi^a(x) - \frac{1}{2} g_0^2 f^{abc} f^{ade} \varphi^{\dagger b}(x) \varphi^c(x) \varphi^{\dagger d}(x) \varphi^e(x) \\ &+ \sqrt{2} g_0 f^{abc} \bar{\psi}^a(x) \left(P_+ \varphi^b(x) - P_- \varphi^{\dagger b}(x) \right) \psi^c(x) \biggr], \end{aligned} \tag{B.18}$$

$$S_{\mathrm{gf}} = \frac{\lambda_0}{2g_0^2} \int d^D x\, \partial_\mu A^a_\mu(x) \partial_\nu A^a_\nu(x), \tag{B.19}$$

$$S_{c\bar{c}} = -\frac{1}{g_0^2} \int d^D x\, \bar{c}^a(x) \partial_\mu D_\mu c^a(x). \tag{B.20}$$

Vertices that can be read off from this action are listed below.

- gauge field three-point vertex
  $-\frac{1}{g_0^2} \int d^D x\, f^{abc} \partial_\alpha A^a_\beta(x) A^b_\alpha(x) A^c_\beta(x)$
- gauge field four-point vertex
  $-\frac{1}{4g_0^2} \int d^D x\, f^{abc} f^{ade} A^b_\alpha(x) A^d_\alpha(x) A^c_\beta(x) A^e_\beta(x)$
- gauge-gaugino-gaugino three-point vertex
  -$\int d^D x\ f^{abc} \bar{\psi}^a(x) A^b_\alpha(x) \gamma_\alpha \psi^c(x)$
- scalar-gaugino-gaugino three-point vertex (Yukawa interaction)
  -$\sqrt{2} g_0 \int d^D x\, f^{abc} \bar{\psi}^a(x) (P_+ \varphi^b(x) - P_- \varphi^{b\dagger}(x)) \psi^c(x)$
- scalar-gauge-gauge three-point vertex
  $-\int d^D x\, f^{abc} \partial_\alpha \varphi^{a\dagger}(x) A^b_\alpha(x) \varphi^c(x) + h.c.$
- scalar-gauge-gauge-gauge four-point vertex
  -$\int d^D x\, f^{abc} f^{ade} A^b_\alpha(x) A^d_\alpha(x) \varphi^{c\dagger}(x) \varphi^e(x)$
- scalar field four-point vertex
  $+\frac{1}{2} g_0^2 \int d^D x\, f^{abc} f^{ade} \varphi^{b\dagger}(x) \varphi^c(x) \varphi^{d\dagger}(x) \varphi^e(x)$
- gauge-ghost-ghost three-point vertex
  $+\frac{1}{g_0^2} \int d^D x\ f^{abc} \bar{c}^a(x) \partial_\alpha \left(A^b_\alpha(x) c^c(x) \right)$

These vertices are denoted by black blobs in figures in Appendix D. Here, operators at the vertices are multiplied by a minus sign, because we consider the functional integral with the weight e^{-S}.

The tree-level propagators that connects the above vertices and external fields are (in the Feynman gauge, $\lambda_0 = 1$),

$$\left\langle B^a_\mu(t,x) B^b_\nu(s,y) \right\rangle_0 = \delta^{ab}\delta_{\mu\nu} \int \frac{d^D p}{(2\pi)} \frac{e^{-(t+s)p^2}}{p^2} e^{ip(x-y)}, \tag{B.21}$$

$$\left\langle \chi^a(t,x) \bar{\chi}^b(s,y) \right\rangle_0 = \delta^{ab} \int \frac{d^D p}{(2\pi)} \frac{e^{-(t+s)p^2}}{i\not{p}} e^{ip(x-y)}, \tag{B.22}$$

$$\left\langle \phi^a(t,x) \phi^{b\dagger}(s,y) \right\rangle_0 = \delta^{ab} \int \frac{d^D p}{(2\pi)} \frac{e^{-(t+s)p^2}}{p^2} e^{ip(x-y)}. \tag{B.23}$$

In Chap. 4, we calculate flow Feynman diagrams in Appendix D by employing the above Feynman rules. For this, we need the integration formulas in Appendix E.

Appendix C

The ringed flow fields

Unlike the gauge field, the fermion and the scalar fields require the wave function renormalization even after the flow [15]. The required renormalization factors are regularization-dependent and not quite convenient for our purpose of a universal representation of composite operators. To avoid this, we introduce the following "ringed fields". For the flowed fermion fields, the ringed fields are defined by [19],

$$\mathring{\chi}(t,x) \equiv \sqrt{\frac{-2\dim(G)}{(4\pi)^2 t^2 \left\langle \bar{\chi}^a(t,x) \overleftrightarrow{\not{D}} \chi^a(t,x) \right\rangle}}\, \chi(t,x), \tag{C.1}$$

$$\mathring{\bar{\chi}}(t,x) \equiv \sqrt{\frac{-2\dim(G)}{(4\pi)^2 t^2 \left\langle \bar{\chi}^a(t,x) \overleftrightarrow{\not{D}} \chi^a(t,x) \right\rangle}}\, \bar{\chi}(t,x), \tag{C.2}$$

where $\overleftrightarrow{D}_\mu \equiv D_\mu - \overleftarrow{D}_\mu$. The factor $\left\langle \bar{\chi}^a(t,x) \overleftrightarrow{\not{D}} \chi^a(t,x) \right\rangle$ in the denominator cancels the wave function renormalization factor of χ and $\bar{\chi}$ and makes $\mathring{\chi}$, $\mathring{\bar{\chi}}$ UV finite. The correlator $\left\langle \bar{\chi}^a(t,x) \overleftrightarrow{\not{D}} \chi^a(t,x) \right\rangle$ in dimensional regularization $D = 4 - 2\epsilon$ in one-loop level is calculated as [19]

$$\begin{aligned} &\left\langle \bar{\chi}^a(t,x) \overleftrightarrow{\not{D}} \chi^a(t,x) \right\rangle \\ &= \frac{-2\dim(G)}{(4\pi)^2 t^2} \left\{ (8\pi t)^\epsilon + \frac{g_0^2}{(4\pi)^2} C_2(G) \left[-\frac{4}{\epsilon} - 8\ln(8\pi t) - \frac{3}{2} + \ln(432) \right] \right\}. \end{aligned} \tag{C.3}$$

Similarly, for the flowed scalar field, the ringed variable is defined by [20]

$$\mathring{\phi}(t,x) \equiv \sqrt{\frac{\dim(G)}{2(4\pi)^2 t \left\langle \phi^{\dagger a}(t,x)\phi^a(t,x) \right\rangle}}\, \phi(t,x), \tag{C.4}$$

$$\mathring{\phi}^\dagger(t,x) \equiv \sqrt{\frac{\dim(G)}{2(4\pi)^2 t \left\langle \phi^{\dagger a}(t,x)\phi^a(t,x) \right\rangle}}\, \phi^\dagger(t,x). \tag{C.5}$$

The denominator $\left\langle \phi^{\dagger a}(t,x)\phi^a(t,x) \right\rangle$ in dimensional regularization in one-loop level is obtained from calculation of diagrams E01–E07 in Appendix D. The results are summarized in Table C.1. These

Table C.1: Contribution of E01–E07 to $\left\langle \phi^{\dagger a}(t,x)\phi^a(t,x)\right\rangle$ in units of $\frac{\dim(G)}{2(4\pi)^2 t}\frac{g_0^2}{(4\pi)^2}C_2(G)$.

Diagram	
E01	$\frac{1}{C_2(G)}$
E02	$\frac{2}{\epsilon}+4\ln(8\pi t)+6$
E03	$\frac{2}{\epsilon}+4\ln(8\pi t)+6$
E04	$-2-4\ln 2+6\ln 3$
E05	$12\ln 2-6\ln 3$
E06	$-\frac{4}{\epsilon}-8\ln(8\pi t)-6$
E07	$-\frac{2}{\epsilon}-4\ln(8\pi t)-7$

yield

$$\begin{aligned}&\left\langle \phi^{\dagger a}(t,x)\phi^a(t,x)\right\rangle\\&=\frac{\dim(G)}{2(4\pi)^2 t}\left\{\frac{1}{1-\epsilon}(8\pi t)^{\epsilon}+\frac{g_0^2}{(4\pi)^2}C_2(G)\left[-\frac{2}{\epsilon}-4\ln(8\pi t)-3+8\ln 2\right]\right\}.. \end{aligned}\tag{C.6}$$

In the calculation of the two loop diagrams in E01–E07 in the D-dimensional spacetime, we sometimes encounter the Feynman parameter integrals that cannot be calculated analytically; we need some trick. For example, in the calculation of the diagram E03, we have following integrations:

$$\begin{aligned}&4C_2(G)\dim(G)g_0^2\times\frac{2}{(4\pi)^D(D-2)}\\&\quad\times\int_0^t ds\int_0^{\infty}du\,\frac{[(2t-s)(s+u)+(s+u)s+s(2t-s)]^{1-D/2}}{2s+u}\end{aligned}\tag{C.7}$$

and

$$\begin{aligned}&2C_2(G)\dim(G)g_0^2\times\frac{2}{(4\pi)^D(D-2)}\\&\quad\times\int_0^t ds\int_0^{\infty}du\,\frac{[(2t-s)(s+u)+(s+u)s+s(2t-s)]^{1-D/2}}{2t+u}.\end{aligned}\tag{C.8}$$

First, we re-scale the integration variables so that the structure of possible divergences becomes manifest: Equation (C.7) becomes

$$4C_2(G)\dim(G)g_0^2 \times \frac{2}{(4\pi)^D(D-2)} \times \int_0^1 ds \int_0^\infty du\, t^{3-D} s^{1-D/2} \frac{[(2-s)(1+u)+(1+u)s+(2-s)]^{1-D/2}}{2+u} \tag{C.9}$$

and while Eq. (C.8) becomes

$$2C_2(G)\dim(G)g_0^2 \times \frac{2}{(4\pi)^D(D-2)} \times \int_0^1 ds \int_0^\infty du\, t^{3-D} s^{2-D/2} \frac{[(2-s)(1+u)+(1+u)s+(2-s)]^{1-D/2}}{2+su}. \tag{C.10}$$

For the first integral, we see that it diverges as $D \to 4$ at $s = 0$. Since the integral for any D cannot be computed analytically, we proceed as follows: First, we "model" the singularity in the integrand $f(s,u)$ by a simpler function $g(s,u)$ such that whose integral can be computed exactly for and D while the integral of the difference $f(s,u) - g(s,u)$ is finite for $D \to 4$. We can choose

$$g(s,u) = t^{3-D} s^{1-D/2} \frac{[2(1+u)+2]^{1-D/2}}{2+u}. \tag{C.11}$$

Then, the integral of $g(s,u)$ for $D = 4 - 2\epsilon$ is

$$\frac{1}{(4\pi)^4}\frac{1}{4t}\left[\frac{1}{\epsilon} + 2 + 2\ln(8\pi t)\right], \tag{C.12}$$

while the finite integration of the difference $f(s,u) - g(s,u)$ in $D = 4$ can be computed as

$$\begin{aligned} &\int_0^1 ds \int_0^\infty du\, t^{3-D} s^{1-D/2} \frac{[(2-s)(1+u)+(1+u)s+(2-s)]^{1-D/2}}{2+u} \\ &- \int_0^1 ds \int_0^\infty du\, t^{3-D} s^{1-D/2} \frac{[2(1+u)+2]^{1-D/2}}{2+u} \\ &\overset{\text{if } D=4}{=} \frac{1}{(4\pi)^4}\frac{1}{4t}(1 - 6\ln 2 + 3\ln 3). \end{aligned} \tag{C.13}$$

In this way, Eq. (C.7) is evaluated as

$$\begin{aligned} &4C_2(G)\dim(G)g_0^2 \times \frac{2}{(4\pi)^D(D-2)} \\ &\times \int_0^1 ds \int_0^\infty du\, t^{3-D} s^{1-D/2} \frac{[(2-s)(1+u)+(1+u)s+(2-s)]^{1-D/2}}{2+u} \\ &= 4C_2(G)\dim(G)g_0^2 \left[\frac{1}{(4\pi)^4}\frac{1}{4t}(1 - 6\ln 2 + 3\ln 3) + \frac{1}{(4\pi)^4}\frac{1}{4t}\left[\frac{1}{\epsilon} + 2 + 2\ln(8\pi t)\right]\right] \\ &= C_2(G)\dim(G)g_0^2 \frac{1}{(4\pi)^4}\frac{1}{t}\left[\frac{1}{\epsilon} + 2\ln(8\pi t) + 3 - 6\ln 2 + 3\ln 3\right]. \end{aligned} \tag{C.14}$$

On the other hand, Eq. (C.8) does not diverge at $s = 0$ for $D \to 4$ and we can set $D = 4$ to yield

$$\int_0^1 ds \int_0^\infty du\, t^{3-D} s^{2-D/2} \frac{[(2-s)(1+u)+(1+u)s+(2-s)]^{1-D/2}}{2+su}$$
$$\stackrel{\text{if } D=4}{=} \frac{1}{2t}(6\ln 2 - 3\ln 3). \tag{C.15}$$

Therefore, Eq. (C.8) is

$$2C_2(G)\dim(G)g_0^2 \times \frac{2}{(4\pi)^D(D-2)}$$
$$\times \int_0^1 ds \int_0^\infty du\, t^{3-D} s^{2-D/2} \frac{[(2-s)(1+u)+(1+u)s+(2-s)]^{1-D/2}}{2+su}$$
$$\stackrel{\text{if } D=4}{=} C_2(G)\dim(G)g_0^2 \frac{1}{(4\pi)^4}\frac{1}{t}(6\ln 2 - 3\ln 3) + \mathcal{O}(\epsilon). \tag{C.16}$$

Summing these two results, the contribution of the diagram E03 is given by

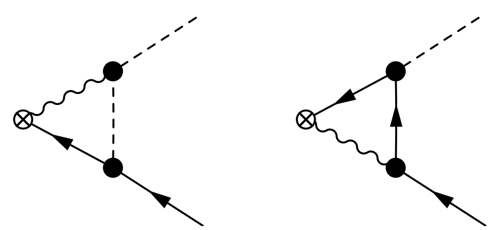
$$C_2(G)\dim(G)g_0^2 \frac{1}{(4\pi)^4}\frac{1}{t}\left[\frac{1}{\epsilon} + 2\ln(8\pi t) + 3\right]. \tag{C.17}$$

Other entries in the table C.1 can be obtained in a similar way,

Appendix D

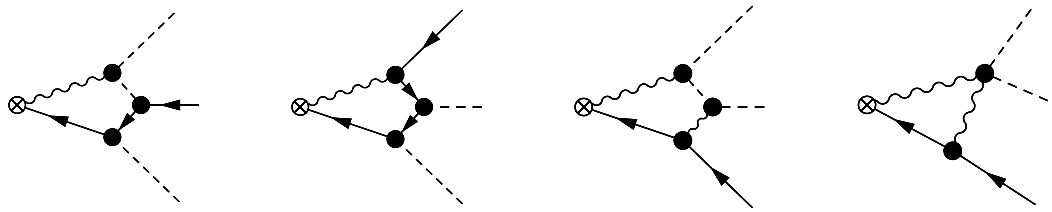
(Flow) Feynman diagrams with scalar fields

In this Appendix, we present the Feynman diagrams which are necessary in the computations in Chap. 4. The Feynman rules for drawing and calculating these diagrams are summarized in Appendix B. The ordinary vertices are denoted by black blobs and the flow vertices are denoted by white blobs. The wavy lines and the straight arrowed lines indicate the flow propagators for the gauge field and the fermion field, respectively. The broken lines represent the flow propagator for the scalar field. The doubled wavy lines, the doubled straight arrowed lines, and the doubled broken lines indicate the heat kernels for the gauge, the fermion, and the scalar fields, respectively. The x-marks represent composite operators under consideration.



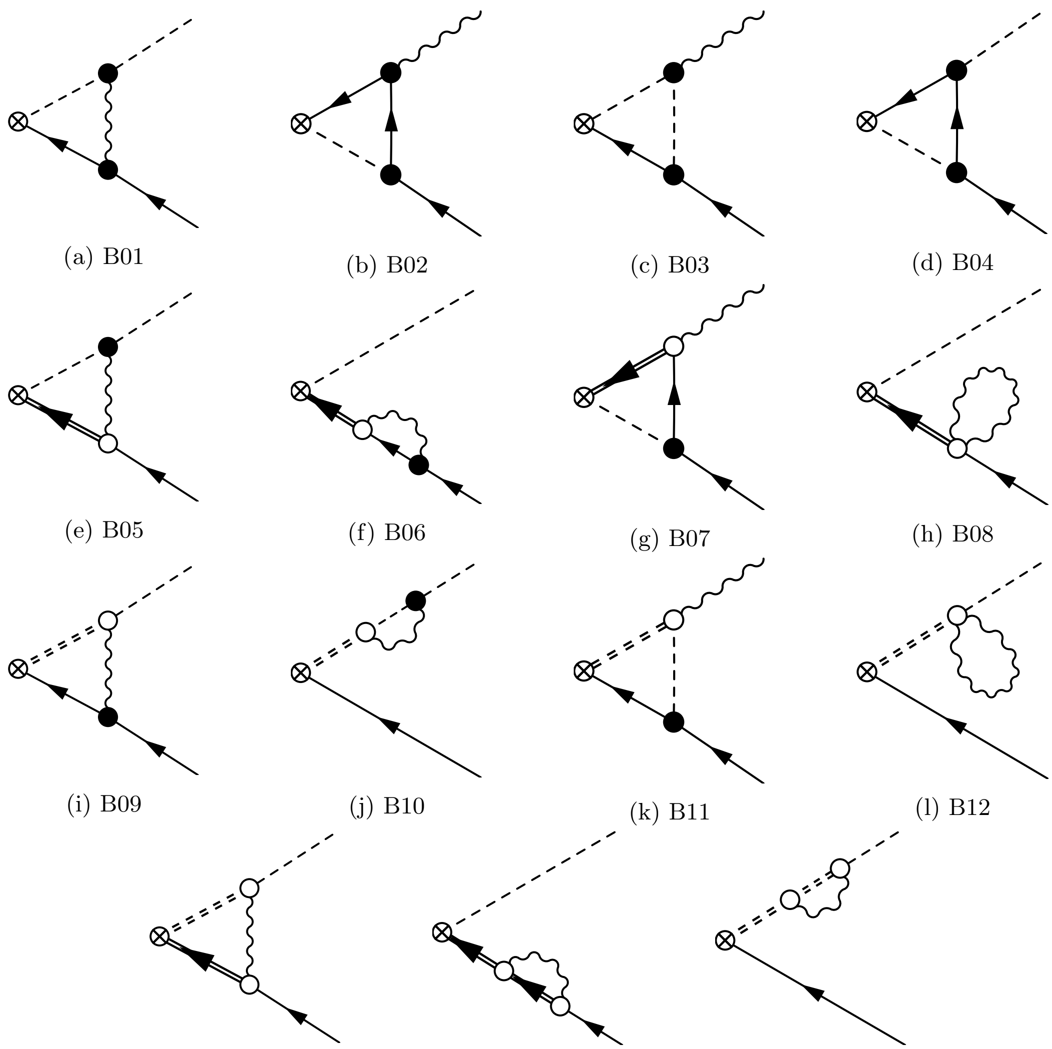
(a) A01 (b) A02

Figure D.1



(a) A03 (b) A04 (c) A05 (d) A06

Figure D.2



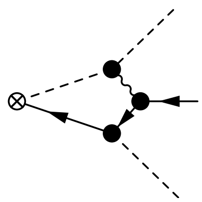
(a) B01 (b) B02 (c) B03 (d) B04

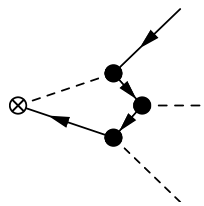
(e) B05 (f) B06 (g) B07 (h) B08

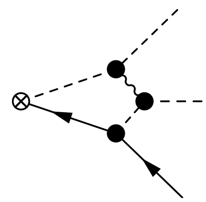
(i) B09 (j) B10 (k) B11 (l) B12

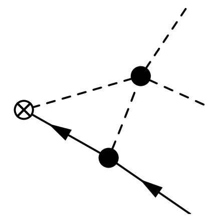
(m) B13 (n) B14 (o) B15

Figure D.3


(a) B16


(b) B17


(c) B18


(d) B19

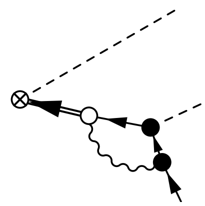
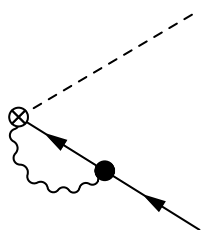
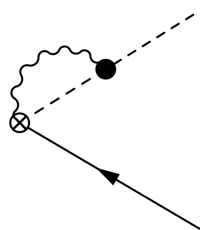
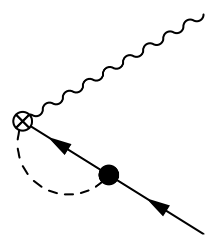
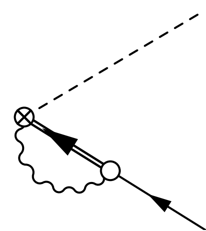

(e) B20

Figure D.4


(a) C01


(b) C02


(c) C03


(d) C04

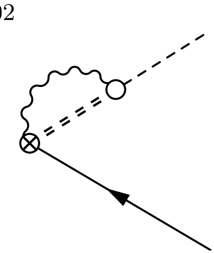
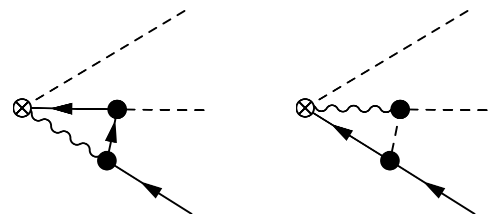
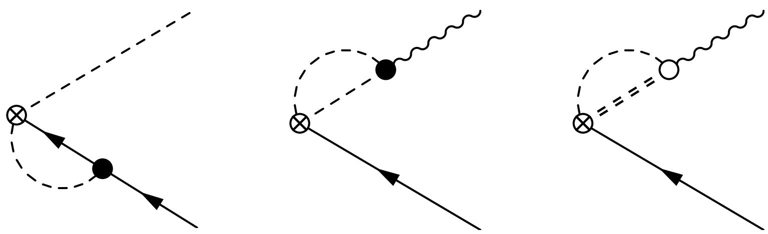

(e) C05

Figure D.5



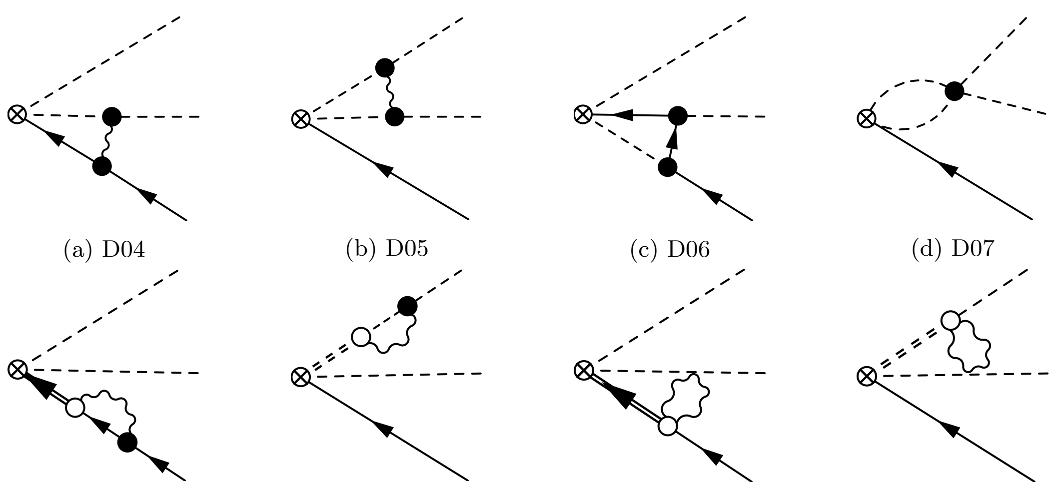
(a) C06 (b) C07

Figure D.6



(a) D01 (b) D02 (c) D03

Figure D.7



(a) D04 (b) D05 (c) D06 (d) D07

(e) D08 (f) D09 (g) D10 (h) D11

Figure D.8

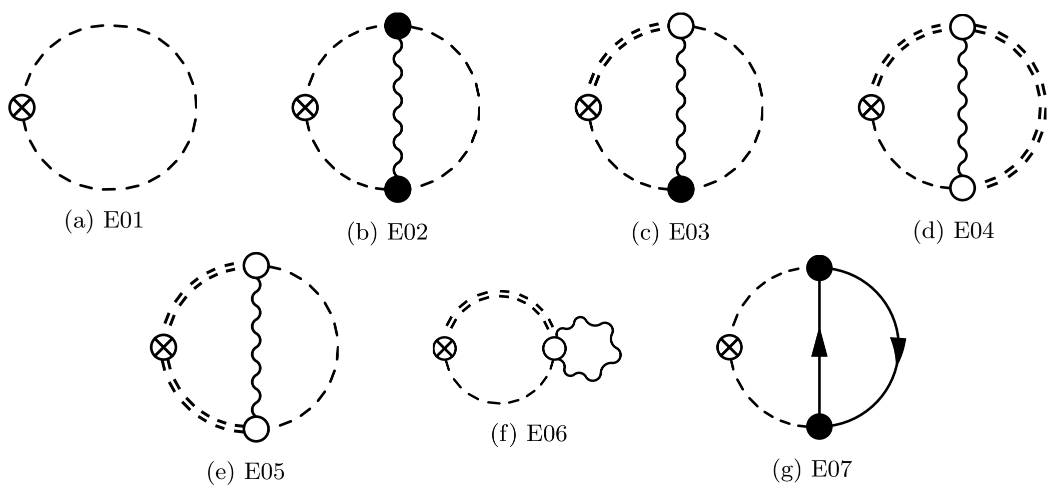


Figure D.9

Appendix E

Integration formulas

In this Appendix, we list some integration formulas that are used in the calculations of the flowed Feynman diagrams. Note our abbreviation, $\int_p \equiv \int \frac{d^D p}{(2\pi)^D}$ for the momentum integration.

$$\int_l e^{-sl^2} = \frac{1}{s^2}\frac{1}{(4\pi)^2}(4\pi s)^{2-D/2}, \tag{E.1}$$

$$\int_l e^{-sl^2}\frac{1}{l^2} = \frac{1}{s}\frac{1}{(4\pi)^2}\frac{2}{D-2}(4\pi s)^{2-D/2}, \tag{E.2}$$

$$\int_l e^{-sl^2}\frac{1}{(l^2)^2} = \frac{1}{(4\pi)^2}\frac{4}{(D-2)(D-4)}(4\pi s)^{2-D/2}, \tag{E.3}$$

$$\int_l e^{-sl^2} l_\mu l_\nu = \frac{1}{2}s\delta_{\mu\nu}\frac{1}{(4\pi)^2}(4\pi s)^{-D/2-2}, \tag{E.4}$$

$$\int_l e^{-sl^2} l_\mu l_\nu l_\rho l_\sigma = \frac{1}{4}(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\frac{1}{(4\pi)^2}(4\pi s)^{-D/2-2}. \tag{E.5}$$

The first one is just the D dimensional Gaussian integration. The following two are obtained by integrating Eq. (E.1) by s. For the last two follow from

$$\int_l f(l^2) l_\mu l_\nu = \frac{1}{D}\int_l f(l^2) l^2 \delta_{\mu\nu}, \tag{E.6}$$

$$\int_l f(l^2) l_\mu l_\nu l_\rho l_\sigma = \frac{1}{D(D+2)}\int_l f(l^2) l^2 (\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}). \tag{E.7}$$

We have also used the following double integration formula in the calculation of the two-loop diagrams in Appendix C

$$\int_k\int_l \frac{e^{-sk^2-ul^2-v(k+l)^2}}{k^2} = \frac{1}{(4\pi)^D(D/2-1)(u+v)}(su+uv+vs)^{1-D/2}. \tag{E.8}$$

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