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Abstract

In this study we discuss the all-pairwise multiple comparison for checking differences among normal variances. Although Imada (2018) determined two kinds of conservative critical values for pairwise comparison for a specified significance level using Bonferroni's inequality and the improved Bonferroni's inequality respectively, we discuss how to determine the critical value for pairwise comparison satisfying a specified significance level exactly. Finally, we give some numerical results regarding critical values and power of the test intended to compare the exact critical value and the conservative critical values.

Key Words and Phrases: Iterated integration, Power of the test, Single step procedure.

1. Introduction

Assume there are independent normal random variables $X_1, X_2, \ldots, X_K$ and X_k is distributed according to normal $N(\mu_k, \sigma_k^2)$ for $k = 1, 2, \ldots, K$. For testing whether $\mu_1 = \mu_2 = \cdots = \mu_K$ or not by the analysis of variance the assumption $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_K^2$ is necessary. The assumption is also necessary for multiple comparison procedures proposed by Dunnett (1955) and Tukey (1953) for checking specific differences among $\mu_1, \mu_2, \ldots, \mu_K$. When the hypothesis $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_K^2$ is rejected, we occasionally want to find the pair σ_i^2, σ_j^2 satisfying $\sigma_i^2 \neq \sigma_j^2$. Imada (2018) discussed the multiple comparison with a control for comparing σ_1^2 with $\sigma_2^2, \sigma_3^2, \ldots, \sigma_K^2$ simultaneously and the all-pairwise multiple comparison for $\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2$ (cf. Dunnett (1955) and Tukey (1953)) . For the multiple comparison with a control Imada (2018) determined the critical value for pairwise comparison satisfying a specified significance level exactly and formulated the power of the test under a specified alternative hypothesis. For the all-pairwise multiple comparison Imada (2018) determined two kinds of conservative critical values for pairwise comparison for a specified significance level using Bonferroni's inequality and the improved Bonferroni's inequality respectively and calculated the power of the test by Monte Carlo simulation. The aim of this study is to determine the critical value for pairwise comparison of the all-pairwise multiple comparison satisfying a specified significance level exactly. We give some numerical results regarding critical values and power of the test intended to compare the exact critical value with the conservative critical values.

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2. All-pairwise multiple comparison for normal variances

We consider the all-pairwise multiple comparison for $\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2$ using a sample x_{k1}, x_{k2}, ..., x_{kn_k} from $N(\mu_k, \sigma_k^2)$ for $k = 1, 2, \ldots, K$. Intended to compare σ_k^2 and σ_l^2 for $1 \leq k < l \leq K$ we set up a null hypothesis and its alternative hypothesis as

$$H_{k,l} : \sigma_k^2 = \sigma_l^2 \quad \text{vs.} \quad H_{k.l}^A : \sigma_k^2 \neq \sigma_l^2$$

and consider the simultaneous test of all $H_{k,l}$s. Letting

$$\nu_k^2 = \frac{\sum_{i=1}^{n_k}(x_{ki} - \bar{x}_k)^2}{n_k - 1}$$

for $k = 1, 2, \ldots, K$, we use the statistic

$$F_{k,l} = \frac{\nu_l^2}{\nu_k^2}$$

for testing $H_{k,l}$. If $n_1, n_2, \ldots, n_K$ are unbalanced, it is preferable to set up appropriate critical values for each $H_{k,l}$. However, we set up common critical values for all $H_{k,l}$s for simplicity. We specify a lower critical value c_1 and an upper critical value c_2 satisfying $0 < c_1 < c_2$. If $F_{k,l} < c_1$ or $c_2 < F_{k,l}$, we reject $H_{k,l}$. Otherwise, we retain $H_{k,l}$. Since

$$F_{k,l} = \frac{\nu_l^2}{\nu_k^2} < c_1 \Leftrightarrow c_1^{-1} < F_{k,l}^{-1} = \frac{\nu_k^2}{\nu_l^2}$$

and

$$c_2 < F_{k,l} = \frac{\nu_l^2}{\nu_k^2} \Leftrightarrow F_{k,l}^{-1} = \frac{\nu_k^2}{\nu_l^2} < c_2^{-1},$$

we restrict c_1, c_2 as

$$c_2 = c_1^{-1} = c > 1.$$

Then, we obtain

$$c^{-1} < F_{k,l} = \frac{\nu_l}{\nu_k} < c \quad \Leftrightarrow \quad c^{-1} < F_{k,l}^{-1} = \frac{\nu_k}{\nu_l} < c \tag{1}$$

for all (k, l)s satisfying $1 \leq k < l \leq K$. The probability that at least one hypothesis among $H_{k,l}$s is rejected is

$$P(\min_{1 \leq k < l \leq K} F_{k,l} < c^{-1} \text{ or } c < \max_{1 \leq k < l \leq K} F_{k,l}).$$

We want to determine c so that

$$P(\min_{1 \leq k < l \leq K} F_{k,l} < c^{-1} \text{ or } c < \max_{1 \leq k < l \leq K} F_{k,l}) = \alpha \tag{2}$$

for a specified significance level α under the assumption that $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_K^2$.

2.1. Conservative critical values

Imada (2018) determined two kinds of conservative critical values using Bonferroni's inequality and the improved Bonferroni's inequality respectively. By Bonferroni's inequality we obtain

$$P(\min_{1\le k<l\le K} F_{k,l} < c^{-1} \text{ or } c < \max_{1\le k<l\le K} F_{k,l}) \le \sum_{1\le k<l\le K} P(F_{k,l} < c^{-1} \text{ or } c < F_{k,l}). \quad (3)$$

Letting $f_{k,l}(x)$ be the probability density function of F-distribution with degrees of freedom $(n_l - 1, n_k - 1)$, we obtain

$$\sum_{1\le k<l\le K} P(F_{k,l} < c^{-1} \text{ or } c < F_{k,l}) = \sum_{1\le k<l\le K} \left\{1 - \int_{c^{-1}}^{c} f_{k,l}(x)dx\right\}.$$

If we determine c so that

$$\sum_{1\le k<l\le K} \left\{1 - \int_{c^{-1}}^{c} f_{k,l}(x)dx\right\} = \alpha,$$

we obtain the conservative critical value by (3).

Next, by the improved Bonferroni's inequality we obtain

$$\begin{aligned} &P(\min_{1\le k<l\le K} F_{k,l} < c^{-1} \text{ or } c < \max_{1\le k<l\le K} F_{k,l}) \\ &\le \sum_{1\le k<l\le K} P(F_{k,l} < c^{-1} \text{ or } c < F_{k,l}) \\ &- \sum_{l=1}^{K-2} \sum_{k=1}^{K-l-1} P(F_{k,k+l} < c^{-1} \text{ or } c < F_{k,k+l}, F_{k+1,k+l+1} < c^{-1} \text{ or } c < F_{k+1,k+l+1}) \\ &- \sum_{l=1}^{K-2} P(F_{K-l,K} < c^{-1} \text{ or } c_2 < F_{K-l,K}, F_{1,l+2} < c^{-1} \text{ or } c < F_{1,l+2}). \end{aligned} \quad (4)$$

Let

$$\lambda_{k_1,k_2} = \frac{n_{k_2} - 1}{n_{k_1} - 1}$$

for each pair (k_1, k_2) chosen from $1, 2, \ldots, K$ and $g_k(x)$ be the probability density function of χ^2-distribution with degrees of freedom $n_k - 1$ for $k = 1, 2, \ldots, K$. Imada (2018) gave the formulation of

$$P(F_{k,k+l} < c^{-1} \text{ or } c < F_{k,k+l}, F_{k+1,k+l+1} < c^{-1} \text{ or } c < F_{k+1,k+l+1})$$

for $k \ge 1, l \ge 1, k + l \le K - 1$ as follows. If $l = 1$,

$$P(F_{k,k+1} < c^{-1} \text{ or } c < F_{k,k+1}, F_{k+1,k+2} < c^{-1} \text{ or } c < F_{k+1,k+2})$$

$$= \int_0^\infty g_{k+1}(x_{k+1}) \left\{1 - \int_{c^{-1}\lambda_{k,k+1}^{-1}x_{k+1}}^{c\lambda_{k,k+1}^{-1}x_{k+1}} g_k(x_k)dx_k\right\} \left\{1 - \int_{c^{-1}\lambda_{k+1,k+2}x_{k+1}}^{c\lambda_{k+1,k+2}x_{k+1}} g_{k+2}(x_{k+2})dx_{k+2}\right\} dx_{k+1}.$$

If $l \ge 2$,

$$P(F_{k,k+l} < c^{-1} \text{ or } c < F_{k,k+l}, F_{k+1,k+l+1} < c^{-1} \text{ or } c < F_{k+1,k+l+1})$$

$$= \left\{1 - \int_{c^{-1}}^{c} f_{k,k+l}(x)dx\right\}\left\{1 - \int_{c^{-1}}^{c} f_{k+1,k+l+1}(x)dx\right\}.$$

Furthermore, Imada (2018) gave the formulation of

$$P(F_{K-l,K} < c^{-1} \text{ or } c < F_{K-l,K}, F_{1,l+2} < c^{-1} \text{ or } c < F_{1,l+2})$$

for $1 \leq l \leq K-2$ as follows. If $l = K-2$,

$$P(F_{2,K} < c^{-1} \text{ or } c < F_{2,K}, F_{1,K} < c^{-1} \text{ or } c < F_{1,K})$$

$$= \int_0^\infty g_K(x_K)\left\{1 - \int_{c^{-1}\lambda_{2,K}^{-1}x_K}^{c\lambda_{2,K}^{-1}x_K} g_2(x_2)dx_2\right\}\left\{1 - \int_{c^{-1}\lambda_{1,K}^{-1}x_K}^{c\lambda_{1,K}^{-1}x_K} g_1(x_1)dx_1\right\}dx_K.$$

If K is even and $l = K/2-1$,

$$P(F_{K/2+1,K} < c^{-1} \text{ or } c < F_{K/2+1,K}, F_{1,K/2+1} < c^{-1} \text{ or } c < F_{1,K/2+1})$$

$$= \int_0^\infty g_{K/2+1}(x_{K/2+1})\left\{1 - \int_{c^{-1}\lambda_{K/2+1,K}x_{K/2+1}}^{c\lambda_{K/2+1,K}x_{K/2+1}} g_k(x_K)dx_K\right\}$$
$$\times\left\{1 - \int_{c^{-1}\lambda_{1,K/2+1}^{-1}x_{K/2+1}}^{c\lambda_{1,K/2+1}^{-1}x_{K/2+1}} g_1(x_1)dx_1\right\}dx_{K/2+1}.$$

In other cases

$$P(F_{K-l,K} < c^{-1} \text{ or } c < F_{K-l,K}, F_{1,l+2} < c^{-1} \text{ or } c < F_{1,l+2})$$
$$= \left\{1 - \int_{c^{-1}}^{c} f_{K-l,K}(x)dx\right\}\left\{1 - \int_{c^{-1}}^{c} f_{1,l+2}(x)dx\right\}.$$

Therefore, we obtain the formulation of the right hand of (4). If we determine c so that the right hand of (4) may be equal to α, we obtain the less conservative critical value compared to the critical value determined by Bonferroni's inequality.

2.2. Determination of the exact critical value

We determine the critical value c satisfying (2) exactly. (2) is equivalent to

$$P(c^{-1} < F_{k,l} < c \text{ for each pair } (k,l) \text{ satisfying } 1 \leq k < l \leq K) = 1 - \alpha.$$

We formulate $P(c^{-1} < F_{k,l} < c$ for each pair (k,l) satisfying $1 \leq k < l \leq K)$. By (1) we obtain

$$c^{-1} < F_{k,l} < c \text{ for each pair } (k,l) \text{ satisfying } 1 \leq k < l \leq K$$

$$\Leftrightarrow \quad c^{-1}\nu_l < \nu_k < c\nu_l \text{ for each pair } (k,l) \text{ chosen from } 1,2,\ldots,K$$

$$\Leftrightarrow \quad c^{-1}\max\{\nu_1,\nu_2,\ldots,\nu_K\} < \nu_k < c\min\{\nu_1,\nu_2,\ldots,\nu_K\} \text{ for } k = 1,2,\ldots,K. \quad (5)$$

The event defined by (5) and

$$\min\{\nu_1,\nu_2,\ldots,\nu_K\} = \nu_1, \quad \max\{\nu_1,\nu_2,\ldots,\nu_K\} = \nu_2$$

is equal to the event defined by

$$\nu_1 \le \nu_k \le \nu_2,\ c^{-1}\nu_2 < \nu_k < c\nu_1 \ \text{ for } k = 1, 2, \ldots, K. \tag{6}$$

(6) is equivalent to

$$\nu_1 \le \nu_2 \le c\nu_1, \quad \nu_1 = \max\{c^{-1}\nu_2, \nu_1\} < \nu_k < \min\{c\nu_1, \nu_2\} = \nu_2 \ \text{ for all } k \ne 1, 2.$$

Therefore, the event (5) is divided into $K(K-1)$ events

$$\nu_{k_1} \le \nu_{k_2} \le c\nu_{k_1},\ \nu_{k_1} \le \nu_l \le \nu_{k_2} \text{ for } l \ne k_1, k_2. \tag{7}$$

Here k_1, k_2 are chosen arbitrarily from $1, 2, \ldots, K$. Letting

$$v_k = \frac{(n_k - 1)\nu_k^2}{\sigma^2} \text{ for } k = 1, 2, \ldots, K$$

where $\sigma^2 = \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_K^2$, v_k is distributed according to χ^2 distribution with degrees of freedom $n_k - 1$. (7) is equivalent to

$$\lambda_{k_1,k_2} v_{k_1} \le v_{k_2} \le c\lambda_{k_1,k_2} v_{k_1},\ \lambda_{k_1,l} v_{k_1} \le v_l \le \lambda_{k_2,l} v_{k_2} \text{ for } l \ne k_1, k_2.$$

Therefore, we obtain

$$\begin{aligned} &P(c^{-1} < F_{k,l} < c \ \text{ for each pair } (k,l) \text{ satisfying } 1 \le k < l \le K) \\ &= \sum_{k_1,k_2} \int_0^\infty \int_{\lambda_{k_1,k_2} x_1}^{c\lambda_{k_1,k_2} x_1} \left\{ \prod_{l \ne k_1,k_2} \int_{\lambda_{k_1,l} x_1}^{\lambda_{k_2,l} x_2} g_l(x) dx \right\} g_{k_2}(x_2) dx_2 g_{k_1}(x_1) dx_1. \end{aligned}$$

If $n_1 = n_2 = \cdots = n_K = n$,

$$\begin{aligned} &P(c^{-1} < F_{k,l} < c \ \text{ for each pair } (k,l) \text{ satisfying } 1 \le k < l \le K) \\ &= K(K-1) \int_0^\infty \int_{x_1}^{cx_1} \left\{ \int_{v_1}^{x_2} g(x) dx \right\}^{K-2} g(x_2) dx_2 g(x_1) dx_1 \end{aligned}$$

where $g(x)$ denotes the probability density function of χ^2 distribution with degrees of freedom $n - 1$.

3. Numerical examples

We discussed how to determine the critical value for pairwise comparison satisfying a specified significance level exactly in Section 2. In this section we compare the exact critical value with the conservative critical values determined by Bonferroni's inequality and the improved Bonferroni's inequality through numerical examples regarding the critical values and the power of the test.

Let $K = 5$. We set up two types of sample sizes

$$\text{Sam.1} : (20, 20, 20, 20, 20),\ \text{Sam.2} : (15, 25, 20, 25, 15).$$

Let $\alpha = 0.05$. Table 1 gives exact critical values and conservative critical values determined by Bonferroni's inequality and the improved Bonferroni's inequality. They are the upper critical values. Here E means exact critical values. B and I-B mean conservative critical values determined by Bonferroni's inequality and the improved Bonferroni's inequality respectively. Table 2 gives Type I error using the conservative critical values given in Table 1. The results of Table 2 were obtained by Monte Carlo simulation with 10,000,000 times of experiments.

Table 1: Critical values

	B	I-B	E
Sam.1	3.862	3.831	3.659
Sam.2	4.163	4.131	3.881

Table 2: Type I error using the conservative critical value of Table 1

	B	I-B
Sam.1	0.0371	0.0388
Sam.2	0.0353	0.0369

Next, we consider the power of the test. We set up four types of $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2)$ as follows.

Case 1. $\sigma_1^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = 1,\ \sigma_2^2 = \gamma,$
Case 2. $\sigma_1^2 = \sigma_4^2 = \sigma_5^2 = 1,\ \sigma_2^2 = \sigma_3^2 = \gamma,$
Case 3. $\sigma_1^2 = \sigma_4^2 = \sigma_5^2 = 1,\ \sigma_2^2 = \gamma,\ \sigma_3^2 = \gamma^2,$
Case 4. $\sigma_1^2 = \sigma_5^2 = 1,\ \sigma_2^2 = \gamma,\ \sigma_3^2 = \gamma^2,\ \sigma_4^2 = \gamma^3.$

Herein $\gamma = 0.4, 0.5$. We focus on the all pairs power defined by Ramsey (1978). In Case 1 the power is the probability that H_{12},H_{23},H_{24},H_{25} are rejected. In Case 2 the power is the probability that H_{12},H_{13},H_{24},H_{25},H_{34},H_{35} are rejected. In Case 3 the power is the probability that H_{12},H_{13},H_{23},H_{24},H_{25},H_{34},H_{35} are rejected. In Case 4 the power is the probability that H_{12},H_{13},H_{14},H_{23},H_{24},H_{25},H_{34},H_{35},H_{45} are rejected. Tables 3 to 6 give the power of Cases 1 to 4. The results of Tables 3 to 6 were obtained by Monte Carlo simulation with 1,000,000 times of experiments. The power when $\gamma = 0.4$ is uniformly higher than that when $\gamma = 0.5$. The power decreases as the number of hypotheses which should be rejected increases.

Table 3: Power in Case 1

γ		0.4			0.5	
	B	I-B	E	B	I-B	E
Sam. 1	0.620	0.627	0.668	0.224	0.229	0.264
Sam. 2	0.559	0.565	0.618	0.212	0.216	0.257

Table 4: Power in Case 2

γ		0.4			0.5	
	B	I-B	E	B	I-B	E
Sam. 1	0.519	0.527	0.576	0.118	0.123	0.151
Sam. 2	0.429	0.437	0.503	0.081	0.085	0.115

Table 5: Power in Case 3

γ	0.4			0.5		
	B	I-B	E	B	I-B	E
Sam. 1	0.534	0.544	0.598	0.066	0.071	0.097
Sam. 2	0.446	0.456	0.534	0.043	0.046	0.073

Table 6: Power in Case 4

γ	0.4			0.5		
	B	I-B	E	B	I-B	E
Sam. 1	0.494	0.505	0.567	0.026	0.029	0.048
Sam. 2	0.394	0.405	0.493	0.011	0.013	0.028

4. Conclusions

In this study we discussed the all-pairwise multiple comparison procedures for checking differences among normal variances. Specifically, we derived the formula for determining the critical value for pairwise comparison satisfying a specified significance level exactly. Then we gave some numerical results regarding critical values and power of the test intended to compare the exact critical value and conservative critical values. Although the test using the exact critical value is more powerful compared to those using the conservative critical values for the single step procedure, we should discuss the multi-step procedures for obtaining higher power. Specifically, we should construct the closed testing procedure referring to Marcus *et al.* (1976) and construct the sequentially rejective step down procedure referring to Shaffer (1986) and Holland and Copenhaver (1987). We should compare them through numerical examples regarding the power of the test.

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